Module 3F2: Systems and Control
EXAMPLES PAPER 1 - STATE-SPACE MODELS

Solutions

1.

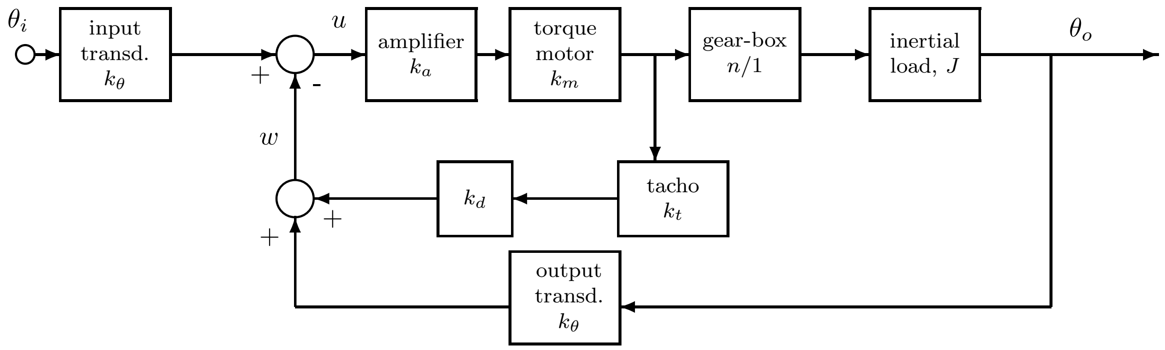


Figure 1:

Note that this is a 'connection diagram' and is not a block diagram in the usual sense. In particular the motor output shaft is connected to a gear box and tachometer, but the motor's output is really torque. An assumption in a block diagram is that the input/output relation of each block is independent of the rest of the system. The underlying equation of motion is:

$$J\ddot{\theta}_o = \text{ torque on } \textbf{output} \text{ shaft} = n \times \text{ motor torque } = nk_m k_a u$$

and together with noting that the tacho measures $n\dot{\theta}_o$, this gives the following block diagram:

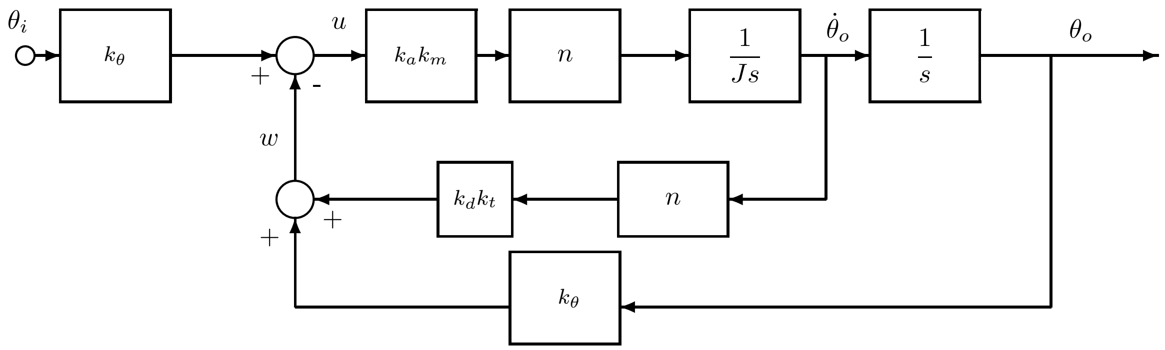


Figure 2:

(a) Defining the state variable as $\underline{x} = \begin{bmatrix} \theta_o \\ \dot{\theta}_o \end{bmatrix}$, the state equations are

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = \ddot{\theta}_o = \frac{1}{J} nk_m k_a u, \quad u = k_\theta \theta_i - w, \quad w = k_d k_t n x_2 + k_\theta x_1$$

Substituting the numerical values and putting in matrix form gives:

$$\begin{aligned} \dot{\underline{x}} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 2.5 \end{bmatrix} u; \quad u = 10\theta_i - w, \\ w &= \begin{bmatrix} 10 & 4k_d \end{bmatrix} \underline{x}, \quad \theta_o = \begin{bmatrix} 1 & 0 \end{bmatrix} \underline{x} \end{aligned}$$

The closed-loop equations are thus:

$$\begin{aligned}\underline{\dot{x}} &= \begin{bmatrix} 0 & 1 \\ -25 & -10k_d \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 25 \end{bmatrix} \theta_i \\ \theta_0 &= \begin{bmatrix} 1 & 0 \end{bmatrix} \underline{x}\end{aligned}$$

(b) The transfer function of the state-space model

$$\dot{x} = Ax + Bu, \qquad y = Cx + Du$$

is $G(s) = C(sI - A)^{-1}B + D$ (from the Lecture Notes). In this case we have $A = \begin{bmatrix} 0 & 1 \\ -25 & -10k_d \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 25 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$, and $D = 0$. Now

$$\begin{aligned}(sI - A)^{-1} &= \begin{bmatrix} s & -1 \\ 25 & s + 10k_d \end{bmatrix}^{-1} \\ &= \frac{\begin{bmatrix} s + 10k_d & 1 \\ -25 & s \end{bmatrix}}{s(s + 10k_d) + 25}\end{aligned}$$

so

$$\begin{aligned}G(s) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \frac{\begin{bmatrix} s + 10k_d & 1 \\ -25 & s \end{bmatrix}}{s(s + 10k_d) + 25} \begin{bmatrix} 0 \\ 25 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} \frac{\begin{bmatrix} 25 \\ 25s \end{bmatrix}}{s(s + 10k_d) + 25} \\ &= \frac{25}{s^2 + 10k_d s + 25}\end{aligned}$$

Check from the block-diagram shown above.

(c) (i) The poles are (a subset of) the eigenvalues of the matrix A, ie they are the roots of $\det(sI - A) = s(s + 10k_d) + 25 = s^2 + 10k_d s + 25$. From the formula for the roots of a quadratic polynomial, the roots are

$$-5k_d \pm 5\sqrt{k_d^2 - 1} = -5k_d \pm j\sqrt{1 - k_d^2}$$

(ii) The poles are the roots of the denominator in (b). Clearly this is the same polynomial as $\det(sI - A)$.

