2010

1) Mass on a spring ✓
2) Generalised Forces for trolley ✓
3) Particle in polar coordinates ✓
4) Gravity - generalised force or potential energy ✓
   Mass on a spring ✓
   Pendulum with force applied
5) Helicopter rotor blade ✓
6) Double lumped mass pendulum ✓

2009

1) Mass on spring including gravity
2) Generalised Forces for trolley
3) Simple pendulum with applied force
4) Particle in polar coordinates.
5) Gravity : potential energy or generalised force?
6) Compound pendulum system
7) Helicopter rotor blade

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Paper 3C5: Lagrangian Mechanics

(R.S. Langley, Room 328, RSL21@eng.cam.ac.uk)

1. Introduction

The analysis of a dynamic system requires three stages:-

1. The identification of the system degrees of freedom
2. The formulation of the equations of motion (one for each degree of freedom)
3. The solution of the equations of motion

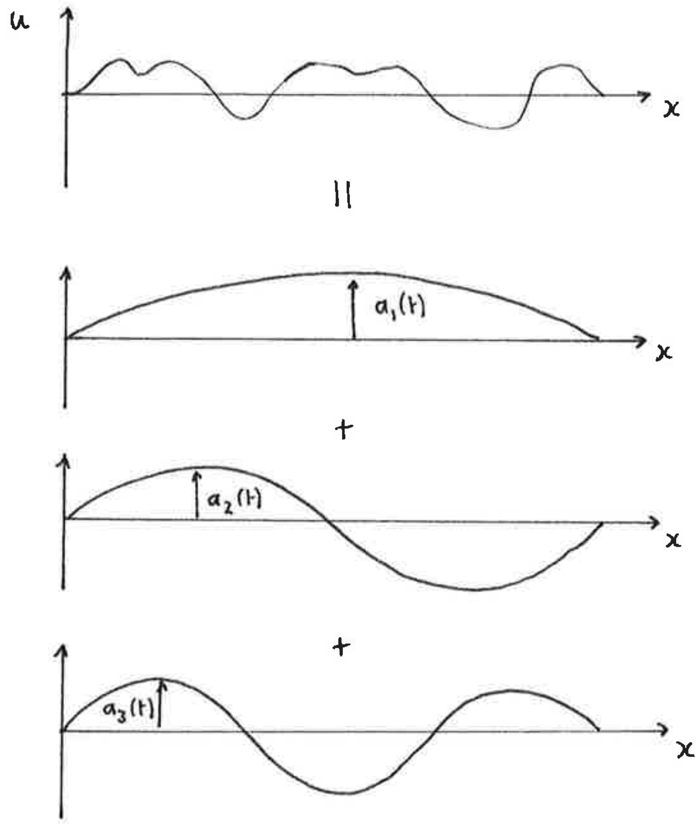
Although step (2) is fairly straight forward for a simple dynamic systems (for example a mass-spring system) it can be extremely difficult to derive the equations of motion of a complex system (for example an aircraft wing, or an offshore structure). There is a direct analogy with static analysis here: for a statically determinate system the equilibrium equations can be derived simply by resolving forces, whereas a statically indeterminate system requires a more sophisticated approach, such as the principle of minimum potential energy. The dynamic equivalent of the principle of minimum potential energy is known as *Lagrange's Equation*, after the French/Italian mathematician Joseph-Louis Lagrange (1736-1813). This part of the course provides an introduction to Lagrange's Equation and its applications.

2. Lagrange's Equation

2.1 Degrees of Freedom and Generalised Coordinates

The degrees of freedom of a dynamic system are those quantities required to describe the configuration of the system – for example a mass in space has six degrees of freedom, corresponding to three displacements and three rotations. The degrees of freedom may also be more abstract – for example the displacements of a string of length L might be represented by a Fourier series in the form

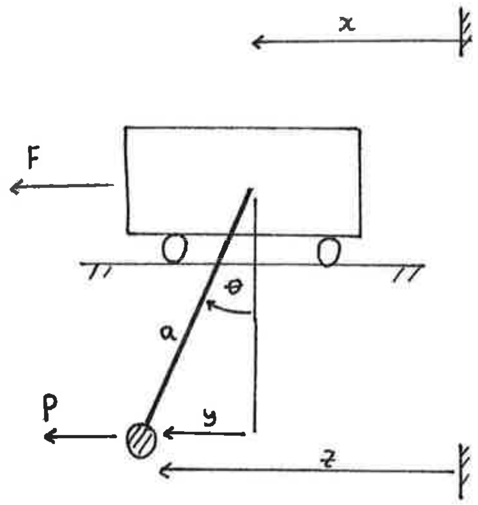
$$u(x,t) = \sum_n a_n(t) \sin\left(\frac{n\pi x}{L}\right).$$



In this case the degrees of freedom are the Fourier amplitudes $a_n(t)$. Degrees of freedom of this type (and also of the more straight forward type) are known as *generalised coordinates*, and they are usually labelled q_1, q_2,..., q_N. Thus the generalised coordinates are the parameters needed to specify the system configuration.

In this course we shall consider *holonomic systems*: in this case the generalised coordinates can be varied independently in a general way without violating any physical constraints that act on the system. In contrast, a *non-holonomic* system typically has non-integrable velocity constraints and the motion cannot be described simply in terms of independent generalised coordinates; a rolling system is an example of this type of system.

It can be noted that the generalised coordinates used to describe a system are generally non-unique, in that the motion of a system can be described in several different ways. This is illustrated below for a trolley/pendulum system.



Generalised coordinate options:-

X Y (a) $q_1 = x \quad q_2 = y$

X θ (b) $q_1 = x \quad q_2 = \theta$

X z (c) $q_1 = x \quad q_2 = z$

z Y (d) $q_1 = z \quad q_2 = y$

z θ (e) $q_1 = z \quad q_2 = \theta$

2.2 Lagrange's Equation

Lagrange's equation states that

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{q}_i}\right] - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i \qquad i = 1, 2 \ldots N$$

(Note static case $T = 0 \Rightarrow$ Minimum potential energy)

where T is the *kinetic energy* of the system, V is the *potential energy*, and Q_i is called the *generalised force* associated with the ith generalised coordinate. Note that there is a Lagrange equation for each generalised coordinate, so we have N equations for the N unknowns. A proof of Lagrange's equation is given in the Appendix. The equation can also be written in the form

NB/ $\frac{\partial V}{\partial \dot{q}_i} = 0$

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_i}\right] - \frac{\partial L}{\partial q_i} = Q_i \qquad i = 1, 2 \ldots N$$

where $L=T-V$ is known as the *Lagrangian*.

