UNIVERSITY OF CAMBRIDGE

3F1, Signals and Systems

PART II.3: Design of IIR filters

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Module A

Discretization by response matching

Impulse invariance

$G_c(s)$ Laplace transform continuous-time filter. The impulse response of the corresponding *impulse invariance* digital filter G(z) (with sampling T) is equal to the impulse response of the $G(s)$ sampled at $t = kT$.

$$G(z) = \mathcal{Z}\left(\mathcal{L}^{-1}\left(G_c(s)\right)_{t=kT}\right)$$

- For <u>bandlimited</u> filters the digital filter frequency response will closely approximate the continuous-time frequency response.
- Preserves stability ($\Re(\beta_q) < 0 \rightarrow |e^{\beta_q T}| < 1$).

Example:

$$G_c(s) = \frac{\alpha}{s-\beta} \overset{\mathcal{L}^{-1}}{\rightarrow} \alpha e^{\beta t} \xrightarrow{\text{sample}} \alpha e^{\beta T k} \overset{\mathcal{Z}}{\rightarrow} \frac{\alpha}{1 - e^{\beta T} z^{-1}} = G(z)$$

Step response invariance

Take a continuous time filter/system with Laplace transfer function $G_c(s)$. The corresponding *step response invariance* digital filter (with sampling period T) is a digital filter whose step response is equal to the step response of the continuous time filter sampled at $t = kT$.

$$G(z) = \frac{z-1}{z} \mathcal{Z}\left(\mathcal{L}^{-1}\left(\frac{G_c(s)}{s} \right)_{t=kT} \right)$$

$$G_c(s) \overset{\text{step}}{\rightarrow} \frac{G_c(s)}{s} \overset{\mathcal{L}^{-1}}{\rightarrow} y(t) \overset{\text{sample}}{\rightarrow} y(kT) \overset{\mathcal{Z}}{\rightarrow} Y(z) \overset{\frac{1}{\text{step}}}{\rightarrow} \frac{z-1}{z} Y(z) = G(z)$$

1. Find the step response of the continuous system
2. Sample at time $t = kT$ and take the z-transform
3. Multiply by $(z-1)/z$.

Ramp invariance

Take a continuous time filter/system with Laplace transfer function $G_c(s)$. The corresponding *ramp response invariance* digital filter (with sampling period T) is a digital filter whose ramp response is equal to the ramp response of the continuous time filter sampled at $t = kT$.

$$G(z) = \frac{(z-1)^2}{Tz} \mathcal{Z}\left(\mathcal{L}^{-1}\left(\frac{G_c(s)}{s^2}\right)_{t=kT} \right)$$

<u>Note</u>: any waveform invariance can be considered. The digital filter will preserve the properties of the continuous filter response to that particular waveform.

Module B

Discretization by algebraic transformations

Algebraic transformations are used to map filters that are not bandlimited (avoiding aliasing) at the cost of introducing distortion between the frequency response of the continuous-time filter and the frequency response of the digital filter.

$$H(z) = H_c(s)_{s=\psi(z)} \text{ where } \psi(\cdot) \text{ is given by}$$

Euler's method or Forward difference

$$s = \frac{z-1}{T} \qquad \text{(intuition } \dot{x} \simeq \frac{x(t+T) - x(t)}{T}\text{)}$$

Backward difference

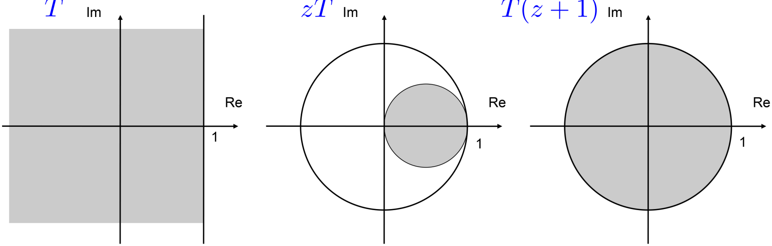
$$s = \frac{z-1}{zT} \qquad \text{(intuition } \dot{x} \simeq \frac{x(t) - x(t-T)}{T}\text{)}$$

Bilinear transformation or Tustin's transformation

$$s = \frac{2}{T}\frac{z-1}{z+1} \qquad \text{(or simply } \frac{z-1}{z+1}, \text{no normalization factor)}$$

Each of these transformations corresponds to a certain mapping between $s-$plane and $z-$plane. Below, the shaded region shows the set of points in the $z-$plane which corresponds to the left half of the s-plane (stable regions)

$$s = \frac{z-1}{T} \qquad s = \frac{z-1}{zT} \qquad s = \frac{2(z-1)}{T(z+1)}$$



Forward Difference — Backward Difference — Tustin

Backward difference and Tustin transformations applied to stable continuous systems result in stable discrete time systems (all the left plane poles get mapped into the unit disk). Not necessarily true for Euler's method.

