3F4: Data Transmission

Handout 5: Passband modulation: QAM and FSK

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A PAM signal carrying the information symbols $X_1, X_2, \ldots$ is

$$x_b(t) = \sum_k X_k p(t - kT)$$

- $x_b(t)$ is a *baseband* signal: its bandwidth is the same as that of the pulse $p(t)$

  If the channel is a baseband channel, e.g., ethernet cable, we can directly transmit $x_b(t)$

- But many channels are *passband* — we are only allowed to transmit our signal over a fixed frequency band centred around a carrier frequency f_c.

  E.g., a wireless channel may have $f_c = 2$ GHz and channel bandwidth $= 10$ MHz

Upconverted PAM

A natural way to "up-convert" a PAM signal to passband is to multiply with a carrier:

$$x(t) = x_b(t)\cos(2\pi f_c t) = \left[\sum_k X_k p(t-kT)\right]\cos(2\pi f_c t)$$

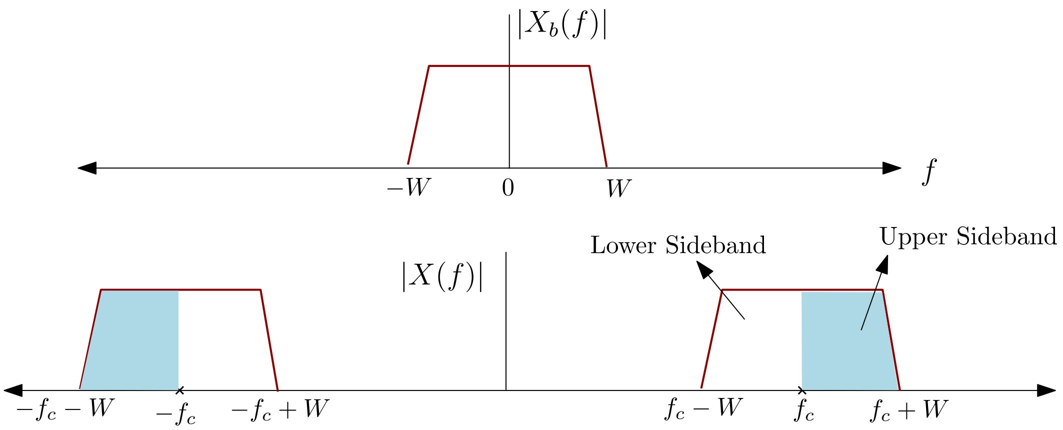
However, this is not an efficient use of bandwidth.

To see why, note that in PAM, both the pulse $p(t)$ and the information symbols $\{X_k\}$ are real-valued. Therefore:

- The spectrum of baseband signal $x_b(t) = \sum_k X_k p(t-kT)$ satisfies $X_b(-f) = X_b^*(f)$.
- That is, the spectrum $X_b(f)$ for $f < 0$ is completely determined by the spectrum for $f \geq 0$.
- The passband signal $x(t)$ has spectrum

$$X(f) = \frac{1}{2}\left[X_b(f-f_c) + X_b(f+f_c)\right]$$

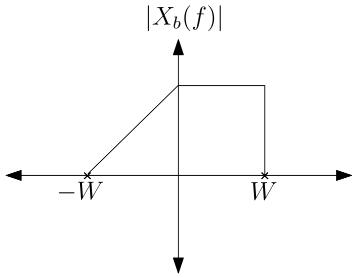
Due to $X_b(-f) = X_b^*(f)$, the lower sideband in $X(f)$ will be completely determined by the upper one (and vice versa).

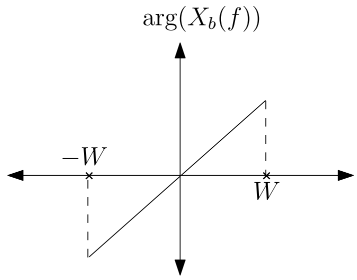


- Recall that bandwidth of a signal is the range of *positive* frequencies for which it is non-zero.
- If the bandwidth of the PAM signal is W, then the passband signal has bandwidth $2W$
- With up-converted PAM, we are transmitting information at the same rate as PAM, but using twice the bandwidth

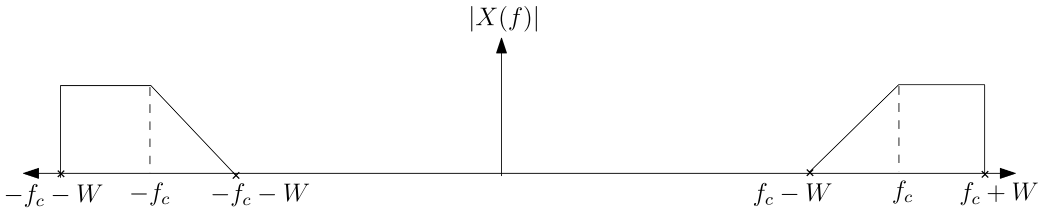
What we want is for $x(t)$ to be a real-valued passband signal, but carrying information on *both* sidebands.

