Part IIA Paper 3C6 Ex. Sheet 2 Solutions

1. $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$ so $M = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}$

$V = \frac{1}{2}k_1 x^2 + \frac{1}{2}k_2 y^2 + \frac{1}{2}s(x-y)^2$

So $K = \begin{bmatrix} k_1+s & -s \\ -s & k_2+s \end{bmatrix}$.

So eigenvalue equation is $\begin{vmatrix} k_1+s-\omega^2 m & -s \\ -s & k_2+s-\omega^2 m \end{vmatrix} = 0$

Define $\omega_1^2 = \frac{k_1+s}{m}$, $\omega_2^2 = \frac{k_2+s}{m}$, $\Omega^2 = \frac{s}{m}$,

then $\begin{vmatrix} \omega_1^2 - \omega^2 & -\Omega^2 \\ -\Omega^2 & \omega_2^2 - \omega^2 \end{vmatrix} = 0$ //.

So $(\omega^2 - \omega_1^2)(\omega^2 - \omega_2^2) - \Omega^4 = 0$

$\therefore \omega^4 - \omega^2(\omega_1^2 + \omega_2^2) + \omega_1^2\omega_2^2 - \Omega^4 = 0$

$\therefore \omega^2 = \frac{1}{2}\left\{\omega_1^2 + \omega_2^2 \pm \sqrt{(\omega_1^2+\omega_2^2)^2 - 4\omega_1^2\omega_2^2 + 4\Omega^4}\right\}$

$= \frac{1}{2}\left\{\omega_1^2 + \omega_2^2 \pm \sqrt{(\omega_1^2-\omega_2^2)^2 + 4\Omega^4}\right\}$

So (i) $\omega_1^2 = \omega_2^2 \Rightarrow \omega^2 = \omega_1^2 \pm \Omega^2$

(ii) $(\omega_1^2 - \omega_2^2)^2 \gg 4\Omega^4 \Rightarrow \omega^2 \simeq \omega_1^2, \omega_2^2$.

Mode shapes: $(k_1+s)x - sy = \omega^2 m x$

$\therefore \omega_1^2 x - \Omega^2 y = \omega^2 x$

$\therefore \frac{y}{x} = \frac{\omega_1^2 - \omega^2}{\Omega^2}$

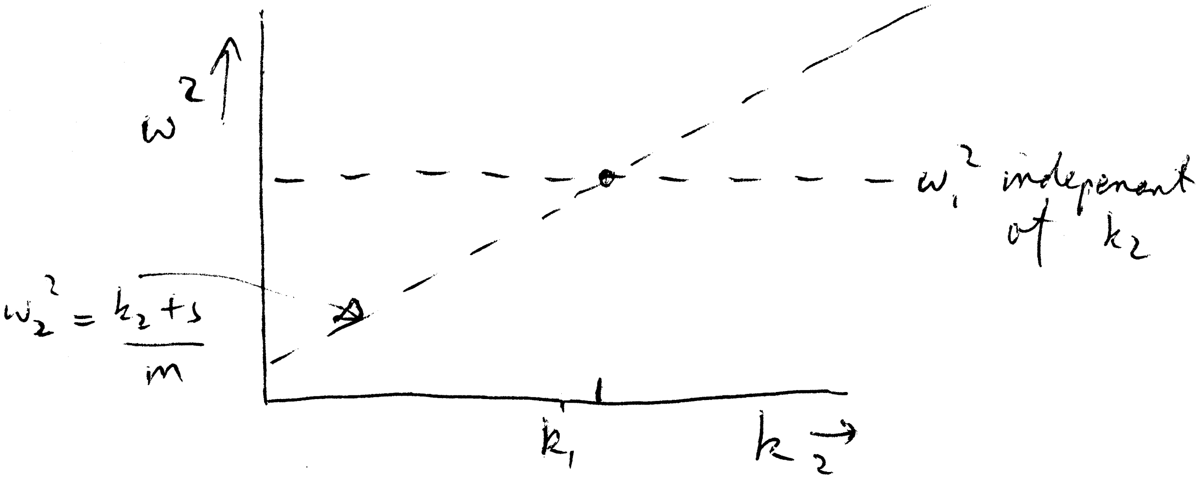
So (i) modes are $\frac{y}{x} = \mp 1$ as $\omega^2 = \omega_1^2 \pm \Omega^2$

(ii) $\omega^2 = \omega_1^2 \to \frac{y}{x} = 0$, $\omega^2 = \omega_2^2 \gg \omega_1^2$ or $\ll \omega_1^2$, so $\frac{y}{x} \to \infty$