The generalised forces Q_i are caused by the external forces acting on the system, and they can be found by employing *virtual work*. The total work δW done by all the external forces during a small (virtual) change δq_i (i=1,2,...,N) in the configuration is

$$\delta W = \sum_i Q_i \delta q_i$$

We can find Q_i by considering the work done by a change δq_i alone and equating this to $Q_i\, \delta q_i$.

Example: Generalised forces associated with the trolley/pendulum system

Examples: Lagrange's equations for a mass-spring system and a simple pendulum.

Generalised Forces For trolley/pendulum

(a) $q_1 = x \quad q_2 = y$: $\delta W = F\delta x + P(\delta x + \delta y)$

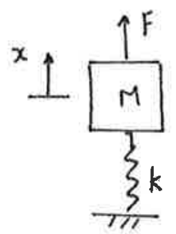
$$\underline{Q_1 = F+P} \qquad \underline{Q_2 = P}$$

(b) $q_1 = x \quad q_2 = \theta$: $\delta W = F\delta x + P(\delta x + a\delta\theta\cos\theta)$

$$\underline{Q_1 = F+P} \qquad \underline{Q_2 = aP\cos\theta}$$

Other cases are similar

Lagrange's Equation for a Mass/Spring System



$T = \frac{1}{2}M\dot{x}^2$

$V = \frac{1}{2}kx^2 + Mgx$ (spring energy, gravitational potential)

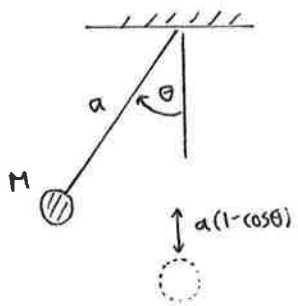
$\delta W = F\delta x \Rightarrow Q = F$

Lagrange $(q_1 = x)$ $\quad \frac{d}{dt}\left[\frac{\partial T}{\partial \dot{x}}\right] - \frac{\partial T}{\partial x} + \frac{\partial V}{\partial x} = Q$

$$\frac{d}{dt}[M\dot{x}] + kx + Mg = F$$

$$\underline{M\ddot{x} + kx = F - Mg}$$

Lagrange's Equation for a Simple Pendulum



$T = \frac{1}{2}M(a\dot{\theta})^2$

$V = Mga(1-\cos\theta)$

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{\theta}}\right] - \frac{\partial T}{\partial \theta} + \frac{\partial V}{\partial \theta} = 0$$

$\underline{Ma^2\ddot{\theta} + Mga\sin\theta = 0}$ (small angles $Ma^2\ddot{\theta} + Mga\theta = 0$)

3. Conservation of Momentum and Energy

3.1 Conservation of Momentum

If one of the generalised coordinates is *not* subjected to either an external force (Q_i=0) or a potential induced force ($\partial V/\partial q_i$ =0), and if the kinetic energy is independent of the coordinate ($\partial T/\partial q_i$ =0) then the Lagrange equation reduces to

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{q}_i}\right] = 0 \quad \Rightarrow \quad \frac{\partial T}{\partial \dot{q}_i} = \text{constant}$$

The term $\partial T/\partial \dot{q}_i$ is known as the *generalised momentum* associated with the *i*th generalised coordinate, and it can be seen that this momentum is *conserved* under the stated conditions.

Consider the two previous examples: for the mass/spring system

$$T = \tfrac{1}{2}M\dot{x}^2 \quad \Rightarrow \quad \frac{\partial T}{\partial \dot{x}} = M\dot{x}$$

Clearly the generalised momentum is equal to the linear momentum of the system. For the pendulum

$$T = \tfrac{1}{2}Ma^2\dot{\theta}^2 \quad \Rightarrow \quad \frac{\partial T}{\partial \dot{\theta}} = Ma^2\dot{\theta}$$

So in this case the generalised momentum is equal to the angular momentum of the system.

3.2 Conservation of Energy

In the absence of external forcing (Q_i=0), conservation of energy can be demonstrated. If the *i*th Lagrange equation is multiplied by $\dot{q}_i$ and then a sum is taken over *i*, then the following result ensues

$$\sum_i \dot{q}_i \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \sum_i \dot{q}_i \frac{\partial T}{\partial q_i} + \sum_i \dot{q}_i \frac{\partial V}{\partial q_i} = 0\,.$$

Now if neither T nor V depend explicitly on time, and if V is independent of the system velocities, then the time derivatives of T or V can be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}(T)=\sum_i \ddot{q}_i \frac{\partial T}{\partial \dot{q}_i}+\sum_i \dot{q}_i \frac{\partial T}{\partial q_i}, \qquad \frac{\mathrm{d}}{\mathrm{d}t}(V)=\sum_i \dot{q}_i \frac{\partial V}{\partial q_i}.$$

Thus the modified Lagrange equation can be written in the form

$$\sum_i \dot{q}_i \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_i}\right)+\sum_i \ddot{q}_i \frac{\partial T}{\partial \dot{q}_i}-\frac{\mathrm{d}T}{\mathrm{d}t}+\frac{\mathrm{d}V}{\mathrm{d}t}=0 \;\Rightarrow\; \frac{\mathrm{d}}{\mathrm{d}t}\sum_i \dot{q}_i \frac{\partial T}{\partial \dot{q}_i}-\frac{\mathrm{d}T}{\mathrm{d}t}+\frac{\mathrm{d}V}{\mathrm{d}t}=0.$$

Now if T is a quadratic function of velocity then $\sum_i \dot{q}_i \frac{\partial T}{\partial \dot{q}_i}=2T$ and it follows that

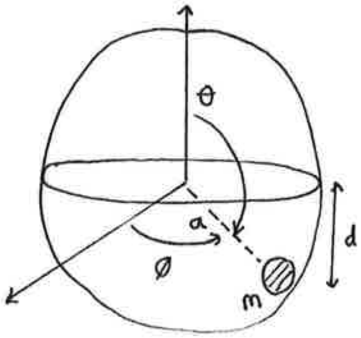
$$\frac{\mathrm{d}}{\mathrm{d}t}(T+V)=0,$$

and hence energy is conserved.

