Handout 4

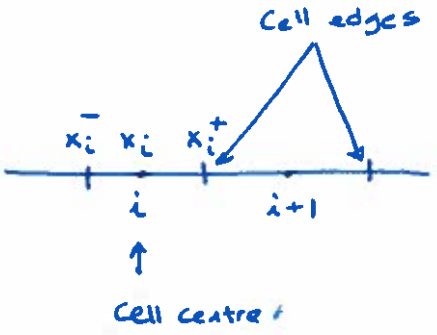
Finite volume scheme

These schemes are based on the idea of a local control volume analysis, applied to the region (the *cell*) between neighbouring nodes. To see how they work let us take the mass conservation equation from the Euler equations described in the previous section

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} = 0$$

Let $\rho u = f$ represent the flux or flow of density. The mass conservation equation becomes

$$\frac{\partial \rho}{\partial t} + \frac{\partial f}{\partial x} = 0 \tag{10}$$



Let us divide the spatial domain x into *cells* with cell centres indexed by i. For a cell i the volume-average of the density can be defined as

$$\bar{\rho}_i(t) = \frac{1}{x_i^+ - x_i^-} \int_{x_i^-}^{x_i^+} \rho(x,t)dx$$

where x_i^+ and x_i^- represent the location of the downstream and upstream *edge*, respectively, of the cell i.

Integrating Eq. (10) over the cell i, we get

$$\int_{x_i^-}^{x_i^+} \frac{\partial \rho}{\partial t} dx + \int_{x_i^-}^{x_i^+} \frac{\partial f}{\partial x} dx = 0$$

Therefore,

$$\Delta x_i \frac{\partial \bar{\rho}}{\partial t} + (f_i^+ - f_i^-) = 0$$

where $\Delta x_i = (x_i^+ - x_i^-)$ and f_i^+ and f_i^- represent cell edge fluxes $f(x_i^+)$ and $f(x_i^-)$ respectively. This equation states the rate of change of mass in the cell (control volume) is equal to the net rate of mass flow out of the cell. If we divide by Δx_i, we get

$$\frac{\partial \bar{\rho}}{\partial t} + \frac{1}{\Delta x_i}(f_i^+ - f_i^-) = 0$$

This represents a finite-volume scheme. If the cells are uniformly spaced and if we take cell averages to find the fluxes, e.g. $f_i^+ = (f_{i+1} + f_i)/2$, then we can show that the finite volume scheme is equivalent to second-order central finite-difference scheme. The finite volume method is, however, much more flexible when the grid becomes non-uniform. For a general three-dimensional case the mass conservation equation can be written as

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{f} = 0$$

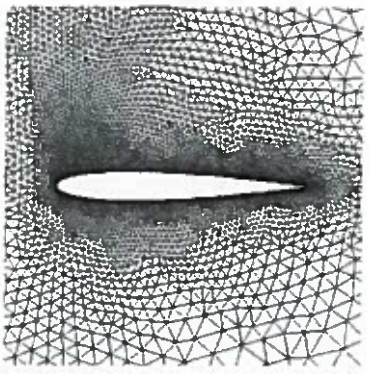
We can divide the domain into finite cells or volumes indexed by i.

Integrating over the volume of one such cell, V_i, we get

$$\int_{V_i} \frac{\partial \rho}{\partial t} dV + \int_{V_i} \nabla \cdot \mathbf{f} dV = 0$$

The first term gives the volume average and applying the divergence theorem on the second term gives

$$V_i \frac{\partial \bar{\rho}}{\partial t} + \oint_{S_i} \mathbf{f} \cdot \mathbf{n} \, dS = 0$$



The fluxes can again be obtained by averaging values from neighbouring cells.

Finite volume schemes work very well on *unstructured* grids. These grids are particularly suited to complex geometries. Boundary conditions are more intuitive in finite-volume schemes. They are also easy to program in a modular way. OpenFoam is a popular finite-volume based CFD software widely used in industry. A major disadvantage of finite-volume method is that it is very difficult to get high-order accuracy schemes.

$$\oint_{S_i} \underline{f} \cdot \underline{\hat{n}} \, ds = \int_1 \underline{f} \cdot \underline{\hat{n}} \, ds + \int_2 \underline{f} \cdot \underline{\hat{n}} \, ds + \int_3 \underline{f} \cdot \underline{\hat{n}} \, ds$$

$$\int_1 \underline{f} \cdot \underline{\hat{n}} \, ds = \int_{Face1} \left(\frac{\underline{f}_{i-1} + \underline{f}_i}{2} \right) \cdot \underline{\hat{n}} \, ds$$

$$\int_2 \underline{f} \cdot \underline{\hat{n}} \, ds = \int_{Face2} \left(\frac{\underline{f}_{i+1} + \underline{f}_i}{2} \right) \cdot \underline{\hat{n}} \, ds$$

$$\int_3 \underline{f} \cdot \underline{\hat{n}} \, ds = \int_{Face3} \left(\frac{\underline{f}_i + \underline{f}_{i+1}}{2} \right) \cdot \underline{\hat{n}} \, ds .$$