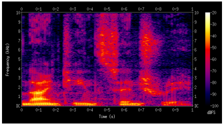
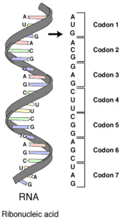
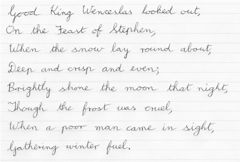
Sequence Modelling

Rich Turner and José Miguel Hernández-Lobato

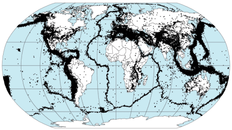
Sequence data

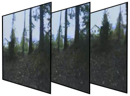


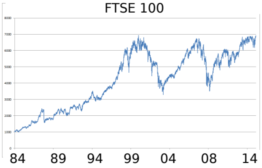


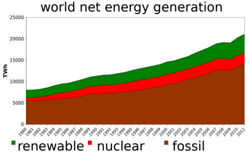


Preliminary Determination of Epicenters
358,214 Events, 1963 - 1998









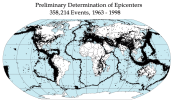
I believe that at the end of the century the use of words and general educated opinion will have altered so much that one will be able to speak of machines thinking without expecting to be contradicted. A. Turing

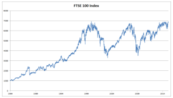
Some images taken from wikipedia

Goals of sequence modelling

Predict future items in sequence

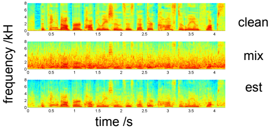
$$p(y_t | y_1, \ldots, y_{t-1})$$





Remove noise from a sequence

$$p(y'_1, \ldots, y'_t | y_1, \ldots, y_t)$$

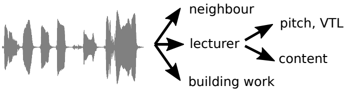


Predict one sequence from another

$$p(y'_1, \ldots, y'_t | y_1, \ldots, y_t)$$

Discover underlying latent variables

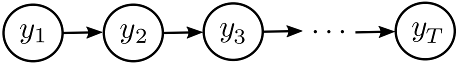
$$p(x_1, \ldots, x_t | y_1, \ldots, y_t)$$



Markov models

First order Markov

$$p(y_1, y_2, y_3, \ldots, y_T) = p(y_1)p(y_2|y_1)p(y_3|y_2)\ldots p(y_T|y_{T-1})$$



Markov models

parameters tied

∞ number of variables
finite number of parameters

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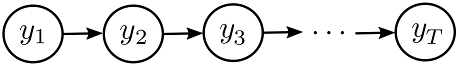

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$$y_1 \to y_2 \to y_3 \to \cdots \to y_T$$

Markov model = conditional independence relationship + product rule

independent of past

future $y_{t+1} \perp y_{1:t-1} | y_t$

given present

$$p(y_{1:T}) = \prod_{t=1}^{T} p(y_t | y_{1:t-1})$$

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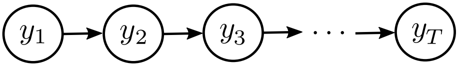

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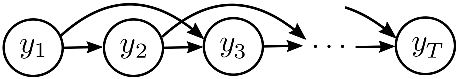

Markov models for discrete data: n-gram models

First order Markov (bi-gram)

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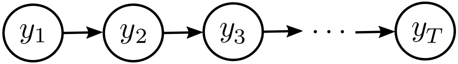


$y_t \in \{1, ..., K\}$

discrete states

$p(y_1 = k) = \pi_k^0$

initial state probabilities

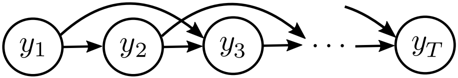
$p(y_t = k|y_{t-1} = l) = T_{k,l}$

transition probabilities
(stochastic matrix)

$\sum_{k=1}^{K} T_{k,l} = 1$

Second order Markov (tri-gram)

$$p(y_1, y_2, y_3, \dots, y_T) = p(y_1)p(y_2|y_1)p(y_3|y_2, y_1)\dots p(y_T|y_{T-1}, y_{T-2})$$



$p(y_t = k|y_{t-1} = l, y_{t-2} = m) = T_{k,l,m}$

n-grams require large
multidimensional arrays

Some questions about n-gram models

First order Markov (bi-gram)

$$y_t \in \{1, ..., K\} \quad p(y_1 = k) = \pi_k^0 \qquad p(y_t = k | y_{t-1} = l) = T_{k,l}$$

Q1. How can we compute the marginal distribution over the second state?

Some questions about n-gram models

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$$p(y_2 = k) = \sum_{l=1}^{K} p(y_2 = k | y_1 = l) p(y_1 = l) = \sum_{l=1}^{K} T_{k,l} \pi^0_l$$

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eigenvectors of transition matrix with eigenvalue = 1

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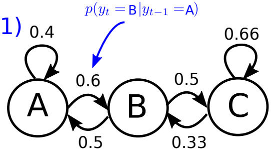
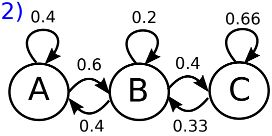
eigenvectors of transition matrix with eigenvalue = 1

Q3. Which transition matrix is most compatible with the following sequence?

ABAAABBABCCCBC

'State Transition Diagrams'

1)


2)


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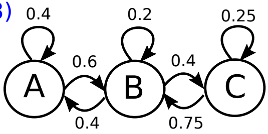

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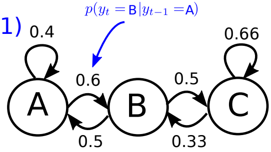
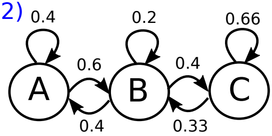
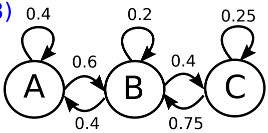
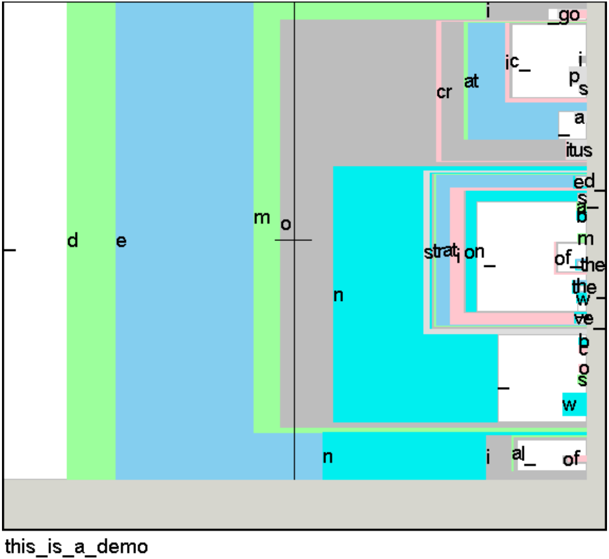
1) 

2) 

3) 

Example application of n-grams: text modelling for dasher



http://www.inference.phy.cam.ac.uk/dasher/ https://www.youtube.com/watch?v=nr3s4613DX8

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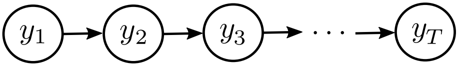

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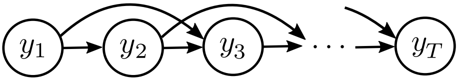

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Markov models for continuous data: Auto-Regressive (AR) Gaussian models

First order Markov (AR(1))

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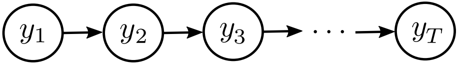


$y_t \in \mathbb{R}^D$ $\qquad p(y_1) = \mathcal{G}(y_1; \mu_0, \Sigma_0)$ $\qquad p(y_t|y_{t-1}) = \mathcal{G}(y_t; \Lambda y_{t-1}, \Sigma)$

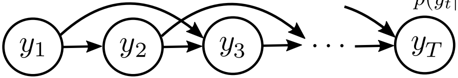
continuous vector states — initial state density — transition density

$$\mathcal{G}(y; \mu, \Sigma) = \frac{1}{(2\pi)^{D/2}\det(\Sigma)^{1/2}} \exp\left\{-\frac{1}{2}(y-\mu)^\top \Sigma^{-1}(y-\mu)\right\}$$

Second order Markov (AR(2))

$$p(y_1, y_2, y_3, \ldots, y_T) = p(y_1)p(y_2|y_1)p(y_3|y_2, y_1)\ldots p(y_T|y_{T-1}, y_{T-2})$$

$$p(y_t|y_{t-1}, y_{t-2}) = \mathcal{G}(y_t; \Lambda_1 y_{t-1} + \Lambda_2 y_{t-2}, \Sigma)$$



joint distribution over all variables
is always multivariate Gaussian

Markov models for continuous data: Auto-Regressive (AR) Gaussian models

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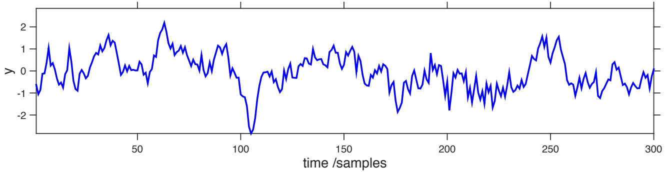
$$y_t \in \mathbb{R}^1 \qquad p(y_t|y_{t-1}) = \mathcal{G}(y_t; \lambda y_{t-1}, \sigma^2) \qquad \lambda = 0.9 \quad \sigma^2 = 0.01$$



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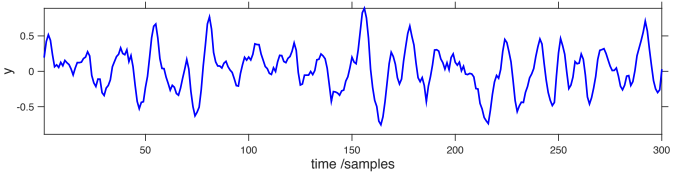
$$[\lambda_1, \lambda_2] = [1.57, -0.78] \qquad \sigma^2 = 0.01$$



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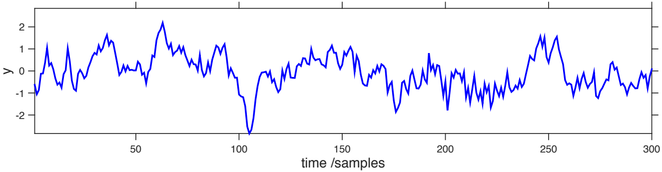

What is the stationary distribution of this process? $p(y_\infty) = \,?$

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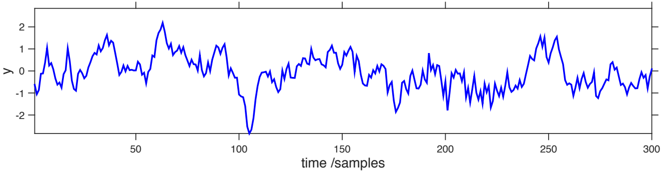

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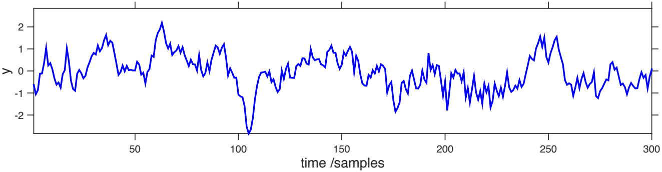

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$$y_t = \lambda y_{t-1} + \sigma\epsilon_t \;\; \epsilon_t \sim \mathcal{G}(0,1)$$

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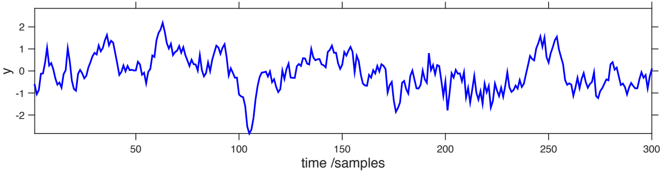

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Mean: $\langle y_t \rangle$

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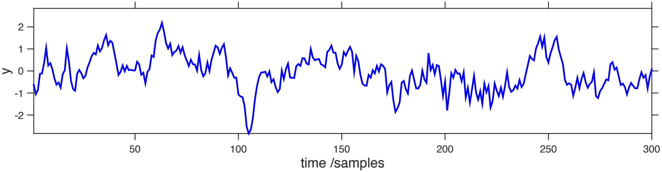

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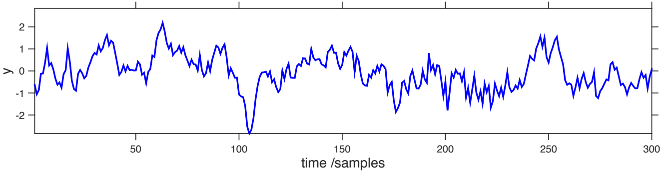
Markov models for continuous data: Auto-Regressive (AR) Gaussian models

First order Markov (AR(1))

$$y_t \in \mathbb{R}^1 \qquad p(y_t|y_{t-1}) = \mathcal{G}(y_t; \lambda y_{t-1}, \sigma^2) \qquad \lambda = 0.9 \quad \sigma^2 = 0.01$$



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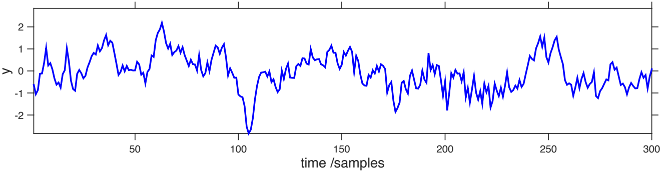

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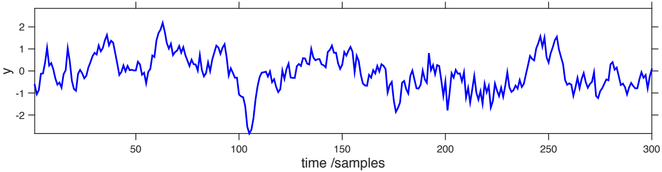

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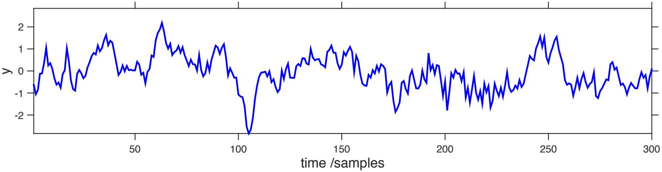

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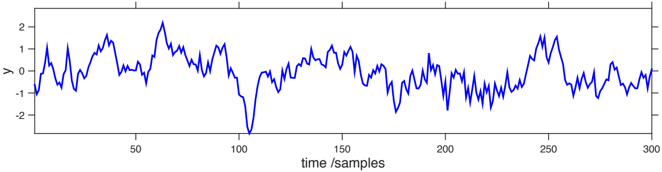

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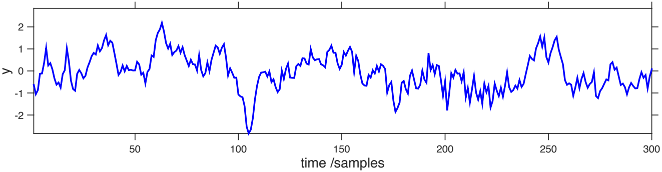

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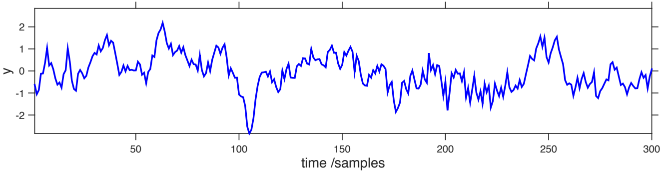

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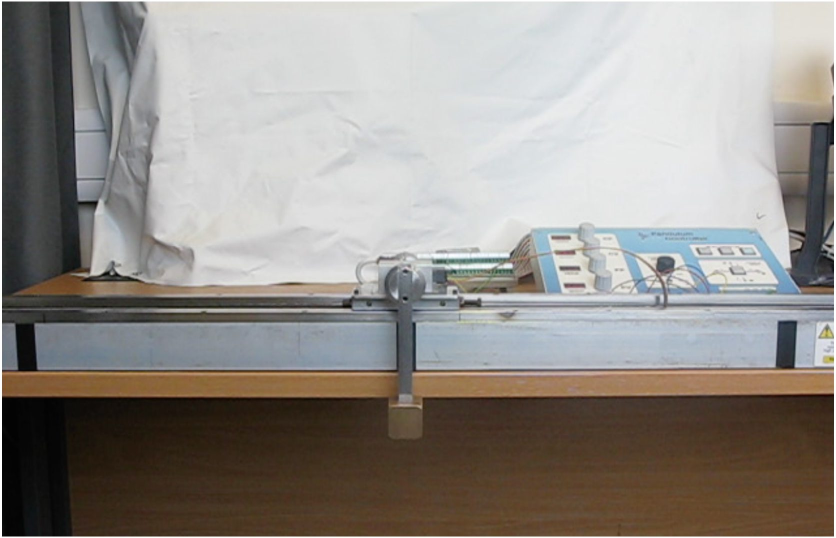
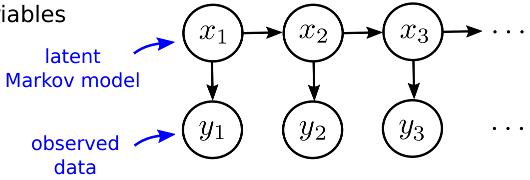
Example application of Markov Models: pendulum swing up control problem



Hidden Markov models

Real data depend on latent variables



ASR

x phonemes/words

y waveform/feature

Computer Vision

x objects, pose, lighting

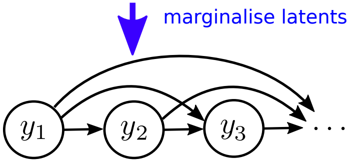
y image pixel intensities

Natural Language Processing

x topics

y words

$$p(y_{1:T}, x_{1:T}) = \prod_{t=1}^{T} p(x_t|x_{t-1})p(y_t|x_t)$$



$p(y_{1:T})$

fully connected

Two prevelant Examples:

Hidden Markov Models (discrete x)
Linear Gaussian State Space Models (Gaussian x and y)

Hidden Markov models: discrete hidden state

Discrete Hidden State

$$x_t \in \{1, ..., K\}$$

$$p(x_t = k | x_{t-1} = l) = T_{k,l}$$

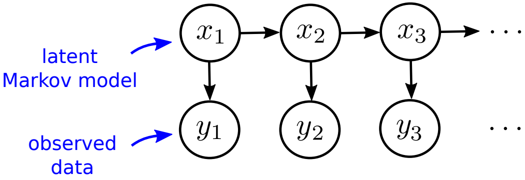
E.g. in examples below $K = 2$

$$T = \begin{bmatrix} 0.9 & 0.1 \\ 0.1 & 0.9 \end{bmatrix}$$

Continuous Observed State

$$p(y_t | x_t = k) = \mathcal{G}(y_t; \mu_k, \Sigma_k)$$

$\mu_1 = 3 \quad \mu_2 = -3 \quad \sigma_1^2 = \sigma_2^2 = 1$



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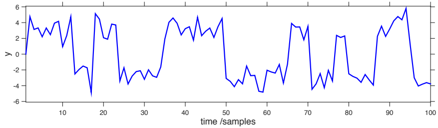

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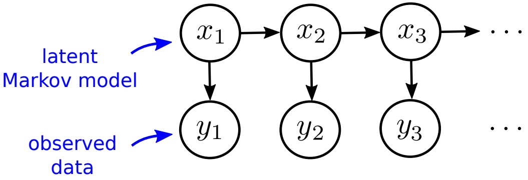

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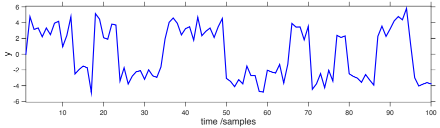

Discrete Observed State

$$p(y_t = l | x_t = k) = S_{l,k}$$

$$S = \begin{bmatrix} 0.5 & 0 \\ 0.5 & 0 \\ 0 & 1 \end{bmatrix}$$

ABBBBAAABAAACCCCBBBBBBCCCCCCCCCC
AAABBBBAABAAABBCCCCCCCCCCCCCCBBA
AACCCCCBABCCCCCCCAABBAABABCCCCC

Hidden Markov models: discrete hidden state

Discrete Hidden State, Continuous Observed State

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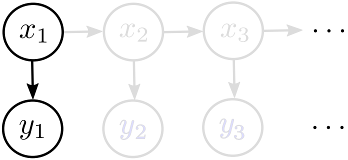
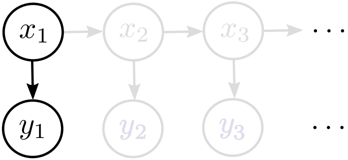


Consider T = 1

Q1: What type of distribution is $p(y_1)$?

Hidden Markov models: discrete hidden state



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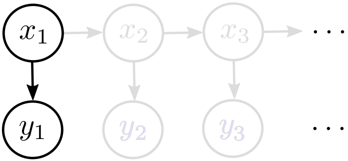

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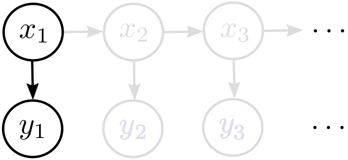

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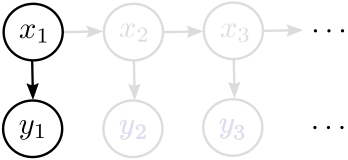

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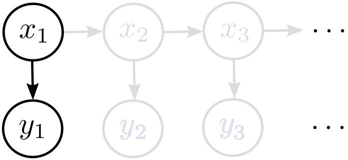

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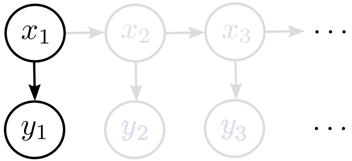

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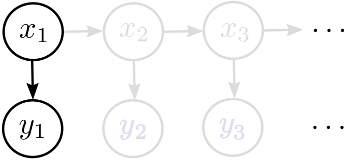

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this HMM = Mixture of Gaussian Models with dynamic cluster assignments

Hidden Markov models: continuous hidden state (LGSSMs)

Continuous Hidden State

$$x_t \in \mathbb{R}^K$$

$$p(x_t|x_{t-1}) = \mathcal{G}(x_t; Ax_{t-1}, Q)$$

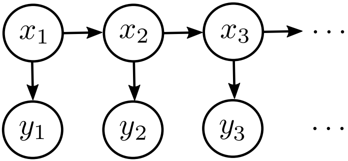
Continuous Observed State

$$y_t \in \mathbb{R}^D$$

$$p(y_t|x_t) = \mathcal{G}(y_t; Cx_t, R)$$

latent Markov model

observed data



$$p(y_{1:T}, x_{1:T}) = \prod_{t=1}^{T} p(x_t|x_{t-1})p(y_t|x_t)$$

E.g. simple example $K = 2 \quad D = 1$

$$A = \lambda \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

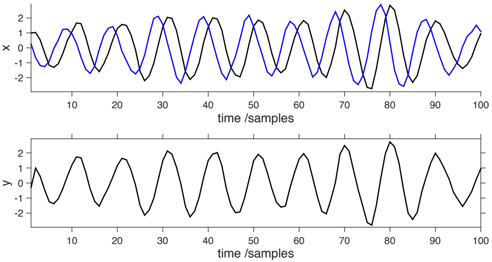
$$\lambda = 0.99 \qquad \theta = 2\pi/10$$

$$Q = (1 - \lambda^2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

dynamics model

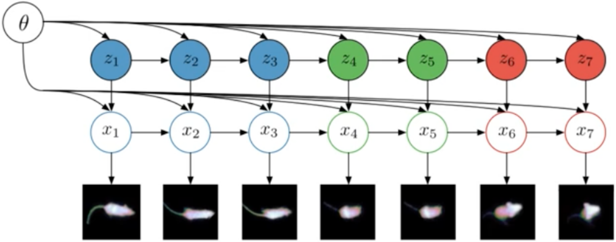
$$C = [1, 0] \qquad R = 0.01$$

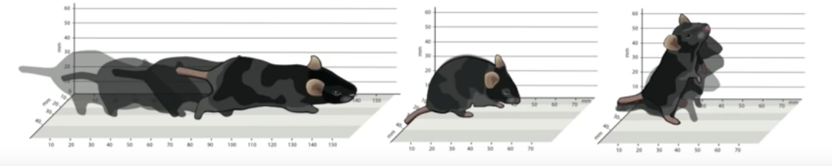
obs. model



Hidden Markov Models for unsupervised high dimensional video understanding

Hidden Markov Models for unsupervised high dimensional video understanding



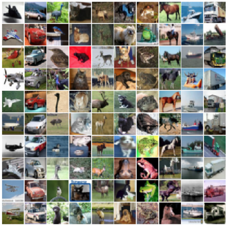


https://www.youtube.com/watch?v=btr1poCYIzw

Connections

Neural auto-regressive models: closely related to Markov models and HMMs

Pixel CNN



2 WaveNet

In this paper we introduce a new generative model operating directly on the raw audio waveform. The joint probability of a waveform $\mathbf{x} = \{x_1, \ldots, x_T\}$ is factorised as a product of conditional probabilities as follows:

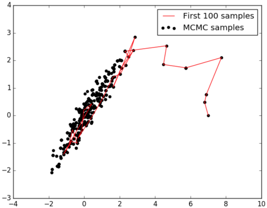
$$p(\mathbf{x}) = \prod_{t=1}^{T} p(x_t \mid x_1, \ldots, x_{t-1}) \tag{1}$$

Each audio sample x_t is therefore conditioned on the samples at all previous timesteps.

MCMC: design Markov chain with desired stationary distribution

Algorithm 1 Metropolis-Hastings algorithm

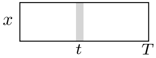
Initialize $x^{(0)} \sim q(x)$
for iteration $i = 1, 2, \ldots$ **do**
  Propose: $x^{cand} \sim q(x^{(i)}|x^{(i-1)})$
  Acceptance Probability:
    $\alpha(x^{cand}|x^{(i-1)}) = \min\left\{1, \frac{q(x^{(i-1)}|x^{cand})\pi(x^{cand})}{q(x^{cand}|x^{(i-1)})\pi(x^{(i-1)})}\right\}$
  $u \sim \text{Uniform}\ (u; 0, 1)$
  if $u < \alpha$ **then**
    Accept the proposal: $x^{(i)} \leftarrow x^{cand}$
  else
    Reject the proposal: $x^{(i)} \leftarrow x^{(i-1)}$
  end if
end for

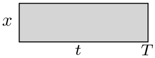


Varieties of Inference

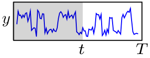
Distributional estimates

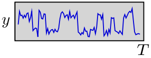
infer single state or sequence?





future data available?





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smoother	$p(x_t \mid y_{1:T})$	$p(x_{1:T} \mid y_{1:T})$

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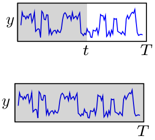

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Point estimates

most probable state @ t

$$x_t^* = \arg\max_{x_t} p(x_t | y_{1:T})$$

most probable sequence

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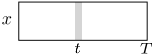

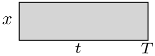

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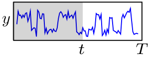

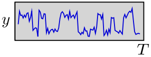

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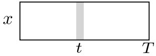

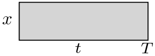

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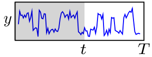

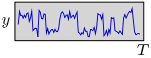

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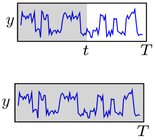

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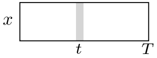
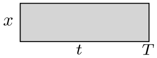
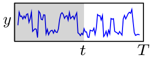
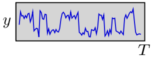




look at this next

future data available?





