Handout 4 – Vibration of bending beams

We have found that the equation of motion for the Euler-Bernoulli bending beam is:

$$\rho A \frac{\partial^2 y}{\partial t^2} + EI \frac{\partial^4 y}{\partial x^4} = f(x,t)$$

To obtain the modes of vibration, we need to find the solution under given boundary conditions. Various boundary conditions are possible at the ends of a beam – notice that, because this is a fourth-order differential equation in x, we would expect the general solution in x to involve four arbitrary constants, and so to determine a particular solution we will need four boundary conditions. Whereas the systems considered so far needed only one boundary condition at each end, for a beam we will require TWO at each end. There are three particular boundary conditions which are useful approximations to common engineering configurations:

(i) Clamped boundary
(or built-in)

(ii) pinned (or hinged) boundary

(iii) free boundary

To find the vibration modes and natural frequencies we need to find the general solution when the time-dependence is harmonic:

$$y(x,t) = U(x)\mathrm{e}^{i\omega t}$$

Substitute into the PDE above without any input force (free vibration) to get:

$$EI\frac{\partial^4 U}{\partial x^4} - \rho A\omega^2 U = 0$$

This is a fourth-order ODE, so the general solution is:

$$U(x) = C_1\mathrm{e}^{ikx} + C_2\mathrm{e}^{-ikx} + C_3\mathrm{e}^{kx} + C_4\mathrm{e}^{-kx}$$

with

$$k^4 = \omega^2\frac{\rho A}{EI}$$

So free sinusoidal vibration has two kinds of motion:

- Harmonic in space: $\mathrm{e}^{i\omega t}\,\mathrm{e}^{-ikx}$ and $\mathrm{e}^{i\omega t}\,\mathrm{e}^{ikx}$
- Exponential in space: $\mathrm{e}^{i\omega t}\,\mathrm{e}^{-kx}$ and $\mathrm{e}^{i\omega t}\,\mathrm{e}^{kx}$

- $\mathrm{e}^{i\omega t}\,\mathrm{e}^{-ikx}$ and $\mathrm{e}^{i\omega t}\,\mathrm{e}^{ikx}$ describe sinusoidal travelling waves. The wave speed is ω/k which represents how fast a given peak of a wave at that frequency is moving. But we know that the group velocity $\mathrm{d}\omega/\mathrm{d}k$ is different, so the overall cluster of waves will travel at another speed. The dispersion equation for a beam is:

  The group velocity is twice the wave velocity, so waves appear at the front of a group then move towards the back of the group over time. See http://www.falstad.com/dispersion/

- $\mathrm{e}^{i\omega t}\,\mathrm{e}^{-kx}$ and $\mathrm{e}^{i\omega t}\,\mathrm{e}^{kx}$ describe something new. These are disturbances on the beam which do not travel, but simply decay exponentially in one direction or the other along the beam. These are known as 'evanescent waves' or 'near fields' because their effect is spatially localised.

See `H4_beam_waves.m`

To find the vibration modes and natural frequencies we are looking for standing wave solutions, so we can use the real form for the mode shapes:

We could equally have used the complex version, but the constants C would have been complex while the constants D will be real. This choice also adds the constraint that the solution is a standing wave, but that is exactly what we are looking for so there is no loss of generality.

When we apply the boundary conditions we will need to know the derivatives of $U(x)$ up to the third derivative:

$$\begin{array}{lcllllll}
U(x) & = & & (& +D_1 \cos kx & +D_2 \sin kx & +D_3 \cosh kx & +D_4 \sinh kx) \\
U'(x) & = & k & (& -D_1 \sin kx & +D_2 \cos kx & +D_3 \sinh kx & +D_4 \cosh kx) \\
U''(x) & = & k^2 & (& -D_1 \cos kx & -D_2 \sin kx & +D_3 \cosh kx & +D_4 \sinh kx) \\
U'''(x) & = & k^3 & (& +D_1 \sin kx & -D_2 \cos kx & +D_3 \sinh kx & +D_4 \cosh kx)
\end{array}$$

Modes of a pinned-pinned beam

therefore:

So that natural frequencies are

$$\omega_n = k_n^2\sqrt{\frac{EI}{\rho A}} = \left(\frac{n\pi}{L}\right)^2\sqrt{\frac{EI}{\rho A}}$$

and the mode shapes are

$$U_n(x) = \sin(k_n x) = \sin\frac{n\pi x}{L}$$

So the mode shapes are identical to a stretched string, but the natural frequencies are not equally spaced, as $\omega_n \propto n^2$.

