3F1 Signals and Systems

(15) Continuous time random signals

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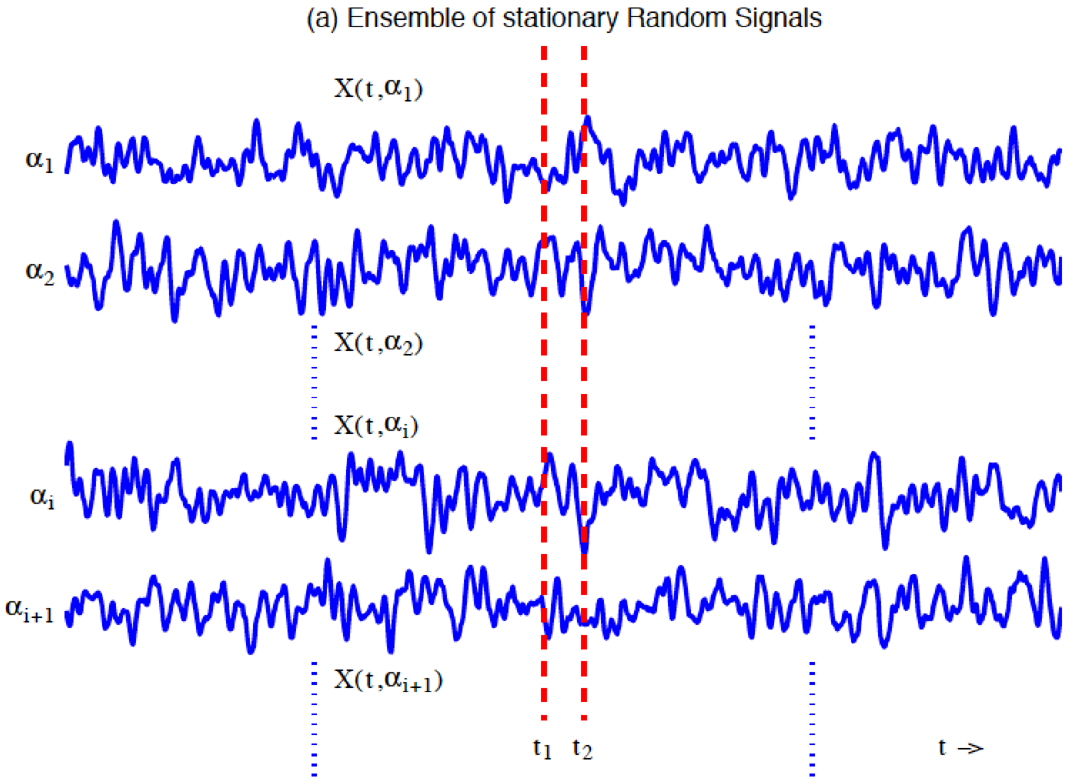
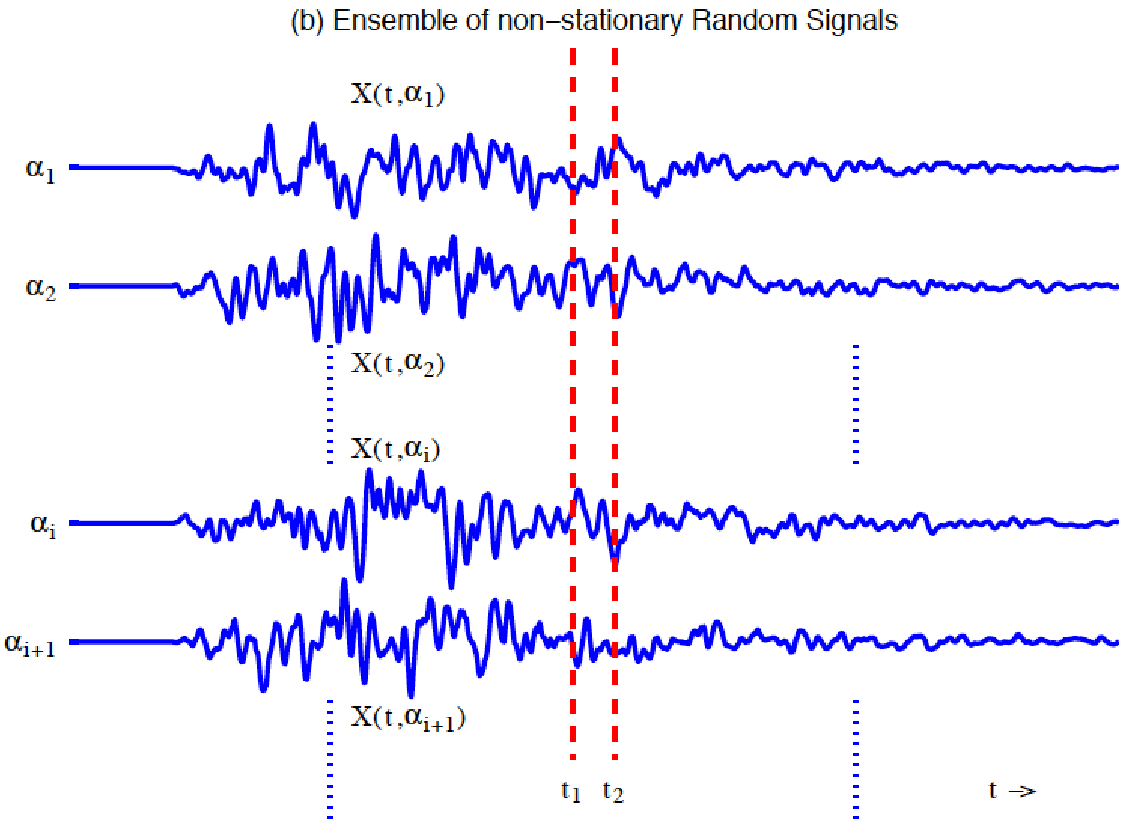
A **random signal** is a time-dependent random variable, $X(t)$. We can formalize this notion by considering the act of drawing randomly from a set of possible signal waveforms, which we call an **Ensemble**. The ensemble of random signals together with the associated probability distributions is known as a **Random Process**.

