Module 3F1 – Signals and Systems
Examples Paper 3F1/2
SOLUTIONS

9. (a) The z-transform reads $H(z) = 1 + 2z^{-1} + z^{-2}$ thus,

$$H(e^{j\theta}) = 1 + 2e^{-j\theta} + e^{-j2\theta} = e^{-j\theta}(e^{j\theta} + 2 + e^{-j\theta}) = e^{-j\theta}(2 + 2\cos(\theta)) \ .$$

The term $e^{-j\theta}$ is linear phase, corresponding to a delay of one sample. Lowpass.

(b) From the z transform $H(z) = -1 + 2z^{-1} - z^{-2}$ we have

$$H(e^{j\theta}) = -1+2e^{-j\theta}-e^{-j2\theta} = e^{-j\theta}(-e^{j\theta}+2-e^{-j\theta}) = H(e^{j\theta}) = e^{-j\theta}(2-2\cos(\theta)) \ .$$

Highpass.

(c) One has $H(z) = -1 + 2z^{-2} - z^{-4}$ thus

$$H(e^{j\theta}) = -1 + 2e^{-j2\theta} - e^{-j4\omega} = e^{-j2\theta}(-e^{j2\theta} + 2 - e^{-j2\theta}) = e^{-j2\theta}(2 - 2\cos(2\theta)) \ .$$

The filter is a Bandpass.

(d) From the z-transform $H(z) = 1 + 2z^{-1} + 2z^{-2} + z^{-3}$ thus

$$\begin{aligned} H(e^{j\theta}) &= 1 + 2e^{-j\theta} + 2e^{-j2\theta} + e^{-j3\theta} \\ &= e^{-j3\theta/2}(e^{j3\theta/2} + 2e^{j\theta/2} + 2e^{-j\theta/2} + e^{-j3\theta/2}) \\ &= e^{-j3\theta/2}(2\cos(3\theta/2) + 4\cos(\theta/2)) \ . \end{aligned}$$

Lowpass filter.

