Engineering Tripos Part IIA

Module 3M1: Mathematical Methods

Examples Paper 1 Linear Algebra

Solutions to a number of questions are very short (1–2 lines). These are marked by ($\mathcal{S}$). Questions for which you are encouraged to use a computer are marked by ‡.

Revision

Look over the concepts of rank, row space, column space, nullspace and left-nullspace from Part IB.

Complex matrices

1. ($\mathcal{S}$) For the matrix

$$\boldsymbol{A} = \begin{bmatrix} 4+4i & 2-i \\ -3+2i & 4+i \end{bmatrix}$$

compute:

   a) $\det \boldsymbol{A}$;

   b) $\det \boldsymbol{A}^H$; and

   c) $\boldsymbol{A}^{-1}$.

2. ($\mathcal{S}$) What can you say about the diagonal entries of a Hermitian matrix?

Eigenvalues

3. Compute the eigenvalues of the rotation matrix

$$\boldsymbol{Q} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

4. For a Hermitian matrix $\boldsymbol{M} \in \mathbb{C}^{n\times n}$:

   a) ($\mathcal{S}$) Prove that all eigenvalues are real; and

   b) For the case of distinct eigenvalues, prove that the eigenvectors are orthogonal.

5. ($\mathcal{S}$) For a matrix $\boldsymbol{A} \in \mathbb{C}^{m\times n}$, show that the matrix $\boldsymbol{A}^H\boldsymbol{A}$ is positive semi-definite, i.e. $\boldsymbol{x}^H\boldsymbol{A}^H\boldsymbol{A}\boldsymbol{x} \geq 0$ for all $\boldsymbol{x} \in \mathbb{C}^n$. What can you conclude on the eigenvalues of $\boldsymbol{A}^H\boldsymbol{A}$?

6. a) ($\mathcal{S}$) Show that the eigenvalues of the matrices $\boldsymbol{A} \in \mathbb{C}^{n\times n}$ and $\boldsymbol{A}^T$ are the same.

   b) ($\mathcal{S}$) Show that the eigenvalues of $\boldsymbol{A} \in \mathbb{C}^{n\times n}$ are the complex conjugates of the eigenvalues of $\boldsymbol{A}^H$.

   c) ($\mathcal{S}$) For a matrix $\boldsymbol{A} \in \mathbb{C}^{n\times m}$ and a matrix $\boldsymbol{B} \in \mathbb{C}^{m\times n}$, show that the non-zero eigenvalues of $\boldsymbol{AB}$ and $\boldsymbol{BA}$ are the same.

Norms

7. ($\mathcal{S}$) Show that for a vector $\boldsymbol{x} \in \mathbb{C}^n$:
$$\|\boldsymbol{x}\|_\infty \leq \|\boldsymbol{x}\|_1 \leq n\|\boldsymbol{x}\|_\infty.$$

8. Using the definition of an operator norm of a matrix, prove that:
   a) $\|\boldsymbol{AB}\| \leq \|\boldsymbol{A}\|\|\boldsymbol{B}\|$; and
   b) $\|\boldsymbol{A}+\boldsymbol{B}\| \leq \|\boldsymbol{A}\| + \|\boldsymbol{B}\|$.

9. For $\boldsymbol{A} \in \mathbb{C}^{m\times n}$, show that:
   a) $\|\boldsymbol{A}\|_1 = \max_{1\leq j\leq n} \|\boldsymbol{a}_j\|_1$, where $\boldsymbol{a}_j$ is the jth column vector of $\boldsymbol{A}$; and
   b) $\|\boldsymbol{A}\|_\infty = \max_{1\leq i\leq m} \|\boldsymbol{a}_i^\star\|_1$, where $\boldsymbol{a}_i^\star$ is the ith row vector of $\boldsymbol{A}$.

   Hint: Consider the definition of an operator norm for vectors with unit norm, i.e. $\|\boldsymbol{x}\|_1 = 1$ *and* $\|\boldsymbol{x}\|_\infty = 1$*, respectively.*

10. Compute the l_2 norm of the matrix
$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$$
and verify that $\|\boldsymbol{Ax}\|_2 \leq \|\boldsymbol{A}\|_2$ for all $\boldsymbol{x} \in \mathbb{R}^2$ that satisfy $\|\boldsymbol{x}\|_2 = 1$.

