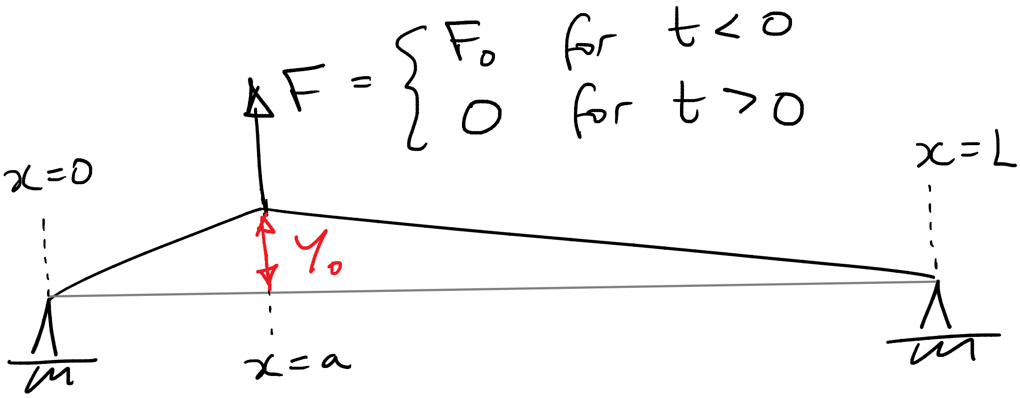
Handout 2 – Transient response, forced response and transfer functions

Transient response from initial conditions (plucked string)

It is common to excite a string into motion by plucking or striking it (e.g. a guitar string or a piano wire respectively). We will consider an idealised pluck: pulling a string into a triangular shape then suddenly letting go at $t = 0$.



After release there are no external forces acting on the string, so the response must satisfy the equation of free motion, i.e. it must consist of a combination of the vibration modes:

$$y(x,t) = \sum_n c_n U_n(x) \mathrm{e}^{i\omega_n t} = \sum_n c_n \sin\left(\frac{n\pi x}{L}\right) \mathrm{e}^{in\Omega t} \qquad \Omega = \frac{\pi c}{L}$$

The coefficients c_n represent how much the nth mode contributes to the overall response: note that they may be complex as they contain phase and amplitude information.

At $t = 0$ the string is at rest, and is in the shape of a triangle, so we need to match the coefficients to its Fourier Series representation. It is more convenient to use $c_n = a_n + ib_n$, giving:

$$y(x,t) = \sum_n \left(a_n \cos(n\Omega t) - b_n \sin(n\Omega t)\right) \sin\left(\frac{n\pi x}{L}\right)$$

So at $t = 0$ it's a triangle:

$$y(x,0) = \begin{cases} y_0 x/a & 0 \leqslant x \leqslant a \\ \dfrac{y_0(L-x)}{(L-a)} & a \leqslant x \leqslant L \end{cases}$$

and it's at rest:

$$\left.\frac{\partial y}{\partial t}\right|_{t=0} = 0$$

The second condition (initially at rest) requires:

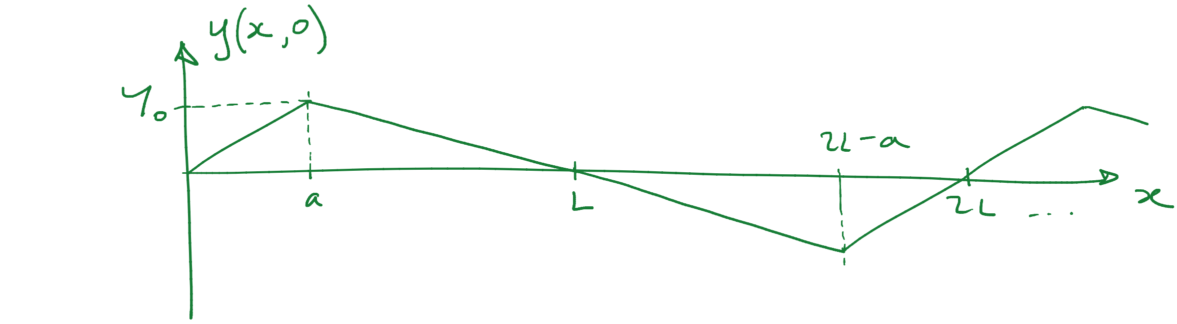
$$\left.\frac{\partial y}{\partial t}\right|_{t=0} = \sum_n \left(-n\Omega a_n \sin(n\Omega t) - n\Omega b_n \cos(n\Omega t)\right) \sin\left(\frac{n\pi x}{L}\right)\Bigg|_{t=0} = 0$$