Example: **first order low pass filter** $G_c(s) = \dfrac{1}{s+1}$. Sampling T.

Forward (possibly unstable)

$$G(z) = G_c\left(\frac{z-1}{T}\right) = \frac{T}{z+(T-1)} \qquad \text{pole } |T-1| > 1?$$

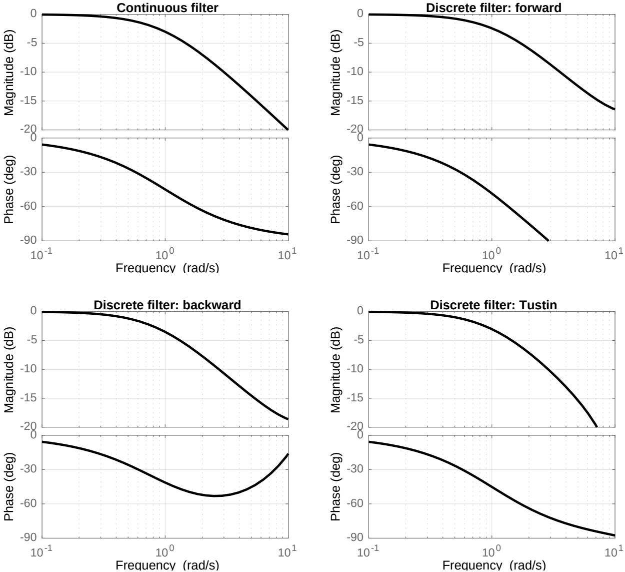
Backward (stable)

$$G(z) = G_c\left(\frac{z-1}{zT}\right) = \frac{\frac{T}{1+T}z}{z - \frac{1}{(1+T)}} \qquad \text{pole } \left|\frac{1}{1+T}\right| < 1$$

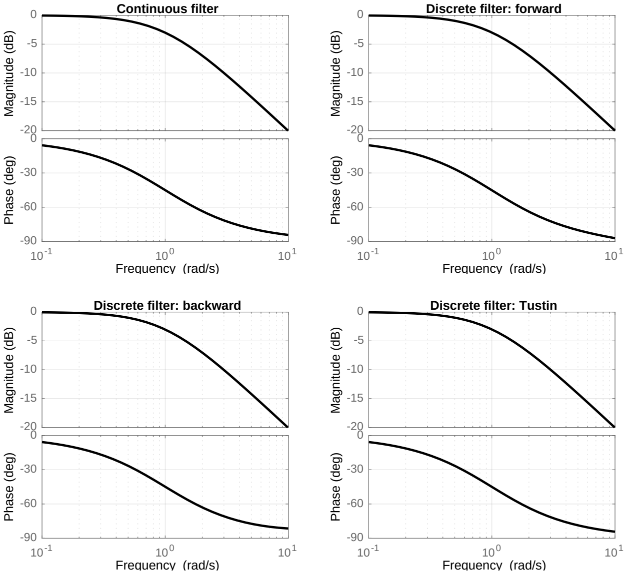
Tustin (stable)

$$G(z) = G_c\left(\frac{2}{T}\frac{z-1}{z+1}\right) = \frac{\frac{T}{T+2}(z+1)}{z+\frac{T-2}{T+2}} \qquad \text{pole } \left|\frac{T-2}{T+2}\right| < 1$$

Frequency response distortion of discretized filters.
Sampling $T = 0.25$ ($\omega_{max} = \frac{\pi}{T} = 4\pi$ rad/s).



The frequency response is recovered as $T \to 0$.
Sampling $T = 0.01$ ($\omega_{max} = \frac{\pi}{T} = 100\pi$ rad/s).

