Module 3F2: Systems and Control
EXAMPLES PAPER 3
OBSERVERS AND STATE-FEEDBACK

Solutions

1. (a) **Suppose the bias is on the star tracker (measurement of θ) rather than on the rate gyro (measurement of $\dot{\theta}$). Show that the technique shown in the Notes, based on appending the bias to the state vector, won't work in this case, because the system is not observable.**

We have $J\ddot{\theta} = u$, $y_1 = \theta + b_\theta$, and $y_2 = \dot{\theta}$. Let the state vector be $\underline{x} = [\theta, \dot{\theta}, b_\theta]^T$, then, since b_θ is constant, the state-space form of the model is

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 1/J \\ 0 \end{bmatrix} u, \qquad \underline{y} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \underline{x} + 0u$$

Observability test: $n = 3$ so

$$Q = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This has only 2 linearly independent rows (or columns) so $\text{rank}(Q) = 2 < 3 = n$, and hence the system is not observable.

(b) **Suppose that there is a constant but unknown disturbance torque d acting on the satellite: $J\ddot{\theta} = u + d, \quad \dot{d} = 0$, and the star tracker measures θ correctly (without bias). Show that, if d is appended to the state vector then the system is observable (using the star tracker output only), and hence that d can be estimated.**

In this case take the state vector to be $\underline{x} = [\theta, \dot{\theta}, d]^T$. We use only the start tracker output: $y = \theta$. So the state space equations are:

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1/J \\ 0 & 0 & 0 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 1/J \\ 0 \end{bmatrix} u, \qquad y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \underline{x} + 0u$$

Observability test: $n = 3$ so

$$Q = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/J \end{bmatrix}$$

Clearly $\text{rank}(Q) = 3 = n$ in this case, so this system is observable. It is therefore possible to construct a stable observer, such that $\hat{x}_3 \to d$ as $t \to \infty$.

2. **Derive a set of state-space equations in standard form for the systems shown in Figures 1 and 2 and investigate their controllability and observability as a parameter α varies. For values of α when either controllability or observability is lost determine the set of states that can be reached from the origin and the set of initial conditions that do not affect the output. Relate these results to pole/zero cancellations in the transfer functions.**

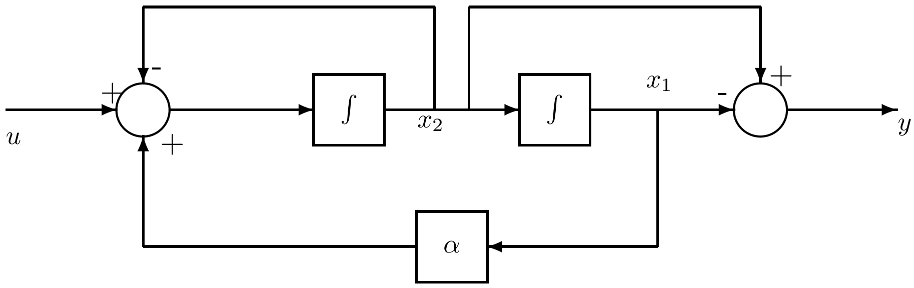


Figure 1:

The state equation for Figure 1 is:

$$\begin{aligned}\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} &= \underbrace{\begin{bmatrix}0 & 1\\ \alpha & -1\end{bmatrix}}_{A}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \underbrace{\begin{bmatrix}0\\ 1\end{bmatrix}}_{B} u\\ y &= \underbrace{\begin{bmatrix}-1 & 1\end{bmatrix}}_{C}\begin{bmatrix}x_1\\ x_2\end{bmatrix}\end{aligned}$$

The controllability matrix, $P = \begin{bmatrix}B & AB\end{bmatrix} = \begin{bmatrix}0 & 1\\ 1 & -1\end{bmatrix}$, which has rank 2 since $\det(P) = -1 \neq 0 \Rightarrow$ system is controllable.

Observability matrix, $Q = \begin{bmatrix}C\\ CA\end{bmatrix} = \begin{bmatrix}-1 & 1\\ \alpha & -2\end{bmatrix}$, which has $\det(Q) = 2 - \alpha \neq 0$ if $\alpha \neq 2$. $\Rightarrow$ system is observable if $\alpha \neq 2$ (when Q has rank 2).

Unobservable states when $\alpha = 2$ and the system is not observable, satisfy $Q\underline{x} = \underline{0}$, i.e. $\underline{x}_0 = \begin{bmatrix}a\\ a\end{bmatrix}$ i.e. $\underline{x}_0$ is in the range space of $\begin{bmatrix}1\\ 1\end{bmatrix}$.

The transfer function is,

$$\begin{aligned}G(s) &= C\,(sI - A)^{-1}\,B = \begin{bmatrix}-1 & 1\end{bmatrix}\begin{bmatrix}s & -1\\ -\alpha & s+1\end{bmatrix}^{-1}\begin{bmatrix}0\\ 1\end{bmatrix}\\ &= \frac{1}{s^2 + s - \alpha}\begin{bmatrix}-1 & 1\end{bmatrix}\begin{bmatrix}1\\ s\end{bmatrix} = \frac{-1+s}{s^2 + s - \alpha}\\ \text{at } \alpha = 2, G(s) &= \frac{(s-1)}{(s-1)(s+2)} = \frac{1}{s+2}\end{aligned}$$

and there is a pole/zero cancellation at $s = 1$ due to the unobservable mode.

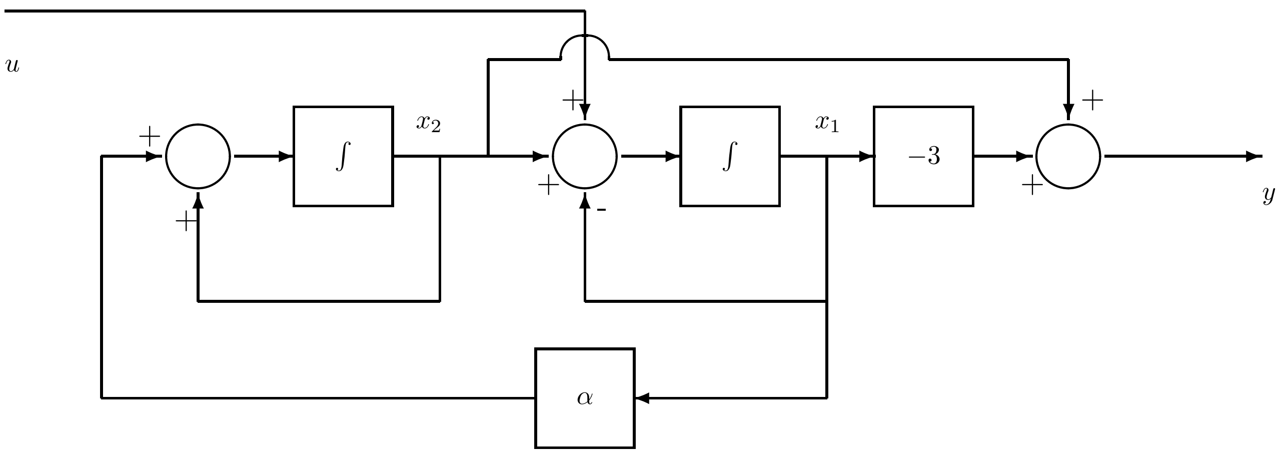


Figure 2:

The state equation for Figure 2 is:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \underbrace{\begin{bmatrix}-1 & 1\\ \alpha & 1\end{bmatrix}}_{A}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \underbrace{\begin{bmatrix}1\\ 0\end{bmatrix}}_{B} u, \quad y = \underbrace{\begin{bmatrix}-3 & 1\end{bmatrix}}_{C}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

The controllability matrix, $P = \begin{bmatrix}B & AB\end{bmatrix} = \begin{bmatrix}1 & -1\\ 0 & \alpha\end{bmatrix}$, $\det(P) = \alpha \Rightarrow$ system is controllable unless $\alpha = 0$, when the reachable states are range(P) = range $\begin{bmatrix}1\\ 0\end{bmatrix}$.

