Cambridge University Engineering Dept. Third year

Module 3F2: Systems and Control

LECTURE NOTES 1: STATE-SPACE SYSTEMS

Contents

K. Glover Revised by G.Vinnicombe
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1 State Space Descriptions Of Dynamical Systems

1.1 States

The essence of a dynamical system is its memory, i.e. the present output, $y(t)$ depends on past inputs, $u(\tau)$, for $\tau \leq t$.

Whereas in a static system $y(t)$is a function of $u(\tau)$ at $\tau = t$ only.

Three sets of variables define a dynamical system -

1. The inputs, $u_1(t)$, $u_2(t)$, ...$u_m(t)$ (input vector $\underline{u}(t)$)
2. The state variables, $x_1(t)$, ...$x_n(t)$ (state vector $\underline{x}(t)$)
3. The outputs, $y_1(t)$, $y_2(t)$, ...$y_p(t)$, (output vector $\underline{y}(t)$)

The states may depend on $\underline{u}(\tau)$ and $\underline{x}(\tau)$ for $\tau \leq t$. However for any $t_o < t_1$, $\underline{x}(t_1)$ can always be determined from $\underline{x}(t_o)$ and $\underline{u}(\tau)$, $t_o \leq \tau \leq t_1$. [NB this is the definition of system state] That is $\underline{x}(t_o)$ summarizes the effect on the future of inputs and states prior to t_o.

The output, $\underline{y}$ at time t, is a memoryless function of $\underline{x}(t)$ and $\underline{u}(t)$.

The most general class of dynamical system that we will consider is described by the set of first order ordinary differential equations -

$$\mathcal{S} \left\{ \begin{array}{rcll} \dfrac{dx_i}{dt} & = & f_i(x_1(t), x_2(t), ...x_n(t), u_1(t), ...u_m(t), t), & i = 1, 2, ...n. \\ y_j(t) & = & g_j(x_1(t), ...x_n(t), u_1(t), ...u_m(t), t) & j = 1, 2, ...p. \end{array} \right.$$

$\mathcal{S}$ is the standard form for a **state-space dynamical system model**. Or a vector form -

$$\mathcal{S} \left\{ \begin{array}{rcl} \underline{\dot{x}}(t) & = & \underline{f}(\underline{x}(t), \underline{u}(t), t) \\ \underline{y}(t) & = & \underline{g}(\underline{x}(t), \underline{u}(t), t) \end{array} \right.$$

Note that $\mathcal{S}$ has only first order ode's, but we can use a standard technique to convert high order o.d.e's to first order vector o.d.e.'s, by the use of auxiliary variables.

Ex:

$\dddot{y} + 6y\ddot{y} + 5(\dot{y})^3 + 12\sin(y) = \cos(t)$

a state variable is given by: $x_1 = y$, $x_2 = \dot{y}$, $x_3 = \ddot{y}$, when,

$$\left\{ \begin{array}{rcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & x_3 \\ \dot{x}_3 & = & -6x_1x_3 - 5x_2^3 - 12\sin(x_1) + \cos(t) \end{array} \right.$$

which is in the form

$$\underline{\dot{x}}(t) = \underline{f}(\underline{x}(t), t)$$

where $\underline{f}$ is a vector valued function of a vector, i.e.

$$\underline{f}(\underline{x}, t) = \left[\begin{array}{l} x_2 \\ x_3 \\ -6x_1x_3 - 5x_2^3 - 12\sin(x_1) + \cos(t) \end{array} \right]$$

For linear time-invariant dynamical systems we use the standard form:

$$\mathcal{S}\begin{cases} \underline{\dot{x}}(t) &= A\underline{x}(t) + B\underline{u}(t) \\ \underline{y}(t) &= C\underline{x}(t) + D\underline{u}(t) \end{cases}$$

1.2 General Guidelines For State-Space Modelling

Method 1: Choose system states, $x_1, x_2, ...x_n$ by considering the independent 'energy storage devices' or 'memory elements',

e.g.

electrical circuits - current in L or voltage on C.

mechanics - positions and velocities of masses (linear and angular).

chemical engineering - temperature, pressure, volume, concentration.

Then use the basic physical laws derive expressions involving $\dot{x}_i$, e.g.

- $V = Ldi/dt, i = Cdv/dt$
- $m \times \text{acc}^n$ = force; $\dfrac{d(\text{pos}^n)}{dt}$ = velocity.
- $\dfrac{d(\text{volume})}{dt}$ = flow = function of pressure.

Solve for $\underline{\dot{x}} = \underline{f}(\underline{x}, \underline{u}, t)$.

Method 2: Choose state variables as successive time derivatives.

Example: Original ODE: $J\ddot{\theta} + B\dot{\theta} = M$.
Let $x_1 = \theta$, $x_2 = \dot{\theta}$. Then

In general for n'th-order ODE in θ define

$$x_1 = \theta, \quad x_2 = \frac{d\theta}{dt}, \quad \ldots, \quad x_n = \frac{d^{n-1}\theta}{dt^{n-1}}$$

1.3 Ideal Operational Amplifier Circuits

In a negative feedback configuration the amplifier acts so as to make the +ve and -ve inputs have essentially equal voltages.

$$\begin{cases} C_1\dot{x}_1 = (V_i - x_1)/R_1; \quad C_2\dot{x}_2 = x_1/R_2 \\ V_o = x_1 + x_2 \end{cases}$$

Linear operation until the amplifier *saturates* when quite different equations may hold.

If the amplifier output saturates at $\pm V_s$, then for $|x_1 + x_2| \geq V_s$

we will have: $C_2\dot{x}_2 = (\pm V_s - x_2)/R_2$.

1.4 Velocity Fields In The State-Space

Consider the free motion of the time invariant dynamical system -

$$\underline{\dot{x}} = \underline{f}(\underline{x}) \quad \Longrightarrow \quad \underline{x}(t+\delta t) \cong \underline{x}(t) + \underline{f}(\underline{x}(t))\delta t$$

This implies a velocity field in the state space which can give a good qualitative idea of the system's behaviour in simple cases.

<u>Ex</u> n=2

The velocity field can be sketched by drawing an arrow in the direction $\underline{f}(\underline{x})$ for many values of $\underline{x}$. Equilibrium points (or *singular* points) where $\underline{f}(\underline{x}_e) = \underline{0}$, may be stable or unstable.