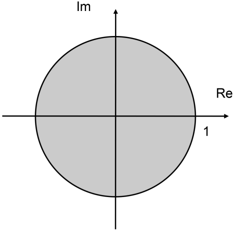


Module C

Bilinear transform in detail

Bilinear transform: stability is preserved

$$s = \psi(z) = \frac{z-1}{z+1}$$



Solve for z: $z = \psi^{-1}(s) = \dfrac{1+s}{1-s}$

For $s = \lambda + j\omega$:

$$|z|^2 = zz^* = \psi^{-1}(\lambda + j\omega)(\psi^{-1}(\lambda + j\omega))^* = \frac{(1+\lambda)^2 + \omega^2}{(1-\lambda)^2 + \omega^2}$$

If $\lambda = 0$ then

$$|z|^2 = \frac{1+\omega^2}{1+\omega^2} = 1 \ \rightarrow \ \text{unit circle}$$

If $\lambda < 0$ then $(1+\lambda)^2 < (1-\lambda)^2$ thus

$$|z|^2 = \frac{(1+\lambda)^2 + \omega^2}{(1-\lambda)^2 + \omega^2} < 1 \ \rightarrow \ \text{inside unit circle}$$

Bilinear transform: frequency warping

$$s = \psi(z) = \frac{z-1}{z+1}$$

Analog prototype filter: $G_c(s)$. Frequency response $G_c(j\omega)$.
Digital filter: $G(z) = G_c(\psi(z))$.

The normalized frequency response of the digital filter ($|\theta| \leq \pi$) is given by

$$G(e^{j\theta}) = G_c(\psi(e^{j\theta}))$$

where

$$\psi(e^{j\theta}) = \frac{e^{j\theta}-1}{e^{j\theta}+1} = \frac{e^{j\theta/2}-e^{-j\theta/2}}{e^{j\theta/2}+e^{-j\theta/2}} = \frac{j\sin(\theta/2)}{\cos\theta/2} = j\tan(\theta/2)$$

that is

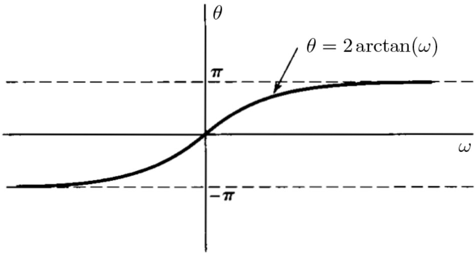
$$G(e^{j\theta}) = G_c(j\tan(\theta/2)) \qquad \text{frequency warping}$$

$$G(e^{j\theta}) = G_c(j\tan(\theta/2))$$

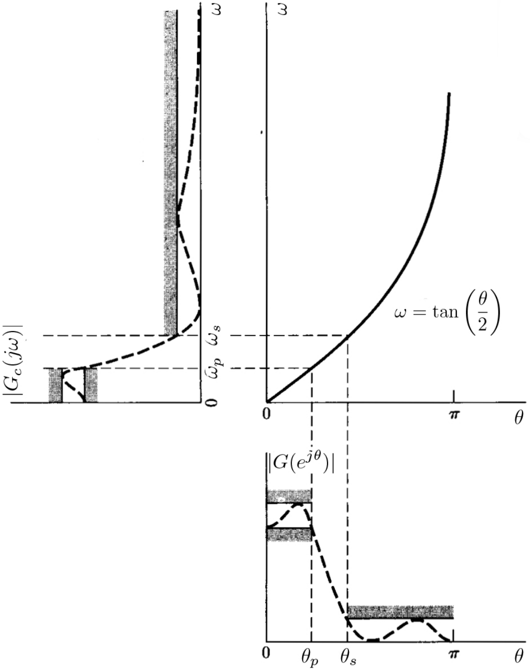
Inverse relation: the frequency response of the analog filter at ω is mapped into the frequency response of the digital filter at $\theta = 2\arctan(\omega)$.

$$\omega = \tan(\theta/2) \to \theta = 2\arctan(\omega) \qquad \text{frequency warping}$$

$$G_c(j\omega) = G\left(e^{j2\arctan(\omega)}\right)$$

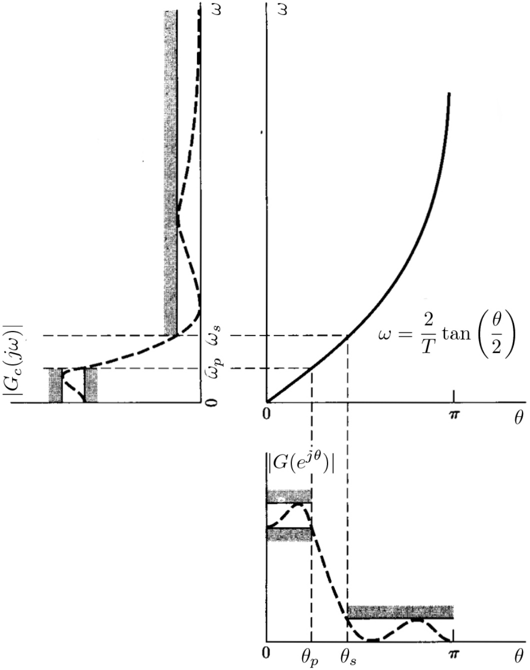


$$s = \frac{z-1}{z+1}$$



ω
ω
$\omega = \tan\left(\frac{\theta}{2}\right)$
$|G_c(j\omega)|$
ω_s
ω_p
0
0
π
θ
$|G(e^{j\theta})|$
0
θ_p
θ_s
π
θ

$$s = \frac{2}{T}\frac{z-1}{z+1}$$



Example (low pass filter design)

Design a first order lowpass digital filter with -3dB frequency of 1kHz and a sampling frequency of 8kHz.