Modes of a free-free beam

At $x = 0$:

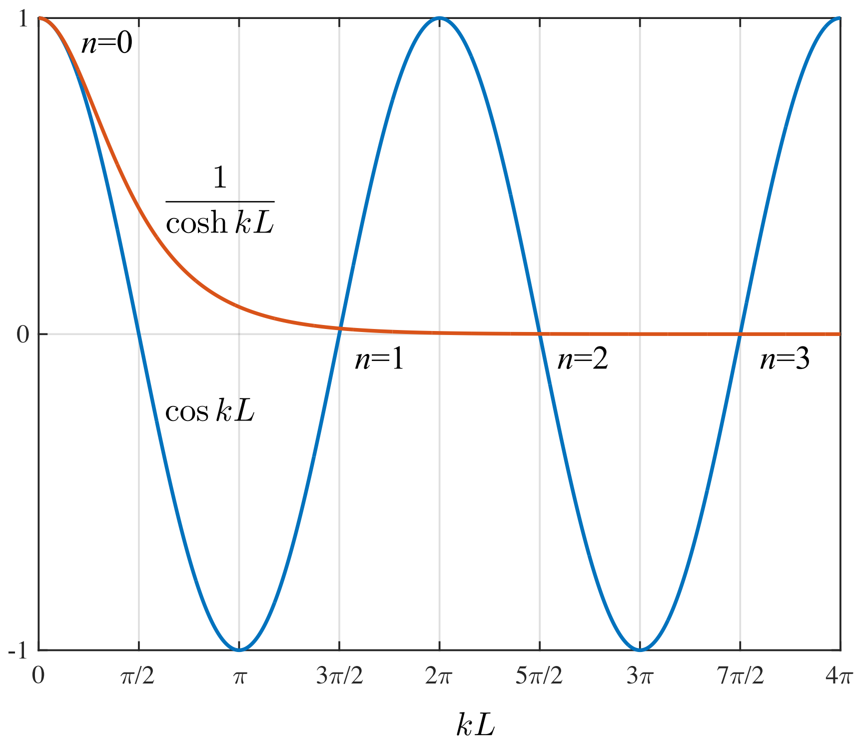
At $x = L$:

$$\begin{array}{rclcrrrrr} U''(L) & = & 0 & \Longrightarrow & -D_1\cos kL & -D_2\sin kL & +D_1\cosh kL & +D_2\sinh kL & =0 \\ U'''(L) & = & 0 & \Longrightarrow & D_1\sin kL & -D_2\cos kL & +D_1\sinh kL & +D_2\cosh kL & =0 \end{array}$$

giving

This time we cannot find an analytic solution, but we can use a graphical approach. Rearrange the final equation into the form: $\cos kL = \dfrac{1}{\cosh kL}$

and look for intersections of the two curves:



the n=0 solution corresponds to rigid body motion of the free-free beam: the wavenumber and natural frequency are both zero, and the structure can translate or rotate freely

So the approximate non-zero solutions are

$$k_n \approx \left(n + \frac{1}{2}\right) \frac{\pi}{L}$$

giving the approximate non-zero natural frequencies

$$\omega_n = k_n^2 \sqrt{\frac{EI}{\rho A}} \approx \left(\frac{(n + 1/2)\,\pi}{L}\right)^2 \sqrt{\frac{EI}{\rho A}}$$

A more accurate answer can be obtained by a numerical approach, e.g. Newton-Raphson.

