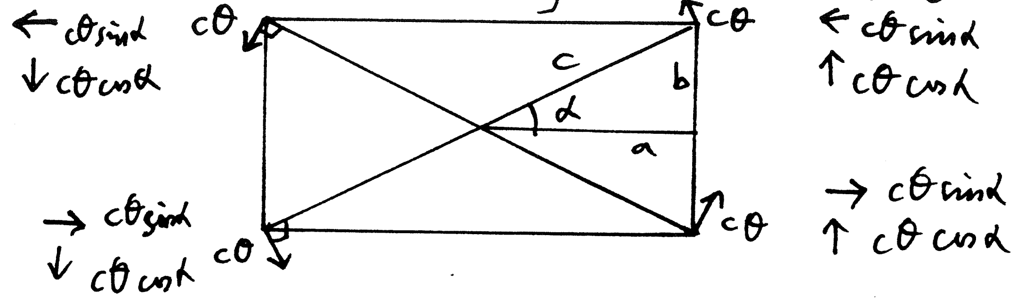
Part IIA Module 3C6 Ex sheet 1 Solutions

1 (a) Motion induced at corners by small rotation θ:



where $c^2 = a^2 + b^2$, $\tan\alpha = b/a$.

But of course $c\sin\alpha = b$, $c\cos\alpha = a$.

$$\begin{aligned}
\text{So } V = {} & \tfrac{1}{2}k(b\theta - x)^2 + \tfrac{1}{2}k(a\theta - y)^2 && \text{(top LH)}\\
& + \tfrac{1}{2}k(b\theta + x)^2 + \tfrac{1}{2}k(a\theta - y)^2 && \text{(bottom LH)}\\
& + \tfrac{1}{2}k(b\theta + x)^2 + \tfrac{1}{2}k(a\theta + y)^2 && \text{(bottom RH)}\\
& + \tfrac{1}{2}k(b\theta - x)^2 + \tfrac{1}{2}k(a\theta + y)^2 && \text{(top RH)}\\
= {} & 2k(b^2\theta^2 + x^2) + 2k(a^2\theta^2 + y^2)
\end{aligned}$$

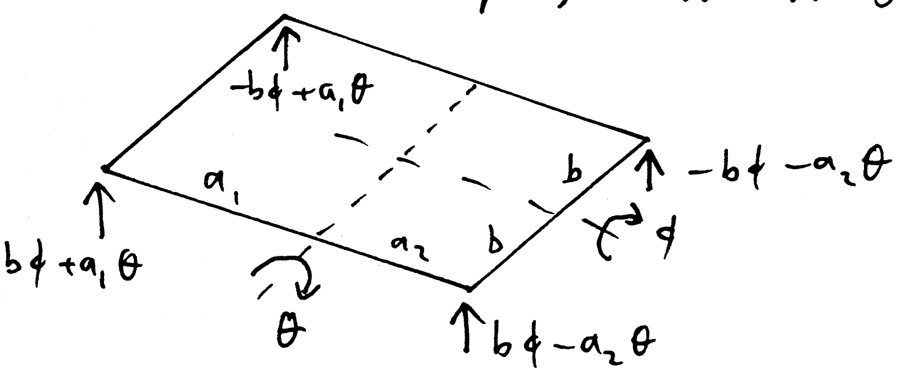
So $K = 4k\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a^2+b^2 \end{bmatrix}$ for vector $\begin{bmatrix} x \\ y \\ \theta \end{bmatrix}$

$T = \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}m\dot{y}^2 + \tfrac{1}{2}I\dot{\theta}^2$

so $M = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & I \end{bmatrix}$, where $I = \frac{1}{3}m(a^2+b^2)$

(b) Vertical motion at wheels for small rotations θ, ϕ:

(car)



1 contd

So $V = \frac{1}{2}k(z+b\phi+a_1\theta)^2 + \frac{1}{2}k(z+b\phi-a_2\theta)^2 + \frac{1}{2}k(z-b\phi-a_2\theta)^2 + \frac{1}{2}k(z-b\phi+a_1\theta)^2$

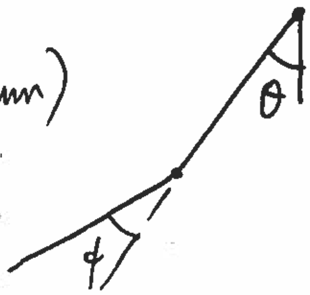
$= 2kz^2 + 2kb^2\phi^2 + k\theta^2(a_1^2+a_2^2) + 2kz\theta(a_1-a_2)$

So $K = 2k\begin{bmatrix} 2 & a_1-a_2 & 0 \\ a_1-a_2 & a_1^2+a_2^2 & 0 \\ 0 & 0 & 2b^2 \end{bmatrix}$ for vector $\begin{bmatrix} z \\ \theta \\ \phi \end{bmatrix}$

$T = \frac{1}{2}m\dot{z}^2 + \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}J\dot{\phi}^2$

So $M = \begin{bmatrix} m & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & J \end{bmatrix}$.