Now $\frac{\omega_1^2}{\omega_2^2} = \frac{k_1+s}{k_2+s} \simeq \frac{k_1}{k_2}$

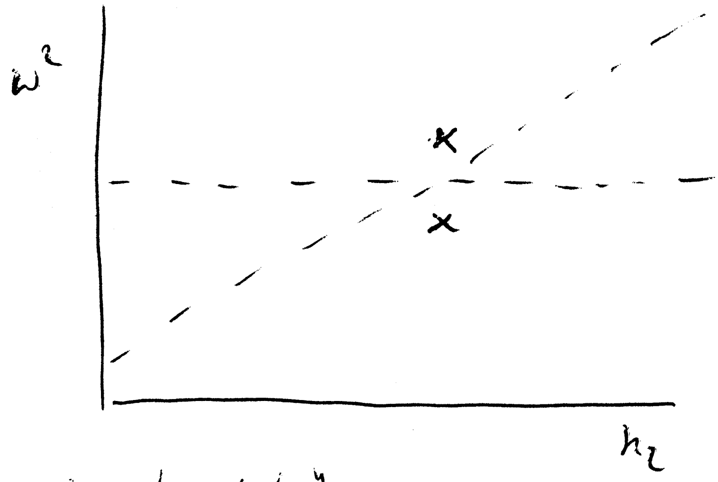
To sketch the curves:

First draw in the lines corresponding to case (ii)

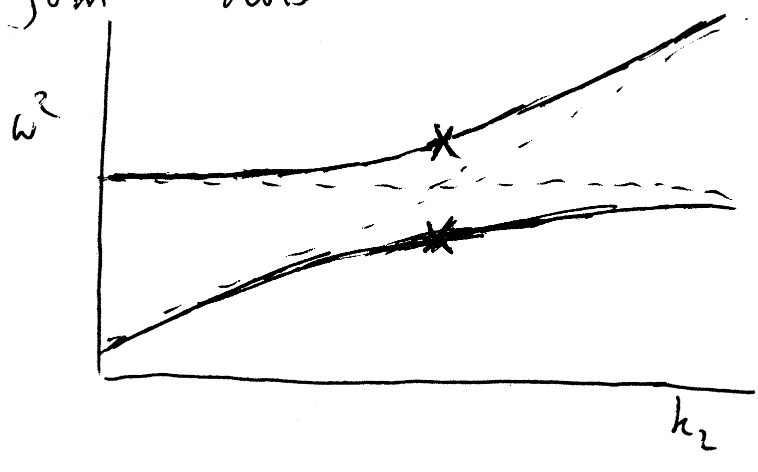


Expect the true graph to follow these lines <u>except</u> near the crossing point at $k_1 = k_2$.

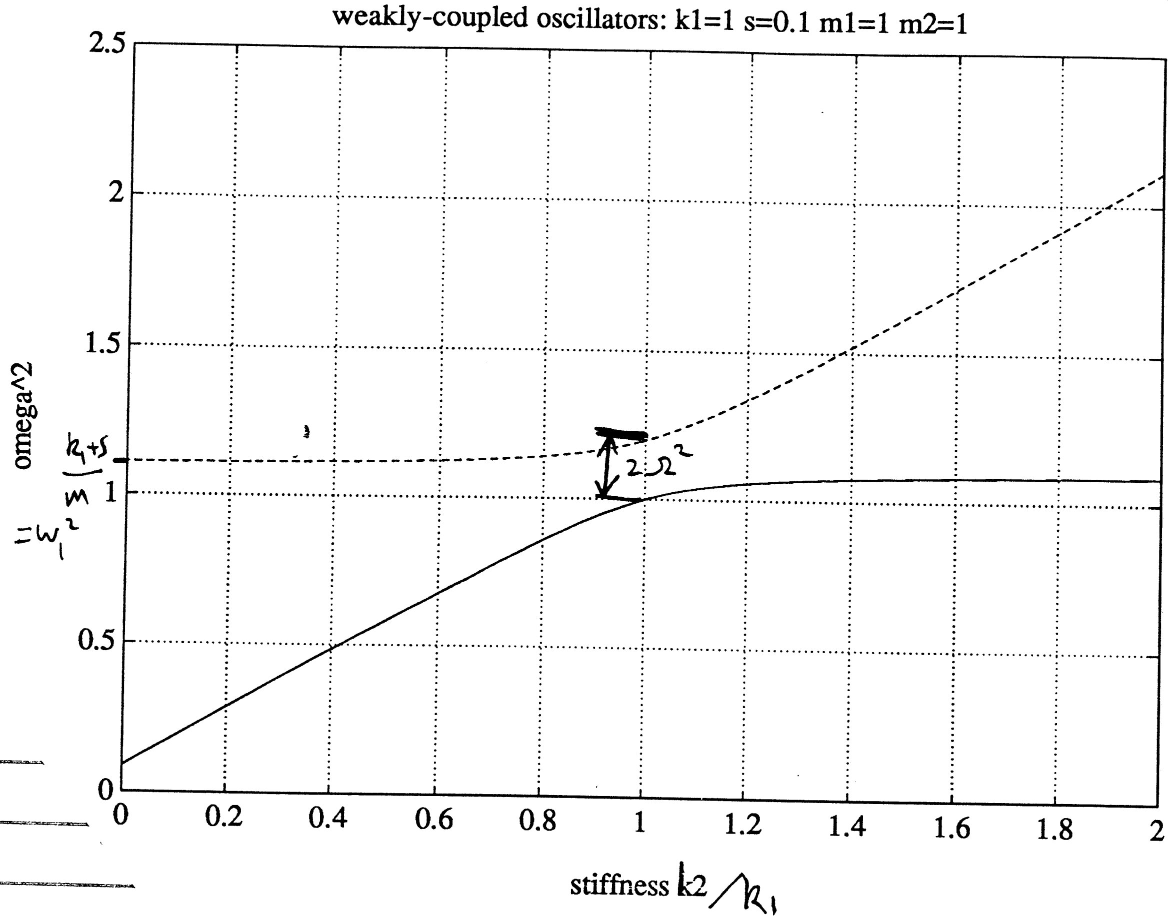
The solution to case (i) shows that the true solutions lie above and below this crossing point:



So now "join the dots"



1 cont. Correct computed version:



	Modes	Modes	Modes
Upper:	$\begin{bmatrix}1\\0\end{bmatrix}$	$\begin{bmatrix}1\\-1\end{bmatrix}$	$\begin{bmatrix}0\\1\end{bmatrix}$
lower:	$\begin{bmatrix}0\\1\end{bmatrix}$	$\begin{bmatrix}1\\1\end{bmatrix}$	$\begin{bmatrix}1\\0\end{bmatrix}$

Note they have swapped

If dissipation occurs in the spring S, the pattern of damping factors will be:

Upper:	Both the same	High	Both the same
Lower:		Zero	

2. $T = \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} m \dot{x}_2^2 + \dots + \frac{1}{2} m \dot{x}_N^2$

$$\therefore M = \begin{bmatrix} m & & & \\ & m & & \\ & & m & \\ & & & \ddots \\ & & & & n \end{bmatrix}$$

$V = \frac{1}{2} k x_1^2 + \frac{1}{2} k x_2^2 + \dots$ (leaf springs)
$+ \frac{1}{2} s (x_1 - x_2)^2 + \frac{1}{2} s (x_2 - x_3)^2 + \dots$ (coupling springs)

$$\therefore K = \begin{bmatrix} k+2s & -s & & & -s \\ -s & k+2s & -s & 0 & \\ & -s & k+2s & -s & \\ & 0 & & \ddots & \\ -s & & & & k+2s \end{bmatrix}$$

Try $x_j = \cos \lambda j$.
The system is circular, so this only makes sense if $\cos \lambda N = 1 = \cos \lambda 0$ so that it joins up.
i.e. $\lambda = \frac{2n\pi}{N}$, $n = 0, 1, 2 \dots$ are possible values.

Now substitute in $K \underline{x} = \omega^2 M \underline{x}$.
Since the system is so symmetric, we only need to look at one typical row: e.g. jth row.
Then $-s \cos \lambda (j-1) + (k+2s) \cos \lambda j - s \cos \lambda (j+1) = \omega^2 m \cos \lambda j$

But $\cos \lambda (j-1) + \cos \lambda (j+1) = 2 \cos \lambda j \cos \lambda$ (D. Book)
So a factor $\cos \lambda j$ cancels all through, leaving

$$-2s \cos \lambda + (k+2s) = \omega^2 m$$

$$\therefore \omega^2 = \frac{k}{m} + \frac{2s}{m}(1 - \cos \lambda) \qquad (1)$$

Doesn't depend on j, so if this equation is satisfied, all rows of $K \underline{x} = \omega^2 M \underline{x}$ are satisfied.
$\therefore \underline{x}$ is a mode provided $\lambda = \frac{2n\pi}{N}$, with natural frequency given by (1).

2 cont.

How many different modes can be found this way?

Since $\cos(\pi + \lambda j) = -\cos(\lambda j)$, and $\cos(\lambda j + 2\pi) = \cos(\lambda j)$, the only distinct vectors arise from

$$\lambda = 0, \frac{2\pi}{N}, \frac{4\pi}{N} \ldots \frac{2\pi}{N}\left(\frac{N}{2}\right)$$

or $\frac{N-1}{2}$ if N odd

This only gives roughly $N/2$ different modes. However, for nearly all these values of λ, as well as a mode $\cos \lambda j$, there is also a mode $\sin \lambda j$ having the same natural frequency. In total, there are always exactly N modes of this kind.

For N even:

$$\begin{cases} \lambda = 0 & 1 \text{ mode} \\ \lambda = 2\pi/N & 2 \text{ modes} \\ \lambda = 4\pi/N & 2 \text{ modes} \\ \vdots & \\ \lambda = \pi & 1 \text{ mode} \end{cases}$$

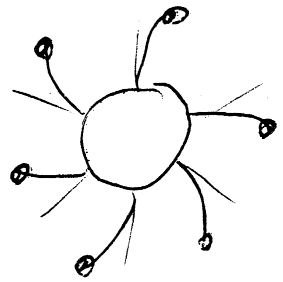
For N odd:

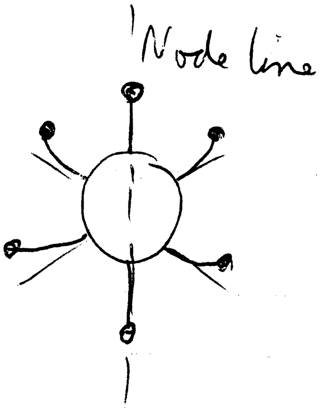
$$\begin{cases} \lambda = 0 & 1 \text{ mode} \\ \lambda = 2\pi/N & 2 \text{ modes} \\ \vdots & \\ \lambda = \left(\frac{N-1}{N}\right)\pi & 2 \text{ modes} \end{cases}$$

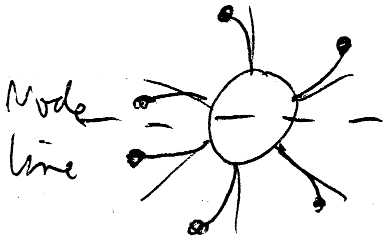
So all natural frequencies satisfy (1), so they all lie between $w = \sqrt{\frac{k}{m}}$ and $w = \sqrt{\frac{k+4s}{m}}$

2 Cont.

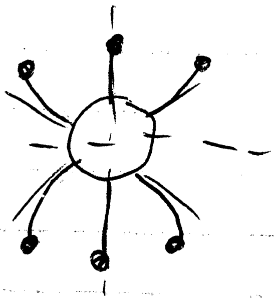
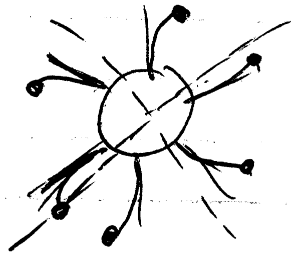
For $N=6$, the modes are:

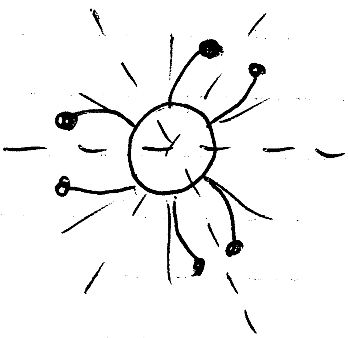
No nodal lines  $\lambda = 0$

One nodal line 



$\lambda = \pi/3$

2 nodal lines   $\lambda = 2\pi/3$

3 nodal lines  $\lambda = \pi$

3. $$y = a_1 \frac{x}{L} + a_2 \frac{x^2}{L^2} + a_3 \frac{x^3}{L^3}$$

At $x=0$, $y=0$ already.
At $x=L$, $y=0 \Rightarrow a_1 + a_2 + a_3 = 0$

So $$y = a_1\left(\frac{x}{L} - \frac{x^3}{L^3}\right) + a_2\left(\frac{x^2}{L^2} - \frac{x^3}{L^3}\right)$$

Now $$V = \frac{1}{2} P \int_0^L y'^2 \, dx = \frac{1}{2} P \int_0^L \left[\frac{a_1}{L}\left(1 - \frac{3x^2}{L^2}\right) + \frac{a_2}{L}\left(\frac{2x}{L} - \frac{3x^2}{L^2}\right)\right]^2 dx$$

$$= \frac{1}{2}\frac{P}{L^2}\left[a_1^2 \int \left(1 - \frac{3x^2}{L^2}\right)^2 dx + 2a_1 a_2 \int \left(1 - \frac{3x^2}{L^2}\right)\left(\frac{2x}{L} - \frac{3x^2}{L^2}\right) dx + a_2^2 \int \left(\frac{2x}{L} - \frac{3x^2}{L^2}\right)^2 dx\right]$$

Multiply out tediously, do the integrals...

$$= \frac{1}{2}\frac{P}{L}\left[a_1^2 \cdot \frac{4}{5} + 2a_1 a_2 \cdot \frac{3}{10} + a_2^2 \cdot \frac{2}{15}\right]$$

$$\therefore \quad K = \frac{P}{L}\begin{bmatrix} 4/5 & 3/10 \\ 3/10 & 2/15 \end{bmatrix}$$

$$T = \frac{1}{2} m \int_0^L \dot{y}^2 \, dx = \frac{1}{2} m \int_0^L \left[\dot{a}_1^2 \left(\frac{x}{L} - \frac{x^3}{L^3}\right)^2 + 2\dot{a}_1 \dot{a}_2 \left(\frac{x}{L} - \frac{x^3}{L^3}\right)\left(\frac{x^2}{L^2} - \frac{x^3}{L^3}\right) + \dot{a}_2^2 \left(\frac{x^2}{L^2} - \frac{x^3}{L^2}\right)^2\right] dx$$

... more algebra ...

$$= \frac{1}{2} mL \left[\frac{8}{105} \dot{a}_1^2 + \frac{11}{420} \cdot 2\dot{a}_1 \dot{a}_2 + \frac{1}{105} \dot{a}_2^2\right]$$

3 cntd

$$\therefore \quad M = \frac{mL}{420}\begin{bmatrix} 32 & 11 \\ 11 & 4 \end{bmatrix}$$

Now $\quad K\underline{u} = \omega^2 M\underline{u}$

$$\rightarrow \frac{P}{30L}\begin{bmatrix} 24 & 9 \\ 9 & 4 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \omega^2 . \frac{mL}{420}\begin{bmatrix} 32 & 11 \\ 11 & 4 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

$$\therefore \quad \frac{1}{30}\begin{bmatrix} 24 & 9 \\ 9 & 4 \end{bmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \Omega^2 . \frac{1}{420}\begin{bmatrix} 32 & 11 \\ 11 & 4 \end{bmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

with $\quad \omega^2 = \frac{P\,\Omega^2}{mL^2}$

The Matlab code:

```
M = (1/420)*[32 , 11; 11 , 4];
K = (1/30) * [24 , 9; 9 , 4];
eig(K,M)
```

will print the values of Ω^2. Result is 10, 42
(Could also do this by hand, of course).

