# Examples Paper 3: Unconstrained Optimization

*Tripos standard questions are marked **

*Problem Formulation*

1. An engineer has been asked to design a can that is closed at one end (and open at the other) using the smallest area of sheet metal for a specified interior volume of 600 cm$^3$. The can is a right circular cylinder with interior height $h$ and radius $r$. The ratio of height to diameter must not be less than 1.0 and not greater than 1.5. The height cannot be more than 20 cm. Formulate the design optimization problem. You can assume that the thickness of the sheet metal is much less than other dimensions.

2. A beam of rectangular cross-section with breadth $b$ and depth $d$ is subjected to a maximum bending moment of $M$ and a maximum shear force of $V$. The maximum bending stress in the beam is calculated as
$$\sigma = \frac{6M}{bd^2}$$
and the average shear stress in the beam is calculated as
$$\tau = \frac{3V}{2bd}$$
The allowable maximum bending stress is $\sigma_a$ and the allowable maximum average shear stress is $\tau_a$. It is also desired that the depth of the beam should not exceed twice its breadth. Formulate the design problem for minimum cross-sectional area.

*3. A chain is suspended from two hooks that are a distance $L$ apart on a horizontal line. The chain itself consists of $n$ links of stiff steel. Each link has length $d$ (measured inside) and a uniform mass $m$. Formulate the optimization problem to determine the equilibrium shape of the chain. (Hint: You want to minimize the potential energy of the chain subject to appropriate geometrical constraints.)

*Optimality Criteria*

4. A first-order low-pass filter is needed to separate a signal from white noise. The filter is of the form
$$S(t) + n(t) \rightarrow \boxed{G(s)} \rightarrow x(t)$$
The optimal filter is obtained by minimizing its mean-square error. It can be shown that for a simple first-order low-pass filter separating a Gauss-Markov signal from white noise, the mean-square error is given by
$$E(e^2) = \frac{\sigma^2 \beta T}{1 + \beta T} + \frac{A}{2T}$$
where $\sigma$, $\beta$ and $A$ are positive constants.
Find the value of the (non-negative) parameter $T$ which minimizes $E$ and show that $\sigma > \sqrt{A\beta/2}$ is a necessary condition for a minimum.

*5. The annual power requirements of an electricity supply company are given by $h(x)$ as shown in Figure 1.

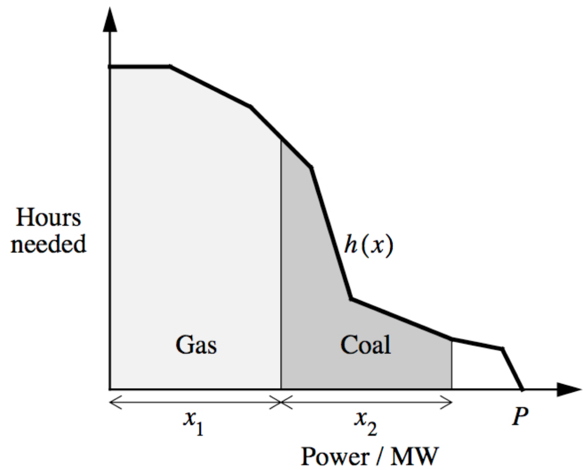


Figure 1.

The company can meet its requirements by installing gas and coal-fired plants and by purchasing from the central grid (for satisfying peak demand only). Let $x_1$ and $x_2$ signify the capacities of the gas and coal plants respectively. Let $b_1$ and $b_2$ be their yearly capital costs (per MW of capacity), $c_1$ and $c_2$ be their operational costs (per MWhr generated) and $c_3$ be the price of the power purchased from the grid (per MWhr).

(a) Determine the electricity company's total cost function, which it aims to minimize, stating any constraints.

(b) Assuming that the optimal solution is interior to the constraints (i.e. this can be treated as an unconstrained problem), find the necessary conditions governing the optimal solution. How could you ensure that these conditions govern a minimum?

*Line search*

6. A shaft with a hole is subject to a bending moment (see Figure 2). There is some flexibility as to the required size of hole, but the restrictions are that $0.05 \leq a/d \leq 0.15$.


Figure 2.

From stress theory, for a given bending moment and shaft diameter, the maximum stress $\sigma_{\max}$ on the shaft is given by

$$\sigma_{\max} = \frac{C}{(a/d)^{0.148}} \left[ \frac{1}{\pi/32 - (a/d)/6} \right]$$

(a) Using the Golden Section method, find the ratio $a/d$ that minimizes the maximum stress on the shaft (to an accuracy of 0.02) for a given value of bending moment $M$.

