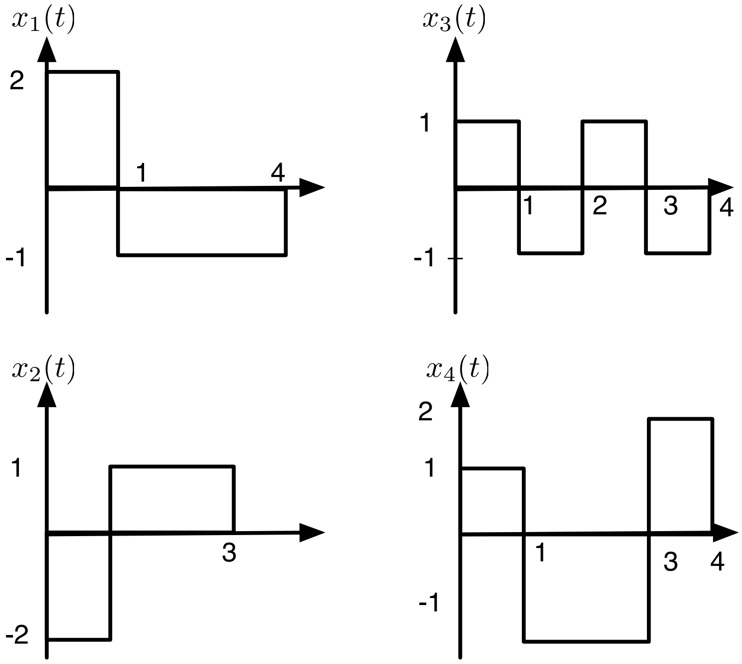
3F4: Data Transmission (Lent 2019)
Examples Paper 1

1. *Signal Space*: Consider the four waveforms $x_i(t)(\cdot), \ldots, x_4(t)$ shown below.



(a) Determine the dimensionality of the set of waveforms and a set of orthonormal basis functions by inspection.

(b) Determine an orthonormal basis for the set of waveforms using the Gram-Schmidt procedure.

(c) Represent the four waveforms by vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$, first using the orthonormal basis functions obtained in part (a), and then the basis functions in part (b).

(d) The distance between any two waveforms $x_i(t), x_j(t)$ is defined as

$$d_{ij} = \left(\int (x_i(t) - x_j(t))^2 dt \right)^{\frac{1}{2}}.$$

Show that $d_{ij} = \|\mathbf{x}_i - \mathbf{x}_j\|$. (Note that $\|\mathbf{x}\|$ denotes the Euclidean norm of the vector $\mathbf{x}$.)

(e) Use part (c) to determine the minimum distance between any pair of waveforms shown above.

2. *Fourier transform properties*: Denote the Fourier transform of $g(t)$ by

$$\mathcal{F}[g(t)] := G(f) = \int_{t \in \mathbb{R}} g(t) e^{-\mathrm{j}2\pi f t}\, dt.$$

Show the following properties:

(a) $\mathcal{F}[g(t-T)] = G(f)e^{-\mathrm{j}2\pi f T}$, where T is any constant.

(b) $\mathcal{F}[g(t)\cos(2\pi f_0 t)] = \frac{1}{2}[G(f - f_0) + G(f + f_0)]$, where f_0 is any constant.