1 (c)
(Pendulum)



$V = mga(1-\cos\theta)$ (top rod)
$+mg[2a(1-\cos\theta)+a(1-\cos(\theta+\phi))]$ (bottom rod)

$\simeq mga\frac{\theta^2}{2} + mga\left[\theta^2 + \frac{(\theta+\phi)^2}{2}\right]$

So $K = mga\begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix}$ for vector $\begin{bmatrix} \theta \\ \phi \end{bmatrix}$

For small θ, ϕ, motion at the centre of mass of each rod is predominantly horizontal, with magnitudes
$\begin{cases} \leftarrow a\dot{\theta} & \text{(top rod)} \\ \leftarrow 2a\dot{\theta} + a(\dot{\theta}+\dot{\phi}) & \text{(bottom rod)} \end{cases}$

So $T = \frac{1}{2}m(a\dot{\theta})^2 + \frac{1}{2}\cdot\frac{1}{3}ma^2\dot{\theta}^2 + \frac{1}{2}m[3a\dot{\theta}+a\dot{\phi}]^2 + \frac{1}{2}\cdot\frac{1}{3}ma^2(\dot{\phi}+\dot{\theta})^2$

So $M = \frac{ma^2}{3}\begin{bmatrix} 32 & 10 \\ 10 & 4 \end{bmatrix}$

1 (d) discs $V = \frac{1}{2}k(\theta-\phi)^2 + \frac{1}{2}k(\phi-\psi)^2$

So $K = k\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$ for vector $\begin{bmatrix} \theta \\ \phi \\ \psi \end{bmatrix}$

$$T = \tfrac{1}{2}I\dot\theta^2 + \tfrac{1}{2}J\dot\phi^2 + \tfrac{1}{2}I\dot\psi^2$$

So $M = \begin{bmatrix} I & 0 & 0 \\ 0 & J & 0 \\ 0 & 0 & I \end{bmatrix}$

In (b), gravity is just a DC force: it shifts the equilibrium position, but it contributes no oscillatory restoring force to the vibration. It can be ignored here. In (c), gravity provides an AC force, and is an essential part of the vibration dynamics so it must be included

2 Obvious modes: (a) has both matrices diagonal, so modes consist of uncoupled motion in x, y and θ respectively.

(b) ϕ is uncoupled from z, θ, so one mode is pure rotation about the back/front centre line.

(d) No decoupled generalised coordinates here, but the system is symmetric, and also unconstrained against rigid rotation!

So (i) rigid body mode $[1 \ 1 \ 1]^t$ at natural frequency 0

(ii) symmetric mode of the form $[\alpha \ \beta \ \alpha]^t$

(iii) antisymmetric mode of the form $[-\gamma, 0, \gamma]^t$.

2 contd (c) Can't say anything about this case. Notice that here neither K nor M is diagonal.

So the modes are found as follows:

(a) (i) $\begin{bmatrix}1\\0\\0\end{bmatrix}$ at frequency $2\sqrt{\frac{k}{m}}$

(ii) $\begin{bmatrix}0\\1\\0\end{bmatrix}$ at frequency $2\sqrt{\frac{k}{m}}$

These two occur at the same frequency: called a "degeneracy"

(iii) $\begin{bmatrix}0\\0\\1\end{bmatrix}$ at frequency $\sqrt{\dfrac{4(a^2+b^2)k}{\frac{1}{3}m(a^2+b^2)}} = \sqrt{\dfrac{12k}{m}}$

(b) (i) $\begin{bmatrix}0\\0\\1\end{bmatrix}$ at frequency $\sqrt{\dfrac{4kb^2}{J}}$.

Now solve 2-DOF problem $2k\begin{bmatrix}2 & a_1-a_2\\ a_1-a_2 & a_1^2+a_2^2\end{bmatrix}\begin{bmatrix}z\\ \theta\end{bmatrix} = \omega^2\begin{bmatrix}m & 0\\ 0 & I\end{bmatrix}\begin{bmatrix}z\\ \theta\end{bmatrix}$.

