3M1 Examples Paper 4 Solutions

1. $$P = \begin{bmatrix} .75 & .25 \\ .25 & .75 \end{bmatrix} = \begin{bmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{bmatrix}$$

P is symmetric so it shouldn't matter which initial state we consider. Probability of starting in 1 and ending in 2 $= (P^2)_{12}$

$$P^2 = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix}\begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix} = \begin{bmatrix} \frac{5}{8} & \frac{3}{8} \\ 3/8 & 5/8 \end{bmatrix} \Rightarrow (P^2)_{12} = 3/8$$

e-values of P

$$(\lambda - 3/4)^2 = \left(\tfrac{1}{4}\right)^2 \Rightarrow \lambda - 3/4 = \pm\tfrac{1}{4} \quad \lambda = 1 \text{ or } \tfrac{1}{2}$$

$\lambda = 1$

$$\begin{bmatrix} a & b \end{bmatrix}\begin{bmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{bmatrix} = \begin{bmatrix} a & b \end{bmatrix} \Rightarrow \frac{3a}{4} + \frac{b}{4} = a \Rightarrow a = b$$

$\therefore$ e-vector $= \left(\tfrac{1}{2} \quad \tfrac{1}{2}\right)$

$\lambda = \frac{1}{2}$

$$\begin{bmatrix} a & b \end{bmatrix}\begin{bmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{bmatrix} = \tfrac{1}{2}\begin{bmatrix} a & b \end{bmatrix} \Rightarrow \frac{3a}{4} + \frac{b}{4} = \frac{a}{2} \Rightarrow a = -b$$

$\therefore$ e-vector $= \left(\tfrac{1}{2} \quad -\tfrac{1}{2}\right)$

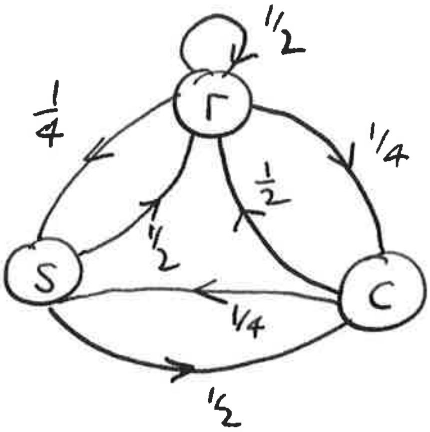
Now $\underline{\lambda}^0 = \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix} + \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$

$$\therefore \underline{\lambda}^n = \underline{\lambda}^0 P^n = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix} P^n + \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \end{bmatrix} P^n$$

$$= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix} + \left(\frac{1}{2}\right)^n \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

$$= \left[\frac{1}{2}\left(1 + \frac{1}{2^n}\right), \frac{1}{2}\left(1 - \frac{1}{2^n}\right)\right]$$

2.



Probabilities not explicitly given worked out by symmetry & $\sum_{\text{possibilities}} () = 1$

$$\therefore P = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

Eigenvalues of P are given by

$$\begin{vmatrix} \frac{1}{2}-\lambda & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & -\lambda & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2}-\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad \left(\tfrac{1}{2}-\lambda\right)\left[\lambda\left(\lambda-\tfrac{1}{2}\right)-\tfrac{1}{8}\right] - \tfrac{1}{4}\left[\tfrac{1}{2}\left(\tfrac{1}{2}-\lambda\right)-\tfrac{1}{8}\right] + \tfrac{1}{4}\left[\tfrac{1}{8}+\tfrac{\lambda}{4}\right] = 0$$

$$\Rightarrow \quad -\lambda^3 + \lambda^2\left[\tfrac{1}{2}+\tfrac{1}{2}\right] + \lambda\left[-\tfrac{1}{4}+\tfrac{1}{8}+\tfrac{1}{8}+\tfrac{1}{16}\right] + \left[-\tfrac{1}{16}-\tfrac{1}{16}+\tfrac{1}{32}+\tfrac{1}{32}\right] = 0$$

ie $-\lambda^3 + \lambda^2 + \frac{\lambda}{16} - \frac{1}{16} = 0$ & $\lambda = 1$ is a root as expected

so that $(\lambda-1)\left(-\lambda^2+\frac{1}{16}\right) = 0$

$$\Rightarrow \lambda = 1 \text{ or } \pm\tfrac{1}{4}$$

For $\lambda = 1$

$$\begin{bmatrix} r & s & c \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & -1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{2} \end{bmatrix} = 0 \quad \Rightarrow \quad \begin{cases} -\frac{r}{2}+\frac{s}{2}+\frac{c}{4} = 0 \\ \frac{r}{4} - s + \frac{c}{4} = 0 \end{cases}$$

Subtracting $\Rightarrow -\frac{3r}{4} + \frac{3s}{2} = 0 \quad \Rightarrow \quad r = 2s$

& then $\frac{c}{4} = \frac{r}{2} - \frac{s}{2} \Rightarrow c = 2s \qquad \therefore$ e-vector $= \left[\frac{2}{5} \ \frac{1}{5} \ \frac{2}{5}\right]$

As $n \to \infty$, there is only one e-value of magnitude unity, so a stationary distribution will be reached.