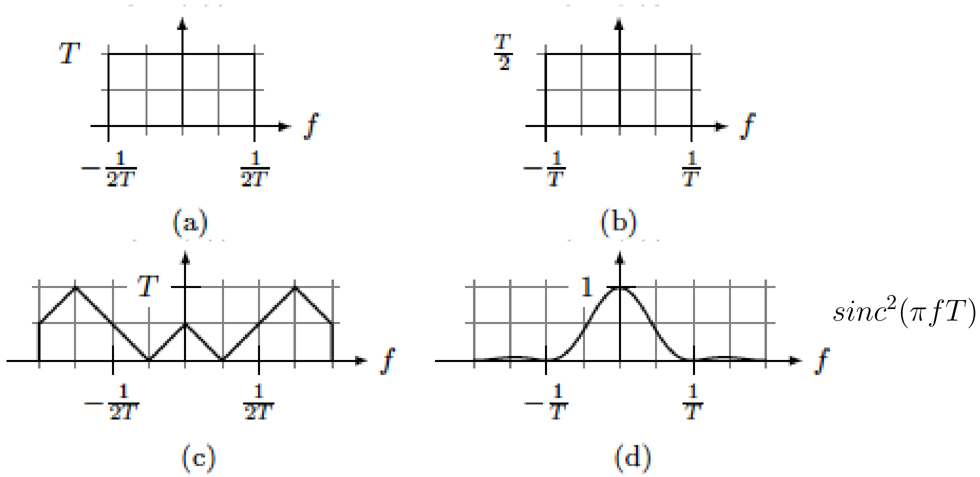
(c) $\mathcal{F}[g(-t)] = G(-f)$.

(d) $\mathcal{F}[g^*(t)] = G^*(-f)$, where * denotes the complex conjugate.

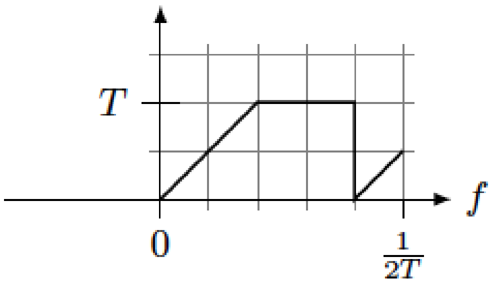
(e) Use (c) and (d) to show that if $g(t)$ is real and even, then its Fourier transform $G(f)$ is also real and even.

3. *Nyquist pulse criterion.*

   (a) Determine whether each of the four functions $G(f)$ in the figure below satisfies the Nyquist pulse criterion for symbol time T. The function in part (d) is $\text{sinc}^2(\pi f T)$. (Note that $G(f)$ is the frequency response of the overall filter.)



   (b) The figure below shows part of the plot of an overall PAM filter $G(f)$. The overall filter impulse response $g(t) = p(t) \star p(-t)$, where $p(t)$ is the transmit filter, and $p(-t)$ is the (matched) received filter. Complete the plot of $G(f)$, such that $G(f)$ satisfies the Nyquist pulse criterion, and $G(f) = 0$ for $|f| > \frac{1}{T}$.



4. *Power spectral density of PAM signal*: Calculate the power spectral density $S_x(f)$ of the transmitted waveform $x(t) = \sum_i X_i p(t - iT)$ in the following cases:

   (a) The symbols $\{X_i\}_{i=-\infty}^{\infty}$ are independent and uniformly distributed in $\{\pm\sqrt{\mathcal{E}}\}$, and $p(t) = \frac{1}{\sqrt{T}}\text{sinc}(\pi t/T)$.

   (b) The pulse is the same as in part (a), but the symbols are generated by an encoder as

$$X_i = \sqrt{2\mathcal{E}}(B_i - B_{i-2}),$$

   where $\{B_i\}_{i=-\infty}^{\infty}$ are a sequence of independent bits, uniformly distributed over $\{0, 1\}$.

   Remark: Note that the PSD in part (b) vanishes at $f = 0$. This is desirable if the channel blocks very low frequencies (including DC signals). This encoding technique is called *correlative encoding* or *partial response signalling.*

5. *Detection with non-uniform symbol probabilities*: Consider BPSK modulation with symbols $\{+A, -A\}$ over the discrete-time AWGN channel

$$Y = X + N$$

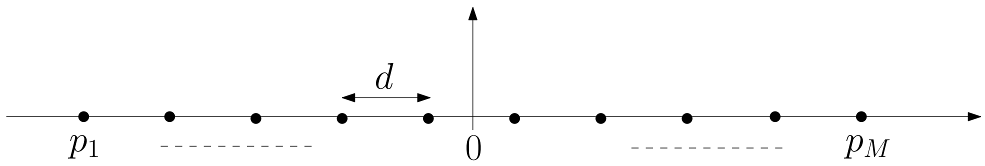
   where N is Gaussian noise $\sim \mathcal{N}(0, N_0/2)$. Suppose that $P(X = A) = p$ and $P(X = -A) = 1 - p$.