Observability matrix, $Q = \begin{bmatrix}C\\ CA\end{bmatrix} = \begin{bmatrix}-3 & 1\\ 3+\alpha & -2\end{bmatrix}$, which has $\det(Q) = 3 - \alpha \neq 0$ if $\alpha \neq 3$. $\Rightarrow$ system is observable if $\alpha \neq 3$ (when Q has rank 2).

Unobservable states when $\alpha = 3$ and the system is not observable, satisfy $Q\underline{x} = \underline{0}$, i.e. $\underline{x}_0$ is in the range space of $\begin{bmatrix}1\\ 3\end{bmatrix}$.

The transfer function is,

$$\begin{aligned}
G(s) &= C\,(sI - A)^{-1}\,B = \begin{bmatrix}-3 & 1\end{bmatrix}\begin{bmatrix}s+1 & -1\\ -\alpha & s-1\end{bmatrix}^{-1}\begin{bmatrix}1\\ 0\end{bmatrix}\\
&= \frac{1}{s^2 - 1 - \alpha}\begin{bmatrix}-3 & 1\end{bmatrix}\begin{bmatrix}s-1\\ \alpha\end{bmatrix} = \frac{-3s + 3 + \alpha}{s^2 - 1 - \alpha}\\
\text{at } \alpha = 0, \quad G(s) &= \frac{-3(s-1)}{(s-1)(s+1)} = \frac{-3}{s+1} \quad \text{not controllable}\\
\text{at } \alpha = 3, \quad G(s) &= \frac{-3(s-2)}{(s-2)(s+2)} = \frac{-3}{s+2} \quad \text{not observable}
\end{aligned}$$

3. **The inverted pendulum laboratory experiment has the linearized equations:**

$$\underline{\dot{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \omega_0^2 - \omega_1^2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \omega_0^2 & 0 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} u$$

(a) Show that the system is controllable.

The controllability matrix, $P = \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \omega_o^2 - \omega_1^2 \\ 1 & 0 & \omega_o^2 - \omega_1^2 & 0 \\ 0 & 1 & 0 & \omega_o^2 \\ 1 & 0 & \omega_o^2 & 0 \end{bmatrix}$, and

$$\begin{aligned} \det(P) &= (-1)\begin{vmatrix} 1 & \omega_o^2 - \omega_1^2 & 0 \\ 0 & 0 & \omega_o^2 \\ 1 & \omega_o^2 & 0 \end{vmatrix} + (-1)(\omega_o^2 - \omega_1^2)\begin{vmatrix} 1 & 0 & \omega_o^2 - \omega_1^2 \\ 0 & 1 & 0 \\ 1 & 0 & \omega_o^2 \end{vmatrix} \\ &= -\left[\omega_o^2(\omega_o^2 - \omega_1^2) - \omega_o^4\right] - \left[\omega_o^2 - (\omega_o^2 - \omega_1^2)\right](\omega_o^2 - \omega_1^2) \\ &= \omega_1^4 = (g/L)^2 \neq 0 \\ &\Rightarrow \text{ system is controllable.} \end{aligned}$$

(b) **Will this system be observable if just one of the states is measured, i.e. if $y = x_i$ for each of the four cases $i = 1, 2, 3, 4$?**

(i) $y = x_1 = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}}_{C} \underline{x}$. So observability matrix,

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \omega_o^2 - \omega_1^2 & 0 \\ 0 & 0 & 0 & \omega_o^2 - \omega_1^2 \end{bmatrix}, \quad \det(Q) = (\omega_o^2 - \omega_1^2)^2 \neq 0, \ \Rightarrow \text{system observable from } x_1.$$

(ii) $y = x_2 = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}}_{C} \underline{x}$. So observability matrix,

$$Q = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \omega_o^2 - \omega_1^2 & 0 \\ 0 & 0 & 0 & \omega_o^2 - \omega_1^2 \\ 0 & 0 & \omega_o^2(\omega_o^2 - \omega_1^2) & 0 \end{bmatrix}, \ \Rightarrow \left\{ \begin{array}{l} \text{system not observable from } x_2 \text{ since the first column of } Q \text{ is zero,} \\ \text{and the unobservable states will be the range space of } \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{array} \right. .$$

(ii) $y = x_3 = \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}}_{C} \underline{x}$. So observability matrix,

$$Q = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \omega_o^2 & 0 \\ 0 & 0 & 0 & \omega_o^2 \end{bmatrix}, \ \Rightarrow \left\{ \begin{array}{l} \text{system not observable from } x_3 \text{ since the first two columns of } Q \text{ are zero,} \\ \text{and the unobservable states will be the range space of } \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \end{array} \right. .$$

(ii) $y = x_3 = \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}}_{C} \underline{x}$. So observability matrix,

$$Q = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & \omega_o^2 & 0 \\ 0 & 0 & 0 & \omega_o^2 \\ 0 & 0 & \omega_o^4 & 0 \end{bmatrix}, \ \Rightarrow \left\{ \begin{array}{l} \text{system not observable from } x_4 \text{ since the first two columns of } Q \text{ are zero,} \\ \text{and the unobservable states will be the range space of } \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \end{array} \right. .$$

Note that if only the carriage position, x_1, is measured then

* differentiating x_1 once gives carriage velocity x_2,
* differentiating x_1 twice will give $\dot{x}_2 = (\omega_o^2 - \omega_1^2)x_3 + u$ and hence x_3 and the pendulum angle can be found,
* differentiating x_1 three times will give an equation for x_4.

and hence the estimates of x_3 and especially x_4 may be very noise sensitive.

