4F7-STATISTICAL SIGNAL ANALYSIS

Examples Paper

Exercise 1. The ARMA(2,2) model is a scalar valued stochastic process satisfying

$$X_n = a_1 X_{n-1} + a_2 X_{n-2} + b_0 W_n + b_1 W_{n-1}$$

where $\{W_n\}$ is the driving noise sequence. Express this process in state-space form.

Exercise 2. Write the Gaussian AR(P) (or the ARMA(P,0)) model in state-space form. Assuming the model is in stationarity, find the Gaussian probability density function of the variables $(X_0, \dots, X_{P-1})$. When $P = 2$, find the mean and covariance matrix of this probability density function.

Exercise 3. Prove the results

$$K\left[aX + bU + c \mid Y_{1:n}\right] = aK\left[X \mid Y_{1:n}\right] + bK\left[U \mid Y_{1:n}\right] + c$$

and

$$K\left[X \mid Y_{1:n}\right] = K\left[X \mid Y_{1:n-1}\right] + K\left[X \mid Y_n\right] - \mathbb{E}(X),$$

provided $\mathrm{Cov}(Y_i, Y_n) = 0$ for $i < n$, stated in the lecture notes.

Exercise 4. Consider the state-space model

$$X_n = X_{n-1}$$

$$Y_n = X_n + V_n$$

where $\{V_n\}_n \sim \mathrm{WN}\,(0, r)$, $\mathbb{E}(X_1) = 0$, $\mathbb{E}(X_1^2) = \sigma$. Moreover, X_1 and $\{V_n\}_n$ are uncorrelated. Find $K\left[X_n \mid Y_{1:n}\right]$ and compare the mean square error of this estimate to that of the sample average. Find the limiting mean square error of $K\left[X_n \mid Y_{1:n}\right]$ as $n \to \infty$.

Exercise 5. Consider the Gaussian ARMA(2,2) model in state-space form. Assuming $X_{-2} = X_{-1} = W_{-1} = 0$, find $p(y_0, \ldots, y_n)$ using the Kalman filter.

At a casino a fair die is used but occasionally switches to a biased die. The fair die has probability $1/6$ for each number turning up but the biased die has the following outcome and probability pairs,

$$\{(1, 0.1), (2, 0.1), (3, 0.1), (4, 0.1), (5, 0.1), (6, 0.5)\}$$

that is 6 turns up with probability $1/2$ while the remaining numbers turn up with equal probability. After each role, the next die used is selected with the following probabilities:

$$\Pr(\text{next is fair} \mid \text{current is fair}) = 0.95,$$

$$\Pr(\text{next is biased} \mid \text{current is fair}) = 0.05,$$

$$\Pr(\text{next is fair} \mid \text{current is biased}) = 0.1,$$

$$\Pr(\text{next is biased} \mid \text{current is biased}) = 0.9.$$

The player only observes the outcome of each roll the die and there are no visible distinctions between the fair and the biased die.

Exercise 6. Write down the hidden Markov model that describes the series of outcomes $\{Y_1, Y_2, \ldots\}$ observed by the player.

Exercise 7. Write down the probability of the sequence $(x_1, y_1, \ldots, x_T, y_T)$ where x_n is the type of die used for the nth roll and y_n is the corresponding outcome.

Exercise 8. Let π_n be the column vector of probabilities of the conditional probability mass function of X_n given $Y_1 = y_1, \ldots, Y_n = y_n$. Find $p(x_{n+1} \mid y_{1:n})$ and define the matrix P such that $p(x_{n+1} \mid y_{1:n}) = \pi_n^{\mathrm{T}} P$. Given y_{n+1}, find π_{n+1} and give the matrix B such that

$$\pi_{n+1}^{\mathrm{T}} = \frac{\pi_n^{\mathrm{T}} P B}{\pi_n^{\mathrm{T}} P B \mathbf{1}}$$

where $\mathbf{1}$ is the column vector of ones.

Exercise 9. Given $\beta_n(x_n) = p(y_{n+1}, \ldots, y_T \mid x_n)$, for $n \leq T-1$, find $\beta_{n-1}(x_{n-1})$ and express the relationship between β_{n-1} and β_n using the matrices P and B defined in the previous question. How should β_T be defined so that β_{T-1} and β_T has the same relationship? (Note that β_T has no interpretation as a probability mass function and is merely a suitable initialisation of the procedure.)

Exercise 10. Find an expression for the smoother $p(x_n \mid y_{1:T})$, $n < T$, using π_n and β_n.

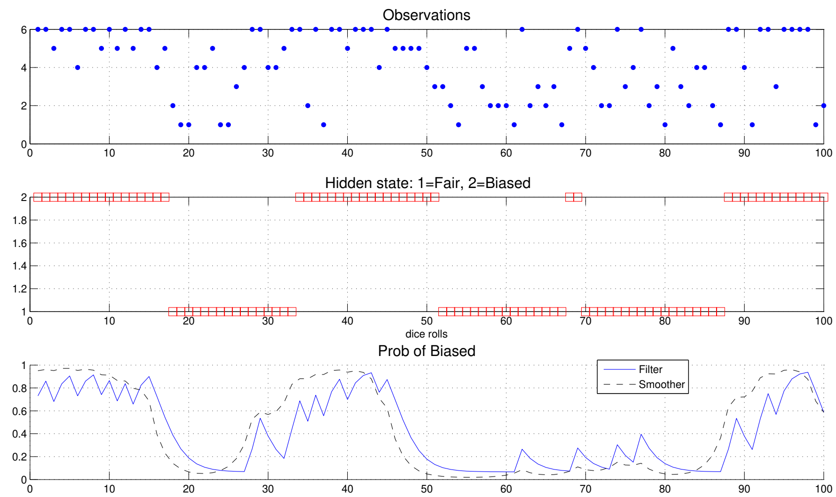

FIGURE 0.1. (Top to bottom.) Observed die rolls, true die used, filtered and smooth estimates of the hidden state.

Exercise 11. Extrapolating from this hidden Markov model with a hidden state process that takes two possible values, give a complete definition of a hidden Markov model with a hidden states process with values in $\{1, \ldots, n\}$ and an observed process with values in $\{1, \ldots, m\}$. (Note that the derived equations for the filter and smoother should also apply to this more general hidden Markov model.)

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