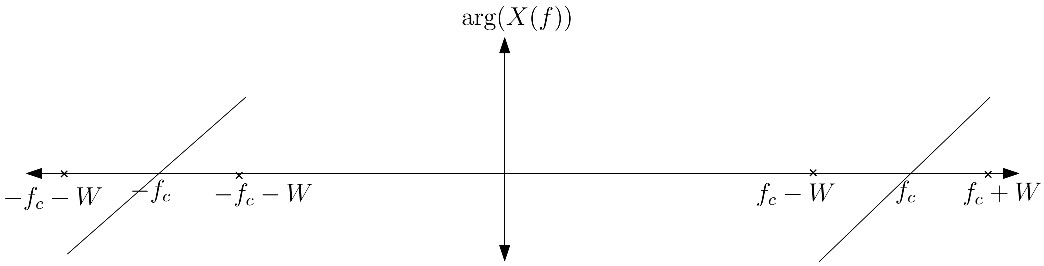
We can do this by starting with a complex baseband signal $x_b(t)$. For example,





But we still want the passband $x(t)$ to be real, i.e., $X(-f) = X^*(f)$





Quadrature Amplitude Modulation (QAM)

The baseband waveform is

$$x_b(t) = \sum_k X_k p(t - kT),$$

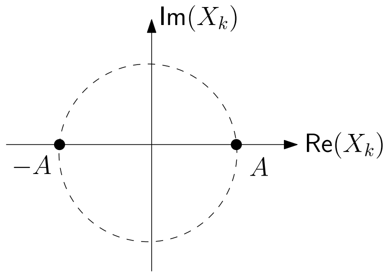
but now the constellation from which the symbols X_k are drawn can be *complex-valued*.

The passband waveform to be transmitted over the channel still has to be real. The QAM signal is generated as:

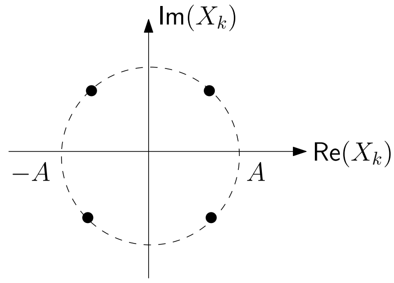
$$\begin{aligned} x(t) &= \text{Re}\left[\sqrt{2}\, x_b(t)\, e^{j2\pi f_c t}\right] \\ &= \text{Re}(x_b(t))\, \sqrt{2}\cos(2\pi f_c t) - \text{Im}(x_b(t))\, \sqrt{2}\sin(2\pi f_c t) \\ &= \sum_k p(t - kT)\, \left[\mathbf{Re}(\mathbf{X_k})\, \sqrt{2}\cos(2\pi f_c t) - \mathbf{Im}(\mathbf{X_k})\, \sqrt{2}\sin(2\pi f_c t)\right] \end{aligned}$$

- The carriers $\cos(2\pi f_c t)$ and $\sin(2\pi f_c t) = \cos(2\pi f_c t - \pi/2)$ are called the 'in-phase' and 'quadrature' components.
- For this reason, QAM is also called I-Q modulation
- The $\sqrt{2}$ factor makes the carriers have unit power

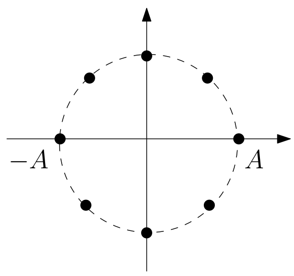
Some typical QAM Constellations



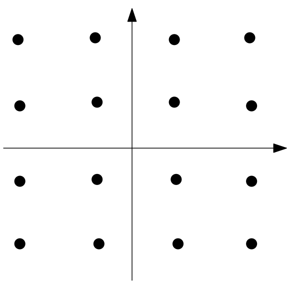
BPSK



QPSK



8-PSK

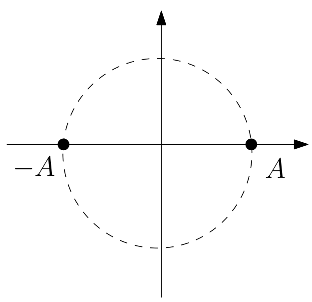


16-QAM

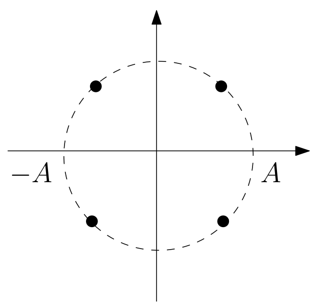
In "Phase Shift Keying" (PSK), the magnitude of X_k is constant, and the information is in the phase of the symbol.