<u>Ex.</u>

$$\begin{aligned} -L\frac{dx_1}{dt} &= x_1 R + x_2 \\ C\frac{dx_2}{dt} &= x_1 \\ \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} -R/L & -1/L \\ 1/C & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned}$$

i) Let $R = 1/2,\ L = 1,\ C = 1$ (in consistent units) then

$$A = \begin{bmatrix} -\frac{1}{2} & -1 \\ 1 & 0 \end{bmatrix}$$

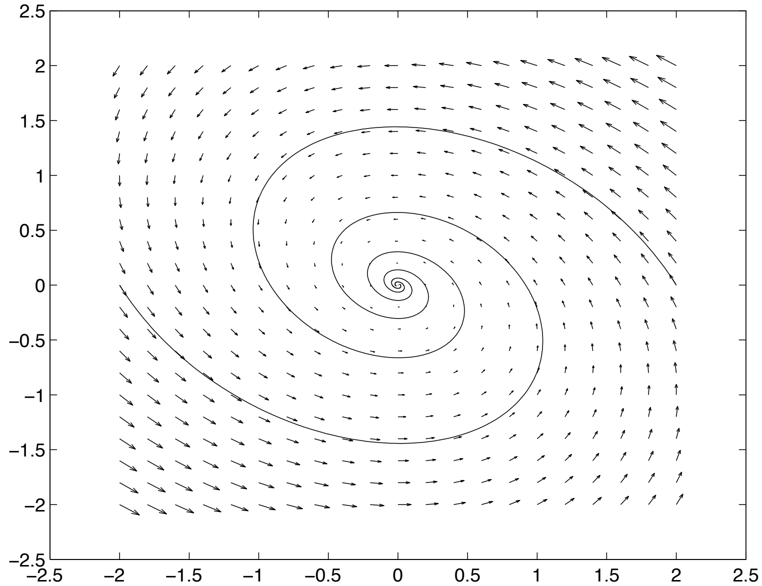

Figure 1: State space trajectories: underdamped RLC network

(ii) Let $R = 4, L = 1, C = 1 \Rightarrow A = \begin{bmatrix} -4 & -1 \\ 1 & 0 \end{bmatrix}$

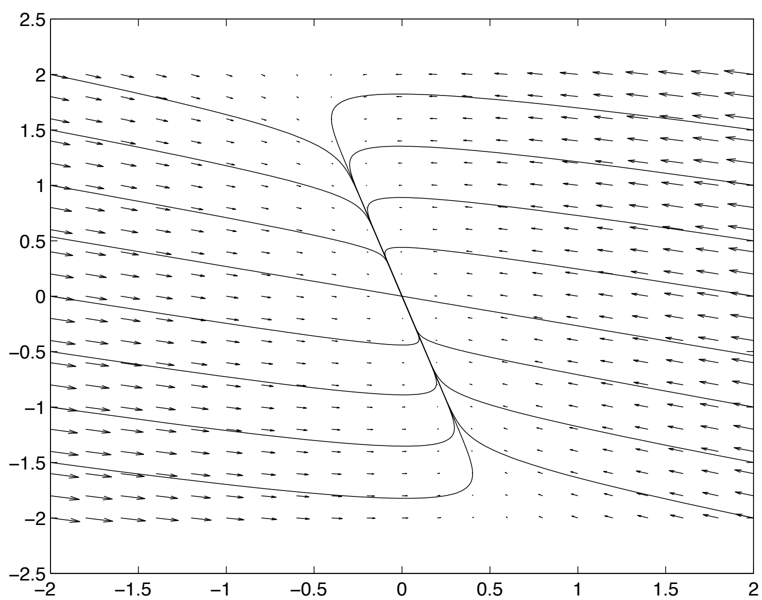

Figure 2: State space trajectories: overdamped RLC network

The MATLAB code to produce these two sets of state-plane trajectories is as follows:

```
% A=[-.5 -1; 1 0]; % for underdamped example
A=[-4 -1; 1 0];  % for overdamped example

hold off

[x,y]=meshgrid(-2:.2:2,-2:.2:2);
xdot=A(1,1)*x+A(1,2)*y;
ydot=A(2,1)*x+A(2,2)*y;

quiver(x,y,xdot,ydot); drawnow; hold on

for x0=[-2 2],
for y0=[-2 -1.5 -1 -4+2*sqrt(3)   0 .5 1 1.5];
xx=[[x0;y0*sign(x0)],zeros(2,1000)];
for i=1:1000; xx(:,i+1)= (eye(2)+.02*A)*xx(:,i); end;
plot(xx(1,:)',xx(2,:)','r')
drawnow
end
end
```

2 Linearizing Nonlinear Dynamical Systems

2.1 Linearizing the standard form

Suppose we have a nonlinear (time invariant) dynamical system in state space form -

$$\mathcal{S}\begin{cases} \underline{\dot{x}} &= \underline{f}(\underline{x},\underline{u}) \\ \underline{y} &= \underline{g}(\underline{x}) \end{cases}$$

Let $\underline{x}_e$ be an equilibrium state for the system when $\underline{u}(t) = \underline{u}_e$ (constant)

i.e. $\underline{f}(\underline{x}_e, \underline{u}_e) = \underline{0}$ and also let $\underline{y}_e = \underline{g}(\underline{x}_e)$.

Now consider small perturbations from this equilibrium, let

$$\underline{x}(t) = \underline{x}_e + \underline{\delta x}(t), \quad \underline{u}(t) = \underline{u}_e + \underline{\delta u}(t), \quad \underline{y}(t) = \underline{y}_e + \underline{\delta y}(t)$$

A Taylor Series expansion of the i-th equation of $\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u})$ gives

$$
\begin{aligned}
\dot{x}_i &= \dot{x}_{ei} + \dot{\delta x_i} \\
&= f_i(\underline{x}_e, \underline{u}_e) + \left.\frac{\partial f_i}{\partial x_1}\right|_{\underline{x}_e, \underline{u}_e} \delta x_1 + \left.\frac{\partial f_i}{\partial x_2}\right|_{\underline{x}_e, \underline{u}_e} \delta x_2 + ... + \left.\frac{\partial f_i}{\partial x_n}\right|_{\underline{x}_e, \underline{u}_e} \delta x_n \\
&\quad + \left.\frac{\partial f_i}{\partial u_1}\right|_{\underline{x}_e, \underline{u}_e} \delta u_1 + ... + \left.\frac{\partial f_i}{\partial u_m}\right|_{\underline{x}_e, \underline{u}_e} \delta u_m + \text{Remainder}
\end{aligned}
$$