Notation: formally, we can write $X(t, \alpha)$ to represent an instance of a random process, where α represents a draw from some set, $\mathcal{A}$. The random process is then denoted $\{X(t, \alpha)\}$. In practice, we usually drop the dependence on α for convenience.

Examples of Random Processes $X(t, \alpha)$ are shown in figures (a) and (b), where t is time and α is an index to the various members of the ensemble.



(a) Ensemble of stationary Random Signals
X(t,α1)
α1
α2
X(t,α2)
X(t,αi)
αi
αi+1
X(t,αi+1)
t1
t2
t ->



(b) Ensemble of non–stationary Random Signals
X(t,α1)
α1
α2
X(t,α2)
X(t,αi)
αi
αi+1
X(t,αi+1)
t1
t2
t ->

The first example (a) shows several samples from a process whose statistics do not vary with time (a stationary process), while the second example (b) shows a time-dependent process (different instances of similar speech sounds).

- The members of the ensemble can be the result of different random events, such as different instances of the sound 'ah' during the course of this lecture. In this case is discrete.
- Alternatively the ensemble members are often just different portions of a single random signal. If the signal is a continuous waveform, then may also be a continuous variable, indicating the starting point of each ensemble waveform.

If we consider the process $\{X(t)\}$ at one particular time $t = t1$, then we have a random variable $X(t1)$.

Correlations and covariance

Correlations and covariance characterise the similarity of signals at different points in time.

