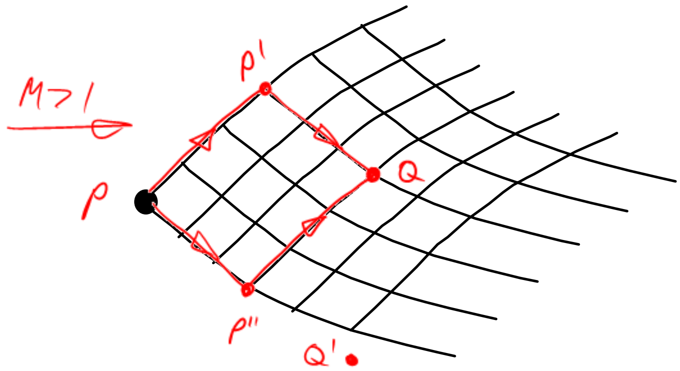
Two important properties of hyperbolic equations dominate their method of solution.

a) **Region of influence of P**

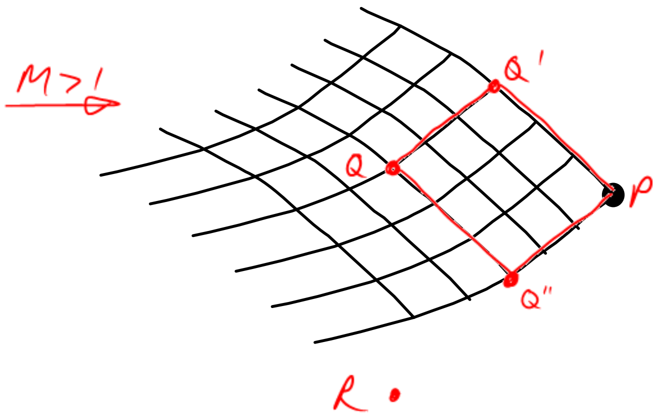


The solution for the flow at P only influences those points <u>downstream</u> of the characteristics running through P.

∴ Flow change at P
↳ change at P' & P''
↳ changes at Q

N.B. Q' is <u>not</u> influenced

b) **Domain of dependence of P**



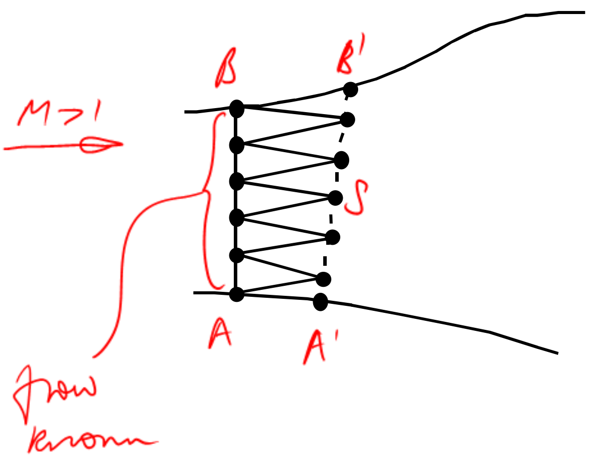
Conversely, only those points upstream of the characteristics through P can effect the solution at P (provided that the flow at P is supersonic)

∴ Q can influence P

R cannot

Numerical solution by the method of characteristics exploits these properties embodied in the characteristic equations to numerically solve for the flow by sweeping downstream.

<u>(i) Lattice method</u>

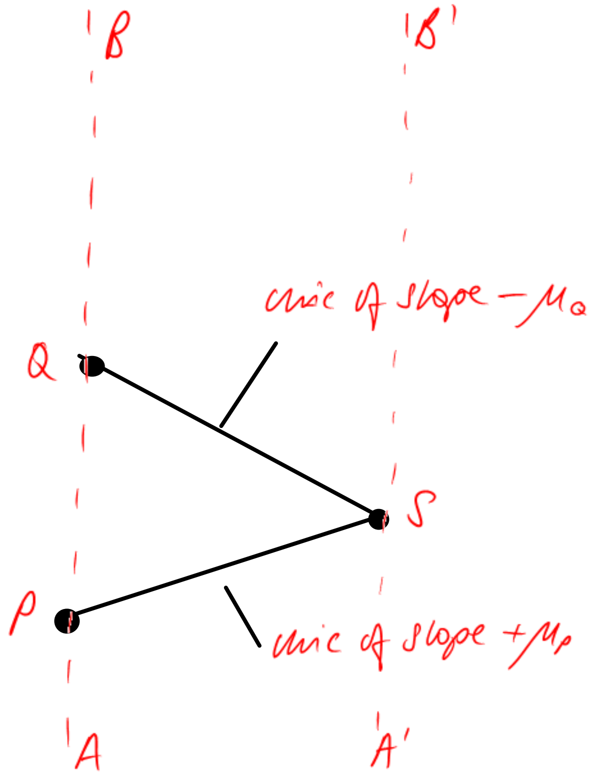


Having divided AB into a number of portions, characteristics can be drawn through all the nodal points. Where characteristics intersect defines the new nodal points at the downstream calculation curve A'B'.