2. **Control system compensators can be implemented using op-amp circuits. For each of the circuits in Figure 3,**

   (a) **Taking the capacitor voltages as internal states write down the state-space equations.**

   (i) By considering the current flowing onto the capacitor we obtain,

   $$C\dot{v}_o = -v_e/R,$$

   and defining $x(t) = v_o(t)$ we get the state space equations:

   $$\begin{aligned}\dot{x}(t) &= 0.x(t) - \frac{1}{CR}v_e(t)\\ y(t) &= 1.x(t)\end{aligned}$$

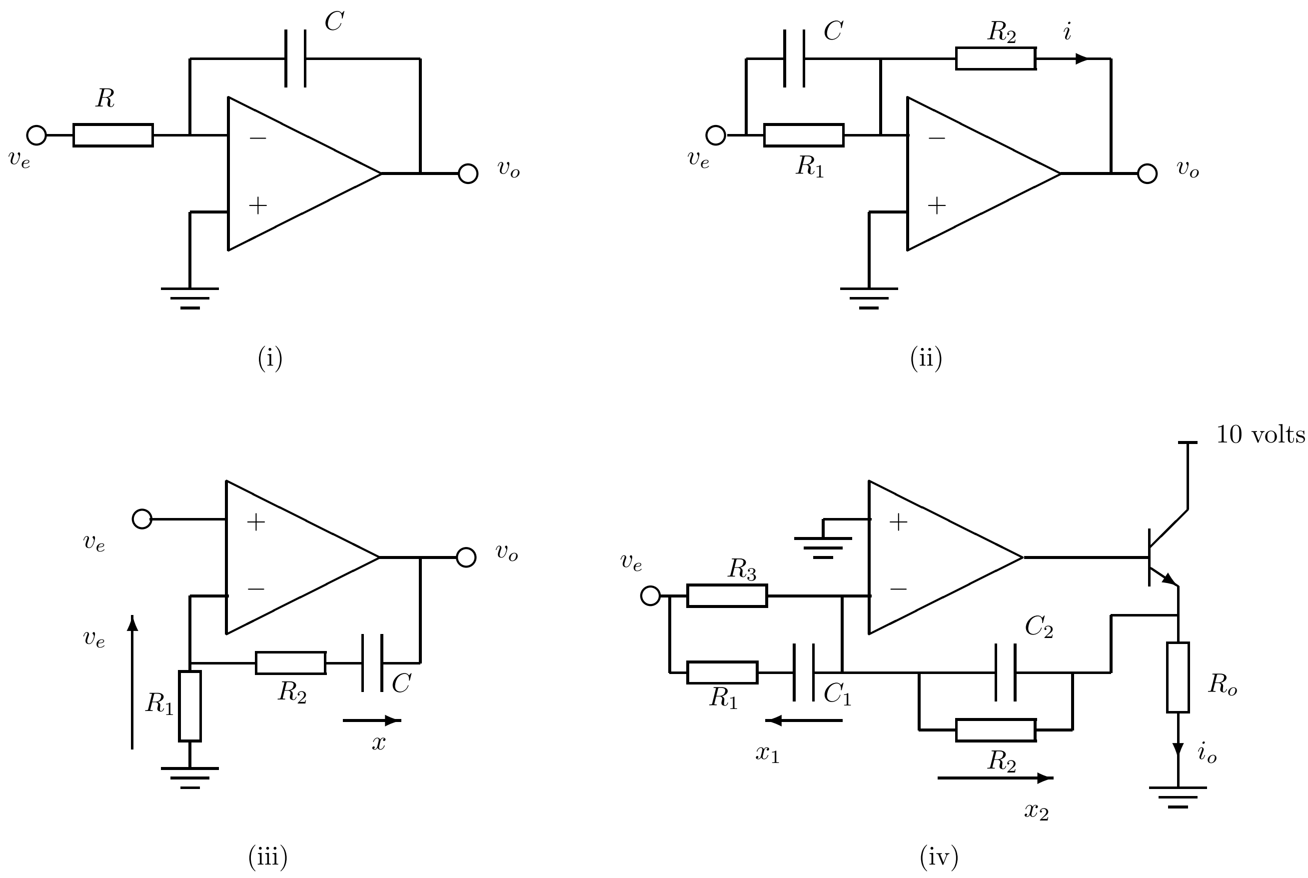


Figure 3:

   (ii) Remembering that no current flows into the input to the op amp, then the current, i, flowing through R_2 is given by,

   $$i = \frac{v_e}{R_1} + C\frac{dv_e}{dt}, \quad \text{and} \quad v_o = -R_2 i$$

   Hence

   $$v_o = -\frac{R_2}{R_1}v_e - R_2C\frac{dv_e}{dt}$$

   but the standard state space form will not allow a differentiator. Hence this circuit does not have a state space form. However if any series resistance is included with the capacitor it will have a state space form.

(iii) The voltage across R_1 is equal to v_e and hence the current flowing onto C is v_e/R_1, giving

$$\begin{aligned} C\dot{x} &= v_e/R_1 \\ v_o &= (R_1+R_2)\frac{v_e}{R_1} + x \\ \dot{x} &= \frac{1}{CR_1}v_e \\ v_o &= x + \frac{(R_1+R_2)}{R_1}v_e \end{aligned}$$

(iv) Calculating the currents flowing onto each capacitor gives:

$$\begin{aligned} C_1\dot{x}_1 &= \frac{v_e - x_1}{R_1} \\ C_2\dot{x}_2 &= -\frac{v_e}{R_3} + \frac{x_1 - v_e}{R_1} - \frac{x_2}{R_2} \\ i_o &= \frac{x_2}{R_o} \end{aligned}$$

and in vector form this gives:

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} -\frac{1}{R_1C_1} & 0 \\ \frac{1}{R_1C_2} & -\frac{1}{R_2C_2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{R_1C_1} \\ -\frac{1}{C_2R_3} - \frac{1}{C_2R_1} \end{bmatrix} v_e \\ i_o &= \begin{bmatrix} 0 & \frac{1}{R_o} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned}$$

(b) **Hence calculate the transfer functions, poles and zeros; and state the type of controller that each implements.**

Using the formula $G(s) = C(sI - A)^{-1}B + D$ (in (i), (iii) and (iv)) gives:

(i) $G(s) = -\frac{1}{CRs}$ with a pole at $s = 0$. Integral action controller.

(ii) $G(s) = -\frac{R_2}{R_1} - R_2Cs$, with a zero at $s = -\frac{1}{R_1C}$. Proportional plus derivative action controller.

(iii) $G(s) = \frac{R_1+R_2}{R_1} + \frac{1}{CR_1s} = \frac{(R_1+R_2)Cs+1}{CR_1s}$, with a pole at $s = 0$ and a zero at $s = -\frac{1}{(R_1+R_2)C}$.