4. **Consider the linearized two link manipulator of Question 6 Examples Paper 1.**

(a) **Is it controllable from**

(i) T_1 **and** T_2,
From the previous examples paper we have the linearised state equation:

$$\underline{\dot{x}} = A\underline{x} + B\underline{u}, \quad \text{where } , A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \alpha^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\beta & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ -0.1 & 0.25 \\ 0 & 0 \\ 0.25 & -1.625 \end{bmatrix}$$

where $\alpha^2 = 6\sqrt{3}$ and $\beta = 15\sqrt{3} = 5\alpha^2/2$.
The controllability matrix,

$$\begin{aligned} P &= \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & -0.1 & 0.25 & \cdots \\ -0.1 & 0.25 & 0 & 0 & \cdots \\ 0 & 0 & 0.25 & -1.625 & \cdots \\ 0.25 & -1.625 & 0 & 0 & \cdots \end{bmatrix} \end{aligned}$$

The first four columns of P are linearly independent since $\det \begin{bmatrix} -0.1 & 0.25 \\ 0.25 & -1.625 \end{bmatrix} = 0.1 \neq 0$.
(Note that $\underline{z}^T P = \underline{0} \Rightarrow \begin{bmatrix} z_2 & z_4 \end{bmatrix} \begin{bmatrix} -0.1 & 0.25 \\ 0.25 & -1.625 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$ and
$\begin{bmatrix} z_1 & z_3 \end{bmatrix} \begin{bmatrix} -0.1 & 0.25 \\ 0.25 & -1.625 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$ and so $\underline{z} = \underline{0}$.)
Therefore P has rank 4 and the system is controllable if both inputs are available.

(ii) T_1 **alone** ($T_2 = T_{2e}$)**?** In this case we just have, $\underline{b}_1$, the first column of B, and the controllability matrix is,

$$P_1 = \begin{bmatrix} \underline{b}_1 & A\underline{b}_1 & A^2\underline{b}_1 & A^3\underline{b}_1 \end{bmatrix} = \begin{bmatrix} 0 & -0.1 & 0 & -0.1\alpha^2 \\ -0.1 & 0 & -0.1\alpha^2 & 0 \\ 0 & 0.25 & 0 & 0.1\beta \\ 0.25 & 0 & 0.1\beta & 0 \end{bmatrix}$$

$$\begin{aligned} \det(P_1) &= -\det \begin{bmatrix} 0.25 & 0 & 0.1\beta & 0 \\ -0.1 & 0 & -0.1\alpha^2 & 0 \\ 0 & 0.25 & 0 & 0.1\beta \\ 0 & -0.1 & 0 & -0.1\alpha^2 \end{bmatrix} = \det \begin{bmatrix} 0.25 & 0.1\beta & 0 & 0 \\ -0.1 & -0.1\alpha^2 & 0 & 0 \\ 0 & 0 & 0.25 & 0.1\beta \\ 0 & 0 & -0.1 & -0.1\alpha^2 \end{bmatrix} \\ &= \det \begin{bmatrix} 0.25 & 0.1\beta \\ -0.1 & -0.1\alpha^2 \end{bmatrix}^2 = (0.01\beta - 0.025\alpha^2)^2 \\ &= (0.01 \times 2.5\alpha^2 - 0.025\alpha^2)^2 = 0 \end{aligned}$$

$\Rightarrow$ system is **not** controllable from the first input alone.

(iii) T_2 **alone** ($T_1 = T_{1e}$)**?** In this case we just have, $\underline{b}_2$, the second column of B, and the controllability matrix is,

$$
\begin{aligned}
P_2 &= \begin{bmatrix} \underline{b}_2 & A\underline{b}_2 & A^2\underline{b}_2 & A^3\underline{b}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0.25 & 0 & 0.25\alpha^2 \\ 0.25 & 0 & 0.25\alpha^2 & 0 \\ 0 & -1.625 & 0 & -0.25\beta \\ -1.625 & 0 & -0.25\beta & 0 \end{bmatrix} \\
\det(P_2) &= \det \begin{bmatrix} -1.625 & -0.25\beta \\ 0.25 & 0.25\alpha^2 \end{bmatrix}^2 = (0.0625\beta - 0.40625\alpha^2)^2 = (0.15625\alpha^2 - 0.40625\alpha^2)^2 \\
&= 0.25^2 \times 6^2 \times 3 = 6.75 \neq 0
\end{aligned}
$$

$\Rightarrow$ system **is** controllable from the second input alone.

(b) **What implication can you draw on the achievable closed-loop behaviour for this system for these three cases? Include a discussion of the achievable steady-state conditions.**
Can conclude:

- closed loop poles can be assigned using state feedback to T_2 or to (T_1 and T_2), but not to T_1 alone.
- if only T_2 is used then any state can be achieved at a time instant but cannot necessarily be maintained. In particular in the steady state,

$$
\begin{aligned}
\dot{\underline{x}} &= \underline{0} = A\underline{x} + \underline{b}_2 u_2 \\
\Rightarrow x_2 &= x_4 = 0 \text{ and } \begin{bmatrix} \alpha^2 & 0.25 \\ -\beta & -1.625 \end{bmatrix} \begin{bmatrix} x_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\Rightarrow x_1 &= 0, \ u_2 = 0, \quad \text{but } x_3 \text{ can be arbitrary.}
\end{aligned}
$$

  Hence we cannot maintain a state with $x_1 \neq 0$ in equilibrium.
- if both T_1 and T_2 are both available then can maintain any values for x_1 and x_3.

5. (a) **In lectures we defined the *controllability Gramian* $W_c(t)$ as**

$$W_c(t) = \int_0^t e^{A\tau} BB^T e^{A^T\tau} d\tau.$$

Assume that the system is asymptotically stable; then $W_c(\infty) = \lim_{t\to\infty} W_c(t)$ exists. By considering

$$\frac{d}{d\tau}\left\{ e^{A\tau} BB^T e^{A^T\tau} \right\}$$

show that

$$AW_c(\infty) + W_c(\infty)A^T = -BB^T$$

Since $\frac{d}{dt}e^{At} = Ae^{At} = e^{At}A$, we have:

$$\frac{d}{d\tau}\left\{ e^{A\tau} BB^T e^{A^T\tau} \right\} = Ae^{A\tau} BB^T e^{A^T\tau} + e^{A\tau} BB^T e^{A^T\tau} A$$

Therefore

$$
\begin{aligned}
AW_c(\infty) + W_c(\infty)A^T &= A\int_0^\infty e^{A\tau} BB^T e^{A^T\tau} d\tau + \int_0^\infty e^{A\tau} BB^T e^{A^T\tau} d\tau A^T \\
&= \int_0^\infty \left[Ae^{A\tau} BB^T e^{A^T\tau} + e^{A\tau} BB^T e^{A^T\tau} A^T \right] d\tau \\
&= \left[e^{A\tau} BB^T e^{A^T\tau} \right]_0^\infty \\
&= 0 - BB^T
\end{aligned}
$$

since $e^{A\tau} \to 0$ as $\tau \to \infty$, since the system is assumed to be asymptotically stable.

(b) **For the system shown in Figure 1, with $\alpha = -1$, find $W_c(\infty)$. (Note that $W_c(\infty)$ is symmetric.)**
Since $\alpha = -1$, using the solution to Q.2, we have

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

A has eigenvalues at the roots of $s^2 + s + 1$, so the system is asymptotically stable, as assumed in (a).
Let

$$W_c(\infty) = \begin{bmatrix} w_{11} & w_{12} \\ w_{12} & w_{22} \end{bmatrix}$$

(note how symmetry has been used here). From (a) we know that

$$AW_c(\infty) + W_c(\infty)A^T = -BB^T$$

so

$$\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} w_{11} & w_{12} \\ w_{12} & w_{22} \end{bmatrix} + \begin{bmatrix} w_{11} & w_{12} \\ w_{12} & w_{22} \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} = -\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Multiplying this out and equating elements on each side:
(1,1): $2w_{12} = 0 \Rightarrow w_{12} = 0$.
(1,2): $w_{22} - w_{11} - w_{12} = 0 \Rightarrow w_{22} = w_{11}$.
(2,2): $-w_{12} - w_{22} - w_{12} - w_{22} = -1 \Rightarrow w_{22} = 1/2$.