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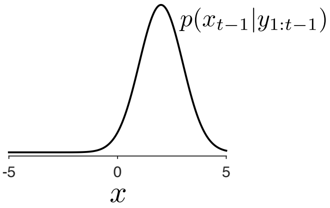

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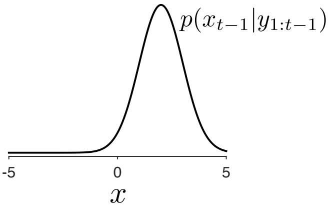

diffuse via dynamics

$$p(x_{t-1}|y_{1:t-1})$$

sum for discrete hidden state

$$p(x_t|y_{1:t-1}) = \int p(x_t|x_{t-1})p(x_{t-1}|y_{1:t-1})\mathrm{d}x_{t-1}$$

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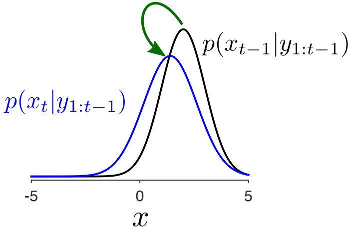

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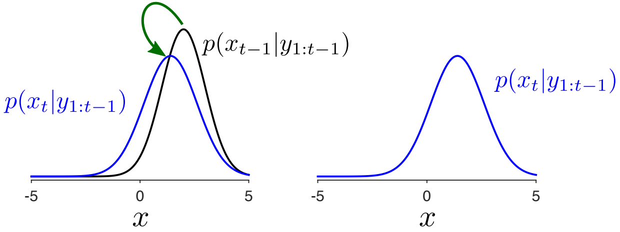

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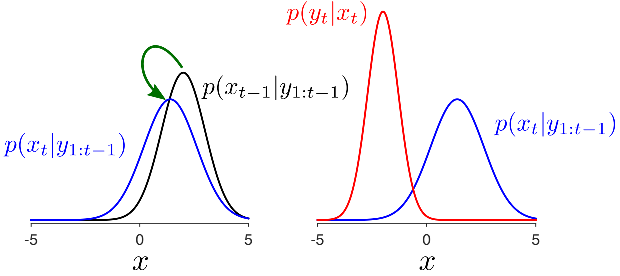
Bayes' Rule

Inference: Kalman Filter



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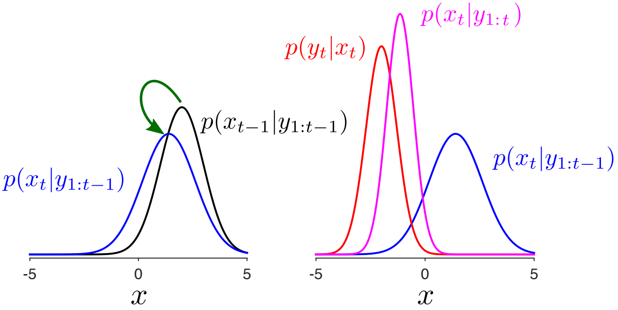

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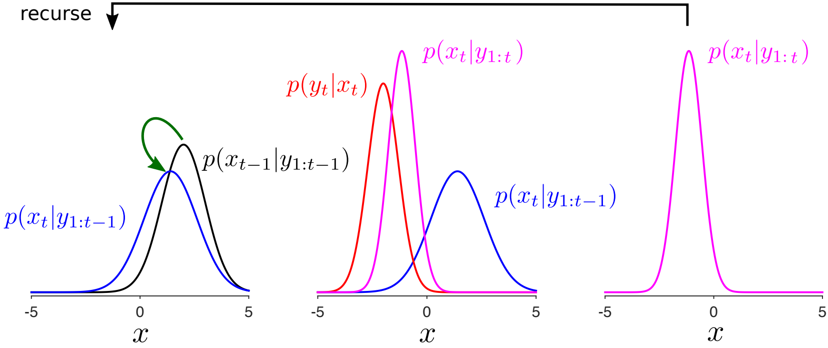

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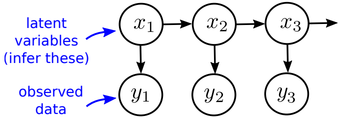
Inference: Derivation of General Filtering Equations

Model



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Inference: Derivation of General Filtering Equations

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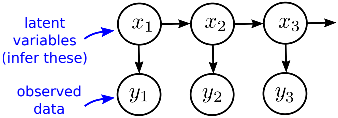

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$$p(A|B,C) = \frac{1}{p(B|C)}\, p(B|A,C)p(A|C)$$

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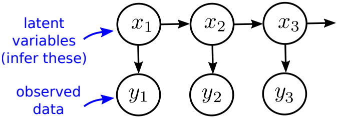

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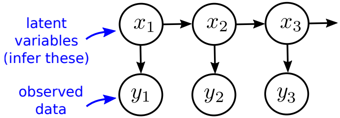

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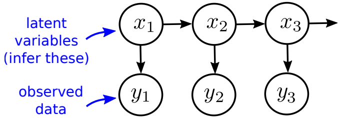

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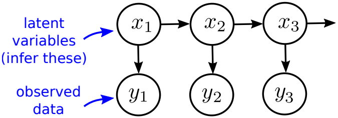

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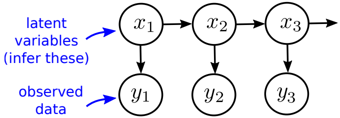

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conditional independence from model

$y_t \perp y_{1:t-1}|x_t$

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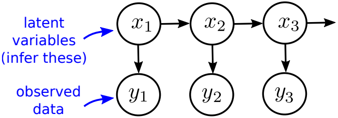

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$$= \frac{1}{p(y_t|y_{1:t-1})}\, p(y_t|x_t)p(x_t|y_{1:t-1})$$

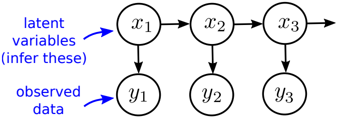
conditional independence from model $y_t \perp y_{1:t-1}|x_t$

$$\propto p(y_t|x_t)p(x_t|y_{1:t-1})$$

constant of proportionality $p(y_t|y_{1:t-1})$ (see learning)

Inference: Derivation of General Filtering Equations

Model



$$p(y_{1:T}, x_{1:T}) = \prod_{t=1}^{T} p(x_t|x_{t-1})p(y_t|x_t)$$

$+$

Rules of probability

product rule

$$p(A|B,C) = \frac{1}{p(B|C)}\, p(B|A,C)p(A|C)$$

sum rule

$$p(A|C) = \sum_B p(A,B|C)$$

$= \ ?$

Inference

$$\begin{aligned} p(x_t|y_{1:t}) &= p(x_t|y_t, y_{1:t-1}) \\ &= \frac{1}{p(y_t|y_{1:t-1})}\, p(y_t|x_t, y_{1:t-1})p(x_t|y_{1:t-1}) \\ &= \frac{1}{p(y_t|y_{1:t-1})}\, p(y_t|x_t)p(x_t|y_{1:t-1}) \\ &\propto p(y_t|x_t)p(x_t|y_{1:t-1}) \end{aligned}$$

product rule $A = x_t \;\; B = y_t \;\; C = y_{1:t-1}$

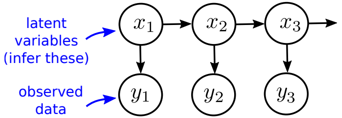
conditional independence from model $y_t \perp y_{1:t-1}|x_t$

constant of proportionality $p(y_t|y_{1:t-1})$ (see learning)

$$p(x_t|y_{1:t-1})$$

Inference: Derivation of General Filtering Equations

Model



$$p(y_{1:T}, x_{1:T}) = \prod_{t=1}^{T} p(x_t|x_{t-1})p(y_t|x_t)$$

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Rules of probability

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$= \ ?$

Inference

$$
\begin{aligned}
p(x_t|y_{1:t}) &= p(x_t|y_t, y_{1:t-1}) \\
&= \frac{1}{p(y_t|y_{1:t-1})}\, p(y_t|x_t, y_{1:t-1})p(x_t|y_{1:t-1}) \\
&= \frac{1}{p(y_t|y_{1:t-1})}\, p(y_t|x_t)p(x_t|y_{1:t-1}) \\
&\propto p(y_t|x_t)p(x_t|y_{1:t-1})
\end{aligned}
$$

product rule $A = x_t \;\; B = y_t \;\; C = y_{1:t-1}$

conditional independence from model $y_t \perp y_{1:t-1}|x_t$

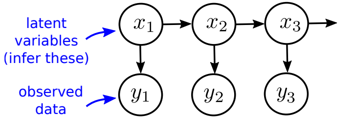
constant of proportionality $p(y_t|y_{1:t-1})$ (see learning)

$$p(x_t|y_{1:t-1}) = \int p(x_t, x_{t-1}|y_{1:t-1})\mathrm{d}x_{t-1}$$

sum rule $A = x_t \;\; B = x_{t-1} \;\; C = y_{1:t-1}$

Inference: Derivation of General Filtering Equations

Model



$$p(y_{1:T}, x_{1:T}) = \prod_{t=1}^{T} p(x_t|x_{t-1})p(y_t|x_t)$$

+

Rules of probability

product rule

$$p(A|B,C) = \frac{1}{p(B|C)} p(B|A,C)p(A|C)$$

sum rule

$$p(A|C) = \textstyle\sum_B p(A,B|C)$$

= ?

Inference

$$p(x_t|y_{1:t}) = p(x_t|y_t, y_{1:t-1})$$

$$= \frac{1}{p(y_t|y_{1:t-1})} p(y_t|x_t, y_{1:t-1})p(x_t|y_{1:t-1})$$

product rule $A = x_t \; B = y_t \; C = y_{1:t-1}$

$$= \frac{1}{p(y_t|y_{1:t-1})} p(y_t|x_t)p(x_t|y_{1:t-1})$$

conditional independence from model $y_t \perp y_{1:t-1}|x_t$

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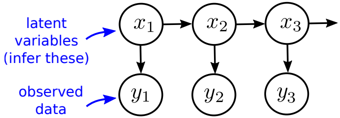
sum rule $A = x_t \; B = x_{t-1} \; C = y_{1:t-1}$

$$= \int p(x_t|x_{t-1}, y_{1:t-1})p(x_{t-1}|y_{1:t-1})\mathrm{d}x_{t-1}$$

product rule

Inference: Derivation of General Filtering Equations

Model



$$p(y_{1:T}, x_{1:T}) = \prod_{t=1}^{T} p(x_t|x_{t-1}) p(y_t|x_t)$$

\+

Rules of probability

product rule

$$p(A|B,C) = \frac{1}{p(B|C)} p(B|A,C) p(A|C)$$

sum rule

$$p(A|C) = \sum_B p(A,B|C)$$

= ?

Inference

$$p(x_t|y_{1:t}) = p(x_t|y_t, y_{1:t-1})$$

$$= \frac{1}{p(y_t|y_{1:t-1})} p(y_t|x_t, y_{1:t-1}) p(x_t|y_{1:t-1})$$

product rule $A = x_t \;\; B = y_t \;\; C = y_{1:t-1}$

$$= \frac{1}{p(y_t|y_{1:t-1})} p(y_t|x_t) p(x_t|y_{1:t-1})$$

conditional independence from model $y_t \perp y_{1:t-1}|x_t$

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constant of proportionality $p(y_t|y_{1:t-1})$ (see learning)

$$p(x_t|y_{1:t-1}) = \int p(x_t, x_{t-1}|y_{1:t-1}) \mathrm{d}x_{t-1}$$

sum rule $A = x_t \;\; B = x_{t-1} \;\; C = y_{1:t-1}$

$$= \int p(x_t|x_{t-1}, y_{1:t-1}) p(x_{t-1}|y_{1:t-1}) \mathrm{d}x_{t-1}$$

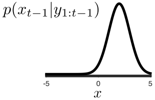
product rule

$$= \int p(x_t|x_{t-1}) p(x_{t-1}|y_{1:t-1}) \mathrm{d}x_{t-1}$$

conditional independence from model

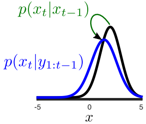
Inference: Kalman Filter

$$p(x_{t-1}|y_{1:t-1})$$



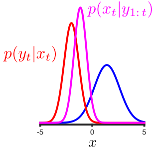
diffuse via
dynamics

$$\color{blue}{p(x_t|y_{1:t-1})} = \int \color{green}{p(x_t|x_{t-1})}\color{black}{p(x_{t-1}|y_{1:t-1})\mathrm{d}x_{t-1}}$$



combine
with
likelihood

$$\color{magenta}{p(x_t|y_{1:t})} \propto \underbrace{\color{blue}{p(x_t|y_{1:t-1})}}_{\text{prior}}\underbrace{\color{red}{p(y_t|x_t)}}_{\text{likelihood}}$$

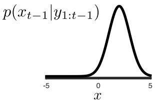


Inference: Kalman Filter

$$p(x_{t-1}|y_{1:t-1}) = \mathcal{G}(x_{t-1}; \mu_{t-1}^{t-1}, V_{t-1}^{t-1})$$

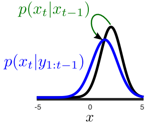
most recent data used in prediction

variable being predicted



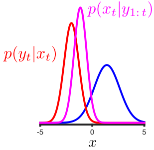
diffuse via dynamics

$$p(x_t|y_{1:t-1}) = \int p(x_t|x_{t-1}) p(x_{t-1}|y_{1:t-1}) \mathrm{d}x_{t-1}$$



combine with likelihood

$$p(x_t|y_{1:t}) \propto \underbrace{p(x_t|y_{1:t-1})}_{\text{prior}} \underbrace{p(y_t|x_t)}_{\text{likelihood}}$$

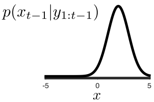


Inference: Kalman Filter

$$p(x_{t-1}|y_{1:t-1}) = \mathcal{G}(x_{t-1}; \mu_{t-1}^{t-1}, V_{t-1}^{t-1})$$

most recent data used in prediction

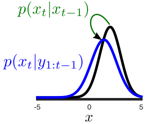
variable being predicted



diffuse via dynamics

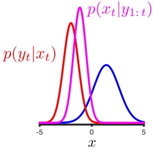
$$p(x_t|y_{1:t-1}) = \int p(x_t|x_{t-1})p(x_{t-1}|y_{1:t-1})\mathrm{d}x_{t-1}$$

$$p(x_t|y_{1:t-1}) = \mathcal{G}(x_t; \mu_t^{t-1}, V_t^{t-1})$$



combine with likelihood

$$p(x_t|y_{1:t}) \propto \underbrace{p(x_t|y_{1:t-1})}_{\text{prior}}\underbrace{p(y_t|x_t)}_{\text{likelihood}}$$

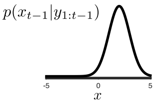


Inference: Kalman Filter

$$p(x_{t-1}|y_{1:t-1}) = \mathcal{G}(x_{t-1}; \mu_{t-1}^{t-1}, V_{t-1}^{t-1})$$

most recent data used in prediction

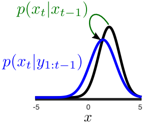
variable being predicted



diffuse via dynamics

$$p(x_t|y_{1:t-1}) = \int p(x_t|x_{t-1}) p(x_{t-1}|y_{1:t-1}) \mathrm{d}x_{t-1}$$

diffuses toward 0

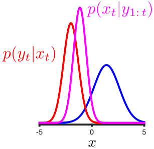


$$p(x_t|y_{1:t-1}) = \mathcal{G}(x_t; \mu_t^{t-1}, V_t^{t-1}) \qquad \mu_t^{t-1} = A\mu_{t-1}^{t-1}$$

$$V_t^{t-1} = AV_{t-1}^{t-1}A^\top + Q$$

combine with likelihood

variance inflates



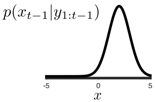
$$p(x_t|y_{1:t}) \propto \underbrace{p(x_t|y_{1:t-1})}_{\text{prior}} \underbrace{p(y_t|x_t)}_{\text{likelihood}}$$

Inference: Kalman Filter

most recent data used in prediction

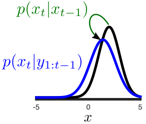
$$p(x_{t-1}|y_{1:t-1}) = \mathcal{G}(x_{t-1}; \mu_{t-1}^{t-1}, V_{t-1}^{t-1})$$

variable being predicted



diffuse via dynamics

$$p(x_t|y_{1:t-1}) = \int p(x_t|x_{t-1}) p(x_{t-1}|y_{1:t-1}) \mathrm{d}x_{t-1}$$



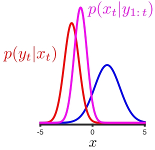
diffuses toward 0

$$p(x_t|y_{1:t-1}) = \mathcal{G}(x_t; \mu_t^{t-1}, V_t^{t-1}) \qquad \mu_t^{t-1} = A\mu_{t-1}^{t-1}$$

$$V_t^{t-1} = AV_{t-1}^{t-1}A^\top + Q$$

variance inflates

combine with likelihood



$$p(x_t|y_{1:t}) \propto \underbrace{p(x_t|y_{1:t-1})}_{\text{prior}} \underbrace{p(y_t|x_t)}_{\text{likelihood}}$$

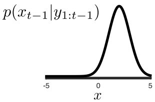
$$p(x_t|y_{1:t}) = \mathcal{G}(x_t; \mu_t^t, V_t^t)$$

Inference: Kalman Filter

most recent data used in prediction

$$p(x_{t-1}|y_{1:t-1}) = \mathcal{G}(x_{t-1}; \mu_{t-1}^{t-1}, V_{t-1}^{t-1})$$

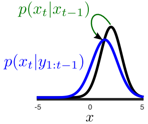
variable being predicted



diffuse via dynamics

$$p(x_t|y_{1:t-1}) = \int p(x_t|x_{t-1}) p(x_{t-1}|y_{1:t-1}) \mathrm{d}x_{t-1}$$

diffuses toward 0



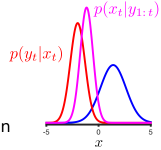
$$p(x_t|y_{1:t-1}) = \mathcal{G}(x_t; \mu_t^{t-1}, V_t^{t-1})$$

$$\mu_t^{t-1} = A\mu_{t-1}^{t-1}$$

$$V_t^{t-1} = AV_{t-1}^{t-1}A^\top + Q$$

variance inflates

combine with likelihood



$$p(x_t|y_{1:t}) \propto \underbrace{p(x_t|y_{1:t-1})}_{\text{prior}} \underbrace{p(y_t|x_t)}_{\text{likelihood}}$$

prediction

correction

$$p(x_t|y_{1:t}) = \mathcal{G}(x_t; \mu_t^t, V_t^t)$$

$$\mu_t^t = \mu_t^{t-1} + K_t(y_t - C\mu_t^{t-1})$$

$$V_t^t = V_t^{t-1} - K_t C V_t^{t-1}$$

Kalman gain

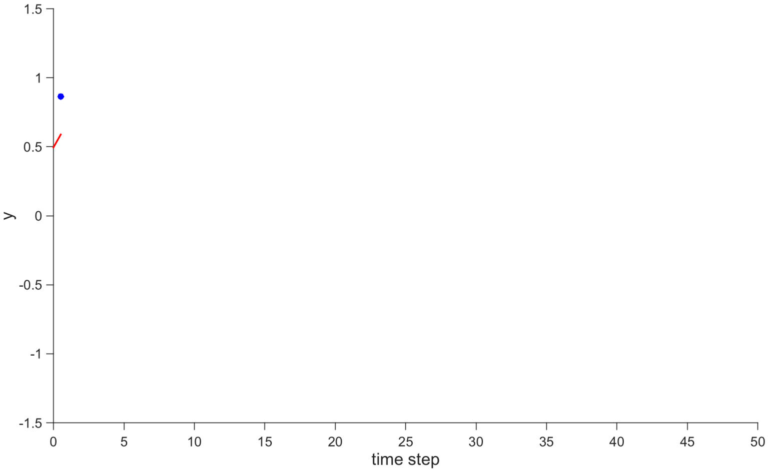
$$K_t = V_t^{t-1} C^\top (C V_t^{t-1} C^\top + R)^{-1}$$

Kalman Filter Demo

- data: $y_t = \sin(\omega t) + \sigma_y \epsilon_t$ where $\sigma_y^2 = 0.1$
- model: $x_t = \lambda x_{t-1} + \sigma \eta$ and $y_t = x_t + \sigma_y \eta'_t$
  where $\lambda = 0.99$ and $\sigma^2 = 1 - \lambda^2$
- demo shows how the Kalman filter processes the data to form estimates of the hidden state at each time point $p(x_t | y_{1:t})$

Kalman Filter Demo

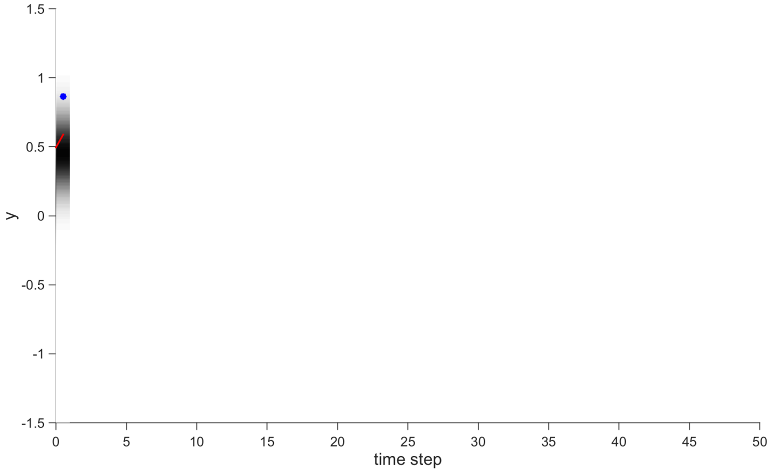
observed noisy data y_t, ground truth sinusoid



observe first data point y_1

Kalman Filter Demo

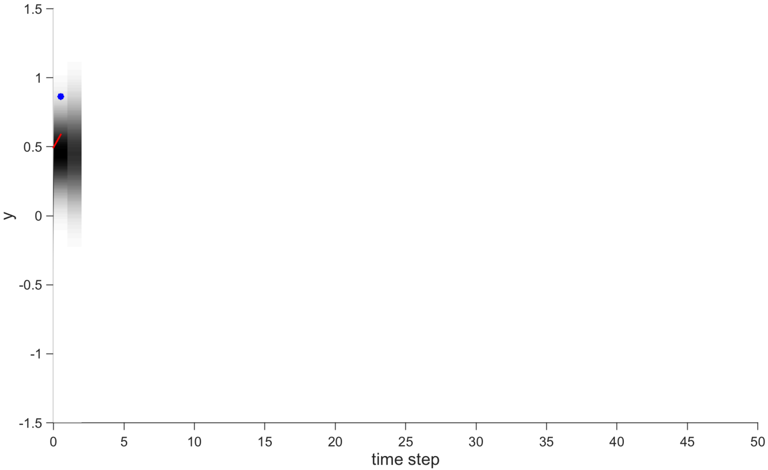
observed noisy data y_t, ground truth sinusoid



posterior over first latent variable $p(x_1|y_1)$

Kalman Filter Demo

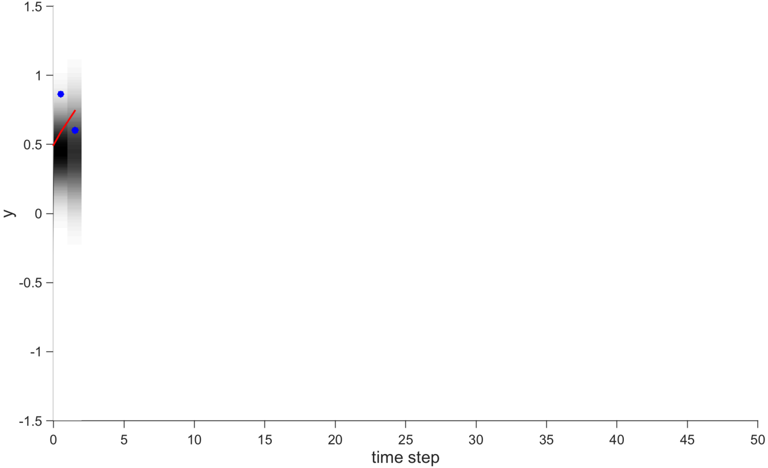
observed noisy data y_t, ground truth sinusoid



prediction for second latent variable $p(x_2|y_1)$

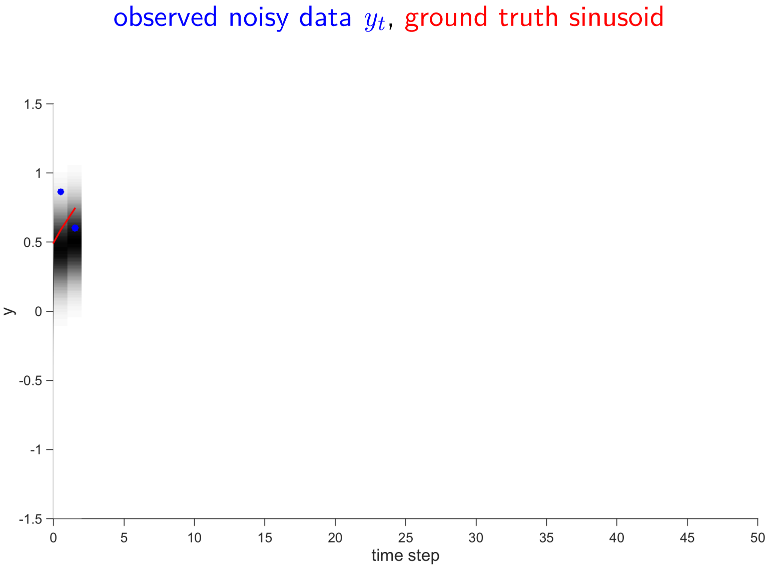
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



observe next data point y_2

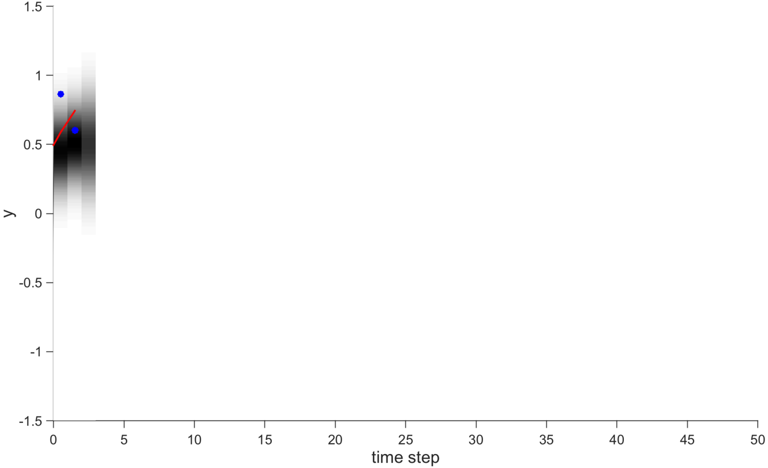
Kalman Filter Demo



form posterior over second latent variable $p(x_2|y_1, y_2)$

Kalman Filter Demo

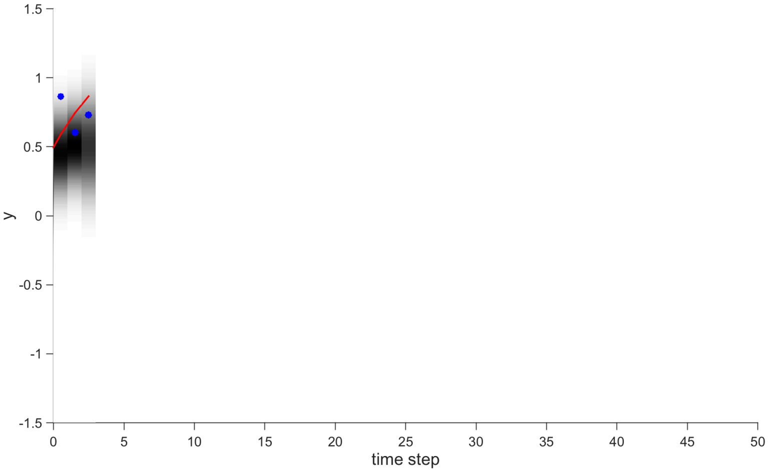
observed noisy data y_t, ground truth sinusoid



prediction for third latent variable $p(x_3|y_1, y_2)$

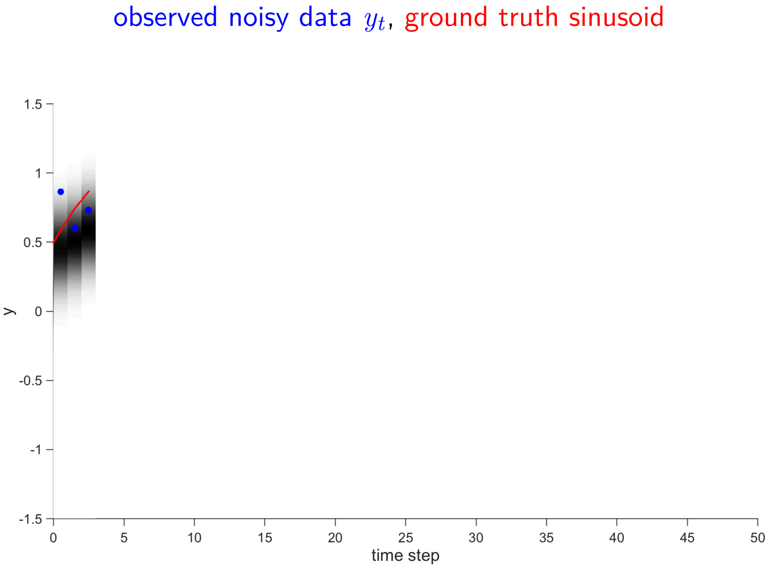
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