Consider the first order analogue lowpass filter

$$G_c(s) = \frac{1}{1 + \frac{s}{\omega_c}}$$

which has gain 1 (0dB) at $s = j0$ and gain 0.5 (-3dB) at $s = j\omega_c$ rad/s (cutoff frequency). Thus, the normalized digital cutoff frequency reads

$$\theta_c = (1000 \cdot 2\pi) \cdot T = \frac{1000 \cdot 2\pi}{8000} = \pi/4$$

The equivalent pre-warped analogue filter cutoff frequency:

$$\omega_c = \tan(\theta_c/2) = \tan(\pi/8) = 0.4142$$

Example (low pass filter design)

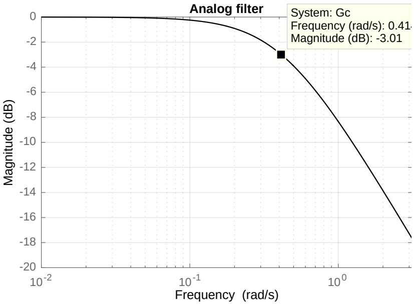
Apply now the bilinear transform $s = \psi(z) = \frac{z-1}{z+1}$.

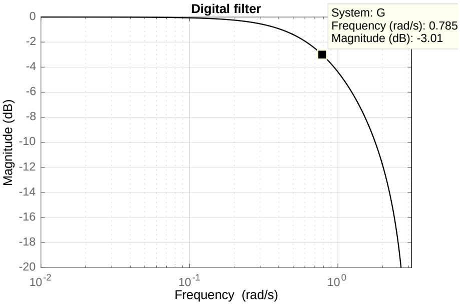
$$
\begin{aligned}
G(z) &= G_c(\psi(z)) = \frac{1}{1+\frac{\psi(z)}{\omega_c}} = \frac{1}{1+\frac{z-1}{(z+1)\omega_c}} \\
&= \frac{(z+1)\omega_c}{(z+1)\omega_c + z - 1} = \frac{(z+1)\omega_c}{(1+\omega_c)z + (\omega_c - 1)} \\
&= \frac{(z+1)\frac{\omega_c}{(1+\omega_c)}}{z + \frac{(\omega_c-1)}{(1+\omega_c)}} = \frac{0.2929(z+1)}{z - 0.4142}
\end{aligned}
$$

whose implementation reads

$$
y_k = 0.4142 y_{k-1} + 0.2929(u_k + u_{k-1})
$$

Example (low pass filter design)





Module D

Band transformations

Transformation between different filter types

Analogue prototypes are typically lowpass. Standard transformation can be used to convert lowpass fprototype into other types.

Assuming a lowpass prototype with cutoff at 1:

- Lowpass to Lowpass:
  set $s = \frac{\bar{s}}{\omega_c}$ to change the cutoff frequency to ω_c.
- Lowpass to Highpass:
  set $s = \frac{\omega_c}{\bar{s}}$ to get highpass with cutoff frequency at ω_c.
- Lowpass to Bandpass:
  set $s = \frac{\bar{s}^2 + \omega_l \omega_u}{\bar{s}(\omega_u - \omega_l)}$ to get bandpass with lower cutoff at ω_l and upper cutoff at ω_u.
- Lowpass to Bandstop:
  set $s = \frac{\bar{s}(\omega_u - \omega_l)}{\bar{s}^2 + \omega_l \omega_u}$ to get bandstop with lower cutoff at ω_l and upper cutoff at ω_u.

Transformation between different filter types

- Lowpass to Lowpass by $s = \frac{\bar{s}}{\omega_c}$

$$\frac{1}{s+1} \quad \rightarrow \quad \frac{1}{\frac{\bar{s}}{\omega_c}+1} = \frac{\omega_c}{\bar{s}+\omega_c}$$

after transformation cutoff at ω_c, $|G(j0)| = 1$, $|G(j\infty)| = 0$.