Thus for small $\underline{\delta x}$ and $\underline{\delta u}$ ($\Rightarrow$ v. small remainder) we get

$$
\dot{\underline{\delta x}} \cong A\underline{\delta x} + B\underline{\delta u}
$$

where

$$
A = \frac{\partial \underline{f}}{\partial \underline{x}}(\underline{x}_e, \underline{u}_e) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \cdots & & \frac{\partial f_2}{\partial x_n} \\ \vdots & & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & & \frac{\partial f_n}{\partial x_n} \end{bmatrix}_{\underline{x}=\underline{x}_e, \underline{u}=\underline{u}_e}
$$

$$
B = \frac{\partial \underline{f}}{\partial \underline{u}}(\underline{x}_e, \underline{u}_e) = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \cdots & \frac{\partial f_1}{\partial u_m} \\ \vdots & & & \vdots \\ \frac{\partial f_n}{\partial u_1} & \cdots & & \frac{\partial f_n}{\partial u_m} \end{bmatrix}_{\underline{x}=\underline{x}_e, \underline{u}=\underline{u}_e}
$$

Similarly $\quad \underline{\delta y} = C\underline{\delta x}$ where $\quad C = \frac{\partial \underline{g}(\underline{x}_e)}{\partial \underline{x}}$.
The linearized system equations are thus

$$
\begin{cases} \dot{\underline{\delta x}} = A\underline{\delta x} + B\underline{\delta u} \\ \underline{\delta y} = C\underline{\delta x} \end{cases}
$$

which will accurately predict the system behaviour for $\underline{x}$ and $\underline{u}$ close to $\underline{x_e}$ and $\underline{u_e}$ respectively.

2.2 Linearizing when the State Equations are Implicit

Quite often a system's equations cannot easily be written as $\underline{\dot{x}} = \underline{f}(\underline{x}, \underline{u})$ but can be written as

$$\underline{F}(\underline{\dot{x}}, \underline{x}, \underline{u}) = \underline{0} \quad (n \text{ equations in the } n \text{ unknowns } \dot{x}_1 ... \dot{x}_n.)$$

The linearized model can be derived without solving for $\underline{\dot{x}}$ as follows. Since $(\underline{x}_e, \underline{u}_e)$ gives an equilibrium we have,

$$\underline{F}(\underline{0}, \underline{x}_e, \underline{u}_e) = \underline{0}$$

Now linearize $\underline{F}$ about $(\underline{0}, \underline{x}_e, \underline{u}_e)$ to get

$$\underbrace{\left.\frac{\partial \underline{F}}{\partial \underline{\dot{x}}}\right|_{(\underline{0}, \underline{x}_e, \underline{u}_e)}}_{L} \dot{\overline{\delta \underline{x}}} + \underbrace{\left.\frac{\partial \underline{F}}{\partial \underline{x}}\right|_{(\underline{0}, \underline{x}_e, \underline{u}_e)}}_{M} \delta \underline{x} + \underbrace{\left.\frac{\partial \underline{F}}{\partial \underline{u}}\right|_{(\underline{0}, \underline{x}_e, \underline{u}_e)}}_{N} \delta \underline{u} \simeq \underline{0}$$

$$L \dot{\underline{\delta x}} + M \underline{\delta x} + N \underline{\delta u} \simeq \underline{0} \quad \Longrightarrow \quad \dot{\underline{\delta x}} \simeq -L^{-1} M \underline{\delta x} - L^{-1} N \underline{\delta u}$$

(N.B. no nonlinear equations to solve except to obtain the equilibrium point)

2.3 Behaviour of Nonlinear Systems

As mentioned above the linearized equations will accurately predict the behaviour of the nonlinear system for $\underline{x}$ and $\underline{u}$ close to their equilibrium values. When the states and inputs are far away from the equilibrium values then the behaviour can be quite different, e.g.

- many equilibria with some stable and some unstable (e.g. inverted pendulum).
- limit cycles.
- divergence.

Example: Van der Pol oscillator

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + (1 - x_1^2)x_2 \end{cases}$$

This has a stable *Limit Cycle* and an unstable equilibrium at $\underline{x} = \underline{0}$.

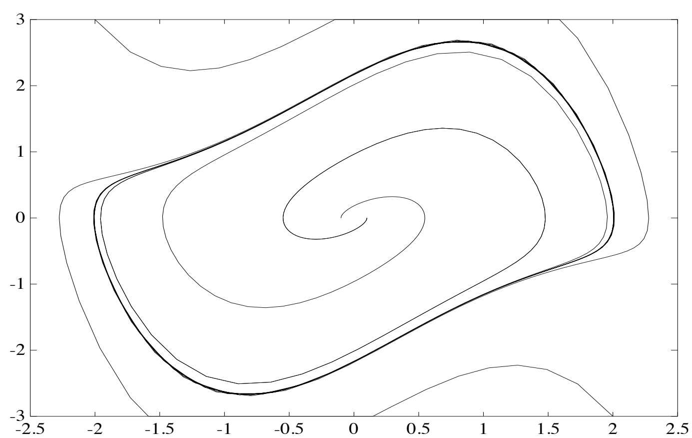

Figure 3: State space trajectories for Van der Pol Oscillator

Note that if we were to change the signs on the $\dot{x}_1$ and $\dot{x}_2$ terms then this will just change the directions of the arrows in the state space. Hence the origin would then be a stable equilibrium and the limit cycle would be unstable (i.e. if perturbed from the limit cycle then it would either decay to the origin or diverge to infinity).

2.3.1 Example

Figure 2 shows a design for a hydraulically actuated table for simulating earthquakes. The table is denoted as ABC, with the point C constrained to move horizontally. DA and EB denote hydraulic rams which are pin-jointed at each end and can produce forces F_1 and F_2, respectively. The equations of motion (which should *not* be verified) are:

$$M\ddot{z} = F_1 \cos\phi_1 + F_2 \sin\phi_2$$

and

$$\begin{aligned}\frac{2}{a}\left(I + \frac{1}{4}Ma^2\cos^2\theta\right)\ddot{\theta} &= M\cos\theta\left(\frac{1}{2}a\dot{\theta}^2\sin\theta - g\right) + \\ &+ \left[\sin(\theta+\phi_1) + \sin\phi_1\cos\theta\right]F_1 + \\ &+ (\cos\theta\cos\phi_2)\,F_2\end{aligned}$$

where

$$\tan\phi_1 = \frac{a\sin\theta}{a + z - \frac{1}{2}a\cos\theta}, \quad \tan\phi_2 = \frac{z}{a + \frac{1}{2}a\sin\theta},$$

M and I are constants, a and z are the lengths shown in Fig. 2, and θ, ϕ_1, ϕ_2 are the angles shown in the figure.

