Engineering Tripos Part IIA Paper 3C6

Linear vibration of continuous systems

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These notes are based on the 3C6 notes written by Prof. Jim Woodhouse.

Background

This course is about linear vibration in continuous structures: taking the ideas of mechanical vibration and linear systems and applying them to structures that have an infinite number of degrees of freedom. The equation of motion of these kind of structures is described by PDE's: in contrast to the 'discrete systems' half of 3C6 where the equation of motion is given by matrices of second-order ODE's. But the distinction is blurred: continuous systems can be approximated by discrete systems if we have 'enough' degrees of freedom.

In some ways discrete systems are 'easier': we can make a discrete Finite Element model of a complicated structure and solve the equations numerically. So why bother with analytic methods for simple continuous structures? For three reasons (at least):

- It lets us do back-of-the-envelope calculations when faced with typical engineering components;
- It provides a reference to check if our complicated Finite Element model is doing what we expect;
- and we get a lot of insight into the fundamental physics of vibration.

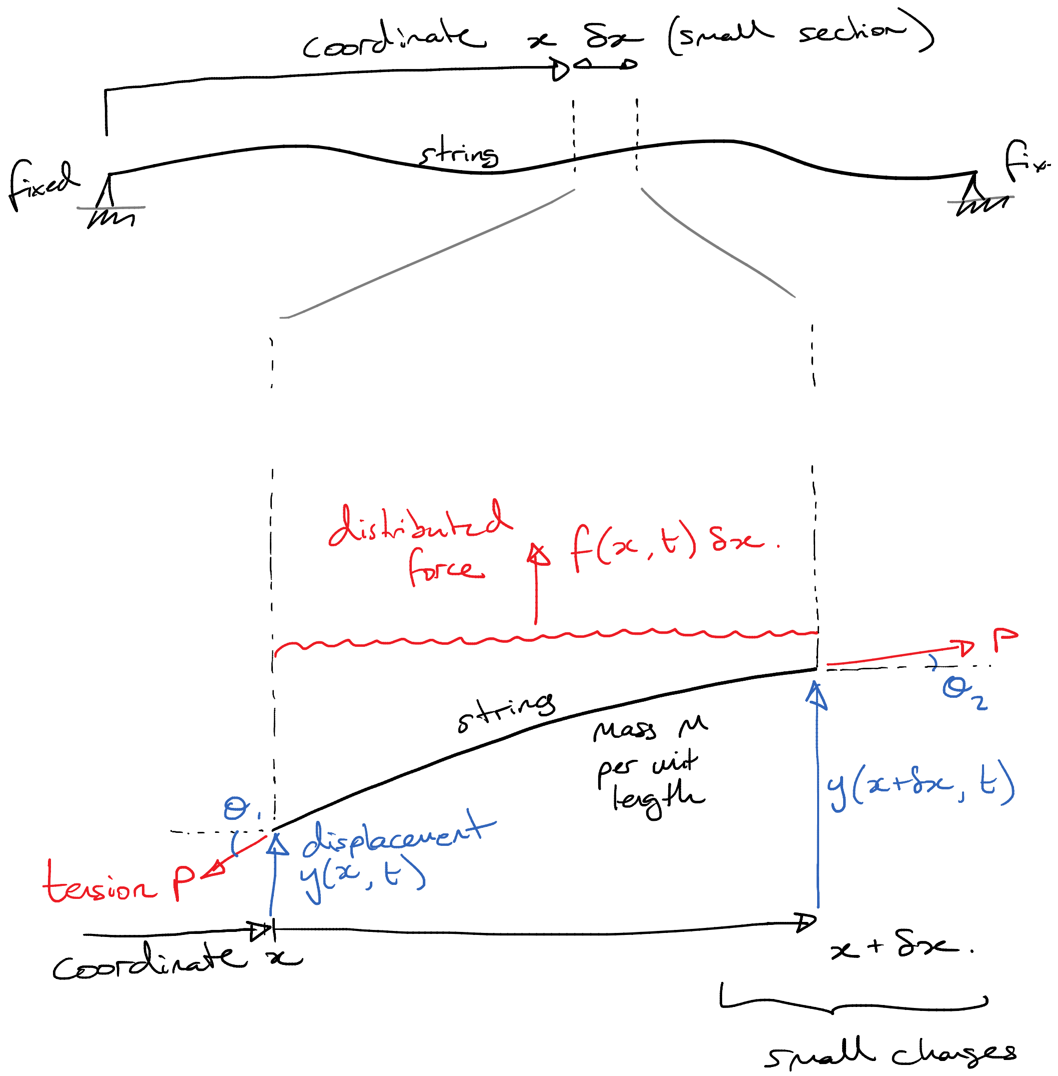
Why do we want to understand vibration? In industry, vibration is almost always unwanted as high amplitude vibration can cause fatigue or stress: the goal is to eliminate or reduce the vibration amplitude, or at least avoid conditions where high amplitudes occur. Just occasionally vibration is exploited in industry: percussive drilling or for structural health monitoring to detect the onset of damage before it is too late. But there is another category where vibration is essential: musical instruments. This course will use both industrial and musical examples to illustrate the principles.