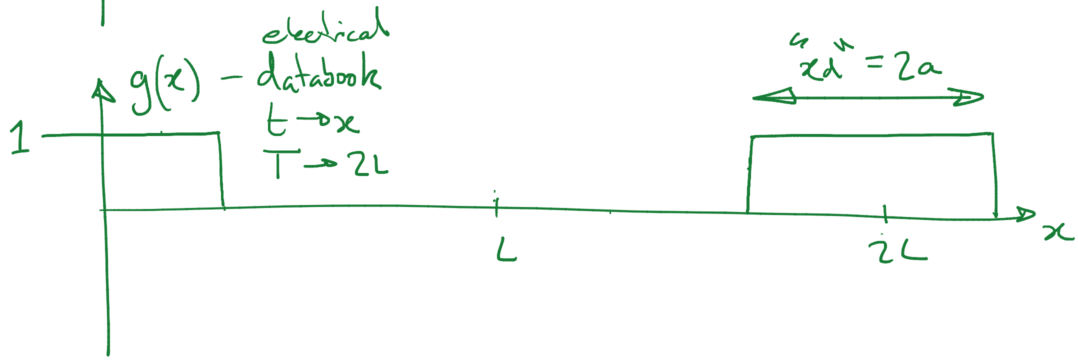
$$\implies \sum_n (n\Omega b_n) \sin\left(\frac{n\pi x}{L}\right) = 0$$

for all values of x. This is only possible if $b_n = 0$ for all values of n.

The first condition then becomes:

$$y(x,0) = \sum_n a_n \sin\left(\frac{n\pi x}{L}\right) = \text{triangle, wavelength} = 2L$$





Then relate our function to the databook:

$$\frac{\partial y}{\partial x} = \frac{Y_0 L}{a(L-a)} g(x) - \frac{Y_0}{L-a}$$

$$= \frac{Y_0 L}{a(L-a)} \frac{2a}{2L}\left[1 + 2\sum_{n=1}^{\infty} \frac{\sin\left(n\pi \frac{2a}{2L}\right)}{n\pi \frac{2a}{2L}} \cos\left(\frac{n2\pi x}{2L}\right)\right] - \frac{Y_0}{L-a}$$

$$= \frac{2Y_0}{L-a} \sum_{n=1}^{\infty} \frac{\sin(n\pi a/L)}{n\pi a/L} \cos\left(\frac{n\pi x}{L}\right)$$

Then integrate to recover $y(x,0)$:

$$y(x,0) = \mathrm{DC} + \frac{2Y_0}{L-a} \sum_{n=1}^{\infty} \frac{\sin(n\pi a/L)}{n^2\pi^2 a/L^2} \sin\left(\frac{n\pi x}{L}\right)$$

Note that the DC term is zero (wavelength $2L$), and the coefficients a_n are:

$$a_n = \frac{2Y_0 L^2 \sin(n\pi a/L)}{n^2\pi^2 a(L-a)}$$

Putting it all together, the motion of the string is given by:

$$y(x,t) = \sum_n \left(a_n \cos(n\Omega t) - b_n \sin(n\Omega t)\right) \sin\left(\frac{n\pi x}{L}\right)$$

$$= \sum_{n=1}^{\infty} \frac{2Y_0 L^2 \sin(n\pi a/L)}{n^2\pi^2 a(L-a)} \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi ct}{L}\right)$$