(iv)

$$\begin{aligned} G(s) &= \begin{bmatrix} 0 & \frac{1}{R_o} \end{bmatrix}\begin{bmatrix} s + \frac{1}{R_1C_1} & 0 \\ -\frac{1}{R_1C_2} & s + \frac{1}{R_2C_2} \end{bmatrix}^{-1}\begin{bmatrix} \frac{1}{R_1C_1} \\ -\frac{1}{C_2}\left(\frac{1}{R_3} + \frac{1}{R_1}\right) \end{bmatrix} \\ &= \frac{1}{\left(s + \frac{1}{R_1C_1}\right)\left(s + \frac{1}{R_2C_2}\right)}\frac{1}{R_o}\left[\frac{1}{R_1C_2}.\frac{1}{R_1C_1} - \left(s + \frac{1}{R_1C_1}\right)\frac{1}{C_2}\left(\frac{1}{R_3} + \frac{1}{R_1}\right)\right] \\ &= \frac{-R_2[1 + sC_1(R_1 + R_3)]}{R_0R_3(1 + sC_1R_1)(1 + sC_2R_2)} \end{aligned}$$

with poles at $s = -\frac{1}{R_1C_1}$, $-\frac{1}{R_2C_2}$, and a zero at $s = -\frac{1}{C_1(R_1+R_3)}$.

3. **For the linearisation worked example given in lectures in section 3.3.1 verify the (1,2) element of the matrix P.**

It is clear from the form of the linearised equation that the (1,2) element of the matrix P needs to be $\left.\frac{\partial\ddot{\theta}}{\partial z}\right|_e$. This is obtained from the second equation (giving $\ddot{\theta}$) and differentiating with respect to z whilst holding other variables at the equilibrium values $\theta_e = 0, \dot{\theta}_e = 0, F_{1e} = -Mg, F_{2e} = \sqrt{2}Mg$. The equilibrium values of ϕ_1 and ϕ_2 are given by $\phi_{1e} = 0$ and $\phi_{2e} = \pi/4$ but they are functions of z so we also need to calculate $\frac{\partial\phi_i}{\partial z}$ which are given by differentiating the $\tan\phi_i$ equations giving:

$$\begin{aligned} \sec^2(\phi_1)\frac{\partial\phi_1}{\partial z} &= -\frac{a\sin\theta}{\left(a + z - \frac{1}{2}a\cos\theta\right)^2} \quad \Rightarrow \left.\frac{\partial\phi_1}{\partial z}\right|_e = 0 \\ \sec^2(\phi_2)\frac{\partial\phi_2}{\partial z} &= \frac{1}{\left(a + \frac{1}{2}a\sin\theta\right)} \quad \Rightarrow \left.\frac{\partial\phi_2}{\partial z}\right|_e = \frac{1}{a}\cos^2(\phi_{2e}) = \frac{1}{2a} \end{aligned}$$

Now differentiating the $\ddot{\theta}$ equation wrt z gives:

$$\begin{aligned}
\frac{2}{a}\left(I+\frac{1}{4}Ma^2\right)\left.\frac{\partial\ddot{\theta}}{\partial z}\right|_e &= 0+[\cos(\theta_e+\phi_{1e})+\cos\phi_{1e}\cos\theta_e]\left.\frac{\partial\phi_1}{\partial z}\right|_e F_{1e}+\cos\theta_e(-\sin\phi_{2e})\left.\frac{\partial\phi_2}{\partial z}\right|_e F_{2e} \\
\Rightarrow \left.\frac{\partial\ddot{\theta}}{\partial z}\right|_e &= \frac{a}{2}\left(I+\frac{1}{4}Ma^2\right)^{-1}\left([2]\times 0+1\left(\frac{-1}{\sqrt{2}}\right)\left(\frac{1}{2a}\right)\sqrt{2}Mg\right) = -\frac{1}{4}Mg\left(I+\frac{1}{4}Ma^2\right)^{-1} \\
&= -\frac{1}{2a}\left(\frac{2I}{Mag}+\frac{a}{2g}\right)^{-1}
\end{aligned}$$

as required.

4. **The following are matrices for state-space models in the form $\dot{x}=Ax+Bu$, $y=Cx+Du$. ('$0_{p,m}$' denotes the $p\times m$ zero matrix.) In each case determine (i) how many inputs, states and outputs there are, (ii) the *dimensions* of the transfer function matrix, and (iii) the transfer function matrix:**

(a)

$$A=\begin{bmatrix}-1&0&0\\0&-2&0\\0&0&-3\end{bmatrix},\quad B=\begin{bmatrix}3\\2\\1\end{bmatrix},\quad C=\begin{bmatrix}4&5&6\end{bmatrix},\quad D=0$$

(b)

$$A=\begin{bmatrix}-1&0\\1&-2\end{bmatrix},\quad B=\begin{bmatrix}2\\3\end{bmatrix},\quad C=\begin{bmatrix}1&2\\0&1\end{bmatrix},\quad D=\begin{bmatrix}1\\1\end{bmatrix}$$

(c)

$$A=-2,\qquad B=\begin{bmatrix}1&2\end{bmatrix},\qquad C=\begin{bmatrix}3\\0\\1\end{bmatrix},\quad D=0_{3,2}$$

In all of these the following apply:
(i) Number of inputs = no. of columns of B (or D).
Number of states = no. of rows (or columns) of A.
Number of outputs = no. of rows of C (or D).
(ii) Dimensions of $G(s)$: no. of rows of C (or D) $\times$ no. of columns of B (or D).
Hence (i) and (ii) follow immediately for (a), (b) and (c).
(iii) Use $G(s)=C(sI-A)^{-1}B+D$ in each case:
(a)

$$\begin{aligned}
G(s) &= \begin{bmatrix}4&5&6\end{bmatrix}\begin{bmatrix}\frac{1}{s+1}&0&0\\0&\frac{1}{s+2}&0\\0&0&\frac{1}{s+3}\end{bmatrix}\begin{bmatrix}3\\2\\1\end{bmatrix}+0 \\
&= \begin{bmatrix}4&5&6\end{bmatrix}\begin{bmatrix}\frac{3}{s+1}\\\frac{2}{s+2}\\\frac{1}{s+3}\end{bmatrix}+0 \\
&= \frac{12}{s+1}+\frac{10}{s+2}+\frac{6}{s+3}
\end{aligned}$$

