Scaling

When faced with an equation like (A.9), which is still not that easy to solve, progress can often be made by looking for "similarity " solutions. This approach is summarised as:

"can we relate solutions corresponding to different data by suitably re-normalising (i.e. re-scaling) the dependent and independent variables.?

If such a relation *is* established, then, given a solution for one set of data (from numerical computation, experiment, etc), we can generate a family of solutions (called "similarity solutions") for a whole host of other sets of data. Sometimes a particular choice of scaling parameters can result in a simpler or "more studied" equation.

The parameters which appear in this problem are thus

U_∞, M_∞, τ, c & the shape function(s) g.

The objective of the hunt for "similarity solutions" is to make substitutions using other scaled non-dimensional variables which reduces this number of parameters.

An obvious choice for scaling x is $\tilde{x} = \dfrac{x}{c}$ which means that the boundary conditions are applied on $0 < \tilde{x} < 1$.

It is not obvious how to scale y, so take $\tilde{y} = \beta \dfrac{y}{c}$ with β still to be determined

The blade boundary condition becomes

$$\frac{\partial \phi}{\partial \tilde{y}} = \frac{c}{\beta} \frac{\partial \phi}{\partial y} = \frac{\tau}{\beta} U_\infty g'(\tilde{x})$$

and clearly the way to scale ϕ is to take $\phi = \dfrac{\tau}{\beta} U_\infty \tilde{\phi}$ so that this boundary condition becomes

$$\frac{\partial \tilde{\phi}}{\partial \tilde{y}} = g'(\tilde{x})$$

With this choice, equation (A.9) becomes

$$\beta^{-2}\left(1-M_\infty^2\right)\frac{\partial^2\tilde{\phi}}{\partial\tilde{x}^2}+\frac{\partial^2\tilde{\phi}}{\partial\tilde{y}^2}=0$$

and the smart move is to take $\beta=\sqrt{1-M_\infty^2}$ (or $\sqrt{M_\infty^2-1}$ if the incoming flow is supersonic).

Recapping, if we make the following substitutions

$$\tilde{\phi}=\frac{\sqrt{1-M_\infty^2}}{\tau U_\infty}\phi\,,\quad \tilde{x}=\frac{x}{c}\quad\text{and}\quad \tilde{y}=\sqrt{1-M_\infty^2}\,\frac{y}{c} \qquad \text{(A.10)}$$