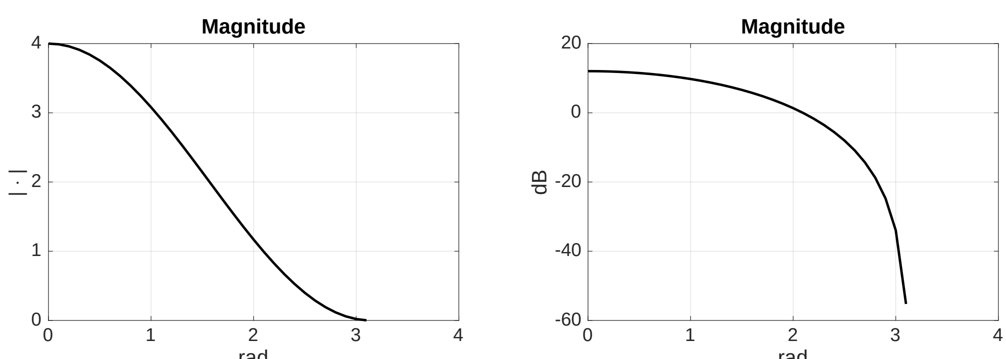


Figure 1: Question 6.a

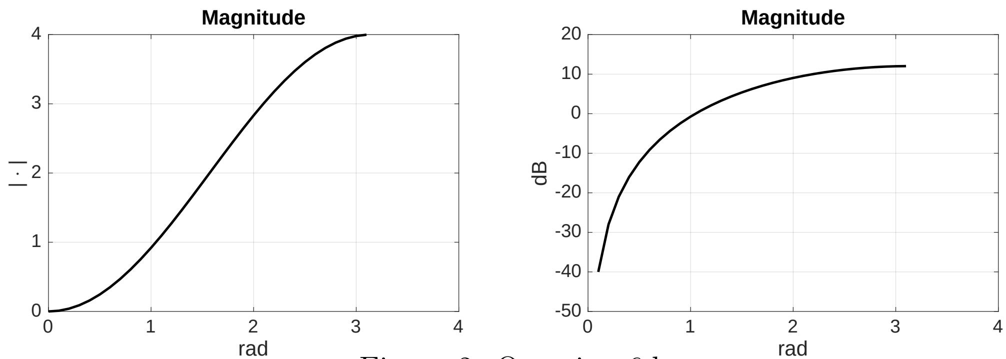


Figure 2: Question 6.b

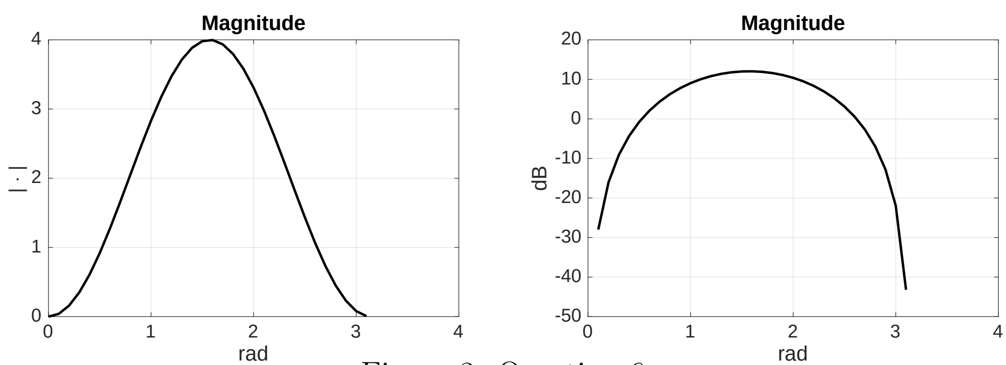


Figure 3: Question 6.c

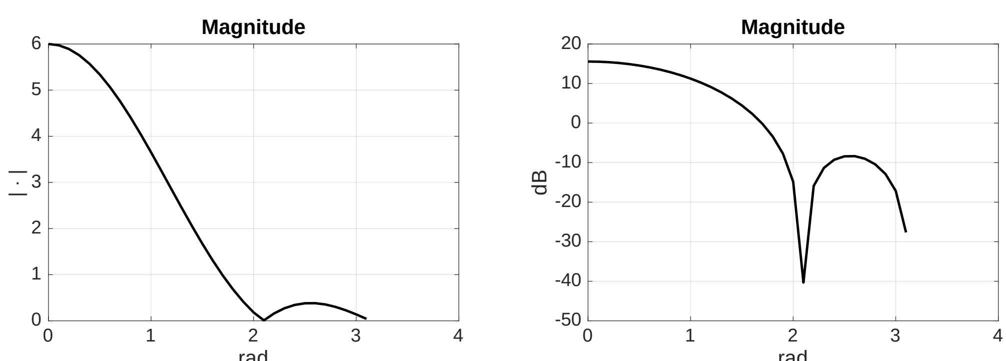


Figure 4: Question 6.d

10. The transfer function of the filter reads

$$H(z) = \frac{-a + z^{-1}}{1 - az^{-1}}.$$

Thus

$$|H(e^{j\theta})| = \left| \frac{-a + e^{-j\theta}}{1 - ae^{-j\theta}} \right| = \left(\frac{a^2 - 2a\cos(\theta) + 1}{a^2 - 2a\cos(\theta) + 1} \right)^{\frac{1}{2}} = 1$$

for all $\theta \in [-\pi, \pi]$ (all frequencies).

11. (a) The impulse response is obtained by antitransform

$$\begin{aligned} h_d(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\theta}) e^{j\theta n} d\theta \\ &= \frac{1}{2\pi} \left(\int_{-0.5\pi}^{-0.4\pi} e^{j\theta n} d\theta + \int_{0.4\pi}^{0.5\pi} e^{j\theta n} d\theta \right) \\ &= \frac{1}{2\pi} \left(\left[\frac{e^{j\theta n}}{jn} \right]_{-0.5\pi}^{-0.4\pi} + \left[\frac{e^{j\theta n}}{jn} \right]_{0.4\pi}^{0.5\pi} \right) \\ &= \frac{1}{2\pi} \left(\frac{e^{-j0.4\pi n}}{jn} - \frac{e^{-j0.5\pi n}}{jn} + \frac{e^{j0.5\pi n}}{jn} - \frac{e^{j0.4\pi n}}{jn} \right) \\ &= \frac{1}{\pi n} (\sin(0.5\pi n) - \sin(0.4\pi n)). \end{aligned}$$