- **Autocorrelation** is defined as:

  $$r_{XX}(t_1, t_2) = E[X(t_1)X(t_2)] = \int\int x_1 x_2 f(x_1, x_2) dx_1 dx_2$$

  where x_1, x_2 are the values of the random process sampled at times t_1, t_2 respectively and f is the corresponding joint pdf.
- **Autocovariance** is autocorrelation with mean subtracted:

$$\begin{aligned} c_{XX}(t_1, t_2) &= E[(X(t_1) - \bar{X}(t_1))(X(t_2) - \bar{X}(t_2))] \\ &= \int\int (x_1 - \bar{X}(t_1))(x_2 - \bar{X}(t_2)) f(x_1, x_2) dx_1 dx_2 \end{aligned}$$

$$
\begin{aligned}
&= \int \int (x_1 - \bar{X}(t_1))(x_2 - \bar{X}(t_2)) f(x_1, x_2) dx_1 dx_2 \\
&= r_{XX}(t_1, t_2) - \bar{X}(t_1) \int x_2 f_2(x_2) dx_2 \\
&- \bar{X}(t_2) \int x_1 f_1(x_1) dx_1 + \bar{X}(t_1)\bar{X}(t_2) \\
&= r_{XX}(t_1, t_2) - \bar{X}(t_1)\bar{X}(t_2)
\end{aligned}
$$

- **Cross correlation** for two different processes, X and Y:

$$r_{XY}(t_1, t_2) = E[X(t_1)Y(t_2)] = \int \int x_1 y_2 f(x_1, y_2) dx_1 dy_2$$

  where f is the joint pdf when x_1 and y_2 are samples of X and Y at times t_1 and t_2.

- **Cross covariance**:

$$
\begin{aligned}
c_{XY}(t_1, t_2) &= E[(X(t_1) - \bar{X}(t_1))(Y(t_2) - \bar{Y}(t_2))] \\
&= r_{XY}(t_1, t_2) - \bar{X}(t_1)\bar{Y}(t_2)
\end{aligned}
$$

In all cases, expectations are over the **whole ensemble**.

Note: for processes $\{X(t, \alpha)\}$ in which X (and Y) depend deterministically on the random variable α (see examples sheet, questions 22, 23), the above formulas simplify. For example, autocorrelation becomes:

$$r_{XX}(t_1, t_2) = E[X(t_1, \alpha)X(t_2, \alpha)] = \int_{\mathcal{A}} x(t_1, \alpha)x(t_2, \alpha)f(\alpha)d\alpha$$

where f is the pdf of α.

Stationarity

Stationarity means that the statistical characteristics of a signal do not change over time. There are different degrees of stationarity:

- **Strict Sense Stationary (SSS)**: a process is Strict Sense Stationary if its probability distribution does not change in time. Formally, X is SSS iff for all finite N and all sets of time points $\{t_1, ..., t_N\}$, the cdf, F, of the vector $(X(t_1), ..., X(t_N))$ is invariant for all time shifts, T:

$$F_{X(t_1),\ldots,X(t_N)}(x_1, ..., x_N) = F_{X(t_1+T),\ldots,X(t_N+T)}(x_1, ..., x_N)$$

- **Wide Sense Stationary (WSS)**: a process is Wide Sense Stationary (or 'weakly stationary') iff:
  1. The mean value is independent of time:

$$E[X(t)] = \mu \qquad \text{for all } t$$

  2. The autocorrelation depends only on $\tau = t_2 - t_1$:

$$E[X(t_1)X(t_2)] = E[X(t_1)X(t_1 + \tau)] = r_{XX}(\tau) \quad \text{for all } t_1$$

Note: $SSS \implies WSS$ but not conversely.