In a constellation with M symbols, each symbol corresponds to $\log_2 M$ bits

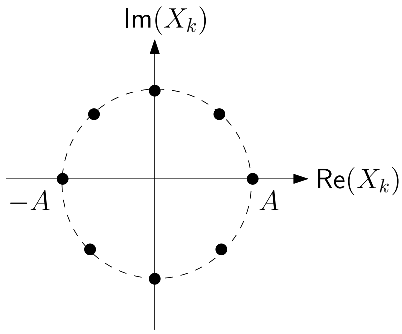
Average Energy per Symbol



BPSK

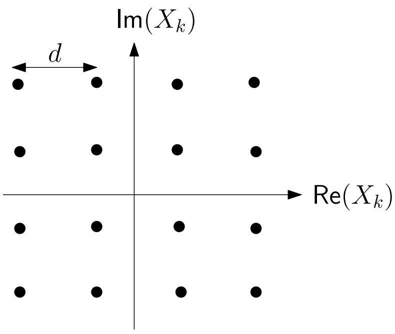


QPSK



8-PSK

For all the PSK constellations, average *symbol* energy $E_s = A^2$



16-QAM

Average energy per symbol for 16-QAM

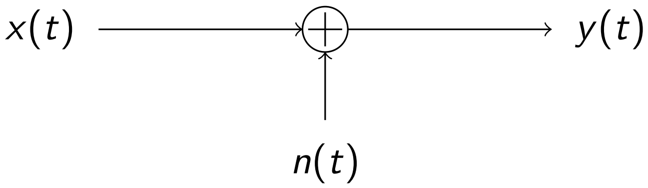
$$E_s = \frac{40d^2}{16} = 2.5d^2$$

Average energy per *bit* $E_b = E_s / \log_2 M$

The transmitted QAM waveform is

$$x(t) = \sum_k p(t - kT) \left[X_k^r \sqrt{2} \cos(2\pi f_c t) - X_k^i \sqrt{2} \sin(2\pi f_c t) \right]$$

where $X_k^r := \mathrm{Re}(X_k)$ and $X_k^i := \mathrm{Im}(X_k)$



At the receiver, we have

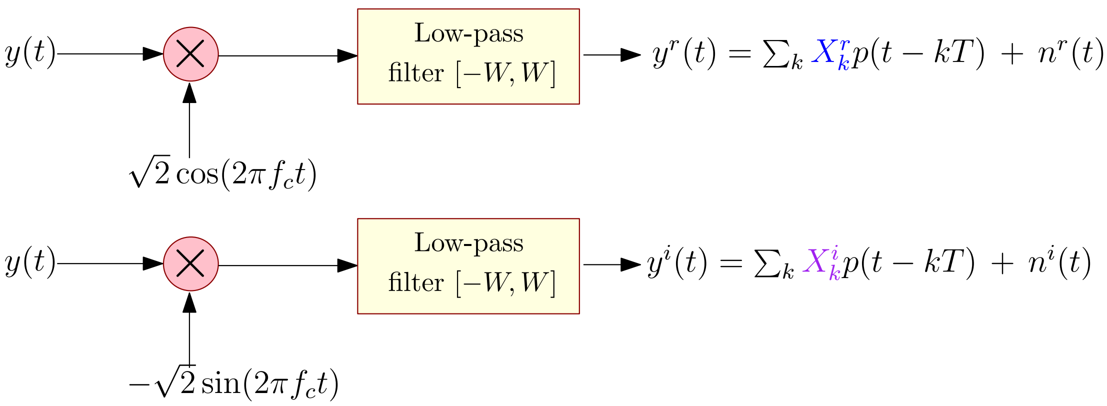
$$y(t) = \sum_k p(t - kT) \left[X_k^r \sqrt{2} \cos(2\pi f_c t) - X_k^i \sqrt{2} \sin(2\pi f_c t) \right] + n(t)$$

Note that:

$$x(t)\sqrt{2}\cos(2\pi f_c t) = \sum_k p(t - kT) \left[X_k^r + X_k^r \cos(4\pi f_c t) - X_k^i \sin(4\pi f_c t) \right]$$

$$-x(t)\sqrt{2}\sin(2\pi f_c t) = \sum_k p(t - kT) \left[X_k^i - X_k^i \cos(4\pi f_c t) - X_k^r \sin(4\pi f_c t) \right]$$

As $p(t)$ is a baseband pulse bandlimited to $[-W, W]$, we can reject the high-frequency components using low-pass filters:



At the output of the low-pass filter, we get

$$y^r(t) = \sum_k X_k^r p(t - kT) + n^r(t)$$

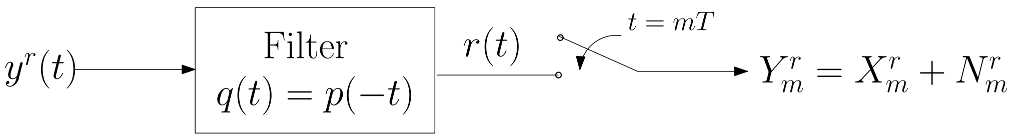
$$y^i(t) = \sum_k X_k^i p(t - kT) + n^i(t)$$

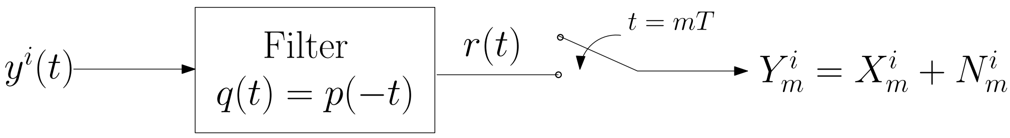
where $n^r(t)$ and $n^i(t)$ are low-pass filtered versions of $n(t)$.

$$y^r(t) = \sum_k X_k^r p(t - kT) + n^r(t)$$
$$y^i(t) = \sum_k X_k^i p(t - kT) + n^i(t)$$

We have therefore reduced the problem to detecting $\{X_k = (X_k^r, X_k^i)\}$ from two PAM-like received waveforms.