So the approximate natural frequencies satisfy

$$\omega^2 = \frac{P}{mL^2} \times \begin{cases} 10 \\ 42 \end{cases}$$

Exact answers are $\omega^2 = \frac{P}{mL^2}\begin{cases} \pi^2 \\ 4\pi^2 \end{cases} = \begin{cases} 9.87 \\ 39.5 \end{cases}$

So pretty close for a 2DoF approximation.

4(a) $$Y_n + X_n = \frac{b_n}{\omega - \omega_n(1+ic_n)} - \frac{b_n}{\omega - \omega_n(-1+ic_n)}$$

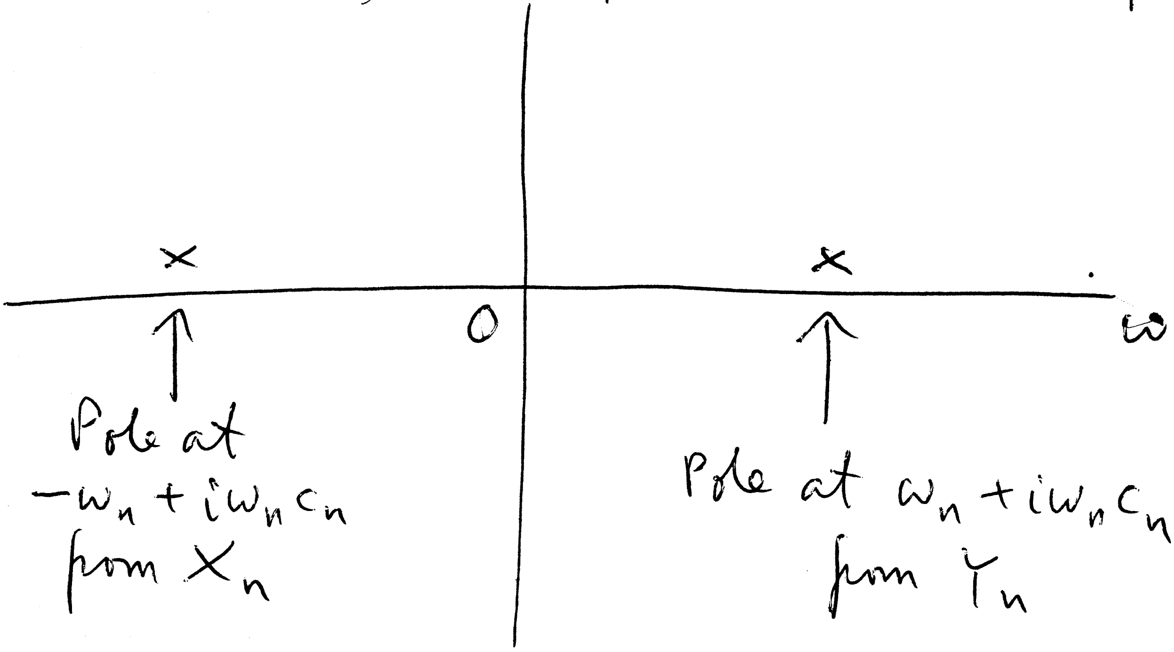
$$= \frac{2b_n\omega_n}{(\omega-\omega_n - i\omega_n c_n)(\omega+\omega_n - i\omega_n c_n)}$$

$$= \frac{2b_n\omega_n}{(\omega - i\omega_n c_n)^2 - \omega_n^2} \simeq \frac{2b_n\omega_n}{\omega^2 - 2i\omega\omega_n c_n - \omega_n^2}$$

$(|c_n|^2 \ll 1)$

$$= H_n \text{ if } b_n = -a_n/2\omega_n$$

Each of X_n, Y_n represents one pole of the transfer function. This division shows that occur in symmetric pairs in the ω Argand plane



For approximate behaviour near the positive frequency ω_n, it will be OK to take Y_n only, and ignore X_n. The pole at negative frequency is further away than the other positive-frequency poles of the system, corresponding to other modes.

