Module 3C5: Dynamics and Vibrations

DYNAMICS

Examples paper 3C5/2 – Applications

Straightforward questions are marked †
*Tripos standard questions are marked **

Gyroscopic motion

1.(a) A symmetrical rotor with principal moments of inertia A, A and C is mounted on two weightless gimbals. Euler angles θ, ϕ and ψ are aligned with the rotor as shown in Fig 1. The rotor is spinning fast (ie ω_3 is large) and a couple Q is applied as shown.

Show that two possible steady-state motions (ie with constant θ) are possible as follows:

$$\dot{\phi} \approx \frac{Q}{C\omega_3 \sin\theta} \quad \text{and} \quad \dot{\phi} \approx \frac{C\omega_3}{A\cos\theta}$$

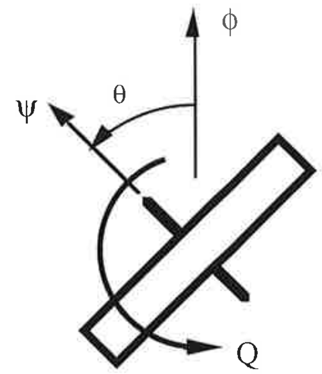

Fig. 1

These motions are called "precession" and "nutation" respectively. Interpret the two motions physically (it may help to note that only one of the motions depends on the applied couple Q. What if Q=0?).

(b) A uniform steel disc, radius 1m and thickness 1cm is spinning at 3000rpm with its spin axis inclined to the vertical. The disc is supported by a ball joint on the spin axis 10cm away from the centre of mass. Calculate the rate of slow precession of the disc under the action of its own weight.

Gyroscopic Instruments

†2. A rate gyro is shown schematically in Fig. 4 as it may be used to determine the steady rate of turning Ω of a turntable. Rotation θ of the gimbal about the **j** axis is resisted by a torsional spring of stiffness k with internal viscous damping λ. About this axis the gimbal has moment of inertia I_G. The rotor has principal moments of inertia I,I,J and is spinning fast with angular speed ω.

(a) Derive an equation of motion for the rate gyro.

(b) How does the rate gyro operate in steady state and what is the frequency of small vibrations about this steady state?

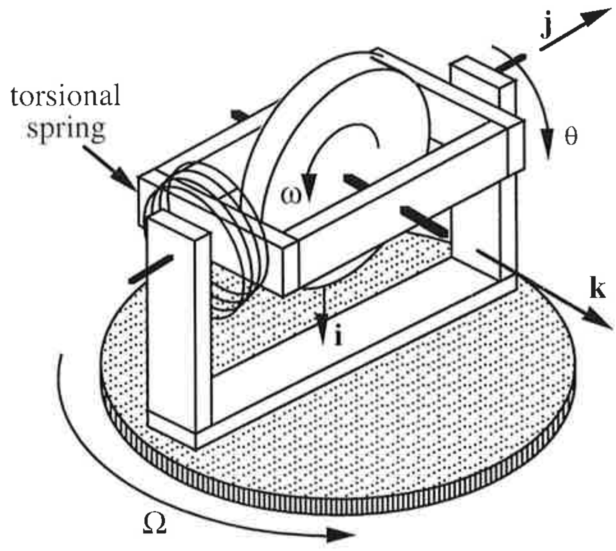

Fig. 4

(c) Explain how this instrument is used in a strap-down inertial-navigation system.

*3. A simplified gyrocompass is shown in Fig. 3a. The gimbal is free to turn through angle θ about its vertical axis **j**. The rotor has principal moments of inertia I,I,J at its centre and it spins fast at a constant rate ω about its axis **k**. The Earth spins at a constant rate Ω and the instrument is situated at latitude λ as shown in Fig. 3b.