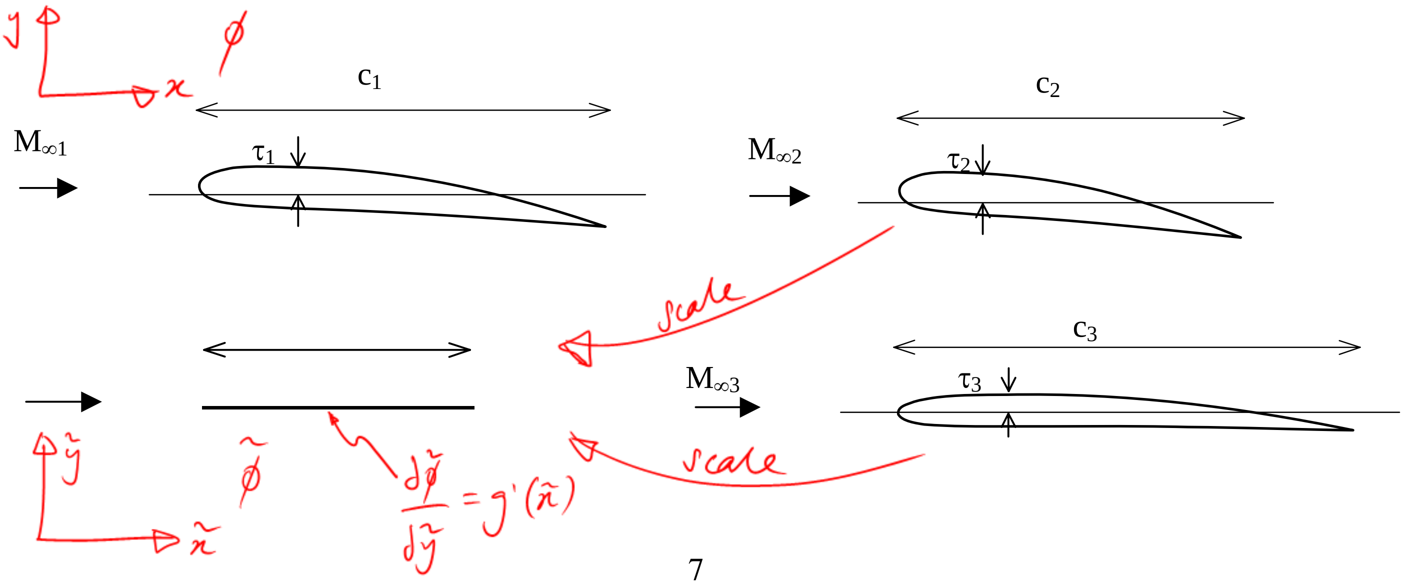
then equation (A.9) becomes

$$\boxed{\begin{gathered}\frac{\partial^2\tilde{\phi}}{\partial\tilde{x}^2}+\frac{\partial^2\tilde{\phi}}{\partial\tilde{y}^2}=0 \qquad \text{(A.11)}\\ \text{with b.c.}\quad \frac{\partial\tilde{\phi}}{\partial\tilde{y}}=g'(\tilde{x})\ \text{on}\ \tilde{y}=0,\ 0<\tilde{x}<1 \qquad \text{(A.12)}\end{gathered}}$$

Thus

the problem for $\tilde{\phi}$ is now completely independent of M_∞, c, τ or U_∞ and hence $\tilde{\phi}$ is independent of these quantities

This means that if we take a family of aerofoils with the same g_U and g_L , but with varying τ and c, (such a family of shapes is said to be *affinely related*), then, for many M_∞'s , if we normalise the flow variables as in (A.10), we will obtain the same $\tilde{\phi}$.



Significance

This commonality of $\tilde{\phi}$, has consequences for real flow quantities. Thus, for example, taking the linearised x-component of the (Euler) momentum equation

$$\frac{\partial p}{\partial x} = -\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) \rightarrow \frac{\partial p}{\partial x} = -\rho_\infty U_\infty \frac{\partial u}{\partial x} = -\rho_\infty U_\infty \frac{\partial^2 \phi}{\partial x^2}$$

$$\Rightarrow \quad p = -\rho_\infty U_\infty \frac{\partial \phi}{\partial x} + f(y)$$

and far upstream $\frac{\partial \phi}{\partial x} = 0$, p is uniform, giving $f(y) = p_\infty$

$$\Rightarrow \quad p - p_\infty = -\rho_\infty U_\infty \frac{\partial \phi}{\partial x}$$

$$\Rightarrow \quad C_p = \frac{p - p_\infty}{\frac{1}{2}\rho_\infty U_\infty^2} = -\frac{2}{U_\infty}\frac{\partial \phi}{\partial x} = -\frac{2}{U_\infty}\frac{\tau U_\infty}{\sqrt{1-M_\infty^2}}\frac{1}{c}\frac{\partial \tilde{\phi}}{\partial \tilde{x}}$$

i.e.

$$C_p = \frac{k\tau}{c\sqrt{1-M_\infty^2}} \qquad \text{(A.12)}$$

where k depends only on g, the non-dimensional shape of the aerofoil, and on $\tilde{x}$, the fraction of chord. Thus, if an aerofoil with thickness and chord τ_1 and c_1, is tested at M_1, and the value of C_p measured, then for other aerofoils in the same family (same value of g) with τ_2 and c_2, tested at M_2,

$$C_{p_2} = \frac{\tau_2}{\tau_1}\frac{c_1}{c_2}\frac{\sqrt{1-M_{\infty_1}^2}}{\sqrt{1-M_{\infty_2}^2}}C_{p_1} \qquad \text{(A.13)}$$

(for the appropriate point on the aerofoil. i.e. at the same value of x/c).

Equation (A.12) indicates how low-speed data can be extended into the compressible range. If case (1) is taken as a low-speed test (i.e. zero Mach number), for an aerofoil having the same thickness-chord ratio and shape (i.e. being geometrically similar), then equation (A.13) becomes

$$\boxed{C_p = \frac{C_{p_0}}{\sqrt{1-M_\infty^2}}}$$

which when integrated over the aerofoil also implies $C_L = \frac{C_{L_0}}{\sqrt{1-M_\infty^2}}$ and $C_m = \frac{C_{m_0}}{\sqrt{1-M_\infty^2}}$

where C_L and C_M are lift and moment coefficients.