Handout 1 – Vibration of 1D systems: strings, columns, shafts, and ducts

Stretched string

The simplest starting point is to consider transverse vibration of an ideal stretched string. By ideal we mean that it's perfectly flexible, is stretched between two fixed points, and is allowed to vibrate transversely at small amplitude.

To obtain the equation of motion, consider a small section of the string:



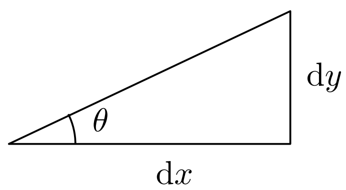
Resolve transversely, sum forces and apply Newton's law, $F = ma$:

$$f(x,t)\delta x + P\sin\theta_2 - P\sin\theta_1 = m\delta x\frac{\mathrm{d}^2 y}{\mathrm{d}t^2}$$

Small angle approximations:

$$\sin\theta_1 \approx \theta_1 \approx \tan\theta_1 = \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_x$$

$$\sin\theta_2 \approx \theta_2 \approx \tan\theta_2 = \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x+\delta x}$$



Substitute:

$$f(x,t)\delta x + P\sin\theta_2 - P\sin\theta_1 = m\delta x\frac{\partial^2 y}{\partial t^2}$$

$$f(x,t)\delta x + P\theta_2 - P\theta_1 = m\delta x\frac{\partial^2 y}{\partial t^2}$$

$$f(x,t)\delta x + P\left\{\left.\frac{\partial y}{\partial x}\right|_{x+\delta_x} - \left.\frac{\partial y}{\partial x}\right|_{x}\right\} = m\delta x\frac{\partial^2 y}{\partial t^2}$$

$$f(x,t) + P\frac{\left\{\left.\frac{\partial y}{\partial x}\right|_{x+\delta_x} - \left.\frac{\partial y}{\partial x}\right|_{x}\right\}}{\delta x} = m\frac{\partial^2 y}{\partial t^2}$$

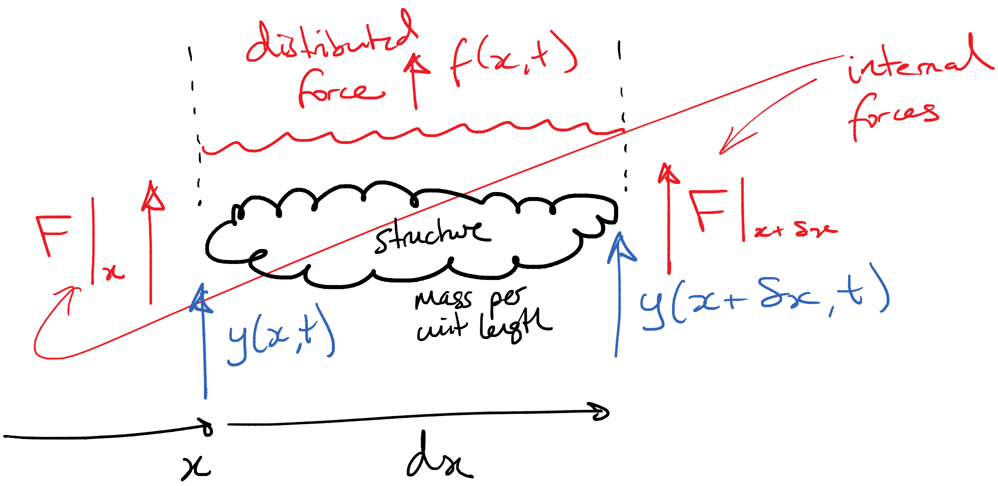
Take limits and rearrange:

$$m\frac{\partial^2 y}{\partial t^2} - P\frac{\partial^2 y}{\partial x^2} = f(x,t)$$

This is the *equation of motion* of an ideal stretched string with external distributed forcing.

Summary of obtaining equations of motion:

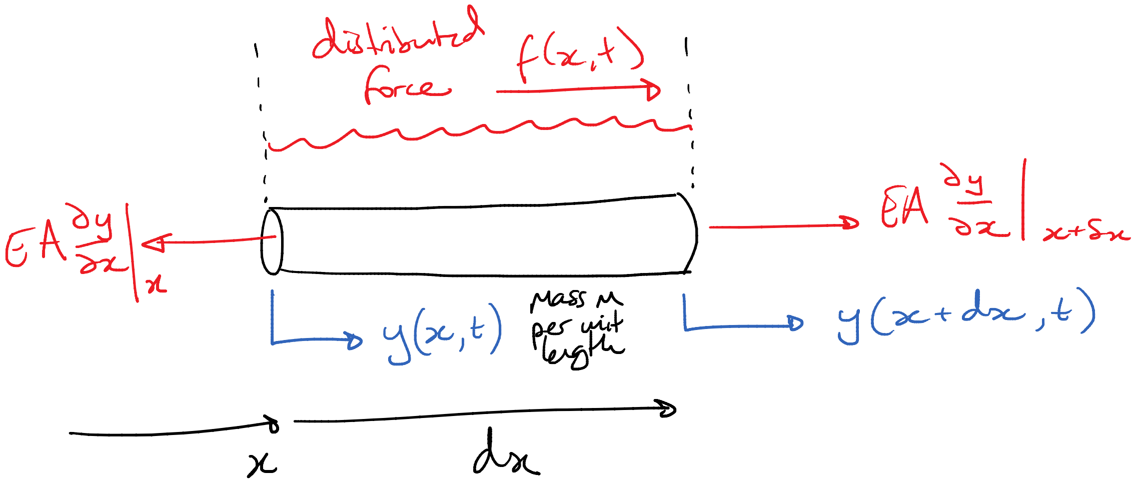
1. Draw a big diagram of a small section of the structure



2. label the coordinates: the start and length of the small section
3. label the displacements from equilibrium: nominal values on the left, small changes on the right
4. identify any distributed external forces
5. work out expressions for the internal forces on both sides of the small section
6. apply $F = ma$ and take limits

Axial vibration of a bar:

1. Draw a big diagram of a small section of the structure



2. label the coordinates: the start and length of the small section
3. label the displacements from equilibrium: nominal values on the left, small changes on the right
4. identify any distributed external forces
5. work out expressions for the internal forces on both sides of the small section
6. apply $F = ma$ and take limits

Work out expressions for the internal forces on both sides:

From the structures databook strain is defined as:

$$\epsilon_{xx} = \frac{\partial y}{\partial x}$$

and in terms of stress:

$$\epsilon_{xx} = \frac{1}{E}\sigma_{xx}$$

where E is the Young's modulus. We want the force acting at each end of the bar, so:

$$F(x) = A\sigma_{xx} = EA\left.\frac{\partial y}{\partial x}\right|_x$$

where A is the cross-sectional area.