observe next data point y_3

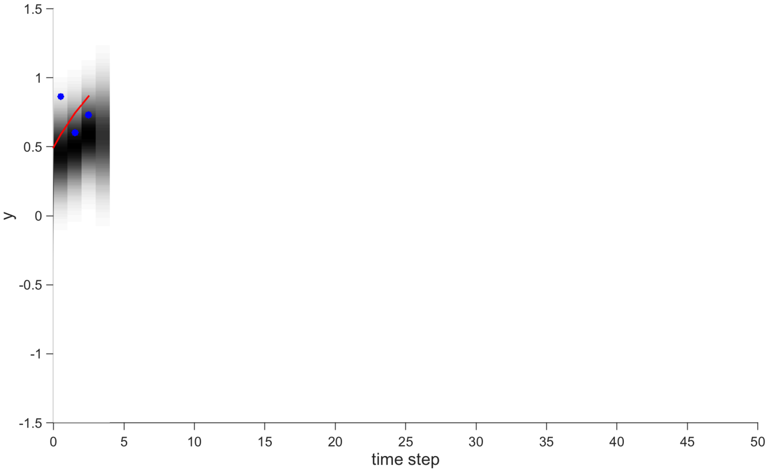
Kalman Filter Demo



form posterior over third latent variable $p(x_3|y_1, y_2, y_3)$

Kalman Filter Demo

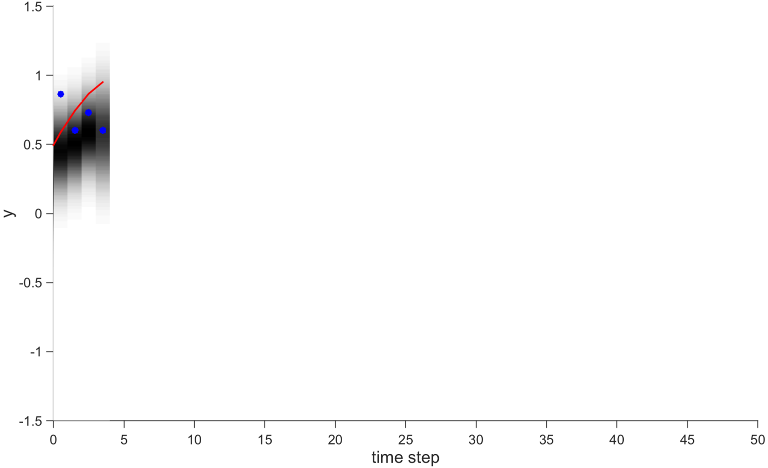
observed noisy data y_t, ground truth sinusoid



prediction for fourth latent variable $p(x_4|y_{1:3})$

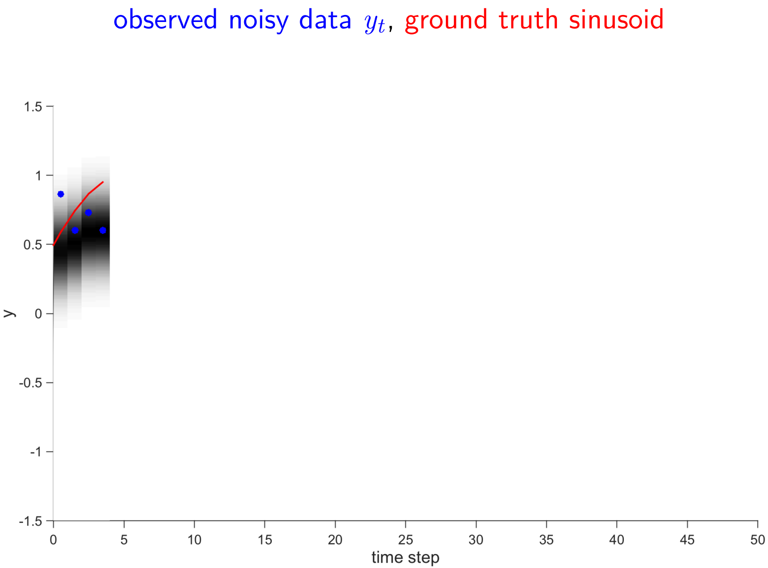
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



observe next data point y_4

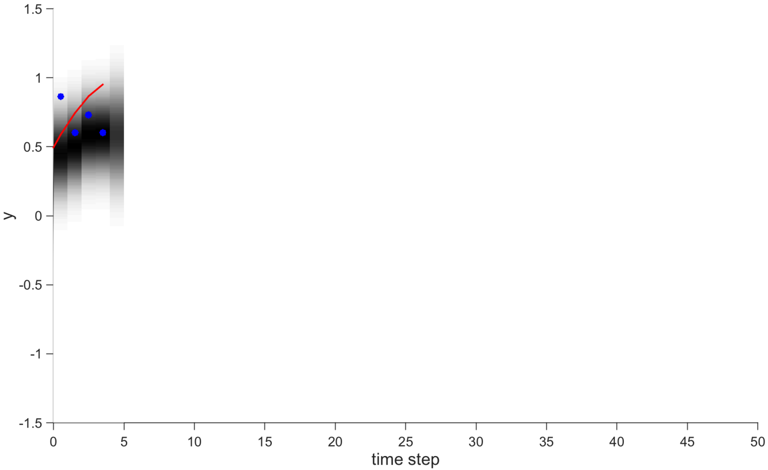
Kalman Filter Demo



form posterior over fourth latent variable $p(x_4|y_{1:4})$

Kalman Filter Demo

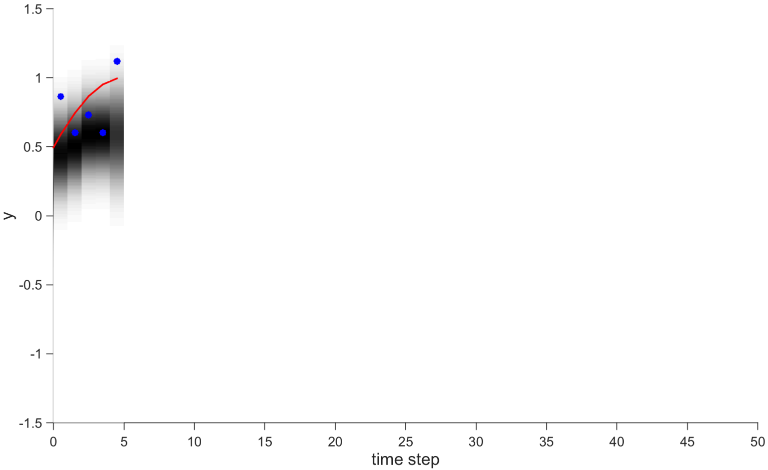
observed noisy data y_t, ground truth sinusoid



prediction for fifth latent variable $p(x_5|y_{1:4})$

Kalman Filter Demo

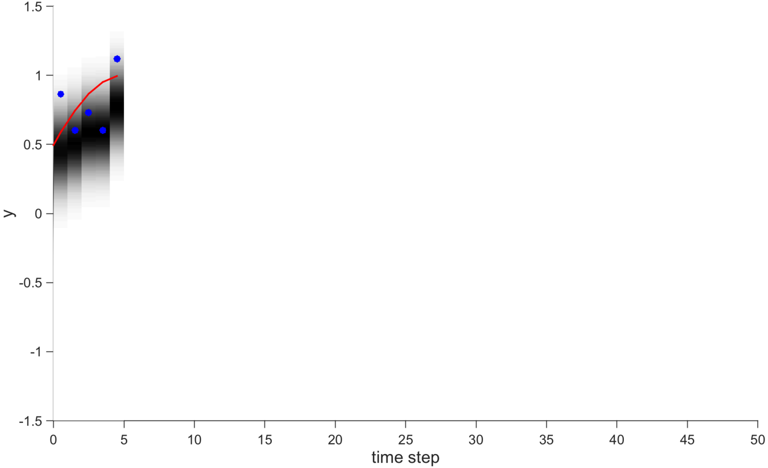
observed noisy data y_t, ground truth sinusoid



observe next data point y_5

Kalman Filter Demo

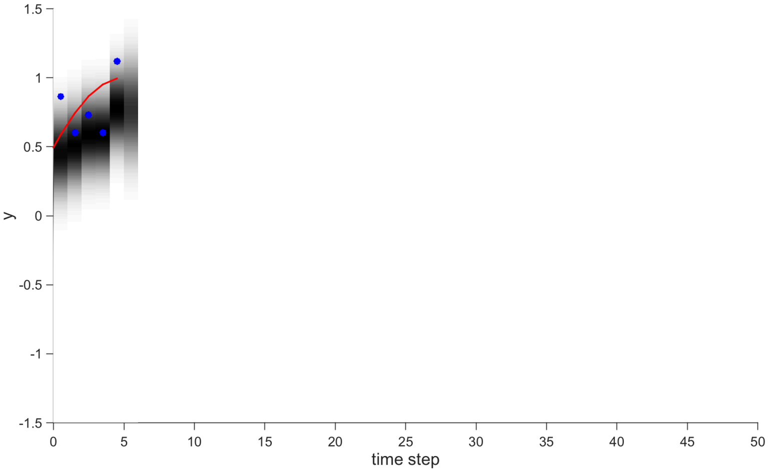
observed noisy data y_t, ground truth sinusoid



form posterior over fifth latent variable $p(x_5|y_{1:5})$

Kalman Filter Demo

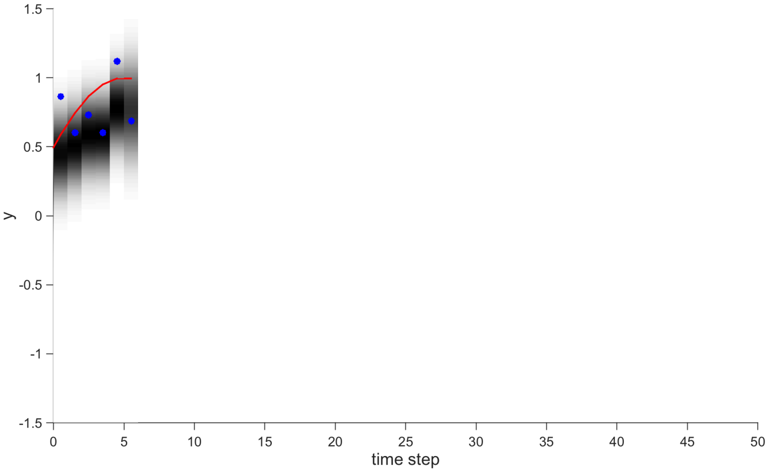
observed noisy data y_t, ground truth sinusoid



prediction for sixth latent variable $p(x_6|y_{1:5})$

Kalman Filter Demo

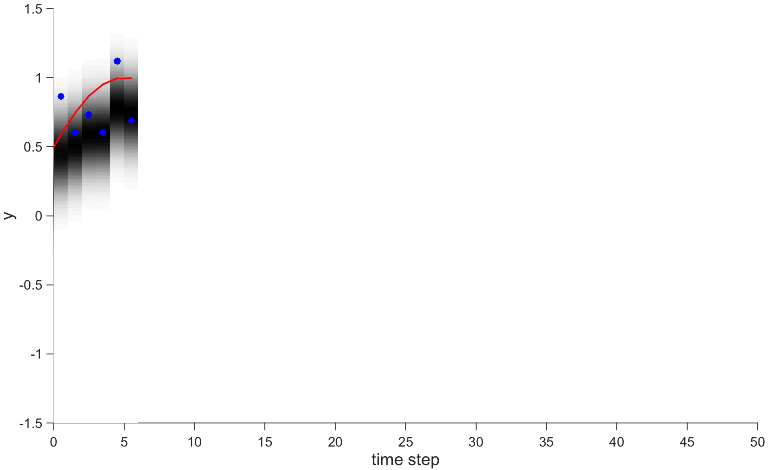
observed noisy data y_t, ground truth sinusoid



observe next data point y_6

Kalman Filter Demo

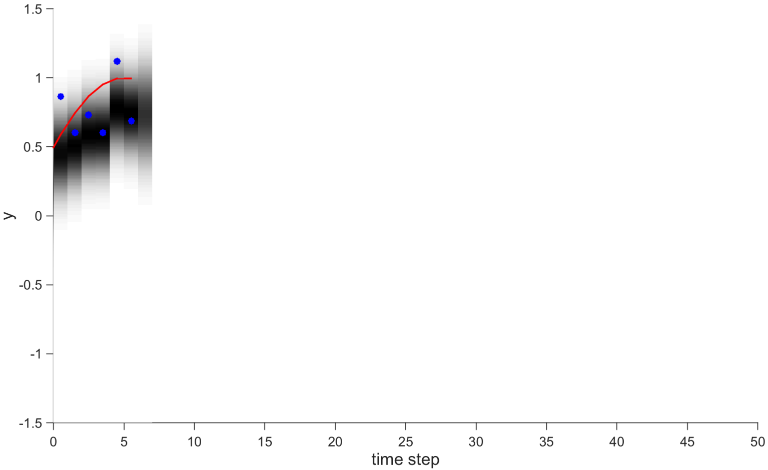
observed noisy data y_t, ground truth sinusoid



form posterior over sixth latent variable $p(x_6|y_{1:6})$

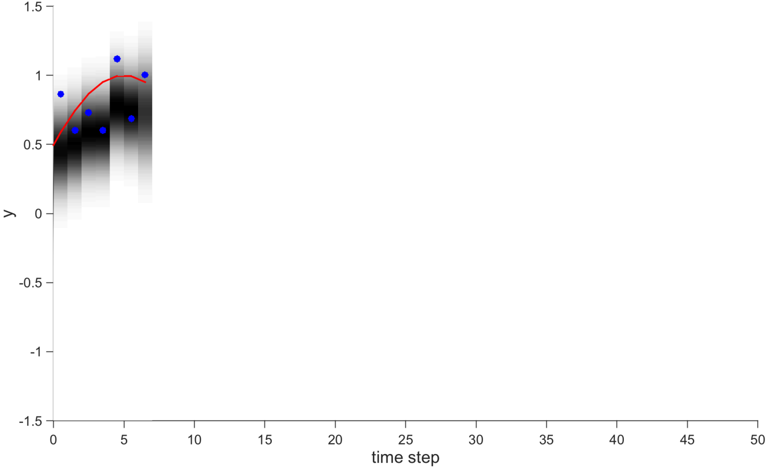
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



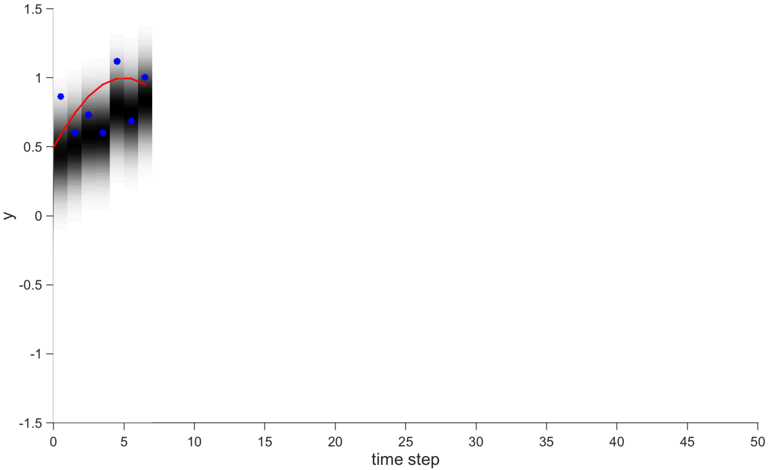
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



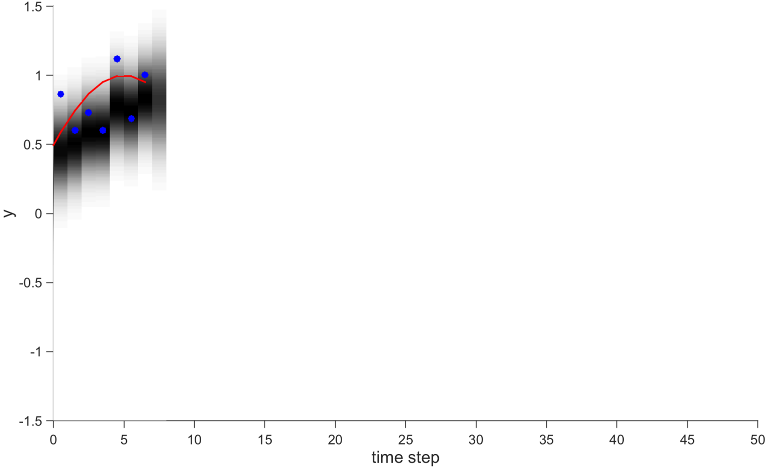
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid

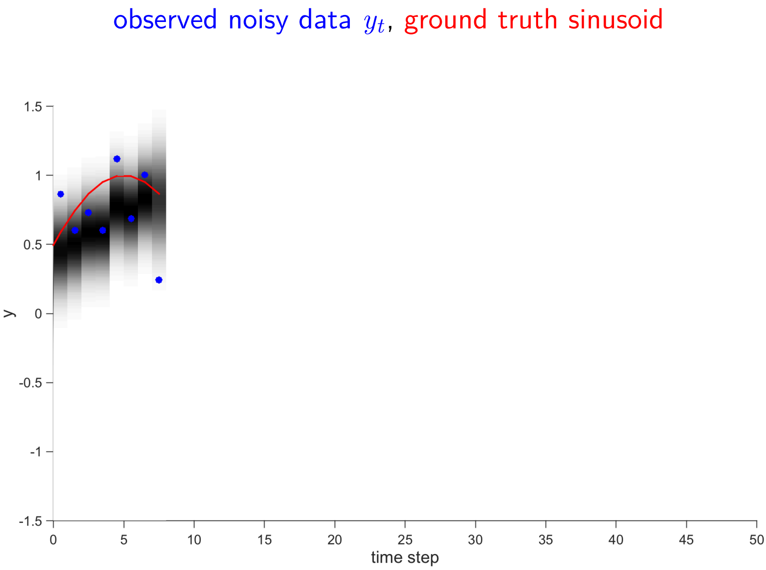


Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid

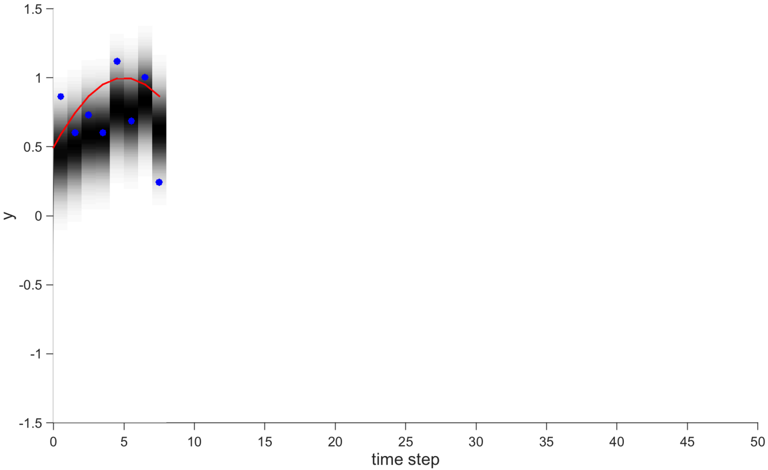


Kalman Filter Demo



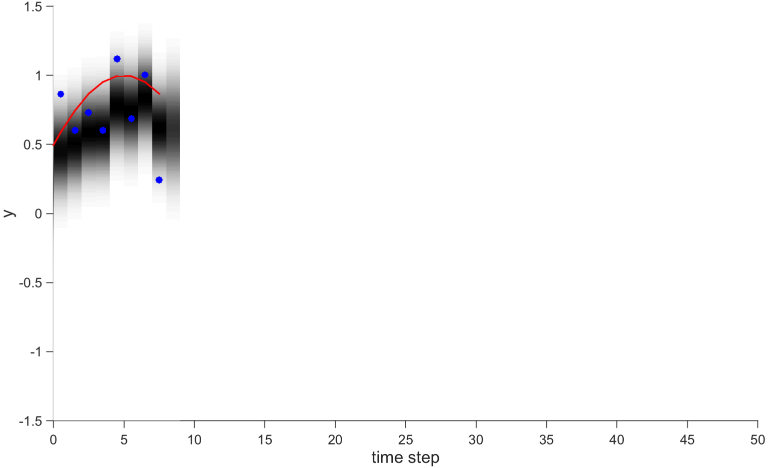
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid

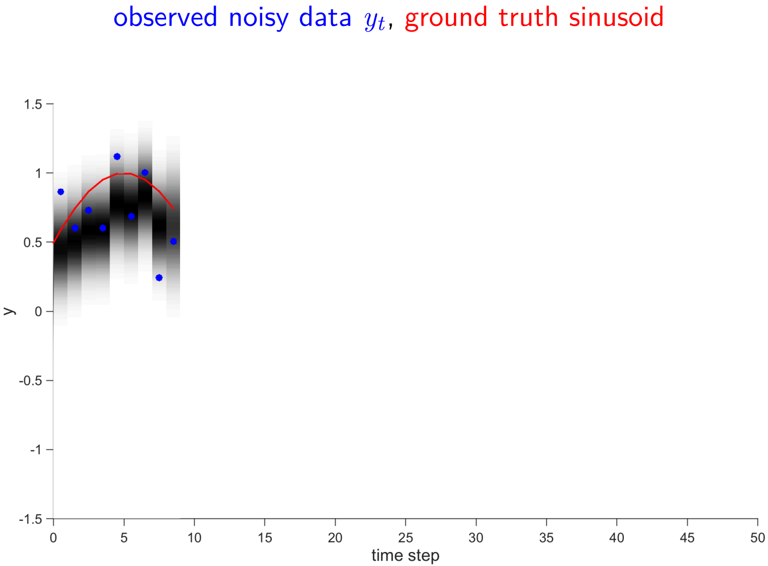


Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid

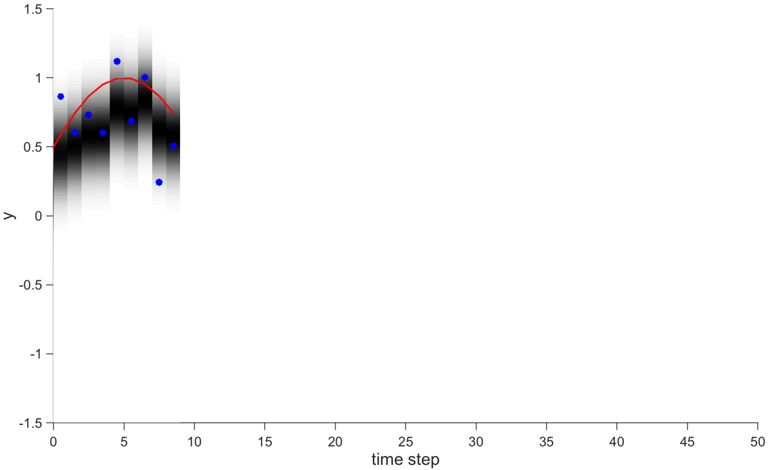


Kalman Filter Demo



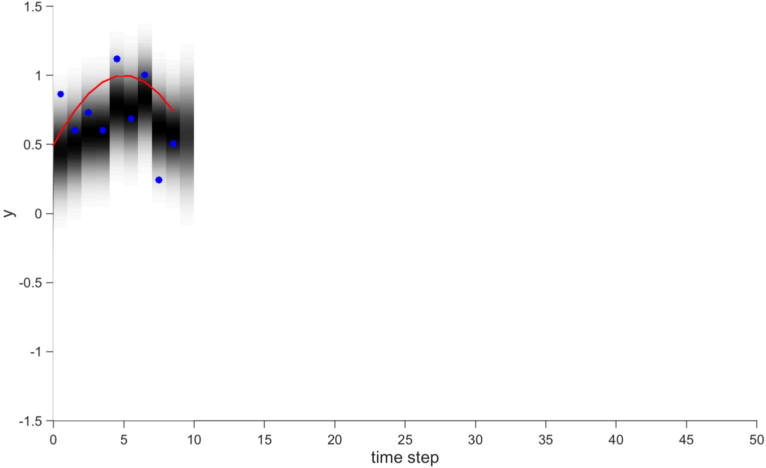
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



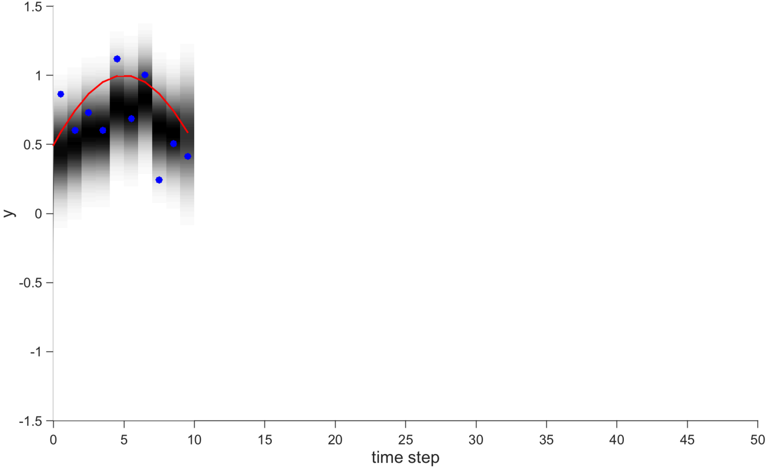
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



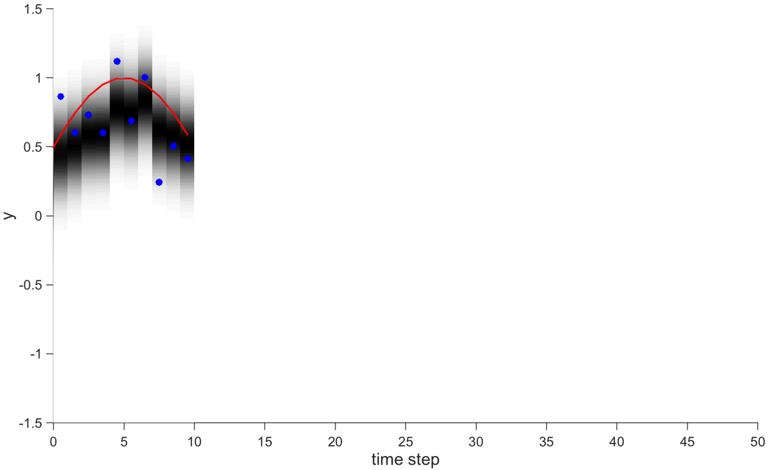
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



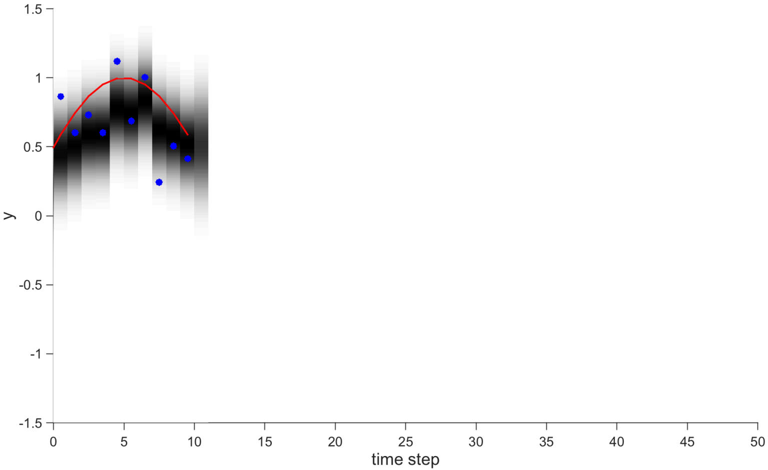
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



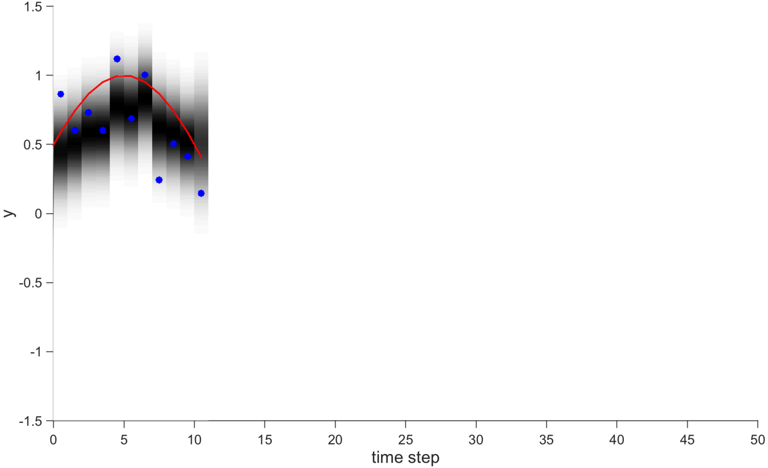
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid

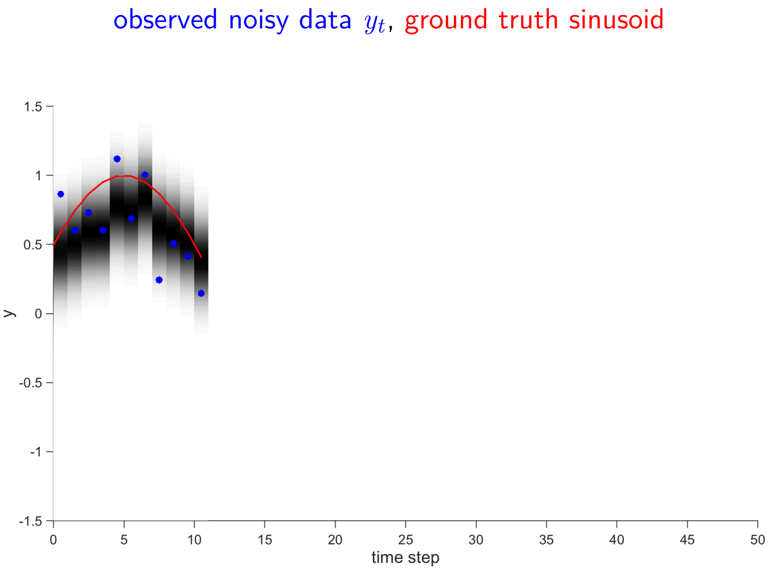


Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



Kalman Filter Demo



Kalman Filter Demo

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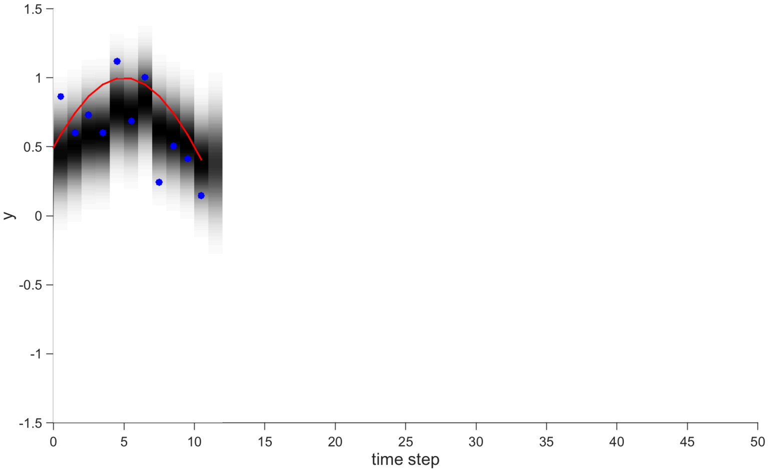
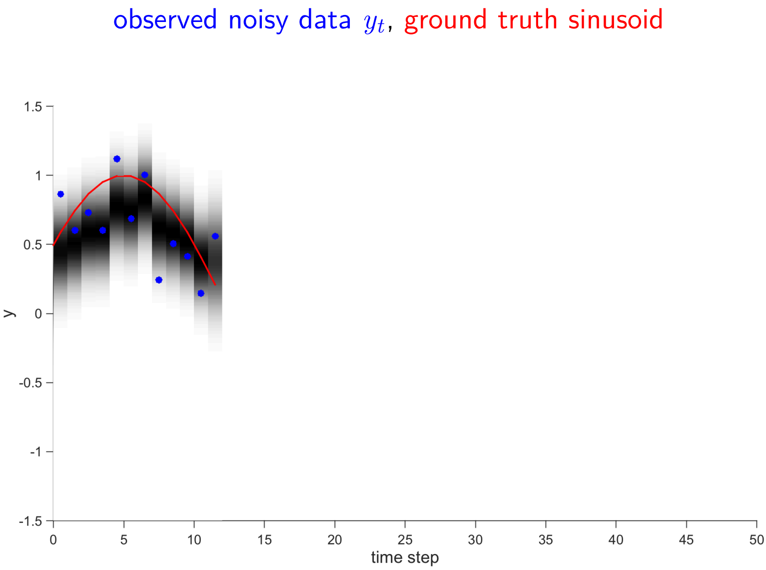
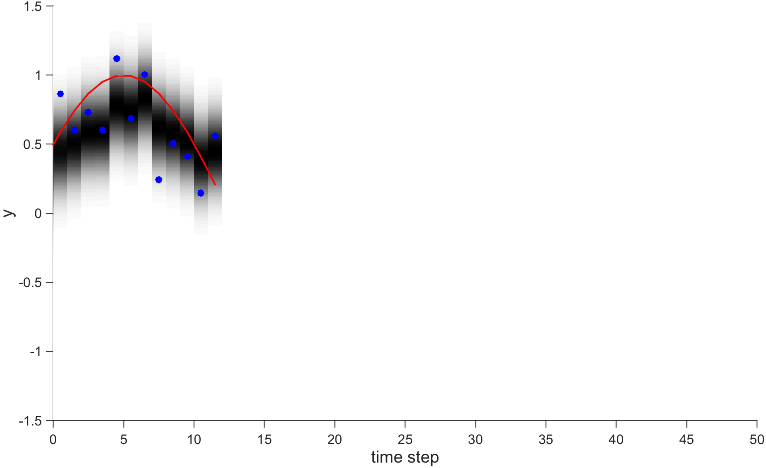


Kalman Filter Demo



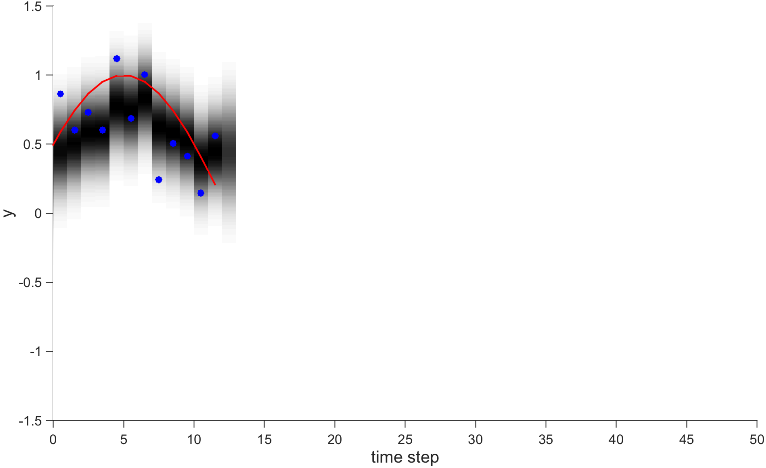
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



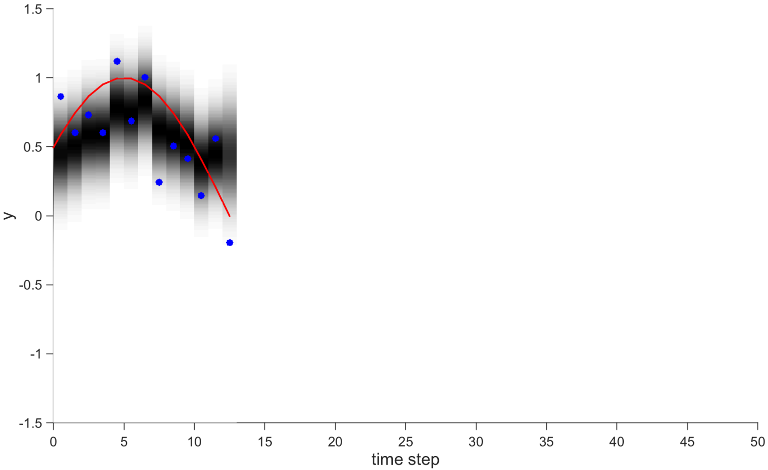
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



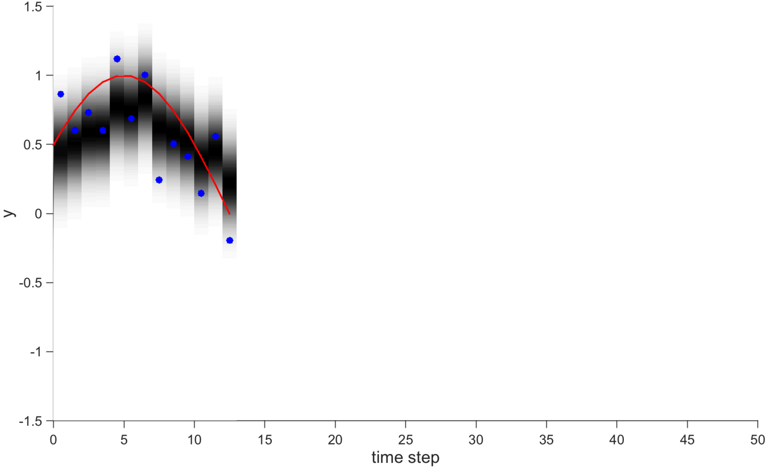
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



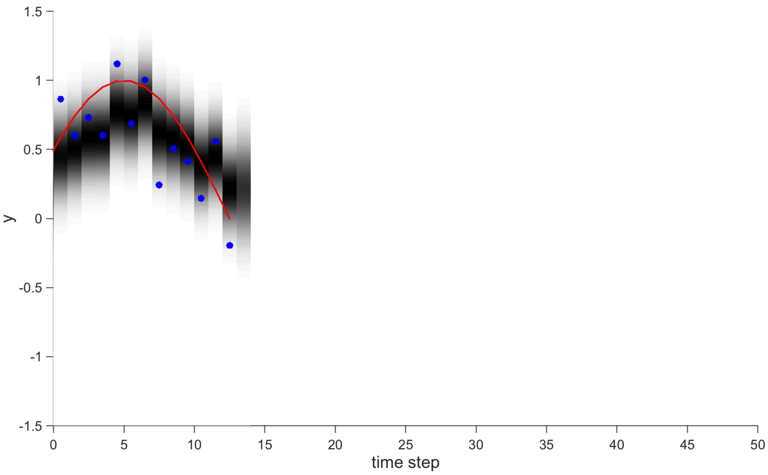
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



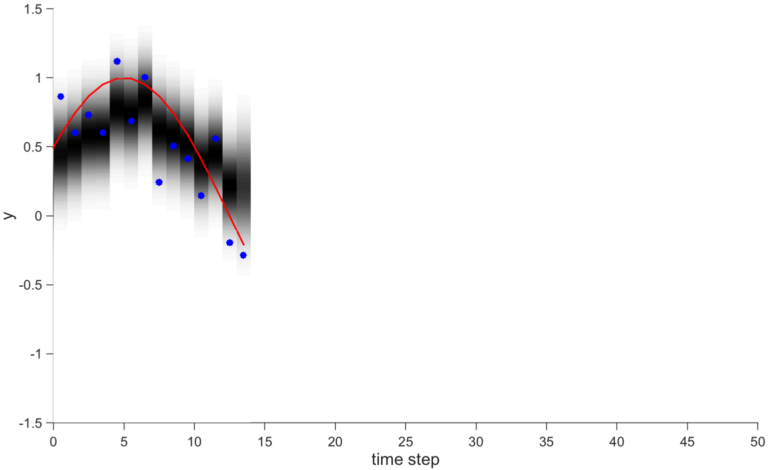
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



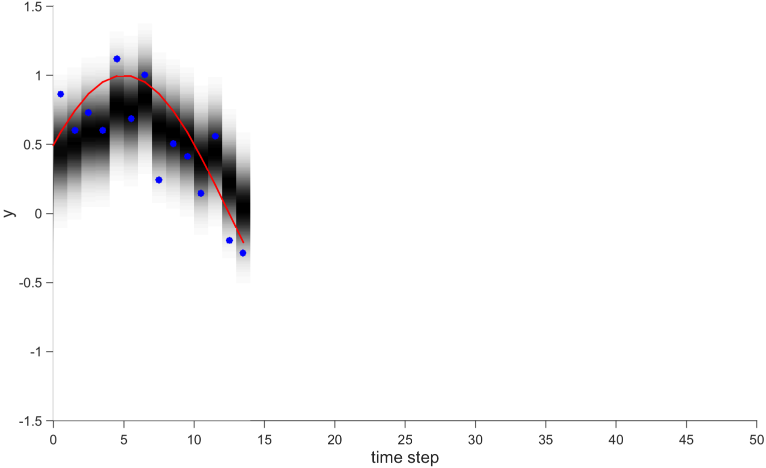
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



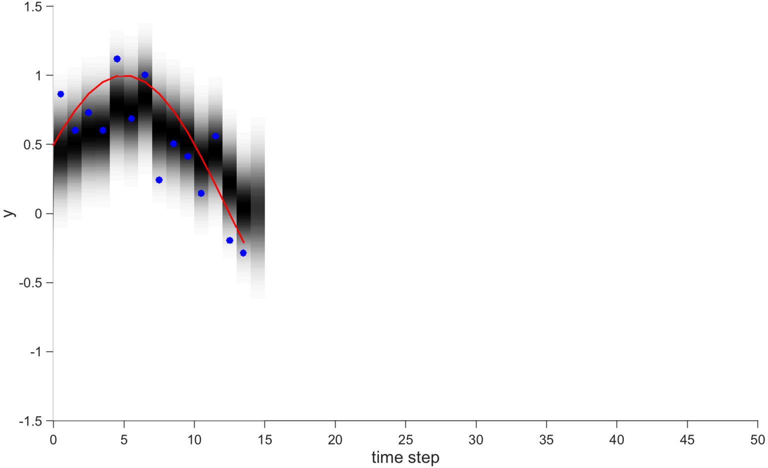
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



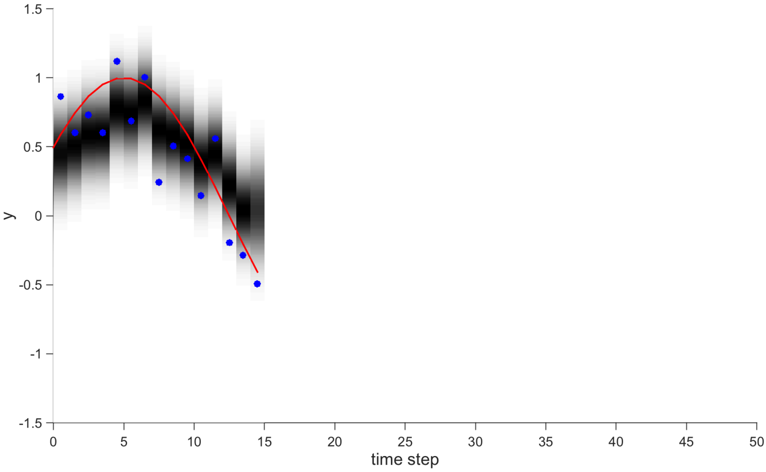
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



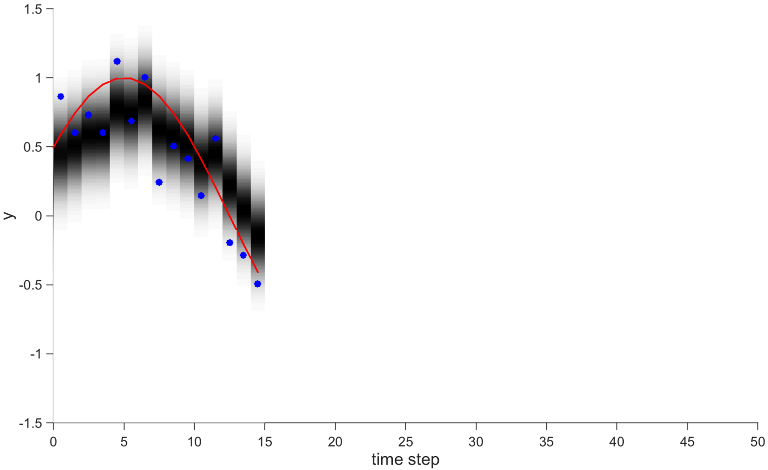
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



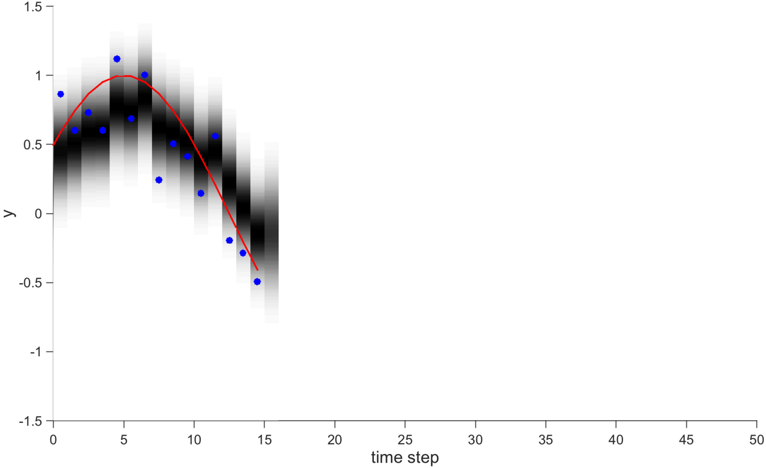
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



Kalman Filter Demo

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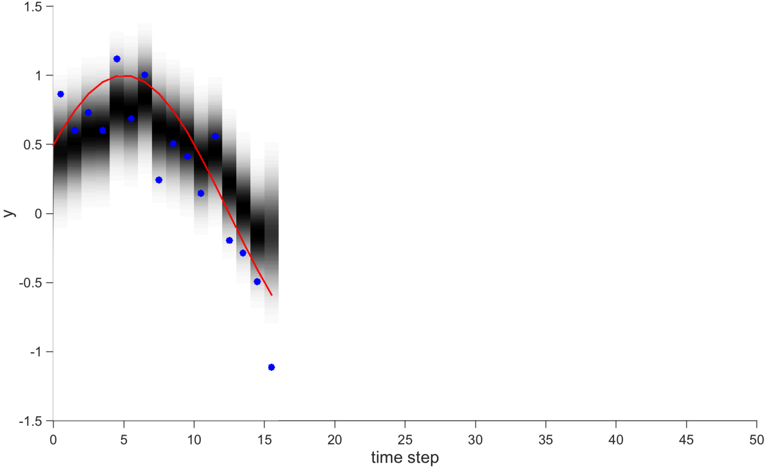
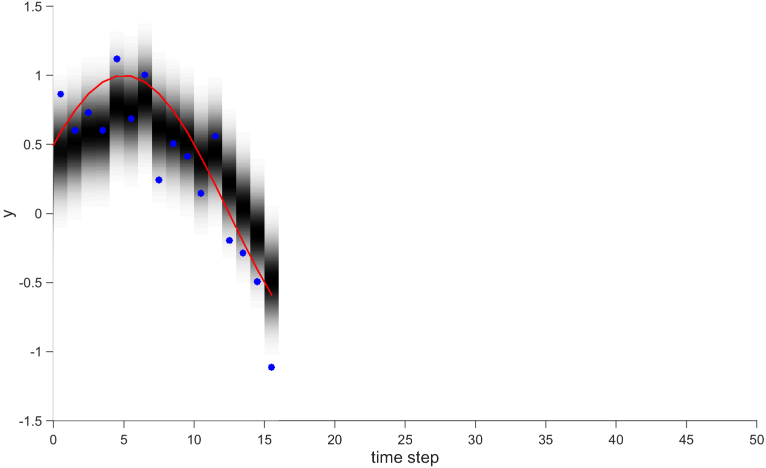


Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



Kalman Filter Demo

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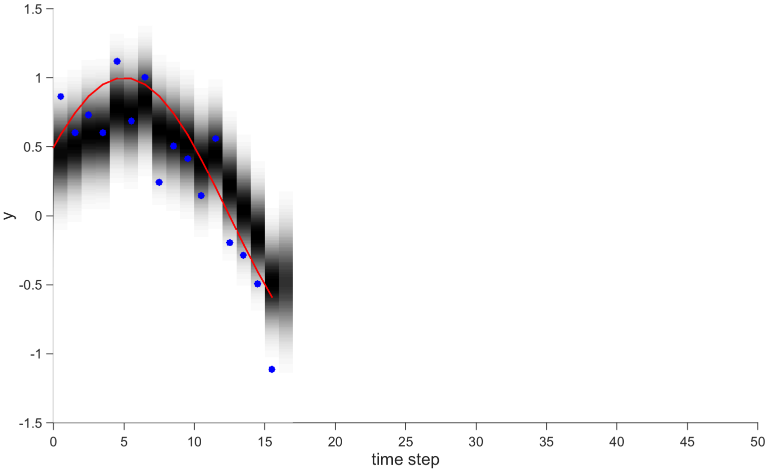
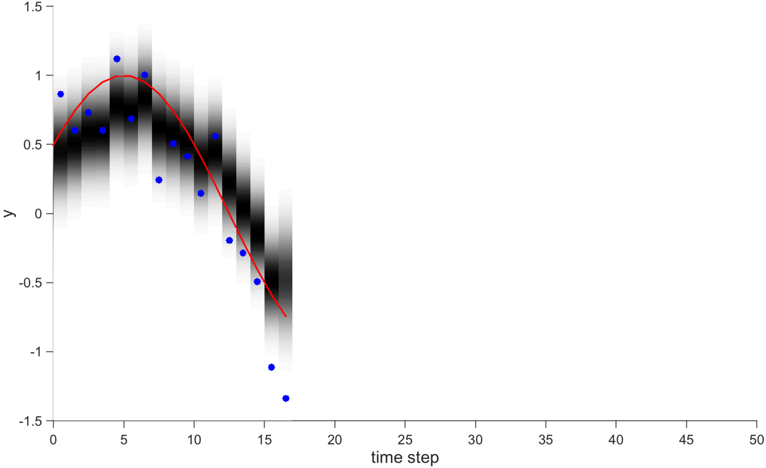


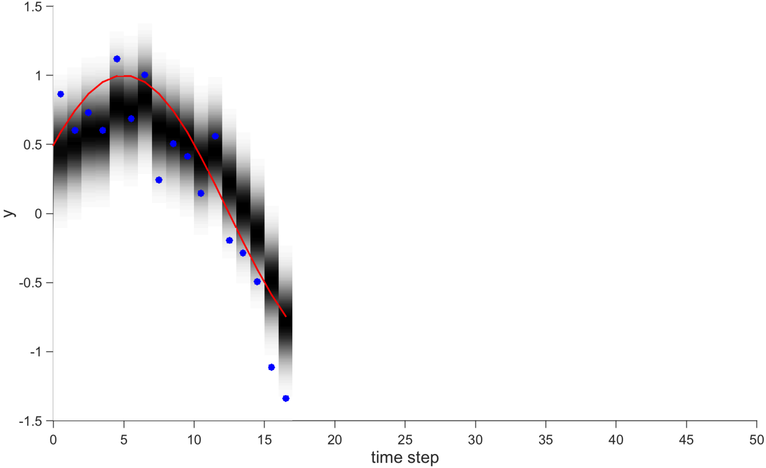
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



Kalman Filter Demo

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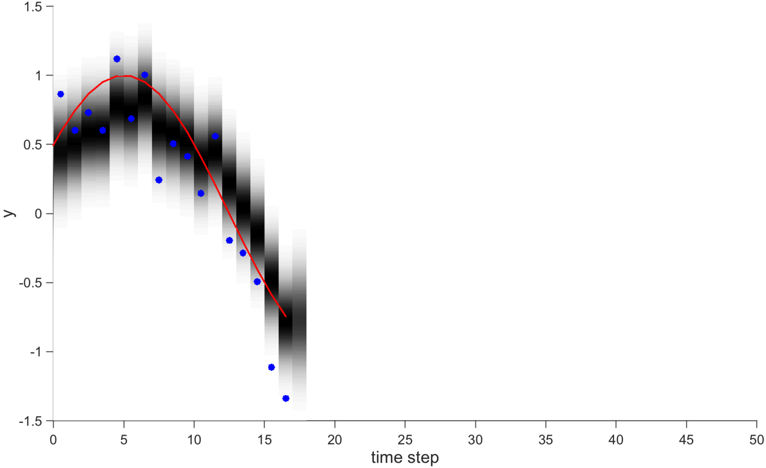
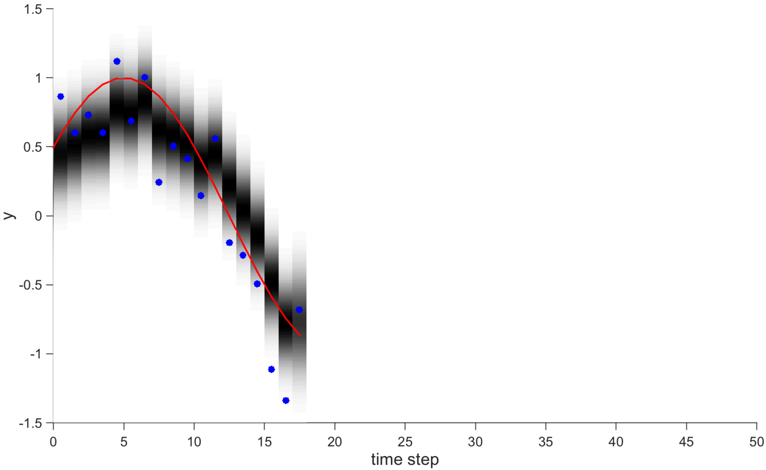


Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



Kalman Filter Demo

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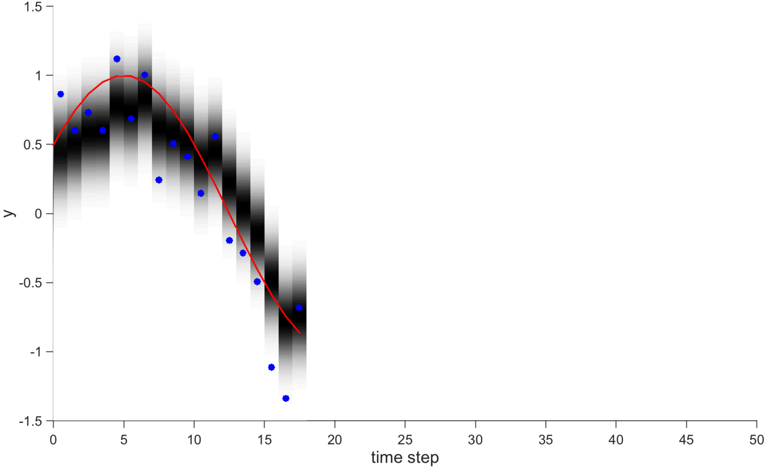

Kalman Filter Demo

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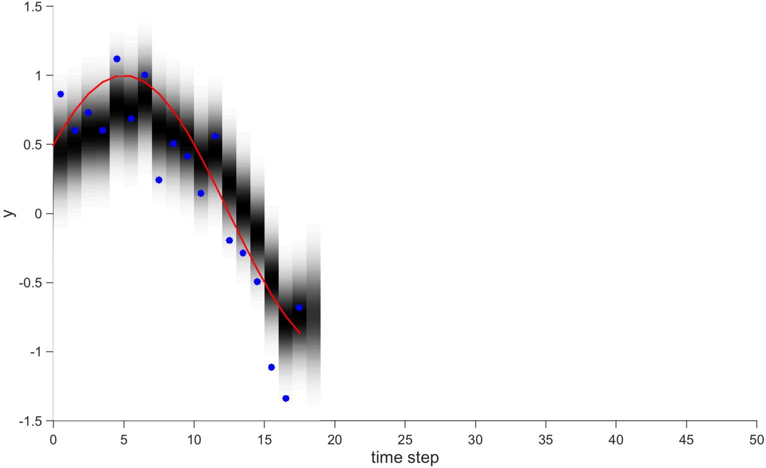

Kalman Filter Demo

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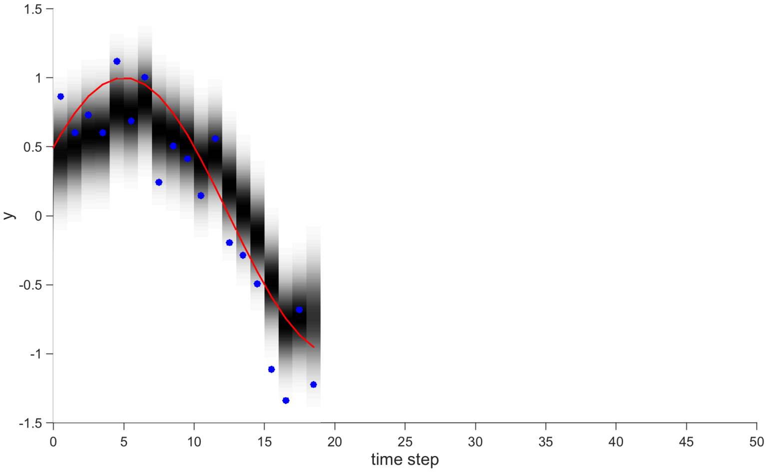
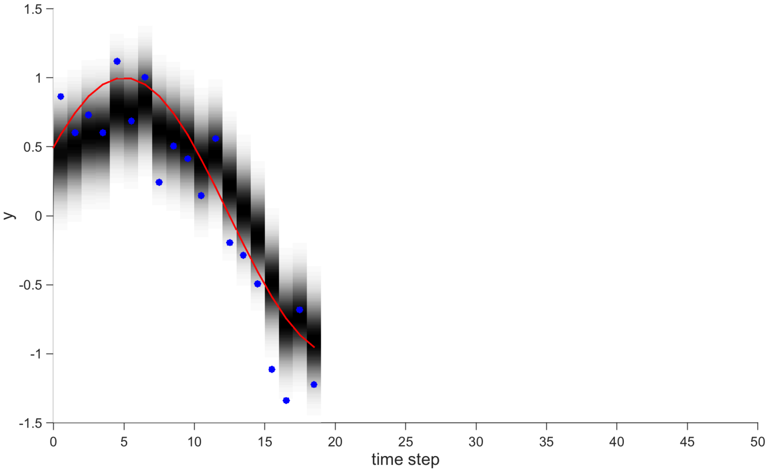


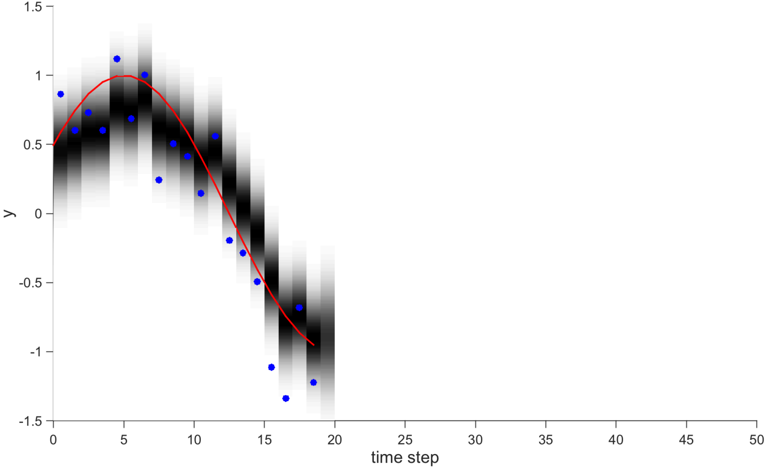
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



