Examples Paper 4: Constrained Optimization

*Tripos standard questions are marked **

Linear Programming

1. Use the Simplex Algorithm to minimise the function

$$f(x) = -2x_1 - 4x_2 - x_3$$

subject to

$$x_1 + 3x_2 \leq 4$$

and

$$x_2 + 4x_3 \leq 3$$

A convenient starting vertex is $(x_1, x_2, x_3) = (0, 0, 0)$ with slack variables (needed to convert the problem to standard form) taking appropriate values.

*2 A furniture manufacturing company offers the following product range:

Bookcases: each bookcase requires three hours of labour, one unit of metal and four units of wood, and brings in a net profit of £19;

Desks: each desk requires two hours of labour, one unit of metal and three units of wood, and brings in a net profit of £15;

Chairs: each chair requires one hour of labour, one unit of metal and three units of wood, and brings in a net profit of £12;

Bedframes: each bedframe requires two hours of labour, one unit of metal, and four units of wood, and brings in a net profit of £17.

225 hours of labour, 117 units of metal and 420 units of wood are available each day. The managament wants all the available labour and materials to be used every day.

Formulate the task of finding the product balance that maximises the company's daily profit as a linear programming problem, and solve this problem using the Simplex Algorithm.

(Treat any integer control variables as real numbers and adjust the final solution appropriately, if required)

3. A factory produces an item for which the demand over the next three months are 150, 200, and 250. Each month the demand can be satisfied by:

- Excess production in an earlier month held in stock for later;
- Production in the current month;
- Excess production in a later month, back-ordered for preceding months.

The production cost per unit in the next three months is £16, £17 and £18. A unit produced and stored until later incurs a storage cost at a rate of £3 per month. Back-ordered items incur a penalty cost of £6 per unit per month. the production capacity in each of the next three months is 200 units. The objective is to devise a minimum cost production plan. Formulate this problem as a linear programming problem.

Nonlinear Constrained Optimisation

*4 A thin-walled cylindrical pressure vessel with spherical ends is to be designed to minimize the total volume of material used in its manufacture. The vessel must contain at least 25 m^3 of gas at a pressure, p, of 3.5 bar. The hoop (circumferential) stress, $\sigma_h = pr/t$, in the cylinder walls must not exceed 200 MPa. A cross-section of the vessel is shown in Figure 1.

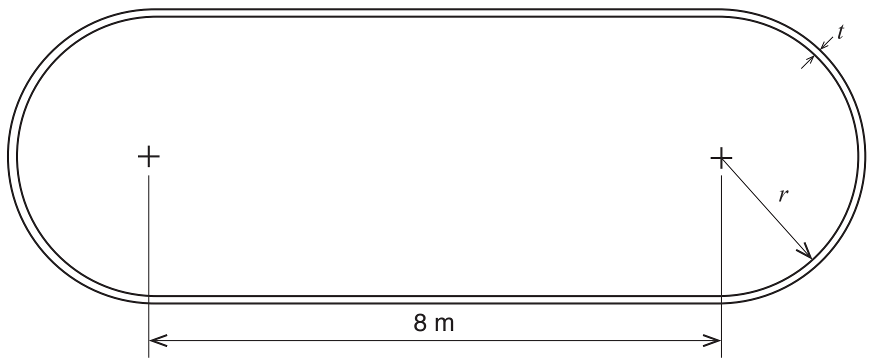

Figure 1.

(a) Graphically identify the feasible region.

(b) Using the Kuhn-Tucker conditions, determine the values of radius, r, and wall thickness, t, which minimize the volume of the material subject to the constraints.

(c) Calculate the sensitivity of the optimal solution to small changes in the right-hand side of the inequality constraints.

5. Derive the equations governing the vertical positions of the links of the hanging chain in question 3 of Examples Paper 3 in terms of the Lagrange multipliers, using $n = 20$ links, and $L = 1.6$ m, $d = 10$ cm and $m = 0.1$ kg. (You may take g, the acceleration due to gravity, to be 10 ms^{-2}.)

6. The Bose-Einstein distribution for the number of particles n_i at energy level E_i can be derived by maximising the probability of the distribution subject to a given number of particles, n, and total energy, E, i.e. by solving the following optimisation problem:

$$\text{maximize } P \text{ subject to } \sum_i n_i = n \text{ and } \sum_i n_i E_i = E,$$

where

$$P = W \prod_i \frac{(g_i + n_i - 1)!}{n_i!(g_i - 1)!},$$

in which g_i is the number of states available at energy E_i and W is a normalization constant.

By considering $\ln(P)$, show that the most probably distribution is given by

$$n_i = \frac{(g_i - 1)}{\exp(\lambda_1 + \lambda_2 E_i) - 1}$$

for some constants λ_1 and λ_2. It may be assumed that $g_i \geq n_i$ and that all the g_i and n_i are large, and hence can be treated as real numbers. Note that for large x,

$$\frac{d}{dx}(\ln(x!)) \approx \ln(x).$$

*7 (a) Describe the penalty function and barrier methods for constrained optimization.

(b) Consider the following problem

$$\text{minimize } f(x) \text{ subject to } g_j(x) = b_j \quad j = 1, \ldots, n,$$

where f and g_j are differentiable.

When a penalty function of the form $\mu(g(x) - b)^T(g(x) - b)$ is used, the exact optimal solution is obtained for a particular value of μ. Explain why this can only happen if the gradient of f is zero at the optimal solution.

(c) It is desired to minimize a function $F(x, y)$ subject to the condition that x and y are integers. The penalty function method is suggested with a penalty term proportional to $\sin^2(\pi x) + \sin^2(\pi y)$. Discuss the effectiveness of this procedure.

Answers

1. $(x_1, x_2, x_3) = (4, 0, 0.75), f(x^*) = -8.75$
2. 39 desks, 9 chairs 69 bedframes
3. Check with your supervisor
4. $r = 928$ mm, $t = 1.624$ mm; $\mu_1 = 4.67 \times 10^{-10}$, $\mu_2 = 3.83 \times 10^{-3}$
5. $$y_i = \frac{0.1(\lambda_1 - 20 + i - 0.5)}{\sqrt{\lambda_2^2 + (\lambda_1 - 20 + i - 0.5)^2}} \text{ for } i = 1, \ldots, 20$$

Suitable past Tripos questions

Part IIB 2007 Paper 4M13 Q4
Part IIB 2008 Paper 4M13 Q4
Part IIB 2009 Paper 4M13 Q4
Part IIB 2010 Paper 4M13 Q3,Q4
Part IIB 2011 Paper 4M13 Q4
Part IIA 2012 Paper 3M1 Q3
Part IIA 2013 Paper 3M1 Q3
Part IIA 2015 Paper 3M1 Q2 parts (a) to (c), Q3
Part IIA 2016 Paper 3M1 Q3