(a) What conditions are satisfied at an equilibrium? Determine values F_{1e} and F_{2e} of the forces F_1 and F_2, which will give an equilibrium position $\theta = \theta_e$ and $z = z_e$, if $\theta_e = 0$ and $z_e = a$.

(b) The linearised equations about the equilibrium $(\theta_e = 0, z_e = a)$ are:

$$\dot{\underline{x}} = A\underline{x} + B\underline{u}$$

where

$$\underline{x} = \left[\theta, z-a, \dot{\theta}, \dot{z}\right]^T, \ \underline{u} = \left[F_1 - F_{1e}, F_2 - F_{2e}\right]^T, \ A = \begin{bmatrix} 0 & I_2 \\ P & 0 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ Q \end{bmatrix},$$

$$P = \begin{bmatrix} \dfrac{-13}{12\tau^2} & -\dfrac{1}{2a\tau^2} \\ -\dfrac{g}{4} & \dfrac{g}{2a} \end{bmatrix}, \ Q = \begin{bmatrix} 0 & \dfrac{1}{\sqrt{2}Mg\tau^2} \\ \dfrac{1}{M} & \dfrac{1}{\sqrt{2}M} \end{bmatrix}, \ \tau^2 = \frac{2I}{Mag} + \frac{a}{2g},$$ and I_2 is the 2×2 identity matrix.

Verify that the term $-\frac{g}{4}$, which appears in P, is correct. (Do *not* verify any other terms. Assume that the nonlinear equations are correct.)

(c) Is the linearised system of part (b) controllable from $\underline{u}$? Is it controllable from u_1 (the first element of $\underline{u}$) alone?

(d) Comment on the difference in the achievable behaviour of this system when only u_1 is available for control, and when the complete vector $\underline{u}$ is available.

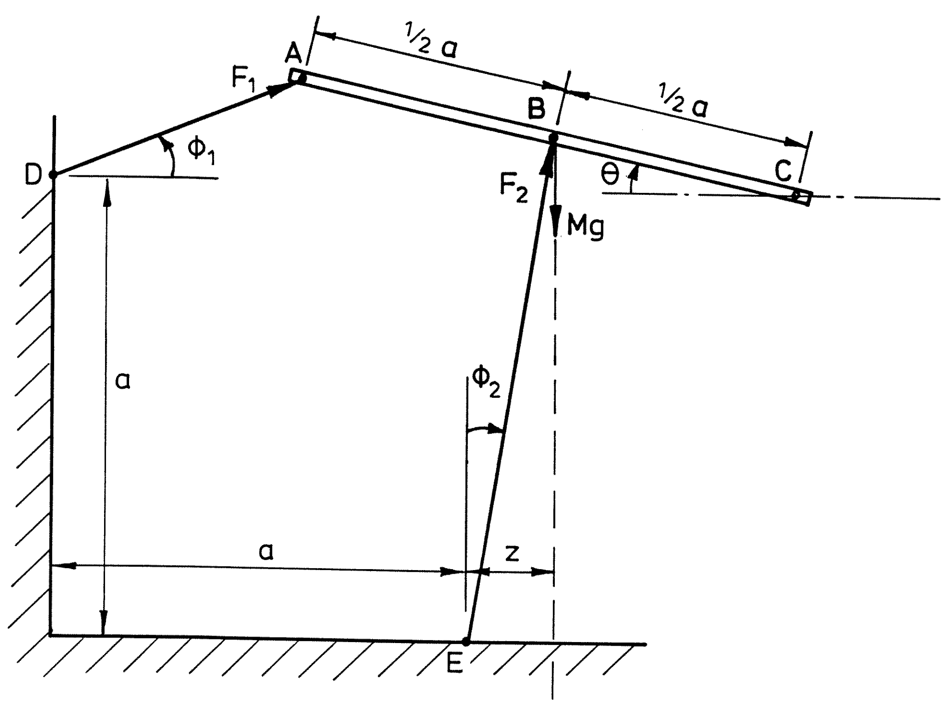

Fig. 2

3 Solutions of Linear State Equations

3.1 Using Laplace Transforms

Taking Laplace Transforms of

$$\begin{aligned}\underline{\dot{x}}(t) &= A\underline{x}(t) + B\underline{u}(t); \quad \underline{x}(0) = \underline{x}_0 \\ \underline{y}(t) &= C\underline{x}(t) + D\underline{u}(t)\end{aligned}$$

gives

$$\begin{aligned}s\underline{X}(s) - \underline{x}_0 &= A\underline{X}(s) + B\underline{U}(s) \\ (sI - A)\underline{X}(s) &= \underline{x}_0 + B\underline{U}(s) \\ \underline{X}(s) &= (sI - A)^{-1}\underline{x}_0 + (sI - A)^{-1}B\underline{U}(s) \\ \underline{Y}(s) &= \underbrace{C(sI - A)^{-1}\underline{x}_0}_{\text{initial condition response}} + \underbrace{(D + C(sI - A)^{-1}B)\underline{U}(s)}_{\text{input response}}\end{aligned}$$

For $\underline{x}_0 = \underline{0}$, $\underline{Y}(s) = \underbrace{(D + C(sI - A)^{-1}B)}_{G(s)}\underline{U}(s)$

and

$$\boxed{G(s) = D + C(sI - A)^{-1}B}$$

is called the **transfer function matrix**.

The i, j^{th} entry of $G(s)$ gives the transfer function from u_j to y_i.

3.2 Transfer function poles

Poles are values of s at which the transfer function becomes infinite:

$$\|G(p)\| = \infty \Rightarrow p \text{ is a pole of } G(s)$$

This can only happen when the matrix $(sI - A)$ becomes singular, ie when

$$\det(sI - A) = 0$$

namely at the eigenvalues of A.

Later we will see that eigenvalues of A are not always poles of $G(s)$.

