Continuous State-Space Systems

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Stochastic Processes: Handout 2

IIA Module 3M1: Mathematical Methods

Continuous State Space Systems

- Previous lectures have discussed finite-state models
  - finite number of (discrete) states
  - discrete time intervals, transition matrix governs changes over time
- This lecture will extend this form to
  - continuous-time, discrete processes
  - continuous-time, continuous-space processes
- Interested in answering similar questions as the discrete state case
  - how do distributions change over time?
  - does the system reach equilibrium?
  - how sensitive is the final state to the initial state?

Birth Process (Yule-Furry Process)

- Consider the following set-up:
  - "birth"-rate for a cell is λ per unit time
  - $n(t)$ is the number of cells at time instance t
  - initially have $n(0) = n_0$ "cells"

- The process is now continuous in time
  - still Markovian - $n(t)$ describes state of the system
  - interested what happens to $n(t)$

$$n(t + \Delta t) = n(t) + n(t)\lambda\Delta t; \quad \text{as } \Delta t \to 0 \quad \frac{dn(t)}{dt} = \lambda n(t)$$

  - standard solution

$$n(t) = n_0 \exp(\lambda t)$$

  - but the numbers of cells needs to be integer ... this is expected value

Birth Process - Probabilistic Approach

- Let's link back to the Markov Process
  - require probability of n cells at time t $P(N(t)=n)=P_n(t)$
  - how does the distribution evolve over time?

- Again consider time slot $t \rightarrow t+\Delta t$ (assumed very small)

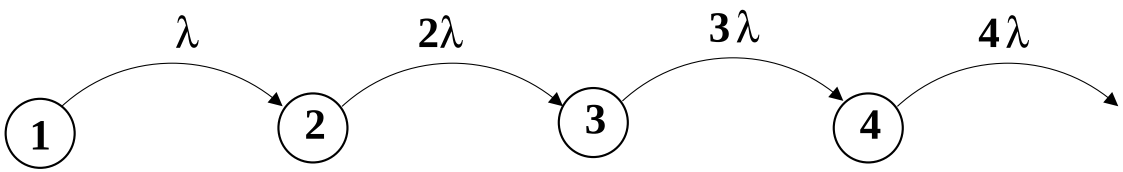
$$P_n(t+\Delta t)=P_n(t)(1-n\lambda\Delta t)+P_{n-1}(t)((n-1)\lambda\Delta t)$$

  - stay in the same state: no birth
  - birth occurs: move from previous state
  - ignored multiple events - often written $o(\Delta t)$, as $\Delta t \rightarrow 0, o(\Delta t) \rightarrow 0$

- Take the limiting condition $\Delta t \rightarrow 0$

$$\frac{dP_n(t)}{dt}=-n\lambda P_n(t)+(n-1)\lambda P_{n-1}(t)$$

Birth Process Chain



- For discrete time we had the transition matrix
  - for continuous time there's the transition rate matrix, $\mathbf{Q}$

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{x}(t)\mathbf{Q}$$

where (note rows sum to zero - probability mass conserved)

$$\mathbf{Q} = \begin{bmatrix} -\lambda & \lambda & 0 & 0 & \cdots \\ 0 & -2\lambda & 2\lambda & 0 & \cdots \\ 0 & 0 & -3\lambda & 3\lambda & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix}$$

Birth Process Solution (reference)

- Now need to solve the problem (no need to derive/remember this!)
  - assuming that at $t = 0$ there are n_0 cells

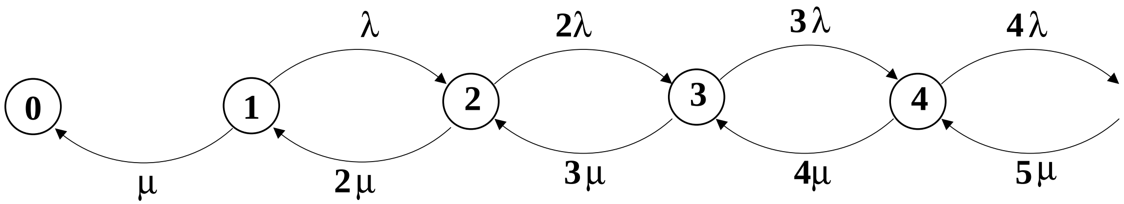
$$P_n(t) = \left(\begin{array}{c} n-1 \\ n-n_0 \end{array} \right) \exp(-\lambda n_0 t)(1-\exp(-\lambda t))^{n-n_0}$$

where

$$\left(\begin{array}{c} n \\ k \end{array} \right) = \frac{n!}{k!(n-k)!}$$

- No steady-state distribution - keeps growing ...