The analysis presented is for *subsonic* flow, but clearly, a very similar one, with $\beta = \sqrt{1 - M_\infty^2}$ replaced by $\beta = \sqrt{M_\infty^2 - 1}$ and

$$\frac{\partial^2 \tilde{\phi}}{\partial \tilde{x}^2} - \frac{\partial^2 \tilde{\phi}}{\partial \tilde{y}^2} = 0$$

for supersonic cases and, rather than getting Laplace's equation, we get the wave equation.

We derived these equations for 2D, but clearly they are also valid for 3D, provided the object under study produces relatively small variations in the flow

$$(1 - M_\infty^2)\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

These relations do not hold for *mixed* supersonic and subsonic flow: the whole quality and nature of the flow changes when the subsonic and supersonic regions change size. This is the *transonic* region and is the subject of next year's Aerodynamics course (4A7).

Glauert's Compressibility Correction is often referred to as the "Prandtl-Glauert Similarity Rule" despite the fact that the two men did not collaborate on the work and, unlike Glauert, Prandtl published neither the result nor its derivation. Those interested by this historical quirk are encouraged to read Anderson's *"A History of Aerodynamics"*.

Thus Glauert demonstrated that, by his elegantly simple correction, low-speed (incompressible) test data could be reliably modified and thus used for airplane design at the appropriate Mach Number. His derivation was made public in the Proceedings of the Royal Society in the same year that Kinkead was killed attempting to exceed M=0.4.

The Schneider Trophy races continued. The rules stated that races would continue until one country had won three competitions in succession; the contest would then be judged to be over and the winner declared. A Supermarine *S.6* won the 1929 race, and a Supermarine *S.6B* in 1931: Britain had won the Trophy outright.

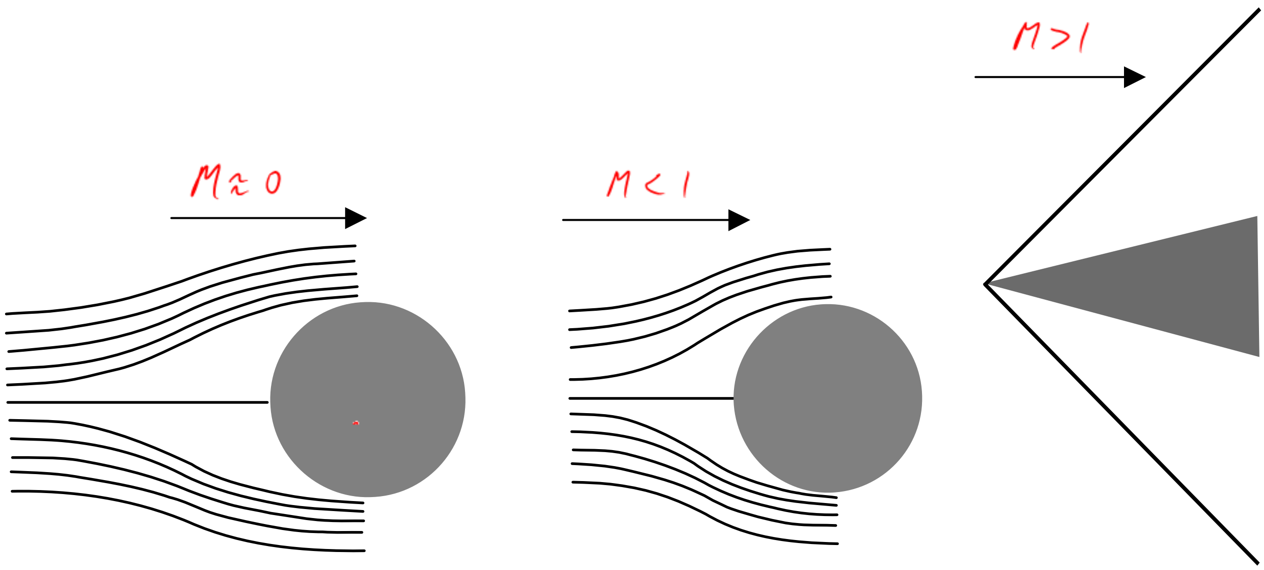
On 29 September 1931, a modified Supermarine *S.6B* set a new absolute airspeed record of 407.5mph – i.e at a Mach number of ~0.53 (and over 40mph faster than a production Mk 1A *Spitfire* the first of which left the factory in June 1938).