(b) One considers

$$h_s(n) = h_d(n) w(n)$$

for $-100 \leq n \leq 100$. Then

$$h(n) = h_s(n - 100)$$

will be the impulse response of the filter for $0 \leq n \leq 200$

(c) Sharp discontinuity of the rectangular window results in side-lobe interference independent of the filter's order and shape, windows with no abrupt discontinuity such as Hamming window is used to reduce this effect.

12. (i) Forward difference: $s = \dfrac{z-1}{T}$

$$\Rightarrow H(z) = \frac{a}{\frac{z-1}{T} + a} = \frac{aT}{z - 1 + aT}$$

For stability need $aT < 2$.

(ii) Backward difference: $s = \dfrac{z-1}{zT}$

$$\Rightarrow H(z) = \frac{a}{\frac{z-1}{zT} + a} = \frac{aTz}{(1 + aT)z - 1}$$

Stable for all $aT > 0$.

(iii) Tustin: $s = \dfrac{2}{T}\dfrac{z-1}{z+1}$

$$\Rightarrow H(z) = \frac{a}{\frac{2}{T}\frac{z-1}{z+1} + a} = \frac{aT(z+1)}{z(2+aT) + aT - 2}$$

Pole at $z = \dfrac{2-aT}{2+aT}$. Stable for all $aT > 0$.

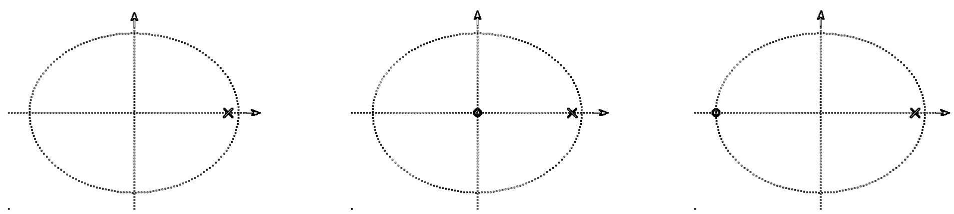
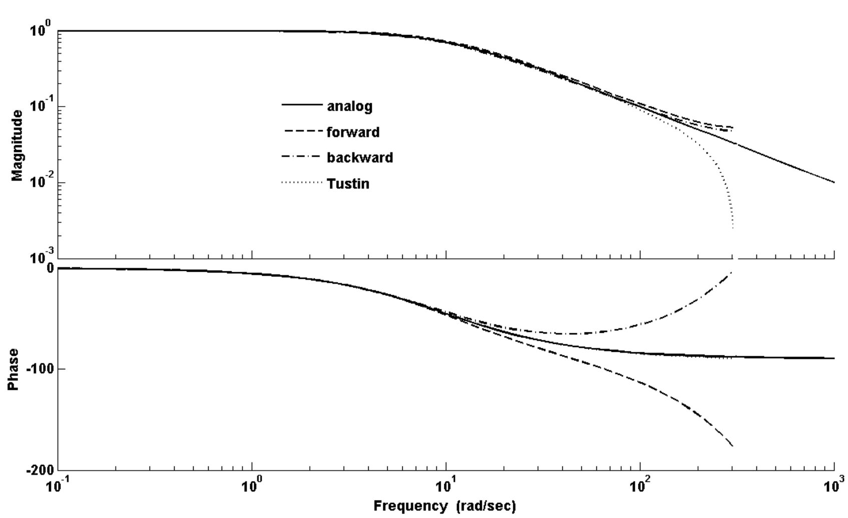


Figure 5: Poles/zeros placement for forward, backward and Tustin transforms.