Birth-Death Processes



- Introduce death-rate for a cell μ per unit time
  - repeating the probabilistic birth rate analysis (for $n > 1$)

$$\begin{aligned} P_n(t+\Delta t) &= P_n(t)(1 - n\lambda\Delta t - n\mu\Delta t) + P_{n-1}(t)((n-1)\lambda\Delta t) \\ &\quad + P_{n+1}(t)((n+1)\mu\Delta t) \end{aligned}$$

  - stay in the same state: no birth/death
  - birth occurs: move from previous state
  - death occurs: move from next state

- See examples paper for attributes

Applications

- Range of extensions
  - fixed (time/state independent) birth rate Poisson Process
  - use more general rate transition matrix (currently $n\lambda$)
- This form of continuous-time process has a range of applications
  - Yule studied this for evolution (mutations)
  - Furry used the model for radioactive transmutations
  - populations of bacteria
  - queueing systems
  - etc etc
- So far only considered discrete state-space models ...

Random Walk

- Consider a random walk - at each time take step ± 1
  - probability of direction at time k, $\zeta_k \in \{-1, 1\}$, is uniform $P(\zeta_k = 1) = \frac{1}{2}$
  - after n steps the position, X_n is given by

$$X_n = \sum_{k=1}^{n} \zeta_k$$

- What is the distribution of X_N as $N \rightarrow \infty$?
  - at each instance the average step is zero, the step variance is 1
  - from the central limit theorem the distribution is Gaussian, $\mathcal{N}(0, N)$

- Now taking small step δ in direction ζ_k every δ seconds
  - at time instance $t = N\delta$ where N is very large
  - from above location, W_t, is Gaussian distributed, $\mathcal{N}(0, N\delta) = \mathcal{N}(0, t)$
  - in the limit this is Brownian motion

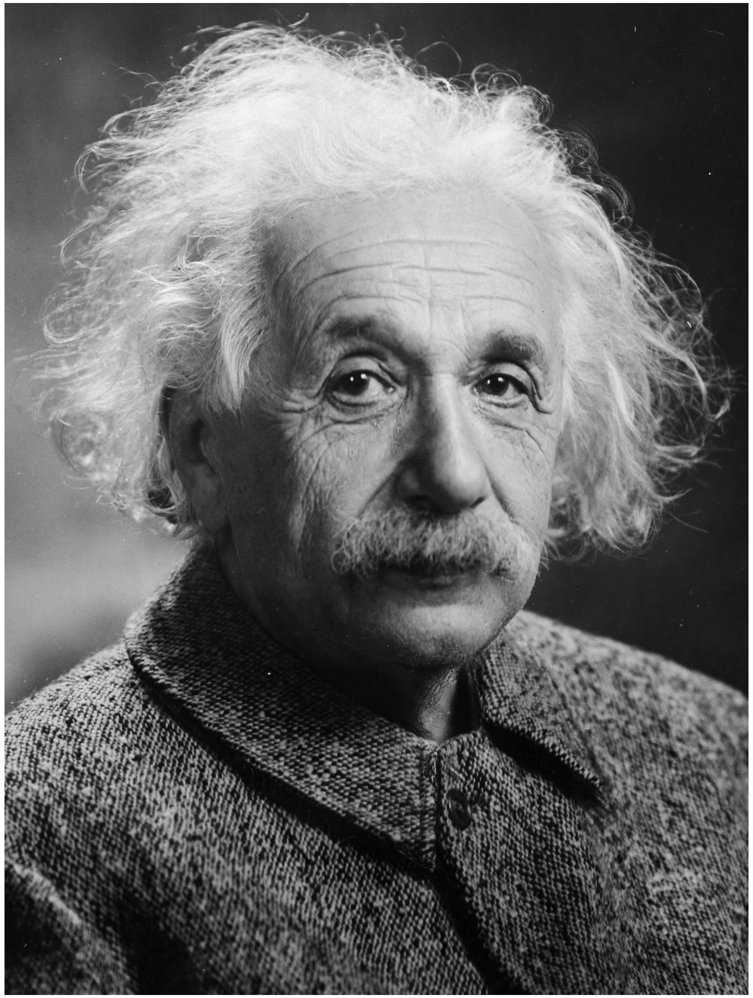
Brownian Motion

- Consider Brownian Motion
  - random motion of particles in a fluid resulting from collisions
- If we know the physics behind the interaction of particles, can model collisions
  - but there can be, for example, 10^{21} collisions per second!
- If we don't care about an individual particle, what can we do