Hence we have the important result:

$$\boxed{\text{Poles of } G(s) \subset \text{eigenvalues of } A}$$

Analytical expression for transfer function matrix

It can be shown that, for any matrix M,

$$M^{-1} = \frac{1}{\det M}\begin{bmatrix} M_{11} & M_{21} & \dots & M_{n1} \\ M_{12} & & & \\ \vdots & & & \\ M_{1n} & \dots & & M_{nn} \end{bmatrix}$$

where M_{ij} is called the cofactor of m_{ij} given by

$$M_{ij} = (-1)^{i+j} \det(M \text{ with } i\text{-th row and } j\text{-th column deleted})$$

Hence $\quad (sI - A)^{-1} = \dfrac{1}{\alpha(s)} N(s) \quad$ where $\quad \begin{aligned} \alpha(s) &= \det(sI - A) \\ &= s^n + \alpha_1 s^{n-1} + \cdots + \alpha_{n-1} s + \alpha_n \end{aligned}$

and $N(s) = N_1 s^{n-1} + N_2 s^{n-2} + \cdots + N_{n-1} s + N_n$

The transfer function can therefore be written as $\quad G(s) = \frac{1}{\alpha(s)}\left(CN(s)B + D\alpha(s)\right)$ with $(CN(s)B + D\alpha(s))$ being a matrix of polynomials in s.

Cayley-Hamilton Theorem

$$\boxed{A^n + \alpha_1 A^{n-1} + \cdots + \alpha_n I = 0}$$

since

$$\alpha(s)I = (sI - A)\left(N_1 s^{n-1} + N_2 s^{n-2} + \cdots + N_n\right)$$

Equating coefficients of s^k and premultiplying by A^k for $k = n, .., 1, 0$ gives

$$\begin{array}{ccccc}
s^n & : & A^n I & = & A^n N_1 \\
s^{n-1} & : & A^{n-1}\alpha_1 I & = & A^{n-1}N_2 - A^{n-1}AN_1 \\
\vdots & & & \vdots & \\
s & : & A\alpha_{n-1} I & = & AN_n - A^2 N_{n-1} \\
s^0 & : & \alpha_n I & = & -AN_n \\
\hline
\end{array}$$

Adding these equalities gives the result.

Hence *any* power of A is a linear combination of $I, A, A^2, \ldots, A^{n-1}$ — only!

3.3 Initial Condition Response of the State

For $\underline{u}(t) = \underline{0}$ we have $\dot{\underline{x}}(t) = A\underline{x}(t)$ and hence,

$$\underline{X}(s) = (sI - A)^{-1}\, \underline{x}_0$$

$\Longrightarrow$

$$\underline{x}(t) = \mathcal{L}^{-1}\left((sI - A)^{-1}\right)\underline{x}_0 = \Phi(t)\underline{x}_0$$

where

$$\begin{aligned}
\Phi(t) &= \mathcal{L}^{-1}\left\{(sI - A)^{-1}\right\} \\
&= \mathcal{L}^{-1}\left\{\sum_{k\geq 0} A^k s^{-(k+1)}\right\} \\
&= I + At + A^2t^2/2! + \cdots + A^k t^k/k! + \cdots \\
&\overset{def}{=} e^{At}
\end{aligned}$$

Note that

$$\frac{d}{dt}\Phi(t) = Ae^{At} = e^{At}A$$

(check by differentiating series expansion of $\Phi(t)$).

Hence with $\underline{x}(t) = \Phi(t)\underline{x}_0$,

$$\begin{aligned}
\frac{d}{dt}\underline{x}(t) = \frac{d}{dt}\left\{\Phi(t)\underline{x}_0\right\} &= Ae^{At}\underline{x}_0 = A\left\{\Phi(t)\underline{x}_0\right\} = A\underline{x}(t) \\
\Phi(0) &= I \\
\underline{x}(0) &= \underline{x}_0
\end{aligned}$$

and the differential equation and initial condition are satisfied as required.

$\Phi(t)$ is called the **state transition matrix**.

Properties of e^{At}

(1) **Change of state coordinates**

If

$$\begin{aligned}
A &= T^{-1}\bar{A}T \\
\text{then}\quad A^2 &= T^{-1}\bar{A}TT^{-1}\bar{A}T = T^{-1}\bar{A}^2T \\
A^k &= T^{-1}\bar{A}^kT \\
\text{hence}\quad e^{At} &= \sum_{k=0}^{\infty} A^k t^k/k! \\
&= \sum_{k=0}^{\infty} T^{-1}\bar{A}^kTt^k/k! \\
&= T^{-1}\left(\sum_{k=0}^{\infty}\bar{A}^kt^k/k!\right)T
\end{aligned}$$

$$\Rightarrow \boxed{e^{At} = T^{-1}e^{\bar{A}t}T}$$

Why is this 'change of coordinates'?

If $\underline{\dot{x}}(t) = A\underline{x}(t)$ and $\underline{z} = T\underline{x}$, then

$$\begin{aligned}
\underline{\dot{z}}(t) &= T\underline{\dot{x}}(t) \\
&= TA\underline{x}(t) \\
&= TAT^{-1}\underline{z}(t) \\
&= \bar{A}\underline{z}(t) \\
\underline{z}(t) &= e^{\bar{A}t}\underline{z}(0) \\
\underline{x}(t) &= T^{-1}e^{\bar{A}t}T\underline{x}(0)
\end{aligned}$$

Special case: Eigenvectors as coordinate axes

Recall that for W the matrix of eigenvectors of A, then the defining relations,

$$\begin{aligned}
A\underline{w}_1 &= \underline{w}_1\lambda_1 \\
A\underline{w}_2 &= \underline{w}_2\lambda_2 \\
&\vdots \\
A\underline{w}_n &= \underline{w}_n\lambda_n.
\end{aligned}$$

can be written as: $$A\underbrace{\left[\underline{w}_1, \underline{w}_2, ..., \underline{w}_n\right]}_{W} = \underbrace{\left[\underline{w}_1, \underline{w}_2, ..., \underline{w}_n\right]}_{W}\underbrace{\begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & \\ \vdots & & \ddots & \\ 0 & & \dots & \lambda_n \end{bmatrix}}_{\Lambda}$$

Hence for *non-defective* A we have the eigenvalue/eigenvector decomposition:

$$\boxed{A = W\Lambda W^{-1}}$$

So that for $T = W^{-1}$ we have,

$$\bar{A} = TAT^{-1} = W^{-1}AW = \Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} = \text{ diag } \{\lambda_i\}$$

$$\begin{aligned} e^{\Lambda t} = \sum_{k=0}^{\infty} \Lambda^k t^k / k! &= \text{diag } \{\sum \lambda_i^k t^k / k!\} \\ &= \text{diag } \{e^{\lambda_i t}\}. \end{aligned}$$

This gives one way of evaluating e^{At}.