close-up

$$\nu_Q + \theta_Q = \nu_S + \theta_S$$

$$\nu_P - \theta_P = \nu_S - \theta_S$$

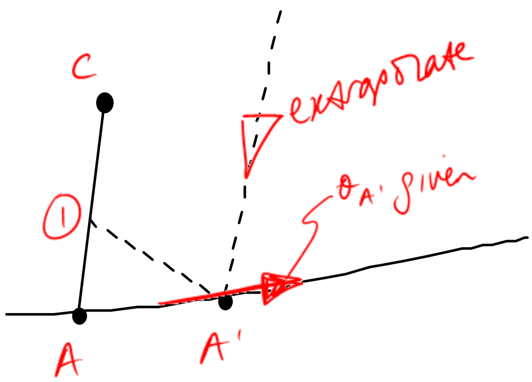


Hence

$$\nu_S = \frac{1}{2}(\nu_Q + \nu_P) + \frac{1}{2}(\theta_Q - \theta_P) \qquad (1.11)$$

$$\theta_S = \frac{1}{2}(\nu_Q - \nu_P) + \frac{1}{2}(\theta_Q + \theta_P) \qquad (1.12)$$

To complete the calculation, at the downstream line A'B', boundary conditions must be applied at A' and B' themselves. There is a certain amount of ambiguity as to how this should be done. What follows is one way.



$$\nu_{A'} + \theta_{A'} = \nu_1 + \theta_1$$

First extrapolate a curve through the known portion of A'B' to find the position of A'. One property is usually known at each of the points A'.

At a solid boundary $\theta_{A'}$ is known.

If the boundary is one of constant pressure (e.g. the edge of a jet) then p, and hence M, are known

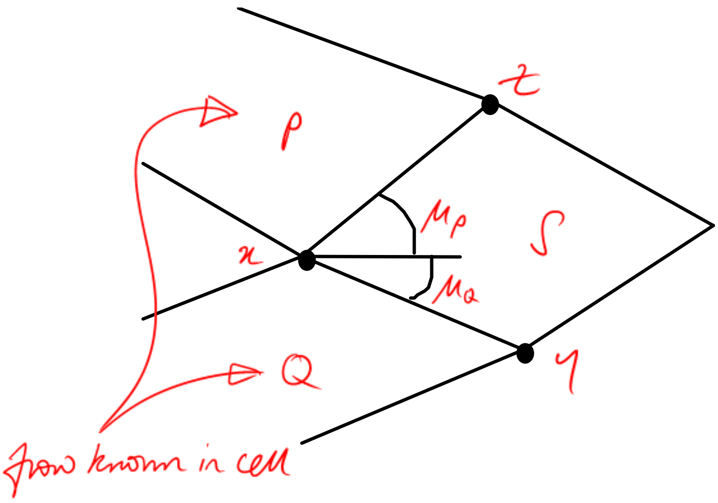
i.e. $\nu_{A'}$ is known

To find the other flow variable at A', it is necessary to estimate the characteristic emanating from somewhere along CA which passes through A'. Guess a point on CA and find where it meets the wall. Repeat till find correct one. The Riemann invariant along this line determines the remaining unknown at A'.

The calculation continues to the next downstream line.

Numerical packages based on characteristics use the lattice method. Since equations (1.11) and (1.12) are *algebraic* equations, they give can be solved *exactly*. The only error is in the local slopes of the characteristics i.e. the position of the lattice points or cells. If nodes get too far apart or cells become too large, one simply interpolates more points on AB and sweeps to A'B' again. There is no danger of numerical instability, false dissipation or dispersion, etc as there is with the solution of partial differential equations by finite difference or finite volume techniques. Phenomenal accuracy can, therefore, be obtained. In fact, solutions obtained by the method of characteristics are often treated as *exact* solutions against which to check those obtained by other methods.