consider the number of particles per unit volume

- Start by simplifying the system
  - consider each dimension independently - just consider x dimension
  - overall form obtained by simply multiplying dimension together
- Brownian motion is also called a Wiener process in stochastic processes

Albert Einstein



"Albert Einstein Head" Photograph by Oren Jack Turner, Princeton

Particle Density

- Let the particle density at time t and position x be

$$f(x,t)$$

  - clearly this function will vary with position x
  - and time t

- It is possible to define the state of the system
  - probability of next state only depends on current state
  - only need current state at time t, no need to consider previous states
  - it's a Markov Process

Particle Density - Taylor Series Expansion

- First consider a Taylor series at time instance t with position x
  - consider a small shift in the position x to $x + \Delta$

$$f(x+\Delta,t) \approx f(x,t) + \Delta\frac{\partial f(x,t)}{\partial x} + \frac{\Delta^2}{2!}\frac{\partial^2 f(x,t)}{\partial x^2} + \mathcal{O}(\Delta^3)$$

  - smoothness assumption in the particle density equation

- What we care about is the evolution over time of the particle density function
  - consider a (very) small change in time from t to $t + \tau$

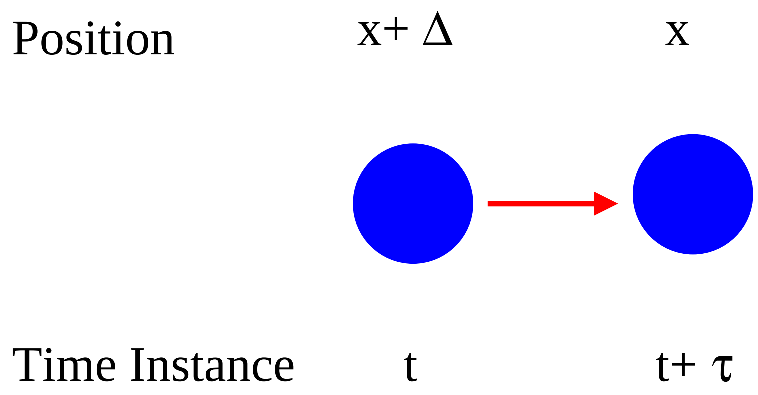
$$f(x,t+\tau) \approx f(x,t) + \tau\frac{\partial f(x,t)}{\partial t} + \mathcal{O}(\tau^2)$$

  - but system too complicated to get this derivative

Movement of One Particles

- If we know the status of the system at time t and position x
  - change in the density at time $t+\tau$ results from particles moving to position x from time t to $t+\tau$

- How to characterise this movement:



  - if a particle is at position $x+\Delta$ at time t
  - needs to be at position x at time $t+\tau$

- Need to the probability of a particle moving distance $-\Delta$ in time τ

Expected Movement of Particles

- Given the system at time instance t ($f(x,t)$) we can write

$$f(x,t+\tau) = \int_{-\infty}^{\infty} f(x+\Delta,t)p(-\Delta)d\Delta$$

  - where $p(\Delta)$ is the probability of a particle moving Δ in time τ
  - continuous form of Chapman-Kolmogorov

- Assume that $p(\Delta)$ is symmetric and combining with the Taylor Series expansion

$$\begin{aligned} f(x,t+\tau) &= \int_{-\infty}^{\infty} f(x+\Delta,t)p(\Delta)d\Delta \\ &\approx \int_{-\infty}^{\infty} \left(f(x,t) + \Delta\frac{\partial f(x,t)}{\partial x} + \frac{\Delta^2}{2!}\frac{\partial^2 f(x,t)}{\partial x^2} \right) p(\Delta)d\Delta \end{aligned}$$

Expected Movement of Particles (cont)