So $$\begin{vmatrix}4k - m\omega^2 & 2k(a_1-a_2)\\ 2k(a_1-a_2) & 2k(a_1^2+a_2^2) - I\omega^2\end{vmatrix} = 0$$

$$\therefore\ 8k^2(a_1^2+a_2^2) - 4kI\omega^2 - 2km(a_1^2+a_2^2)\omega^2 + mI\omega^4 - 4k^2(a_1-a_2)^2 = 0$$

$$\therefore\ mI\left(\frac{\omega^2}{2k}\right)^2 - \left(2I + m(a_1^2+a_2^2)\right)\left(\frac{\omega^2}{2k}\right) + (a_1+a_2)^2 = 0$$

Roots of this give the two natural frequencies. Mode shapes then determined by

$$2z + (a_1-a_2)\theta = \frac{\omega^2}{2k} m z$$

i.e. $$\frac{z}{\theta} = \frac{a_1-a_2}{\frac{\omega^2}{2k}m - 2}$$

2 contd (d) (i) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ at frequency 0

(ii) Try $K \begin{bmatrix} \alpha \\ \beta \\ \alpha \end{bmatrix} = \omega^2 M \begin{bmatrix} \alpha \\ \beta \\ \alpha \end{bmatrix}$

$\rightarrow k \begin{bmatrix} \alpha - \beta \\ 2\beta - 2\alpha \\ \alpha - \beta \end{bmatrix} = \omega^2 \begin{bmatrix} I\alpha \\ J\beta \\ I\alpha \end{bmatrix}$

So require $\begin{cases} k(\alpha-\beta) = \omega^2 I \alpha & (1) \\ 2k(\beta - \alpha) = \omega^2 J \beta \end{cases}$

Divide : $-2 = \frac{\beta}{\alpha}\frac{J}{I}$, so $\beta = -2\alpha \frac{I}{J}$

So from (1), $k(1 + 2I/J) = \omega^2 I$

$\therefore \omega^2 = \frac{k}{IJ}(J + 2I)$

So mode is $\begin{bmatrix} J \\ -2I \\ J \end{bmatrix}$ at frequency $\sqrt{\frac{k}{IJ}(J+2I)}$

(iii) Similarly, try $K \begin{bmatrix} -\gamma \\ 0 \\ \gamma \end{bmatrix} = \omega^2 M \begin{bmatrix} -\gamma \\ 0 \\ \gamma \end{bmatrix}$

$\therefore \begin{bmatrix} -k\gamma \\ 0 \\ k\gamma \end{bmatrix} = \omega^2 \begin{bmatrix} -I\gamma \\ 0 \\ I\gamma \end{bmatrix}$

So mode is $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ at frequency $\sqrt{\frac{k}{I}}$

2 contd (c) Need to solve $\omega^2 \frac{ma^2}{3}\begin{bmatrix}32 & 10\\ 10 & 4\end{bmatrix}\begin{bmatrix}\theta\\ \phi\end{bmatrix} = mga\begin{bmatrix}4 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}\theta\\ \phi\end{bmatrix}$

So $\begin{bmatrix}4 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}\theta\\ \phi\end{bmatrix} = \lambda\begin{bmatrix}32 & 10\\ 10 & 4\end{bmatrix}\begin{bmatrix}\theta\\ \phi\end{bmatrix}, \quad \lambda = \frac{\omega^2 a}{3g}$

So $\begin{vmatrix}4-32\lambda & 1-10\lambda\\ 1-10\lambda & 1-4\lambda\end{vmatrix} = 0$

$\therefore\ (4-32\lambda)(1-4\lambda) - (1-10\lambda)^2 = 0$

$\therefore\ 4 - 48\lambda + 128\lambda^2 - 1 + 20\lambda - 100\lambda^2 = 0$

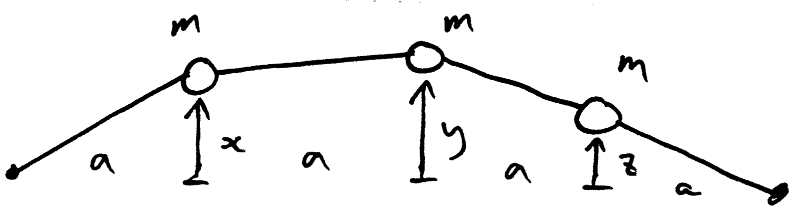
$\therefore\ 28\lambda^2 - 28\lambda + 3 = 0$

Roots give the natural frequencies ω, then the mode shapes satisfy

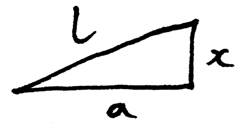
$$4\theta + \phi = \lambda(32\theta + 10\phi)$$

i.e. $$\frac{\theta}{\phi} = \frac{10\lambda - 1}{4 - 32\lambda}.$$

3.