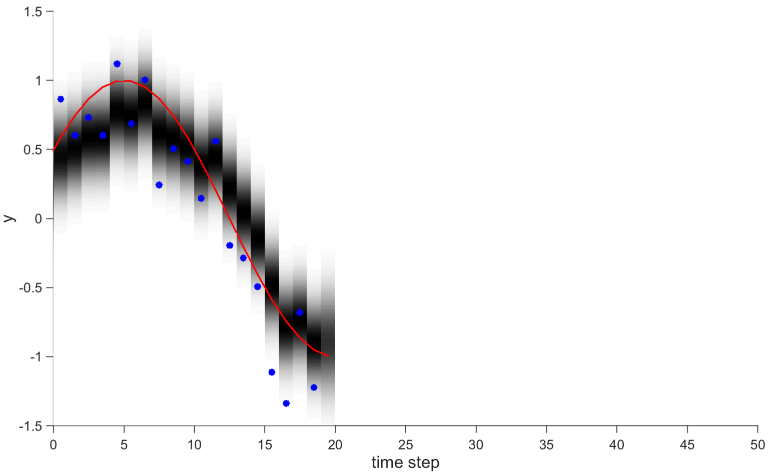
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



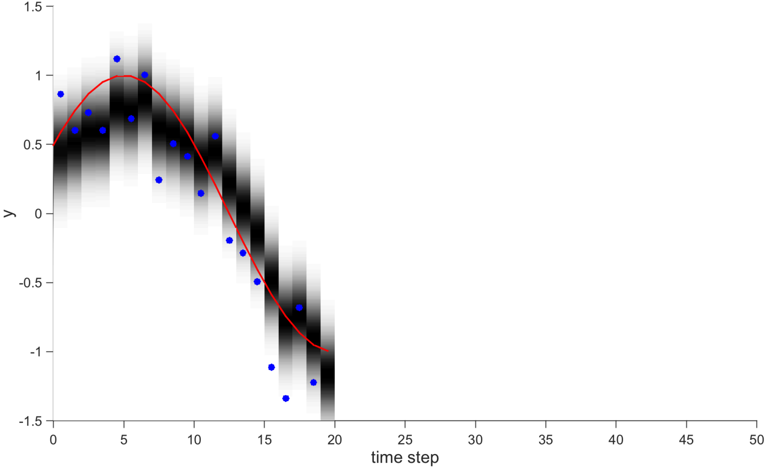
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



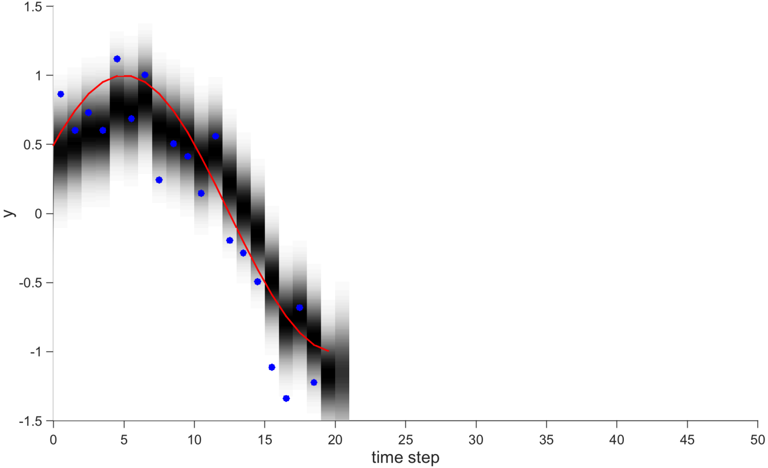
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



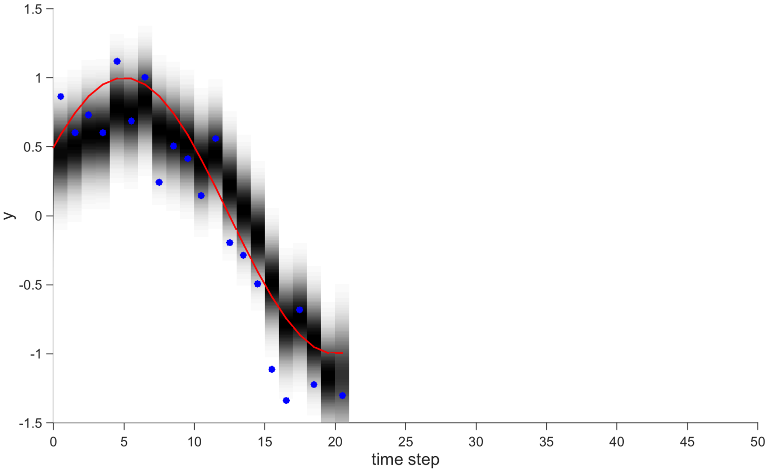
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



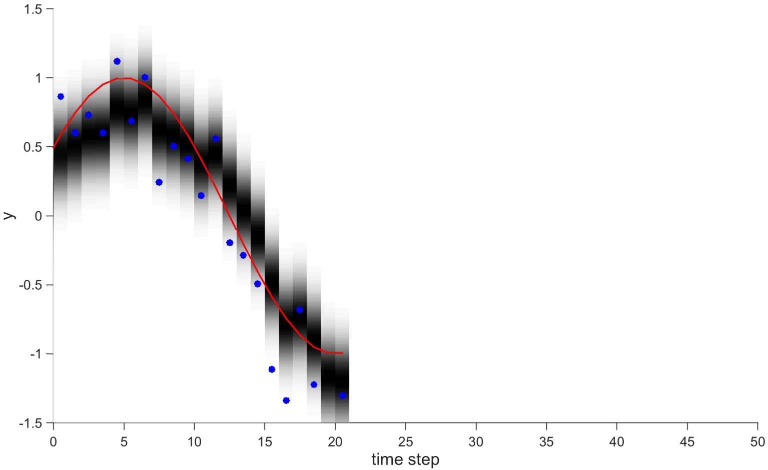
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



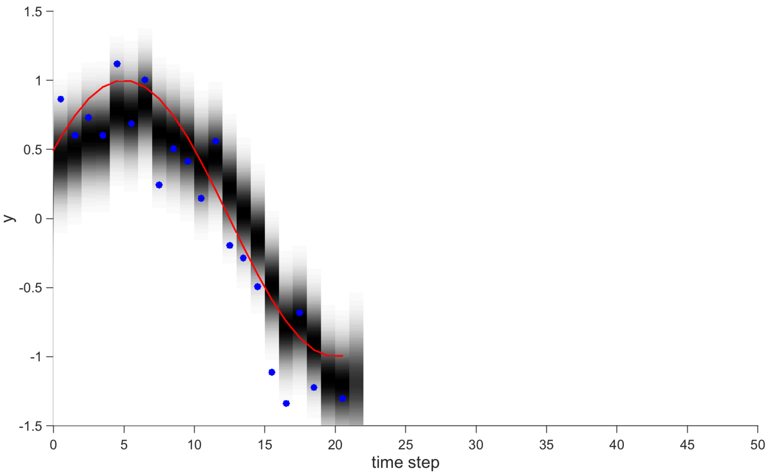
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



Kalman Filter Demo

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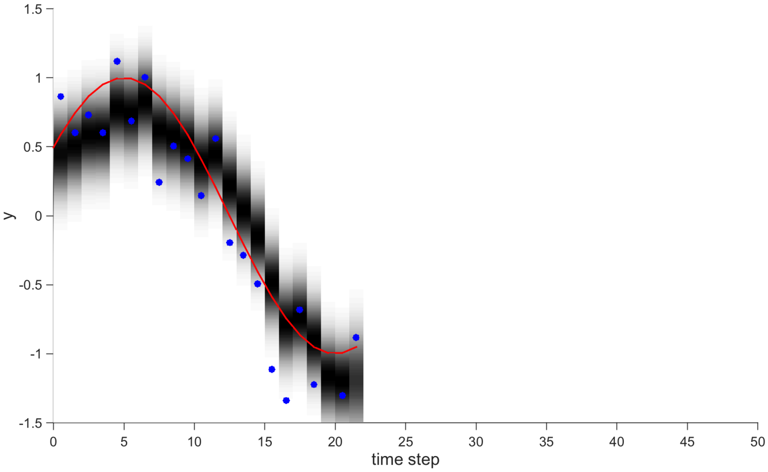

Kalman Filter Demo

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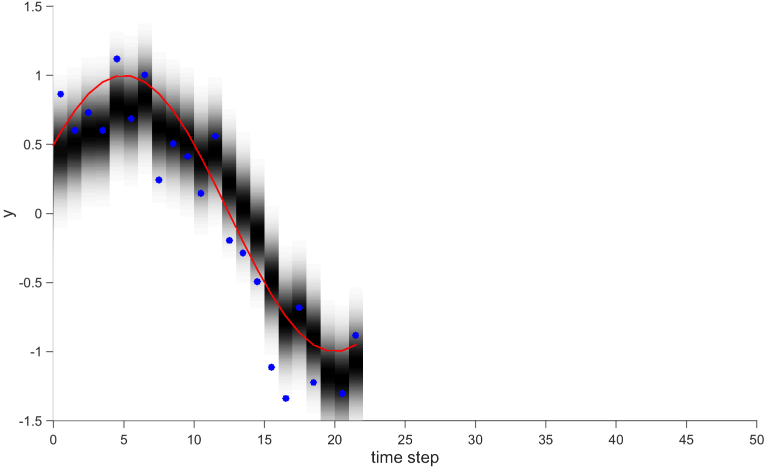

Kalman Filter Demo

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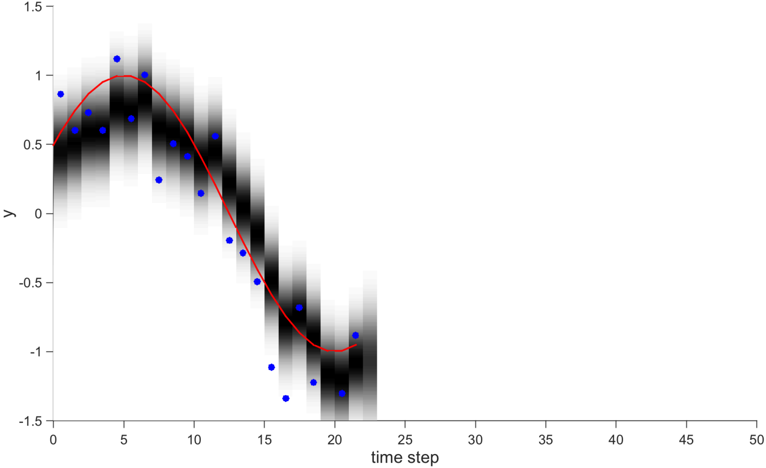

Kalman Filter Demo

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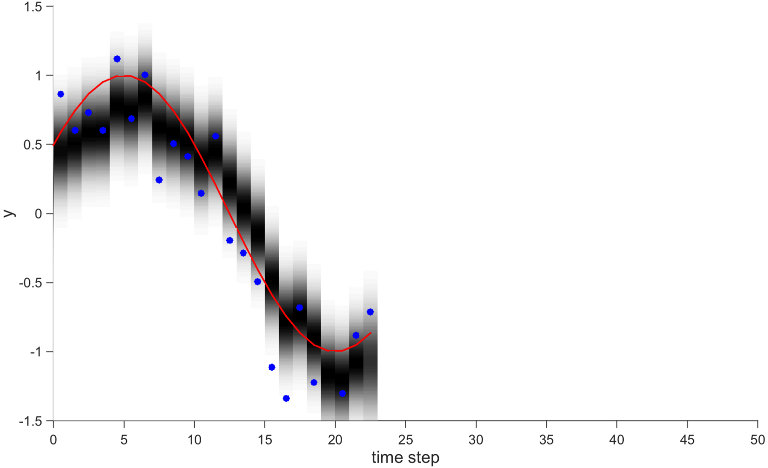

Kalman Filter Demo

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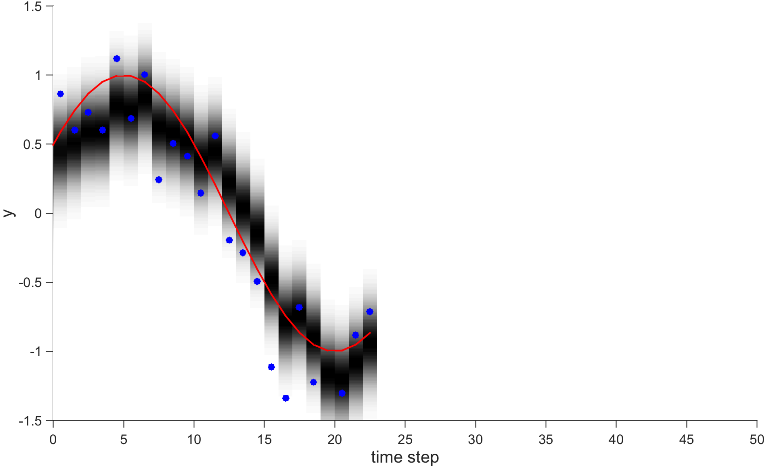

Kalman Filter Demo

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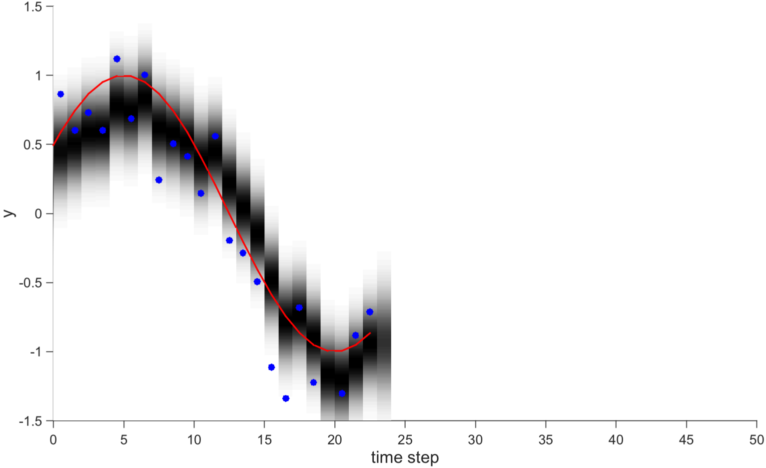

Kalman Filter Demo

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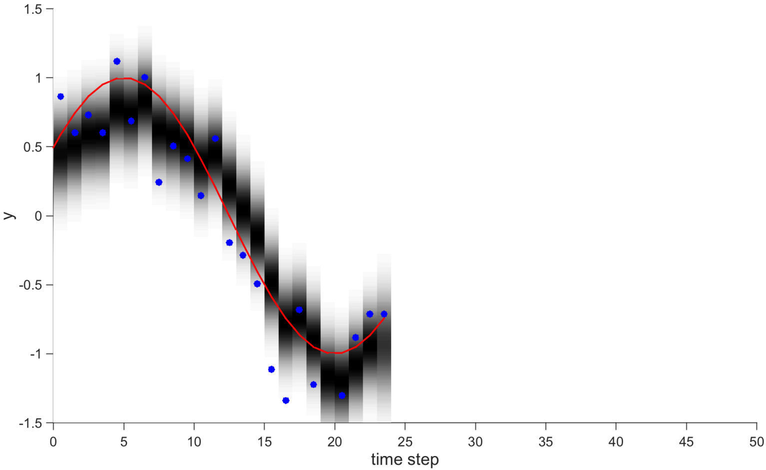

Kalman Filter Demo

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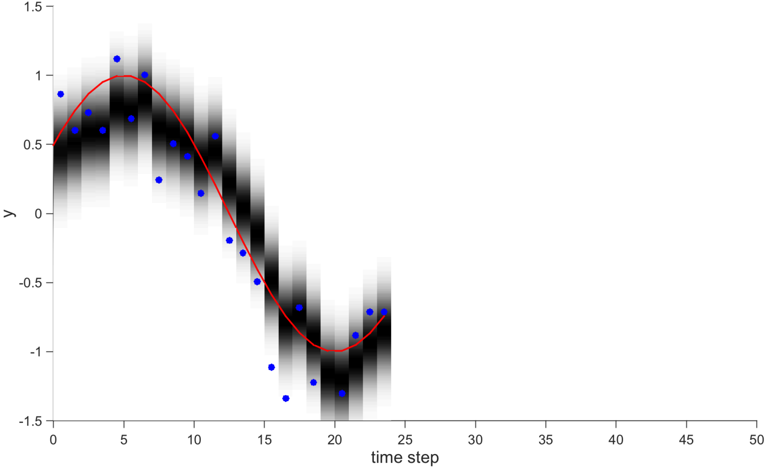

Kalman Filter Demo

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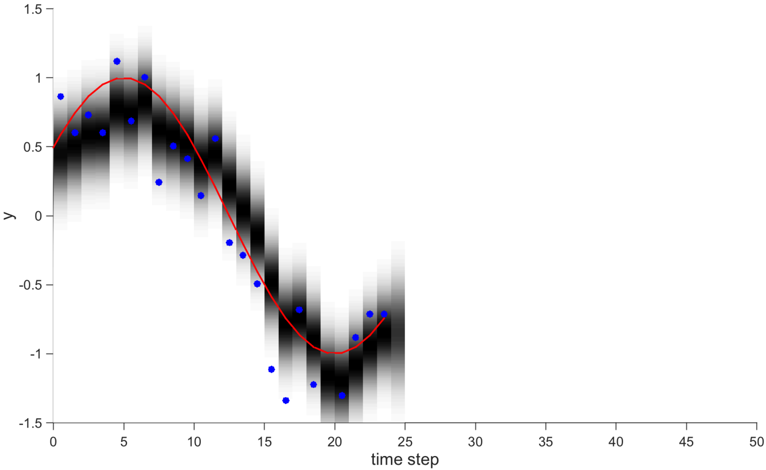

Kalman Filter Demo

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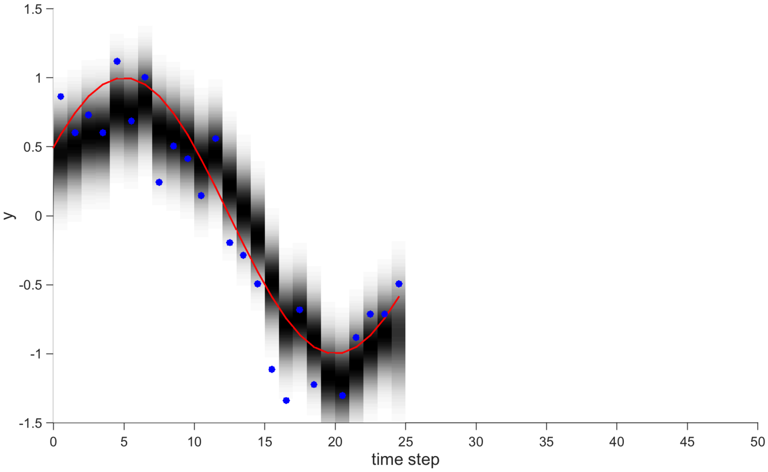

Kalman Filter Demo

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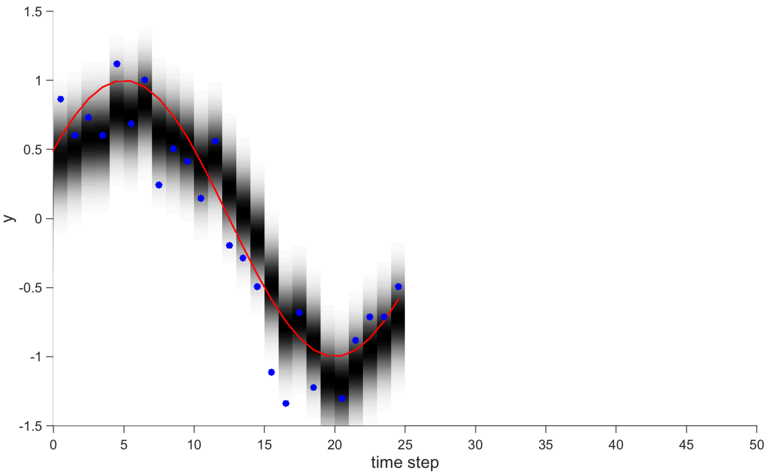

Kalman Filter Demo

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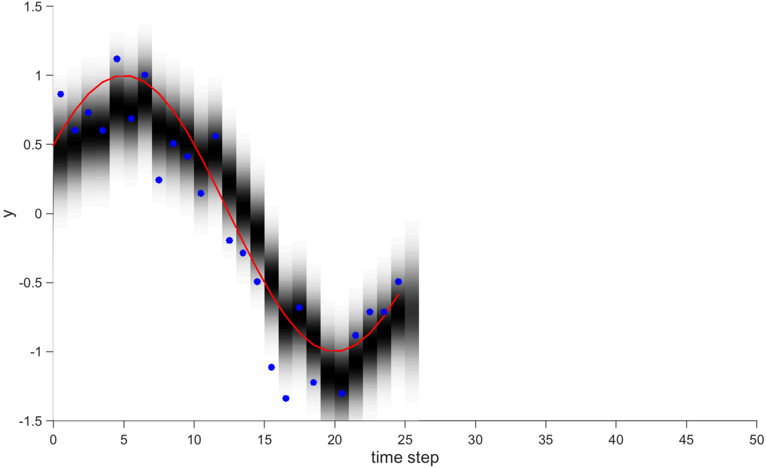

Kalman Filter Demo

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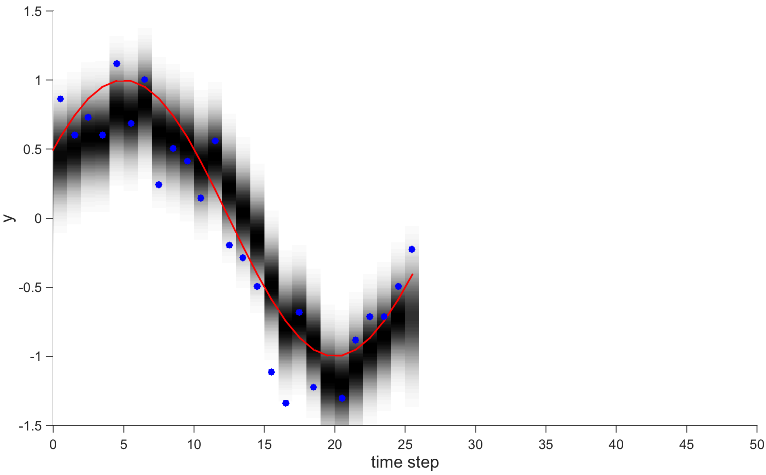

Kalman Filter Demo

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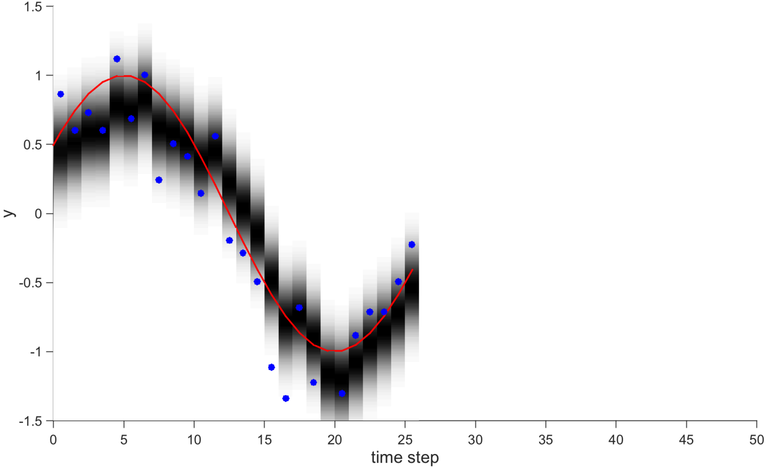

Kalman Filter Demo

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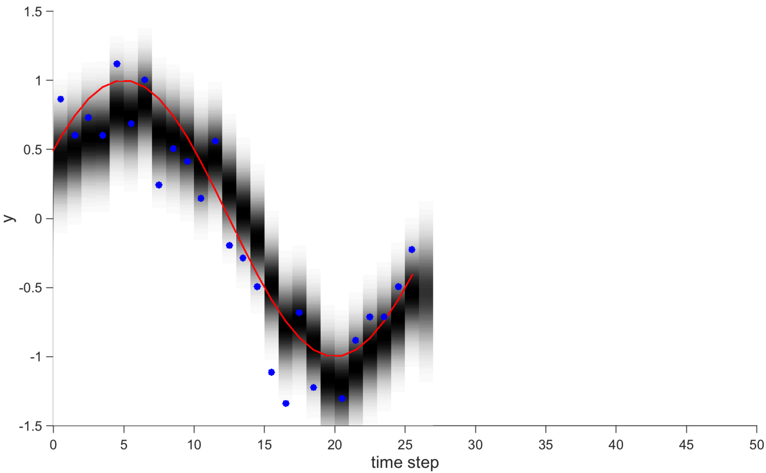
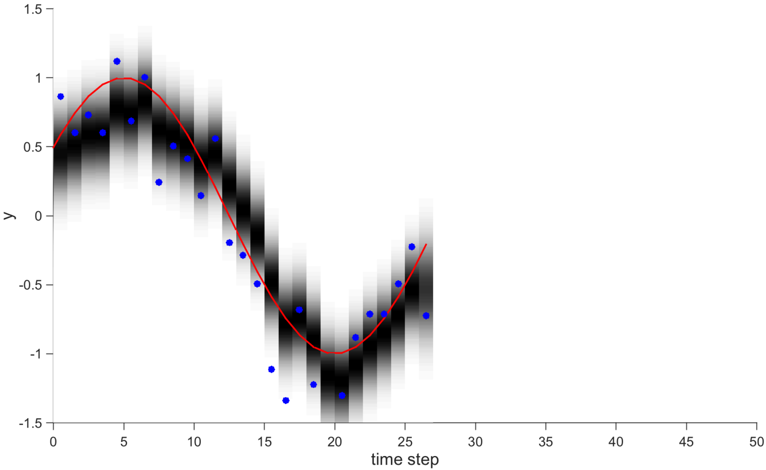


Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



Kalman Filter Demo

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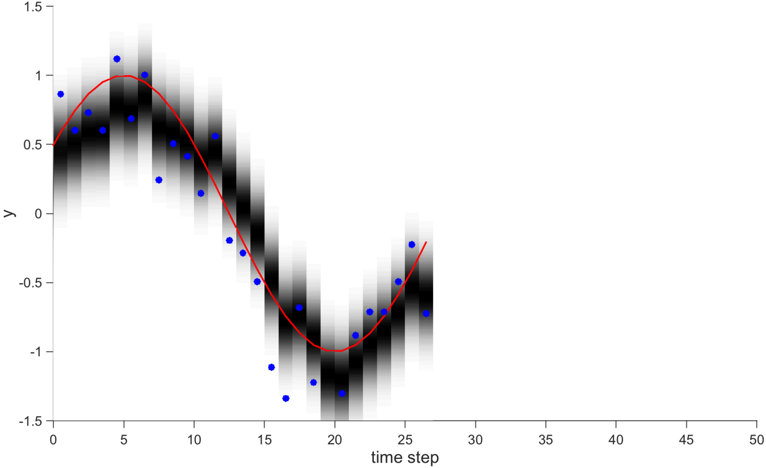

Kalman Filter Demo

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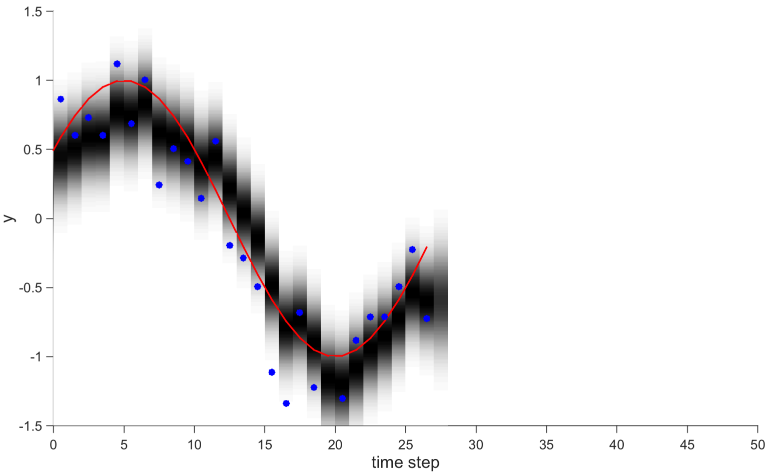

Kalman Filter Demo

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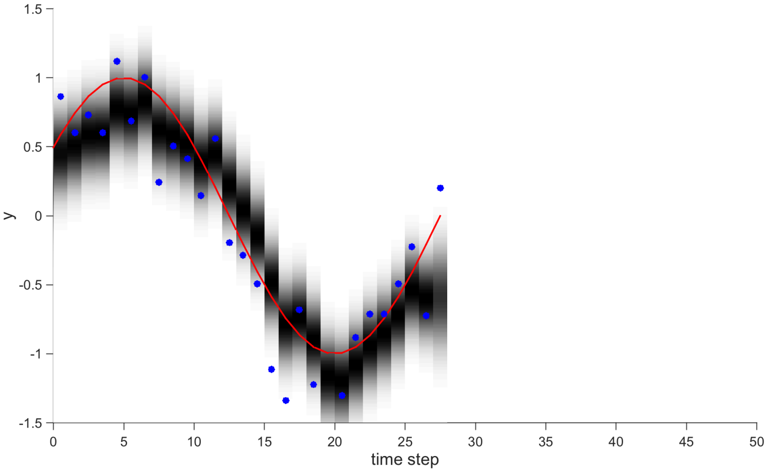