6. **Design an observer for Question 1(b), such that the state estimation error decays with a time constant of 1 sec.**

There are 3 states and 1 output measurement, so the observer gain matrix must have 3 rows and 1 column. So let $L = [\ell_1, \ell_2, \ell_3]^T$. Then

$$A - LC = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1/J \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} \ell_1 \\ \ell_2 \\ \ell_3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -\ell_1 & 1 & 0 \\ -\ell_2 & 0 & 1/J \\ -\ell_3 & 0 & 0 \end{bmatrix}$$

and hence the observer's eigenvalues are given by the roots of the characteristic polynomial $\det[sI - (A - LC)]$ or

$$\begin{aligned} \det\begin{bmatrix} s+\ell_1 & -1 & 0 \\ \ell_2 & s & -1/J \\ \ell_3 & 0 & s \end{bmatrix} &= s[(s+\ell_1)s + \ell_2] + \frac{1}{J}(0 + \ell_3) \\ &= s^3 + \ell_1 s^2 + \ell_2 s + \frac{\ell_3}{J} \end{aligned}$$

The state estimation errors are to decay with time constant 1 sec, so place all 3 eigenvalues at -1 sec^{-1}. (Another possibility would be to make the real parts -1, but choose a conjugate pair of imaginary parts for two of the eigenvalues, etc.) That means we want the characteristic polynomial to be

$$(s+1)^3 = s^3 + 3s^2 + 3s + 1$$

Comparing coefficients, we see that we need $\ell_1 = 3$, $\ell_2 = 3$, and $\ell_3 = J$.

7. **Design a state-feedback controller for the system of Figure 1, which places both closed-loop poles at -10 sec^{-1}. (Assume that both state variables are available for feedback.)**

   We have 2 state variables and 1 input, so the state feedback matrix should have 1 row and 2 columns. So let $K = [k_1, k_2]$, and $u = -K\underline{x}$. Then we have

$$\begin{aligned}\dot{\underline{x}} &= A\underline{x} - BK\underline{x} = (A - BK)\underline{x} \\ &= \left(\begin{bmatrix} 0 & 1 \\ \alpha & -1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} \right) \underline{x} \\ &= \begin{bmatrix} 0 & 1 \\ \alpha - k_1 & -1 - k_2 \end{bmatrix} \underline{x}\end{aligned}$$

   Now *either* use "trace = sum of eigenvalues" and "determinant = product of eigenvalues":

$$\begin{aligned}\text{trace}(A - BK) &= -1 - k_2 = -20 \Rightarrow k_2 = 19 \\ \det(A - BK) &= k_1 - \alpha = 100 \Rightarrow k_1 = 100 + \alpha\end{aligned}$$

   or find characteristic polynomial and compare it with the required one:

$$\det[sI - (A - BK)] = s(s + 1 + k_2) + (k_1 - \alpha) = s^2 + (1 + k_2)s + (k_1 - \alpha)$$

   But we want the characteristic polynomial to be

$$(s + 10)^2 = s^2 + 20s + 100$$

   so comparing coefficients gives $1 + k_2 = 20$ and $k_1 - \alpha = 100$, as before.

8. **A "pirate ship" at a fairground is controlled by a suitable electric motor. An idealised and simplified model of the system is given by:**

$$I\ddot{\theta} = -MgL\theta + \tau$$

   where θ is the angle (of the ship's mast to the vertical), $I = L^2M$, $g \simeq 10$ms^{-2}, $L = 10$m, and τ (Nm) is the geared motor torque. It is desired to minimise the losses in the motor which is assumed to be proportional to,

$$J = \int_0^{t_1} \tau^2(t)dt.$$

   Show that the minimum possible value of J when moving the ship from rest to $\theta(t_1) = \pi/4$, $\dot{\theta}(t_1) = 0$, with $t_1 = n\pi$ s ($n = 1, 2, \cdots$) is given by,

$$J_{\textbf{min}} = \frac{\pi I^2}{8n}$$

   when the optimal input is

$$\tau_{\textbf{opt}} = \frac{(-1)^{n+1} I \sin(t)}{2n}.$$

   To write down the state equation take the state as $\underline{x} = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$ and scale the input as $u = \tau/I$. Then

$$I\ddot{\theta} = -MgL\theta + \tau \quad \Rightarrow \ddot{\theta} = -\frac{g}{L}\theta + \frac{1}{I}\tau \Rightarrow \dot{\underline{x}} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{A} \underline{x} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u$$

   and

$$J = \int_0^{t_1} \tau^2(t)\, dt = I^2 \int_0^{t_1} u^2(t)\, dt = I^2 \tilde{J} \quad \text{where } \tilde{J} = \int_0^{t_1} u^2(t)\, dt$$

The minimum value of $\tilde{J} = \underline{x}(t_1)^T W_c(t_1)^{-1} \underline{x}(t_1)$ where

$$\begin{aligned}
W_c(t_1) &= \int_0^{t_1} e^{At} B B^T e^{A^T t}\, dt \\
e^{At} B &= \mathcal{L}^{-1}\left\{(sI - A)^{-1} B\right\} = \mathcal{L}^{-1}\left\{\begin{bmatrix} s & -1 \\ 1 & s \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\begin{bmatrix} 1 \\ s \end{bmatrix}\right\} = \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix}
\end{aligned}$$

Now for $t_1 = n\pi$, $\underline{x}(t_1) = \begin{bmatrix} \pi/4 \\ 0 \end{bmatrix}$, and $\underline{x}(0) = \underline{0}$ we have

$$\begin{aligned}
\Rightarrow W_c(n\pi) &= \int_0^{n\pi} \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix} \begin{bmatrix} \sin(t) & \cos(t) \end{bmatrix} dt \\
&= \frac{1}{2}\int_0^{n\pi} \begin{bmatrix} 1-\cos(2t) & \sin(2t) \\ \sin(2t) & 1+\cos(2t) \end{bmatrix} dt = \frac{1}{2}\begin{bmatrix} t - \frac{1}{2}\sin(2t) & -\frac{1}{2}\cos(2t) \\ -\frac{1}{2}\cos(2t) & t + \frac{1}{2}\sin(2t) \end{bmatrix}\Bigg|_0^{n\pi} = \frac{1}{2}\begin{bmatrix} n\pi & 0 \\ 0 & n\pi \end{bmatrix}; \\
W_c(n\pi)^{-1} &= \frac{2}{n\pi}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
\Rightarrow J_{\text{min}} &= I^2 \tilde{J}_{\text{min}} = I^2 \times \frac{2}{n\pi} \times \left(\frac{\pi}{4}\right)^2 = \frac{\pi I^2}{8n}
\end{aligned}$$

To achieve this the optimal value of u is given by,

$$\begin{aligned}
u(t) &= u_0(t) = B^T e^{A^T(t_1 - t)} W_c(t_1)^{-1} \underline{x}_1 \\
&= \begin{bmatrix} \sin(n\pi - t) & \cos(n\pi - t) \end{bmatrix} \frac{2}{n\pi}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \pi/4 \\ 0 \end{bmatrix} = \frac{1}{2n}(\sin(n\pi)\cos(t) - \cos(n\pi)\sin(t)) \\
&= \frac{(-1)^{n+1}}{2n}\sin(t) \\
\tau_{\text{opt}} &= I\frac{(-1)^{n+1}}{2n}\sin(t)
\end{aligned}$$

i.e. the optimal input excites the system at its resonant frequency.