   (a) Derive the detection rule that that minimises the probability of detection error. Sketch the decision regions when $p = 2/3$ and $A/N_0 = 1/4$.

(b) Obtain the probability of detection error, first in terms of p, A, N_0, then express in terms of p and E_b/N_0.

6. *M-ary Pulse Amplitude Modulation (PAM)*: Consider the M-ary PAM constellation shown in the figure below. For $M \geq 2$, the constellation consists of M symbols $\{p_1, \ldots, p_M\}$ on the real line, symmetric around 0 and with equal spacing d between symbols. That is,

$$p_i = (2i - 1 - M)\frac{d}{2}, \quad i = 1, \ldots, M$$



Suppose that we use this constellation to signal over the discrete-time AWGN channel

$$Y = X + N$$

where the Gaussian noise N is distributed $\sim \mathcal{N}(0, N_0/2)$. Assuming all the constellation symbols are equally likely:

(a) Sketch the decision regions that minimise the probability of detection error.

(b) Obtain the probability of error when p_1 or p_M is sent.

(c) Obtain the probability of error when p_i is sent, for $2 \leq i \leq M-1$. Combine this with part (b) to obtain an expression for the overall probability of error P_e.

(d) Show that the average symbol energy E_s is $\frac{(M^2-1)d^2}{12}$. (Induction may be useful.)

(e) Express the probability of error P_e in terms of $\frac{E_b}{N_0}$. For fixed E_b/N_0, how does P_e vary as M increases? Is this what you'd expect?

7. *Differential encoding*: In a twisted pair cable (often used in Ethernet and telephone networks), there is a pair of wires such that electric field components that build up along the wires alternate polarity, and tend to cancel out one another. This mitigates the effect of electromagnetic interference.

Differential encoding is a technique for encoding the information in such a way that the decoding process is not affected by polarity. The differential encoder takes the data sequence $\{B_i\}_{i=1}^n$, here assumed to have independent and uniformly distributed components taking values in $\{0, 1\}$, and produces the symbol sequence $\{X_i\}_{i=1}^n$, according to the following encoding rule:

$$X_i = \begin{cases} X_{i-1}, & B_i = 0, \\ -X_{i-1}, & B_i = 1, \end{cases}$$

(We fix X_0 to be either $\sqrt{\mathcal{E}}$ or $-\sqrt{\mathcal{E}}$, by convention). Suppose that the symbol sequence is used to generate a PAM signal $x(t) = \sum_{i=1}^n X_i\, p(t - iT)$, where $p(t)$ is orthogonal to its shifts by integer multiples of T. The signal is transmitted over an AWGN channel with noise power spectral density $N_0/2$. The received signal is passed through a matched filter with impulse response $p(-t)$. Let Y_m denote the filter output sampled at time mT.

(a) Determine $R_X[k] = \mathbb{E}[X_i X_{i+k}]$, for $k \in \mathbb{Z}$, assuming an infinite sequence $\{X_i\}_{i=-\infty}^{\infty}$. Hence specify the power spectral density of $x(t)$ if the pulse $p(t) = \frac{1}{\sqrt{T}}\text{sinc}(\pi t/T)$.