(b)

$$\begin{aligned}
G(s) &= \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s+1 & 0 \\ -1 & s+2 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
&= \frac{\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s+2 & 0 \\ 1 & s+1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}}{(s+1)(s+2)} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
&= \frac{\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2(s+2) \\ 3s+5 \end{bmatrix}}{(s+1)(s+2)} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
&= \frac{\begin{bmatrix} 8s+14 \\ 3s+5 \end{bmatrix}}{(s+1)(s+2)} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
&= \frac{\begin{bmatrix} s^2+11s+16 \\ s^2+6s+7 \end{bmatrix}}{(s+1)(s+2)}
\end{aligned}$$

(c)

$$G(s) = \frac{\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix}}{s+2} + 0_{3,2} = \frac{\begin{bmatrix} 3 & 6 \\ 0 & 0 \\ 1 & 2 \end{bmatrix}}{s+2}$$

5. **Consider the state-space equation,**

$$\dot{\underline{x}}(t) = A\underline{x}(t), \quad \underline{x}(0) = \underline{x}_0.$$

(a) **Show that if $\underline{x}_0$ is an eigenvector of A then $\underline{x}(t) = e^{\lambda t}\underline{x}_0$ if λ is the corresponding eigenvalue.**
$\underline{x}_o$ is an eigenvector of A so $A\underline{x}_o = \lambda \underline{x}_o$. With $\underline{x}(t) = e^{\lambda t}\underline{x}_o$, $\dot{\underline{x}} = \lambda e^{\lambda t}\underline{x}_o$, and $A\underline{x}(t) = e^{\lambda t}A\underline{x}_o = e^{\lambda t}\lambda\underline{x}_o = \dot{\underline{x}}$. Also the initial condition, $\underline{x}(0) = e^0\underline{x}_o = \underline{x}_o$, is satisfied.

(b) **If**

$$A = \begin{bmatrix} 0 & 1 \\ -k & -2 \end{bmatrix}$$

calculate the state transition matrix for $k = -3, 0, 1$ and 5, and verify that part (a) holds for all eigenvectors of A. Are there any non-zero equilibrium states?

$(sI - A)^{-1} = \begin{bmatrix} s & -1 \\ k & s+2 \end{bmatrix}^{-1} = \frac{1}{s^2+2s+k} \begin{bmatrix} s+2 & 1 \\ -k & s \end{bmatrix}$

Eigenvalues of A: $\lambda_1 = -1 - \sqrt{1-k}$, $\lambda_2 = -1 + \sqrt{1-k}$

Eigenvectors of A: $\underline{w_1} = \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix}$, $\underline{w_2} = \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}$

$$\begin{aligned}
e^{At} &= \mathcal{L}^{-1}\left[(sI-A)^{-1}\right] \\
\underline{k=-3:} \quad e^{At} &= \mathcal{L}^{-1}\frac{\begin{bmatrix} s+2 & 1 \\ 3 & s \end{bmatrix}}{(s+3)(s-1)} = \mathcal{L}^{-1}\frac{\begin{bmatrix} -1 & 1 \\ 3 & -3 \end{bmatrix}}{(-4)(s+3)} + \mathcal{L}^{-1}\frac{\begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix}}{(4)(s-1)} \\
&= \frac{e^{-3t}}{4}\begin{bmatrix} 1 \\ -3 \end{bmatrix}\begin{bmatrix} 1 & -1 \end{bmatrix} + \frac{e^t}{4}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\begin{bmatrix} 3 & 1 \end{bmatrix} \\
e^{At}\underline{w}_1 &= e^{At}\begin{bmatrix} 1 \\ -3 \end{bmatrix} \stackrel{\checkmark}{=} e^{-3t}\begin{bmatrix} 1 \\ -3 \end{bmatrix}; \; e^{At}\underline{w}_2 = e^{At}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \stackrel{\checkmark}{=} e^{t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\underline{k=0:}\quad e^{At} &= \mathcal{L}^{-1}\frac{\begin{bmatrix} s+2 & 1 \\ 0 & s \end{bmatrix}}{s(s+2)} = \mathcal{L}^{-1}\frac{\begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{bmatrix}}{(s)} + \mathcal{L}^{-1}\frac{\begin{bmatrix} 0 & \frac{-1}{2} \\ 0 & 1 \end{bmatrix}}{(s+2)} \\
&= \begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 1 & \frac{1}{2} \end{bmatrix} + e^{-2t}\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \end{bmatrix} \\
e^{At}\underline{w}_1 &= e^{At}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \overset{\checkmark}{=} e^{0t}\begin{bmatrix} 1 \\ 0 \end{bmatrix};\ e^{At}\underline{w}_2 = e^{At}\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} \overset{\checkmark}{=} e^{-2t}\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} \\
\underline{k=1:}\quad e^{At} &= \mathcal{L}^{-1}\frac{\begin{bmatrix} s+2 & 1 \\ -1 & s \end{bmatrix}}{(s+1)^2} = \mathcal{L}^{-1}\frac{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}{(s+1)} + \mathcal{L}^{-1}\frac{\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}}{(s+1)^2} \\
&= e^{-t}\begin{bmatrix} 1+t & t \\ -t & 1-t \end{bmatrix}
\end{aligned}$$

The repeated eigenvalue, $\lambda_1 = \lambda_2 = -1$, with just one eigenvector, $\underline{w}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.
$e^{At}\underline{w}_1 = e^{At}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \overset{\checkmark}{=} e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

$$\begin{aligned}
\underline{k=5:}\quad e^{At} &= \mathcal{L}^{-1}\frac{\begin{bmatrix} s+2 & 1 \\ -5 & s \end{bmatrix}}{(s+1)^2+2^2} = e^{-t}\begin{bmatrix} \cos 2t + \frac{1}{2}\sin 2t & \frac{1}{2}\sin 2t \\ -\frac{5}{2}\sin 2t & \cos 2t - \frac{1}{2}\sin 2t \end{bmatrix} \\
\lambda_1 &= -1+j2,\quad \lambda_2 = -1-j2 \\
e^{At}\begin{bmatrix} 1 \\ -1+j2 \end{bmatrix} &= e^{-t}\begin{bmatrix} \cos 2t + j\sin 2t \\ (-1+j2)(\cos 2t + j\sin 2t) \end{bmatrix} \overset{\checkmark}{=} e^{-t}e^{j2t}\begin{bmatrix} 1 \\ -1+j2 \end{bmatrix}
\end{aligned}$$

The equilibrium states, $\underline{x}_e$, satisfy $A\underline{x}_e = \underline{0}$ which implies that $\underline{x}_e = \underline{0}$ unless $\det(A) = 0$. Now $\det(A) = k$ and hence when $k = 0$, $\underline{x}_e = \begin{bmatrix} \alpha \\ 0 \end{bmatrix}$ is an equilibrium state for any α.