As an aside let's look at the case of a general t_1, when

$$\begin{aligned}
W_c(t_1) &= \frac{1}{2}\begin{bmatrix} t - \frac{1}{2}\sin(2t) & -\frac{1}{2}\cos(2t) \\ -\frac{1}{2}\cos(2t) & t + \frac{1}{2}\sin(2t) \end{bmatrix}_0^{t_1} = \frac{1}{2}\begin{bmatrix} t_1 - \frac{1}{2}\sin(2t_1) & \frac{1}{2}(1-\cos(2t_1)) \\ \frac{1}{2}(1-\cos(2t_1)) & t_1 + \frac{1}{2}\sin(2t_1) \end{bmatrix} \\
\det W_c(t_1) &= \frac{1}{4}\left(t_1^2 - \frac{1}{4}\sin^2(2t_1) - \frac{1}{4}(1-\cos(2t_1))^2\right) = \frac{1}{4}\left(t_1^2 - \frac{1}{4} - \frac{1}{4} + \frac{1}{2}\cos(2t_1)\right) = \frac{1}{4}\left(t_1^2 - \sin^2(t_1)\right) \\
\Rightarrow \tilde{J} &= \frac{2(t_1 + \frac{1}{2}\sin(2t_1))}{(t_1^2 - \sin^2(t_1))} \times \frac{\pi^2}{16}
\end{aligned}$$

Again u_0 will be a sinusoid of suitable magnitude and phase.

9. **(State feedback and observers).**
A system satisfies the state equation

$$\begin{aligned} \dot{\underline{x}} &= A\underline{x} + Bu \\ y &= C\underline{x} + Du \end{aligned}$$

where

$$A = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}, \; B = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 1 \end{bmatrix}, \; D = 0$$

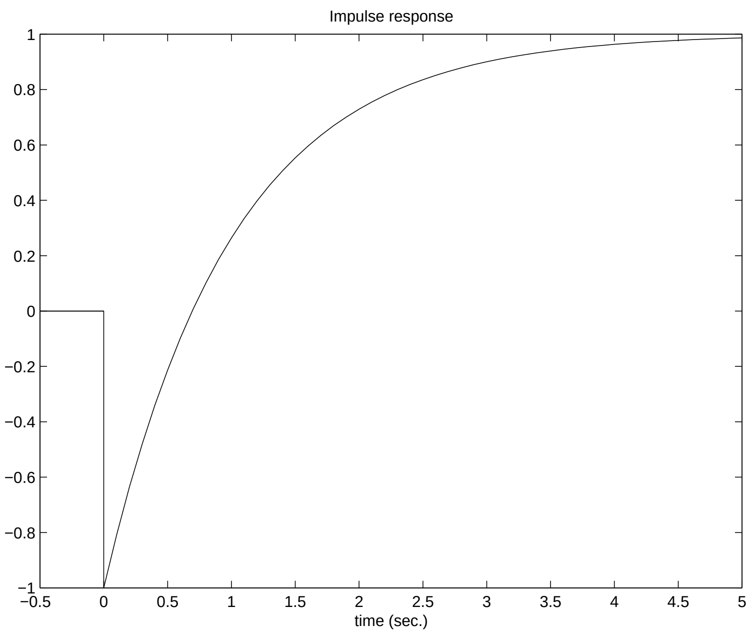
and it is desired to design a feedback controller so that $y(t)$ tracks to an external reference signal $r(t)$ as quickly as possible with satisfactory stability margins.

(a) **Verify that the transfer function is**

$$G(s) = \frac{(1-s)}{s(s+1)}$$

with impulse response $1 - 2e^{-t}$. (Note the zero at $s = +1$ makes the response initially go in the 'wrong' direction).

$$\begin{aligned} G(s) &= C(sI-A)^{-1}B + D = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} s & 0 \\ 0 & s+1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \frac{1}{s} - \frac{2}{s+1} = -\frac{(s-1)}{s(s+1)} \\ g(t) &= \mathcal{L}^{-1}G(s) = 1 - 2e^{-t} \end{aligned}$$



(b) **Design a state feedback controller**

$$u = -F\underline{x} + Lr$$

so that the closed-loop poles are both at $-\beta$ and $y(t) \to r$ when r is a step. Calculate the resulting response of $y(t)$ to a step change in r.

$$\begin{aligned} u &= -F\underline{x} + Lr \\ \dot{\underline{x}} &= (A - BF)\underline{x} + BLr; \quad y = C\underline{x} \\ A - BF &= \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 \\ -2 \end{bmatrix} \begin{bmatrix} f_1 & f_2 \end{bmatrix} = \begin{bmatrix} -f_1 & -f_2 \\ 2f_1 & -1 + 2f_2 \end{bmatrix} \end{aligned}$$

Closed loop poles are both to be placed at $-\beta$ and given by:

$$\begin{aligned}
\det(sI - A + BF) &= (s+f_1)(s+1-2f_2) + 2f_1 f_2 \\
&= s^2 + (1+f_1-2f_2)s + f_1 \\
&= (s+\beta)^2 = s^2 + 2\beta + \beta^2 \\
&\Rightarrow\ f_1 = \beta^2,\ f_2 = \frac{1+f_1-2\beta}{2} = \frac{1}{2}(1-\beta)^2 \\
F &= \begin{bmatrix} \beta^2 & \frac{1}{2}(1-\beta)^2 \end{bmatrix}
\end{aligned}$$