Ergodicity
If we can exchange *ensemble averages* for *time averages*, we say a process is **Ergodic**. This means the ensemble of a process is simply composed of all possible time shifts of a single random signal. We define the time average as:

$$\langle X(t) \rangle_T = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} X(t) dt$$

This leads to the following properties of WSS Ergodic processes, X

- **mean ergodic**:

$$\langle X(t) \rangle_T = E[X(t)] = \int x f_{X(t)}(x) dx$$

- **correlation ergodic**:

$$\begin{aligned} \langle X(t)X(t+\tau) \rangle_T &= r_{XX}(\tau) \\ &= E[X(t)X(t+\tau)] \\ &= \int\int x_1 x_2 f_{X(t),X(t+\tau)}(x_1, x_2) dx_1 dx_2 \end{aligned}$$

and similarly for other correlation/covariance functions. Ergodicity greatly simplifies the measurement of WSS processes. In reality, no process is truly stationary (therefore cannot be Ergodic) but many noise processes are approximately stationary for finite periods. Thus we may estimate quantities such as the autocorrelation function:

$$r_{XX}(\tau) \approx \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} X(t)X(t+\tau) dt$$

Spectral properties of Random Signals

The autocorrelation function (ACF) of an ergodic random signal tells us how correlated it is with itself as a function of time shift and, as you will verify in the example sheet (Q 24), such an ACF is symmetric:

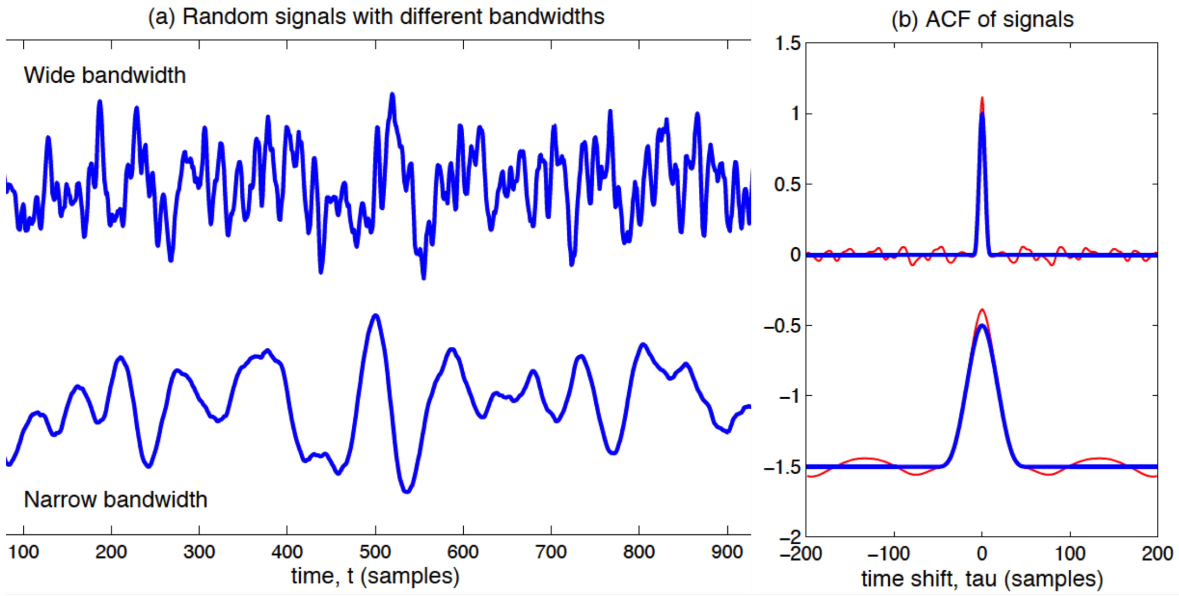
$$r_{XX}(\tau) = r_{XX}(-\tau)$$

Moreover, as τ becomes large, $X(t)$ and $X(t+\tau)$ will tend to become decorrelated. If X has mean $\mu = 0$, r_{XX} will tend to zero (otherwise it tends to μ^2).

Hence, the ACF **has a maximum at 0**. Its **width** tells us **how slowly the signal fluctuates in time**. The Figure below illustrates this for two different random signals.