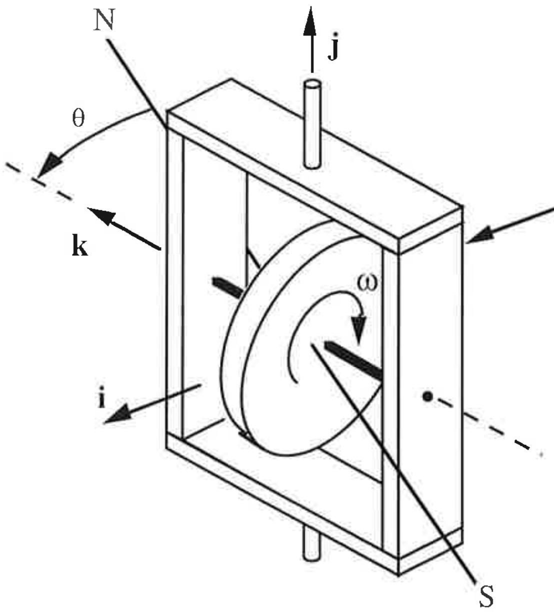

Fig. 3a

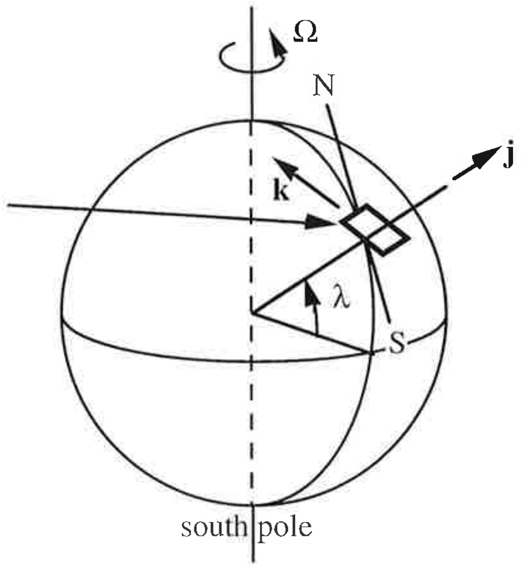

Fig. 3b

(a) Determine the components of the Earth's spin Ω in the gimbal-mounted axis system **i**, **j**, **k**. Hence write down expressions for the components of angular velocity of the reference frame.

(b) Use the Gyroscope Equations (as derived in 1(a) above) to obtain a equations of motion for the gyrocompass (neglect friction and gimbal inertia). Which of Q_1, Q_2 and Q_3 are zero?

(c) Show that there is only one stable steady-state solution and it occurs when the gyrocompass aligns itself with true north. What is the magnitude of the non-zero couple in this steady state?

(d) What is the period of small oscillations about true north? Calculate this period for the gyroscope used in the G7 minor experiment ($I \approx 0.08\text{kgm}^2$, $J \approx 0.02\text{kgm}^2$, $\omega \approx 440\text{rad/s}$, $\lambda \approx 52°$). How small a friction couple acting about the **j** axis will prevent oscillations when $\theta = 0.1\text{rad}$?

<u>Rolling bodies</u>

4. A thin uniform circular coin of mass m and radius a rolls on a table without slip.

(a) Identify Euler angles for the rolling coin system. How many parameters are necessary to specify completely the position and orientation of the coin? How many parameters are needed to stop the coin moving?

(b) When the angle θ between the coin and the table is small, show that a steady wobbling motion is possible and find an expression for the wobbling frequency in terms of θ. Neglect energy losses and assume that the centre of the coin is motionless.

(c) Calculate the slow rate of turning of the coin as viewed from above.

5. The same coin of Q5 is rolled with a large forward speed V and with θ close to $\pi/2$. Find an expression for the radius of the path of the centre of the coin in terms of the forward speed and the inclination α of the coin, $\alpha = \pi/2 - \theta$. Assume steady state. (Hint: calculate the reaction at the point of contact with the table given that G moves on a circular path).

(Notes for Q6, Q7 and Q8: These should be solved using the moment of momentum equation about a moving point contact P. This is because the normal and frictional reaction at the point P have no moment about P and so do not need to be calculated. If you have time, try also using moment of momentum about G and check that you get the same answers.)

†6. (a) A conventional football (*ie* a spherical shell) is initially at rest on the ground. A footballer kicks the ball such that its initial velocity is V and there is no initial spin. What is the final speed of the ball when it is eventually rolling without slip?

(b) A child is playing with a hula-hoop (*ie* a thin ring) of radius a, tossing it, rolling it and spinning it always with the plane of the hoop vertical. In one game, the hoop is projected horizontally with speed V and with backspin ω. It skids along the ground and eventually rolls back to the child with no slip and with the same speed V as it had to start with. What must have been the value of the backspin ω ?