INTRODUCTION TO 2D SUPERSONIC FLOW

The early research by Glauert and others into compressibility aided designers as airplane speeds continued to rise though the 1930s and particularly through the years of the Second World War: though the ground speed increases were themselves significant, much greater increases in Mach number occurred due to such speeds being attained at higher (therefore colder) altitudes (due the use of supercharging to increase the power available from aircraft piston engines in lower density air).

It was only a matter of time before humans made a serious attempt to fly supersonically…

Terminology



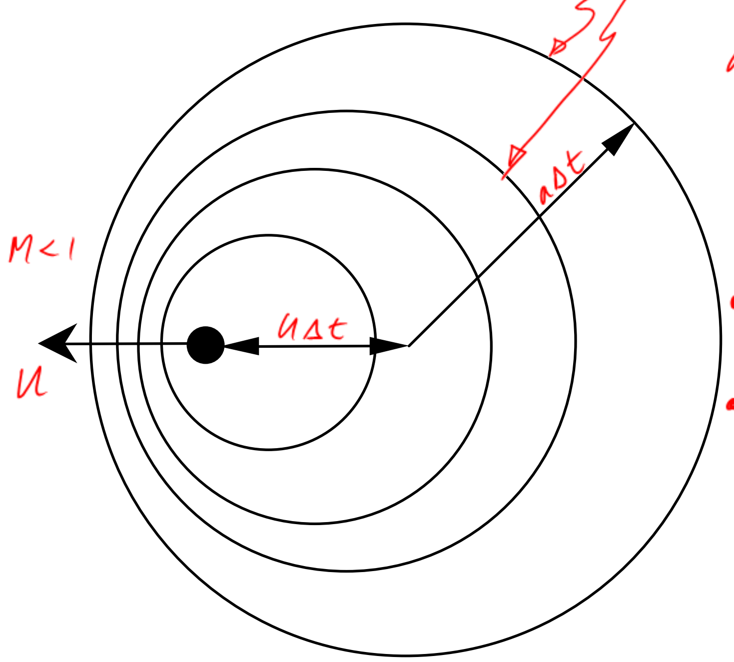
Subsonic flow - information propagates upstream from body giving "advance warning". Streamtube areas adjust so that density changes corresponding to variations in velocity are small - flow is essentially incompressible. As M increases, there is less advance warning and the density changes become greater.

Supersonic flow - information cannot propagate upstream. The flow is uniform until it reaches the body, when it undergoes large changes in pressure, velocity and density.