(c) **For the circuit of 2(iv) determine intial states, $\underline{x}_0$, such that the resulting responses with $v_e(t) = 0$ will be $\underline{x}(t) = e^{-t/R_2C_2}\underline{x}_0$ and $\underline{x}(t) = e^{-t/R_1C_1}\underline{x}_0$.**

$A = \begin{bmatrix} -\frac{1}{R_1C_1} & 0 \\ \frac{1}{R_1C_2} & -\frac{1}{R_2C_2} \end{bmatrix}$ and we want the response of $\dot{\underline{x}} = A\underline{x}$. This will be $\underline{x}(t) = e^{\lambda t}\underline{x}_o$ if $A\underline{x}_o = \lambda\underline{x}_o$, i.e. $\underline{x}_o$ is an eigenvector of A with corresponding eigenvalue λ.

$$\begin{aligned}
\lambda_1 &= -\frac{1}{R_1C_1},\quad \underline{w}_1 = \begin{bmatrix} \frac{1}{R_2C_2} - \frac{1}{R_1C1} \\ \frac{1}{C_2R_1} \end{bmatrix} \\
\lambda_2 &= -\frac{1}{R_2C_2},\quad \underline{w}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\end{aligned}$$

6. **A system's dynamical behaviour is defined by the state-space equation set**

$$\frac{d\underline{x}}{dt} = \underbrace{\begin{bmatrix} 0 & 1 & 3 \\ -1 & 0 & 1 \\ 0 & 0 & 3 \end{bmatrix}}_{A}\underline{x} + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{B}u,\qquad y = \underbrace{\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}}_{C}\underline{x}.$$

(a) **Find a change of state variables described by**

$$\underline{z} = T^{-1}\underline{x}$$

where T is a complex nonsingular matrix such that the state equations for $\underline{z}$ are in diagonal form, and find the appropriately transformed state equations.

To diagonalize A we need T to be a matrix of eigenvectors, so firstly find the eigenvalues and eigenvectors:

$$
\begin{aligned}
\det(\lambda I - A) &= \det \begin{bmatrix} \lambda & -1 & -3 \\ 1 & \lambda & -1 \\ 0 & 0 & \lambda - 3 \end{bmatrix} = (\lambda^2+1)(\lambda-3) \\
\lambda_1 = +j : \underline{0} &= (\lambda_1 I - A)\underline{w}_1 = \begin{bmatrix} j & -1 & -3 \\ 1 & j & -1 \\ 0 & 0 & j-3 \end{bmatrix} \underbrace{\begin{bmatrix} 1 \\ j \\ 0 \end{bmatrix}}_{\underline{w}_1} \\
\lambda_2 = -j : \underline{w}_2 &= \begin{bmatrix} 1 \\ -j \\ 0 \end{bmatrix} \\
\lambda_3 = 3 : \underline{0} &= \begin{bmatrix} 3 & -1 & -3 \\ 1 & 3 & -1 \\ 0 & 0 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}_{\underline{w}_3}
\end{aligned}
$$

$$
\begin{aligned}
T &= \begin{bmatrix} 1 & 1 & 1 \\ j & -j & 0 \\ 0 & 0 & 1 \end{bmatrix} ; \; T^{-1} = \frac{1}{2} \begin{bmatrix} 1 & -j & -1 \\ 1 & j & -1 \\ 0 & 0 & 2 \end{bmatrix} \\
T^{-1}AT &= \Lambda = \begin{bmatrix} j & 0 & 0 \\ 0 & -j & 0 \\ 0 & 0 & 3 \end{bmatrix}, \; T^{-1}B = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \overline{B} \\
CT &= \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} = \overline{C}
\end{aligned}
$$

For $\underline{z} = T^{-1}\underline{x}$ we have the state equations:

$$\dot{\underline{z}} = \Lambda \underline{z} + \overline{B}\underline{u}, \quad \underline{y} = \overline{C}\underline{z}$$

Note that this 'diagonal' state space realisation is not unique since we could introduce additional diagonal scaling of the states.

(b) **Determine the system's state transition matrix.**

$$
\begin{aligned}
e^{At} &= Te^{\Lambda t}T^{-1} \qquad \text{(from lecture notes)} \\
&= \begin{bmatrix} 1 & 1 & 1 \\ j & -j & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e^{jt} & 0 & 0 \\ 0 & e^{-jt} & 0 \\ 0 & 0 & e^{3t} \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & -j & -1 \\ 1 & j & -1 \\ 0 & 0 & 2 \end{bmatrix} \\
&= \begin{bmatrix} e^{jt} & e^{-jt} & e^{3t} \\ je^{jt} & -je^{-jt} & 0 \\ 0 & 0 & e^{3t} \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & -j & -1 \\ 1 & j & -1 \\ 0 & 0 & 2 \end{bmatrix} \\
&= \begin{bmatrix} \cos t & \sin t & -\cos t + e^{3t} \\ -\sin t & \cos t & \sin t \\ 0 & 0 & e^{3t} \end{bmatrix}
\end{aligned}
$$

(c) **If** $\underline{x}(0) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ **and the input** $u(t) = 1$ **for** $t \geq 0$**, find the resulting output** $y(t)$ **for** $t \geq 0$.

Substituting the expression for e^{At} into the general expression for $y(t)$ gives:

$$
\begin{aligned}
y(t) &= Ce^{At}\underline{x}(0) + \int_0^t Ce^{A(t-\tau)}Bu(\tau)\, d\tau \\
&= -\cos t + e^{3t} + \int_0^t \cos(t-\tau)\, d\tau \\
&= -\cos t + e^{3t} + [-\sin(t-\tau)]_0^t \\
&= e^{3t} - \cos t + \sin t.
\end{aligned}
$$

(d) **Repeat parts (b) and (c) using Laplace transforms.**

$$(sI-A)^{-1} = \begin{bmatrix} s & -1 & -3 \\ 1 & s & -1 \\ 0 & 0 & s-3 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{s}{s^2+1} & \frac{1}{s^2+1} & \frac{3s+1}{(s-3)(s^2+1)} \\ \frac{-1}{s^2+1} & \frac{s}{s^2+1} & \frac{(s-3)}{(s-3)(s^2+1)} \\ 0 & 0 & \frac{1}{s-3} \end{bmatrix}$$

$$\text{and } \frac{3s+1}{(s-3)(s^2+1)} = \frac{1}{s-3} - \frac{s}{s^2+1}$$

and this gives e^{At} as above.
In (c),

$$Y(s) = C(sI-A)^{-1}\underline{x}(0) + C(sI-A)^{-1}B \times \frac{1}{s} = \left(\frac{1}{s-3} - \frac{s}{s^2+1}\right) + \frac{s}{s^2+1} \times \frac{1}{s}$$

$\Rightarrow y(t)$ as above.