<u>Example</u>: Lagrange's equation for a ball in a sphere

<u>Example</u>: Lagrange's equation for a compound pendulum

Lagrange example: Ball in a Sphere



Initial velocity $= v$
in horizontal direction

Generalised coordinates are : Latitude θ
Longitude ϕ

Circumferential velocity $= a\sin\theta.\dot{\phi}$

Longitudinal velocity $= a\dot{\theta}$

$$\Rightarrow \quad T = \tfrac{1}{2}m\left[a^2\dot{\theta}^2 + a^2\sin^2\theta.\dot{\phi}^2\right]$$

$$V = -mgd = -mga\cos(\pi-\theta) = mga\cos\theta \quad \text{(gravitational potential)}$$

Equations of motion could be derived from Lagrange's equation - however, we will look at momentum and energy.

Momentum $\left.\begin{array}{l} V \text{ is independent of } \phi \\ T \text{ is independent of } \phi \end{array}\right\}$ $\dfrac{\partial T}{\partial \dot{\phi}} = \text{constant} \Rightarrow ma^2\sin^2\theta.\dot{\phi} = \text{constant}$

At $t=0$ $\dot{\theta}=0$ and $\dot{\phi} = \dfrac{v}{a\sin\theta_0}$

$$\underline{\sin^2\theta.\dot{\phi} = \left(\tfrac{v}{a}\right)\sin\theta_0}$$

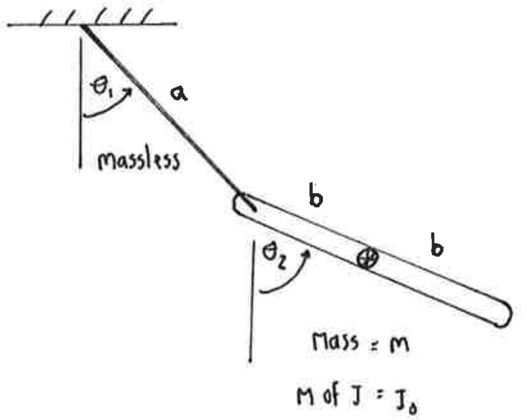
Conservation of angular momentum about the N-S axis.

Energy The system is conservative, so $T+V = \text{constant}$

$$\tfrac{1}{2}m\left[a^2\dot{\theta}^2 + a^2\sin^2\theta.\dot{\phi}^2\right] + mga\cos\theta = \text{const} = \tfrac{1}{2}ma^2\sin^2\theta_0\left(\frac{v}{a\sin\theta_0}\right)^2 + mga\cos\theta_0$$

$$\underline{\dot{\theta}^2 + \sin^2\theta.\dot{\phi}^2 + 2\left(\tfrac{g}{a}\right)\cos\theta = \left(\tfrac{v}{a}\right)^2 + 2\left(\tfrac{g}{a}\right)\cos\theta_0}$$

Lagrange example: Double Pendulum



Horizontal position of c.o.m. = $a\sin\theta_1 + b\sin\theta_2$

Horizontal velocity : $a\dot{\theta}_1\cos\theta_1 + b\dot{\theta}_2\cos\theta_2$

Vertical position of c.o.m = $a\cos\theta_1 + b\cos\theta_2$

Vertical velocity : $-a\dot{\theta}_1\sin\theta_1 - b\dot{\theta}_2\sin\theta_2$

$$T = \tfrac{1}{2}m\left[(a\dot{\theta}_1\cos\theta_1 + b\dot{\theta}_2\cos\theta_2)^2 + (a\dot{\theta}_1\sin\theta_1 + b\dot{\theta}_2\sin\theta_2)^2 \right] + \tfrac{1}{2}I_0\dot{\theta}_2^2$$

$$\underline{T = \tfrac{1}{2}m\left[a^2\dot{\theta}_1^2 + b^2\dot{\theta}_2^2 + 2ab\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2)\right] + \tfrac{1}{2}I_0\dot{\theta}_2^2}$$

$$\underline{V = -mg(a\cos\theta_1 + b\cos\theta_2)}$$

<u>For θ_1</u>

$$\frac{\partial T}{\partial\dot{\theta}_1} = ma^2\dot{\theta}_1 + mab\dot{\theta}_2\cos(\theta_1-\theta_2)$$

$$\frac{d}{dt}\left[\frac{\partial T}{\partial\dot{\theta}_1}\right] = ma^2\ddot{\theta}_1 + mab\ddot{\theta}_2\cos(\theta_1-\theta_2) - mab\dot{\theta}_2(\dot{\theta}_1-\dot{\theta}_2)\sin(\theta_1-\theta_2)$$

$$\frac{\partial T}{\partial\theta_1} = -mab\dot{\theta}_2\sin(\theta_1-\theta_2)\dot{\theta}_1$$

$$\frac{\partial V}{\partial\theta_1} = mga\sin\theta_1$$

Lagrange $\Rightarrow$ $ma^2\ddot{\theta}_1 + mab\ddot{\theta}_2\cos(\theta_1-\theta_2) + mab\dot{\theta}_2^2\sin(\theta_1-\theta_2) + mga\sin\theta_1 = 0$ ← stop here

$$\underline{a\ddot{\theta}_1 + b\ddot{\theta}_2\cos(\theta_1-\theta_2) + b\dot{\theta}_2^2\sin(\theta_1-\theta_2) + g\sin\theta_1 = 0}$$

<u>For θ_2</u>

$$\frac{\partial T}{\partial\dot{\theta}_2} = mb^2\dot{\theta}_2 + mab\dot{\theta}_1\cos(\theta_1-\theta_2) + I_0\dot{\theta}_2$$

$$\frac{d}{dt}\left[\frac{\partial T}{\partial\dot{\theta}_2}\right] = mb^2\ddot{\theta}_2 + mab\ddot{\theta}_1\cos(\theta_1-\theta_2) - mab\dot{\theta}_1(\dot{\theta}_1-\dot{\theta}_2)\sin(\theta_1-\theta_2) + I_0\ddot{\theta}_2$$

$$\frac{\partial T}{\partial\theta_2} = +mab\dot{\theta}_1\sin(\theta_1-\theta_2)\dot{\theta}_2$$

$$\frac{\partial V}{\partial \theta_2} = mgb\sin\theta_2$$

Lagrange $\Rightarrow$ $mb^2\ddot{\theta}_2 + mab\ddot{\theta}_1\cos(\theta_1-\theta_2) - mab\dot{\theta}_1^2\sin(\theta_1-\theta_2) + J_0\ddot{\theta}_2 + mgb\sin\theta_2 = 0$ $\longleftarrow$ stop here

