3F1 Signals and Systems

(2) The Z transform

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Michaelmas Term

For continuous time signals, $x(t)$, the Laplace transform is defined as:

$$\bar{x}(s) = \int_{0^+}^{\infty} x(t)e^{-st}\,\mathrm{d}t$$

Analogously, we define a transform for a discrete time signal, $\{x(kT)\}_{k\geq 0}$, as:

$$\sum_{k=0}^{\infty} x(kT)e^{-skT}$$

Then the **Z transform** of the signal $\{x_k\}_{k\geq 0}$ is defined as:

$$\bar{x}(z) = \mathcal{Z}[x_k] = \sum_{k=0}^{\infty} x_k z^{-k}$$

Notation: throughout this course we will use bars (e.g. $\bar{x}(z)$) or capitals (e.g. $X(z)$) for transformed objects.

Example

Consider the discrete time signal defined by $x_k = p^k$ for $k \geq 0$.
The Z transform is:

$$\begin{aligned}
\bar{x}(z) &= \sum_{k=0}^{\infty} p^k z^{-k} \\
&= \sum_{k=0}^{\infty} (pz^{-1})^k \qquad \text{(geometric series)!} \\
&= \frac{1}{1-(pz^{-1})}
\end{aligned}$$

which converges provided $|pz^{-1}| < 1$

Properties of the Z transform

Let $\{x_k\}$, $\{y_k\}$ be discrete time signals whose Z transforms exist.

1. **Linearity** For any scalars α, β:

$$\mathcal{Z}[\alpha\{x_k\} + \beta\{y_k\}] = \alpha\mathcal{Z}[\{x_k\}] + \beta\mathcal{Z}[\{y_k\}]$$

2. **Time delay** Define the time delay operation: $\{x_k\} \mapsto \{x_{k-1}\}$

$$\begin{array}{ccccc} x_0, & x_1, & x_2, & x_3, & \dots \\ \downarrow & \downarrow & \downarrow & \downarrow & \\ x_{-1}, & x_0, & x_1, & x_2, & \dots \end{array}$$

Then:

$$\begin{aligned}
\mathcal{Z}[\{x_{k-1}\}] &= \sum_{k=0}^{\infty} x_{k-1} z^{-k} \\
&= x_{-1} + \sum_{k=1}^{\infty} x_{k-1} z^{-k} \\
&= x_{-1} + \sum_{i=0}^{\infty} x_i z^{-i-1} \qquad (i = k-1) \\
&= x_{-1} + z^{-1} \sum_{i=0}^{\infty} x_i z^{-i} = x_{-1} + z^{-1}\bar{x}(z)
\end{aligned}$$

Thus, for $x_{-1} = 0$,

$$\mathcal{Z}[\{x_{k-1}\}] = z^{-1}\mathcal{Z}[\{x_k\}]$$

z^{-1} is the **time-delay operator**.

3. **Time advance** $\{x_k\} \mapsto \{x_{k+1}\}$

$$\begin{aligned}
\mathcal{Z}[\{x_{k+1}\}] &= \sum_{k=0}^{\infty} x_{k+1} z^{-k} \\
&= \sum_{i=1}^{\infty} x_i z^{-i+1} \qquad (i = k+1) \\
&= -zx_0 + z\sum_{i=0}^{\infty} x_i z^{-i} \\
&= -zx_0 + z\bar{x}(z)
\end{aligned}$$

(x_0 is missing! So "put it back & take away")

z is the **time-advance** operator.

4. **Scaling**

$$\begin{aligned}
\mathcal{Z}[\{r^k x_k\}] &= \sum_{k=0}^{\infty} x_k r^k z^{-k} \\
&= \sum_{k=0}^{\infty} x_k (r^{-1}z)^{-k} \\
&= \bar{x}(r^{-1}z)
\end{aligned}$$

5. **Initial Value Theorem**

$$\begin{aligned}
\lim_{z\to\infty} \bar{x}(z) &= \lim_{z\to\infty} \sum_{k=0}^{\infty} x_k z^{-k} \\
&= \lim_{z\to\infty} \left(x_0 + \frac{x_1}{z} + \frac{x_2}{z^2} + \dots \right) \\
&= x_0
\end{aligned}$$

4. **Convolution**

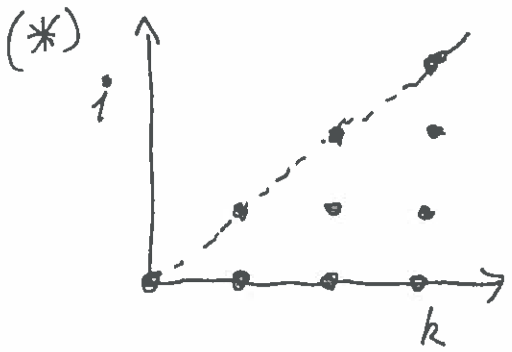
$$\{x_k\} \star \{y_k\} = \sum_{i=0}^{k} x_i y_{k-i} = \sum_{i=0}^{k} x_{k-i} y_i$$