See `H2_transient_response.m`

Implications for stringed instruments (e.g. guitar strings)

- Resonances are harmonic: all the natural frequencies are integer multiples of the fundamental:

  $$\omega_1 = \pi c/L$$

  The total response is the sum of these harmonics, so it must be periodic with frequency ω_1. This is why stringed instruments give a musical note: we hear a periodic waveform as a note with a definite pitch. See `H2_notes_clangs.m`

- The frequency content (one factor that affects 'tone quality') varies with the position of plucking, as that affects the relative contribution of the modes to the response.

- A real string will have energy dissipation so the plucked note will decay with time, each mode at its own particular rate. In other words each mode has some damping.

- If the ends were actually fixed, the guitar would be silent. A real instrument has some compliance at the ends, which allows weak coupling to the guitar body: causing it to vibrate and radiate sound. This is one of the sources of energy loss from the string.

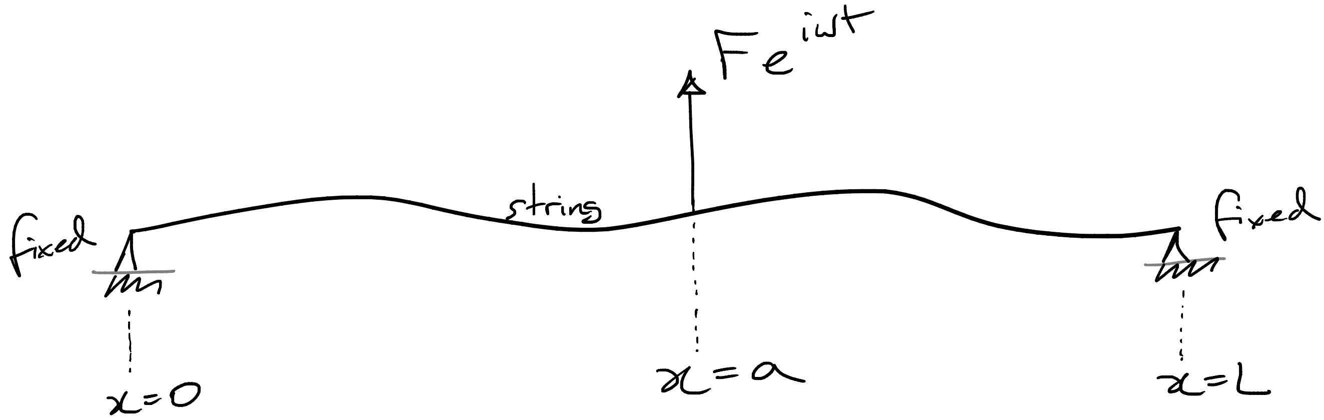
- A real string has some bending stiffness so the behaviour is more complicated, and the natural frequencies are not exactly harmonics.

- A real pluck is made with a finger or plectrum, so the corner of the initial triangle is not perfectly sharp, and it is not released instantaneously. The rounded corner means that in the Fourier Series representation of the initial conditions the high frequencies contribute less as they have a shorter wavelength on the string. This gives the string a more mellow / softer sound. Wide pickups on electric guitars have a similar filtering effect, compared with narrow ones.

Forced harmonic response

We will continue to use the string example as it demonstrates all the principles, and it's the easiest one to visualise. In a typical vibration measurement, a force is applied at a point and the vibration amplitude and phase are observed at another point. The result is the frequency-dependent *transfer function* between two points.

Apply a purely sinusoidal force at $x = a$:



The string must satisfy the free equation of motion everywhere except at the point where the force is applied, so we can write an expression for the response in terms of the general solution. However, we need to use separate expressions for the two parts of the string on either side of the input force, then look at what happens near the force point to see how to fit these together.