(Another way is to evaluate $\mathcal{L}^{-1}(sI - A)^{-1}$.)

2) **Semigroup property** — Don't worry about the fancy name.

$$\begin{aligned} e^{A(t_1+t_2)} &= e^{At_1}e^{At_2} = e^{At_2}e^{At_1} \\ \text{since } \underline{x}(t_1 + t_2) &= \Phi(t_1 + t_2)\underline{x}(0) \\ &= \Phi(t_2)\underline{x}(t_1) \\ &= \Phi(t_2)\Phi(t_1)\underline{x}(0) \quad \text{for all } \underline{x}(0) \end{aligned}$$

NB: This only works because $(At_1)(At_2) = (At_2)(At_1)$.
For arbitrary matrices A and B, $e^{A+B} \neq e^A e^B$.

3) **Inverse**

$$I = e^{A.0} = e^{A(t-t)} = e^{At}e^{-At} \quad \Rightarrow \quad \boxed{(e^{At})^{-1} = e^{-At}}$$

4) **Derivative** — Repeated here for completeness.

$$\frac{d}{dt}e^{At} = Ae^{At} = e^{At}A$$

5) **Integral**

$$\begin{aligned}
\int_0^t e^{A\tau} d\tau &= \int_0^t \sum_{k=0}^{\infty} A^k \tau^k / k! \, d\tau \\
&= \sum_{k=0}^{\infty} \left[A^k \tau^{k+1}/(k+1)! \right]_0^t \\
&= A^{-1} \left\{ \sum_{k=0}^{\infty} A^{k+1} t^{k+1}/(k+1)! - 0 \right\} \\
&= \underline{A^{-1}e^{At} - A^{-1}} \quad \text{if} \quad \det(A) \neq 0.
\end{aligned}$$

If $\det(A) = 0$ then the above formula is not valid and the integration needs to be done directly. e.g. $A = 0$.

3.4 Example: Rotating Rigid Body

Let I_1, I_2, I_3 be the moments of inertia of a rigid body rotating in free space, about its 3 principal axes, and w_1, w_2, w_3 the corresponding angular velocities. Then in the absence of externally applied torques EULER'S EQUATIONS OF MOTION are:-

$$\left.\begin{aligned}
I_1 \dot{w}_1 &= (I_2 - I_3) w_2 w_3 \\
I_2 \dot{w}_2 &= (I_3 - I_1) w_3 w_1 \\
I_3 \dot{w}_3 &= (I_1 - I_2) w_1 w_2
\end{aligned}\right\} \text{nonlinear state space equations}$$

This is a lossless system since the Kinetic Energy,

$$\begin{aligned}
V(w_1, w_2, w_3) &= \frac{1}{2} I_1 w_1^2 + \frac{1}{2} I_2 w_2^2 + \frac{1}{2} I_3 w_3^2 \\
\frac{dV}{dt} &= I_1 w_1 \dot{w}_1 + I_2 w_2 \dot{w}_2 + I_3 w_3 \dot{w}_3 \\
&= w_1 w_2 w_3 \left[(I_2 - I_3) + (I_3 - I_1) + (I_1 - I_2) \right] \\
&= 0
\end{aligned}$$

$\Rightarrow$ if the trajectory starts on an ellipsoid $V(w_1, w_2, w_3) =$ constant, it stays on it.

In addition conservation of angular momentum implies that,

$$\begin{aligned}
\frac{d}{dt} \left\{ I_1^2 w_1^2 + I_2^2 w_2^2 + I_3^2 w_3^2 \right\} &= 2 w_1 w_2 w_3 \left[I_1(I_2 - I_3) + I_2(I_3 - I_1) + I_3(I_1 - I_2) \right] \\
&= 0
\end{aligned}$$

Suppose $I_1 = 6$, $I_2 = 2$, $I_3 = 5$ then

$$\left.\begin{array}{rcl} \dot{w}_1 & = & -\frac{1}{2}w_2w_3 \\ \dot{w}_2 & = & -\frac{1}{2}w_3w_1 \\ \dot{w}_3 & = & \frac{4}{5}w_1w_2 \end{array}\right\}$$

Equilibrium Solutions satisfy: $\dot{w}_1 = \dot{w}_2 = \dot{w}_3 = 0 \;\Rightarrow\; w_2w_3 = w_1w_3 = w_1w_2 = 0 \;\Rightarrow$

$$\begin{array}{cccccc} either & (a) & w_2 = w_3 = 0 & \& & w_1 = \bar{w}_1 \\ or & (b) & w_3 = w_1 = 0 & \& & w_2 = \bar{w}_2 \\ or & (c) & w_1 = w_2 = 0 & \& & w_3 = \bar{w}_3 \\ or & (d) & w_1 = w_2 = w_3 = 0. & & \end{array}$$

The linearized equations are

$$\underline{\dot{\delta w}} \simeq \frac{\partial f}{\partial \underline{w}}\underline{\delta w} = \begin{bmatrix} 0 & -\frac{1}{2}w_3 & -\frac{1}{2}w_2 \\ -\frac{1}{2}w_3 & 0 & -\frac{1}{2}w_1 \\ \frac{4}{5}w_2 & \frac{4}{5}w_1 & 0 \end{bmatrix}\underline{\delta w}.$$

<u>case (a)</u>

$$\Rightarrow \underline{\dot{\delta w}} \cong \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2}\bar{w}_1 \\ 0 & \frac{4}{5}\bar{w}_1 & 0 \end{bmatrix}\underline{\delta w}.$$

$$\Rightarrow \delta w_1(t) \simeq \delta w_1(0) \& \frac{d}{dt}\begin{bmatrix} \delta w_2 \\ \delta w_3 \end{bmatrix} \cong \begin{bmatrix} 0 & -\frac{1}{2}\bar{w}_1 \\ \frac{4}{5}\bar{w}_1 & 0 \end{bmatrix}\begin{bmatrix} \delta w_2 \\ \delta w_3 \end{bmatrix}$$

and the state trajectories are ellipses (with the ratio of the principal axes $= \sqrt{8/5} \simeq 1.26$.

