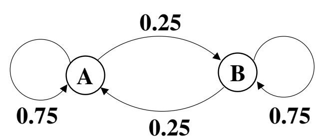
Engineering Tripos Part IIA THIRD YEAR

Module 3M1: MATHEMATICAL METHODS

Examples Paper 2

Straightforward questions are marked †
Tripos standard (but not necessarily Tripos length) questions are marked ∗

1. † A digital system has two states and transition probabilities for one time increment changes are shown in the diagram below.



   (a) Write down the transition matrix for this system, where A is state 1 and B is state 2.

   (b) Find the probability that after 2 time increments the system is in the opposite state to that at the beginning.

   (c) If a large number of runs are performed, with the system starting each time in state A, represented by $\mathbf{x}^{(0)} = [1, 0]$, show that the distribution expected after n steps is given by

$$\mathbf{x}^{(n)} = \left[\frac{1}{2}(1 + 2^{-n}), \frac{1}{2}(1 - 2^{-n}),\right]$$

2. The weather at a particular place is modelled as a Markov Chain with a state space of (rain,sunny, cloudy) with the following properties:

   - if it rains today, the probability that it rains tomorrow is 0.5 and the other two possibilities are equally likely,
   - if it is sunny today, the probability that it is sunny tomorrow is 0 and the other two possibilities are equally likely,
   - if it is cloudy today, the probability of cloudy tomorrow is 0.5 and the other two possibilities are equally likely.

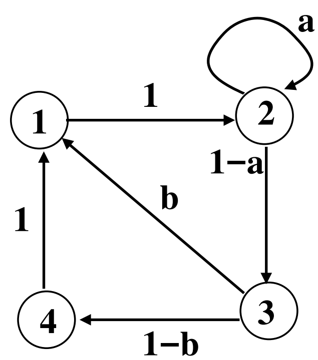
   Denoting rain, sunny, cloudy as states 1,2 and 3 respectively, show that the transition matrix is given by

$$\mathbf{P} = \begin{bmatrix} 0.5 & 0.25 & 0.25 \\ 0.5 & 0.0 & 0.5 \\ 0.25 & 0.25 & 0.5 \end{bmatrix}; \quad \text{where} \quad P(i \to j) = p_{ij}$$

Find the eigenvalues of **P** and hence show that, after a sufficient amount of time, the chain will reach a stationary distribution. Estimate the probability that, in 10 days time, the weather will be rain, sunny or cloudy.

3. Based on the model of the weather described in question 2, if it is sunny today, how many days can one expect to wait until it is sunny again?

4. A Markov process is governed by the state space diagram shown.



   (a) Explain why every state will be aperiodic if $a \neq 0$.

   (b) For the case when $a = b = 0.5$ find the stationary distribution. Is the process regular ergodic in this case?

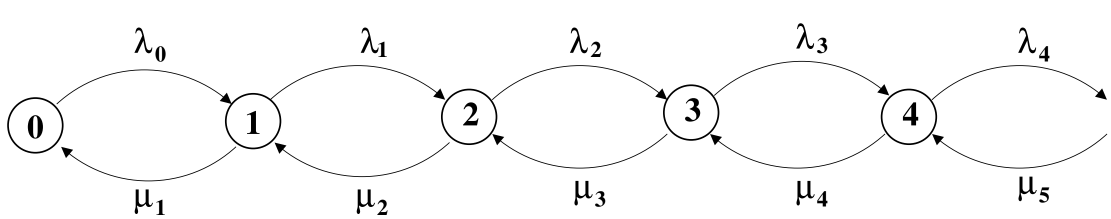
   (c) When $a = 0$ and $b = 0$ find the period of each of the states.

   (d) When $a = 0$ and $b = 1$ find the period of each of the states. If the period does not exist describe the nature of the state.

5. ∗ The number of bacteria is described by a birth-death process.

   - λ_i is the birth rate per unit time when there are i bacteria
   - μ_i is the death rate per unit time when there are i bacteria

   The first few states in the state-space are shown below.



   (a) Derive the transition rate matrix for this system.

(b) The steady state distribution for a birth-death process is given by $[P_0, \dots, P_n, \dots, P_\infty]$. This will occur when

$$\frac{dP_0(t)}{dt} = 0; \quad \frac{dP_n(t)}{dt} = 0; \ \forall n$$

Show that for $i > 0$

$$P_i = \left(\frac{\prod_{j=0}^{i-1} \lambda_i}{\prod_{j=1}^{i} \mu_j} \right) P_0$$

Hence find the steady state distribution.

(c) What happens if $\lambda_n = n\lambda$, where λ is the birth-rate per cell?

6. An Ornstein-Uhlenbeck process is one in which the Fokker-Planck equation for the probability density function $p(x,t)$ is given by

$$\frac{\partial}{\partial t} p(x,t) = \frac{\partial}{\partial x} \left(\beta x p(x,t) \right) + \frac{\partial^2}{\partial x^2} \left(\alpha p(x,t) \right)$$

For a particular initial condition the following solution is obtained

$$p(x,t) = \mathcal{N}(x; 0, f(t))$$

and

$$f(t) = \frac{\alpha(1 - \exp(-2\beta t))}{\beta}$$

(a) When $\beta = 0$ show that

$$f(t) = 2\alpha t$$

and this satisfies the differential equation. What is the process in this case?

(b) What was the initial condition for $\beta = 0$ and $\beta \neq 0$ as $t \to 0$?

(c) Discuss what happens as $t \to \infty$ for the two cases $\beta = 0$ and $\beta \neq 0$. Give a physical interpretation of the difference between the two processes.

7. Importance sampling is to be used for the following integration

$$V = \int_{-\infty}^{\infty} f(x) p(x) dx$$

where $p(x)$ is a valid PDF. It is not possible to draw samples from $p(x)$, so samples $x^{(1)}, \dots, x^{(N)}$ are drawn from a second distribution $q(x)$. Unfortunately for both $p(x)$ and $q(x)$ it is not possible to compute the normalisation terms. Thus

$$p(x) = \frac{1}{Z_p} p^\star(x); \quad q(x) = \frac{1}{Z_q} q^\star(x)$$

Z_p and Z_q cannot be computed (but are required to ensure that both $p(x)$ and $q(x)$ are valid PDFs). Show that the integral can then be approximated by

$$V = \int_{-\infty}^{\infty} f(x)\frac{p(x)}{q(x)}q(x)dx \approx \frac{\sum_{i=1}^{N} f(x^{(i)})w^{(i)}}{\sum_{i=1}^{N} w^{(i)}}$$

where

$$w^{(i)} = \frac{p^{\star}(x^{(i)})}{q^{\star}(x^{(i)})}$$

and $x^{(i)}$ is draw from $q(x)$. What are the requirements for the approximation to converge to V as $N \to \infty$?.

Answers

1. $\begin{bmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{bmatrix}$, 3/8

2. Eigenvalues 1.0, ± 0.25

After 10 days, probabilities of the various states are approximately [0.4, 0.2, 0.4]

3. 5 days.

4. (b) Stationary distribution of the regular ergodic process is

$$\begin{bmatrix} 0.2222 & 0.4444 & 0.2222 & 0.1111 \end{bmatrix}$$

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