- Lowpass to Highpass by $s = \frac{\omega_c}{\bar{s}}$

$$\frac{1}{s+1} \quad \rightarrow \quad \frac{1}{\frac{\omega_c}{\bar{s}}+1} = \frac{\bar{s}}{\bar{s}+\omega_c}$$

after transformation cutoff at ω_c, $|G(j0)| = 0$, $|G(j\infty)| = 0$.

- try the others...

Module E

Matlab and Python code

Useful Matlab code:

▸ Matlab: IIR filter design

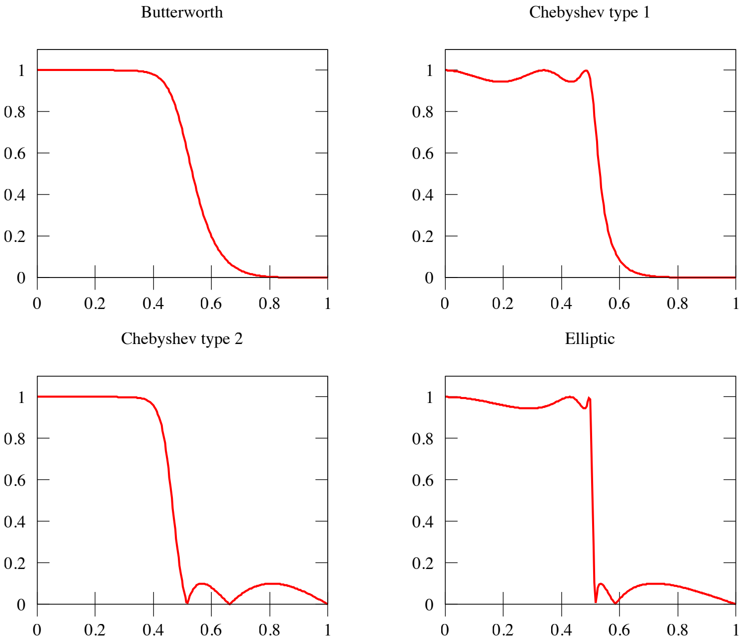
▸ Matlab: analog filters

▸ Matlab: analog filters comparison

Useful Python code:

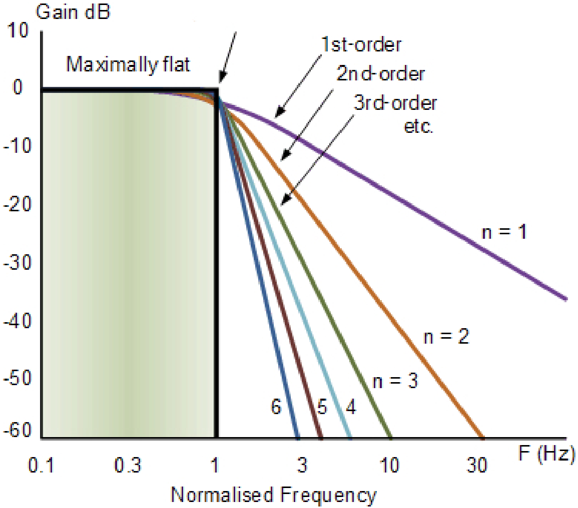
▸ Python: scipy.signal for IIR filter design

Appendix - Classical analog prototypes


Butterworth
Chebyshev type 1
Chebyshev type 2
Elliptic
0
0.2
0.4
0.6
0.8
1

Butterworth: Nth-order lowpass, $G_c(s)$ satisfies

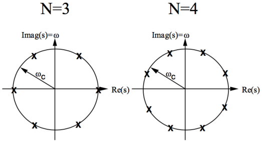
$$G_c(s)G_c(-s) = \frac{1}{1 + \left(\frac{s}{j\omega c}\right)^{2N}}$$



Butterworth: Nth-order lowpass, $G_c(s)$ satisfies

$$G_c(s)G_c(-s) = \frac{1}{1 + \left(\frac{s}{j\omega c}\right)^{2N}}$$

- Unit DC gain, $G_c(j0) = 1$.
- -3dB cutoff frequency at $s = j\omega_c$.
- $G_c(s)G_c(-s)$ poles satisfies $\left(\frac{s}{j\omega c}\right)^{2N} = -1$, i.e. $s = j\omega_c e^{\frac{j(2k+1)\pi}{2N}}$



- If p_i is a root of $G_c(s)$ then $-p_i$ is a root of $G_c(-s)$. Thus the poles of $G_c(s)$ are those roots lying in the left half plane (stability), so that

$$G_c(s) = \prod_{i=1}^{P} \frac{1}{s + p_i}$$

- Matlab routine `[B,A] = butter(N,Wn)` designs digital Butterworth filters (using bilinear transform) ▸ Matlab: butter .