Now the transfer function from r to y is given by,

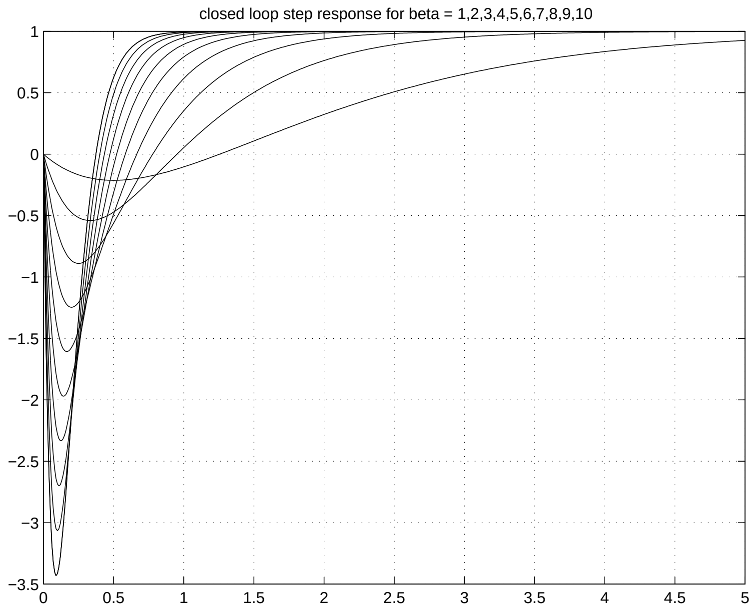
$$\begin{aligned}
Y(s) &= C(sI - A + BF)^{-1} BLR(s) \\
&= \begin{bmatrix} 1 & 1 \end{bmatrix} \frac{\begin{bmatrix} s+1-2f_2 & -f_2 \\ 2f_1 & s+f_1 \end{bmatrix}}{(s+\beta)^2} \begin{bmatrix} 1 \\ -2 \end{bmatrix} LR(s) \\
&= \left(s+1-2f_2+2f_1+(-f_2+s+f_1)(-2)\right) LR(s) / \left(s+\beta\right)^2 \\
&= \frac{(-s+1)}{(s+\beta)^2} LR(s) \\
&\Rightarrow\ L = \beta^2 \text{ to get unity steady state gain from } r \text{ to } y
\end{aligned}$$

Step response of closed loop system will be,

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1} \frac{(1-s)\beta^2}{s(s+\beta)^2} = \mathcal{L}^{-1} \left\{ \frac{1}{s} + \frac{\frac{(1+\beta)\beta^2}{(-\beta)}}{(s+\beta)^2} + \frac{E}{s+\beta} \right\}
\end{aligned}$$

to find E consider $s \times (..)$ and take limit as $s \to \infty$ giving $0 = 1 + 0 + E$ so $E = -1$.

$$\begin{aligned}
&= \mathcal{L}^{-1} \left\{ \frac{1}{s} + \frac{(1+\beta)\beta}{(s+\beta)^2} - \frac{1}{s+\beta} \right\} \\
&= 1 - e^{-\beta t}\left(1 + \beta(1+\beta)t\right)
\end{aligned}$$



(c) **Design a state observer with gain matrix, K, so that the poles of the observer are both at $-\alpha$. Calculate the state estimation error if $\underline{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and the observer state, $\underline{\hat{x}}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.**

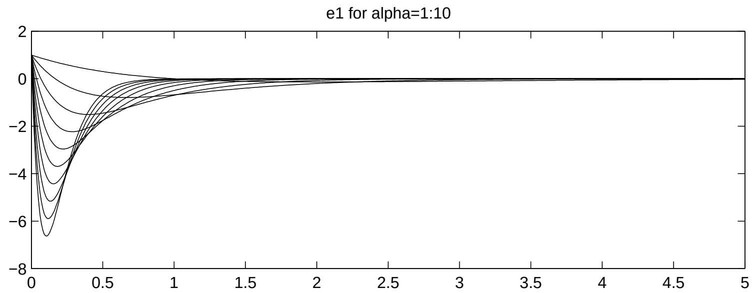
Observer equations are:

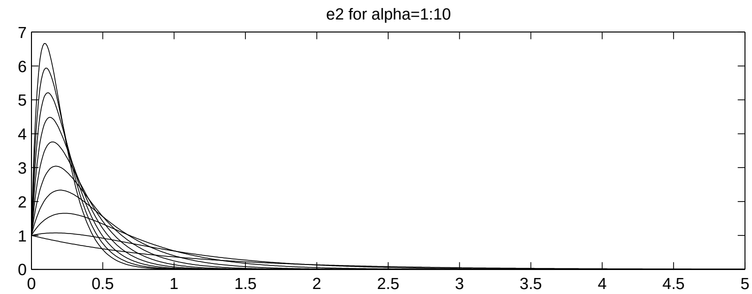
$$\begin{aligned}
\dot{\underline{\hat{x}}} &= A\underline{\hat{x}} + K(y - C\underline{\hat{x}}) + Bu \\
\dot{\underline{x}} &= A\underline{x} + Bu \\
\underline{e} &= \underline{x} - \underline{\hat{x}}; \quad \dot{\underline{e}} = (A - KC)\underline{e}
\end{aligned}$$

Now the poles of the observer are given by the eigen values of $(A - KC)$ which are the roots of the characteristic polynomial:

$$\begin{aligned}
\det(sI - A + KC) &= \det\left[\begin{bmatrix} s & 0 \\ 0 & s+1 \end{bmatrix} + \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}\begin{bmatrix} 1 & 1 \end{bmatrix}\right] \\
&= (s + k_1)(s + 1 + k_2) - k_2 k_1 = s^2 + (1 + k_1 + k_2)s + k_1 \\
&= (s + \alpha)^2 = s^2 + 2\alpha s + \alpha^2 \\
&\Rightarrow k_1 = \alpha^2, \quad k_2 = -(1-\alpha)^2, \quad K = \begin{bmatrix} \alpha^2 \\ -(1-\alpha)^2 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\underline{e}(t) &= e^{(A-KC)t}\underline{e}(0) \\
(sI - A + KC)^{-1} &= \begin{bmatrix} s + k_1 & k_1 \\ k_2 & s + 1 + k_2 \end{bmatrix}^{-1} = \frac{1}{(s+\alpha)^2}\begin{bmatrix} s + 1 + k_2 & -k_1 \\ -k_2 & s + k_1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{s+\alpha} & 0 \\ 0 & \frac{1}{s+\alpha} \end{bmatrix} + \frac{1}{(s+\alpha)^2}\begin{bmatrix} 1 + k_2 - \alpha & -k_1 \\ -k_2 & k_1 - \alpha \end{bmatrix} \\
\Rightarrow e^{(A-KC)t} &= e^{-\alpha t}\begin{bmatrix} 1 + \alpha(1-\alpha)t & -\alpha^2 t \\ (1-\alpha)^2 t & 1 - \alpha(1-\alpha)t \end{bmatrix} \\
\underline{e}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} &\Rightarrow \underline{e}(t) = e^{-\alpha t}\begin{bmatrix} 1 + \alpha(1-\alpha)t & -\alpha^2 t \\ (1-\alpha)^2 t & 1 - \alpha(1-\alpha)t \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = e^{-\alpha t}\begin{bmatrix} 1 + \alpha(1-2\alpha)t \\ 1 + (1-2\alpha)(1-\alpha)t \end{bmatrix}
\end{aligned}$$





(d) **Show that if estimated state feedback is now used then the controller equation will be**

$$\begin{aligned}
\dot{\underline{\hat{x}}} &= (A - BF - KC)\,\underline{\hat{x}} + Ky + BLr \\
u &= -F\underline{\hat{x}} + Lr
\end{aligned}$$

and observe that the step response from r to y will be the same as in (b) if $\underline{x}(0) = \underline{\hat{x}}(0) = \underline{0}$.