4 cont

(b) $|Y_n|$ at $w = w_n$ is $\dfrac{|b_n|}{w_n c_n}$

So want w where $|Y_n| = \dfrac{|b_n|}{\sqrt{2}\, w_n c_n}$

i.e. $\dfrac{b_n^2}{2w_n^2 c_n^2} = \dfrac{b_n^2}{(w-w_n)^2 + w_n^2 c_n^2}$

$$\therefore \ (w-w_n)^2 = w_n^2 c_n^2$$
$$\therefore \ w = w_n(1 \pm c_n) \quad //$$

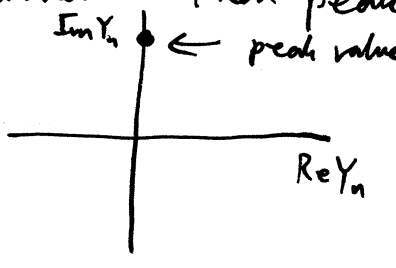
(c) Phase at peak: Y_n is pure imaginary, so phase is $\pm \pi/2$ depending on whether $b_n < 0$ or $b_n > 0$.

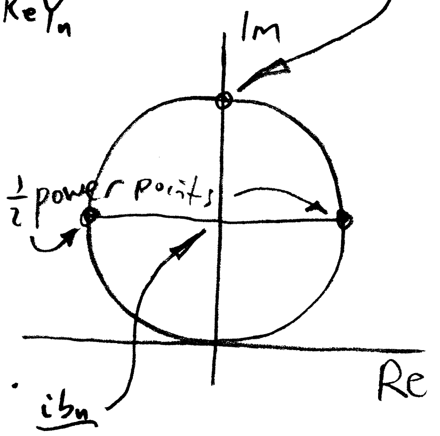
Phase of $Y_n = \pm\dfrac{\pi}{2} \pm \pi/4$ requires $\dfrac{\mathrm{Re}\, Y_n}{\mathrm{Im}\, Y_n} = \pm 1$

i.e. $\dfrac{w - w_n}{w_n c_n} = \pm 1$

$\rightarrow \ w = w_n \pm w_n c_n$, as in (a).

(d) Think of $b_n > 0$, for definiteness. Then peak value has phase $\dfrac{\pi}{2}$:





As $w \to \infty$, $Y_n \to 0$.
Expression for $\mathrm{Re}\, Y_n = \dfrac{w - w_n}{(w-w_n)^2 + c_n^2 w_n^2}$
is antisymmetric with respect to $\begin{cases} w < w_n \\ w > w_n \end{cases}$.

So expect centre of circle to be at $\frac{1}{2}\dfrac{ib_n}{w_n c_n}$.

So want $[\mathrm{Re}\, Y_n]^2 + \left[\mathrm{Im}\, Y_n - \dfrac{b_n}{2w_n c_n}\right]^2 = \dfrac{1}{4w_n^2 c_n^2}$.

Substitute the expressions, and it comes out immediately.

5 Begin with the general solution for a freely vibrating beam : $y(z,t) = Y(z)\, e^{i\omega t}$

$$Y(z) = A\cos\alpha z + B\sin\alpha z + C\cosh\alpha z + D\sinh\alpha z$$

$$\text{where} \quad \alpha^4 = \frac{m\omega^2}{EI} \qquad (1)$$

The usual boundary conditions are :

pinned end : $Y = 0$ and $\frac{d^2Y}{dz^2} = 0$ (Moment)

Clamped end : $Y = 0$ and $\frac{dY}{dz} = 0$ (slope)

free end : $\frac{d^2Y}{dz^2} = 0$ and $\frac{d^3Y}{dz^3} = 0$ (Shear)

$$Y'(z) = \alpha(-A\sin + B\cos + C\sinh + D\cosh)$$
$$Y''(z) = \alpha^2(-A\cos - B\sin + C\cosh + D\sinh)$$
$$Y'''(z) = \alpha^3(A\sin - B\cos + C\sinh + D\cosh)$$

Now solve for the special cases :

(i) 

$z = 0$

$z = L$

$Y = 0 \therefore \quad A\cos + B\sin - A\cosh - B\sinh = 0$

$Y' = 0 \therefore \quad -A\sin + B\cos - A\sinh - B\cosh = 0$

$A = -C$

$B = -D$

(as for (i))

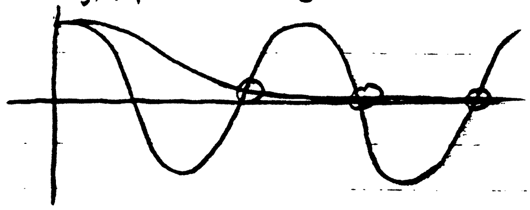
$$\therefore \begin{vmatrix} \cos - \cosh & \sin - \sinh \\ -(\sin + \sinh) & \cos - \cosh \end{vmatrix} = 0$$

$$\therefore (\cos - \cosh)^2 + (\sin + \sinh)(\sin - \sinh) = 0$$

$$\therefore \cos^2 + \sin^2 - 2\cos\cosh + \cosh^2 - \sinh^2 = 0$$

$$\therefore \cos\cosh = 1$$

$$\therefore \boxed{\cos\alpha L = \operatorname{sech}\alpha L}$$

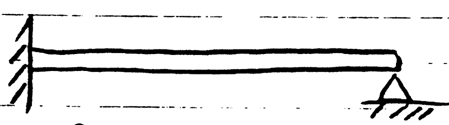


Approx solns $\alpha L = \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2} \ldots$