Potential energy of one section of string:



$$L = \sqrt{a^2+x^2} = a\left(1+\frac{x^2}{a^2}\right)^{1/2}$$
$$\simeq a\left(1+\frac{1}{2}\frac{x^2}{a^2}\right) \text{ when } |x| \ll a$$

So increase in length is $\simeq \dfrac{x^2}{2a}$

So work done against tension $T \simeq \dfrac{Tx^2}{2a}$.

So total potential energy is $V = \dfrac{T}{2a}\left\{x^2 + (y-x)^2 + (z-y)^2 + z^2\right\}$

3 contd So $$K = \frac{T}{a}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Kinetic energy is $T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$

So $$M = m\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

System is symmetric, so modes are either symmetric $\begin{bmatrix} x \\ y \\ x \end{bmatrix}$

or antisymmetric $\begin{bmatrix} x \\ 0 \\ -x \end{bmatrix}$.

Try them in turn: (i) $K\begin{bmatrix} x \\ y \\ x \end{bmatrix} = \omega^2 M \begin{bmatrix} x \\ y \\ x \end{bmatrix}$

$$\therefore \frac{T}{a}\begin{bmatrix} 2x - y \\ 2y - 2x \\ 2x - y \end{bmatrix} = \omega^2 m \begin{bmatrix} x \\ y \\ x \end{bmatrix}$$

So require $\begin{cases} 2x - y = \lambda x \\ 2y - 2x = \lambda y \end{cases}$, $\lambda = \dfrac{\omega^2 ma}{T}$

ie $\begin{cases} (2-\lambda)x = y \\ 2x = (2-\lambda)y \end{cases}$

So $2 - \lambda = \dfrac{2}{2-\lambda}$

$\therefore 4 - 4\lambda + \lambda^2 = 2$

$\therefore \lambda^2 - 4\lambda + 2 = 0$

$\therefore \lambda = 2 \pm \sqrt{4-2} = 2 \pm \sqrt{2} = 3.414, 0.586$

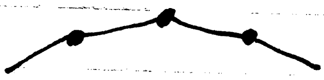
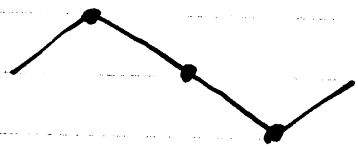
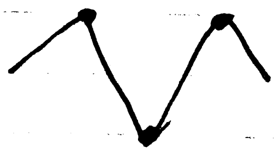
Then mode shapes are $\dfrac{y}{x} = 2 - \lambda = \mp\sqrt{2}$

3 contd (ii) $K\begin{bmatrix} x \\ 0 \\ -x \end{bmatrix} = \omega^2 M \begin{bmatrix} x \\ 0 \\ -x \end{bmatrix}$

$\therefore \frac{T}{a}\begin{bmatrix} 2x \\ 0 \\ -2x \end{bmatrix} = \omega^2 m \begin{bmatrix} x \\ 0 \\ -x \end{bmatrix}$

so need $\frac{2T}{a} = \omega^2 m$, or $\lambda = \frac{\omega^2 m a}{T} = 2$

So modes are:

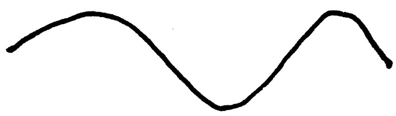




$\lambda = 0.586$ $\quad$ 2 $\quad$ 3.414

Frequency ratios $\quad 1 \quad 1.85 \quad 2.41$

For a continuous string we would have





with ratios $\quad 1 \quad 2 \quad 3$

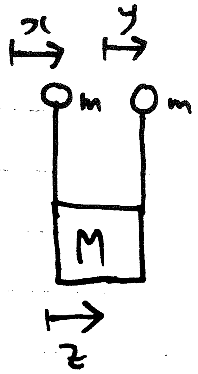
so recognisably close.