- Expanding out (and exploiting symmetry of $p(\Delta)$)

$$
\begin{aligned}
f(x,t+\tau) &\approx f(x,t)\int_{-\infty}^{\infty} p(\Delta)d\Delta + \frac{\partial f(x,t)}{\partial x}\int_{-\infty}^{\infty} \Delta p(\Delta)d\Delta \\
&\quad + \frac{\partial^2 f(x,t)}{\partial x^2}\int_{-\infty}^{\infty} \frac{\Delta^2}{2!}p(\Delta)d\Delta \\
&= f(x,t) + \frac{\partial^2 f(x,t)}{\partial x^2}\int_{-\infty}^{\infty} \frac{\Delta^2}{2!}p(\Delta)d\Delta
\end{aligned}
$$

  - exploited definition of a PDF (equates to one)
  - exploited symmetry of the PDF (equates to zero)

- Note: symmetry will mean that all odd higher terms integrate to zero

Brownian Motion - Differential Equation

- Equating the two expressions

$$\begin{aligned} f(x,t+\tau) &\approx f(x,t) + \tau\frac{\partial f(x,t)}{\partial t} \\ &\approx f(x,t) + \frac{\partial^2 f(x,t)}{\partial x^2}\int_{-\infty}^{\infty}\frac{\Delta^2}{2!}p(\Delta)d\Delta \end{aligned}$$

  - exploiting the fact that both τ and Δ are small values yields

$$\tau\frac{\partial f(x,t)}{\partial t} = \frac{\partial^2 f(x,t)}{\partial x^2}\int_{-\infty}^{\infty}\frac{\Delta^2}{2!}p(\Delta)d\Delta$$

- Rearranging yields an example of Fokker-Planck equation

$$\frac{\partial f(x,t)}{\partial t} = D\frac{\partial^2 f(x,t)}{\partial x^2}, \quad D = \frac{1}{2\tau}\int_{-\infty}^{\infty}\Delta^2 p(\Delta)d\Delta$$

Brownian Motion

- Brownian motion governed by simple diffusion equation

$$\frac{\partial f(x,t)}{\partial t} = D\frac{\partial^2 f(x,t)}{\partial x^2}, \quad D = \frac{1}{2\tau}\int_{-\infty}^{\infty} \Delta^2 p(\Delta) d\Delta$$

  - for example the same equation governs heat diffusion
  - at the heart of the analysis - Brownian Motion is a Markov Process

- Need to obtain value of D
  - D can be obtained by measurement of physical properties
  - no need to consider τ or $p(\Delta)$!

- Final solution will depend on initial conditions
  - let's look at a particular solution for Brownian motion

Brownian Motion - Example Solution

- Would like to get solutions for the Brownian motion differential equation

$$\frac{\partial p(x,t)}{\partial t} = \alpha \frac{\partial^2 p(x,t)}{\partial x^2}$$

  - α constant with time and position
  - take initial condition as $p(x,0) = \delta(x)$ (everything at the origin)

- Standard differential equations solutions - solving (2nd year maths)

$$p(x,t) = X(x)T(t), \quad p(x,t) = A(k)\exp(-\alpha k^2 t)\exp(ikx)$$

  - this is satisfied by any k, so general solution

$$p(x,t) = \int_{-\infty}^{\infty} A(k)\exp(-\alpha k^2 t)\exp(ikx)dk$$

Brownian Motion - Example Solution

- Need to satisfy the initial condition at $t = 0$, hence

$$\delta(x) = p(x,0) = \int_{-\infty}^{\infty} A(k)\exp(ikx)dk = \int_{-\infty}^{\infty} A(-\tilde{k})\exp(-i\tilde{k}x)d\tilde{k}$$

  - by noting that this is the Fourier Transform, $\mathcal{F}(.)$, of $A(-\tilde{k})$

$$A(-\tilde{k}) = \mathcal{F}^{-1}\left\{\delta(x)\right\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(x)\exp(i\tilde{k}x)dx = \frac{1}{2\pi}$$