Apply $F = ma$:

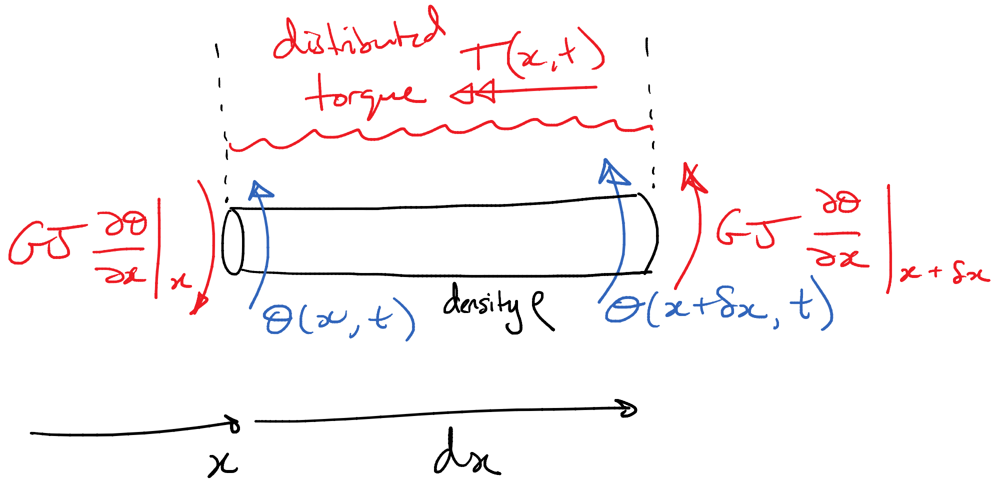
$$EA\left(\left.\frac{\partial y}{\partial x}\right|_{x+\delta x} - \left.\frac{\partial y}{\partial x}\right|_x\right) + f(x,t)\,dx = m\,dx\,\frac{\partial^2 y}{\partial t^2}$$

Take limits:

$$m\frac{\partial^2 y}{\partial t^2} - EA\frac{\partial^2 y}{\partial x^2} = f(x,t)$$

Torsional vibration of a shaft:

1. Draw a big diagram of a small section of the structure



2. label the coordinates: the start and length of the small section
3. label the displacements from equilibrium: nominal values on the left, small changes on the right
4. identify any distributed external forces
5. work out expressions for the internal forces on both sides of the small section
6. apply $F = ma$ and take limits

Work out expressions for the internal forces on both sides

From the structures databook that the internal torque T is given by

$$T = GJ\phi$$

where G is the shear modulus of the material, J is the polar moment of area of the cross section, and ϕ is the twist per unit length, i.e.