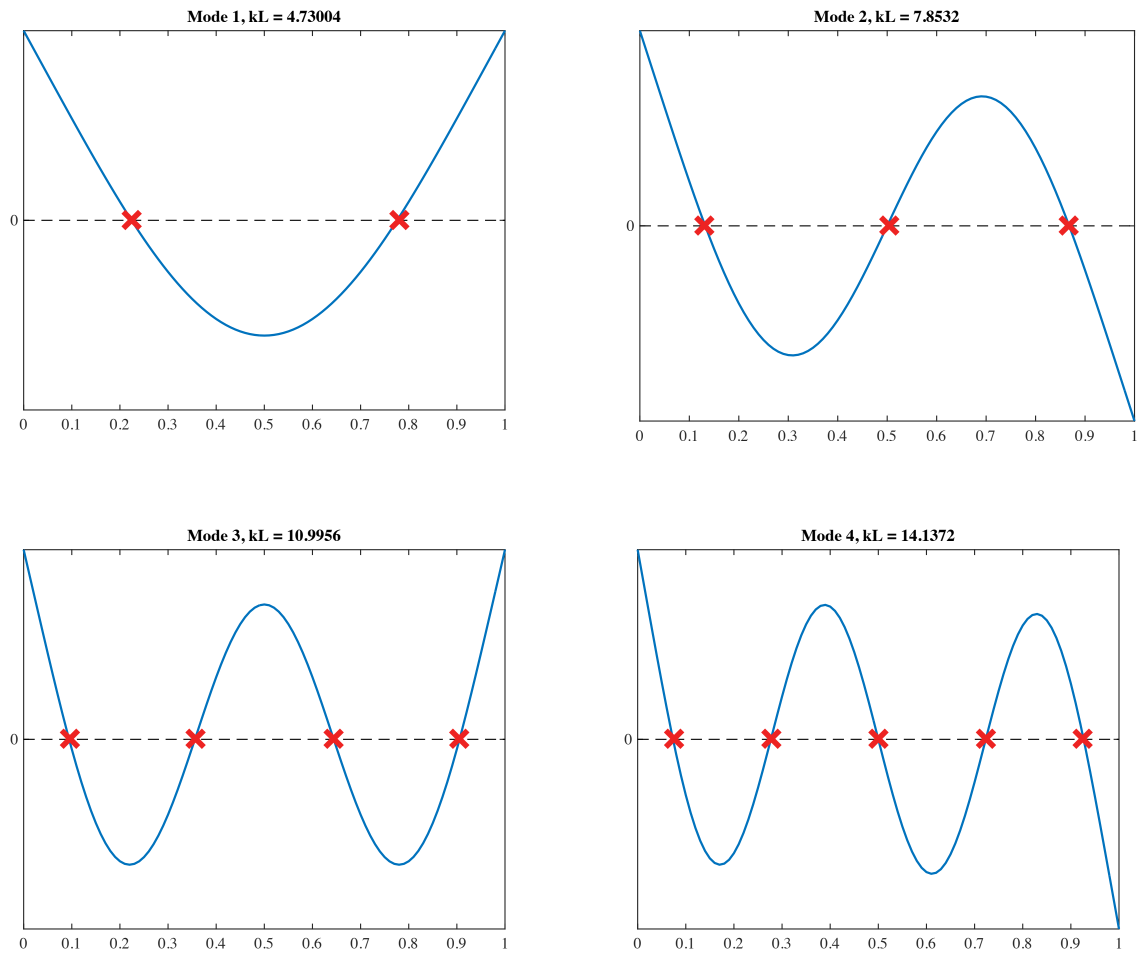
To obtain a given mode shape, then substitute the solution for k into the expression for the mode shape

$$U(x) \quad = \quad (\quad +D_1 \cos kx \quad +D_2 \sin kx \quad +D_3 \cosh kx \quad +D_4 \sinh kx)$$

Use $\quad D_1 = D_3, \quad D_2 = D_4$

and $\quad D_2 = D_1 \dfrac{\cosh kL - \cos kL}{\sin kL - \sinh kL}$

from the boundary conditions, and plot the results numerically. See `H4_coscosh.m`