If we choose receive filter $q(t) = p(-t)$ so that the overall filter $g(t) = p(t) \star p(-t)$ satisfies Nyquist pulse criterion, then:





N_m^r, N_m^i are samples of $n^r(t) \star q(t)$ and $n^i(t) \star q(t)$ at time mT.

Signal space interpretation of matched filter demodulation:

Define the following functions for $k \in \mathbb{Z}$:

$$f_k^r(t) = p(t - kT)\sqrt{2}\cos(2\pi f_c t)$$
$$f_k^i(t) = -p(t - kT)\sqrt{2}\sin(2\pi f_c t)$$

Then the transmitted QAM waveform can be expressed as

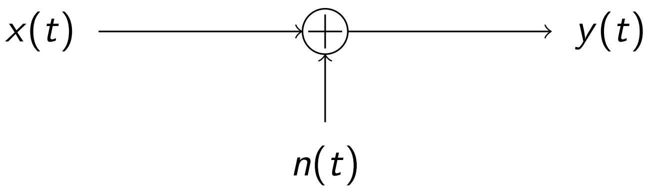
$$x(t) = \sum_k \left[X_k^r f_k^r(t) + X_k^i f_k^i(t)\right]$$

Key Fact

The set of functions $\left\{f_k^r(t), f_k^i(t)\right\}, \ k \in \mathbb{Z}$ is an *orthonormal* set.
(You will prove this in Examples paper 2.)

This suggests a natural receiver structure to extract the QAM symbols from $y(t) = x(t) + n(t)$

At the Receiver

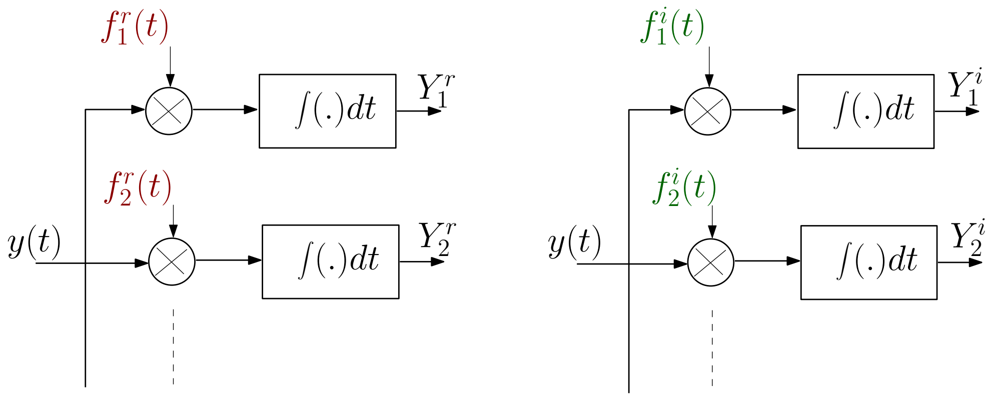


The receiver performs three steps:

1. *Demodulation*: Convert the received waveform $y(t)$ into a discrete-time sequence $Y_1, Y_2, \ldots$ by projecting $y(t)$ onto the elements of the orthonormal set $\{f_k^r(t),\ f_k^i(t)\},\ k \in \mathbb{Z}$.
2. *Detection*: Recover $\hat{X}_1, \hat{X}_2, \ldots$ from $Y_1, Y_2, \ldots$
   ($\hat{X}_1, \hat{X}_2, \ldots$ are points in the complex QAM constellation)
3. Convert $\hat{X}_1, \hat{X}_2, \ldots$ to bits using the assignment of bits to constellation points (easy)

$$y(t) = \sum_k \left[X_k^r f_k^r(t) + X_k^i f_k^i(t)\right] + n(t)$$

Signal-space based QAM demodulator:



For $k, \ell \in \mathbb{Z}$ note that $\langle f_k^r,\ f_\ell^r\rangle = 1$ if $k = \ell$, and 0 otherwise,
$\langle f_k^i,\ f_\ell^i\rangle = 1$ if $k = \ell$, and 0 otherwise, and $\langle f_k^r,\ f_\ell^i\rangle = 0$ for all k, ℓ.

We therefore have

$$Y_k^r = \langle y(t),\ f_k^r(t)\rangle = X_k^r + N_k^r$$
$$Y_k^i = \langle y(t),\ f_k^i(t)\rangle = X_k^i + N_k^i$$

The low-pass filter + matched filter demod in p. 10/11 is just an efficient way of implementing this signal space demodulator

Distribution of noise random variables

The next step is detection of the symbols $\{X_k = (X_k^r, X_k^i)\}$ from $Y_k = (Y_k^r, Y_k^i)$ where

$$Y_k^r = X_k^r + N_k^r, \quad Y_k^i = X_k^i + N_k^i.$$

We need to understand the distribution of the noise rvs

$$N_k^r = \langle n(t), f_k^r(t)\rangle, \quad N_k^i = \langle n(t), f_k^i(t)\rangle, \quad k \in \mathbb{Z}$$

where $n(t)$ is a white Gaussian noise process with PSD $\frac{N_0}{2}$.