Kalman Filter Demo

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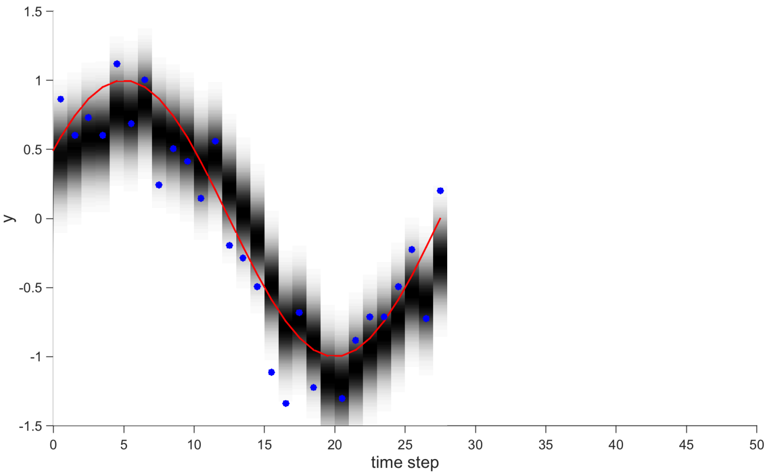

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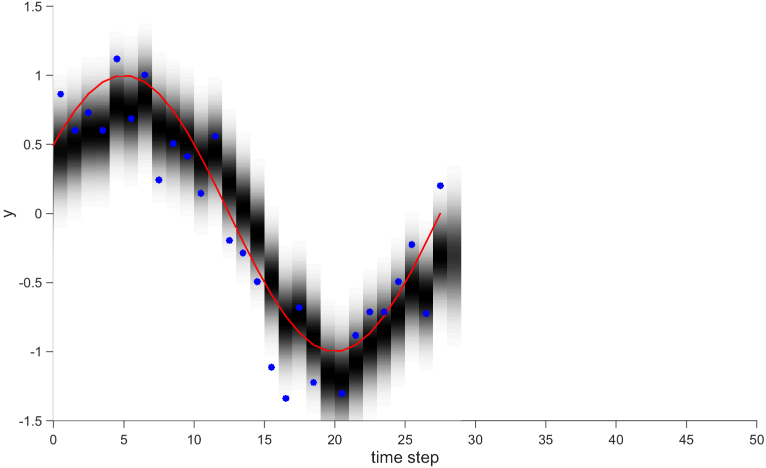

Kalman Filter Demo

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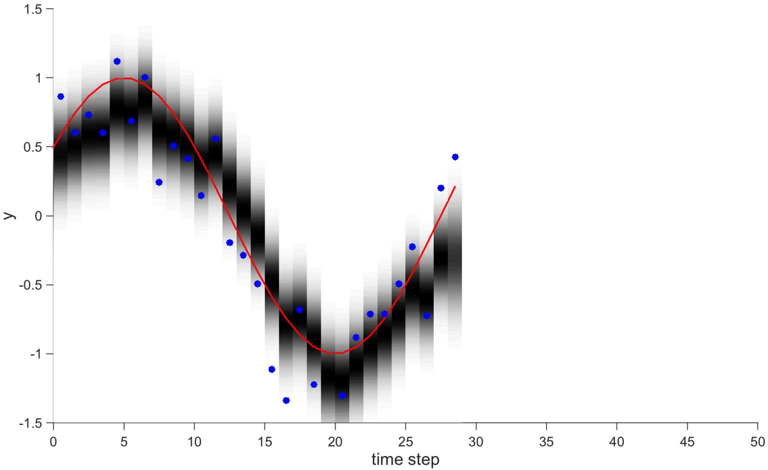

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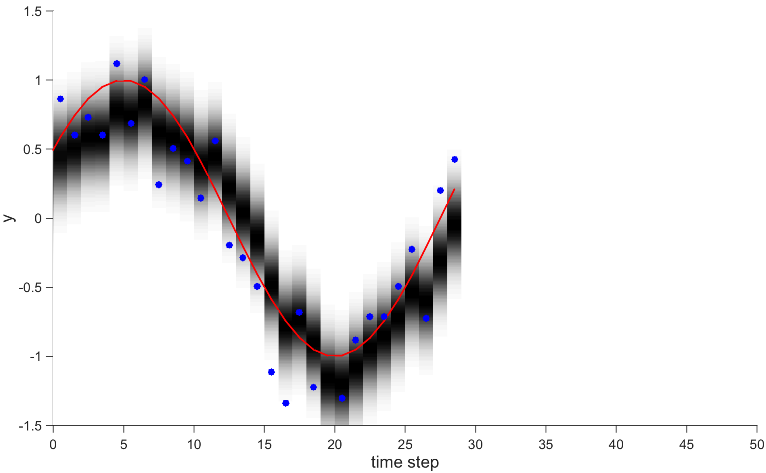

Kalman Filter Demo

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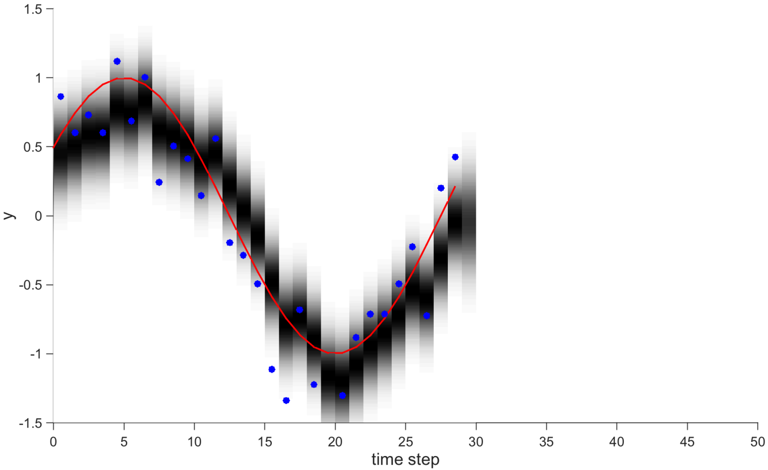

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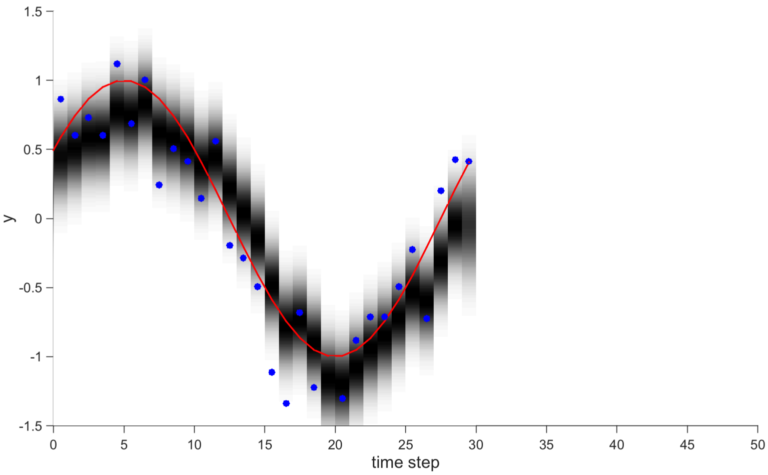

Kalman Filter Demo

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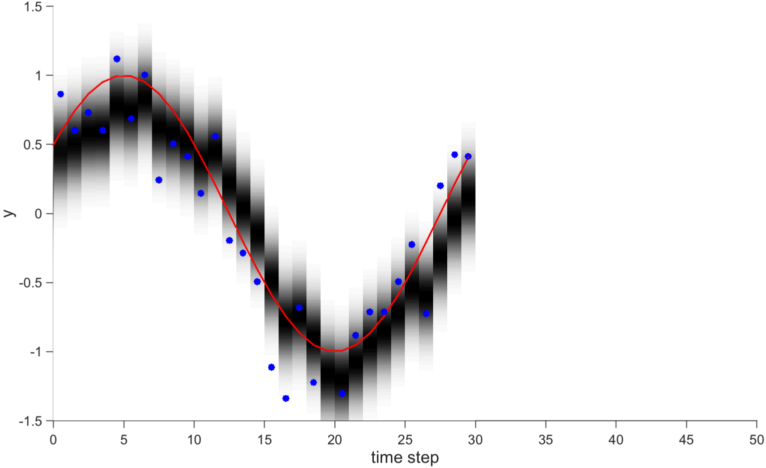

Kalman Filter Demo

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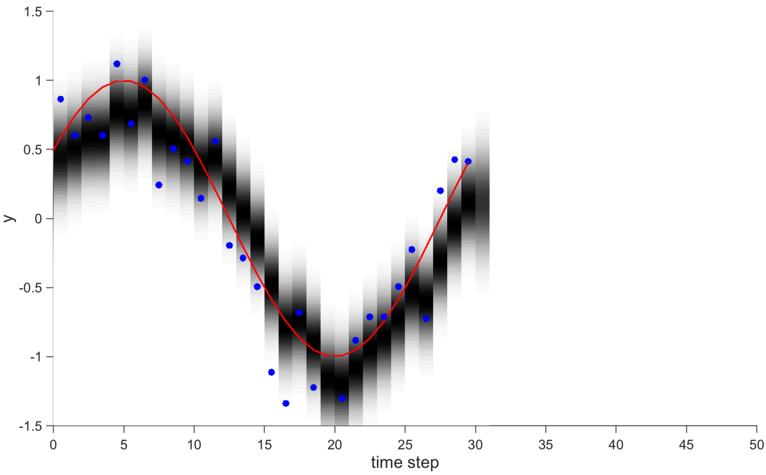

Kalman Filter Demo

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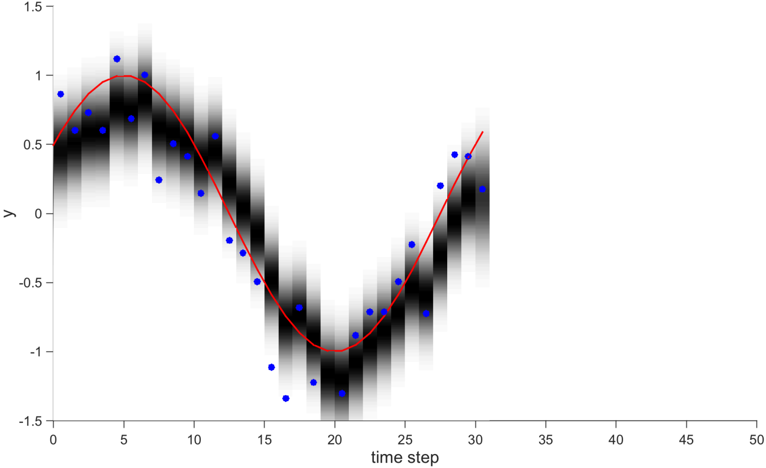

Kalman Filter Demo

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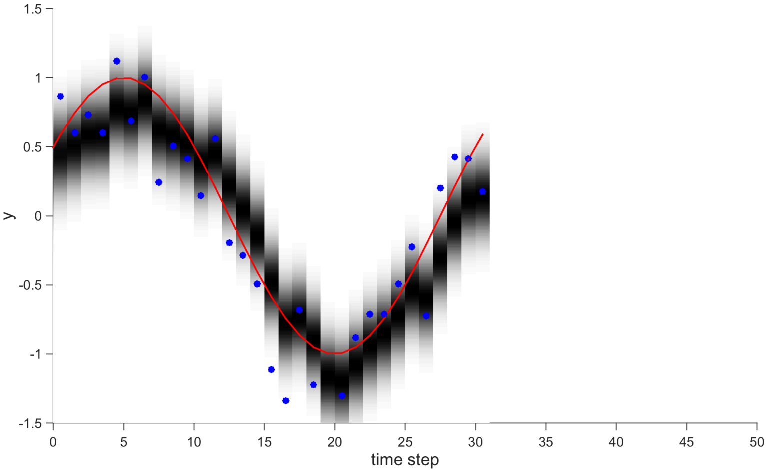

Kalman Filter Demo

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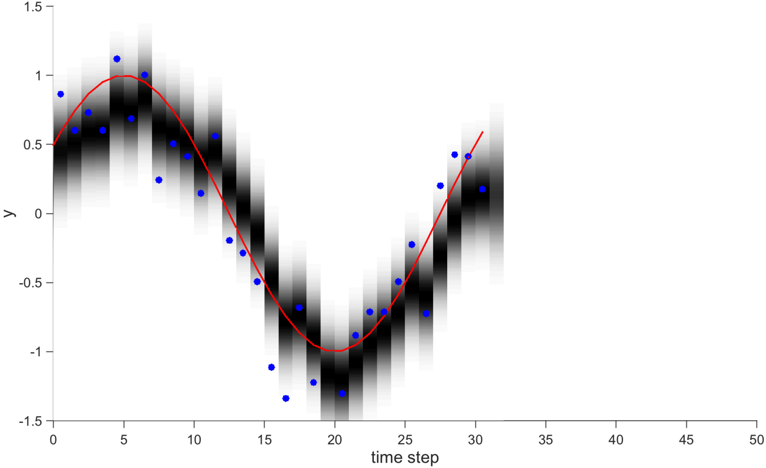

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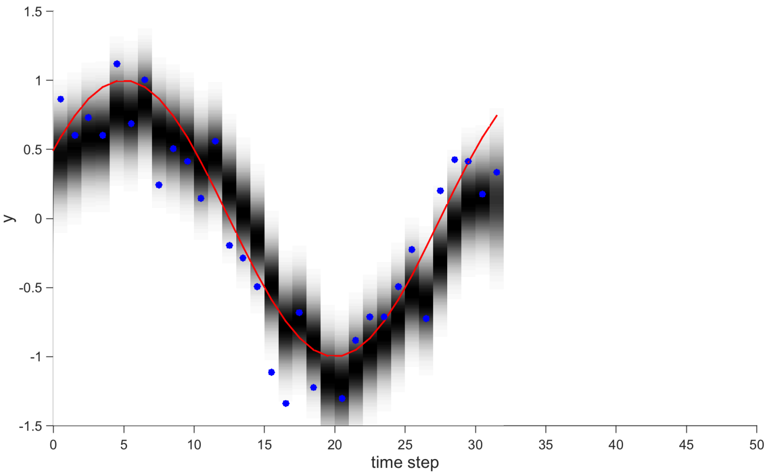

Kalman Filter Demo

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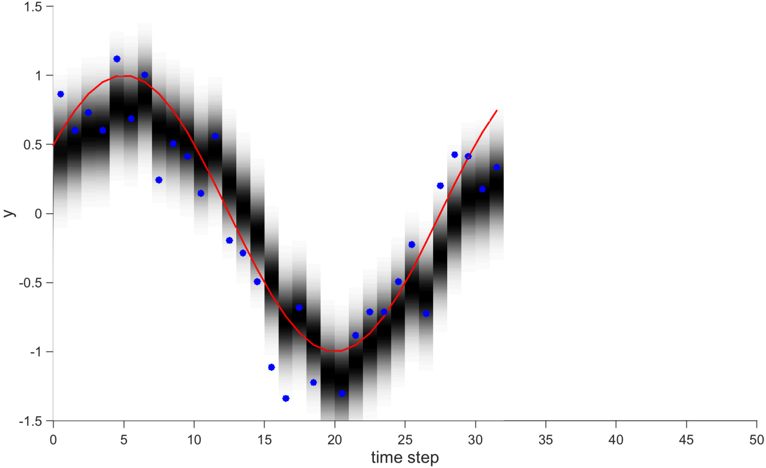

Kalman Filter Demo

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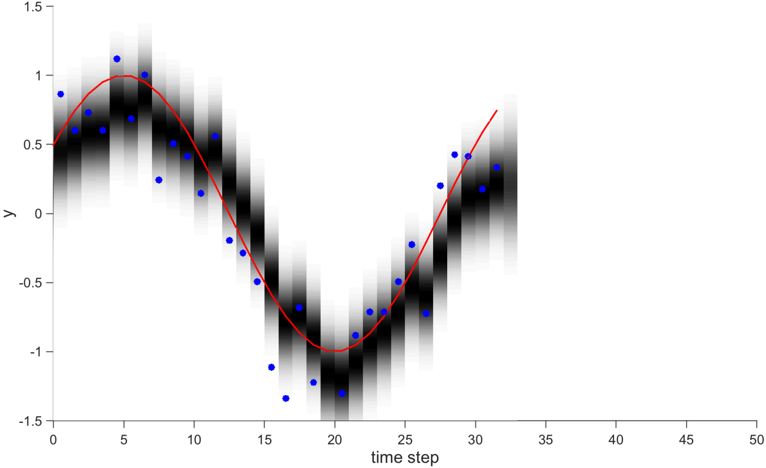

Kalman Filter Demo

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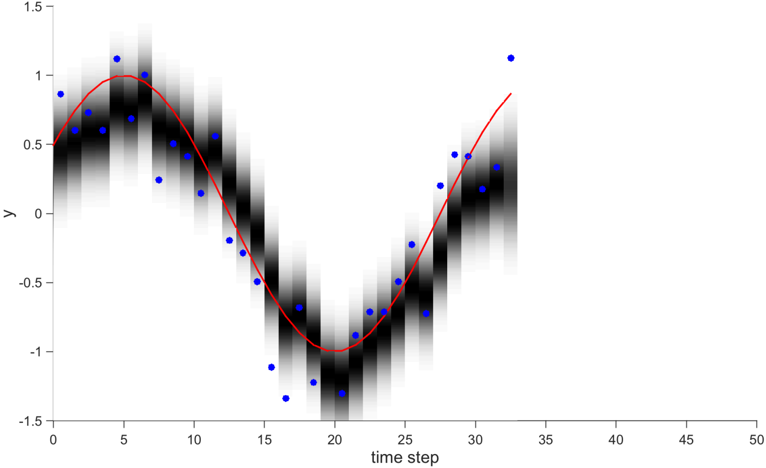

Kalman Filter Demo

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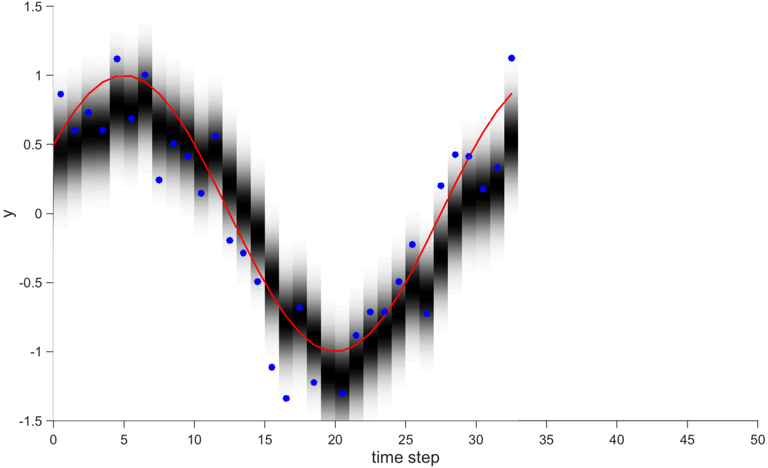

Kalman Filter Demo

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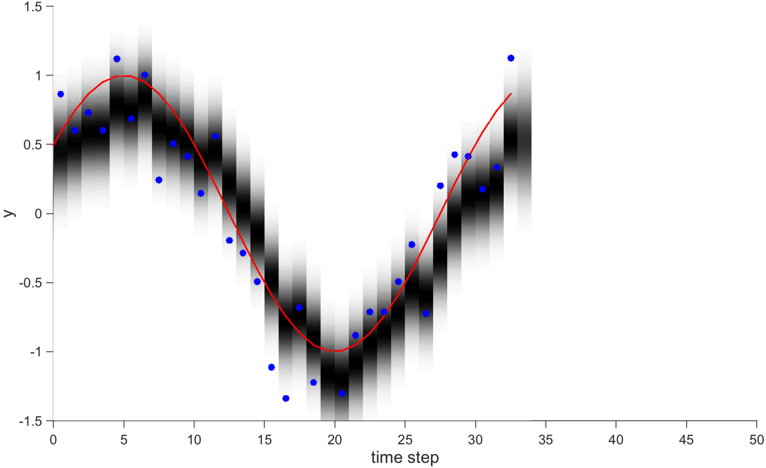

Kalman Filter Demo

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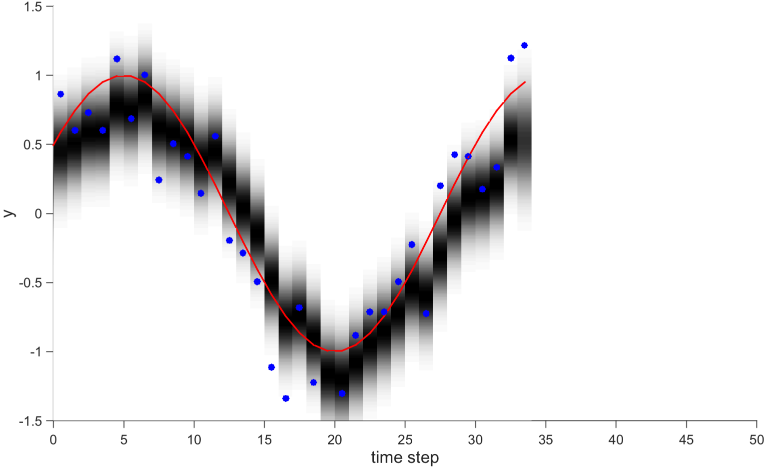

Kalman Filter Demo

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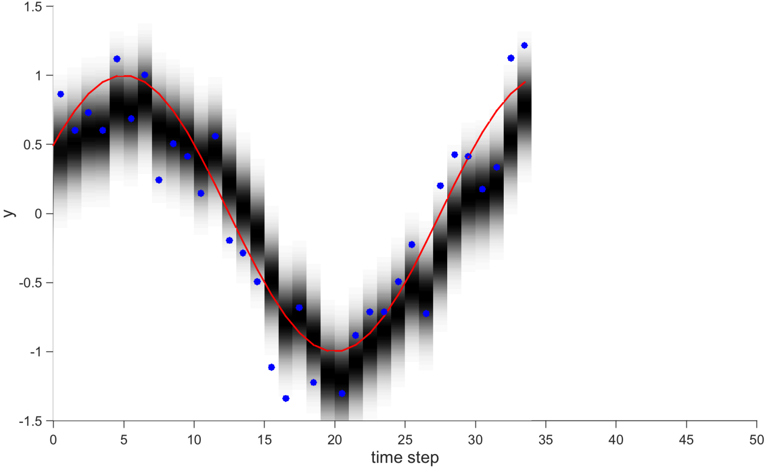

Kalman Filter Demo

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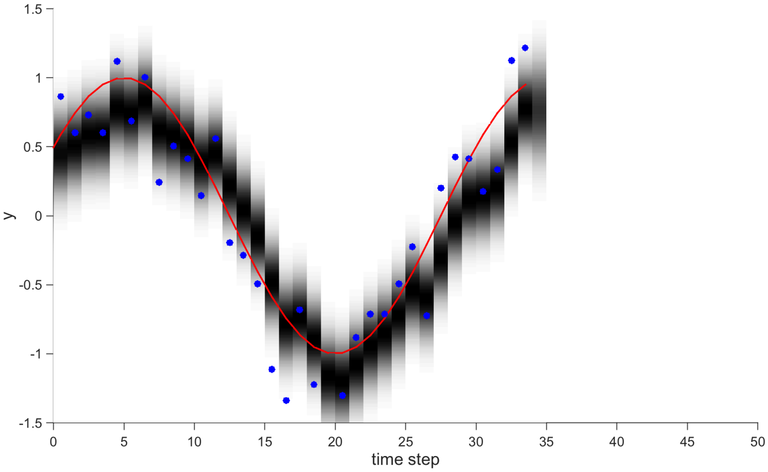

Kalman Filter Demo

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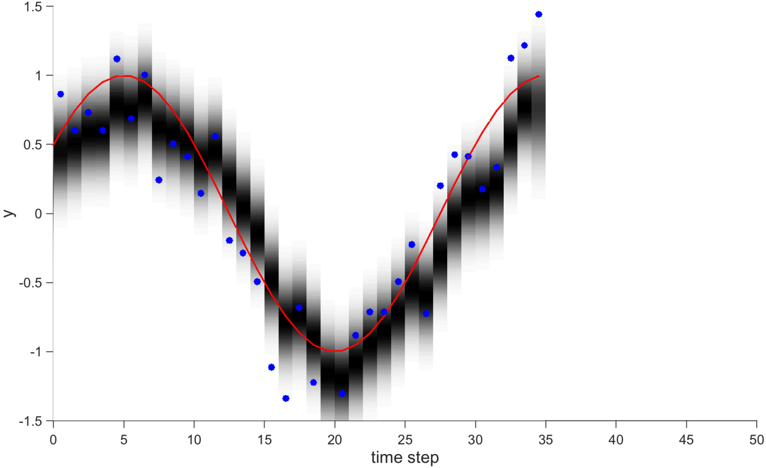

Kalman Filter Demo

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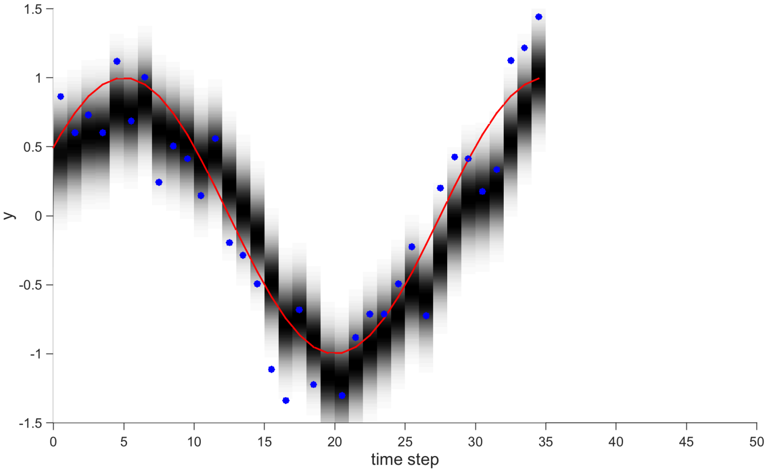

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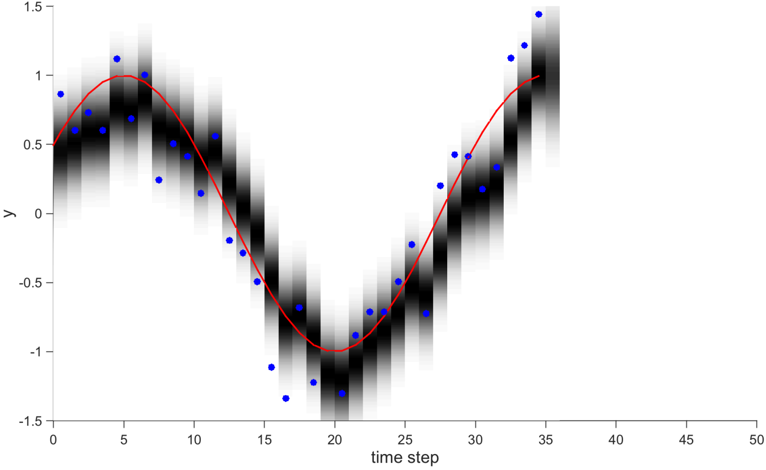

Kalman Filter Demo

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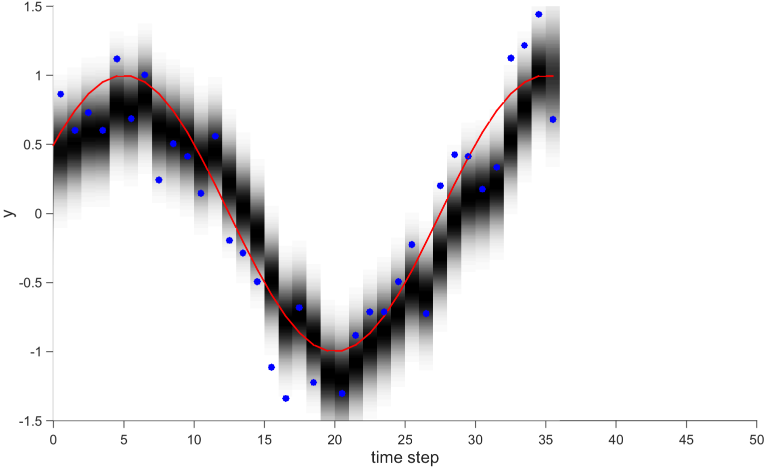

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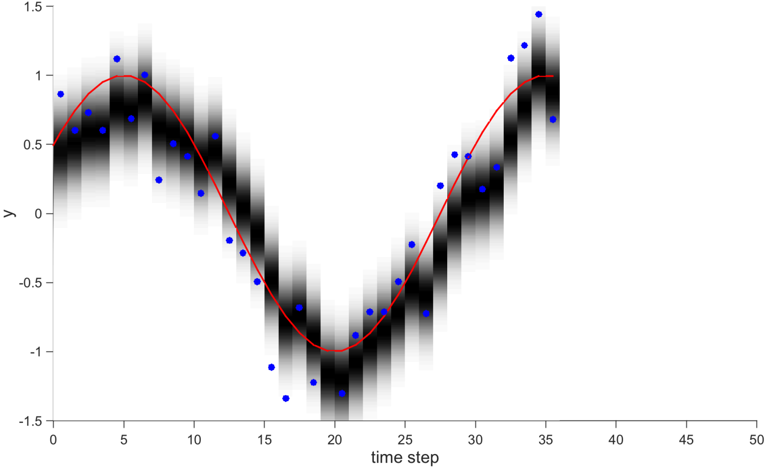