Modes of a clamped-free beam

As a final example we consider the modes of a cantilever beam

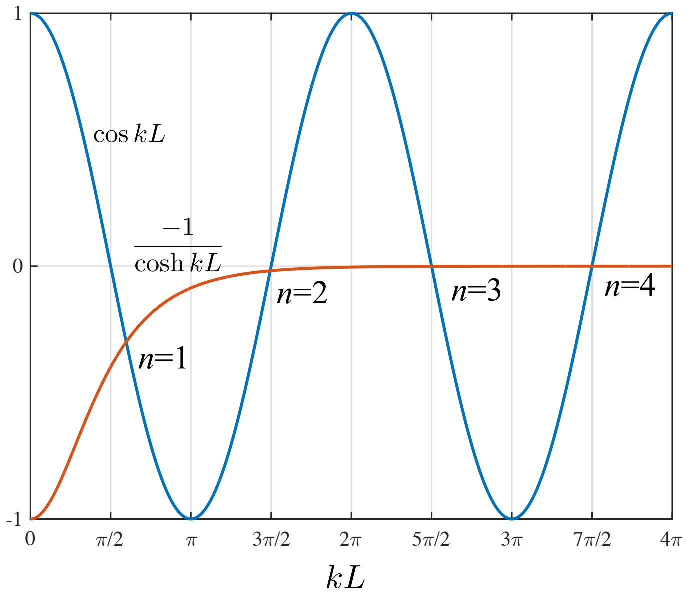
$$
\begin{aligned}
U(0) = 0 &\implies D_1 + D_3 = 0 \\
U'(0) = 0 &\implies D_2 + D_4 = 0
\end{aligned}
$$

$$
\begin{aligned}
U''(L) = 0 &\implies D_1 \left(-\cos kL - \cosh kL\right) + D_2 \left(-\sin kL - \sinh kL\right) = 0 \\
U'''(L) = 0 &\implies D_1 \left(\sin kL - \sinh kL\right) + D_2 \left(-\cos kL - \cosh kL\right) = 0
\end{aligned}
$$

Put into matrix form and set the determinant to zero to give

$$
\begin{aligned}
\left(\cos^2 kL + \cosh kL\right)^2 &= -\left(\sin kL - \sinh kL\right)\left(\sin kL + \sinh kL\right) \\
\cos^2 kL + 2\cos kL \cosh kL + \cosh^2 kL &= -\sin^2 kL + \sinh^2 kL \\
\cos kL \cosh kL &= -1
\end{aligned}
$$

This is very similar to the free-free case, and we can use a similar graphical reconstruction:



there is no n=0 solution this time, as there are no rigid body modes

Try typing into Matlab: `fsolve(@(kL) cos(kL)*cosh(kL)+1,pi/2)`

Approximate mode shapes (these could be computed numerically by adapting the script `H4_coscosh.m` – try it!)

Other simple continuous systems in brief

Tensioned membrane, e.g. drum skin (‘2D string’)

Bending plate, e.g. car panel (‘2D beam’)

Pressure in an acoustic volume, e.g. sound propagation (‘3D string’)

Summary

Wave propagation in beams is more complicated than for structures which satisfy the wave equation. The equation of motion for an Euler-Bernoulli beam is fourth-order in space, resulting in four independent terms for the standing wave free vibration solution.

$$U(x) = D_1 \cos kx + D_2 \sin kx + D_3 \cosh kx + D_4 \sinh kx$$

The most common boundary conditions are: free, pinned and clamped which can be applied in any combination to each end of the beam. Each boundary results in two expressions, giving four equations that can be solved to find the coefficients of the four terms.

A pinned-pinned beam has the same mode shapes as a stretched string, but the frequencies are spaced with mode number squared rather than linearly.

The natural frequencies of a free-free beam cannot be found analytically, but a graphical construction allows (mostly) good approximations to be found.

The natural frequencies of a cantilever beam also cannot be found analytically, but a similar construction allows approximations to be found that are not so good for the first couple of modes.