$J_0 = \frac{1}{3}mb^2$ so that finally

$$\underline{\tfrac{4}{3}b\ddot{\theta}_2 + a\ddot{\theta}_1\cos(\theta_1-\theta_2) - a\dot{\theta}_1^2\sin(\theta_1-\theta_2) + gb\sin\theta_2 = 0}$$

4. Analysis of Small Amplitude Vibrations

Often in dynamic analysis the concern is with small amplitude linear vibrations, and in this case the Lagrange equations can be used to produce a particularly concise form of the equations of motion. Firstly it can be noted that the potential energy V can be expanded as a Taylor series in the form

$$V(q_1, q_2, .., q_N) = V(0,0,..,0) + \sum_i \left.\frac{\partial V}{\partial q_i}\right|_0 q_i + \frac{1}{2}\sum_i\sum_j \left.\frac{\partial^2 V}{\partial q_i \partial q_j}\right|_0 q_i q_j + \ldots$$

Now the potential energy can always be redefined by the addition (or subtraction) of a constant, and so without loss of generality it can be assumed that $V(0,0,..,0)=0$. Furthermore, if there are no applied loads and the system is in static equilibrium at the point $q_i=0$, then $\partial V/\partial q_i=0$ at this point (from the principle of minimum potential energy). Thus for small q_i the potential energy can be approximated in the form

$$V(q_1, q_2, .., q_N) \approx \frac{1}{2}\sum_i\sum_j \left.\frac{\partial^2 V}{\partial q_i \partial q_j}\right|_0 q_i q_j = \frac{1}{2}\sum_i\sum_j V_{ij} q_i q_j ,$$

so that it is a *quadratic function* of the generalised coordinates.

Although not true in every case (a spinning system is one exception), the kinetic energy of a linear or linearised system can generally be expressed as a quadratic function of the generalised velocities so that

$$T(\dot{q}_1, \dot{q}_2, .., \dot{q}_N) \approx \frac{1}{2}\sum_i\sum_j \left.\frac{\partial^2 T}{\partial \dot{q}_i \partial \dot{q}_j}\right|_0 \dot{q}_i \dot{q}_j = \frac{1}{2}\sum_i\sum_j T_{ij} \dot{q}_i \dot{q}_j .$$

The application of Lagrange's equation then yields

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{q}_i}\right] - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = 0 \quad \Rightarrow \quad \sum_j T_{ij}\ddot{q}_j + \sum_j V_{ij} q_j = 0 \qquad i = 1,2,3 \ldots N.$$

This set of equations can be written in matrix form

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{0},$$

where

$$M = \begin{pmatrix} T_{11} & T_{12} & \cdots & T_{1N} \\ T_{21} & & & \\ \vdots & & & \\ T_{N1} & T_{N2} & \cdots & T_{NN} \end{pmatrix} \qquad K = \begin{pmatrix} V_{11} & V_{12} & \cdots & V_{1N} \\ V_{21} & & & \\ \vdots & & & \\ V_{N1} & V_{N2} & \cdots & V_{NN} \end{pmatrix}$$

$$\underset{\sim}{q} = \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_N \end{pmatrix} \qquad \underset{\sim}{0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

M is known as the <u>*mass matrix*</u> and **K** is known as the <u>*stiffness matrix.*</u> It can be noted that the kinetic and potential energies can be expressed in terms of these matrices in the concise form

$$T = (1/2)\dot{\mathbf{q}}^{\mathrm{T}}\mathbf{M}\dot{\mathbf{q}}, \qquad V = (1/2)\mathbf{q}^{\mathrm{T}}\mathbf{K}\mathbf{q}.$$

The equations of motion can be solved by assuming simple harmonic motion, so that

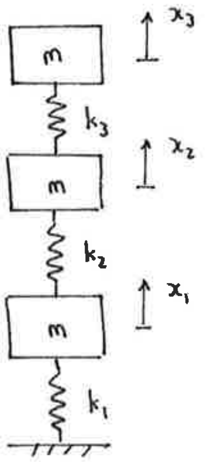
$$\underset{\sim}{q} = \underset{\sim}{q}_0 e^{i\omega t} \quad \Rightarrow \quad (-\omega^2 M + k)\underset{\sim}{q}_0 = \underset{\sim}{0} \quad \Rightarrow \quad |-\omega^2 M + k| = 0$$

We then have an eigen-problem, in which the eigenvalues are the natural frequencies ω^2 and the eigenvectors are the mode shapes $\mathbf{q}_0$.

<u>Example</u>: Three mass system

<u>Example</u>: Two masses on a taut string

Three Mass System



Neglect gravity (ie assume displacements are about the static equilibrium position).

$$T = \tfrac{1}{2} m \left(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2 \right)$$

$$V = \tfrac{1}{2} k_1 x_1^2 + \tfrac{1}{2} k_2 (x_2 - x_1)^2 + \tfrac{1}{2} k_3 (x_3 - x_2)^2$$

$$\left.\begin{aligned} \frac{\partial T}{\partial \dot{x}_1} &= m\dot{x}_1 \qquad \frac{d}{dt}\left[\frac{\partial T}{\partial \dot{x}_1}\right] = m\ddot{x}_1 \\ \frac{\partial V}{\partial x_1} &= k_1 x_1 - k_2 (x_2 - x_1) \end{aligned}\right\} \quad m\ddot{x}_1 + (k_1 + k_2) x_1 - k_2 x_2 = 0 \qquad \text{— (1)}$$

$$\left.\begin{aligned} \frac{\partial T}{\partial \dot{x}_2} &= m\dot{x}_2 \qquad \frac{d}{dt}\left[\frac{\partial T}{\partial \dot{x}_2}\right] = m\ddot{x}_2 \\ \frac{\partial V}{\partial x_2} &= k_2 (x_2 - x_1) - k_3 (x_3 - x_2) \end{aligned}\right\} \quad m\ddot{x}_2 + (k_2 + k_3) x_2 - k_2 x_1 - k_3 x_3 = 0 \qquad \text{— (2)}$$