(this same question could have been asked of a billiards player putting backspin on the cue ball)

7. The uniform solid ball shown in Fig. 5 has radius a and mass m . It rolls without slip on a horizontal table that is rotating at angular speed Ω about a fixed vertical shaft through O. The components of velocity of the centre of mass G of the ball are $(\dot{x}, \dot{y}, \dot{z})$ in the fixed and *non-rotating* reference frame **i, j, k** as shown. The shaft axis is parallel to **k**. Likewise, the **i, j, k** components of angular velocity ω of the ball are $(\omega_1, \omega_2, \omega_3)$.

Point P is the point of contact between ball and table and the position vector OP is $x\,\mathbf{i} + y\,\mathbf{j}$.

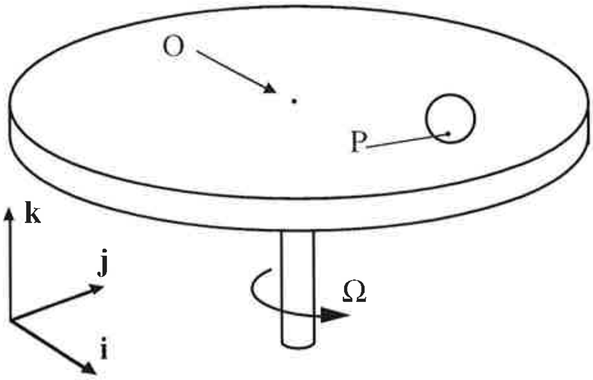

Fig. 5

(a) Draw a free-body diagram showing all forces that act on the ball and by using the data-sheet relation $\dot{\mathbf{h}}_P + \dot{\mathbf{r}}_P \times \mathbf{p} = \mathbf{Q}^{(e)}$ show that $\mathbf{h}_P$ is constant (it helps to note that P moves with the same velocity as G).

(b) Find $\mathbf{h}_P$ using the data sheet relation $\mathbf{h}_P = \mathbf{h}_G + (\mathbf{r}_G - \mathbf{r}_P) \times \mathbf{p}$ and hence find two equations relating ω_1, ω_2, $\dot{x}$, and $\dot{y}$.

(c) Use a no-slip condition at P show that $\dot{x} - a\omega_2 + y\Omega = 0$ and $\dot{y} + a\omega_1 - x\Omega = 0$.

(d) Combine the results of (b) and (c) to show that the ball moves on a circle. Show that the ball completes one complete orbit in the same time that it takes the table to turn 3.5 times.

(Note that the result does not depend on the size or mass of the ball. Also note that the centre and radius of the circle on which the ball moves are completely arbitrary and depend only on initial conditions)

*8. A rigid uniform sphere of radius a rolls inside a fixed, rough, vertical cylindrical hole of radius R as shown in Fig. 6. The centre of the sphere is at G and it makes contact with the cylinder at a moving point P. A system of unit vectors **i**, **j**, **k** is defined at P with **k** aligned with the axis of the cylinder, **i** always aligned with GP and **j** always horizontal. The system **i**, **j**, **k** rotates at $\Omega\mathbf{k}$. The height of G above some arbitrary datum is z.

(a) Use a no-slip condition at the contact point P to find the components of angular velocity of the sphere ω_1, ω_2 and ω_3, the velocity of point P and the velocity of the centre of the sphere G in terms of Ω and $\dot{z}$. Find also the couple **Q** acting on the sphere about P.

(b) Use the form of " $\dot{\mathbf{h}} = \mathbf{Q}$ " appropriate for a moving point P to obtain the equation governing the motion z of the centre of the sphere in the vertical direction. Show that this is SHM with frequency $\sqrt{\frac{2}{7}}\,\Omega$.

Comment on the nature of the solution of this equation in relation to the putting of golf balls. Then explain why the golf ball *doesn't* usually pop out again after it is holed.