<u>case (b)</u>

$$\underline{\delta \dot{w}} \simeq \begin{bmatrix} 0 & 0 & -\frac{1}{2}\bar{w}_2 \\ 0 & 0 & 0 \\ \frac{4}{5}\bar{w}_2 & 0 & 0 \end{bmatrix} \underline{\delta w}$$

$$\delta w_2 \simeq \delta w_2(0) \& \frac{d}{dt} \begin{bmatrix} \delta w_1 \\ \delta w_3 \end{bmatrix} \simeq \begin{bmatrix} 0 & -\frac{1}{2}\bar{w}_2 \\ \frac{4}{5}\bar{w}_2 & 0 \end{bmatrix} \begin{bmatrix} \delta w_1 \\ \delta w_2 \end{bmatrix}$$

<u>case (c)</u>

$$\underline{\delta \dot{w}} \simeq \begin{bmatrix} 0 & -\frac{1}{2}\bar{w}_3 & 0 \\ -\frac{1}{2}\bar{w}_3 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \underline{\delta w}$$

$$\delta w_3(t) \simeq \delta w_3(0), \frac{d}{dt} \begin{bmatrix} \delta w_1 \\ \delta w_2 \end{bmatrix} = -\frac{1}{2}\bar{w}_3 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \delta w_1 \\ \delta w_2 \end{bmatrix}$$

and

$$\begin{aligned} \begin{bmatrix} \delta w_1 \\ \delta w_2 \end{bmatrix} &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-\frac{1}{2}\bar{w}_3 t} \left(\frac{\delta w_1(0) + \delta w_2(0)}{2} \right) \\ &\quad + \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{\frac{1}{2}\bar{w}_3 t} \left(\frac{\delta w_1(0) - \delta w_2(0)}{2} \right) \end{aligned}$$

The trajectories in the 3-dimensional state space can thus be sketched as follows on a particular ellipsoid $V(w_1, w_2, w_3) = \text{const}$.

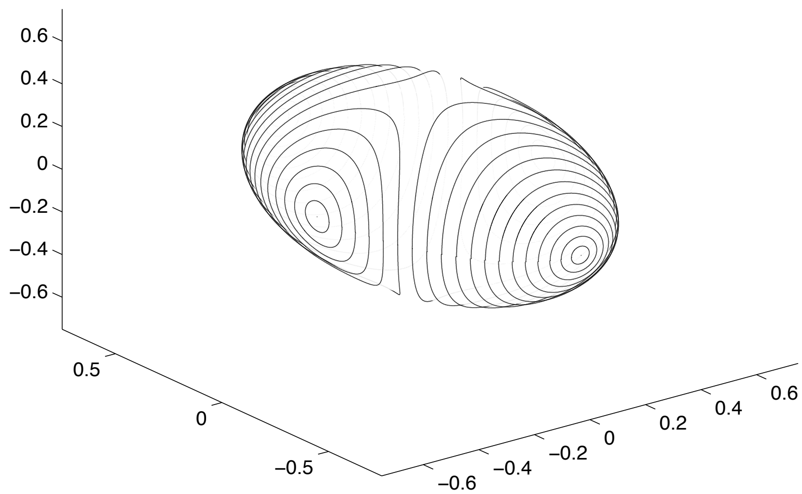

Figure 4: State space trajectories of a rotating rigid body

3.5 Convolution Integral

Consider

$$\underline{\dot{x}} - A\underline{x} = B\underline{u} \qquad (*)$$

Comparing with the scalar case note

$$\begin{aligned} \frac{d}{dt}\left\{e^{-At}\underline{x}(t)\right\} &= \frac{d}{dt}\left\{e^{-At}\right\}\underline{x}(t) + e^{-At}\frac{d\underline{x}}{dt} \\ &= -e^{-At}A\underline{x}(t) + e^{-At}\frac{d\underline{x}}{dt} \end{aligned}$$

now premultiply (*) by e^{-At} to give

$$\begin{aligned} e^{-At}\frac{d\underline{x}}{dt} - e^{-At}A\underline{x} &= e^{-At}B\underline{u} \\ \Rightarrow \frac{d}{dt}\left\{e^{-At}\underline{x}(t)\right\} &= e^{-At}B\underline{u}(t) \\ \Rightarrow e^{-At}\underline{x}(t) - \underline{x}_0 &= \int_0^t e^{-A\tau}B\underline{u}(\tau)d\tau \\ \Rightarrow \underline{x}(t) &= e^{At}\underline{x}_0 + e^{At}\int_0^t e^{-A\tau}B\underline{u}(\tau)d\tau \end{aligned}$$

$$\Rightarrow \boxed{\underline{x}(t) = e^{At}\underline{x}_0 + \int_0^t e^{A(t-\tau)}B\underline{u}(\tau)d\tau}$$

and

$$\boxed{\underline{y}(t) = \underbrace{Ce^{At}\underline{x}_0}_{\text{initial condition response}} \underbrace{+ D\underline{u}(t) + \int_0^t Ce^{A(t-\tau)}B\underline{u}(\tau)\,d\tau}_{\text{input response}}}$$

Let $H(t) = \begin{cases} D\delta(t) + Ce^{At}B & t \geq 0 \\ 0 & t < 0 \end{cases}$

then if $\underline{x}_0 = \underline{0}$,

$$\underline{y}(t) = \int_0^t H(t-\tau)\underline{u}(\tau)d\tau = H(t) * \underline{u}(t)$$

$H(t)$ is called the **impulse response matrix**, since for multiple-input/multiple-output systems, if an impulse is applied at time 0^+ to the j-th input, with the other inputs at zero, then the i-th output will be

$$y_i(t) = \int_0^t h_{ij}(t-\tau)u_j(\tau)d\tau = h_{ij}(t)$$

Note that the transfer function, $G(s) = \mathcal{L}(H(t))$.

3.6 Frequency Response

Consider a linear time-invariant system that is asymptotically **stable**. What is the response due to sinusoidal input at each of the inputs? Let

$$u_j(t) = A_j \cos(\omega_o t + \theta_j), \;\; j = 1, 2, \cdots, m$$

then

$$y_i(t) \rightarrow B_i \cos(\omega_o t + \phi_i) \text{ as } t \rightarrow \infty$$

where

$$B_i e^{j\phi_i} = g_{i1}(j\omega_o)A_1e^{j\theta_1} + g_{i2}(j\omega_o)A_2e^{j\theta_2} + \cdots + g_{im}(j\omega_o)A_me^{j\theta_m}$$

i.e. the sum of the sinusoidal responses from each input. The rate at which the steady state is achieved depends on how quickly the impulse response tends to zero, which in turn depends on the pole positions.