Assuming close to this after 10 days $\left[\left(\frac{1}{2}\right)^{10} \approx 10^{-3}\right] \Rightarrow$ Probs $= \left[\frac{2}{5} \ \frac{1}{5} \ \frac{2}{5}\right]$

3

Let q_i = Expected no of days till next sunny day given that today = i (where $i = (r \text{ or } s \text{ or } c)$).

i.e. $q_i = E(\text{time to wait till sunny} \mid X_0 = i)$

$\therefore$ $q_r = P(X_1 = r \mid X_0 = r) E(\text{time to wait till sunny} \mid X_0 = r, X_1 = r)$

$+ P(X_1 = s \mid X_0 = r) E(\text{'' '' '' '' ''} \mid X_0 = r, X_1 = s)$

$+ P(X_1 = c \mid X_0 = r) E(\text{'' '' '' '' ''} \mid X_0 = r, X_1 = c)$

$$\Rightarrow q_r = \tfrac{1}{2}(1 + q_r) + \tfrac{1}{4}\cdot 1 + \tfrac{1}{4}(1 + q_c) = 1 + \frac{q_r}{2} + \frac{q_c}{4}$$

Similarly

$$q_s = (1 + q_r) P_{s \to r} + 1 . P_{s \to s} + (1 + q_c) P_{s \to c}$$

$$= (1 + q_r)\tfrac{1}{2} + 0 + (1 + q_c)\tfrac{1}{2} = 1 + \frac{q_r}{2} + \frac{q_c}{2}$$

and

$$q_c = (1 + q_r) P_{c \to r} + 1 . P_{c \to s} + (1 + q_c) P_{c \to c}$$

$$= (1 + q_r)\tfrac{1}{2} + \tfrac{1}{4} + (1 + q_c)\tfrac{1}{4} = 1 + \frac{q_r}{2} + \frac{q_c}{4}$$

i.e.

$$\left.\begin{aligned} \frac{q_r}{2} - \frac{q_c}{4} &= 1 \\ q_s - \frac{q_r}{2} - \frac{q_c}{2} &= 1 \\ \frac{3q_c}{4} - \frac{q_r}{2} &= 1 \end{aligned}\right\}$$

adding 1st & 3rd $\Rightarrow \frac{q_c}{2} = 2 \Rightarrow q_c = 4$

3rd $\Rightarrow q_r = 4$

2nd $\Rightarrow \underline{q_s = 5}$

4. (a) If $a \neq 0$ then it is possible for state 2 to self-loop. This means that any path going through state 2 can remain in state 2 an arbitrary number of times. Furthermore all paths must go through state 2. Consider top paths starting and finishing in state 1:

$$\begin{array}{ccccccccccc} 1 & \rightarrow & 2 & \rightarrow & 3 & \rightarrow & 4 & \rightarrow & 1 & & \\ 1 & \rightarrow & 2 & \rightarrow & 2 & \rightarrow & 3 & \rightarrow & 4 & \rightarrow & 1 \end{array}$$

This means that state 1 is aperiodic as the greatest common denominator for these two paths is 1. The same process can be repeated for each of the states. Since all states are aperiodic then the Markov chain is aperiodic if $a \neq 0$.

(b) The simplest approach is to solve the equation

$$\mathbf{x}^{(\infty)} = \mathbf{x}^{(\infty)}\mathbf{P}$$

This yields the solution

$$\left[\begin{array}{cccc} 0.2222 & 0.4444 & 0.2222 & 0.1111 \end{array}\right]$$

The process is regular ergodic.

(c) When $a = 0$ and $b = 0$ all states are periodic with period 4 (by inspection)

(d) When $a = 0$ and $b = 1$ states 1,2 and 3 are periodic with period 3. State 4 is a transient state.

5. (a) For this process. again assuming that the period Δt is sufficiently small that multiple events can be ignored

$$\begin{aligned}
P_n(t+\Delta t) &= P_n(t)(1-\lambda_n\Delta t-\mu_n\Delta t)+P_{n-1}(t)(\lambda_{n-1}\Delta t) \\
&\quad +P_{n+1}(t)(\mu_{n+1}\Delta t) \\
\frac{P_n(t+\Delta t)-P_n(t)}{\Delta t} &= -P_n(t)(\lambda_n+\mu_n)+P_{n-1}(t)\lambda_{n-1}+P_{n+1}(t)\mu_{n+1}
\end{aligned}$$

Rearranging and setting $\Delta t \to 0$ the following equation is obtained

$$\frac{d\mathbf{x}(t)}{dt}=\mathbf{x}(t)\mathbf{Q}$$

wher $\boldsymbol{Q}$ is the transition rate matrix

$$\mathbf{Q}=\begin{bmatrix} -\lambda_0 & \lambda_0 & 0 & \cdots & \cdots \\ \mu_1 & -(\lambda_1+\mu_1) & +\lambda_1 & 0 & \cdots \\ 0 & \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix}$$