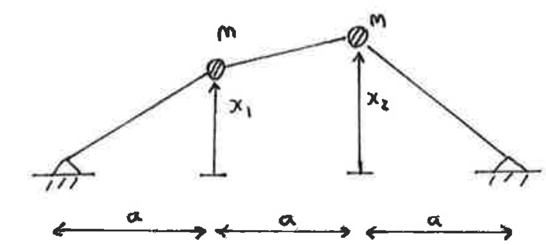
$$\left.\begin{aligned} \frac{\partial T}{\partial \dot{x}_3} &= m\dot{x}_3 \qquad \frac{d}{dt}\left[\frac{\partial T}{\partial \dot{x}_3}\right] = m\ddot{x}_3 \\ \frac{\partial V}{\partial x_3} &= k_3 (x_3 - x_2) \end{aligned}\right\} \quad m\ddot{x}_3 + k_3 (x_3 - x_2) = 0 \qquad \text{— (3)}$$

Now put (1), (2), and (3) in matrix form:-

$$\underbrace{\begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix}}_{\text{This is } \frac{\partial^2 T}{\partial \dot{x}_i \partial \dot{x}_j}} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{pmatrix} + \underbrace{\begin{pmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{pmatrix}}_{\text{This is } \frac{\partial^2 V}{\partial x_i \partial x_j}} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

or $\underline{M\ddot{\underset{\sim}{x}} + k\underset{\sim}{x} = \underset{\sim}{0}}$ $\longrightarrow$ can solve for natural frequencies and mode shapes

Two Masses on a taut string



$x_1 \ll a$

$x_2 \ll a$

The potential energy requires special consideration

Let L_0 = initial unstretched length of string

$L_1 = 3a$ = length of string with applied pre-load P

L = deformed length of string $= L_1 + \Delta$, where Δ is small

$$V = \tfrac{1}{2}k(L-L_0)^2 = \tfrac{1}{2}k(L_1+\Delta-L_0)^2$$

$$= \underbrace{k\Delta(L_1-L_0)}_{k(L_1-L_0)=P} + \underbrace{\tfrac{1}{2}k\Delta^2}_{\text{small} \Rightarrow \text{neglect}} + \underbrace{\tfrac{1}{2}k(L_1-L_0)^2}_{\text{constant} \Rightarrow \text{can be omitted}}$$

$\therefore \; \underline{V = P\Delta}$

Now $L = \sqrt{a^2+x_1^2} + \sqrt{a^2+(x_2-x_1)^2} + \sqrt{a^2+x_2^2}$

$$L = a\left(1+\tfrac{1}{2}\frac{x_1^2}{a^2}\right) + a\left(1+\tfrac{1}{2}\frac{(x_2-x_1)^2}{a^2}\right) + a\left(1+\tfrac{1}{2}\frac{x_2^2}{a^2}\right)$$

$$\Delta = L - 3a = \tfrac{1}{2}\left(\tfrac{1}{a}\right)\left[x_1^2+(x_2-x_1)^2+x_2^2\right] \Rightarrow \underline{V = \left(\frac{P}{2a}\right)\left[x_1^2+(x_2-x_1)^2+x_2^2\right]}$$

$$\underline{T = \tfrac{1}{2}m\dot{x}_1^2 + \tfrac{1}{2}M\dot{x}_2^2}$$

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{x}_1}\right] - \frac{\partial T}{\partial x_1} + \frac{\partial V}{\partial x_1} = 0 \Rightarrow m\ddot{x}_1 + \left(\frac{P}{a}\right)\left[x_1-(x_2-x_1)\right] = 0$$

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{x}_2}\right] - \frac{\partial T}{\partial x_2} + \frac{\partial V}{\partial x_2} = 0 \Rightarrow m\ddot{x}_2 + \left(\frac{P}{a}\right)\left[x_2+(x_2-x_1)\right] = 0$$

$\sqrt{dx^2+dy^2} = \sqrt{1+y'^2}\,dx$

dy

dx

Additional Stretch = $\frac{1}{2}y'^2\,dx$

Total strain = $\varepsilon_0 + \frac{1}{2}y'^2$

$V = \frac{1}{2}P\int_0^L (y')^2\,dx$

$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$

$\sin\frac{6\pi}{3} = -\frac{\sqrt{3}}{2}$

$\frac{3}{4}\times 2 \quad \frac{3}{2}$

In matrix form

$$\begin{pmatrix} m & 0 \\ 0 & M \end{pmatrix}\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \begin{pmatrix} 2P/a & -P/a \\ -P/a & 2P/a \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Eigenvalue problem - put $\underset{\sim}{x} = \underset{\sim}{x_0} e^{i\omega t}$

$$\Rightarrow \quad \begin{vmatrix} -\omega^2 M + 2P/a & -P/a \\ -P/a & -\omega^2 M + 2P/a \end{vmatrix} = 0$$

$$(-\omega^2 M + 2P/a)^2 = (P/a)^2$$

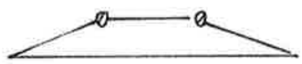
$$-\omega^2 M + 2P/a = \pm P/a$$

$$\underline{\omega^2 = \frac{3P}{aM} \quad \text{or} \quad \omega^2 = \frac{P}{aM}}$$

$$\underset{\text{Mode 2}}{\uparrow} \qquad\qquad \underset{\text{Mode 1}}{\uparrow}$$

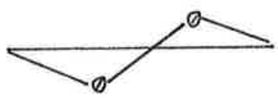
<u>Mode 1</u> $\begin{pmatrix} -M\omega^2 + 2P/a & -P/a \\ \checkmark & \checkmark \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} P/a & -P/a \\ \checkmark & \checkmark \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \underline{x_1 = x_2}$



<u>Mode 2</u> $\begin{pmatrix} -M\omega^2 + 2P/a & -P/a \\ \checkmark & \checkmark \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} -P/a & -P/a \\ \checkmark & \checkmark \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \underline{x_1 = -x_2}$



NB/ Continuous string of total mass $2M$: $\omega = \frac{\pi}{L}\sqrt{\frac{P}{2M/L}} = \pi\sqrt{\frac{P}{6aM}} = 1.28\sqrt{\frac{P}{aM}}$

mass $4M$: $\pi\sqrt{\frac{P}{12aM}} = 0.906\sqrt{\frac{P}{aM}}$ ← we have a lumped mass approx to this