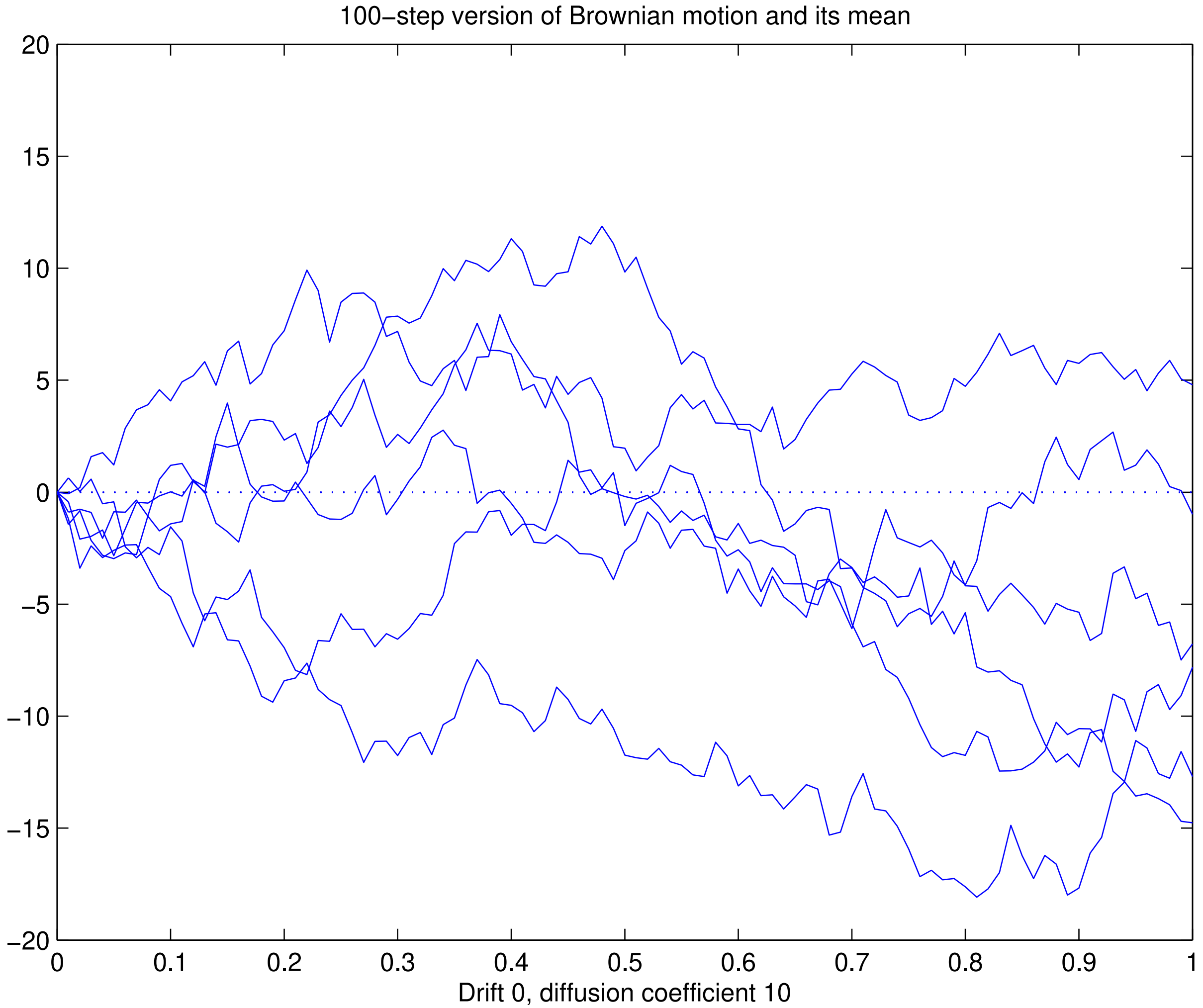
- Thus the final solution is

$$p(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(-\alpha k^2 t)\exp(ikx)dk$$

  - this can be simplified to a Gaussian (zero mean, $\sigma^2 = 2\alpha t$)

$$p(x,t) = \frac{1}{\sqrt{4\alpha\pi t}}\exp\left(-\frac{x^2}{4\alpha t}\right)$$

Example Trajectories - Brownian Motion



- For these plots the diffusion constant $\alpha = 10$

Properties of a One-Dimensional Wiener Process

- At time t, examine the Wiener Process - W_t
  - can consider an instance of a "path" at time t, $w_t^{(i)}$
  - what are the properties when the path is generated from a Wiener Process