Compare absolute frequency of lowest mode: for a string of length $4a$ and mass $3m$, the mass per unit length is $\frac{3m}{4a}$ and the wave speed is $\sqrt{\frac{4aT}{3m}}$

So the fundamental frequency is $\omega = \frac{2\pi}{8a}\sqrt{\frac{4aT}{3m}}$

ie $\lambda = \frac{\omega^2 m a}{T} = \frac{\pi^2}{12} = 0.82$

4.



$$V = \tfrac{1}{2} k (x-z)^2 + \tfrac{1}{2} k (y-z)^2$$

So $K = k \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix}$

$$T = \tfrac{1}{2} m \dot{x}^2 + \tfrac{1}{2} m \dot{y}^2 + \tfrac{1}{2} M \dot{z}^2$$

So $M = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & M \end{bmatrix}$

Modes: (i) Rigid-body mode $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ at $\omega = 0$

(ii) Symmetric mode: try $\begin{bmatrix} a \\ a \\ b \end{bmatrix}$

Then $k \begin{bmatrix} a-b \\ a-b \\ 2b-2a \end{bmatrix} = \omega^2 \begin{bmatrix} ma \\ ma \\ Mb \end{bmatrix}$

So require $\begin{cases} k(a-b) = \omega^2 ma & (1) \\ 2k(b-a) = \omega^2 Mb \end{cases}$

Divide: $-2 = \dfrac{Mb}{ma}$, so $\dfrac{b}{a} = \dfrac{-2m}{M}$

Then from (1), $ka\left[1 + \dfrac{2m}{M}\right] = \omega^2 ma$

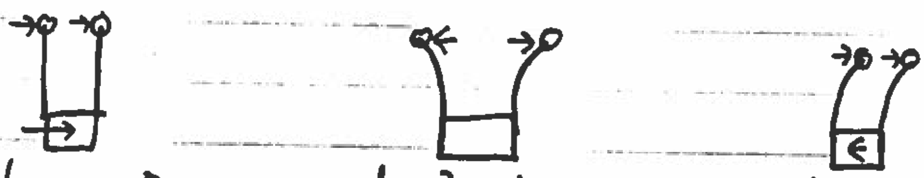
So $\omega^2 = \dfrac{k(2m+M)}{mM}$

(iii) Antisymmetric mode: try $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

Then require $k\begin{bmatrix}1\\-1\\0\end{bmatrix} = \omega^2\begin{bmatrix}m\\-m\\0\end{bmatrix}$

So $\omega^2 = \dfrac{k}{m}$

So modes are:



At $\omega = 0$ — (i)

at $\omega^2 = \dfrac{k}{m}$ — (ii)

at $\omega^2 = \dfrac{k}{m}\,\dfrac{2m+M}{M}$ — (iii)

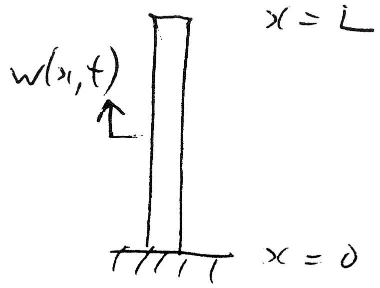
Mode (iii) is effectively damped out when you hold the stalk (mass M) in your fingers. Mode (ii) has no motion there, so it is not damped out and is "the" frequency of the tuning fork.

Note that both modes (ii) + (iii) have zero total linear momentum – this is another way of saying that they are orthogonal to the rigid-body mode (i).

In order to drive the table into vibration and radiate sound, the fork must be applying a vertical force through the stalk. There are two mechanisms for this. The more obvious of the two for the idealised model is that the two masses move along slightly curved arcs, so there must be some net vertical movement of the centre of mass. It is second-order, but non-zero. This results in an oscillating D'Alembert force: but it happens at twice the frequency of the fork, so on its own this effect would produce a sound an octave higher than the direct sound of the fork. The second mechanism is easiest to visualise if you think of a free-free bending beam being "folded up" to make the arms of the fork. As the arms move outwards, there is a bending moment applied to the base of the U-shaped beam, which tends to make it unwind a little, and thus move slightly upwards. That produces a vertical force at the same frequency as the fork vibration. You can hear this effect: a fork placed near the ear produces a very pure sinusoidal tone. Put it on the table, and you hear the same note but with a clearly brighter "tone colour". Your brain is noticing the frequency-doubled component as a harmonic of the fundamental to produce this "bright" effect.