Kalman Filter Demo

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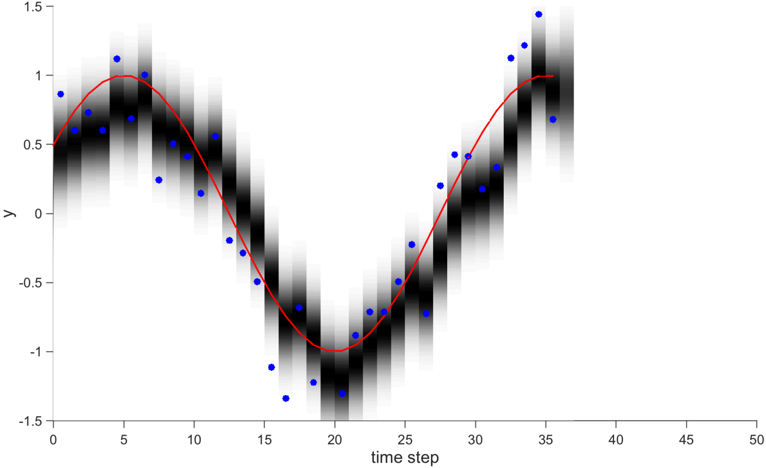

Kalman Filter Demo

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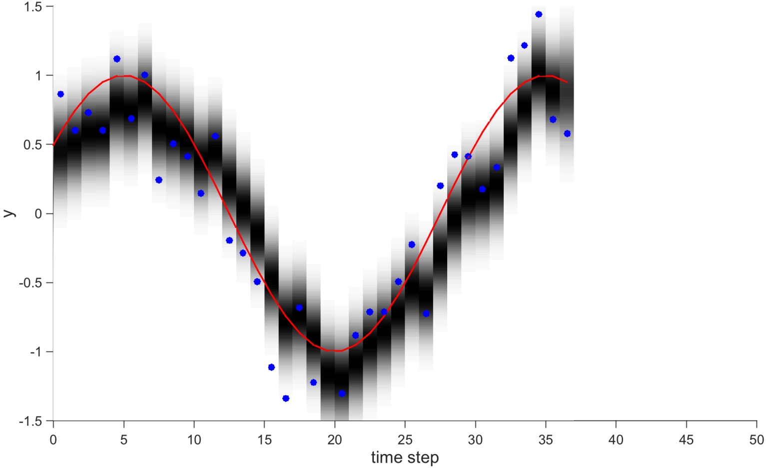

Kalman Filter Demo

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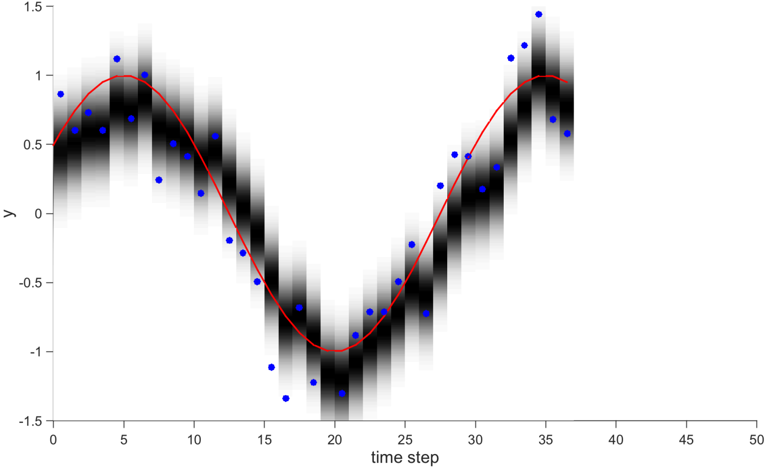

Kalman Filter Demo

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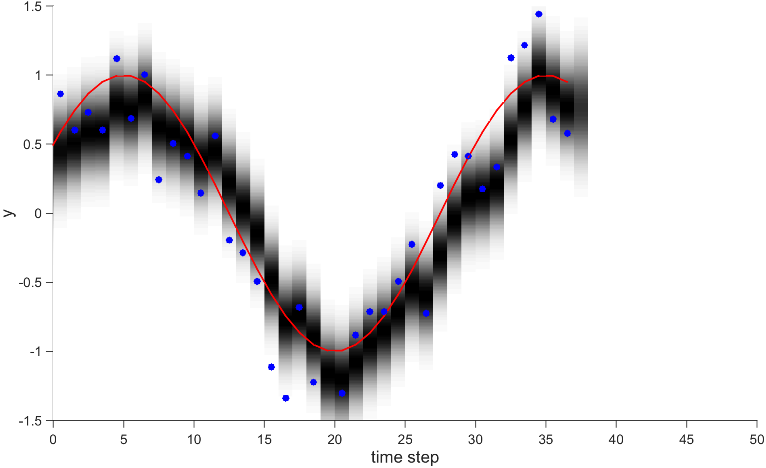

Kalman Filter Demo

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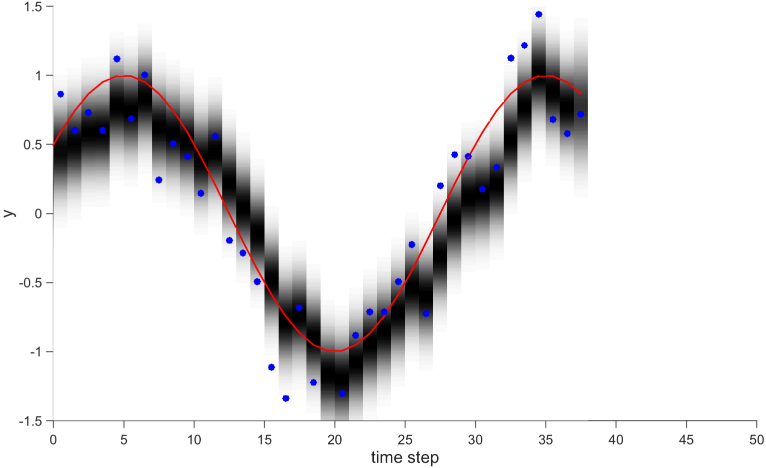

Kalman Filter Demo

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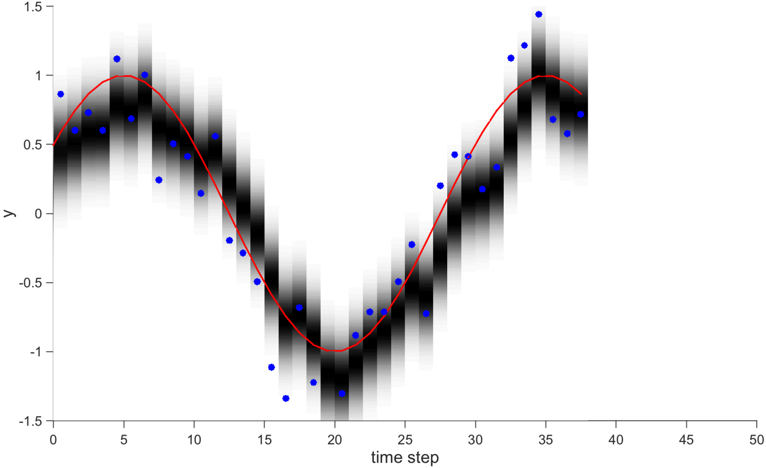

Kalman Filter Demo

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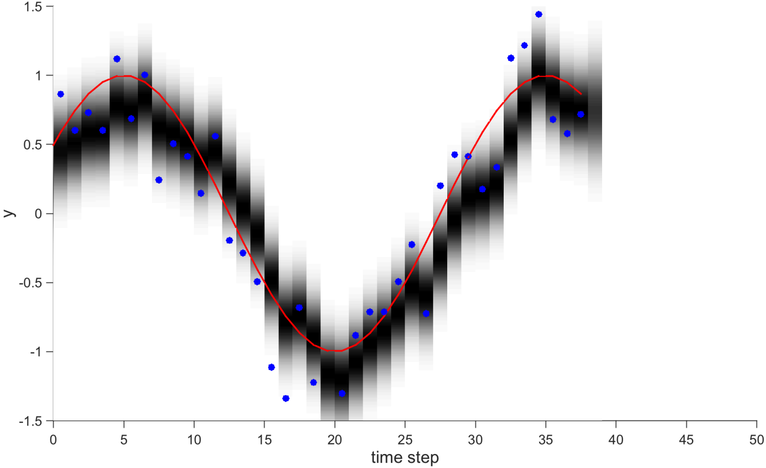

Kalman Filter Demo

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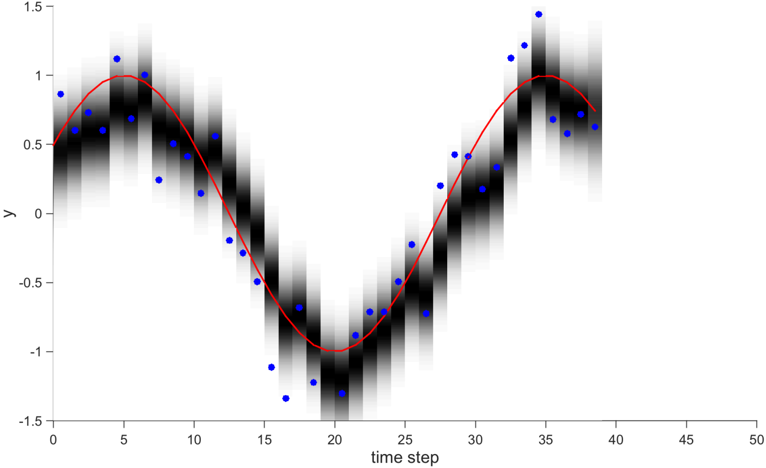

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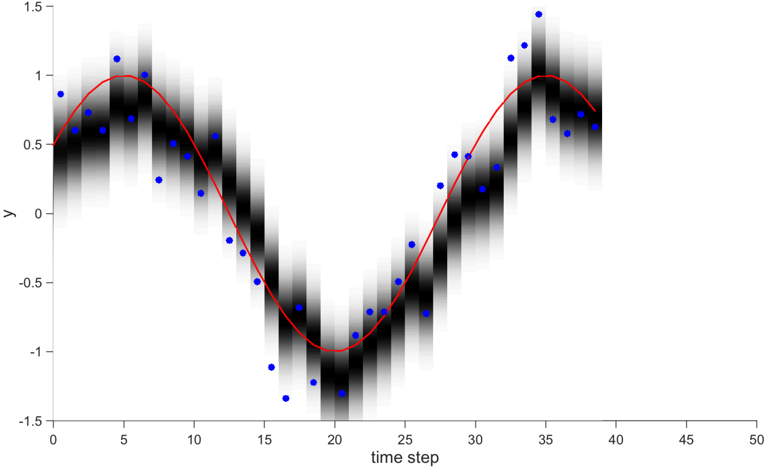

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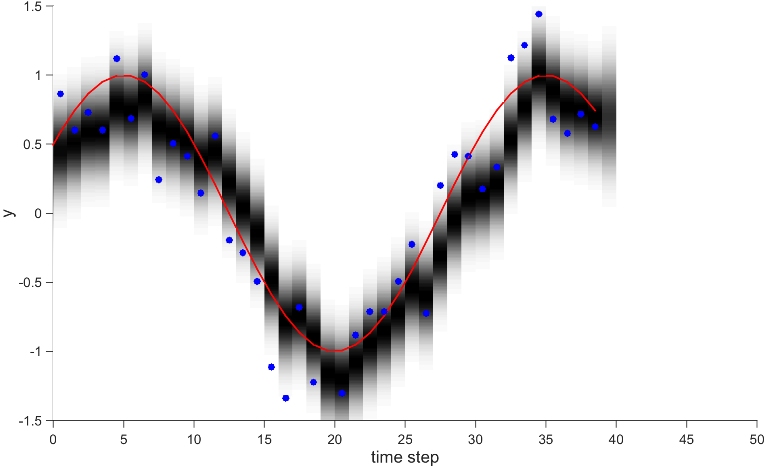
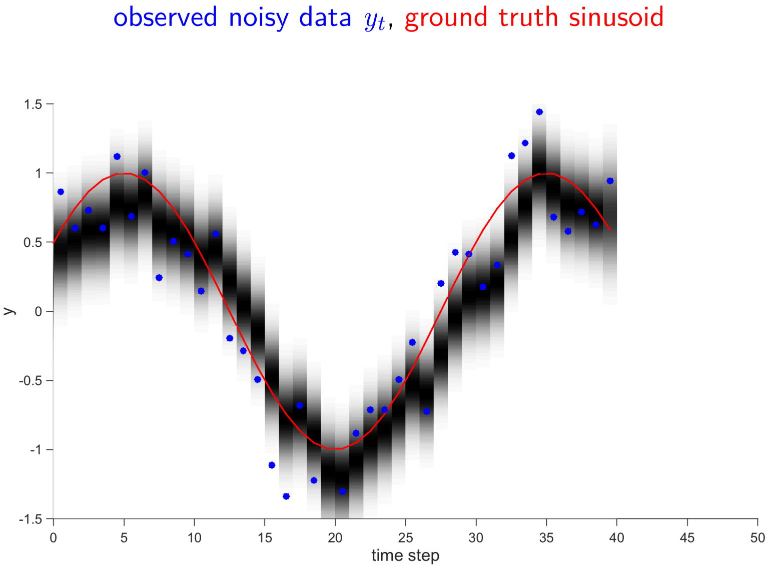


Kalman Filter Demo



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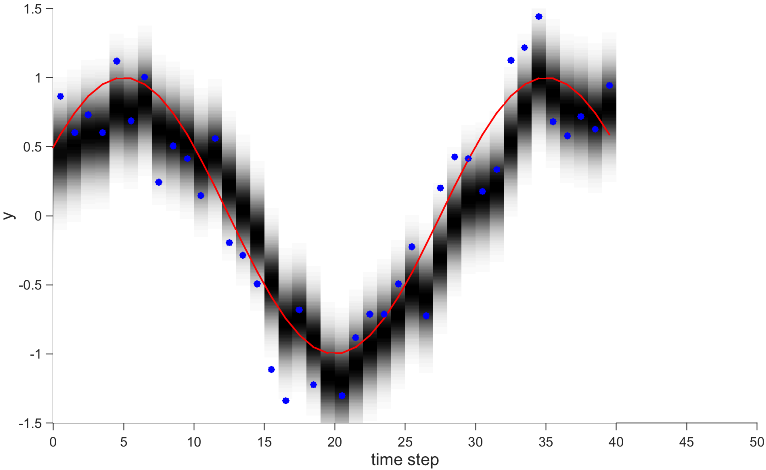

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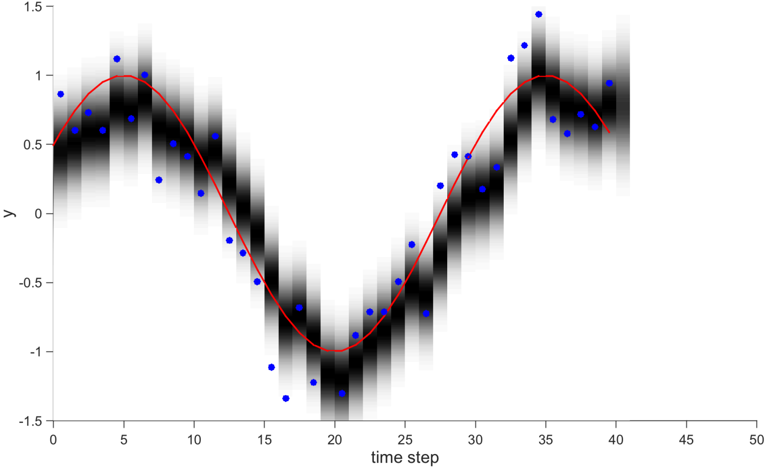

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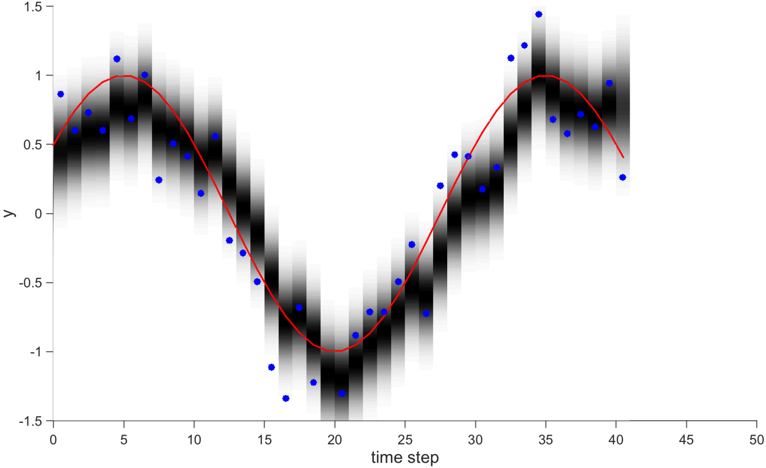

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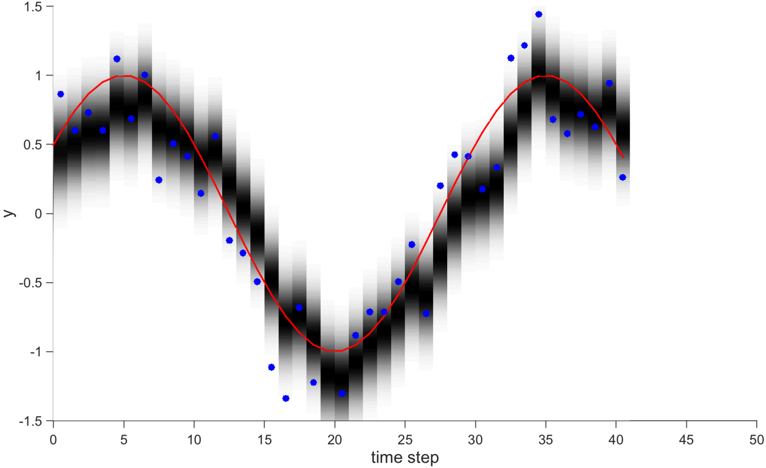

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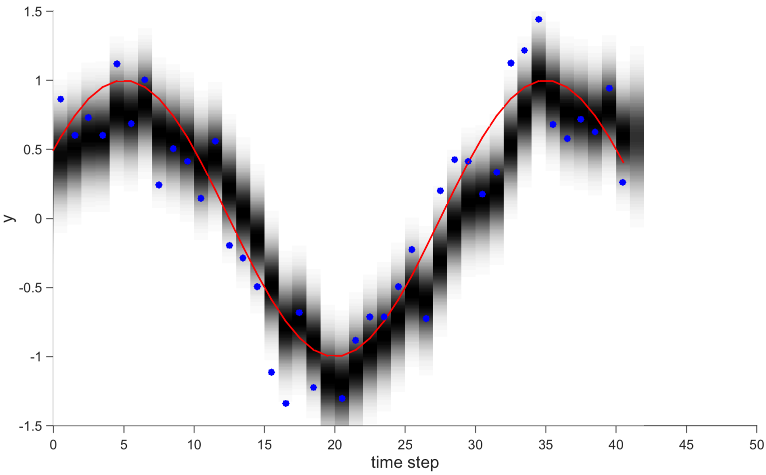

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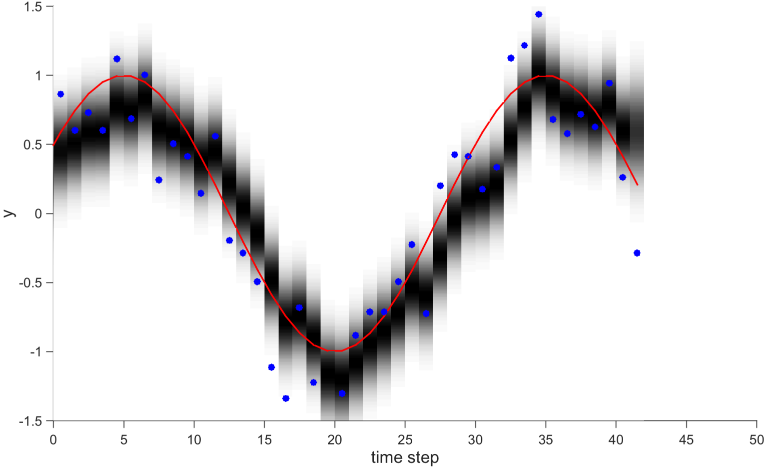

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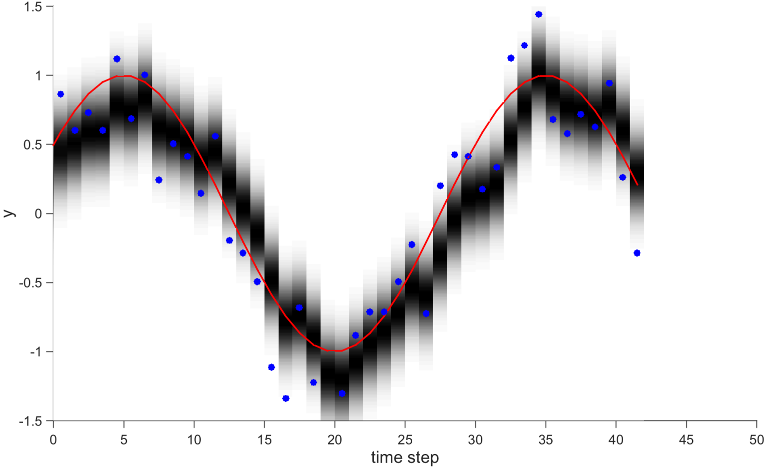

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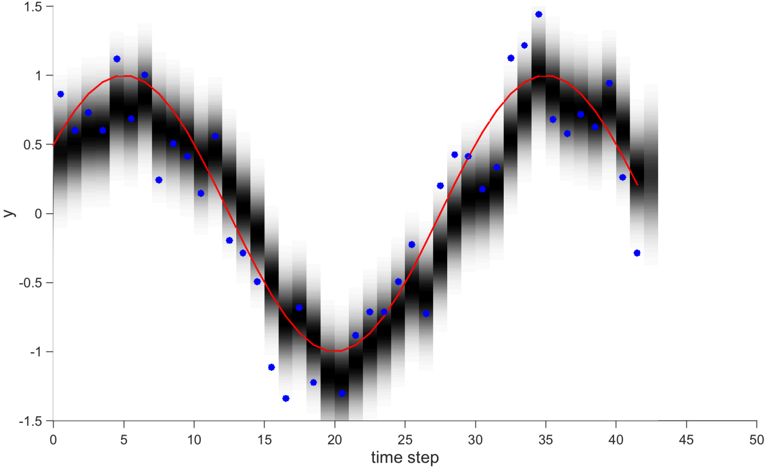

Kalman Filter Demo

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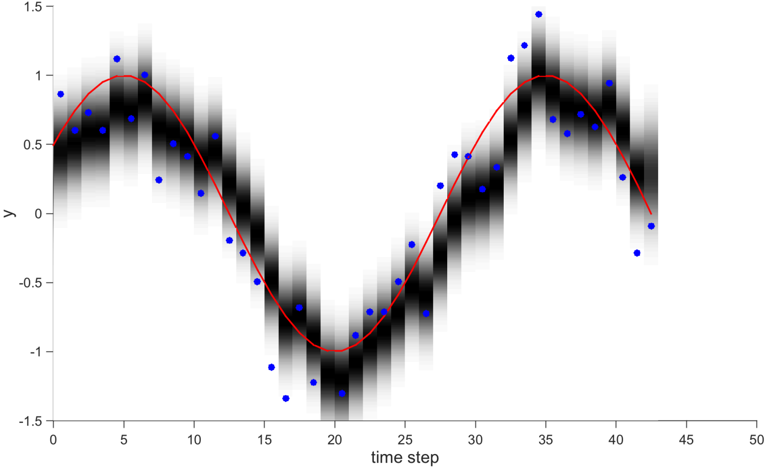

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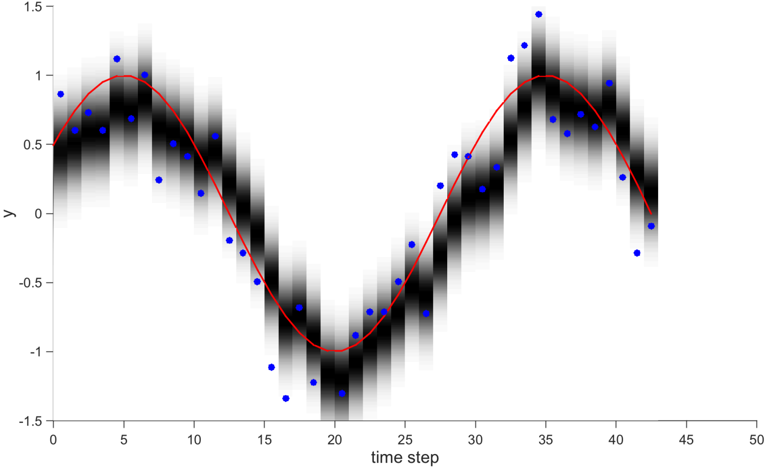

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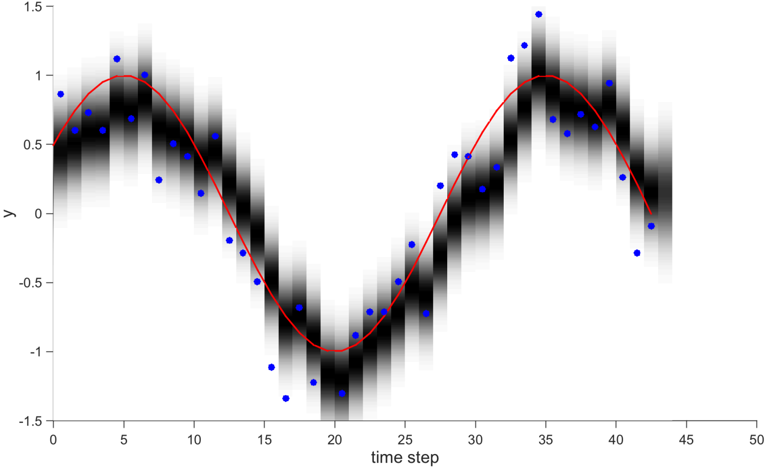

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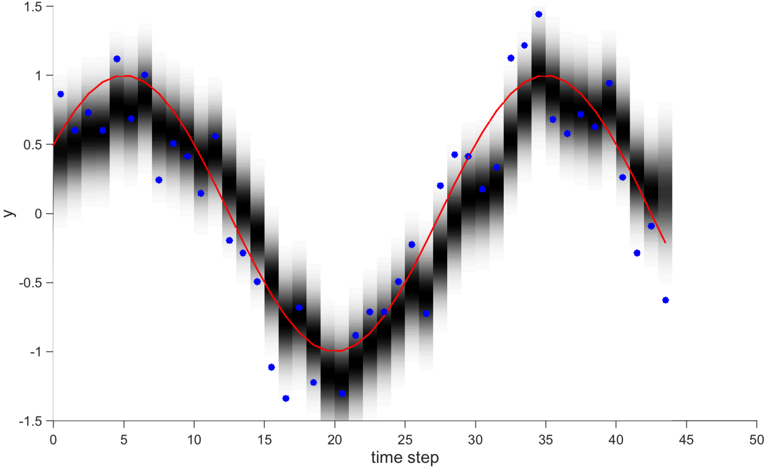

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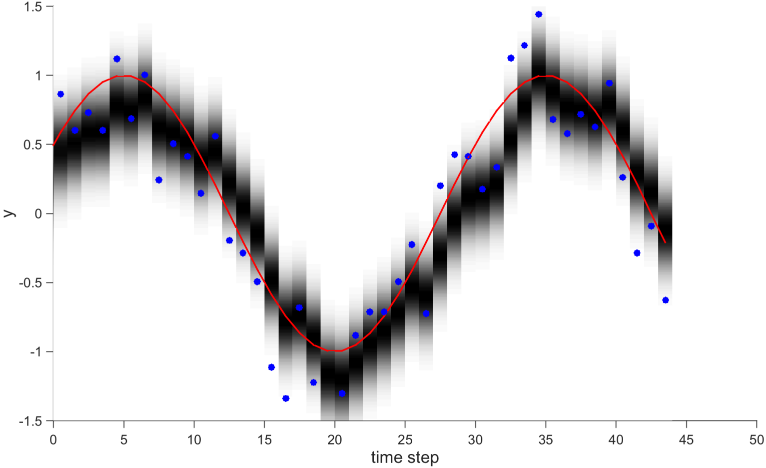