- Properties:
  - Independence: $W_t - W_s$ is independent of $\{W_\tau\}_{\tau \le s}$ for any $0 \le s \le t$
  - Stationarity: the distribution of $W_{t+s} - W_s$ is independent of s
  - Gaussianity: W_t is a Gaussian with

$$\mathcal{E}\left\{W_t\right\} = 0; \quad \mathcal{E}\left\{W_t W_s\right\} = 2\alpha \min(t, s)$$

  - Continuity: W_t is a continuous function with t

- Consider Gaussianity $t \ge s$

$$\mathcal{E}\left\{W_t W_s\right\} = \mathcal{E}\left\{(W_{t-s} + W_s)W_s\right\} = \mathcal{E}\left\{W_s W_s\right\} = 2\alpha s$$

Properties (details)

- Consider the solution derived for Brownian motion

$$p(x,t) = \frac{1}{\sqrt{4\alpha\pi t}}\exp\left(-\frac{x^2}{4\alpha t}\right) = \mathcal{N}(x;0,2\alpha t)$$

- Examine the Gaussianity property
  - mean at time t of Wiener process

$$\mathcal{E}\left\{W_t\right\} = \int xp(x,t)dx = 0$$

  - second element position y at time t, x at time s, $t \geq s$

$$\begin{aligned}\mathcal{E}\left\{W_tW_s\right\} &= \int yxp(y-x,t-s)p(x,s)dydx \\ &= \int x\left(\int yp(y-x,t-s)dy\right)p(x,s)dx = \int x^2p(x,s)dx = 2\alpha s\end{aligned}$$

General Continuous State-Space Systems

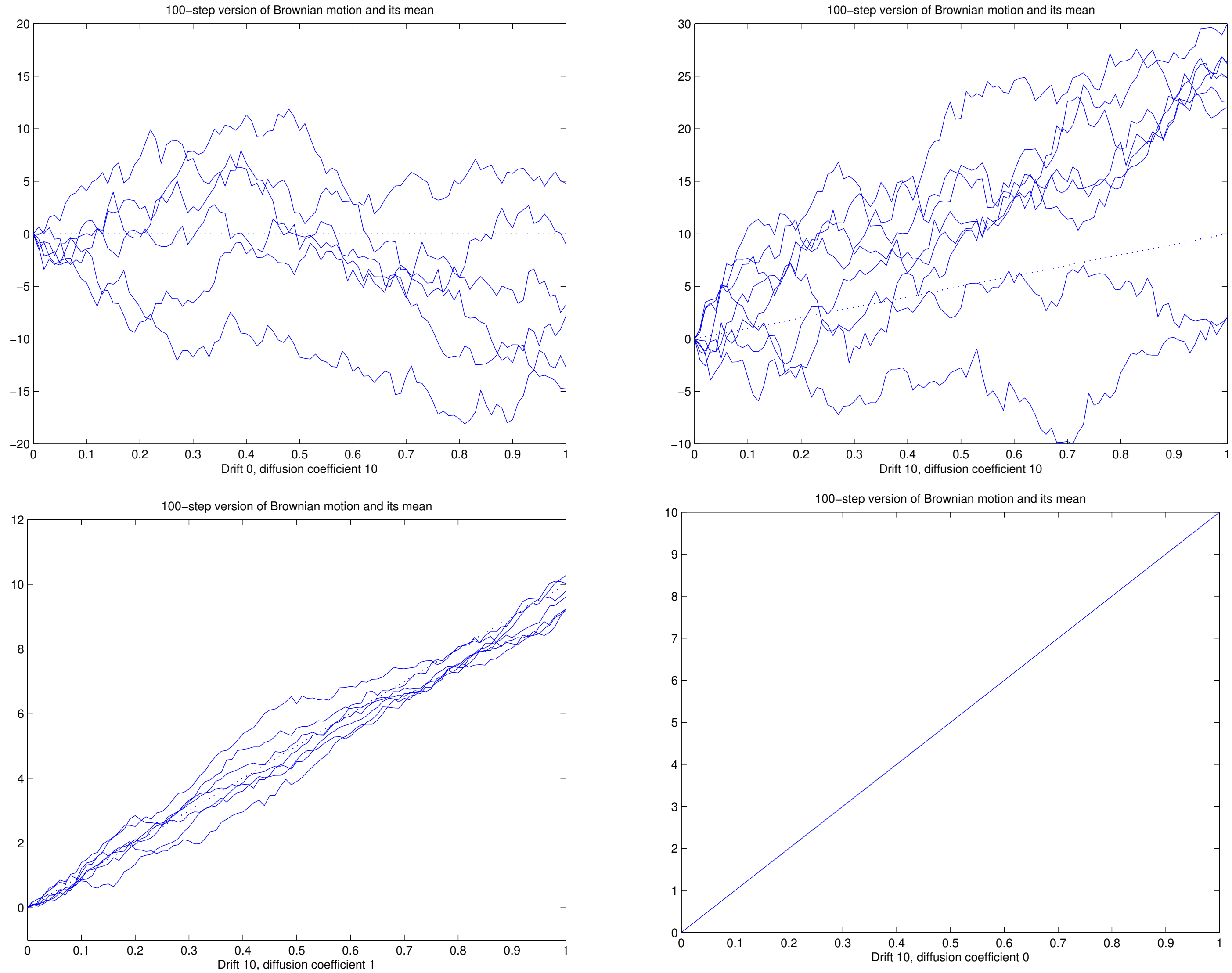
- A range of assumptions have been made in the previous derivation
  - treated the system as discrete (finite small changes in τ) in time
  - distribution $p(\Delta)$ will depend on τ - is there sensitivity to the choice of τ
  - finite Taylor Series expansions were used - but integrated over ∞ range!
  - smoothness assumption of the particle density function - is this really valid
  - assumption that $p(\Delta)$ does not change with position, x