We know that the free solution is sinusoidal, and that $y(0,t) = 0$ and $y(L,t) = 0$. The easiest choice we can make is

$$U = A\sin\left(\underbrace{\frac{\omega x}{c}}_{= kx}\right) \qquad \text{for} \quad 0 < x < a$$

no cosine as $U(0)=0$

$$U = B\sin\left(\frac{\omega(L-x)}{c}\right) \qquad \text{for} \quad a < x < L$$

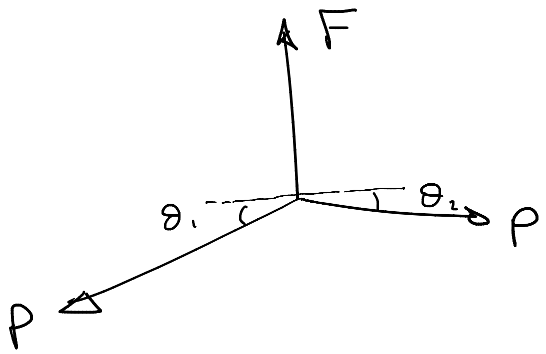
the $(L\text{-}x)$ means this is as if looking at the string from the right, so again no cosine as $U(L) = 0$

We know that at $x = a$:

- The string has no break (*continuity*), so:

$$A\sin\left(\frac{\omega a}{c}\right) = B\sin\left(\frac{\omega(L-a)}{c}\right)$$

- and the forces must balance (apply *balance of forces*):



$$\sin\theta_1 \approx \theta_1 \approx \tan\theta_1 = \left.\frac{\partial y}{\partial x}\right|_{x=a^-}$$

$$\sin\theta_2 \approx \theta_2 \approx \tan\theta_2 = -\left.\frac{\partial y}{\partial x}\right|_{x=a^+}$$

note the sign!

$$\sum F\uparrow \Rightarrow \quad F = P\left.\frac{\partial y}{\partial x}\right|_{a^-} - P\left.\frac{\partial y}{\partial x}\right|_{a^+}$$

substituting in gives:

$$F = P\left\{\frac{A\omega}{c}\cos\left(\frac{\omega a}{c}\right) + \frac{B\omega}{c}\cos\left(\frac{\omega(L-a)}{c}\right)\right\}$$

Use the two equations…

$$A \sin\left(\frac{\omega a}{c}\right) = B \sin\left(\frac{\omega(L-a)}{c}\right)$$

$$F = P\left\{\frac{A\omega}{c}\cos\left(\frac{\omega a}{c}\right) + \frac{B\omega}{c}\cos\left(\frac{\omega(L-a)}{c}\right)\right\}$$

… to find *A* and *B* (after some rearranging):

$$A = \frac{Fc}{\omega P}\frac{\sin\left[\omega(L-a)/c\right]}{\sin(\omega L/c)}, \quad B = \frac{Fc}{\omega P}\frac{\sin\left[\omega a/c\right]}{\sin(\omega L/c)}$$

which are the coefficients for the two sections of the string, giving the solution:

$$U(x) = \begin{cases} \dfrac{Fc}{\omega P}\dfrac{\sin\left[\omega(L-a)/c\right]}{\sin(\omega L/c)}\sin\left[\omega x/L\right] & \text{for} \quad 0 < x < a \\ \dfrac{Fc}{\omega P}\dfrac{\sin\left[\omega a/c\right]}{\sin(\omega L/c)}\sin\left[\omega(L-x)/L\right] & \text{for} \quad a < x < L \end{cases}$$

What does this tell us? It gives the steady-state response at position *x* of the string, given a sinusoidal input force applied at a point *a*. There are four important things to note:

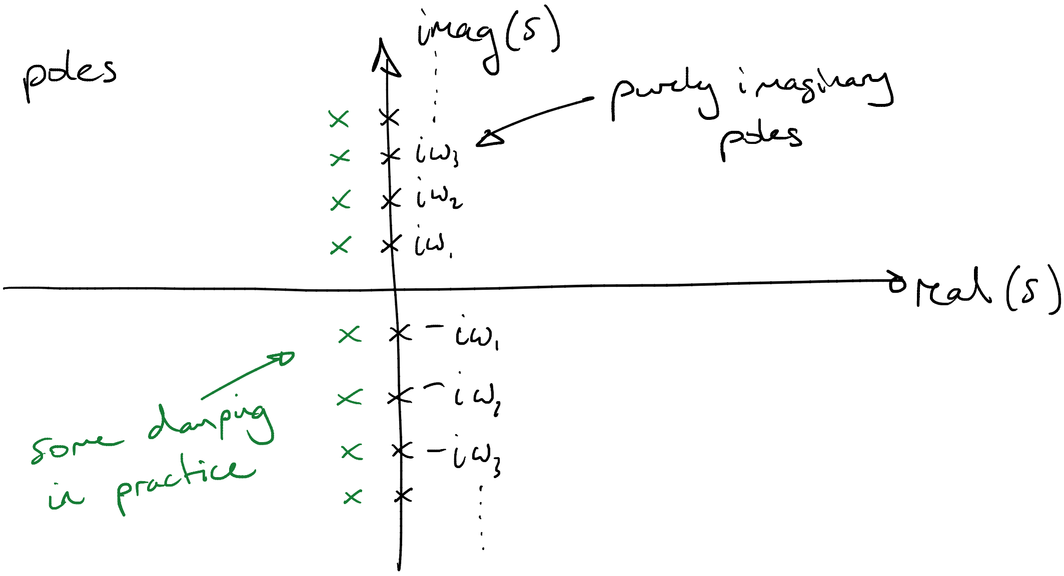
- The quantity $U(x)/F(a)$ is the *transfer function* from the point *a* to the point *x*, and is what you would obtain from from a vibration measurement by applying a force at one place and measuring the displacement at the other. It is often written:

$$\frac{U(x)}{F(a)} = G(a, x, \omega)$$

- The expression is *symmetrical* if you swap *a* and *x*. This is an example of a general reciprocity theorem: in any vibration measurement if you interchange the excitation point and the observation point then you (should) get exactly the same result.

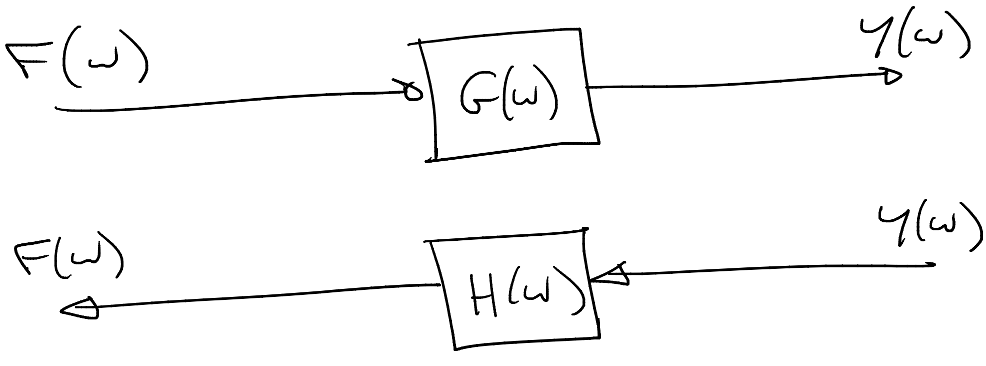
- The displacement *U* goes to infinity at frequencies where $\sin(\omega L/c) = 0$. This is what we expect because these are the natural frequencies ω_n from the previous section, at which the string has resonances. We can only avoid this if the numerator of the transfer function is also zero. This happens when the position of either the input force or the measurement location are at a nodal point of the corresponding mode shape $U_n(x)$. That makes sense: you can't excite a resonance by forcing at a point where there is no motion, nor can you observe a resonance by only looking at a point where there is no motion.

- The resonant frequencies ω_n of the structure give us the *system poles* using $s = i\omega$. So the systems we have considered have an infinite number of poles equally spaced along the imaginary axis. This means that they are marginally stable: if the structure is perturbed then it will continue to vibrate indefinitely as there is no damping. The poles of the system do not depend on the position of the input force or the measurement location, they are an intrinsic property of the structure. In practice there is normally a bit of damping, and the poles all have a negative real part.