$$T(x) = GJ\frac{\partial\theta}{\partial x}$$

Apply $F = ma$

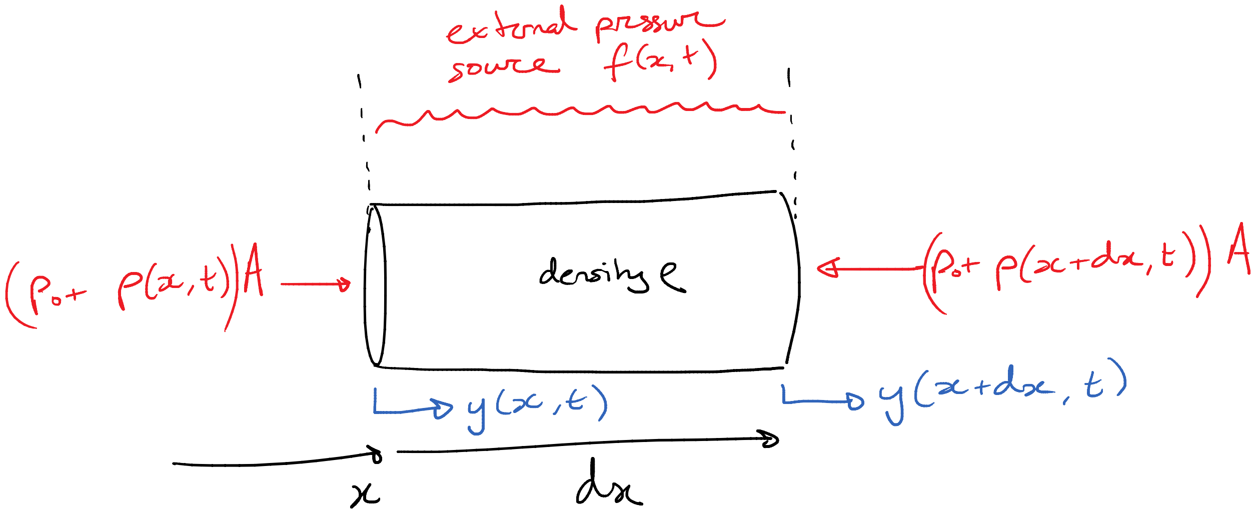
$$GJ\left(\left.\frac{\partial\theta}{\partial x}\right|_{x+\delta x} - \left.\frac{\partial\theta}{\partial x}\right|_{x}\right) + T(x,t)\,dx = \underbrace{\rho\,dx\,J}_{\text{inertia}}\frac{\partial^2\theta}{\partial t^2}$$

Use small value approximations and take limits

$$\rho J\frac{\partial^2\theta}{\partial t^2} - GJ\frac{\partial^2\theta}{\partial x^2} = T(x,t)$$

Acoustic vibration in a duct:

1. Draw a big diagram of a small section of the structure



2. label the coordinates: the start and length of the small section
3. label the displacements from equilibrium: nominal values on the left, small changes on the right
4. identify any distributed external forces
5. work out expressions for the internal forces on both sides of the small section
6. apply $F = ma$ and take limits

Work out expressions for the internal forces on both sides:

The acoustic case is less familiar. Let the total pressure $p_{\text{total}} = p_0 + p$, so that p represents a small change about the ambient pressure. Small changes in pressure p are related to small changes in volume $\mathrm{d}V$ via the bulk modulus K of air:

$$p = -K\frac{\mathrm{d}V}{V}$$

The volume V and small volume change $\mathrm{d}V$ are

$$V = A\mathrm{d}x$$
$$\mathrm{d}V = A\left(y(x + \mathrm{d}x) - y(x)\right)$$

giving

$$p = -K\frac{\partial y}{\partial x}$$

Apply $F = ma$:

$$\left(-K\frac{\partial y}{\partial x}\Big|_{x} + K\frac{\partial y}{\partial x}\Big|_{x+\delta x}\right)A + f(x,t)A\,dx = \rho A dx \frac{\partial^2 y}{\partial t^2}$$

Use small value approximations and take limits

$$\rho\frac{\partial^2 y}{\partial t^2} - K\frac{\partial^2 y}{\partial x^2} = f(x,t)$$

Summary of Equations of Motion

Wave speeds

Transverse vibration of a stretched string:

$$m\frac{\partial^2 y}{\partial t^2} \quad - \quad P\frac{\partial^2 y}{\partial x^2} \quad = \quad f(x,t)$$

$c = \sqrt{P/m}$

Axial vibration of a rod:

$$m\frac{\partial^2 y}{\partial t^2} \quad - \quad EA\frac{\partial^2 y}{\partial x^2} \quad = \quad f(x,t)$$

$c = \sqrt{EA/m} = \sqrt{E/\rho}$

$\simeq 5.2$ km/s in steel

Torsional vibration of a shaft:

$$\rho J\frac{\partial^2 y}{\partial t^2} \quad - \quad GJ\frac{\partial^2 y}{\partial x^2} \quad = \quad T(x,t)$$

$c = \sqrt{G/\rho}$

$\simeq 3.2$ km/s in steel

Acoustic vibration in a duct:

$$\rho\frac{\partial^2 y}{\partial t^2} \quad - \quad K\frac{\partial^2 y}{\partial x^2} \quad = \quad f(x,t)$$

$c = \sqrt{K/\rho}$

$\simeq 340$ m/s in air

Free vibration and travelling waves

Each of the PDEs just describes the behaviour of the system as if it was infinite in extent, i.e. they don't account for the *boundary conditions*. If we set the external force to zero then the equation describes how waves propagate along the structure.