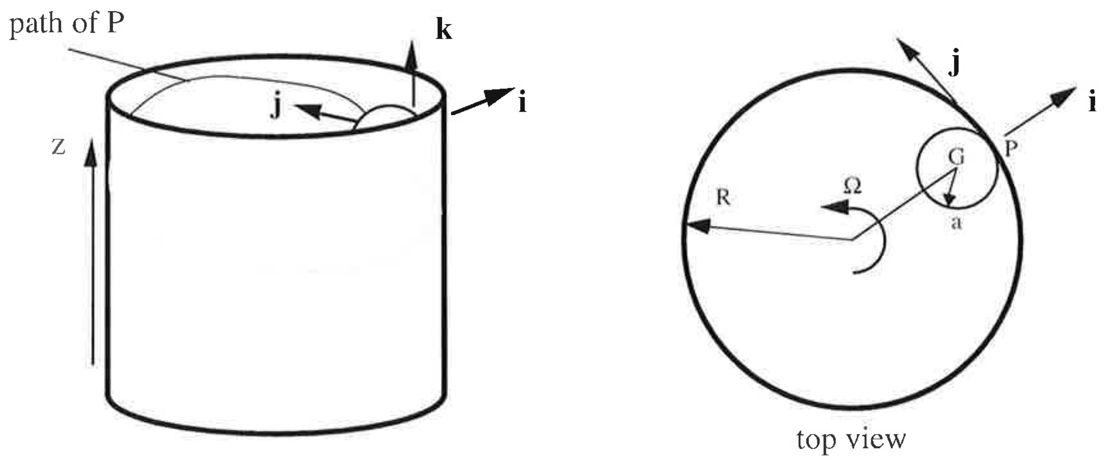

Fig. 6

ANSWERS

1.(a) Precession is driven by a couple; nutation is a free vibration of arbitrary amplitude which just needs a kick to get started (just like a mass on a spring).

(b) 3.58 revs per hour

2.(a) $(I+I_G)\ddot{\alpha} + \lambda\dot{\alpha} + k\alpha \approx J\omega\Omega$ where $\alpha = \pi/2 - \theta$

(b) $\alpha \approx J\omega/k\ \Omega$, damped oscillations at frequency $\sqrt{(k/(I+I_G))}$

(c) three rate gyros and three accelerometers - integrate and correct for variations in g.

3.(a) $\Omega_1 = -\Omega\cos\lambda\sin\theta$, $\Omega_2 = \dot{\theta} + \Omega\sin\lambda$, $\Omega_3 = \Omega\cos\lambda\cos\theta$

(b) $I\ddot{\theta} + J\omega\Omega\cos\lambda\sin\theta = 0$

Q_1 is non-zero (needed to keep the axis of the gyrocompass vertical). $Q_2 = Q_3 = 0$.

(c) $Q_1 = J\omega\Omega\sin\lambda$ (d) $2\pi\sqrt{(I/J\omega\Omega\cos\lambda)}$ 90s ~0.04mNm (very small)

4.(a) five (θ, ϕ, ψ, x and y), three (θ, ϕ and ψ). (b) $2\sqrt{\dfrac{g}{a\theta}}$ (c) $\sqrt{\dfrac{g\theta^3}{a}}$

5. $\dfrac{3}{2}\dfrac{V^2}{g\alpha}$

6.(a) $\frac{3}{5}V$ (b) $\dfrac{3V}{a}$

7. (b) $\frac{2}{5}\omega_1 a - \dot{y} = \text{const}_1$ $\frac{2}{5}\omega_2 a + \dot{x} = \text{const}_2$

8. (a) ω_1 is arbitrary (spin), $\omega_2 = \dot{z}/a$, $\omega_3 = -(R-a)\Omega/a$

$\mathbf{v_P} = R\Omega\mathbf{j} + \dot{z}\mathbf{k}$ $\mathbf{v_G} = (R-a)\Omega\mathbf{j} + \dot{z}\mathbf{k}$ $\mathbf{Q} = -mga\mathbf{j}$

(b) $\ddot{z} + \Omega_1^2 z = \Omega_1^2 z_0 - 5g/7$ where $\Omega_1 = \sqrt{\frac{2}{7}}\Omega$ and z_0 is an arbitrary constant.

Suitable Tripos Questions

2009 !2, Q3 2008 Q1d, Q2, Q3 2007 Q1, Q3 2006 Q1, Q3 2005 Q3
2004 Q1. Q2 2003 Q2. Q3 2002 Q1, Q2 2001 Q1, Q3
2000 Q1, Q2, Q3 1999, Q1, Q3 1998, Q2, Q3 1997, Q2, Q3
1996, Q2, Q3 1995, Q2, Q3

Part II, Paper G7 (Old Tripos) 1994, Q4 1993, Q3, Q4 1992, Q2,Q3,Q4

Note: The syllabus no longer includes consideration of stable platforms and rotor whirl, last examined in June 2002.

HEMH Nov 2011