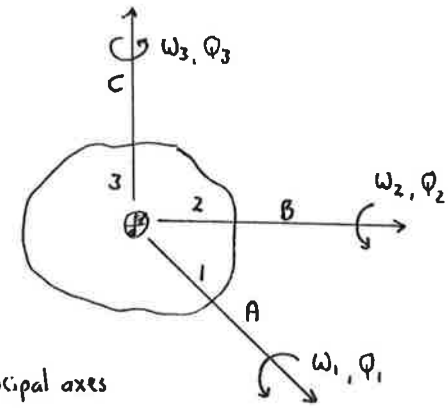
5. Euler's Equations for the Rotation of a Rigid Body

The equations of motion for rotation about the centre of mass of a rigid body have the form

$$A\dot{\omega}_1 - (B - C)\omega_2\omega_3 = Q_1$$

$$B\dot{\omega}_2 - (C - A)\omega_3\omega_1 = Q_2$$

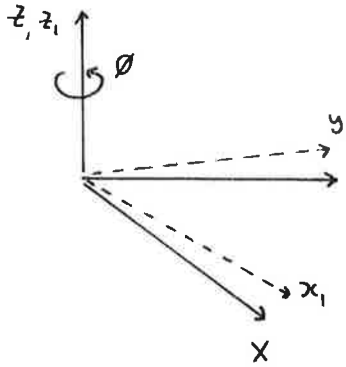
$$C\dot{\omega}_3 - (A - B)\omega_1\omega_2 = Q_3$$



Principal axes

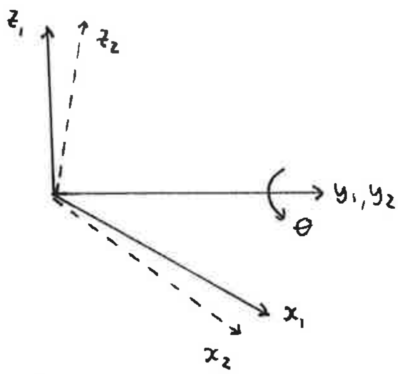
where A, B, and C are the moments of inertia about the principal axes, and ω_i and Q_i are the rotation rates and torques about these axes. To prove these equations by using Lagrange's equation, we need to define a suitable set of generalised coordinates. In doing this we have to allow for the fact that the rotations of the body can be arbitrarily large, which means that we need to specify the order in which the rotations are defined. If we use *Euler angles* ϕ, θ, ψ then the rotation is taken about z, then y, then z again. Thus

NB
$z \to y \to z$



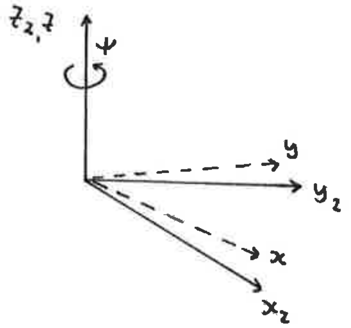
Step 1

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{R_1} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$



Step 2

$$\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}}_{R_2} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$$



+ diagram of Full transformation

Step 3

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{R_3} \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$$

In the above equations, (X,Y,Z) are a set of axes fixed in space, and (x,y,z) are a set of axes fixed in the body. The relation between the various axis systems is clearly

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = R_3 \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = R_3 R_2 \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = R_3 R_2 R_1 \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}.$$

Now the spin rates of the system are as follows: $\dot{\psi}$ about z_2, $\dot{\theta}$ about y_1, and $\dot{\phi}$ about Z. The spin rates (ω_1, ω_2, ω_3) in body axes can thus be written in the form

$$\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = R_3 \begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix} + R_3 R_2 \begin{pmatrix} 0 \\ \dot{\theta} \\ 0 \end{pmatrix} + R_3 R_2 R_1 \begin{pmatrix} 0 \\ 0 \\ \dot{\phi} \end{pmatrix}$$

and hence

$$\begin{aligned} \omega_1 &= \dot{\theta}\sin\psi - \dot{\phi}\sin\theta\cos\psi \\ \omega_2 &= \dot{\theta}\cos\psi + \dot{\phi}\sin\theta\sin\psi \, . \\ \omega_3 &= \dot{\psi} + \dot{\phi}\cos\theta \end{aligned}$$

Now the kinetic energy of the system is

$$T = \frac{1}{2}(A\omega_1^2 + B\omega_2^2 + C\omega_3^2)$$

and Lagrange's equation can be applied to yield the equations of motion. It should be noted however that only ψ represents a rotation about the body axes (z_2 is aligned to z), and hence only this generalised coordinate will yield an equation that is comparable to Euler's equation. The results is

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{\psi}}\right] - \frac{\partial T}{\partial \psi} = Q$$

$$\frac{\partial T}{\partial \dot{\psi}} = \frac{\partial T}{\partial \omega_1}\cancelto{0}{\frac{\partial \omega_1}{\partial \dot{\psi}}} + \frac{\partial T}{\partial \omega_2}\cancelto{0}{\frac{\partial \omega_2}{\partial \dot{\psi}}} + \frac{\partial T}{\partial \omega_3}\frac{\partial \omega_3}{\partial \dot{\psi}} = C\omega_3$$

$$\frac{\partial T}{\partial \psi} = \frac{\partial T}{\partial \omega_1}\frac{\partial \omega_1}{\partial \psi} + \frac{\partial T}{\partial \omega_2}\frac{\partial \omega_2}{\partial \psi} + \frac{\partial T}{\partial \omega_3}\cancelto{0}{\frac{\partial \omega_3}{\partial \psi}}$$

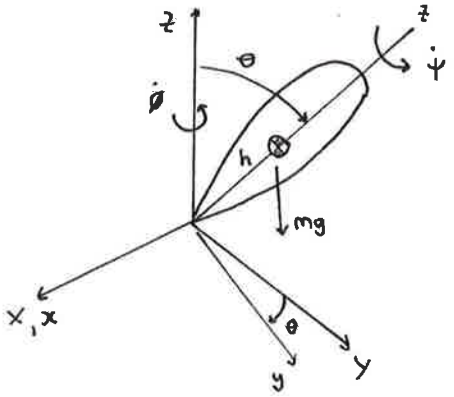
$$= A\omega_1\underbrace{\left[\dot{\theta}\cos\psi + \dot{\phi}\sin\theta\sin\psi\right]}_{\omega_2} + B\omega_2\underbrace{\left[-\dot{\theta}\sin\psi + \dot{\phi}\sin\theta\cos\psi\right]}_{-\omega_1} = (A-B)\omega_1\omega_2$$

Thus Lagrange $\Rightarrow$ $\underline{C\dot{\omega}_3 - (A-B)\omega_1\omega_2 = Q}$

The other two Euler equations can be deduced from symmetry.