Taking the stretched string as an example, consider the equation of motion without external forcing:

$$m\frac{\partial^2 y}{\partial t^2} - P\frac{\partial^2 y}{\partial x^2} = 0$$

We can look for solutions using the 'separation of variables' principle:

$$y(x,t) = U(x)T(t)$$

Knowing that this is a vibration problem (and that all the PDE's are second-order in time):

$$y(x,t) = \text{real}\left\{U(x)\mathrm{e}^{i\omega t}\right\}$$

surely $T(t) = Ce^{i\omega t} + De^{-i\omega t}$?

turns out you can choose $D=0$ without loss of generality

To simplify notation, we just allow $y(x)$ to be complex and keep in mind that the actual solution is the real part. This relaxed notation doesn't cause any problems for linear systems, but you do need to be more careful when dealing with non-linear systems (4C7).

This gives the ODE:

$$PU'' + m\omega^2 U = 0 \qquad \left(U'' \equiv \frac{\mathrm{d}^2 U}{\mathrm{d}x^2}\right)$$

The general solution is

$$U(x) = A\,e^{-ikx} + B\,e^{ikx}$$

where A and B are arbitrary constants. Substituting this into the ODE gives

$$Pk^2 = m\omega^2, \text{ ie } \omega^2 = \frac{Pk^2}{m} = \underbrace{c^2}_{\text{wavespeed}} k^2$$

The wavenumber is just the 'spatial frequency' and the wavelength is

$$\lambda = \frac{2\pi}{k}$$

So the overall solution for $y(x, t)$ is:

$$\begin{aligned} y(x,t) &= U(x)\,e^{i\omega t} \\ &= A\,e^{i(\omega t - kx)} + B\,e^{i(\omega t + kx)} \\ &= \underbrace{f(\omega t - kx)}_{\text{forward}} + \underbrace{g(\omega t + kx)}_{\text{backward}} \end{aligned}$$

which is in the form of a travelling wave of speed c.

$$c = \frac{\omega}{k} = \sqrt{\frac{P}{m}} \qquad \text{(only true for non-dispersive waves…)}$$

So free motion that is sinusoidal in time must be sinusoidal in space and have two components travelling forwards and backwards at speed c. We can now revisit the equations of motion for the other systems that we have looked at and find the wave speed by inspection.

But we haven't used the boundary conditions yet so we can't say anything about the modes.

Boundary conditions and modes

When you fix the ends of a string or leave the end of an organ pipe open then you are imposing a *boundary condition*: fixing the system response at a particular location. The effect is to cause incoming waves to reflect (more on that later) and that allows standing waves to occur at particular frequencies: the shape of a standing wave is the *mode shape* and the frequency is the *natural frequency* or the *resonant frequency*.

So a *vibration mode* of a system is a free motion in which all points in the structure move sinusoidally at a particular frequency with a characteristic *mode shape*: i.e. all points moving in equal or opposite phase with each other. It is a particular case of separation of variables where:

$$y(x,t) = U(x)\mathrm{real}\left\{\mathrm{e}^{i\omega t}\right\}$$

with $U(x)$ real-valued. In other words $A = B^*$, or we could write:

$$U(x) = A' \cos kx + B' \sin kx$$

The most obvious choice of boundary condition for a stretched string is both ends pinned:

$y = 0$ at $x = 0$

and $y = 0$ at $x = L$

This allows us to deduce k and A:

$$y(0,t) = 0, \quad A' \cos k0 + B' \sin k0 = 0 \quad \Longrightarrow \quad A' = 0$$