5. Equation, from Data Sheet, is

$$E \frac{\partial^2 w}{\partial x^2} - \rho \frac{\partial^2 w}{\partial t^2} = 0 \quad (1)$$



At $x = 0$ no motion, so $w = 0$

At $x = L$ no stress, $\therefore$ no strain, $\therefore \frac{\partial w}{\partial x} = 0$

For a mode $w(x,t) = u(x) e^{i\omega t}$

$\therefore$ from (1) $\frac{d^2 u}{dx^2} + \frac{\omega^2}{c^2} u = 0$, $c^2 = \frac{E}{\rho}$

$\rightarrow$ general solution

$$u = B_1 \sin \frac{\omega x}{c} + B_2 \cos \frac{\omega x}{c}$$

$u(0) = 0 \quad \rightarrow \quad B_2 = 0$

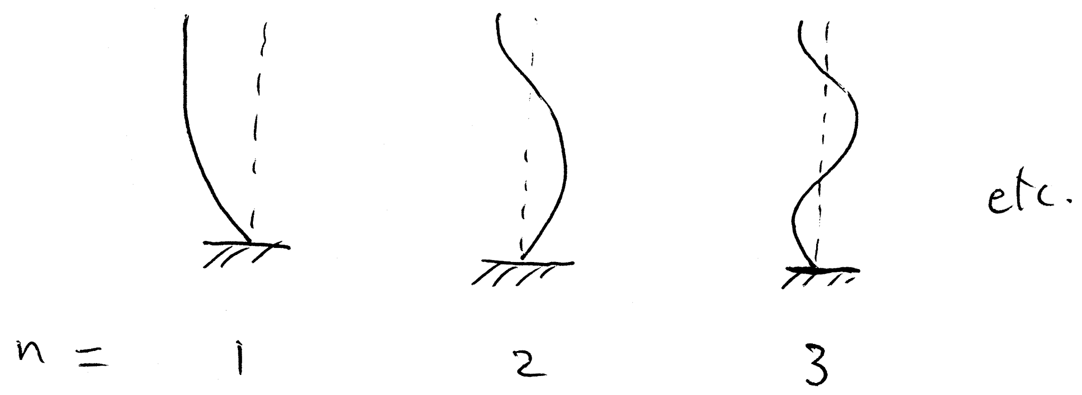
$\frac{du}{dx}(L) = 0 \quad \rightarrow \quad \frac{B_1 \omega}{c} \cos \frac{\omega L}{c} = 0$

$$\therefore \frac{\omega L}{c} = (n - \tfrac{1}{2})\pi \qquad n = 1, 2, 3, \ldots$$

So nth natural frequency $\omega_n = \frac{(n-\frac{1}{2})\pi c}{L}$

Corresponding mode shape $u_n = \sin \frac{(n-\frac{1}{2})\pi x}{L}$

Axial displacement:



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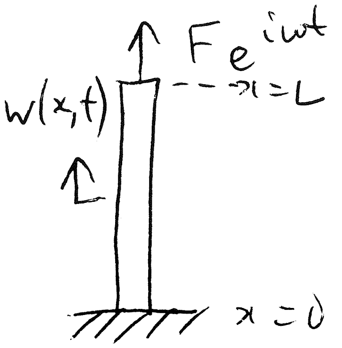

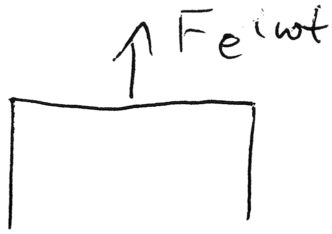

At $x=L$: stress at top $= E \times$ strain $= E\dfrac{\partial w}{\partial x}$

$$\therefore F = EA \frac{\partial w}{\partial x}(L).$$

Now start from general solution for axial motion, derived in Q5:

$$w(x,t) = u(x)\, e^{i\omega t}$$

where $u(x) = B_1 \sin\dfrac{\omega x}{c} + B_2 \cos\dfrac{\omega x}{c}$

$w(0) = 0 \rightarrow B_2 = 0$ as before.

$\dfrac{\partial w}{\partial x}(L) = \dfrac{F}{EA} \rightarrow B_1 \dfrac{\omega}{c} \cos\dfrac{\omega L}{c} = \dfrac{F}{EA}$

$$\therefore B_1 = \frac{Fc}{EA\omega \cos \omega L/c}$$

$\therefore$ solution is

$$u(x) = \frac{Fc}{EA\omega} \frac{\sin \omega x/c}{\cos \omega L/c} \qquad (1)$$

At the top: $u(L) = \dfrac{Fc}{EA\omega} \tan \omega L/c$

so transfer function $H(L,L,\omega) = \dfrac{u(L)}{F} = \dfrac{c}{EA\omega} \tan\dfrac{\omega L}{c}$.