(b) Perform two iterations of Newton's method and, using your results as an example, briefly discuss the advantages and disadvantages of both methods.

*Multidimensional search*

7. For the function

$$f(x_1, x_2, x_3) = x_1^2 + 2x_2^2 + 2x_3^2 + 2x_1x_2 + 2x_2x_3$$

determine the point which minimizes the value of the function using Newton's method, starting from $(2, 4, 10)$

8. For the function

$$f(x_1, x_2) = 25x_1^2 + 20x_2^2 - x_1 - x_2$$

and the starting point $(3, 1)$,

(a) complete two iterations of (i) the steepest descent method, and (ii) the conjugate gradient method.

(b) Show that the search directions at successive iterations are orthogonal for the steepest descent method and conjugate with respect to the Hessian matrix for the conjugate gradient method.

9. The circuit given in Figure 3 below is to provide a voltage that varies with temperature $\theta$ in a prescribed way. $T_1$ and $T_2$ are thermistors whose resistances $r_1(\theta)$ and $r_2(\theta)$ as functions of temperature have been measured. It is desired that the voltage $V_0$ match a desired voltage function $V_d(\theta)$ by design of the resistances $R_1, R_2, R_3$ (note that $R_4$ is to be left fixed).

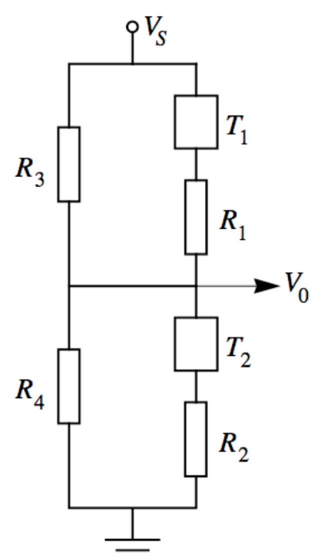


Figure 3.

(a) Formulate the problem as a non-linear least squares problem.

(b) Describe in some detail the calculations required to solve the least-squares problem using (i) the steepest descent method, (ii) Newton's method, and (iii) the Gauss-Newton method, giving the rationale of each method.

**Answers**

1. Check with your supervisor
2. Check with your supervisor
3. Minimize $f(y) = -mg\sum_{i=1}^{n}\left(n-i+\frac{1}{2}\right)y_i$ subject to $\sum_{i=1}^{n} y_i = 0$ and $\sum_{i=1}^{n}\sqrt{d^2-y_i^2} = L$
4. $\frac{\sqrt{A}}{\sigma\sqrt{2\beta}-\beta\sqrt{A}}$
5. (a) Minimize $f(x) = b_1x_1 + b_2x_2 + c_1\int_0^{x_1} h(x)dx + c_2\int_{x_1}^{x_1+x_2} h(x)dx + c_3\int_{x_1+x_2}^{P} h(x)dx$ subject to $x_1 \geq 0$, $x_2 \geq 0$ and $x_1 + x_2 \leq P$.

   (b) $b_1 + (c_1-c_2)h(x_1) + (c_2-c_3)h(x_1+x_2) = 0$
   $b_2 + (c_2-c_3)h(x_1+x_2) = 0$
   $(c_1-c_2)h'(x_1)(c_2-c_3)h'(x_1+x_2) \geq 0$
   $(c_1-c_2)h'(x_1) + (c_2-c_3)h'(x_1+x_2) \geq 0$
   which requires $(c_1-c_2) \leq 0$ and $(c_2-c_3) \leq 0$
6. $a/d = 0.07$
7. $(0, 0, 0)$
8. (i) $(-0.0187, 0.2099), (0.0289, 0.0280)$
   (ii) $(-0.0187, 0.2099), (0.0200, 0.0250)$
9. Minimize $f(R_1, R_2, R_3) = \sum_{i=1}^{N}\left(V_d(\theta_i) - V_S\left[1 + \frac{R_3(R_1+r_1(\theta_i))(R_4+R_2+r_2(\theta_i))}{(R_1+r_1(\theta_i)+R_3)R_4(R_2+r_2(\theta_i))}\right]^{-1}\right)^2$

**Suitable past Tripos questions**
Part IIB 2007 Paper 4M13 Q3
Part IIB 2008 Paper 4M13 Q3
Part IIB 2009 Paper 4M13 Q3
Part IIB 2011 Paper 4M13 Q3
Part IIA 2012 Paper 3M1 Q2
Part IIA 2013 Paper 3M1 Q2
Part IIA 2014 Paper 3M1 Q2
Part IIA 2015 Paper 3M1 Q2 parts (d) and (e)
Part IIA 2016 Paper 3M1 Q2