Comments: As expected from the nature of the s-plane to z-plane mappings, the backward difference and Tustin transformations give stable filters while the forward difference approximation can be unstable. The digital filters only make good sense as approximations of the analog low-pass filter if the sampling frequency ($\frac{1}{T}$ Hz)is much greater than $3dB$ frequency of the filter ($a/2\pi$ Hz) *i.e.* if $aT \ll 2\pi$. In such a case, each approximation shows reasonable fidelity with the desired magnitude behaviour up to at least $\frac{1}{3T}$ Hz. The Tustin approximation gives the closest match to the phase with the added advantage that the magnitude is zero at $\frac{1}{T}$ Hz. The plot below shows the analog filter and the three approximations for $T = 0.01$ sec and $a = 10$.



13. The normalized cutoff frequencies are given by

$$\begin{aligned}\omega_1 &= 6.5084\frac{2\pi}{44.1} \to \Omega_1 = \tan(\omega_1/2) = 0.5\\ \omega_2 &= 7.5861\frac{2\pi}{44.1} \to \Omega_2 = \tan(\omega_2/2) = 0.6.\end{aligned}$$

Thus $\Omega_1\Omega_2 = 0.3$ and $\Omega_2 - \Omega_1 = 0.1$.
Thus we have

$$H(s) = \frac{1}{s+1}.$$

Using the lowpass to bandpass analogue transform

$$s \to \frac{s^2 + \Omega_1\Omega_2}{s(\Omega_2 - \Omega_1)},$$

one obtains

$$H'(s) = \frac{(\Omega_2 - \Omega_1)s}{s^2 + (\Omega_2 - \Omega_1)s + \Omega_1\Omega_2}.$$

$$H'(s) = \frac{0.1s}{s^2 + 0.1s + 0.3}.$$

Using the bilinear transform

$$s \to \frac{1 - z^{-1}}{1 + z^{-1}},$$

one gets (remember to make the leading term of the denominator equal 1)

$$H(z) = H'(s)\big|_{s=\frac{1-z^{-1}}{1+z^{-1}}} = \frac{0.1}{1.4}\frac{1 - z^{-2}}{1 - z^{-1} + 0.857z^{-2}}.$$

The zeros of this filter are ± 1 whereas the poles are given by $0.9257e^{\pm j1.0}$. Hence the resonant frequency is given approximately by

$$44.1 \times 1.0/(2\pi) = 7\text{kH}z.$$

14. Let the poles of the analogue Butterworth lowpass filter be denoted as follows: $p_1 = -0.3827 + j0.9239$, $p_1^* = -0.3827 - j0.9239$ and $p_2 = -0.9239 + j0.3827$ and $p_2^* = -0.9239 - j0.3827$. The transfer transfer function of this filter takes the following form:

$$H(s) = \frac{C}{(s - p_1)(s - p_1^*)(s - p_2)(s - p_2^*)}$$

where C is a constant.

Map digital critical frequency to analogue critical frequency

$$\omega_c = 1 \cdot \frac{2\pi}{8} \rightarrow \Omega_c = \tan\left(\omega_c/2\right) = 0.4142$$

Using lowpass to lowpass mapping to change cutoff frequency

$$s \rightarrow \frac{s}{\Omega_c},$$

$$H'\left(s\right) = \frac{C}{(\frac{s}{\Omega_c} - p_1)(\frac{s}{\Omega_c} - p_1^*)(\frac{s}{\Omega_c} - p_2)(\frac{s}{\Omega_c} - p_2^*)}$$

$$H'\left(s\right) = \frac{C'}{(s-a)(s-a^*)(s-b)(s-b^*)}$$

where $a = p_1\Omega_c$, $a^* = p_1^*\Omega_c$, $b = p_2\Omega_c$ and $b^* = p_2^*\Omega_c$ and hence the analogue filter poles are mapped to $0.4142(-0.3827 \pm j0.9239) = -0.1585 \pm j0.3827$ and $0.4142(-0.9239 \pm j0.3827) = -0.3827 \pm j0.1585$.