With estimated state feedback we have,

$$\begin{aligned}
\text{Observer: } \dot{\underline{\hat{x}}} &= A\underline{\hat{x}} + Bu + K(y - C\underline{\hat{x}}) \\
u &= -F\underline{\hat{x}} + Lr \\
\Rightarrow \quad \dot{\underline{\hat{x}}} &= (A - BF - KC)\underline{\hat{x}} + BLr + Ky
\end{aligned}$$

We still have the observer error dynamics,

$$\begin{aligned}
\dot{\underline{e}} &= (A - KC)\underline{e} \\
\text{so if} \quad \underline{e}(0) &= \underline{x}(0) - \underline{\hat{x}}(0) = \underline{0} \ \text{ then } \ \underline{e}(t) = \underline{0} \ \text{ and } \ \underline{\hat{x}}(t) = \underline{x}(t) \ \text{ for all } \ t.
\end{aligned}$$

Hence $u = -F\underline{x} + Lr$ and closed loop response from r to y is unchanged.

(e) **Now suppose that the system actually satisfies the state equations**

$$\begin{aligned}
\dot{\underline{x}} &= A_a\underline{x} + B_a u \\
y &= C_a\underline{x}
\end{aligned}$$

and the measurement is $(y + v)$ where v is observation noise.
Show that the resulting closed loop equation will be

$$\begin{aligned}
\frac{d}{dt}\begin{bmatrix} \underline{x} \\ \underline{\hat{x}} \end{bmatrix} &= \begin{bmatrix} A_a & -B_aF \\ KC_a & A - BF - KC \end{bmatrix}\begin{bmatrix} \underline{x} \\ \underline{\hat{x}} \end{bmatrix} + \begin{bmatrix} B_aL & 0 \\ BL & K \end{bmatrix}\begin{bmatrix} r \\ v \end{bmatrix} \\
\begin{bmatrix} y \\ u \end{bmatrix} &= \begin{bmatrix} C_a & 0 \\ 0 & -F \end{bmatrix}\begin{bmatrix} \underline{x} \\ \underline{\hat{x}} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ L & 0 \end{bmatrix}\begin{bmatrix} r \\ v \end{bmatrix}
\end{aligned}$$

The observer and controller equations will now be,

$$\begin{aligned}
\dot{\underline{x}} &= A_a\underline{x} + B_a u, \quad y = C_a\underline{x} \\
\dot{\underline{\hat{x}}} &= A\underline{\hat{x}}A\underline{x} + Bu + K(y + \underbrace{v}_{\text{measurement noise}} - C\underline{x}) \\
u &= -F\underline{\hat{x}} + Lr \\
\Rightarrow \quad \dot{\underline{x}} &= A_a\underline{x} - B_aF\underline{\hat{x}} + B_aLr \\
\dot{\underline{\hat{x}}} &= A\underline{\hat{x}} - BF\underline{\hat{x}} + BLr - KC\underline{\hat{x}} + KC_a\underline{x} + Kv \\
\frac{d}{dt}\begin{bmatrix} \underline{x} \\ \underline{\hat{x}} \end{bmatrix} &= \begin{bmatrix} A_a & -B_aF \\ KC_a & A - BF - KC \end{bmatrix}\begin{bmatrix} \underline{x} \\ \underline{\hat{x}} \end{bmatrix} + \begin{bmatrix} B_aL & 0 \\ BL & K \end{bmatrix}\begin{bmatrix} r \\ v \end{bmatrix} \\
\begin{bmatrix} y \\ u \end{bmatrix} &= \begin{bmatrix} C_a & 0 \\ 0 & -F \end{bmatrix}\begin{bmatrix} \underline{x} \\ \underline{\hat{x}} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ L & 0 \end{bmatrix}\begin{bmatrix} r \\ v \end{bmatrix}
\end{aligned}$$

(f) **Show that the open-loop gain of the system with the loop 'broken' at the plant output will be,**

$$H(s) = C_a(sI - A_a)^{-1}B_a \ F(sI - A + BF + KC)^{-1}K$$

and note that this transfer function will determine the stability margins with respect to gain and phase uncertainty in the plant model.

To determine this transfer function we need u as a function of y from the observer with controller, and then y as a function of u from the plant dynamics. The transfer function of the observer comes from its state equation derived in part (d), (including observation noise),

$$U(s) = LR(s) - F\underline{\hat{X}}(s) = LR(s) - F(sI - A + BF + KC)^{-1}(K(Y(s) + V(s)) + BLR(s))$$

The plant equation gives,

$$Y(s) = C_a(sI - A_a)^{-1}B_aU(s)$$

The loop gain is therefore as given taking into account the usual minus sign in the loop which is not normally included in the loop gain. (i.e. so that the critical point on the Nyquist diagram is -1).

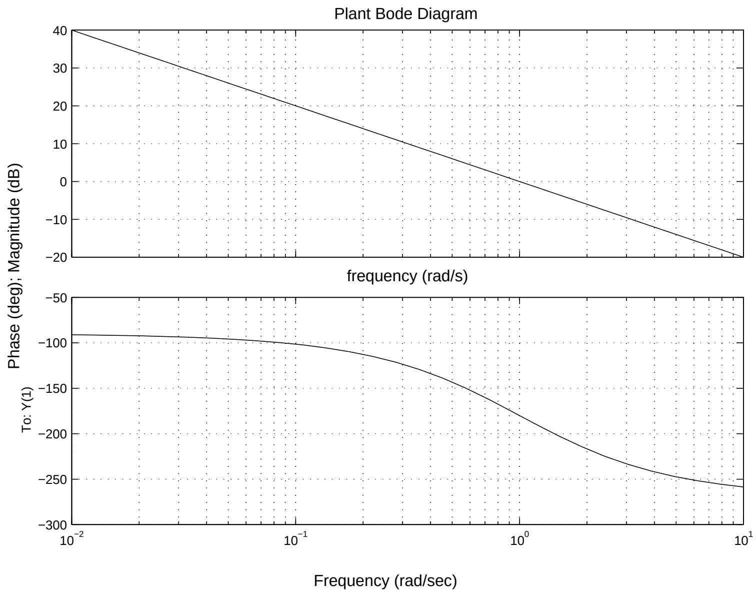
(g) **Investigate the appropriate choices for α and β by considering the speed of the step responses from r to y, the amplitude of the required input, u, and the gain and phase margins. A MATLAB .m file has been written to perform these calculations. It can be invoked from MATLAB as follows**

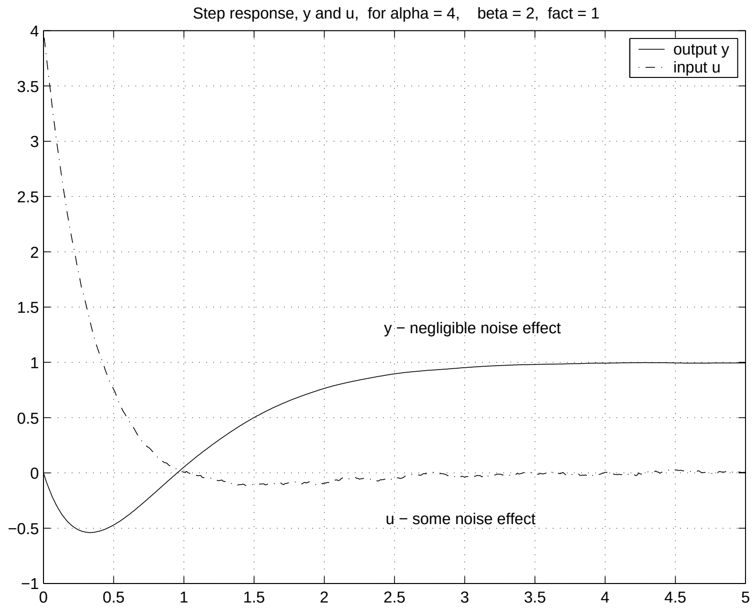
```
>> alpha = 1
>> beta = 1
>> fact = 1
>> Q83F2
```

The gain of $G(s)$ is multiplied by the term `fact`.