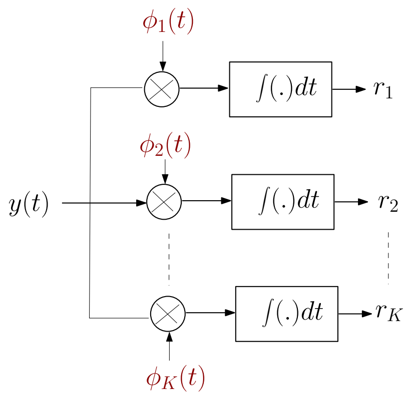
(b) Describe a method to estimate B_i from Y_i and Y_{i-1}, such that the performance is the same if the polarity of the sequence $\{Y_i\}$ is inverted for all i. We ask for a simple decoder, not necessarily maximum-likelihood.

(c) Determine (or estimate) the error probability of your decoder.

8. *Communication using an orthonormal basis*: Let $\{\phi_1(t), \ldots, \phi_K(t)\}$ be a set of K orthonormal functions. A transmitter that wishes to transmit one of M messages first maps each message $i \in \{1, \ldots, M\}$ to a length-K vector $\underline{s}_i = (s_{i,1}, \ldots, s_{i,K})$. Then the transmitted waveform for message i is $x(t) = \sum_{\ell=1}^{K} s_{i,\ell}\phi_\ell(t)$.

The received waveform is $y(t) = x(t) + n(t)$, where $n(t)$ is white Gaussian noise with zero mean and power spectral density $N_0/2$. At the receiver, the demodulator consists of a bank of K correlators (as shown in the figure below), which computes the inner products with each of the K basis functions. The vector output of the demodulator is $\underline{r} = (r_1, \ldots, r_K)$, as shown in the figure. Note that

$$r_1 = \int_{\mathbb{R}} y(t)\phi_1(t)dt, \quad r_2 = \int_{\mathbb{R}} y(t)\phi_2(t)dt, \ldots \quad , r_K = \int_{\mathbb{R}} y(t)\phi_K(t)dt.$$



(a) If message $i \in \{1, \ldots, M\}$ is transmitted, then show that the vector $\underline{r}$ at the demodulator output equals

$$\underline{r} = \underline{s}_i + \underline{n},$$

where $\underline{n} = (n_1, \ldots, n_K)$ is a vector of i.i.d. $\sim N(0, \frac{N_0}{2})$ random variables.

(b) If the M messages have probabilities $\{p_1, \ldots, p_M\}$, then show that the optimal detection rule (which minimises the probability of detection error) is

$$\widehat{m} = \arg\min_{1\leq j\leq M} \left[\frac{\|\underline{r} - \underline{s}_j\|^2}{N_0} + \ln\frac{1}{p_j} \right],$$

where $\widehat{m} \in \{1, \ldots, M\}$ is the decoded message.

(c) Consider a PAM signal $x(t) = \sum_{n=1}^{K} X_n p(t - nT)$, where the transmit pulse $p(t)$ is orthogonal to any shift of itself by an integer multiple of T. Identify the orthonormal basis for the PAM signal, and specify a receiver using a bank of correlators, as in the figure above. With the received signal being $y(t) = x(t) + n(t)$, show that the output of the K correlators is