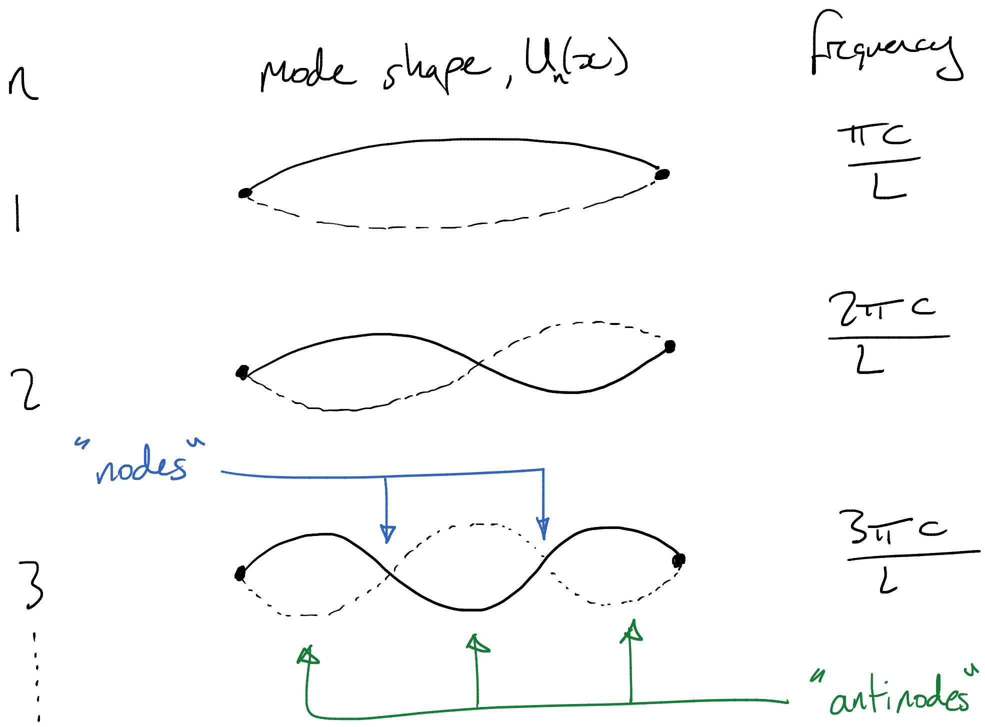
$$y(L,t) = 0, \quad B' \sin kL = 0 \quad \Longrightarrow \quad k = \frac{n\pi}{L}$$

and we know from the travelling wave solution that k_n must be related to ω_n by

$$\omega_n = k_n\sqrt{\frac{P}{m}} = \frac{n\pi}{L}\sqrt{\frac{P}{m}}$$

The constant B' is left undefined as it represents the overall amplitude of vibration, and that depends on the initial conditions. Notice that there are an infinite number of solutions, so there are an infinite number of natural modes of vibration. That makes sense for a continuous structure which can be thought of as having an infinite number of degrees of freedom.

So we end up with a sequence of mode shapes $U_n(x)$ with corresponding natural frequencies:



A mode shape can be multiplied by any constant, corresponding to different amplitudes of vibration. The amplitude will be determined by the initial conditions at the start of the free motion.

Summary

We can find the PDE for vibration of a continuous by considering a small section of the system, resolving forces, applying $F = ma$, and taking limits.

By separation of variables we can find a relationship between the wavenumber k and the frequency $\omega(k)$. This is the dispersion relation and it tells us how waves propagate as if the system was infinitely long.

The examples looked at so far satisfy the wave equation: $\dfrac{\partial^2 y}{\partial t^2} - c^2 \dfrac{\partial^2 y}{\partial x^2} = f(x,t)$

We can use the PDE coefficients to find the wave speed, or get it from the dispersion relation:

$$c = \omega / k$$

When we apply boundary conditions then it is possible to find solutions for free (unforced) vibration of the system: there are an infinite number of possible solutions, each with an associated mode shape $U_n(x)$ and natural frequency ω_n.