7. **Figure 4 represents a two-link manipulator in a vertical plane, to be controlled by the two motors at the joints producing torques T_1 and T_2 as shown. Ignoring frictional and damping terms this particular system satisfies the following differential equations (where the dots over symbols denote differentiation with respect to time):**

$$T_1 = -(14.25 + 4\cos\theta_2)\ddot{\theta}_1 - (1.5 + 2\cos\theta_2)\ddot{\theta}_2 + 120\sin\theta_1 + 20\sin(\theta_1+\theta_2) + 2\dot{\theta}_2(2\dot{\theta}_1 + \dot{\theta}_2)\sin\theta_2 \quad (0.1)$$

$$T_2 = -(1.5 + 2\cos\theta_2)\ddot{\theta}_1 - \ddot{\theta}_2 + 20\sin(\theta_1+\theta_2) - 2\dot{\theta}_1^2\sin\theta_2 \quad (0.2)$$

[T_1 and T_2 in Nm, θ_1 in radians and time in seconds].

(a) **Calculate the torques T_{1e}, T_{2e} required to maintain the system in equilibrium at $\theta_1 = \pi/6$ and $\theta_2 = \pi/3$.**
In equilibrium $\dot{\theta}_1 = \dot{\theta}_2 = 0$, $\ddot{\theta}_1 = \ddot{\theta}_2 = 0$, $\theta_1 = \pi/6$ and $\theta_2 = \pi/3$. Hence

$$T_{1e} = 120\sin(\pi/6) + 20\sin(\pi/2) = 80 \text{ Nm}$$
$$T_{2e} = 20\sin(\pi/2) = 20 \text{ Nm}$$
$$\underline{x} = \begin{bmatrix} \theta_1 - \pi/6 \\ \dot{\theta}_1 \\ \theta_2 - \pi/3 \\ \dot{\theta}_2 \end{bmatrix}, \quad \underline{u} = \begin{bmatrix} T_1 - T_{1e} \\ T_2 - T_{2e} \end{bmatrix}$$

Although the linearization about this equilibrium is given in the question, let's derive it for completeness. Equations (0.1) and (0.2) together with $\dot{x}_1 = x_2$ and $\dot{x}_3 = x_4$ give four equations in the four unknowns $\dot{x}_1, \dot{x}_2, \dot{x}_3, \dot{x}_4$ that we want to linearise. Using the notation of section 3.2 in the notes we have

$$\underline{F}(\underline{\dot{x}}, \underline{x}, \underline{u}) = \begin{bmatrix} \dot{x}_1 - x_2 \\ \dot{x}_3 - x_4 \\ F_3 \\ F_4 \end{bmatrix} = \underline{0}$$

where

$$F_3 = u_1 + T_{1e} + (14.25 + 4\cos(x_3 + \pi/3))\,\dot{x}_2 + (1.5 + 2\cos(x_3+\pi/3))\,\dot{x}_4 - 120\sin(x_1 + \pi/6) - 20\sin(x_1 + x_3 + \pi/2) - 2x_4(2x_2 + x_4)\sin(x_3 + \pi/3)$$

$$F_4 = u_2 + T_{2e} + (1.5 + 2\cos(x_3 + \pi/3))\,\dot{x}_2 + \dot{x}_4 - 20\sin(x_1 + x_3 + \pi/2) + 2x_2^2\sin(x_3 + \pi/3)$$

Linearizing this equation gives:

$$L = \left.\frac{\partial \underline{F}}{\partial \dot{\underline{x}}}\right|_{(\underline{0},\underline{0},\underline{0})} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 14.25 + 4\cos(\pi/3) & 0 & 1.5 + 2\cos(\pi/3) \\ 0 & 1.5 + 2\cos(\pi/3) & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 16.25 & 0 & 2.5 \\ 0 & 2.5 & 0 & 1 \end{bmatrix}$$

$$M = \left.\frac{\partial \underline{F}}{\partial \underline{x}}\right|_{(\underline{0},\underline{0},\underline{0})} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ -120\cos(\pi/6) \\ -20\cos(\pi/2) & 0 & -20\cos(\pi/2) & 0 \\ -20\cos(\pi/2) & 0 & -20\cos(\pi/2) & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ -60\sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$N = \left.\frac{\partial \underline{F}}{\partial \underline{u}}\right|_{(\underline{0},\underline{0},\underline{0})} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

which gives,

$$\dot{\underline{x}} \cong -L^{-1}M\underline{x} - L^{-1}N\underline{u} = A\underline{x} + B\underline{u}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \alpha^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\beta & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ -0.1 & 0.25 \\ 0 & 0 \\ 0.25 & -1.625 \end{bmatrix}$$

and $\alpha^2 = 6\sqrt{3}$ and $\beta = 15\sqrt{3} = 5\alpha^2/2$.

(b) **What are the open-loop poles of this linearized system? Determine an initial condition such that $\underline{x}(t) \to \underline{0}$ as $t \to \infty$.**

Open loop poles = eigenvalues of A which satisfy,

$$0 = \det(\lambda I - A) = \det \begin{bmatrix} \lambda & -1 & 0 & 0 \\ -\alpha^2 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & -1 \\ \beta & 0 & 0 & \lambda \end{bmatrix} = (\lambda^2 - \alpha^2)(\lambda^2)$$

Hence poles are at $\pm\alpha, 0, 0$. Now only one of these is stable, namely $-\alpha$, so that if $\underline{x}(t) \to \underline{0}$ as $t \to \infty$ we must have $\underline{x}_0$ the corresponding eigenvector of A (recall Q.4) i.e.