Note also: for $\tau = 0$:

$$r_{XX}(0) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} X(t)^2 dt = \text{mean power of } X(t)$$



The bold ACF traces are the (true) calculated ACF functions, while the thin (red) traces are numerical estimates based on 4000 waveforms.

Power Spectral Density

As with deterministic signals, we can characterise rate of fluctuation by transforming to the **frequency (spectral) domain** using the **Fourier Transform**:

$$\mathcal{F}_u(\omega) = FT\{u(t)\} = \int_{-\infty}^{\infty} u(t) e^{-j\omega t} dt$$

The **Power Spectral Density** (PSD), $\mathcal{S}_X$ of a random process X is the Fourier transform of the Autocorrelation function:

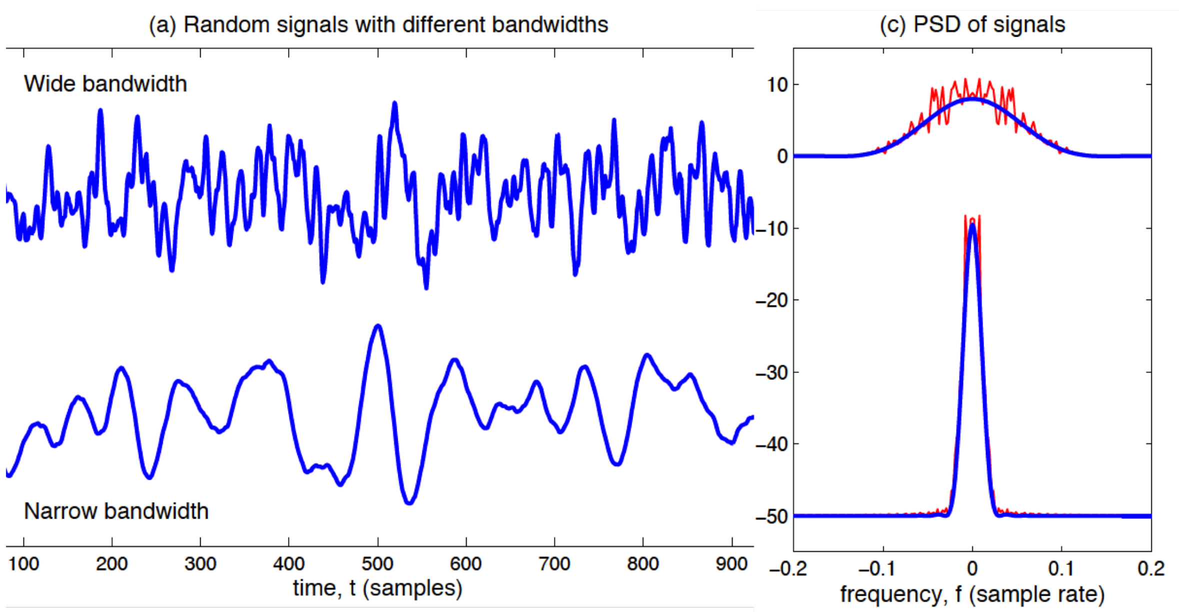
$$\mathcal{S}_X(\omega) = FT\{r_{XX}(\tau)\} = \int r_{XX}(\tau) e^{-j\omega\tau} d\tau$$

Clearly, X must be at least WSS for this to be valid.

Inverting the Fourier Transform, and evaluating the ACF at 0, we see that the mean power of the signal is:

$$r_{XX}(0) = \frac{1}{2\pi} \int \mathcal{S}_X(\omega) d\omega = \int \mathcal{S}_X(2\pi f) df$$

(note: integral is over all frequencies, including negative frequencies!) Thus $\mathcal{S}_X$ has units of power per Hertz.



The bold PDS traces are the (true) calculated PSDs, the thin (red) traces are numerical estimates based on 4000 waveforms.