To give the filter unit DC gain $H(0) = 1$ one needs to put correct scale factor in numerator of transfer function:

$$H\left(s\right) = \frac{aa^*bb^*}{(s-a)(s-a^*)(s-b)(s-b^*)}$$

Applying the bilinear transform

$$s \rightarrow \frac{1-z^{-1}}{1+z^{-1}}$$

one obtains

$$s-a = \frac{1-z^{-1}}{1+z^{-1}} - a = \frac{1-z^{-1}-a-az^{-1}}{1+z^{-1}} = \frac{(1-a)-(1+a)z^{-1}}{1+z^{-1}} = (1-a)\frac{1-\frac{1+a}{1-a}z^{-1}}{1+z^{-1}}$$

Hence

$$H\left(z\right) = \frac{aa^*bb^*\left(1+z^{-1}\right)^4}{(1-a)(1-a^*)(1-b)(1-b^*)(1-Az^{-1})(1-A^*z^{-1})(1-Bz^{-1})(1-B^*z^{-1})}$$

where $A = \frac{1+a}{1-a} = 0.5565 + j0.5141$.

15. Equivalent discrete-time model: (Use step response matching)

$$\begin{aligned}
\text{Step response of continuous time system} &= L^{-1}\left(\frac{1}{s^2(s+1)}\right) \\
&= L^{-1}\left(\frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1}\right) \\
&= t - 1 + e^{-t}
\end{aligned}$$

Z-transform of sampled step response is,

$$\begin{aligned} Y(z) &= Z\left\{kT - 1 + \left(e^{-T}\right)^k\right\} \\ &= \frac{Tz^{-1}}{(1-z^{-1})^2} - \frac{1}{(1-z^{-1})} + \frac{1}{(1-e^{-T}z^{-1})} \end{aligned}$$

Z-plane transfer function of the discrete-time equivalent system is,

$$\begin{aligned} G(z) &= \left(1-z^{-1}\right)Y(z) = \frac{Tz^{-1}}{(1-z^{-1})} - 1 + \frac{1-z^{-1}}{(1-e^{-T}z^{-1})} \\ &= \frac{Tz^{-1}(1-e^{-T}z^{-1}) - (1-z^{-1})(1-e^{-T}z^{-1}) + (1-z^{-1})^2}{(1-z^{-1})(1-e^{-T}z^{-1})} \\ &= \frac{(e^{-T}-1+T)z^{-1} + (1-e^{-T}(1+T))z^{-2}}{(1-z^{-1})(1-e^{-T}z^{-1})} \end{aligned}$$

16. The discrete time system taking $\{u(kT)\}$ to $\{y(kT)\}$ is <u>linear</u> and <u>time-invariant</u>. (it is easy to check these facts). A linear scaling of the input produces the same effect on the output. Adding two inputs together adds the outputs. Shifting $\{u(kT)\}$ to the right by one time period T does the same to $\{y(kT)\}$. Thus, the system has a z-transfer function. To find $H(z)$, we can choose the input at our convenience. The following is a suitable choice (which can be guessed from the term $(z-1)^2/(Tz^2)$ in the answer):

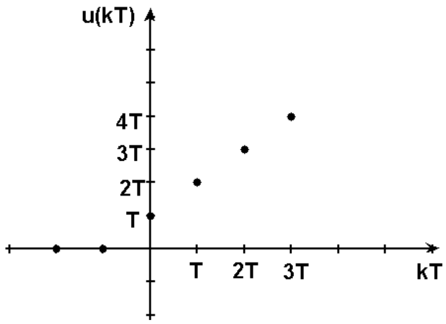
    This gives $u(t) = T + t \Rightarrow U(s) = \dfrac{T}{s} + \dfrac{1}{s^2} = \dfrac{Ts+1}{s^2}$.

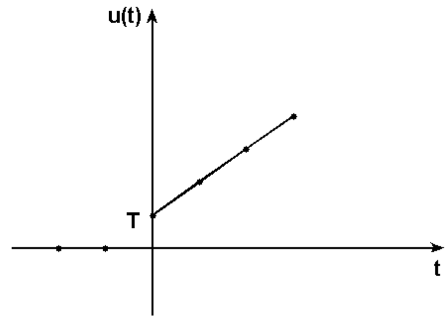
    Hence $\{y(kT)\} = \mathcal{L}^{-1}\left(G(s)\dfrac{Ts+1}{s^2}\right)|_{t=kT}$