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ \alpha^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\beta & 0 & 0 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} 2 \\ -2\alpha \\ -5 \\ 5\alpha \end{bmatrix}}_{\underline{x}_o} = -\alpha \begin{bmatrix} 2 \\ -2\alpha \\ -5 \\ 5\alpha \end{bmatrix} \quad \Rightarrow \quad \underline{x}(t) = e^{-\alpha t}\underline{x}_o \to \underline{0} \text{ as } t \to \infty$$

(c) **Calculate e^{At}. (Note that $\mathcal{L}(\sinh(\alpha t) - \alpha t) = \alpha^3/s^2(s^2 - \alpha^2)$.)** Since A and hence $(sI - A)$ are lower block triangular it is easily verified that:

$$(sI - A)^{-1} = \begin{bmatrix} (sI - A_{11})^{-1} & 0 \\ (sI - A_{22})^{-1}A_{21}(sI - A_{11})^{-1} & (sI - A_{22})^{-1} \end{bmatrix}$$

$$(sI - A_{11})^{-1} = \frac{1}{s^2 - \alpha^2}\begin{bmatrix} s & 1 \\ \alpha^2 & s \end{bmatrix}; (sI - A_{22})^{-1} \begin{bmatrix} \frac{1}{s} & \frac{1}{s^2} \\ 0 & \frac{1}{s} \end{bmatrix}$$

$$(sI - A_{22})^{-1}\begin{bmatrix} 0 & 0 \\ -\beta & 0 \end{bmatrix}(sI - A_{11})^{-1} = -\beta \begin{bmatrix} \frac{1}{s^2} \\ \frac{1}{s} \end{bmatrix} \begin{bmatrix} s & 1 \end{bmatrix} \frac{1}{s^2 - \alpha^2} = \frac{-\beta}{s^2 - \alpha^2}\begin{bmatrix} \frac{1}{s} & \frac{1}{s^2} \\ 1 & \frac{1}{s} \end{bmatrix}$$

The following inverse Laplace transforms can be obtained from the first by integration since each is the previous one multiplied by α/s.

$$\begin{aligned}
\mathcal{L}^{-1}\frac{s}{s^2-\alpha^2} &= \cosh(\alpha t);\;\; \mathcal{L}^{-1}\frac{\alpha}{s^2-\alpha^2} = \sinh(\alpha t) \\
\mathcal{L}^{-1}\frac{\alpha^2}{s(s^2-\alpha^2)} &= \cosh(\alpha t) - 1;\;\; \mathcal{L}^{-1}\frac{\alpha^3}{s^2(s^2-\alpha^2)} = \sinh(\alpha t) - \alpha t
\end{aligned}$$

and substituting these into the expression for $(sI-A)^{-1}$ gives:

$$e^{At} = \begin{bmatrix} \cosh\alpha t & \alpha^{-1}\sinh\alpha t & 0 & 0 \\ \alpha\sinh\alpha t & \cosh\alpha t & 0 & 0 \\ -\beta\alpha^{-2}(\cosh\alpha t - 1) & -\beta\alpha^{-3}(\sinh\alpha t - \alpha t) & 1 & t \\ -\beta\alpha^{-1}\sinh\alpha t & -\beta\alpha^{-2}(\cosh\alpha t - 1) & 0 & 1 \end{bmatrix}, \quad \frac{\beta}{\alpha^2} = \frac{5}{2}$$

(d) **If you can release the system from an initial condition and measure the states how could you measure the $(3,2)$ element of the state transition matrix?**

Let the initial conditions be $T_1 = T_{1e}$ and $T_2 = T_{2e}$, $\theta_1(0) = \pi/6$, $\theta_2(0) = \pi/3$, and with the second link at zero relative angular velocity, $\dot{\theta}_2 = 0$, but $\dot{\theta}_1 = x_{2o}$ the initial (small) angular velocity of the first link. (experimentally this would be quite tricky to set up!). When the system is released from this initial condition the response should be,

$$\underline{x}(t) = e^{At}\begin{bmatrix} 0 \\ x_{2o} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \phi_{12}(t) \\ \phi_{22}(t) \\ \phi_{32}(t) \\ \phi_{42}(t) \end{bmatrix} x_{2o}$$

Hence the element $\phi_{32}(t)$ could be measured by observing the response of the third state, $\theta_2(t) - \pi/3$, for this initial condition response.

(e) **Explain the physical reasons for the difference in response when the system is released from a small displacement in θ_1 and θ_2 .** The response due to changes in $x_1(0)$ is unstable since if θ_1 becomes smaller the torque, T_{1e}, will overcome gravity and make θ_1 continue to decrease, lifting the arm. If θ_1 becomes larger the gravity torque will increase and the arm will accelerate downwards. On the other hand since the second arm is horizontal its torque due to gravity will not change for small changes in θ_2.

(f) **Calculate the transfer function from u_2 to x_3 , (Hint: this only depends on the $(3,2)$ and $(3,4)$ elements of $(sI-A)^{-1}$ and the second column of B), and hence deduce the response of x_3 due to a step input on u_2.**

The transfer function from u_2 to x_3 is,

$$\begin{aligned}
G_{32} &= \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}(sI-A)^{-1}B\begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= \text{3rd row of } (sI-A)^{-1} \times \text{second column of } B \\
&= \begin{bmatrix} \frac{-\beta}{s(s^2-\alpha^2)} & \frac{-\beta}{s^2(s^2-\alpha^2)} & \frac{1}{s} & \frac{1}{s^2} \end{bmatrix}\begin{bmatrix} 0 \\ 0.25 \\ 0 \\ -1.625 \end{bmatrix} \\
&= -\frac{1.625}{s^2} - \frac{0.25\beta}{s^2(s^2-\alpha^2)} = -\frac{1.625}{s^2} - \frac{0.25\beta}{\alpha^2}\left\{-\frac{1}{s^2} + \frac{1}{(s^2-\alpha^2)}\right\} \\
&= -\frac{1}{s^2} - \frac{0.625}{s^2-\alpha^2}
\end{aligned}$$

$\Rightarrow$ Impulse response: $g_{32}(t) = -t - \frac{0.625}{\alpha}\sinh(\alpha t)$,
$\Rightarrow$ response of $x_3(t)$ to a step on $u_2 = -\frac{1}{2}t^2 - \frac{0.625}{\alpha^2}(\cosh(\alpha t) - 1)$.

(g) **Suppose that this system is connected to a digital computer. The ADC measures x_1 and x_3 and the DAC controls u_1 and u_2, all synchronised with sampling period T . Assuming that $u_1(kT) = 0$ for all k, calculate the state-space difference equation relating the number sequences $u_2(kT)$ output by the computer and the sequence $\underline{x}(kT)$.**

Note: In principle students should be able to do this part as an application of the convolution integral. If they find it too difficult, this topic will be covered briefly later in lectures.