Exact

$\alpha L = 4.730,\ 7.854,\ 10.996 \ldots$

which gives natural frequencies as before

(ii) 

$A = -C$

$B = -D$

(as for (i))

$z = L$

$Y = 0 \therefore \quad A\cos + B\sin - A\cosh - B\sinh = 0$

$Y'' = 0 \therefore \quad -A\cos - B\sin - A\cosh - B\sinh = 0$

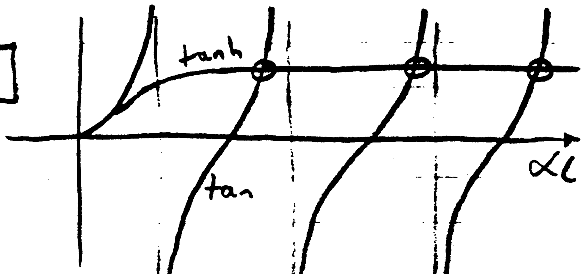
$$\therefore \begin{vmatrix} \cos - \cosh & \sin - \sinh \\ \cos + \cosh & \sin + \sinh \end{vmatrix} = 0$$

$$\therefore \cos\sin - \cosh\sin + \cos\sinh - \cosh\sinh$$
$$-(\cos\sin + \cosh\sin - \cos\sinh - \cosh\sinh) = 0$$

5 contd:

$\therefore \quad \cosh \sin = \cos \sinh$

$\therefore \quad \boxed{\tan \alpha L = \tanh \alpha L}$



Approx:
$\alpha L = \frac{5\pi}{4}, \quad \frac{9\pi}{4}, \quad \frac{13\pi}{4}$

Exact
$\alpha L = 3.927, \quad 7.069, \quad 10.210$

and frequencies from $\quad \omega = (\alpha L)^2 \sqrt{\frac{EI}{m L^4}}$

(iii)

$z = 0$

$y = 0 \quad y'' = 0$

$\left.\begin{aligned} \therefore\ & A = -C \\ \&\ & A = C \end{aligned}\right\} \therefore A = C = 0$

$z = L$

$y'' = 0 \ \therefore \ -B \sin + D \sinh = 0$

$y''' = 0 \ \therefore \ -B \cos + D \cosh = 0$

$\therefore \begin{vmatrix} \sin & -\sinh \\ \cos & -\cosh \end{vmatrix} = 0$

$\therefore \quad \sin \cosh = \cos \sinh$

$\therefore \quad \boxed{\tan \alpha L = \tanh \alpha L} \quad$ as in (iv)

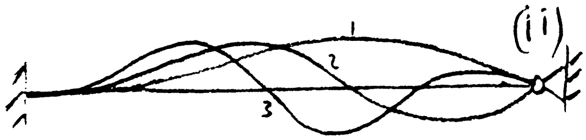
Note: in case (iii) a zero natural frequency is valid and it describes rigid body motions. This should be deduced by "common sense". Good students might check that the maths works this way too if α is not factorized in the early stages.

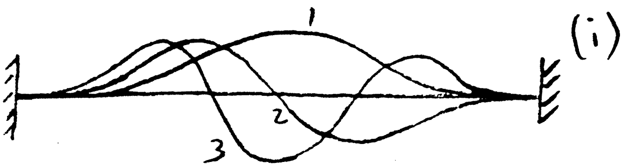
The free–free problem satisfies the DE, and $y'' = y''' = 0$ at both ends. Define now $w = \frac{\partial^2 y}{\partial z^2}$.

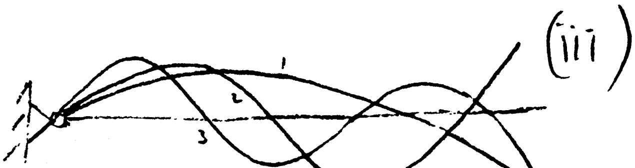
This satisfies the same DE (just take $\frac{\partial^2}{\partial z^2}(DE)$),

but satisfies boundary conditions $w = w' = 0$ so it is the solution of the clamped–clamped problem. So these have the same natural frequencies, and the clamped modes are $\frac{d^2}{dz^2}$(free modes).

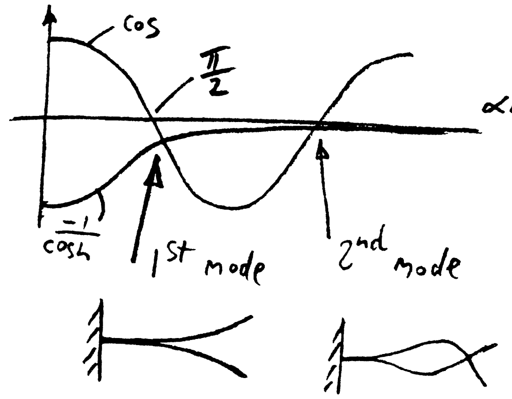
Mode shapes:







6. (i) $\omega_j = (\alpha L)_j^2 \sqrt{\dfrac{EI}{mL^4}}$ where $\cos(\alpha L)\cosh(\alpha L) = -1$