We can apply the result from Handout 4, p.7 for the distribution of projection coefficients of white noise for any orthonormal basis.

Since $\{f_1^r(t), f_2^r(t), \ldots, \; f_1^i(t), f_2^i(t), \ldots\}$ form an orthonormal basis, the random variables

$$N_1^r, N_2^r, \ldots, \; N_1^i, N_2^i, \ldots,$$

are i.i.d. $\sim \mathcal{N}(0, \frac{N_0}{2})$

Optimal detection

We use the notation $Y = (Y^r, Y^i)$, $X = (X^r, X^i)$, $N = (N^r, N^i)$

The optimal rule for detecting a QAM symbol X from $Y = X + N$ is the MAP rule:

$$\hat{X} = \arg\max_{x \in \mathcal{C}} \; P(X = x) \cdot f(Y = y \mid X = x).$$

where the arg max is over all x in the *constellation* $\mathcal{C}$.

- Optimality of MAP is in terms of minimising prob. of error
- Since N^r, N^i are i.i.d. $\sim \mathcal{N}(0, \frac{N_0}{2})$, the likelihood is:

$$f(Y = (y^r, y^i) | X = (x^r, x^i)) = \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(y^r - x^r)^2}{N_0}} \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(y^i - x^i)^2}{N_0}}$$

Equally likely constellation symbols

- If the constellation symbols are equally likely MAP rule reduces to max-likelihood (ML) :

$$\hat{X}_{ML} = \arg\max_{x} f(Y = y \mid X = x)$$
$$= \arg\max_{x = (x^r, x^i)} \frac{1}{\pi N_0} e^{-\frac{(y^r - x^r)^2 + (y^i - x^i)^2}{N_0}}$$

- For i.i.d. Gaussian noise, ML is therefore equivalent to minimum-distance decoding. That is, minimise

$$\|y - x\|^2 = \|y\|^2 + \|x\|^2 - 2(x^r y^r + x^i y^i)$$

- The term $\|y\|^2$ does not affect detection. Therefore, the ML rule becomes

$$\hat{X} = \arg\min_{x} \|x\|^2 - 2x^T y, \quad \text{where} \quad x = \begin{bmatrix} x^r \\ x^i \end{bmatrix}, \; y = \begin{bmatrix} y^r \\ y^i \end{bmatrix}$$

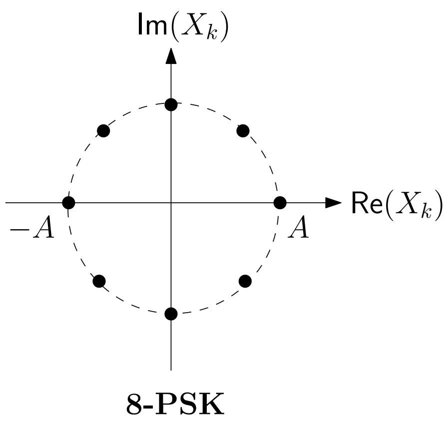
ML decoding examples

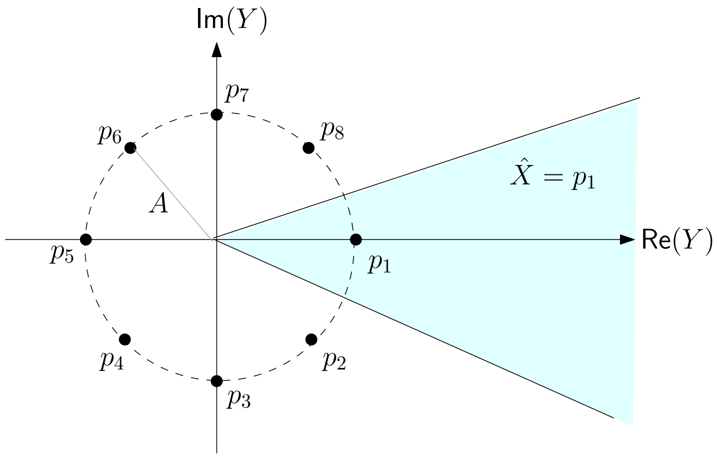
PSK: Let $\theta(x, y)$ be the angle between vectors (x, y).

$$\|y - x\|^2 = \|x\|^2 + \|y\|^2 - 2\|x\|\|y\| \cos(\theta(x, y))$$

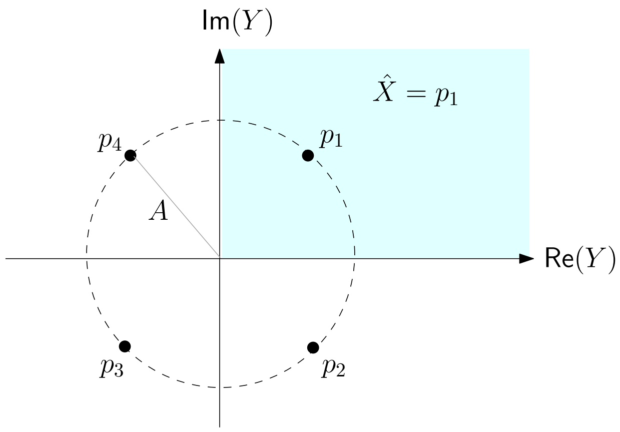
As $\|x\| = A$ for all symbols in PSK, the ML decoding rule is:

$$\hat{X} = \arg\min_{x} \ \theta(x, y)$$

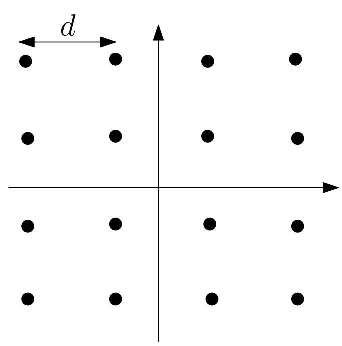




QPSK: This is just 4-PSK, so $\hat{X} = \arg\min_x \ \theta(x, y)$

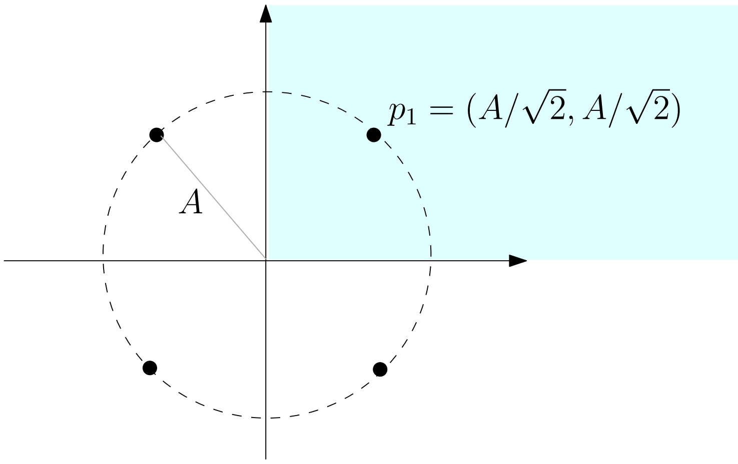


16-QAM: What are the decision regions?



Error Probability Analysis

We will consider QPSK with equally likely symbols here as an example, you will analyse other constellations in Examples Paper 2.



We have

$$P_e = \sum_{i=1}^{4} P(X = p_i)\, P(\hat{X} \neq p_i \mid X = p_i) = \frac{1}{4} \sum_{i=1}^{4} P(\hat{X} \neq p_i \mid X = p_i)$$

Due to symmetry, we can assume that p_1 was the transmitted symbol, i.e., $P_e = P(\hat{X} \neq p_1 \mid X = p_1)$

We have

$$\begin{aligned}
P(\hat{X} \neq p_1 \mid X = p_1) &= P(\{A/\sqrt{2} + N^r < 0\} \cup \{A/\sqrt{2} + N^i < 0\}) \\
&\overset{(a)}{\leq} P(A/\sqrt{2} + N^r < 0) + P(A/\sqrt{2} + N^i < 0) \\
&= P\left(\frac{N^r}{\sqrt{N_0/2}} < \frac{-A/\sqrt{2}}{\sqrt{N_0/2}}\right) + P\left(\frac{N^i}{\sqrt{N_0/2}} < \frac{-A/\sqrt{2}}{\sqrt{N_0/2}}\right) \\
&= Q\left(\sqrt{A^2/N_0}\right) + Q\left(\sqrt{A^2/N_0}\right) \\
&= 2Q(\sqrt{E_s/N_0}) \overset{(b)}{=} 2Q(\sqrt{2E_b/N_0}).
\end{aligned}$$

- Step (a) is obtained using the union bound
- Step (b) is obtained by noting that each QAM symbol carries 2 bits. Therefore $E_s = 2E_b$.
- Using the bound $\mathcal{Q}(x) \leq \frac{1}{2}e^{-x^2/2}$, we see that $P_e \leq e^{-E_b/N_0}$.

QAM: Summary

QAM is a technique to convert bits to a *passband* waveform:

- QAM constellations are complex-valued in general
- The complex baseband waveform is $x_b(t) = \sum_k X_k p(t - kT)$, but what is transmitted is a *real passband* waveform:

$$x(t) = \text{Re}\left[\sqrt{2}x_b(t)\, e^{j2\pi f_c t}\right]$$
$$= \sum_k p(t - kT) \left[\mathbf{Re(X_k)}\sqrt{2}\cos(2\pi f_c t) - \mathbf{Im(X_k)}\sqrt{2}\sin(2\pi f_c t)\right]$$

- Rate $= 1/T$ QAM symbols/s or $\frac{\log_2 \mathbf{M}}{\mathbf{T}}$ bits/s, where M is the constellation size
- Passband bandwidth of QAM waveform $= 2W$, where W is the bandwidth of the (baseband) pulse $p(t)$

At the receiver:

- Demodulator: multiply by carrier + low-pass filter + matched filter (separately for the sine, cosine carriers)
- This demodulator is equivalent to computing inner products of received $y(t)$ with each of the orthonormal basis functions:

$$\left\{p(t - kT)\,\sqrt{2}\cos(2\pi f_c t),\ -p(t - kT)\,\sqrt{2}\sin(2\pi f_c t)\right\}_{k\in\mathbb{Z}}$$

- Then detect X_k from $Y_k = X_k + N_k$ for $k = 0, 1, \ldots$.