    Hint: consider $\boldsymbol{x} = [\cos\theta \ \ \sin\theta]^T$ *and maximise* $\|\boldsymbol{Ax}\|_2$ *over* θ*.*

11. Estimate the condition numbers $\kappa_1(\boldsymbol{A})$, $\kappa_2(\boldsymbol{A})$ and $\kappa_\infty(\boldsymbol{A})$ for the matrix
$$\boldsymbol{A} = \begin{bmatrix} 10^{-4} & 2 \\ 1 & 1 \end{bmatrix}$$
and comment on the suitability of LU decomposition for this matrix.

Least-squares

12. Consider a least-squares problem $\boldsymbol{A}^H\boldsymbol{A}\hat{\boldsymbol{x}} = \boldsymbol{A}^H\boldsymbol{b}$ where $\boldsymbol{A}$ is full rank. The residual vector is defined as $\boldsymbol{r} = \boldsymbol{A}\hat{\boldsymbol{x}} - \boldsymbol{b}$.
    a) Show that
$$\boldsymbol{r}^H\boldsymbol{Az} = \boldsymbol{b}^H\left(\boldsymbol{A}\left(\boldsymbol{A}^H\boldsymbol{A}\right)^{-1}\boldsymbol{A}^H - \boldsymbol{I}\right)\boldsymbol{Az} = 0 \quad \forall \boldsymbol{z} \in \mathbb{C}^n$$
    and explain the significance of this result.

    Hint: consider what the residual vector is orthogonal to.

    b) Show that
$$\boldsymbol{P} = \boldsymbol{A}\left(\boldsymbol{A}^H\boldsymbol{A}\right)^{-1}\boldsymbol{A}^H$$
    is an orhtogonal projection matrix matrix, i.e. $\boldsymbol{P}^2 = \boldsymbol{P}$ and $\boldsymbol{P}^H = \boldsymbol{P}$.

    c) What is the significance of $\boldsymbol{P}$ in a least-squares problem?

13. Show that if the $m \times n$ matrix $\boldsymbol{A}$ has linearly independent rows, then:
    a) the matrix $\boldsymbol{AA}^H$ is invertible; and
    b) $\boldsymbol{A}^+ = \boldsymbol{A}^H\left(\boldsymbol{AA}^H\right)^{-1}$ is a right-inverse of $\boldsymbol{A}$.

Iterative methods

14. If a stationary iterative scheme is used to solve a problem with the matrix

$$\boldsymbol{A} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix},$$

which of the Richardson, Jacobi and Gauss–Seidel methods would you expect to converge and why?

Singular value decomposition

15. ($\mathcal{S}$) If $\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^{H}$ is a singular value decomposition of the square matrix $\boldsymbol{A}$, find an expression for $\boldsymbol{A}^{-1}$ and comment on the significance of the singular values.

16. ‡ For the matrix

$$\boldsymbol{A} = \begin{bmatrix} 1 & 3 \\ 2 & 2 \\ 3 & 1 \end{bmatrix}$$

   a) Find the singular value decomposition of the matrix.
   b) Find the pseudoinverse $\boldsymbol{A}^{+}$.
   c) Show that $\boldsymbol{A}^{+}\boldsymbol{A} = \boldsymbol{I}$ ($\boldsymbol{A}^{+}$ is a left inverse).
   d) Show that $\boldsymbol{A}\boldsymbol{A}^{+} \neq \boldsymbol{I}$.
   e) Find the general form of the vector $\boldsymbol{b}$ for which $\boldsymbol{A}\boldsymbol{A}^{+}\boldsymbol{b} = \boldsymbol{b}$.

17. ‡ Find the singular value decomposition and the pseudoinverse of the matrix

$$\boldsymbol{A} = \begin{bmatrix} 0 & 3 & 0 \\ 2 & 0 & 0 \end{bmatrix}.$$

Show that for every $\boldsymbol{x}$ in the row space of $\boldsymbol{A}$ that $\boldsymbol{A}^{+}\boldsymbol{A}\boldsymbol{x} = \boldsymbol{x}$.

18. ‡ Find the minimum length least-squares solution to

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}.$$

Answers See `https://nbviewer.jupyter.org/github/garth-wells/notebooks-3M1/blob/master/3M1%20Examples%20Paper%201%20Crib.ipynb`