<u>Example</u>: Symmetrical top

Symmetrical Top Example



$\omega_x = -\dot\theta$

$\omega_y = -\dot\phi\sin\theta$

$\omega_z = \dot\psi + \dot\phi\cos\theta$

$T = \frac{1}{2}\left[A\omega_x^2 + B\omega_y^2 + C\omega_z^2\right]$

$T = \frac{1}{2}A\dot\theta^2 + \frac{1}{2}B\dot\phi^2\sin^2\theta + \frac{1}{2}C(\dot\psi + \dot\phi\cos\theta)^2$

with $A = B$ for symmetrical top

$V = mgh\cos\theta$

Consider the θ equation : $\frac{d}{dt}\left[\frac{\partial T}{\partial\dot\theta}\right] - \frac{\partial T}{\partial\theta} + \frac{\partial V}{\partial\theta} = 0$

$$\frac{d}{dt}\left[A\dot\theta\right] - \left[A\dot\phi^2\sin\theta\cos\theta - C\dot\phi\sin\theta(\dot\psi + \dot\phi\cos\theta)\right] - mgh\sin\theta = 0$$

Now suppose we are interested in constant precession where $\underline{\theta = \text{constant}}$. Then $\ddot\theta = 0$ and:-

$$\sin\theta\left\{A\dot\phi^2\cos\theta - C\dot\phi(\dot\psi + \dot\phi\cos\theta) + mgh\right\} = 0$$

Now suppose $\theta \neq 0$ (top not vertical) and that $\dot\psi \gg \dot\theta$ (high spin rate). Then:-

$$-C\dot\phi\dot\psi + mgh = 0$$

$$\underline{\dot\phi = \frac{mgh}{C\omega}} \qquad \omega = \dot\psi = \text{spin rate} \qquad \text{Precession}$$

Alternatively assume mgh is negligably small $\Rightarrow$ $A\cos\theta.\dot\phi - C(\dot\psi + \dot\phi\cos\theta) = 0$

$$\underline{\dot\phi = \frac{C\omega}{(A-C)\cos\theta}} \qquad \text{Nutation}$$

More generally, Lagrange $\longrightarrow$ 3 equations of motion in ψ, ϕ and θ.

Appendix: Proof of Lagrange's Equation

Lagrange's equation can be derived in a number of ways, and the referenced texts give the more standard approaches. Here a fairly brief derivation will be given based on energy considerations.

Firstly, consider the change in kinetic energy between any two times t_0 and t_1:

$$T(t_1)-T(t_0)=[T]_{t_0}^{t_1}=\int_{t_0}^{t_1}\frac{\mathrm{d}T}{\mathrm{d}t}\mathrm{d}t=\int_{t_0}^{t_1}\sum_i\left(\frac{\partial T}{\partial \dot{q}_i}\ddot{q}_i+\frac{\partial T}{\partial q_i}\dot{q}_i\right)\mathrm{d}t. \tag{A1}$$

Now integrating the first term in the integrand by parts yields

$$[T]_{t_0}^{t_1}=\left[\sum_i\left(\frac{\partial T}{\partial \dot{q}_i}\dot{q}_i\right)\right]_{t_0}^{t_1}+\int_{t_0}^{t_1}\sum_i\left[-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_i}\right)+\frac{\partial T}{\partial q_i}\right]\dot{q}_i\,\mathrm{d}t. \tag{A2}$$

We now prove as an aside that

$$\sum_i\left(\frac{\partial T}{\partial \dot{q}_i}\dot{q}_i\right)=2T. \tag{A3}$$

This follows from the fact that if x represents the displacement of the system at a particular point, in a particular direction, then

$$\frac{\partial}{\partial \dot{q}_i}(\dot{x}^2)=2\dot{x}\frac{\partial \dot{x}}{\partial \dot{q}_i}\overset{(*)}{=}2\dot{x}\frac{\partial x}{\partial q_i}\quad\Rightarrow\quad\sum_i\dot{q}_i\frac{\partial}{\partial \dot{q}_i}(\dot{x}^2)=2\dot{x}\sum_i\dot{q}_i\frac{\partial x}{\partial q_i}=2\dot{x}^2.$$

(*) $\dot{x}=\sum_i \frac{\partial x}{\partial q_i}\dot{q}_i$

Since T is proportional to the sum of the velocity squared over the whole system, then it follows that the above result also holds if we replace x^2 by T, thus proving equation (A3). Now it follows from equations (A2) and (A3) that

$$[T]_{t_0}^{t_1}=-\int_{t_0}^{t_1}\sum_i\left[-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_i}\right)+\frac{\partial T}{\partial q_i}\right]\dot{q}_i\,\mathrm{d}t. \tag{A4}$$

Now the change in the potential energy between the two times can be written as

$$[V]_{t_0}^{t_1} = \int_{t_0}^{t_1} \frac{\mathrm{d}V}{\mathrm{d}t}\,\mathrm{d}t = \int_{t_0}^{t_1} \sum_i \left(\frac{\partial V}{\partial q_i} \dot{q}_i \right) \mathrm{d}t \tag{A5}$$

The change in the potential energy plus the change in the kinetic energy must equal the work done by the external forces, and hence it follows that

$$[T+V]_{t_0}^{t_1} = \int_{t_0}^{t_1} \sum_i \left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} \right] \dot{q}_i \,\mathrm{d}t = \int_{t_0}^{t_1} \sum_i Q_i \dot{q}_i \,\mathrm{d}t$$

This result must hold for any time interval and for any possible motion, and thus it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i ,$$

which is Lagrange's equation.

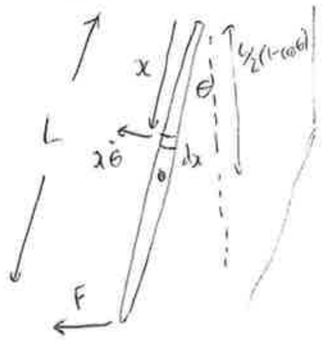
- Comment on force / potential
- Helicopter example
- double mass example
- conservation handout.

Complete first handout
- generalised forces.