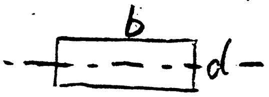


For 1st mode, the solution for αL is approximately $\frac{\pi}{2}$ but more accurately solved by iteration to give

$\alpha L = 1.875$

and for a rectangular section

$I = \dfrac{bd^3}{12}$



and $m = \rho b d$

so $f_1 = \dfrac{1}{2\pi}\omega_1 = \dfrac{1}{2\pi}(1.875)^2\sqrt{\dfrac{E\,bd^3}{12\rho bd\,L^4}} = \underline{\underline{0.162\,\dfrac{dc}{L^2}}}$

where $c = \sqrt{\dfrac{E}{\rho}}$

(ii) For all steels, $E \simeq 200$ GPa $\rho = 7800$ kg/m^3

$\therefore c = \sqrt{\dfrac{200\times10^9}{7800}} = 5064$ m/s which is at the top end of the range

we have $f_1 = 230$ Hz and $L = 0.03$ m

$\therefore d = \dfrac{230 \times .03^2}{.162 \times 5064} = 0.25$ mm

(iii) Each semitone corresponds to a frequency ratio of $\sqrt[12]{2}$ (since 12 semitones is an octave)

$= 1.0595$

and $\dfrac{230}{220} = 1.045$ which is just less than one semitone so the tweezers sound a slightly flat B♭.

7(a)

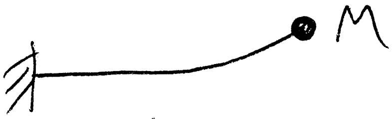
From structures data book, equivalent stiffness at centre of beam is $k = \frac{48EI}{L^3}$

So $\omega^2 = \frac{k}{M} = \frac{48EI}{ML^3}$

From lecture notes, answer for continuous beam is $\omega^2 = \frac{EI}{m}\left(\frac{\pi}{L}\right)^4$

So to get same ω, need $M = \frac{48}{\pi^4} mL = 0.493\,mL$

(b)



Stiffness here is $k = \frac{3EI}{L^3}$

So $\omega^2 = \frac{k}{M} = \frac{3EI}{ML^3}$

Continuous beam has $\omega^2 = \frac{EI}{m}\alpha^4$ with $\alpha = \frac{1.875}{L}$

So to get the same frequency, need

$$M = \frac{3}{(1.875)^4} mL = 0.243\,mL$$

8. (a) For simply support beam, mode shapes are

$$u_j(x) = \sin \frac{j\pi x}{L}$$

So want $\displaystyle\int_0^L \rho A \sin \frac{j\pi x}{L} \sin \frac{k\pi x}{L}\, dx$

$$= \frac{\rho A}{2} \int_0^L \left[\cos \frac{(j-k)\pi x}{L} - \cos \frac{(j+k)\pi x}{L} \right] dx$$

$$= \frac{\rho A}{2} \left[\frac{L}{\pi(j-k)} \sin \frac{(j-k)\pi x}{L} - \frac{L}{\pi(j+k)} \sin \frac{(j+k)\pi x}{L} \right]_0^L$$

$= 0$ if $j \neq k$.

(Fourier series orthogonality: see IA Maths notes)

(b) Equations are

$$\begin{cases} EI \dfrac{d^4 u_j}{dx^4} = m\omega_j^2 u_j \\ EI \dfrac{d^4 u_k}{dx^4} = m\omega_k^2 u_k \end{cases}$$

So

$$\begin{cases} EI \displaystyle\int \frac{d^4 u_j}{dx^4} u_k\, dx = m\omega_j^2 \int u_j u_k\, dx \\ EI \displaystyle\int \frac{d^4 u_k}{dx^4} u_j\, dx = m\omega_k^2 \int u_j u_k\, dx \end{cases}$$

Integrate by parts twice:

$$EI \left\{ \left[\frac{d^3 u_j}{dx^4} u_k \right]_0^L - \left[\frac{d^2 u_j}{dx^2} \frac{du_k}{dx} \right]_0^L + \int_0^L \frac{d^2 u_j}{dx^4} \frac{d^2 u_k}{dx^4}\, dx \right\}$$

$$= m\omega_j^2 \int u_j u_k\, dx$$

plus the corresponding result with $j \leftrightarrow k$.

Subtract:

$$m(\omega_j^2 - \omega_k^2) \int u_j u_k\, dx = EI \left\{ [u_j''' u_k] - [u_j'' u_k'] - [u_k''' u_j] + [u_k'' u_j'] \right.$$

8 cont.

Boundary conditions:

Clamped	$u = 0$	$u' = 0$
Pinned	$u = 0$	$u'' = 0$
Free	$u''' = 0$	$u'' = 0$

So always, <u>either</u> $u = 0$ <u>or</u> $u''' = 0$
and also <u>either</u> $u' = 0$ <u>or</u> $u'' = 0$

So for any combination of these boundary conditions, all four $\left[\sim \right]_0^L$ terms vanish, and so $\int_0^L u_j u_k \, dx = 0$ as required.