Sampled data systems with sampling period, T, will be considered later in the notes. Following the notation in that section we have,

$$\underline{x}((k+1)T) = \Phi \underline{x}(kT) + \underbrace{\int_0^T e^{At} dt B \begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\Gamma} u_2(kT)$$

where

$$\begin{aligned}
\Phi &= e^{AT} \quad \text{as in (c) above} \\
\Gamma &= \int_0^T \begin{bmatrix} \alpha^{-1} 0.25 \sinh(\alpha t) \\ 0.25 \cosh(\alpha t) \\ -0.25\beta\alpha^{-3}(\sinh(\alpha t) - \alpha t) - 1.625t \\ -0.25\beta\alpha^{-2}(\cosh(\alpha t) - 1) - 1.625 \end{bmatrix} dt \\
&= \frac{1}{8} \begin{bmatrix} 2C \\ 2S \\ -5C - 4T^2 \\ -5S - 8T \end{bmatrix}, \quad \begin{aligned} C &= (\cosh \alpha T - 1)/\alpha^2 \\ S &= \sinh(\alpha T)/\alpha \end{aligned}
\end{aligned}$$

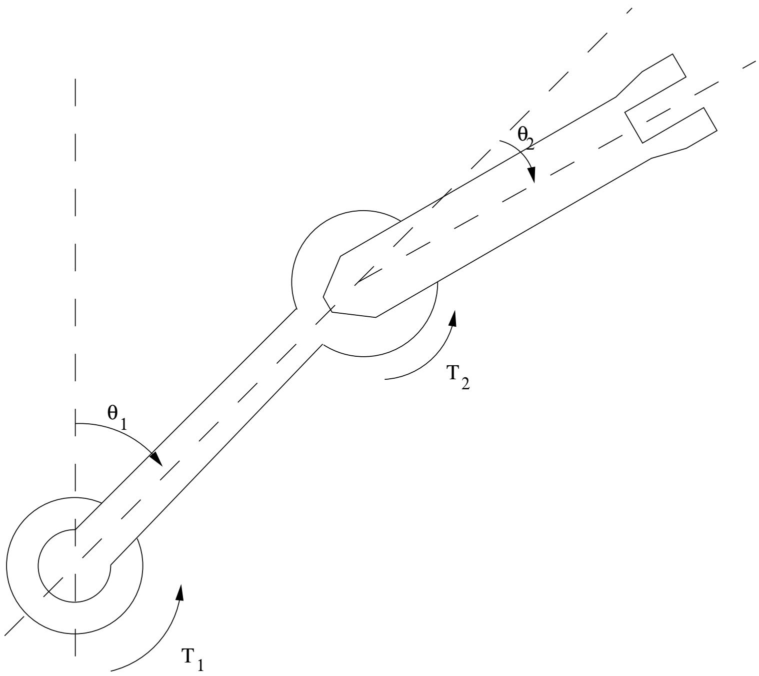


Figure 4: Robot arm

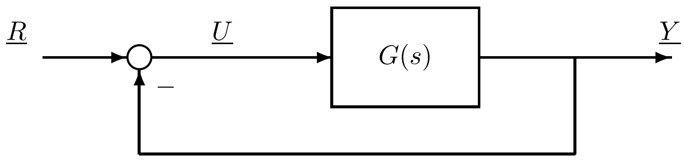
8. (a) **A feedback system is given by**

$$\underline{Y}(s) = G(s)\left(\underline{R}(s) - \underline{Y}(s)\right)$$

where $G(s) = C(sI - A)^{-1} B$. **By considering the corresponding state-equations deduce that** $\underline{Y}(s) = H_{CL}(s)\underline{R}(s)$ **where**

$$H_{CL}(s) = (I + G(s))^{-1} G(s) = C(sI - A + BC)^{-1} B$$

The block diagram will be:



and the state space equations for $G(s)$ are given by:

$$\begin{aligned}
\dot{\underline{x}} &= A\underline{x} + B\underline{u}, \quad \underline{y} = C\underline{x} \\
\underline{u} &= \underline{r} - \underline{y} = \underline{r} - C\underline{x} \\
\Rightarrow \dot{\underline{x}} &= A\underline{x} + B\underline{r} - B(C\underline{x}) \\
&= (A - BC)\underline{x} + B\underline{r} \\
\underline{y} &= C\underline{x} \\
\Rightarrow \underline{Y}(s) &= H_{CL}(s)\underline{R}(s) \text{ with } H_{CL}(s) = C\left(sI - A + BC\right)^{-1} B.
\end{aligned}$$

Optional extra: **Verify this identity using the matrix inversion lemma.**

$$\begin{aligned}
H_{CL} &= (I+G)^{-1}G = \left(I + C(sI-A)^{-1}B\right)^{-1} C(sI-A)^{-1}B \\
&= \left[I - C(sI-A+BC)^{-1}B\right] C(sI-A)^{-1}B \quad \text{by Matrix Inversion Lemma} \\
&= C\left[I - (sI-A+BC)^{-1}BC\right](sI-A)^{-1}B \\
&= C(sI-A+BC)^{-1}\left[(sI-A+BC) - BC\right](sI-A)^{-1}B \\
&= C(sI-A+BC)^{-1}B \quad \text{as required.}
\end{aligned}$$

(b) **In the equations**

$$\begin{aligned}
\dot{\underline{x}}(t) &= A\underline{x}(t) + B\underline{u}(t) \\
\underline{y}(t) &= C\underline{x}(t) + D\underline{u}(t)
\end{aligned}$$

when D is $p \times p$ and invertible, substitute $\underline{u}(t) = D^{-1}\left(\underline{y}(t) - C\underline{x}(t)\right)$ and hence show that the transfer function of the inverse systems is given by,

$$\left(D + C\left(sI - A\right)^{-1}B\right)^{-1} = D^{-1} - D^{-1}C\left(sI - A + BD^{-1}C\right)^{-1}BD^{-1}$$

In this case the state equations will be:

$$\begin{aligned}
\dot{\underline{x}} &= A\underline{x} + B\underline{u}; \quad \underline{y} = C\underline{x} + D\underline{u} \quad \Rightarrow \underline{u} = D^{-1}\underline{y} - D^{-1}C\underline{x} \\
\Rightarrow \dot{\underline{x}} &= A\underline{x} + BD^{-1}\underline{y} - BD^{-1}C\underline{x} = (A - BD^{-1}C)\underline{x} + BD^{-1}\underline{y} \\
\underline{u} &= -D^{-1}C\underline{x} + D^{-1}\underline{y} \\
\Rightarrow \underline{U}(s) &= \left[D^{-1} - D^{-1}C\left(sI - A + BD^{-1}C\right)^{-1}BD^{-1}\right]\underline{Y}(s) \\
&= \left[D + C(sI-A)^{-1}B\right]^{-1}\underline{Y}(s)
\end{aligned}$$

Optional extra: In this case verification is immediate from the Matrix Inversion Lemma.