(b) For steady state we require

$$\frac{d\mathbf{x}(t)}{dt}=\mathbf{x}(t)\mathbf{Q}=\mathbf{0}$$

This means that for element i

$$0=-P_i(\lambda_i+\mu_i)+P_{i-1}\lambda_{i-1}+P_{i+1}\mu_{i+1}$$

Starting with $n=0$

$$\begin{aligned}
0 &= -P_0\lambda_0+P_1\mu_1 \\
P_1 &= \frac{\lambda_0}{\mu_1}P_0
\end{aligned}$$

Now take element $n=1$

$$\begin{aligned}
0 &= -P_1(\lambda_1+\mu_1)+P_0\lambda_0+P_2\mu_2 \\
P_2 &= P_1\frac{(\lambda_1+\mu_1)}{\mu_2}-P_0\frac{\lambda_0}{\mu_2} \\
&= \frac{\lambda_0}{\mu_1\mu_2}(\lambda_1+\mu_1)P_0-P_0\frac{\lambda_0}{\mu_2} \\
&= \frac{\lambda_0\lambda_1}{\mu_1\mu_2}P_0
\end{aligned}$$

By recursion possible to show equality. The complete distribution is required to sum to one, hence

$$
\begin{aligned}
1 &= P_0 + \left(\sum_{i=1}^{\infty}\left(\frac{\prod_{j=0}^{i-1}\lambda_i}{\prod_{j=1}^{i}\mu_j}\right)\right)P_0 \\
P_0 &= \frac{1}{1+\left(\sum_{i=1}^{\infty}\left(\frac{\prod_{j=0}^{i-1}\lambda_i}{\prod_{j=1}^{i}\mu_j}\right)\right)}
\end{aligned}
$$

(c) When $\lambda_n = n\lambda$ this means that $\lambda_0 = 0$. This means that all states are transient other that 0. Hence the steady state distribution is $P_0 = 1$, all other elements 0.

6. (a) As $\beta = 0$ use L'Hopital's rule:

$$
\left\{\frac{\alpha(1-\exp(-2\beta t))}{\beta}\right\}_{\beta\to 0} = 2\alpha t
$$

Consider general case and compute derivatives of $g(x,t) = \log(p(x,t))$ where $p(x,t) = \mathcal{N}(x;0,f(t))$. Simple to show that

$$
\begin{aligned}
\frac{\partial g(x,t)}{\partial t} &= \frac{1}{2f(t)}\left(\frac{x^2}{f(t)}-1\right)\frac{\partial f(t)}{\partial t} \\
\frac{\partial g(x,t)}{\partial x} &= -\frac{x}{f(t)} \\
\frac{\partial^2 g(x,t)}{\partial x^2} &= -\frac{1}{f(t)}
\end{aligned}
$$

The differential equation can then be expressed as

$$
\exp(g(x,t))\frac{\partial g(x,t)}{\partial t} = \left(\beta - \beta x\frac{\partial g(x,t)}{\partial x} + \alpha\frac{\partial^2 g(x,t)}{\partial x^2} + \alpha\left(\frac{\partial g(x,t)}{\partial x}\right)^2\right)\exp(g(x,t))
$$

The equations become

$$
p(x,t)\left(\frac{1}{4\alpha t}\left(\frac{x^2}{2\alpha t}-1\right)\right)2\alpha = -p(x,t)\alpha\left(\frac{1}{2\alpha t}-\frac{x^2}{4\alpha^2 t^2}\right)
$$

Clearly equality is satisfied. this is a Wiener distribution.

(b) For both cases as $t \rightarrow$ the variance tends to zero, $p(x,0) = \delta(x)$.

(c) As $t \to \infty$ for $\beta = 0$ tends to $f(t) \to \infty$ whereas when $\beta > 0$,

$$
f(t) = \frac{\alpha}{\beta}
$$

thus yielding a limiting distribution. For $\beta = 0$ this is a Wirner process with only diffusion. For the Ornstein-Uhlenback process the diffusion can be balanced by the drift term, resulting in a limiting distribution.

7. Consider the expected value of an element the denominator

$$\int \frac{p^\star(x)}{q^\star(x)} q(x) dx = \int \frac{p^\star(x)}{Z_q} dx = \frac{Z_p}{Z_q}$$

Now consider the numerator

$$\int f(x) \frac{p^\star(x)}{q^\star(x)} q(x) dx = \int f(x) \frac{p^\star(x)}{Z_q} dx = V \frac{Z_p}{Z_q}$$

So dividing through yields the required integral.

The underlying assumption here is that $q(x)$ is non-zero for all non-zero values of $p(x)$, otherwise in the limit the correct value of V will not be obtained.

Note though it is not possible to compute Z_q it is possible to draw samples from $q(x)$ as the normalisation term is requiured to draw samples.

M.J.F. Gales
Lent 2017
T.P. Hynes
Lent 2014