Examples : ① Mass spring system done

Gravity as a generalised force or potential?

② Pendulum



$$T = \frac{1}{2}\int_0^L m(x\dot{\theta})^2\,dx = \frac{1}{6}mL^3\dot{\theta}^2 \quad \left[\text{or } \tfrac{1}{2}J\dot{\theta}^2\right]$$

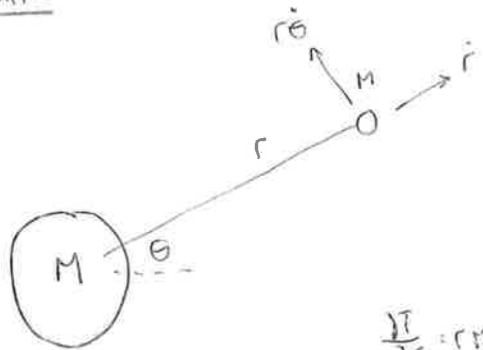
$$U = \frac{L}{2}(1-\cos\theta)mLg$$

$$\boxed{\tfrac{1}{3}mL^3\ddot{\theta} + \frac{mL^2}{2}g\sin\theta = FL\cos\theta}$$

$J\ddot{\theta}$ gravity couple applied moment

$$\delta W = F\,\delta(L\sin\theta) = \underbrace{F\cos\theta\,L}_{Q}\,\delta\theta$$

③ Satellite

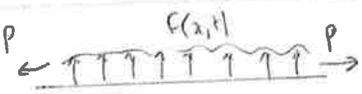


$$T = \tfrac{1}{2}m(r^2\dot{\theta}^2 + \dot{r}^2)$$

$$U = -\frac{GMm}{r}$$

$$\frac{\partial T}{\partial r} = rm\dot{\theta}^2 \quad \frac{\partial T}{\partial \dot{r}} = m\dot{r} \quad \frac{\partial T}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

④ String



$$w = \sum_n q_n(t)\sin\frac{n\pi x}{L} \qquad \dot{w} = \sum_n \dot{q}_n(t)\sin\frac{n\pi x}{L}$$

$$T = \tfrac{1}{2}m\int_0^L \dot{w}^2\,dx$$

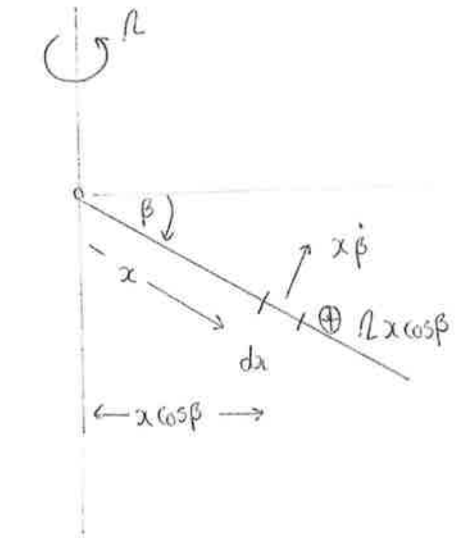
$$\frac{\partial T}{\partial \dot{q}_i} = m\int_0^L \dot{w}\frac{\partial \dot{w}}{\partial \dot{q}_i}\,dx = m\int_0^L \sum_n \dot{q}_n \sin\frac{n\pi x}{L}\sin\frac{i\pi x}{L}\,dx$$

$$= \tfrac{1}{2}Lm\dot{q}_i$$

$$Q_i = \int F(x,t)\sin\frac{n\pi x}{L}\,dx$$

$$U = \tfrac{1}{2}P\int_0^L\left(\frac{\partial w}{\partial x}\right)^2 dx \rightarrow \frac{\partial U}{\partial q_i} = \tfrac{1}{2}P\left(\frac{i\pi}{L}\right)^2 L\, q_i$$

3C5 Helicopter Rotor Blade Example



Length $= L$

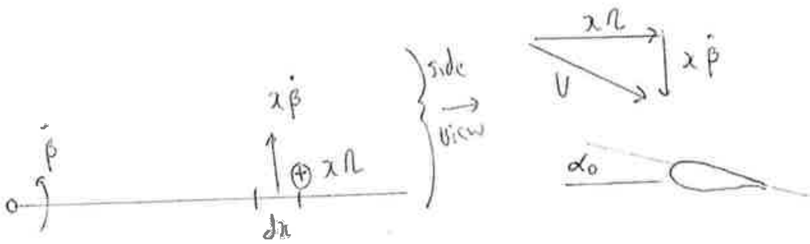
Mass/unit length $= m$

$$\text{Velocity}^2 = (x\dot{\beta})^2 + (\Omega x \cos\beta)^2$$

$$T = \tfrac{1}{2} m \int_0^L \left[x^2\dot{\beta}^2 + x^2\Omega^2\cos^2\beta \right] dx$$

$$\underline{T = \tfrac{1}{6} mL^3 \left[\dot{\beta}^2 + \Omega^2 \cos^2\beta \right]}$$

Generalised Force (take β small) : Lift proportional to ; angle of attack $\times$ velocity^2



$$U \simeq x\Omega$$

$$\alpha \simeq \alpha_0 - \frac{x\dot{\beta}}{x\Omega}$$

$$\text{Lift} = A.dx.\left(\alpha_0 - \frac{x\dot{\beta}}{x\Omega}\right)(x\Omega)^2 = A.dx\left[\alpha_0 x^2\Omega^2 - x\dot{\beta}x\Omega\right] = \hat{L}\,dx$$

Virtual work.

$$\delta W = \int \hat{L}\, x\,\delta\beta\, dx$$

$$Q = \int \hat{L}\, x\, dx$$

$$\underline{Q = \tfrac{1}{3} A\alpha_0 L^3\Omega^2 - \tfrac{1}{3} A\Omega L^3 \dot{\beta}}$$

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{\beta}}\right] - \frac{\partial T}{\partial \beta} = Q$$

$$\tfrac{1}{3} mL^3 \ddot{\beta} + \tfrac{1}{3} mL^3 \sin\beta\cos\beta.\Omega^2 = \tfrac{1}{3} A\alpha_0 L^3 \Omega^2 - \tfrac{1}{3} A\Omega L^3 \dot{\beta}$$

β small $\quad \underline{\ddot{\beta} + c\dot{\beta} + \Omega^2 \beta = d.\alpha_0(t)}$