    Since $u(kT) = T + kT$ we get

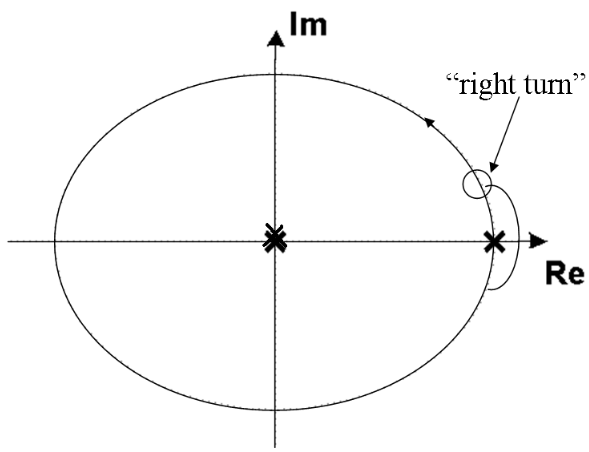
    $$U(z) = \frac{T}{1-z^{-1}} + \frac{Tz^{-1}}{(1-z^{-1})^2} = \frac{T}{(1-z^{-1})^2} = \frac{Tz^2}{(z-1)^2}$$

    The result now follows from: $H(z) = Y(z)/U(z)$.

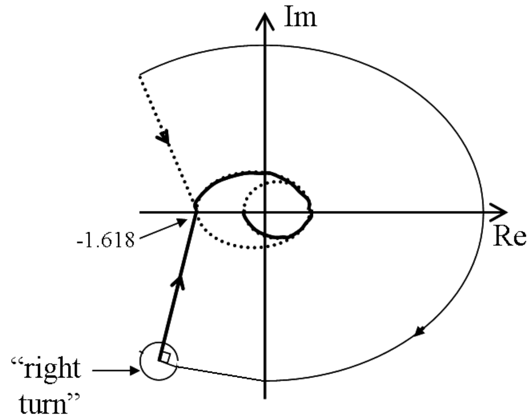




17. (i) $\dfrac{1}{z^2(z-1)}$

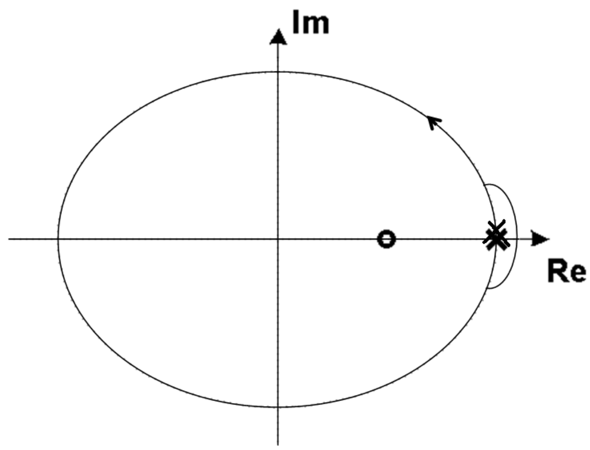


From pole-zero plot, phase is initially $-90°$and decreases to $-540°$. This is diagram 2. There is one pole at $z = 1$, so large semi-circular arc moves through 180°. Direction is obtained from the "right turn"as in indentation.

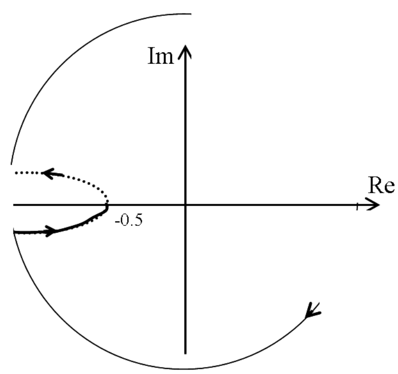


No encirclements of $\dfrac{-1}{k}$ are required for closed loop stability. Hence we require $\dfrac{-1}{k} < -1.618$ which is the same as $0 < k < 0.618$.