1) Time reverse $\{y_x\}$
2) Shift by k
3) Multiply
4) Sum

$$\begin{array}{cccccc} x_0 & x_1 & x_2 & x_3 & \cdots & x_k \\ | & | & | & | & & | \\ y_k & y_{k-1} & y_{k-2} & y_{k-3} & \cdots & y_0 \end{array}$$

$y_k \cdots y_2 \quad y_1 \quad y_0 \xrightarrow{k} $ spaces

$$\mathcal{Z}[\{x_k\} \star \{y_k\}] = \sum_{k=0}^{\infty} \left(\sum_{i=0}^{k} x_{k-i} y_i \right) z^{-k}$$



$$= \sum_{i=0}^{\infty} \sum_{k=i}^{\infty} x_{k-i} y_i z^{-k} \qquad (*) \text{ Reindex!}$$

$$= \sum_{i=0}^{\infty} y_i z^{-i} \sum_{k=i}^{\infty} x_{k-i} z^{-(k-i)}$$

$$= \bar{x}(z) \bar{y}(z)$$

Inversion of the Z transform

We avoid explicitly computing the inverse of the Z transform (there are methods for this that are beyond the scope of this course) and instead identify **standard transforms** by manipulating expressions in the z-domain.

Often, $\bar{x}(z)$ is **rational**, so we can use **partial fractions** in combination with some **standard transforms**:

$$p^k \leftrightarrow \frac{1}{1 - pz^{-1}} \qquad k \leftrightarrow \frac{z^{-1}}{(1 - z^{-1})^2}$$

$$kp^k \leftrightarrow \frac{pz^{-1}}{(1 - pz^{-1})^2}$$

Tip: work in terms of z^{-1}

Example

$$\bar{x}(z) = \frac{z-3}{z^2(z-1)(z-2)}$$

What is $\{x_k\}$, $k \geq 0$?

$$\bar{x}(z) = \frac{z-3}{z^2(z-1)(z-2)} = \frac{z(1-3z^{-1})}{z^4(1-z^{-1})(1-2z^{-1})}$$

$$= z^{-3}\left(\frac{A}{(1-z^{-1})} + \frac{B}{(1-2z^{-1})}\right)$$

$$= z^{-3}\left(\frac{2}{(1-z^{-1})} + \frac{-1}{(1-2z^{-1})}\right)$$

delay by 3

$2(1,1,1\dots)$

$\{2\}_{k\geq 0}$

$\{-2^k\}_{k\geq 0}$

Therefore

$$\boxed{x_k = \begin{cases} 0 & , k = 0,1,2 \\ 2-2^{k-3} & , k \geq 3 \end{cases}}$$

Partial fraction decomp.

. cover-up method.

$A: z \to 1$

$$A = \frac{1-3(1^{-1})}{1-2(1^{-1})} = 2$$

$B: z \to 2$

$$B = \frac{1-3(2^{-1})}{1-(2^{-1})} = -1$$

Or put $x = z^{-1}$

$$\frac{(1-3x)}{(1-x)(1-2x)} = \frac{A}{(1-x)} + \frac{B}{(1-2x)}$$

$$1-3x = (1-2x)A + (1-x)B$$

$x = 0.5 \Rightarrow B = -1$

$x = 1 \Rightarrow A = 2$

Remark: we can use **polynomial long division** to explicitly write down the first few terms of $\{x_k\}$. In the previous example:

$$\bar{x}(z) = \frac{z-3}{z^2(z-1)(z-2)} = \frac{z-3}{z^2(z^2-3z+2)}$$

$$\begin{array}{r|l} & z^{-1} - 2z^{-3} - 6z^{-4} \\ z^2-3z+2 & z-3 \\ & z-3+2z^{-1} \\ \hline & -2z^{-1} \\ & -2z^{-1}+6z^{-2}-4z^{-3} \\ \hline & -6z^{-2}+4z^{-3} \\ & -6z^{-2} \ \ldots \end{array}$$

$$\bar{x}(z) = z^{-2}(z^{-1} - 2z^{-3} - 6z^{-4} + \ldots)$$

$$= z^{-3} - 2z^{-5} - 6z^{-6} + \ldots$$

$$\therefore \{x_k\} = (0, 0, 0, 1, 0, -2, -6, \ldots)$$

$$\begin{array}{r|l} & -3 \\ 1-z^{-1} & 3z^{-1}-2 \\ & 3z^{-1}-3 \\ \hline & 1 \end{array} \Rightarrow -3 + \frac{1}{1-z^{-1}}$$

remainder

Hint: when working with rational functions it helps when

$$\text{degree of numerator}(z^{-1}) < \text{degree of denominator}(z^{-1})$$

If not, we can split off a constant, e.g.

$$\frac{3z^{-1}-2}{1-z^{-1}} = -3 + \frac{1}{1-z^{-1}} \leftrightarrow (-2, 1, 1, 1, \ldots)$$