- The *anti-resonant* frequencies of the structure give us the *zeros* of the transfer function, again using $s = i\omega$. The zeros of the system depend on the position of the input force and the measurement location, they are <u>not</u> an intrinsic property of the structure. In practice there is normally a bit of damping, and the zeros all have a negative real part.

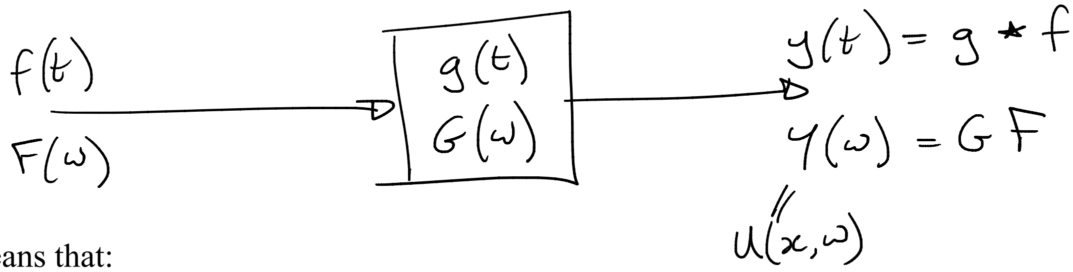
- If we choose the input to be displacement, and the output to be force then the transfer function is turned upside down ($H = 1/G$) and the zeros become poles: peaks become anti-resonances and vice-versa.



- Or if we consider a *driving-point transfer function* $G(x, x, \omega)$, i.e. input and output at the same location, then it can be shown that the zeros of $G(x, x, \omega)$ are the poles of the system if its displacement were constrained at the same location x.

Transient response for arbitrary loads

If we know the transfer function $G(a,x,\omega)$ from point a to x, then we can find the response to any time-varying force applied at the same point:



This means that:

- the impulse response is the inverse Fourier Transform of the transfer function
- the time response to a general input is the convolution of the impulse response with the input force
- the frequency response is the product of the transfer function with the Fourier Transform of the input force

Of course if it's easier, you can find the response to a general input force by first finding the frequency response, then taking the inverse Fourier Transform.

Implications for industrial components (e.g. engine-driven shaft)

- Shafts are ubiquitous in industrial applications: automotive engines, gas turbines, oilwell drills, etc. Torsional vibration of a shaft follows exactly the same equation of motion as the stretched string (just different constants, and perhaps different boundary conditions).

- We can estimate natural frequencies of these shafts very readily when we know the boundary conditions (not always fixed both ends for a shaft!), and we need to be clear about what the 'input' and 'output' are.

- Typically the torque applied by an engine to a shaft is not steady: this means that on top of an average torque there is often a fluctuation that is periodic with frequency proportional to the engine speed. So we either need to avoid operating at speeds where one of the harmonics of the excitation frequency is close to the frequency of a mode, or try to isolate the engine torque fluctuations from the shaft, or try to add damping to the shaft itself.

- The fine details are not normally so important to industry as for musical instruments (just avoid vibration!), which gives added value to back-of-the-envelope calculations.

Summary

We can find the transient response to arbitrary initial conditions by expressing the solution as a combination of modes, and matching coefficients of mode contributions at $t = 0$ to the initial shape.

We can find a transfer function from one point to another for a 1D system by using combinations of the free solution, and joining them up at the point where the force is applied. At the connection we apply *continuity* and *balance of forces*.

The transfer functions tells us the steady-state frequency response from one point to another. The resonances correspond to the system poles and are a property of the whole structure (even if the measurement is point-to-point). Resonances are not observed if the excitation or measurement location are at a node of the corresponding mode.

We can find the transient response to an arbitrary input using the Fourier Transform, its inverse, and the transfer function.