  With Gaussian noise and equally likely symbol, MAP rule reduces to: pick the constellation symbol X_k closest to Y_k

QAM is a very widely used modulation scheme. E.g., 4G LTE uses QPSK/16-QAM/64-QAM, also used in optical fibre communication

Frequency shift keying (FSK)

In QAM the information symbols modulate the amplitudes of two orthogonal sinusoidal carriers.

We now study another passband modulation scheme called FSK where the information modulates the *frequency* of the carrier.

In M-**ary FSK**:

- We transmit one of M messages ($\log_2 M$ bits) in each symbol period T
- To transmit message $i \in \{1, \ldots, M\}$ in any symbol period $[\ell T, (\ell+1)T)$, the FSK waveform is

$$x(t) = \sqrt{E_s}\sqrt{\frac{2}{T}}\cos\left(2\pi\left(f_c + (2i-(M+1))\frac{\Delta_f}{2}\right)t\right),$$

  where $\Delta_f = \frac{1}{2T} = \frac{f_c}{K}$ for some large integer K
- The M symbols are represented by M frequencies, with adjacent frequencies separated by Δ_f:

$$f_c - \frac{(M-1)}{2}\Delta_f, \quad f_c - \frac{(M-3)}{2}\Delta_f, \ldots, \quad f_c + \frac{(M-1)}{2}\Delta_f$$

Orthonormal basis for M-ary FSK

For $i = 1, \ldots, M$, let

$$f_i(t) = \sqrt{\frac{2}{T}}\cos\left(2\pi\left(f_c + (2i-(M+1))\frac{\Delta_f}{2}\right)t\right),$$

for $t \in [\ell T, (\ell+1)T), \ \ell \in \mathbb{Z}$.

Then the set $\{f_1(t), \ldots, f_M(t)\}$ forms an orthonormal basis.

(The proof of this fact is given on the next page.)

Using this fact:

- We can express the FSK signal in terms of the basis functions as $x(t) = \sum_{k=1}^{M} x_k f_k(t)$.
- If the ith message is transmitted, the only non-zero term in the sum is $x_i = \sqrt{E_s}$.
- That is, for message i the projection coefficients are:

$$[x_1, \ldots, x_i, \ldots x_M] = [0, \ldots, \sqrt{E_s}, \ldots, 0]$$

Proof that $\{f_1(t), \ldots, f_M(t)\}$ *is an orthonormal basis*: We will assume that $t \in [0, T)$. The case $t \in [\ell T, (\ell+1)T)$ is very similar. We want to show that $\int_0^T f_i^2(t)dt = 1$ for all $i \in \{1, \ldots, M\}$, and $\int_0^T f_i(t)f_j(t)dt = 0$ for $i \neq j$. We write

$$\int_0^T f_i^2(t)dt = \frac{1}{T}\int_0^T \left[1 + \cos\left(4\pi\left(f_c + (2i - (M+1))\frac{\Delta_f}{2}\right)t\right)\right] dt$$
$$= 1 - \frac{1}{cT} \underbrace{\sin\left(4\pi\left(f_c + (2i - (M+1))\frac{\Delta_f}{2}\right)T\right)}_{=\ 0 \text{ as } f_c T = \frac{K}{2} \text{ and } \Delta_f T = \frac{1}{2}}$$

Similarly, for $i \neq j$

$$\int_0^T f_i(t)f_j(t)dt = \frac{1}{T}\int_0^T \Big[\cos\left(2\pi\left(2f_c + ((i+j) - (M+1))\Delta_f\right)t\right) - \cos\left(2\pi(i-j)\Delta_f t\right)\Big] dt$$

$$= \frac{1}{cT}\underbrace{\sin\left(2\pi\left(2f_c + ((i+j) - (M+1))\Delta_f\right)T\right)}_{0} - \frac{1}{c'T}\underbrace{\sin\left(2\pi(i-j)\ \Delta_f T\right)}_{0}$$

(c, c' are constants whose exact values don't matter)

Demodulation and Detection for M-ary FSK

To transmit message $i \in \{1, \dots, M\}$ in any symbol period $[\ell T, (\ell+1)T)$, the transmitted FSK waveform is

$$x(t) = \sqrt{E_s} f_i(t),$$

where $f_i(t) = \sqrt{\frac{2}{T}} \cos\left(2\pi\left(f_c + (2i - (M+1))\frac{\Delta_f}{2}\right)t\right)$.

At the Rx, we have $y(t) = x(t) + n(t)$.