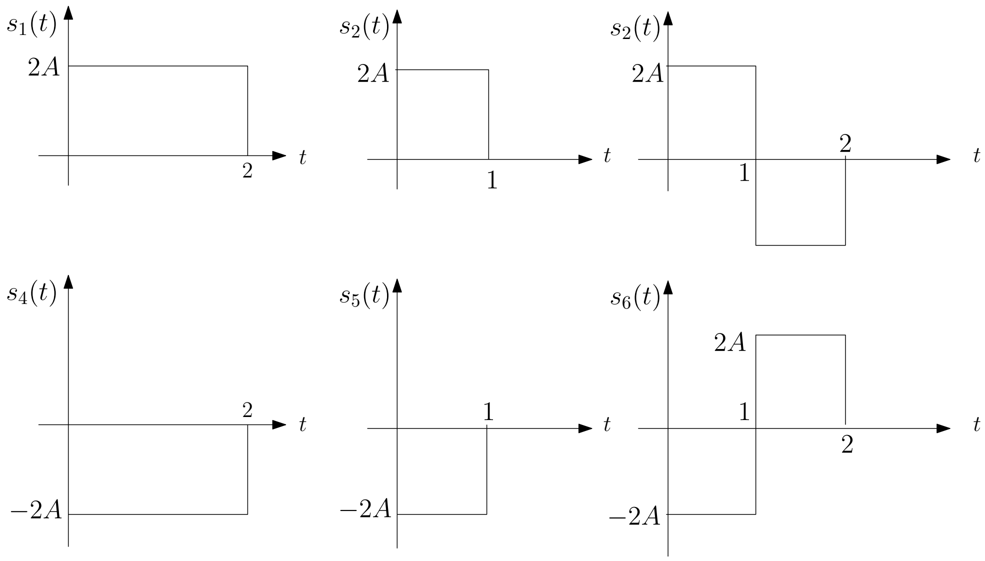
$$r_1 = X_1 + N_1, \quad r_2 = X_2 + N_2, \quad \ldots, \quad X_K = X_K + N_K,$$

where $N_1, \ldots, N_K$ are i.i.d $\sim \mathcal{N}(0, N_0/2)$.

(Hence the above receiver is equivalent to passing the received signal through a filter $p(-t)$ followed by sampling. Matched filter + sampling is just an efficient way of computing the inner products.)

NOTE: All the modulation schemes we study in 3F4 can be put in the framework of this problem, for suitable choices of M, K, and orthonormal basis.

9. *Communication using arbitrary functions.* A transmitter wishes to communicate one of six messages using the following set of waveforms. The received signal is $y(t) = x(t) + n(t)$, where $x(t) = s_i(t)$ if message $i \in \{1, \ldots, 6\}$ is transmitted, and $n(t)$ is white Gaussian noise.



(a) Specify an orthonormal basis for the collection $\{s_1(t), \ldots, s_6(t)\}$. What is the dimension of this collection of waveforms?

(b) Specify a receiver to detect the transmitted signal from $y(t)$, and sketch the decision regions assuming that the signals are equally likely.

Answers to Selected Questions

1. (e) The minimum distance between waveforms is $\sqrt{5}$.

4. (a) $S_x(f) = \mathcal{E}\,\mathbf{1}\{\frac{-1}{2T} \le f \le \frac{1}{2T}\}$; (b) $S_x(f) = 2\mathcal{E}\,\sin^2(2\pi f T)\,\mathbf{1}\{\frac{-1}{2T} \le f \le \frac{1}{2T}\}$.

5. (a) Decode $\hat{X} = A$ when $Y \ge T$ and $\hat{X} = -A$ when $Y < T$, where the threshold $T = \frac{N_0}{4A}\ln\left(\frac{1-p}{p}\right)$. Note that $T = 0$, when $p = \frac{1}{2}$. (b) $P_e = p\,\mathcal{Q}\left(\frac{A-T}{\sqrt{N_0/2}}\right) + (1-p)\,\mathcal{Q}\left(\frac{A+T}{\sqrt{N_0/2}}\right)$; $E_b = A^2$

6. (b) $\mathcal{Q}\left(\frac{d}{\sqrt{2N_0}}\right)$; (c) $2\mathcal{Q}\left(\frac{d}{\sqrt{2N_0}}\right)$, overall $P_e = \frac{2(M-1)}{M}\mathcal{Q}\left(\frac{d}{\sqrt{2N_0}}\right)$.

7. (a) $R_X[k] = \mathcal{E}\mathbf{1}\{k = 0\}$; (b) $\hat{B}_i = 0$, if Y_i and Y_{i-1} have the same sign, else $\hat{B}_i = 1$; (c) $2\mathcal{Q}\left(\sqrt{\frac{\mathcal{E}}{N_0/2}}\right)\left(1 - \mathcal{Q}\left(\sqrt{\frac{\mathcal{E}}{N_0/2}}\right)\right)$