(ii) $\dfrac{4z-2}{3(z-1)^2}$



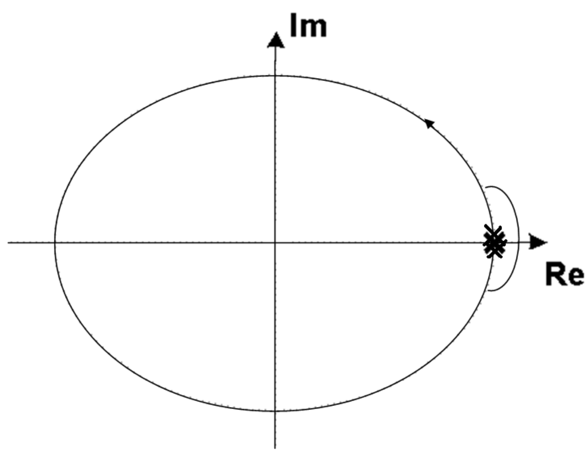
From pole-zero plot, phase is close to $-180°$ for small θ ($z = e^{j\theta}$), and for $\theta = \pi$ ($z = -1$). This is diagram 3. There is a double pole at $z = 1$ so large semi-circular arc moves through $360°$



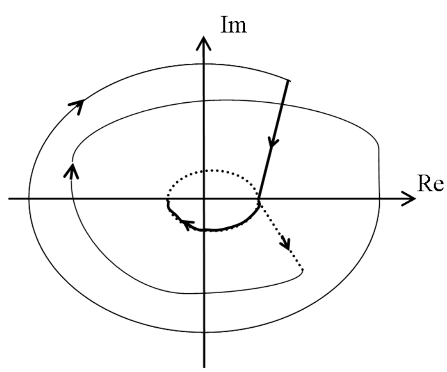
No encirclements of $-1/k$ are required for closed loop stability. Hence we require

$$\frac{-1}{k} < -0.5 \ \textit{i.e.}\ 0 < k < 2$$

(iii) $\dfrac{4}{(z-1)^3}$



From pole-zero plot, phase is initially $-270°$and decreases to $-540°$. This is diagram 1. There is a triple pole at $z = 1$ so large semi-circular arc moves through 540°



No encirclements of $\dfrac{-1}{k}$ are required for closed loop stability. Since all points of real axis are encircled, closed loop is unstable for all k.

18. (a) Using the definition

$$X_{k+N} = \sum_{n=0}^{N-1} x_n e^{-j\frac{2\pi}{N}(k+N)n} = \sum_{n=0}^{N-1} x_n e^{-j\frac{2\pi}{N}kn} e^{-j2\pi n} = \sum_{n=0}^{N-1} x_n e^{-j\frac{2\pi}{N}kn} = X_k$$

(b) From the inverse DFT

$$x_{n+N} = \frac{1}{N}\sum_{k=0}^{N-1} X_k e^{j\frac{2\pi}{N}(n+N)k} = \frac{1}{N}\sum_{k=0}^{N-1} X_k e^{j\frac{2\pi}{N}nk} e^{j2\pi k} = \frac{1}{N}\sum_{k=0}^{N-1} X_k e^{j\frac{2\pi}{N}nk} = x_n$$

(c) Using the definitions (recall that x_n is real)

$$X^*_{N-k} = \left(\sum_{n=0}^{N-1} x_n e^{-j\frac{2\pi}{N}(N-k)n}\right)^* = \sum_{n=0}^{N-1} x_n e^{j\frac{2\pi}{N}(N-k)n} = \sum_{n=0}^{N-1} x_n e^{-j\frac{2\pi}{N}(k-N)n} = X_{k-N} = X_k$$

where the last identity follows by periodicity.

(d)

$$Y_k = \sum_{n=0}^{N-1} y_n e^{-j\frac{2\pi}{N}kn} = \sum_{n=0}^{N-1} X_n e^{-j\frac{2\pi}{N}kn} = \sum_{n=0}^{N-1} X_n e^{j\frac{2\pi}{N}(-k)n} = Nx_{-k} = Nx_{-k+N}$$

where the next to the last identity follows from the definition of inverse DFT. The last identity follows by periodicity.

F. Forni and T. O'Leary, October 2016
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