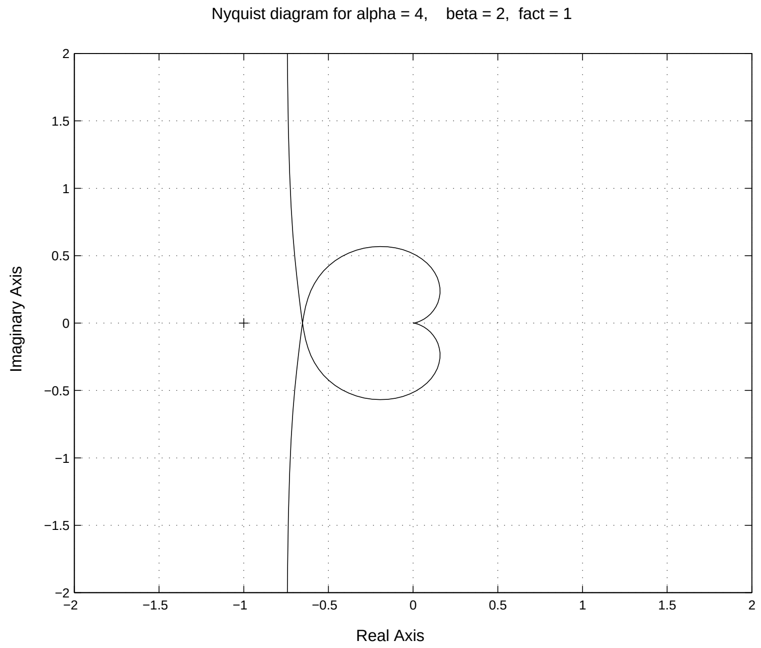
The resulting plots and printout then give the Bode diagram for $G(jw)$; the eigen values of the closed loop A-matrix as in part (e); the step response from r to y in the presence of approximately white noise on v (Normally distributed sequence of standard deviation 0.01, and sampled at 0.01 s); and the Nyquist diagram for the loop gain $H(j\omega)$ as in part (f).

The observer poles are set to $-\alpha$ by choice of K and those of the closed loop system are set to $-\beta$ by choice of the state feedback gain matrix, F. It is usual to choose the obeserver dynamics to be faster than the desired closed loop dynamics. We have therefore chosen $\alpha = 2 \times \beta$. The first design has $\beta = 2$ s^{-1} and the second design is faster with $\beta = 5$ s^{-1} but has significant noise problems and poor stability margins as demonstrated in the following figures.

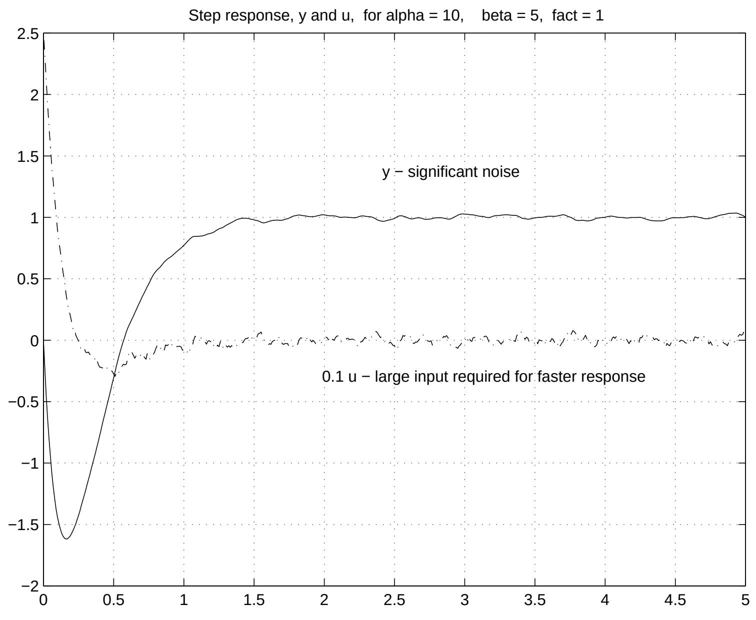


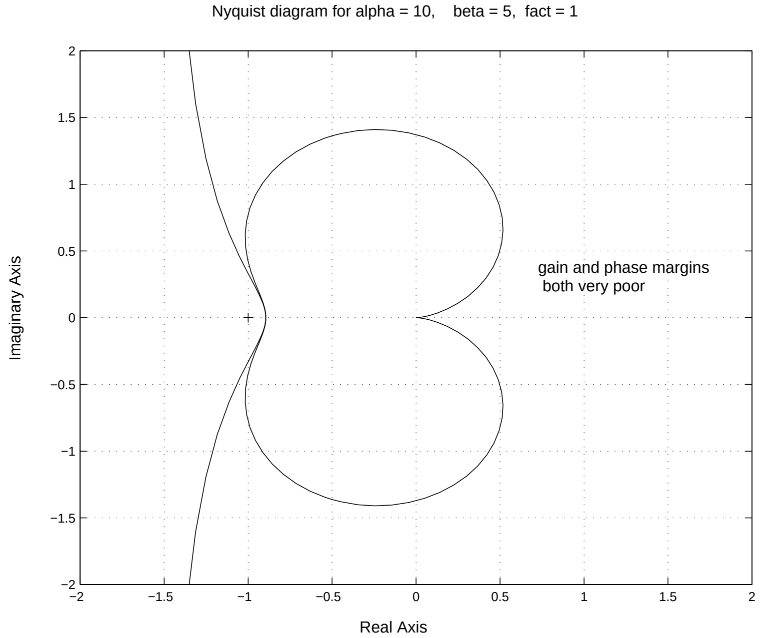

Step response, y and u, for alpha = 4, beta = 2, fact = 1
output y
input u
y − negligible noise effect
u − some noise effect
4
3.5
3
2.5
2
1.5
1
0.5
0
−0.5
−1
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5

The stability margins for these gains are reasonable as given in the following Nyquist diagram.



With the higher gain design we have the following step response:





This design, although faster than the previous design, has poor stability margins. To illustrate this look at the step response if the plant gain is increased by 7 % (`fact = 1.07`) when the closed loop system is nearly unstable.