- Generalising all of these is beyond the scope of this course (1 lecture!)
  - consider multiple dimensions
  - continuity of motion (jumps in the process)

- Simple extension add a drift term - Wiener Process with Drift

$$\frac{\partial p(x,t)}{\partial t} = m\frac{\partial p(x,t)}{\partial x} + D\frac{\partial^2 p(x,t)}{\partial x^2}$$

Example Trajectories with (constant) Drift/Diffusion



Ornstein-Uhlenbeck Process

- This has the form (for a single dimension)

$$\frac{\partial}{\partial t}p(x,t) = \frac{\partial}{\partial x}\left(\beta x p(x,t)\right) + \frac{\partial^2}{\partial x^2}\left(B p(x,t)\right)$$

  – often written without variables in the function - looks simpler!!

$$\frac{\partial}{\partial t}p = \frac{\partial}{\partial x}\left(\beta x p\right) + \frac{\partial^2}{\partial x^2}\left(B p\right)$$

  – βx controls the drift (βx)
  – B controls the diffusion (B)

- Property of this process examined in the examples paper

Summary

- Extended finite discrete Markov Chains to:
  - continuous in time
  - continuous in space
- Yields differential equation - general form has
  - drift for the particle paths
  - diffusion for the particle path
  - jumps (discontinuities)
- Some standard processes are:
  - Birth-Death process - discrete states, continuous time
  - Poisson process - discrete states, continuous time
  - Wiener process - continuous state/time
  - Ornstein-Uhlenbeck process - continuous state/time

Example Solution (further details)

- Need to simplify the equation (slide 18)

$$p(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(-\alpha k^2 t)\exp(ikx)dk = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left(-\alpha k^2 t + ikx\right) dk$$

  - Consider the term in the exponential and re-express

$$-\alpha k^2 t + ikx = -\left(\sqrt{\alpha t}k - \frac{ix}{2\sqrt{\alpha t}}\right)^2 - \frac{x^2}{4\alpha t}$$

- This now looks like a Gaussian (integrated over k) - thus

$$\frac{1}{2\pi}\exp\left(-\frac{x^2}{4\alpha t}\right)\int_{-\infty}^{\infty} \exp\left(-\left(\sqrt{\alpha t}k - \frac{ix}{2\sqrt{\alpha t}}\right)^2\right) dk = \frac{1}{2\pi}\exp\left(-\frac{x^2}{4\alpha t}\right)\sqrt{\frac{\pi}{\alpha t}}$$

- This is a Gaussian distribution

$$p(x,t) = \frac{1}{\sqrt{4\pi\alpha t}}\exp\left(-\frac{x^2}{4\alpha t}\right) = \mathcal{N}(x; 0, 2\alpha t)$$