(ii) Field Method

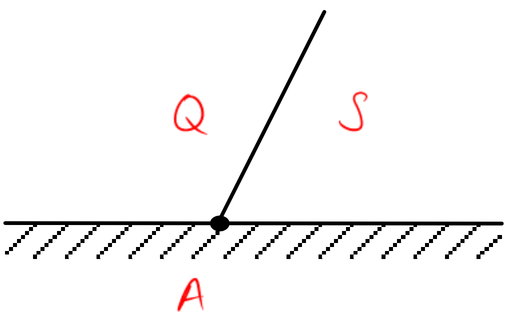


Instead of dealing with the flow properties at *nodes* or lattice points, it is equally plausible to split the flow domain into *cells* bounded by characteristics and to regard flow properties as being representative of conditions in a cell . Suppose flow conditions are known in cell P and cell Q. XY is a line at the Mach angle of the flow in Q (relative to the flow direction in Q), and XZ is at the Mach angle of the flow in P (relative to the flow direction in P).

Again:

$$\left.\begin{aligned} \nu_Q - \theta_Q &= \nu_S - \theta_S \\ \nu_P + \theta_P &= \nu_S + \theta_S \end{aligned}\right\} \Rightarrow \text{solution in } S$$

leading once again to equations (1.11) and (1.12) for properties in S.

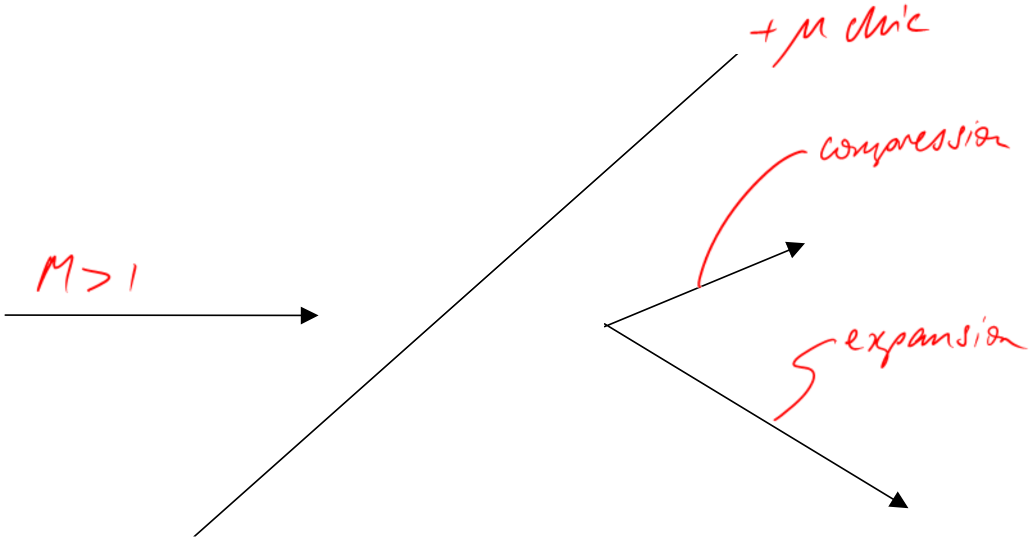


At a boundary, (for example the case of a solid wall),

$$\nu_Q + \theta_Q = \nu_S + \theta_S$$

and θ_S is determined by the slope of the wall

Clearly these two approaches are equivalent. The field method turns out to be much easier to use in hand computations. In the field method, the characteristics themselves are effectively being treated as discrete pressure waves, each of which turns the flow.



Use either:

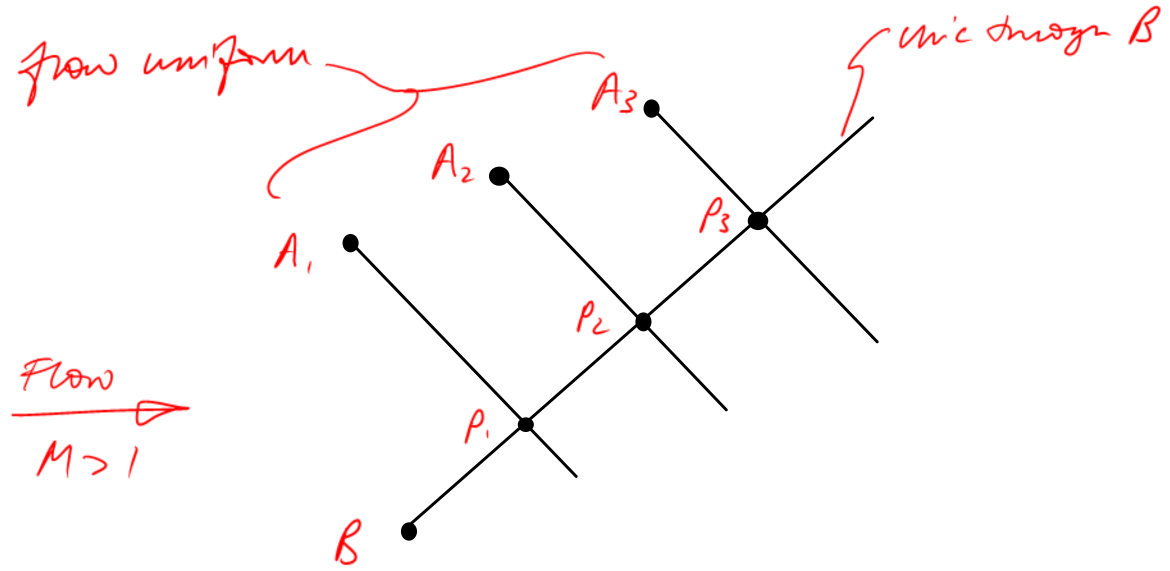
i) strict sign convention

<u>OR</u>

ii) inspection → θ increased
↳ ν decreased
↳ M decreased
↳ compression

angle between characteristic & flow direction decreases for a compression.

SIMPLE EXPANSIONS AND COMPRESSIONS

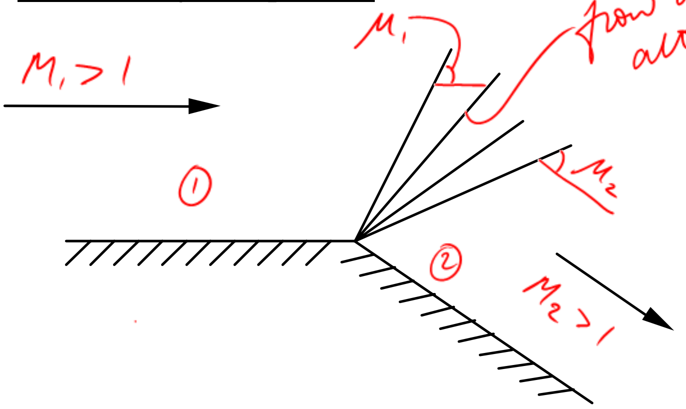


Suppose the points A_1, A_2 and A_3 lie in a region where the flow is uniform (denoted by subscript A). The equations determining the flow at P_1, P_2 and P_3 are each

$$\left.\begin{aligned} \nu_P + \theta_P &= \nu_A + \theta_A \\ \nu_P - \theta_P &= \nu_B - \theta_B \end{aligned}\right\} \text{Same equations for all } P$$

It follows that the solutions at P_1, P_2 and P_3 are *all the same*. $BP_1P_2P_3$ is thus a straight line and all flow properties are uniform along it.

Prandtl-Meyer Expansion



It is conventional not to draw characteristics which emanate from a region of uniform flow (in this case the −μ set), but to draw only those across which there is likely to be change in flow properties.

$\nu_1 + \theta_1 = \nu_2 + \theta_2$

In this case

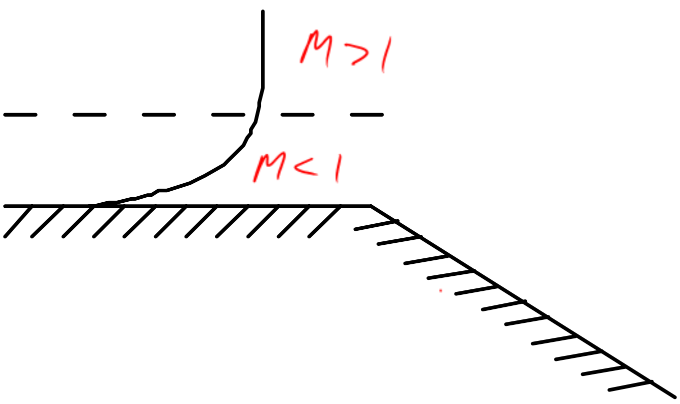
$\nu_1 = \nu(M_1)$, $\theta_1 = 0 \Rightarrow \nu_2 > \nu_1$ (since $\theta_2 < 0$)

The flow thus speeds up , the pressure drops (hence expansion). The first ch'ic of the "fan" is at the Mach angle to the upstream flow and the last is at the Mach angle to the downstream flow.

$M_2 > M_1$, $p_2 < p_1$

Note (i) "expansion" waves spread out.

(ii) static pressure is lower in region 2 than in region 1. i.e. a <u>favourable</u> pressure gradient for the boundary layer. Often the flow <u>will</u> turn a sharp corner. Flow in the boundary layer is mostly <u>sub</u>sonic. Sometimes flow separates, sometimes separates locally then reattaches depending on the state of the boundary layer. Since the flow near the wall is subsonic, pressure variations can feed upstream in this region.



can lead to buffet / shock oscillation ⇒ 4A7