Now $u \rightarrow \infty$ when $\cos \omega L/c = 0$, ie when $\omega = \omega_n$ from Q5.
Only exception, from (1), is if $\sin\dfrac{\omega x}{c} = 0$ also.

This defines the nodal point of the mode $u_n(x)$, as expected. The driving point $x=L$ is always an antinode.

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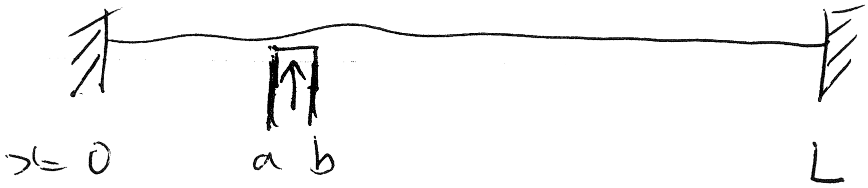

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Method follows section 1.4 of lecture notes.

Try $w(x,t) = \sum_n C_n \sin\frac{n\pi x}{L} e^{in\Omega t}$, $\Omega = \frac{\pi c}{L}$

If $C_n = a_n + ib_n$, $w = \sum (a_n \cos n\Omega t - b_n \sin n\Omega t) \sin\frac{n\pi x}{L}$

At $t = 0$ (i) $w = 0$ so $a_n = 0$ for all n

(ii) $$\frac{\partial w}{\partial t} = \begin{cases} 0 & 0 \le x < a \\ V & a \le x \le b \\ 0 & b < x \le L \end{cases} = f(x) \text{ say}$$

$$\therefore -\sum n\Omega b_n \sin\frac{n\pi x}{L} = f(x)$$

i.e. $-n\Omega b_n$ is the nth Fourier coefficient of $f(x)$.

So from Maths data book,

$$-n\Omega b_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{L}\, dx$$

$$= \frac{2V}{L}\int_a^b \sin\frac{n\pi x}{L}\, dx$$

$$= \frac{2V}{L}\left[-\frac{L}{n\pi}\cos\frac{n\pi x}{L}\right]_a^b$$

$$\therefore b_n = \frac{2V}{n^2\pi\Omega}\left(\cos\frac{n\pi b}{L} - \cos\frac{n\pi a}{L}\right)$$

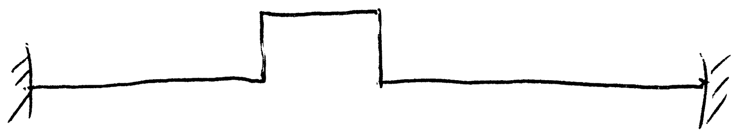
So motion is $w(x,t) = -\sum_n \frac{2V}{n^2\pi\Omega}\left(\cos\frac{n\pi b}{L} - \cos\frac{n\pi a}{L}\right)\sin n\Omega t \sin\frac{n\pi x}{L}$

7 contd.

For a real piano string this will be wrong because:

(1) The hammer will be soft, so that the initial velocity distribution might be more like

rather than

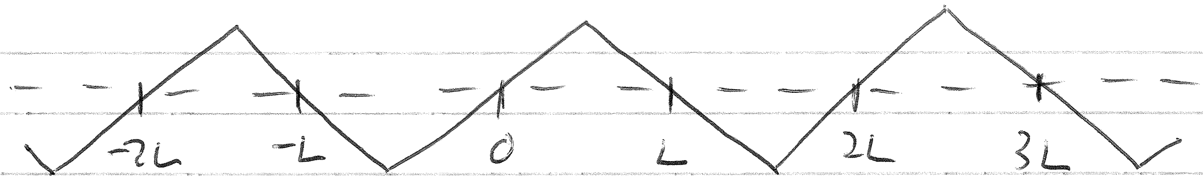


This will make the Fourier series converge faster, and thus reduce the high frequency content of the sound

(2) Real strings have bending stiffness, which will shift the natural frequencies from harmonics

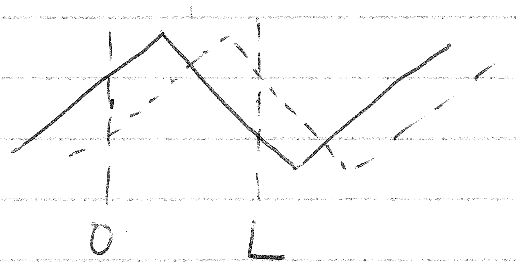
(3) Most notes on a piano have 2 or 3 strings, not just one. These interact because they are all connected to the (non-rigid) soundboard. This produces a mixture of multi-string vibration modes, which have different decay rates. This has a big influence on the sound of a piano note.