3.7 Stability of $\underline{\dot{x}} = A\underline{x}$

Stability of systems is concerned with whether as $t \to \infty$,

$$
\begin{aligned}
\underline{x}(t) \quad &\to \quad 0 \\
&\to \quad \infty \\
&\text{or} \quad \text{remains bounded ?}
\end{aligned}
$$

but since $\underline{x}(t) = e^{At}\underline{x}_0$ the question becomes whether the elements of $e^{At} \to 0, \to \pm\infty$ or remain bounded as $t \to \infty$?

Consider

$$(sI - A)^{-1} = N(s)/\alpha(s)$$

where $\alpha(s)$ is the characteristic polynomial of A,

$$\alpha(s) = \det(sI - A) = s^n + \alpha_1 s^{n-1} + .. + \alpha_n$$

and

$$N(s) = N_1 s^{n-1} + N_2 s^{n-2} + .. + N_n$$

where N_i are $n \times n$ constant matrices. (Recall section 3.2.)

If we factor $\alpha(s)$ as

$$\alpha(s) = (s - \lambda_1)^{n_1}(s - \lambda_2)^{n_2}..(s - \lambda_r)^{n_r}$$

where λ_i will be the eigenvalues of A and $n_1 + n_2 + .. + n_r = n =$ dim (A), then the partial fraction expansion gives

$$(sI - A)^{-1} = \sum_{i=1}^{r}\sum_{k=1}^{n_i} \frac{C_{i,k}}{(s - \lambda_i)^k}$$

for suitable constant matrices $C_{i,j}$.

The inverse transform then gives

$$e^{At} = \sum_{i=1}^{r}\sum_{k=1}^{n_i} \frac{C_{i,k}t^{k-1}e^{\lambda_i t}}{(k-1)!}$$

Let $\lambda_i = \sigma_i + j\omega_i$ then $\left|e^{\lambda_i t}\right| = e^{\sigma_i t}\left|e^{j\omega_i t}\right| = e^{\sigma_i t}$

3 cases

(i) $\sigma_i < 0, \left|t^{k-1}e^{\lambda_i t}\right| \to 0, k = 1, 2, 3, ...$

(ii) $\sigma_i > 0, \left|t^{k-1}e^{\lambda_i t}\right| \to \infty, k = 1, 2, 3, ...$

(iii) $\sigma_i = 0$

(a) $\left|t^{k-1}e^{\lambda_i t}\right| = 1 \qquad k = 1$

(b) $\left|t^{k-1}e^{\lambda_i t}\right| \to \infty \quad k = 2, 3,.$

Hence $\underline{x}(t) \to \underline{0}$ as $t \to \infty$ for all $\underline{x}_0$ if and only if all λ_i satisfy $Re(\lambda_i) < 0$. (i.e. case i)).

Also $\underline{x}(t)$ remains bounded as $t \to \infty$ for all $\underline{x}_0$ if and only if $Re(\lambda_i) \leq 0$ <u>and</u> in partial fraction expansion of $(sI - A)^{-1}$ there are <u>no</u> terms of the form $C_{i,k}/(s - j\omega_i)^k$ with $k \geq 2$. (i.e. case (i) or (iii)(a).).

4 State Space Equations for Composite Systems

4.1 Cascade of Two Systems

Let $G_1(s)$ be realized by the state equation:

$$\begin{cases} \dot{\underline{x}}_1(t) &= A_1\underline{x}_1(t) + B_1\underline{u}(t) \\ \underline{w}(t) &= C_1\underline{x}_1(t) + D_1\underline{u}(t) \end{cases}$$

and $G_2(s)$ be realized by, (input $\underline{w}$, output $\underline{y}$)

$$\begin{cases} \dot{\underline{x}}_2(t) &= A_2\underline{x}_2(t) + B_2\underline{w}(t) \\ \underline{y}(t) &= C_2\underline{x}_2(t) + D_2\underline{w}(t) \end{cases}$$

then $\underline{Y}(s) = G_2(s)G_1(s)\underline{U}(s)$ is realized by

$$\underbrace{\frac{d}{dt}\begin{bmatrix} \underline{x}_1(t) \\ \underline{x}_2(t) \end{bmatrix}}_{\dot{\underline{x}}(t)} = \underbrace{\begin{bmatrix} A_1 & 0 \\ B_2C_1 & A_2 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} \underline{x}_1(t) \\ \underline{x}_2(t) \end{bmatrix}}_{\underline{x}(t)} + \underbrace{\begin{bmatrix} B_1 \\ B_2D_1 \end{bmatrix}}_{B}\underline{u}(t)$$

$$\underline{y}(t) = \underbrace{\begin{bmatrix} D_2C_1 & C_2 \end{bmatrix}}_{C}\underline{x}(t) + \underbrace{D_2D_1}_{D}\underline{u}(t)$$

The cascade realization of a single-input single-output system can hence be obtained if $G(s)$ is factored into second order factors such as

$$G(s) = A\frac{(s^2 + c_1s + d_1)}{(s^2 + a_1s + b_1)}\frac{(s^2 + c_2s + d_2)\dots}{(s^2 + a_2s + b_2)\dots}$$

then the overall system can be realized as the cascade of these factors.

4.2 Parallel combination of two systems

Let $G_1(s)$ be realized by the state equation:

$$\begin{cases} \underline{\dot{x}}_1(t) &= A_1\underline{x}_1(t) + B_1\underline{u}(t) \\ \underline{y}_1(t) &= C_1\underline{x}_1(t) + D_1\underline{u}(t) \end{cases}$$

and $G_2(s)$ be realized by, (input $\underline{u}$, output $\underline{y}_2$)

$$\begin{cases} \underline{\dot{x}}_2(t) &= A_2\underline{x}_2(t) + B_2\underline{u}(t) \\ \underline{y}_2(t) &= C_2\underline{x}_2(t) + D_2\underline{u}(t) \end{cases}$$

then $\underline{Y}(s) = (G_1(s) + G_2(s))\underline{U}(s)$ is realized by

$$\underbrace{\frac{d}{dt}\begin{bmatrix} \underline{x}_1(t) \\ \underline{x}_2(t) \end{bmatrix}}_{\underline{\dot{x}}(t)} = \underbrace{\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} \underline{x}_1(t) \\ \underline{x}_2(t) \end{bmatrix}}_{\underline{x}(t)} + \underbrace{\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}}_{B}\underline{u}(t)$$

$$\underline{y}(t) = \underbrace{\begin{bmatrix} C_1 & C_2 \end{bmatrix}}_{C}\underline{x}(t) + \underbrace{(D_1 + D_2)}_{D}\underline{u}(t)$$