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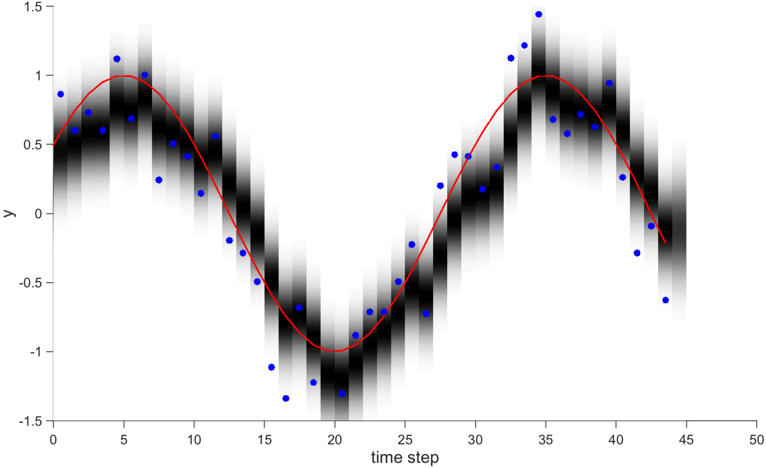

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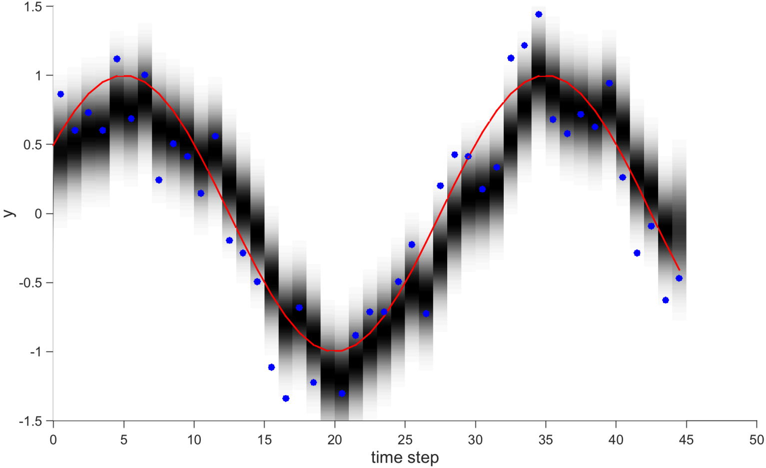

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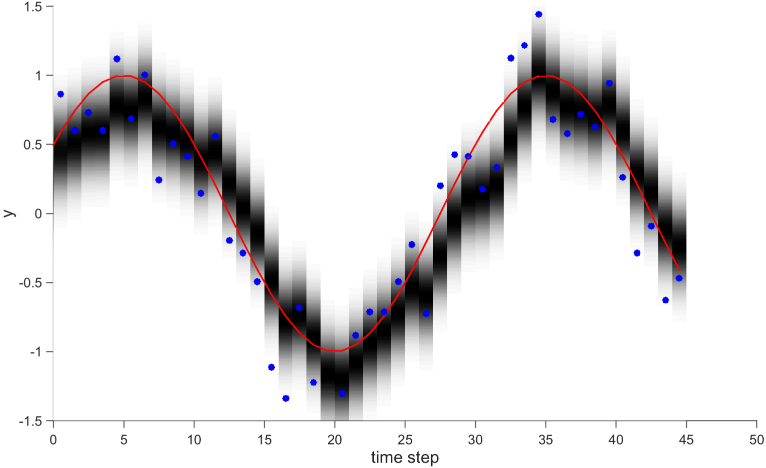

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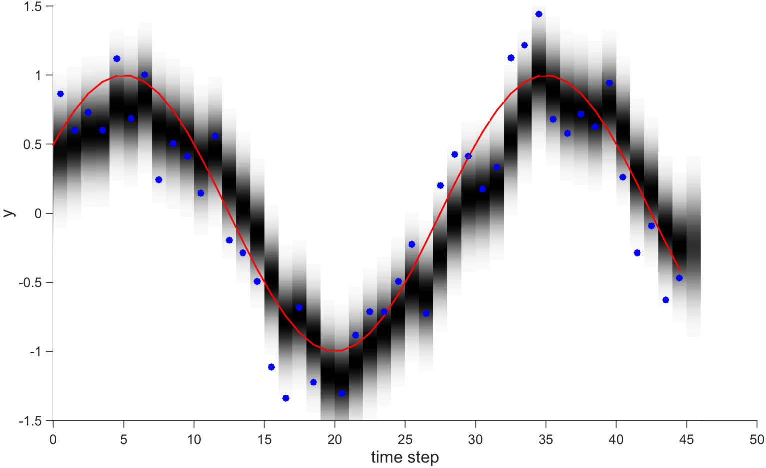

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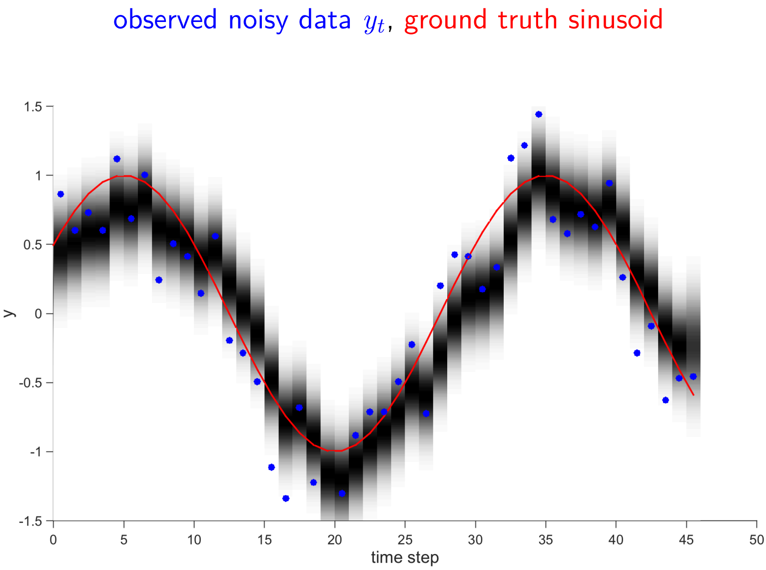

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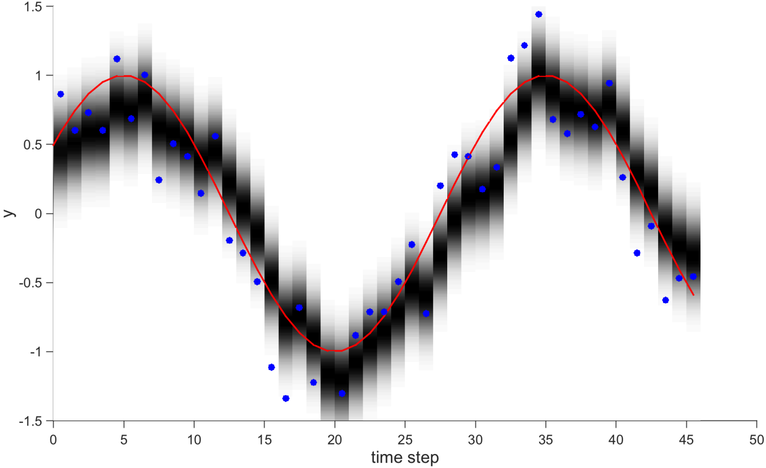

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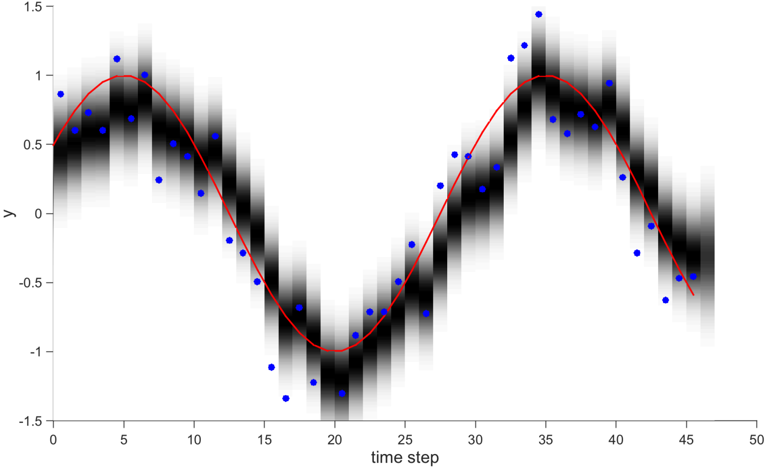

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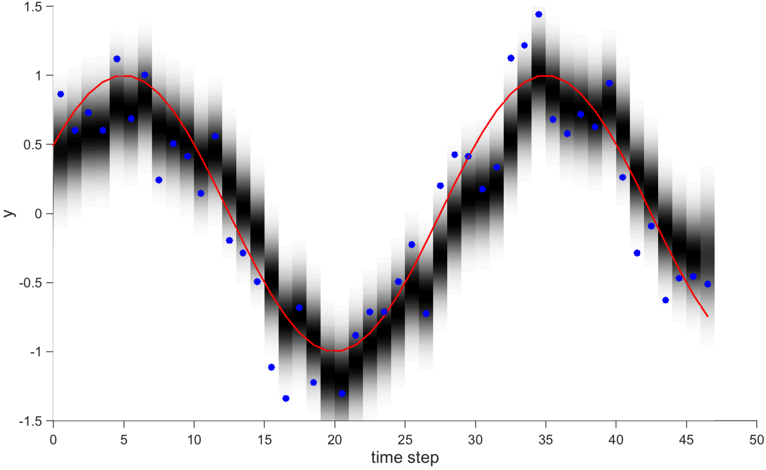

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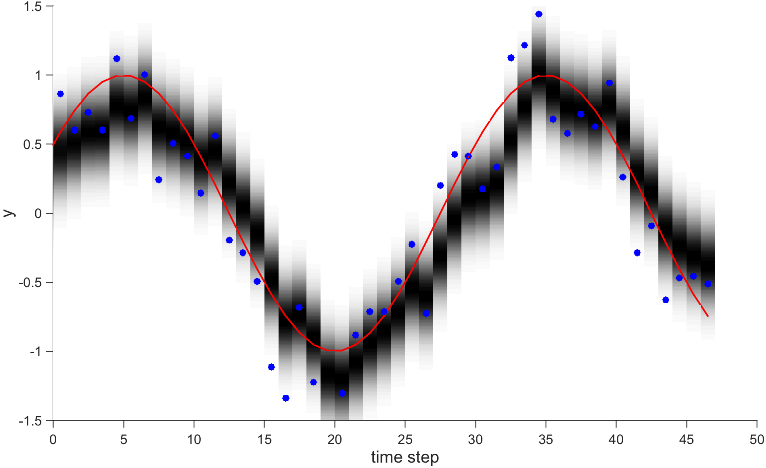

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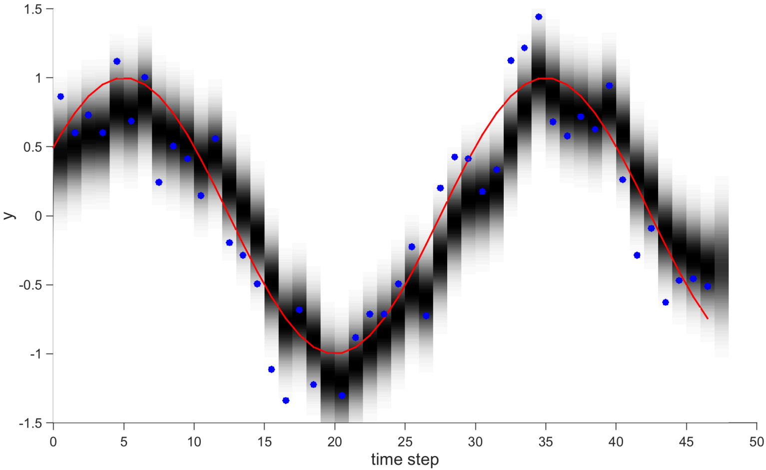

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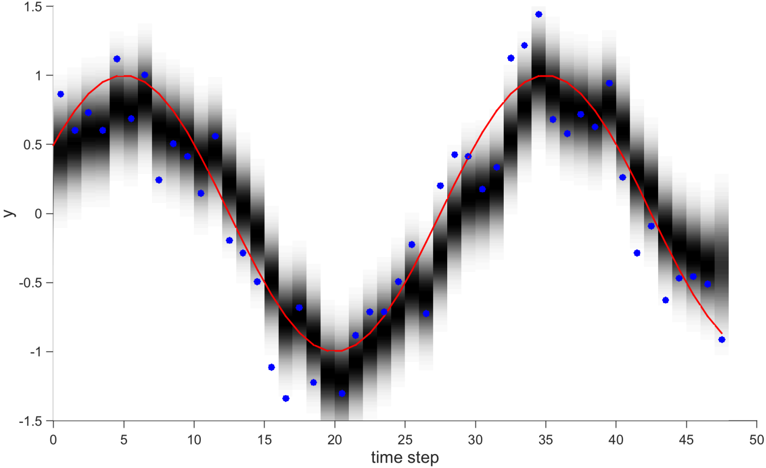

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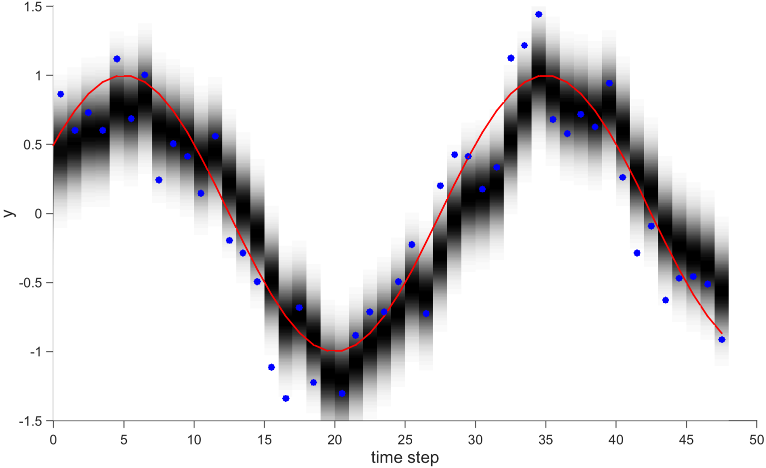

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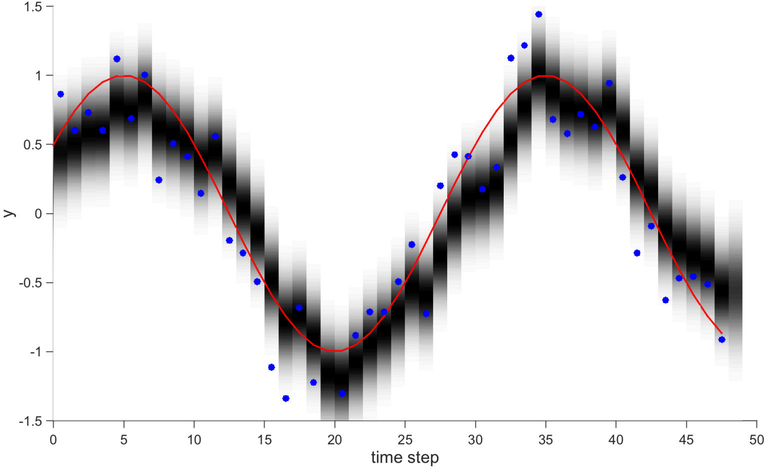
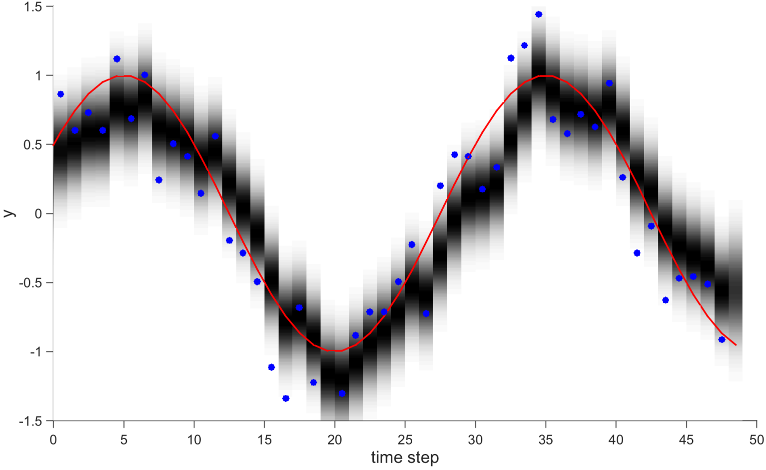


Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



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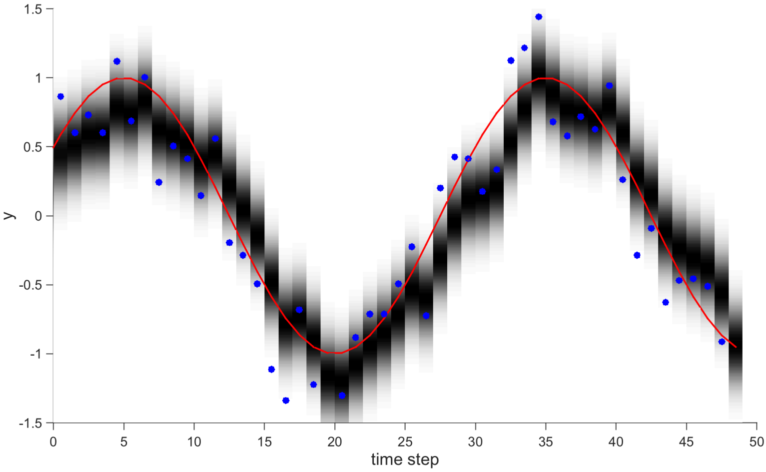

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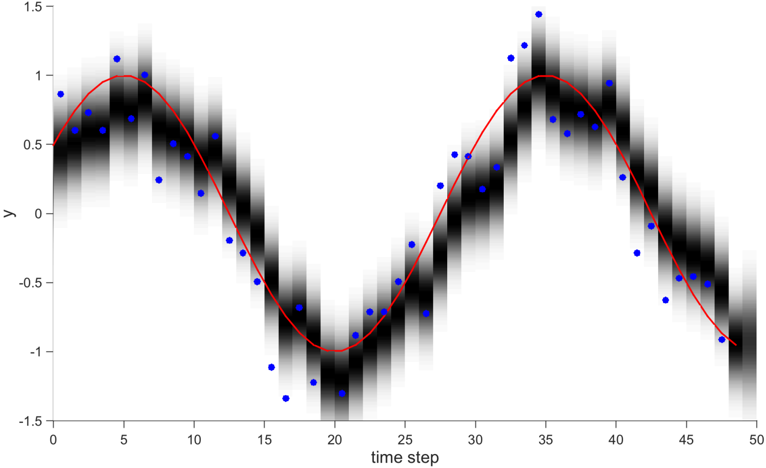

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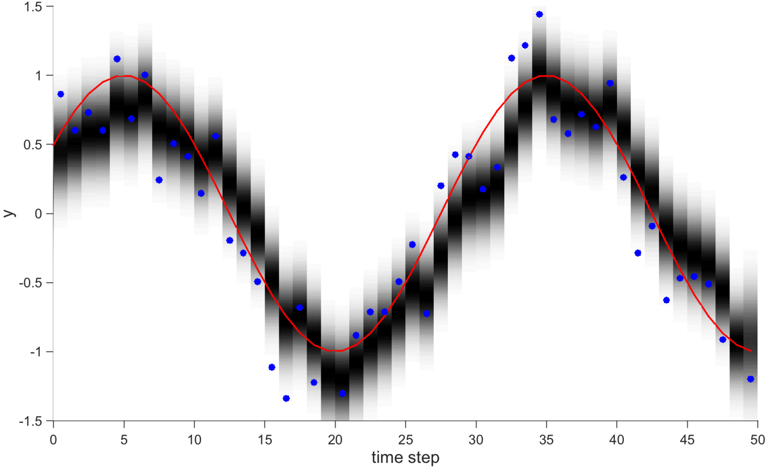

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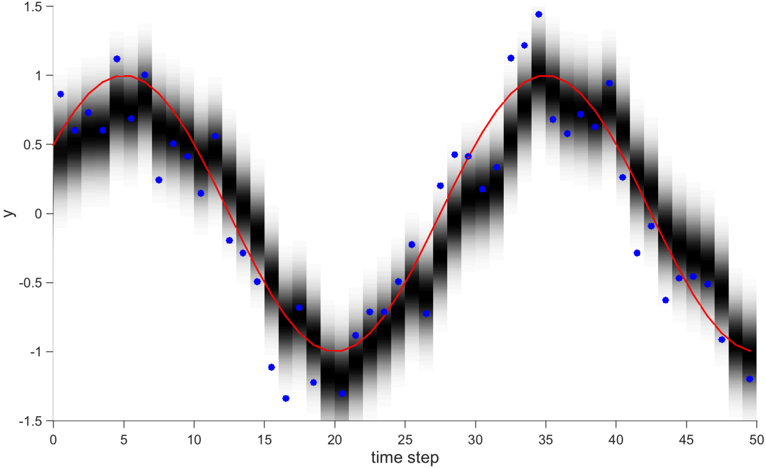
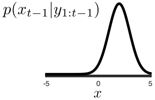


Inference: Forward Algorithm

$$p(x_{t-1} = k | y_{1:t-1})$$



diffuse via
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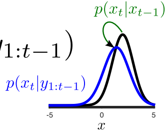

combine
with
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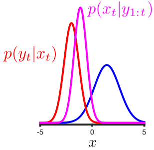
$$p(x_t = k | y_{1:t}) \propto p(x_t = k | y_{1:t-1}) p(y_t | x_t = k)$$

prior likelihood



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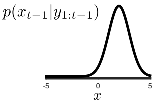
most recent data used in prediction

variable being predicted



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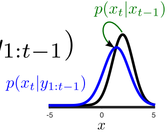

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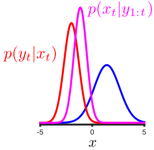

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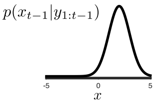

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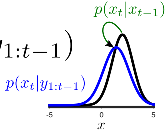

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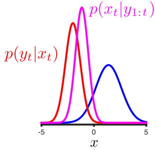

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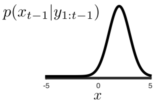

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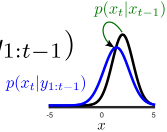

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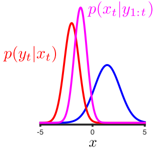

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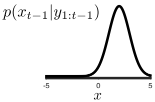

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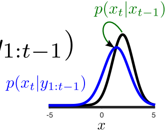

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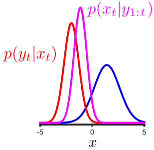

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When implementing, take care with numerical underflow/overflow.

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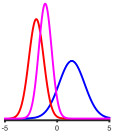

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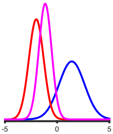

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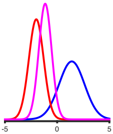

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LGSSM: Kalman Smoother
HMM: Forward-Backward= Algorithm

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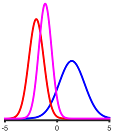

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HMM: Viterbi Decoding

The magic of the Forward Algorithm: Dynamic Programming

What's going on here?

In discrete case, likelihood involves sum over all sequences: $x_{1:T}^{(k)}$

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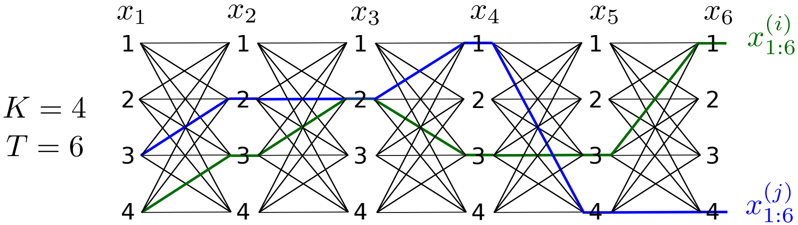

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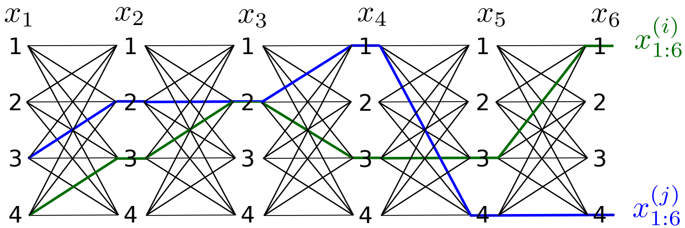

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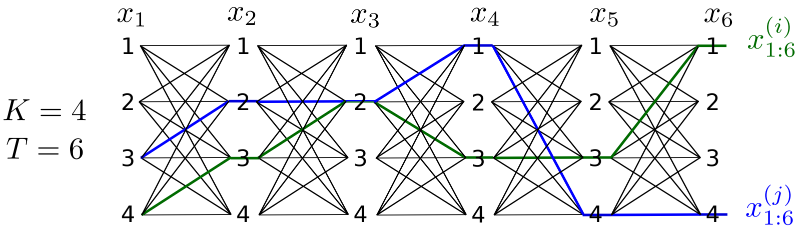

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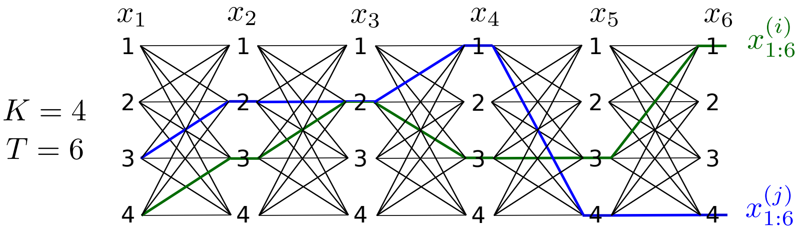

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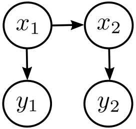

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Markov property means we can forget history of previous states: just remember last one (dynamic programming/belief propagation)

Maximum Likelihood Learning of HMMs: simple once inference is solved

log-likelihood: $$\log p(y_{1:T}|\theta) = \log \int p(y_{1:T}, x_{1:T}|\theta) dx_{1:T}$$

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$$E(\theta; x_{1:T}, y_{1:T}) = \sum_t \left[\log p(y_t|x_t, \theta) + \log p(x_t|x_{t-1}, \theta)\right]$$

$$\frac{\mathrm{d}}{\mathrm{d}\theta} \log p(y_{1:T}|\theta) = \frac{1}{p(y_{1:T}|\theta)} \int \frac{\mathrm{d}}{\mathrm{d}\theta} \exp(\overbrace{\log p(y_{1:T}, x_{1:T}|\theta)}^{E(\theta; x_{1:T}, y_{1:T})}) dx_{1:T}$$

$$\frac{\mathrm{d}}{\mathrm{d}\theta} \log p(y_{1:T}|\theta) = \frac{1}{p(y_{1:T}|\theta)} \int p(y_{1:T}, x_{1:T}|\theta) \frac{\mathrm{d}}{\mathrm{d}\theta} E(\theta; x_{1:T}, y_{1:T}) dx_{1:T}$$

$$\frac{\mathrm{d}}{\mathrm{d}\theta} \log p(y_{1:T}|\theta) = \int p(x_{1:T}|y_{1:T}, \theta) \frac{\mathrm{d}}{\mathrm{d}\theta} E(\theta; x_{1:T}, y_{1:T}) dx_{1:T}$$

Maximum Likelihood Learning of HMMs: simple once inference is solved

log-likelihood: $$\log p(y_{1:T}|\theta) = \log \int p(y_{1:T}, x_{1:T}|\theta) dx_{1:T}$$

gradient of log-likelihood: $$\frac{\mathrm{d}}{\mathrm{d}\theta} \log p(y_{1:T}|\theta) = \frac{1}{p(y_{1:T}|\theta)} \int \frac{\mathrm{d}}{\mathrm{d}\theta} p(y_{1:T}, x_{1:T}|\theta) dx_{1:T}$$

simple form: e.g. quadratic in x for LGSSMs

$E(\theta; x_{1:T}, y_{1:T}) = \sum_t \left[\log p(y_t|x_t, \theta) + \log p(x_t|x_{t-1}, \theta)\right]$

show gradient depends on simple moments of posterior:

$$\frac{\mathrm{d}}{\mathrm{d}\theta} \log p(y_{1:T}|\theta) = \frac{1}{p(y_{1:T}|\theta)} \int \frac{\mathrm{d}}{\mathrm{d}\theta} \exp(\overbrace{\log p(y_{1:T}, x_{1:T}|\theta)}^{E(\theta; x_{1:T}, y_{1:T})}) dx_{1:T}$$

$$\frac{\mathrm{d}}{\mathrm{d}\theta} \log p(y_{1:T}|\theta) = \frac{1}{p(y_{1:T}|\theta)} \int p(y_{1:T}, x_{1:T}|\theta) \frac{\mathrm{d}}{\mathrm{d}\theta} E(\theta; x_{1:T}, y_{1:T}) dx_{1:T}$$

$$\frac{\mathrm{d}}{\mathrm{d}\theta} \log p(y_{1:T}|\theta) = \int p(x_{1:T}|y_{1:T}, \theta) \frac{\mathrm{d}}{\mathrm{d}\theta} E(\theta; x_{1:T}, y_{1:T}) dx_{1:T}$$

$$\frac{\mathrm{d}}{\mathrm{d}\theta} \log p(y_{1:T}|\theta) = \left\langle \frac{\mathrm{d}}{\mathrm{d}\theta} E(\theta; x_{1:T}, y_{1:T}) \right\rangle_{p(x_{1:T}|y_{1:T}, \theta)}$$

requires posterior moments: marginals and pairwise marginals