8. Derivation follows the lecture notes — all details are there. For a mid-point pluck the two functions $f(x+ct)$ and $g(x-ct)$ are symmetrical triangular waves:

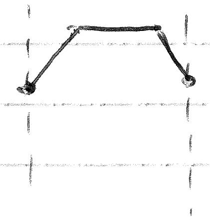


The two copies move out in opposite direction, and the displacement is the sum of the two in the range $0 - L$.

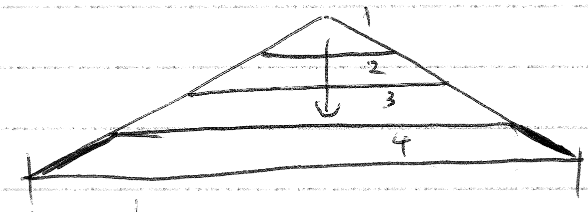
So e.g.



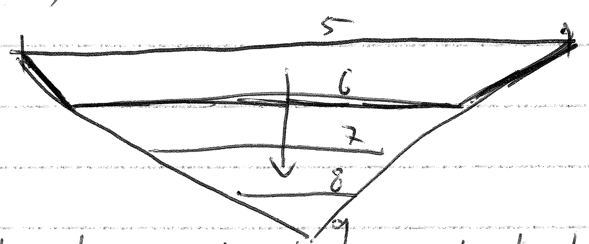
adds to:



The sequence is

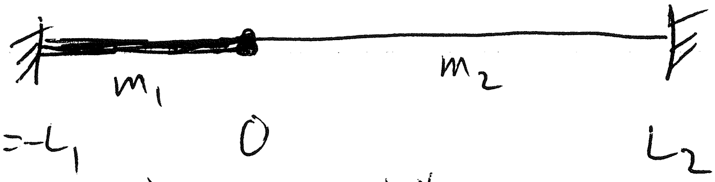


for the first $\frac{1}{4}$-cycle, then



then it all reverses and returns to the initial shape

9.



(a) Let $w(x,t) = u(x)\, e^{i\omega t}$ as usual

Solution for $x<0$ is $u(x) = A \sin \dfrac{\omega(L_1+x)}{c_1}$ so $u(-L_1)=0$

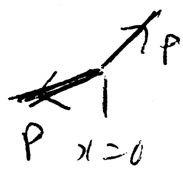
" " $x>0$ is $u(x) = B \sin \dfrac{\omega(L_2-x)}{c_2}$ so $u(L_2)=0$

where $c_1 = \sqrt{P/m_1}$, $c_2 = \sqrt{P/m_2}$.

At $x=0$ (i) string has no break, so $u(x)$ is continuous

(ii) force equilibrium, so

$$P \left.\frac{du}{dx}\right|_{0-} = P \left.\frac{du}{dx}\right|_{0+}$$



ie $\dfrac{du}{dx}$ is also continuous.

(b) From (i), $A \sin \dfrac{\omega L_1}{c_1} = B \sin \dfrac{\omega L_2}{c_2}$

From (ii) $A \dfrac{\omega}{c_1} \cos \dfrac{\omega L}{c_1} = -B \dfrac{\omega}{c_2} \cos \dfrac{\omega L_2}{c_2}$

$$\therefore \frac{A}{B} = \frac{\sin \omega L_2/c_2}{\sin \omega L_1/c_1} = -\frac{c_1}{c_2} \frac{\cos \omega L_2/c_2}{\cos \omega L_1/c_1}$$

$$\therefore \quad c_1 \tan \frac{\omega L_1}{c_1} = -c_2 \tan \frac{\omega L_2}{c_2}$$

Natural frequencies ω_n satisfy this equation.

(c) If $c_1 = c_2$, the equation requires

$$\tan \frac{\omega L_1}{c_1} = -\tan \frac{\omega L_2}{c_2}$$

$$\rightarrow \sin \frac{\omega L_1}{c_1} \cos \frac{\omega L_2}{c_2} = -\sin \frac{\omega L_2}{c_2} \cos \frac{\omega L_1}{c_1}$$

$\rightarrow \sin \dfrac{\omega(L_1+L_2)}{c} = 0$, so $\dfrac{\omega(L_1+L_2)}{c} = n\pi$ as expected.