As the waveforms $\{f_1(t), \dots, f_M(t)\}$ are orthonormal over each symbol period, the demodulator computes:

$$\begin{aligned}
Y_1 &= \langle y(t), f_1(t) \rangle = \int_{t \in \text{symbol period}} y(t) f_1(t) dt \\
Y_2 &= \langle y(t), f_2(t) \rangle \\
&\vdots \\
Y_M &= \langle y(t), f_M(t) \rangle
\end{aligned}$$

We have:

$$\begin{aligned}
Y_1 &= \langle x(t), f_1(t) \rangle + \langle n(t), f_1(t) \rangle = X_1 + N_1 \\
Y_2 &= \langle x(t), f_2(t) \rangle + \langle n(t), f_2(t) \rangle = X_2 + N_2 \\
&\vdots \\
Y_M &= \langle x(t), f_M(t) \rangle + \langle n(t), f_M(t) \rangle = X_M + N_M
\end{aligned}$$

- As before, the noise variables $N_j \sim \mathcal{N}(0, \frac{N_0}{2})$ are i.i.d.
- If message $i \in \{1, \dots, M\}$ is transmitted:

$$X_i = \sqrt{E_s}, \qquad X_j = 0, \ j \neq i$$

Hence $[Y_1, \dots, Y_i, \dots, Y_M] = [N_1, \dots, \sqrt{E_s} + N_i, \dots, N_M]$

What is the optimal detection rule?

M-ary FSK Detection

Assuming all the messages are equally likely, the optimal rule is to pick the message $\hat{m}$ such that:

$$\hat{m} = \arg\max_{1 \le i \le M} \; P(\mathbf{y} \mid \mathbf{X} = \mathbf{x}(i))$$

where:

1. $\mathbf{x}(i)$ is the input vector corresponding to message i, i.e., it is the length-M vector with $\sqrt{E_s}$ in position i and 0 elsewhere
2. Conditioned on the input being $\mathbf{x}(i)$, the received vector $\mathbf{y} = [n_1, \ldots, \sqrt{E_s} + n_i, \ldots, n_M]$

$$P(\mathbf{y} \mid \mathbf{x}(i)) = P(y_1 \mid x_1 = 0) \ldots P(y_i \mid x_i = \sqrt{E_s}) \ldots P(y_M \mid x_M = 0)$$

$$= \frac{1}{(\sqrt{\pi N_0})^M} \; e^{-y_1^2/N_0} \ldots e^{-(y_i - \sqrt{E_s})^2/N_0} \ldots e^{-y_M^2/N_0}$$

$$= \frac{1}{(\sqrt{\pi N_0})^M} \; e^{-(y_1^2 + \ldots + y_M^2)/N_0} \; e^{-E_s/N_0} \; e^{2E_s y_i/N_0}$$

The optimal detection rule is thus $\hat{m} = \arg\max_{1 \le i \le M} \; y_i$

Probability of Detection Error

Due to symmetry, the probability of error is the same regardless of which message $i \in \{1, \ldots, M\}$ was transmitted. So, without loss of generality, assume message 1 was transmitted.

An error occurs if y_1 is *not* the maximum among $[y_1, \ldots, y_M] \Rightarrow$

$$P_e = P\left(\{y_1 \le y_2\} \cup \{y_1 \le y_3\} \cup \ldots \cup \{y_1 \le y_M\}\right)$$

$$= P\left(\{\sqrt{E_s} + n_1 \le n_2\} \cup \{\sqrt{E_s} + n_1 \le n_3\} \ldots \cup \{\sqrt{E_s} + n_1 \le n_M\}\right)$$

$$\le P\left(\{\sqrt{E_s} + n_1 \le n_2\}\right) + \ldots + P\left(\{\sqrt{E_s} + n_1 \le n_M\}\right)$$

$$= (M-1) P\left(n_2 - n_1 \ge \sqrt{E_s}\right)$$

The last line holds because $(n_2 - n_1)$, $(n_3 - n_1)$, …,$(n_M - n_1)$ are each $\mathcal{N}(0, N_0)$ rvs. In Examples Paper 2, you will simplify the above expression and show that

$$P_e \le \frac{1}{2} e^{-\frac{\log_2 M}{2}\left(\frac{E_b}{N_0} - 2\ln 2\right)}, \qquad \text{where } E_s = E_b \log_2 M$$

Thus if $\frac{E_b}{N_0} > 2\ln 2$, the probability of detection error $\downarrow$ as $M \uparrow$

Rate and Bandwidth: QAM vs FSK

Rate: Both M-ary QAM and M-ary FSK have rate $\frac{\log_2 M}{T}$ bits/s

Bandwidth:

- **QAM**: The bandwidth is determined by the baseband pulse $p(t)$. If $p(t)$ is band-limited to $[-W, W]$, then $x(t)$ is bandlimited to $[f_c - W, f_c + W]$, and so has bandwidth **2W**. If $p(t)$ is root raised cosine pulse, W is typically between $\frac{1}{2T}$ and $\frac{1}{T}$
- M-**FSK**: For message $i \in \{1, \ldots, M\}$, the signal $x(t)$ is a cosine with frequency $f_c + (i - \frac{(M+1)}{2})\Delta_f$. Hence the total bandwidth required for M-ary FSK is $(M-1)\Delta_f = \frac{\mathbf{(M-1)}}{\mathbf{2T}}$.

The *bandwidth efficiency* η is defined as rate/bandwidth:

$$\eta_{QAM} \approx \log_2 M \text{ bits/s/Hz}, \qquad \eta_{MFSK} = \frac{2\log_2 M}{M-1} \text{ bits/s/Hz}$$

As M increases, $\eta_{QAM} \uparrow$ but $\eta_{MFSK} \downarrow$. But how does P_e change with M? (Examples Paper 2)