Moving Sources of Sound

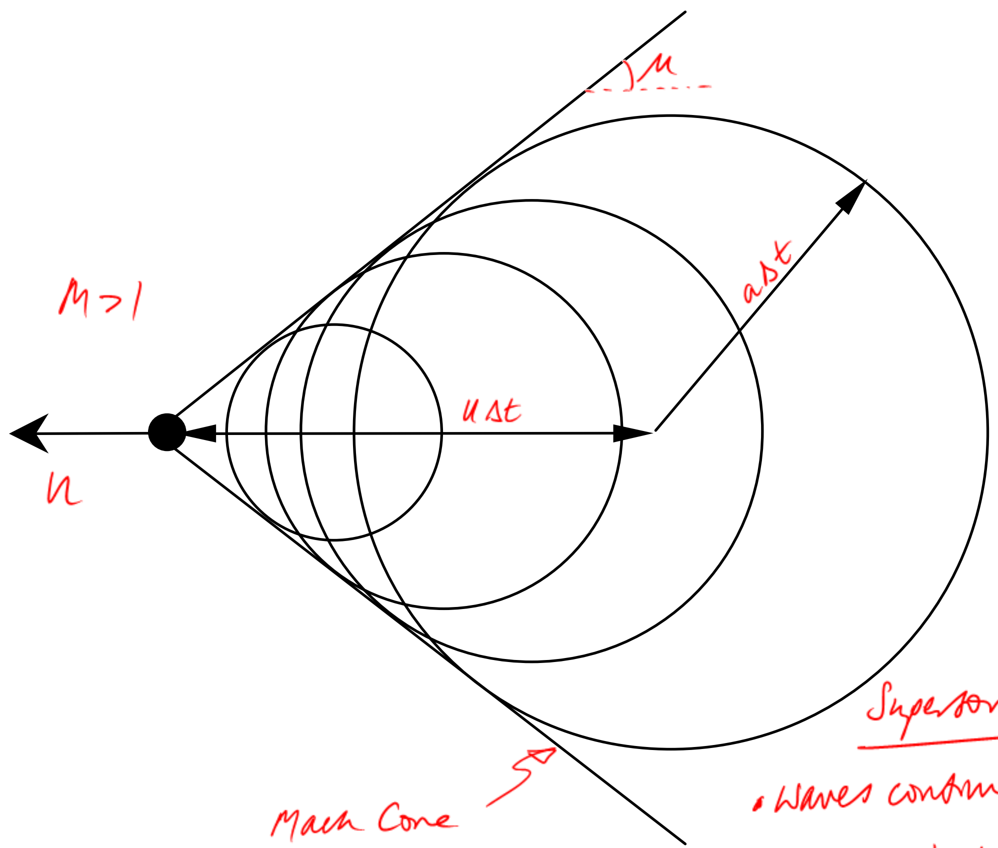
emitted sound waves

a = local speed of sound



Subsonic Source

- waves influence entire domain
- hence can hear source "upstream"



Supersonic Source

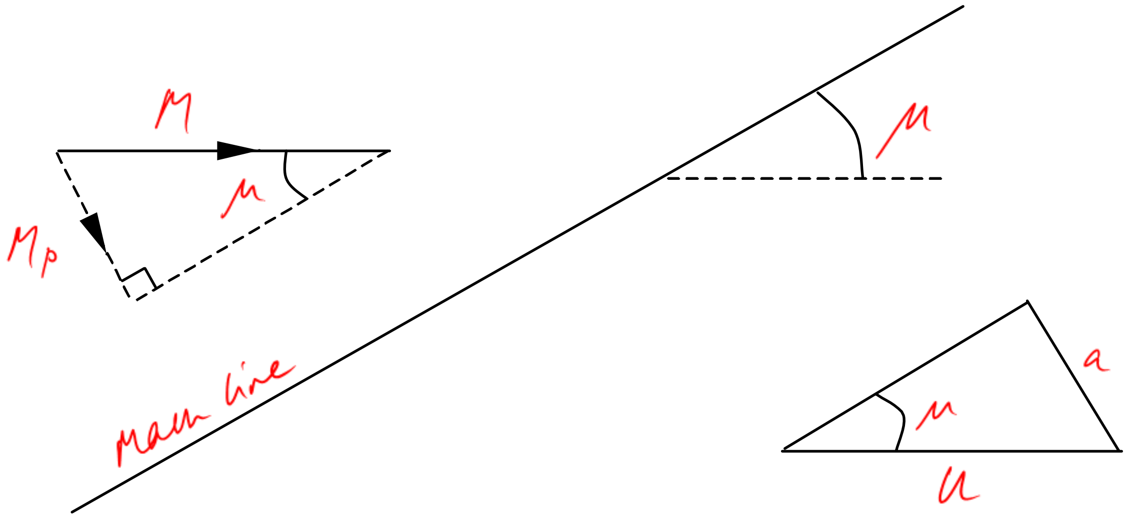
- waves confined to "Mach Cone"
- no knowledge of source upstream

Mach lines

Borders of Mach cone (or wedge in 2-D) are Mach lines

In frame with source stationary, Mach lines are at angle $\mu = sin^{-1}\left(\frac{a\Delta t}{\Delta x}\right) = sin^{-1}\left(\frac{1}{M}\right)$ to flow:

$u\Delta t$



$\perp$ Mach line, flow Mach no $M_p = M\, sin\,\mu = 1$, $V_p = a$

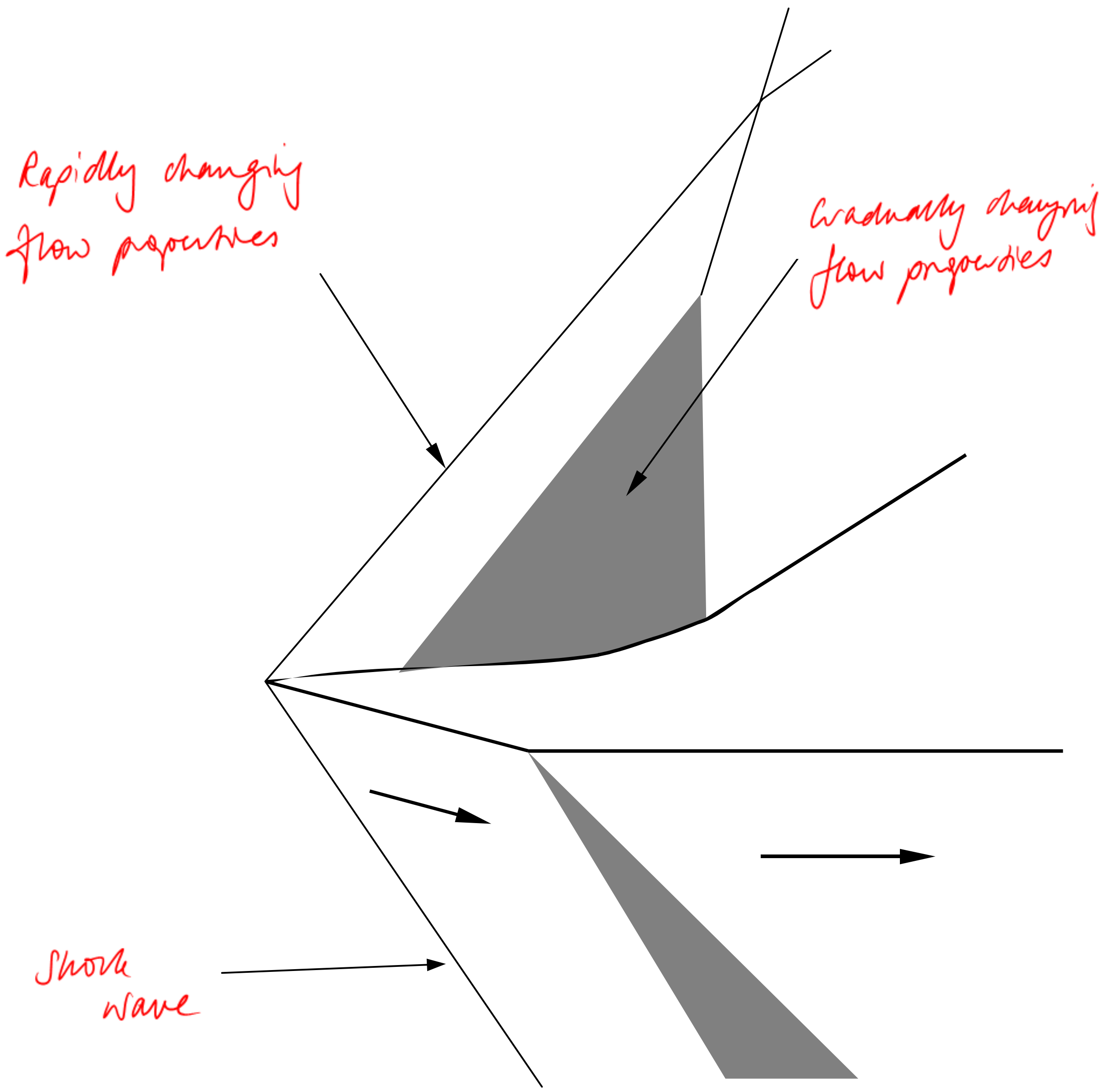
Sound wave stationary $\Leftrightarrow$ sound wave aligned wave on Mach line.

Typical supersonic flow features

Supersonic flowfields generally consist of

(i) large regions where properties change gradually, gradients are small

(ii) localised regions where properties change suddenly.



Analyse general flows by combining techniques for different regions.

Start by analysing flow in smooth regions - Method of characteristics