Engineering Tripos Part IIA THIRD YEAR

Module 3C5: Dynamics and Vibrations

DYNAMICS

Examples paper 3C5/1 – Rigid-Body Dynamics

Straightforward questions are marked †
*Tripos standard questions are marked **

Equations of motion

1. (a) For a collection of particles m_i at positions $\mathbf{r}_i$, show from first principles by summation that

$$\dot{\mathbf{h}}_P + \dot{\mathbf{r}}_P \times \mathbf{p} = \mathbf{Q}^{(e)} \qquad (1)$$

where $\mathbf{h}_P$ is the total moment of momentum about a point P of all particles, $\dot{\mathbf{r}}_P$ is the velocity of P, the total momentum of all particles is $\mathbf{p}$ and the total moment of external forces about P is $\mathbf{Q}^{(e)}$.

(b) The centre of mass G of the collection of particles is at $\mathbf{r}_G$ and the total moment of momentum about G is $\mathbf{h}_G$. Substituting $\mathbf{r}_i - \mathbf{r}_P = (\mathbf{r}_i - \mathbf{r}_G) + (\mathbf{r}_G - \mathbf{r}_P)$ in the summation for $\mathbf{h}_P$ verify the data-sheet result that

$$\mathbf{h}_P = \mathbf{h}_G + (\mathbf{r}_G - \mathbf{r}_P) \times \mathbf{p} \qquad (2)$$

and obtain by differentiation of (2) the result that

$$\dot{\mathbf{h}}_G + (\mathbf{r}_G - \mathbf{r}_P) \times \dot{\mathbf{p}} = \mathbf{Q}^{(e)} \; . \qquad (3)$$

(c) What special results are obtained when:
 (i) point P coincides with G;
 (ii) point P is a fixed point?

2. A trolley of mass m is rolls without friction along a horizontal straight rail as shown in Fig.1. A pendulum bob B of mass m swings on a light string of length a attached to the trolley at P, moving in the vertical plane containing the rail. The displacement of the trolley is x and the angle of inclination of the bob from the vertical is θ. Unit vectors $\mathbf{i}$, $\mathbf{j}$ are shown in the figure.

(a) Obtain expressions for $\mathbf{h}_P$, $\mathbf{r}_P$ and $\mathbf{p}$ as defined above and use equation (1) above to show that

$$-a\ddot{\theta} + \cos\theta\,\ddot{x} = g\sin\theta \; .$$

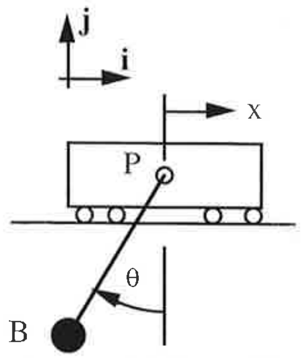
Verify this result by resolving forces at B perpendicular to the string.

(b) Use the result for linear momentum $\dot{\mathbf{p}} = \mathbf{F}^{(e)}$ to show that

$$2\dot{x} - a\dot{\theta}\cos\theta = \text{constant}$$

and find the vertical reaction force R at the wheels of the trolley.

(note: the bob itself has no moment of inertia)



(note: the bob has no "I")

Fig. 1

†3. (a) Calculate the moments of inertia of a solid rectangular cuboid of mass m and sides a , a and $\sqrt{5}a/2$ about axes through the centre of mass and perpendicular to the three sides. (hint: use the perpendicular axis theorem and the Part I Mechanics Data Book).

If the cuboid is thrown into the air, how will its behaviour differ from that of a cylinder?

(b) A square hole of side a/2 and length $\sqrt{5}a/2$ is machined right through the centre of the cuboid perpendicular to the square faces. The mass of the body is now 3m/4. Show that the body will behave like a sphere when tossed in the air.

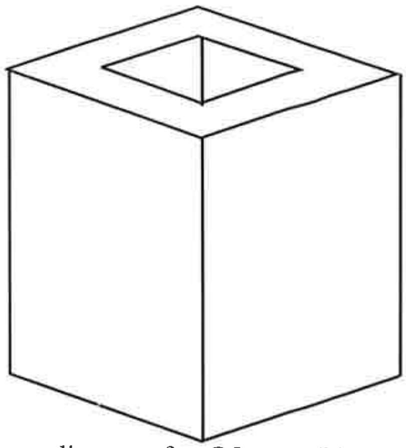


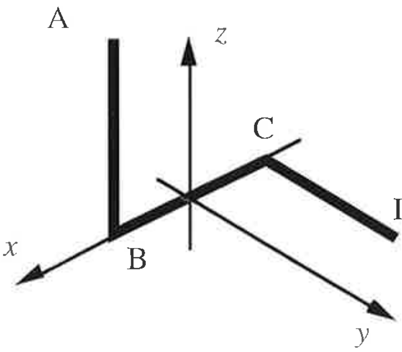
diagram for Q3 part (b)

*4.(a) A rigid body has total mass M and an inertia matrix I_G at its centre of mass G. A point P fixed in the body has coordinates (X, Y, Z) relative to G. Find an expression for the inertia matrix at P in terms of these quantities. You should begin with general expressions such as $I_{yy} = \Sigma m_i(x_i^2 + z_i^2)$ and $I_{xy} = -\Sigma m_i x_i y_i$ etc. and then substitute $x_i = X_i - X$ etc. where (X_i, Y_i, Z_i) are coordinates of the i^{th} particle relative to G.

(b) Determine the principal moments of inertia and the directions of the principal axes at the centre and at a vertex of a uniform solid cube of side a and mass m. (hint: in what sense can the cube to be said to resemble a sphere?)

(c) Such a cube is held with two of its faces horizontal. It is then released so that it begins to turn about one of its upper vertices, which is held fixed. Determine the initial angular acceleration.

5. A uniform wire ABCD of length 6a and mass 3m is bent at right angles at points B and C into three straight sections of equal length. Relative to Cartesian axes OXYZ, the coordinates of the points are as follows: A is (a, 0, 2a); B is (a, 0, 0); C is (-a, 0, 0) and D is (-a, 2a, 0).



Bend one up for yourself out of a paperclip.

Find:

(a) the moments and products of inertia of the wire referred to axes Oxyz;

(b) the coordinates of the mass centre G of the wire;

(c) the moments and products of inertia referred to axes Gx'y'z', where Gx', Gy', Gz' are respectively parallel to Ox, Oy, Oz (see parallel axis theorem, Part I Mechanics Data Book);

(d) the principal moments of inertia at G;

(e) the directions of the principal axes of inertia at G.

3-D dynamics and Euler's Equations

6. A tennis raquet has principal moments of inertia A, B, C at its centre of mass. When it is tossed in the air, spinning about a principal axis, it is found that steady motion about one of the three axes is difficult to achieve.

(a) Derive from first principles Euler's equations of motion of a rigid body in the absence of any externally-applied couple. Begin with the vector expression $\dot{\mathbf{h}}_G = 0$.

(b) Show that steady motion of the racquet is possible about *all three* principal axes.

(c) Show that this motion is stable about only *two* of these axes.

(d) For a tennis racquet use the perpendicular axis theorem to identify which is the unstable axis.

†7. A thin rectangular plate of mass m and sides a and b is fixed to a shaft in the plane of the plate through its centre as shown in Fig. 2. The shaft makes an angle α with the side of length b and is rotating with constant angular velocity Ω in bearings distance L apart.

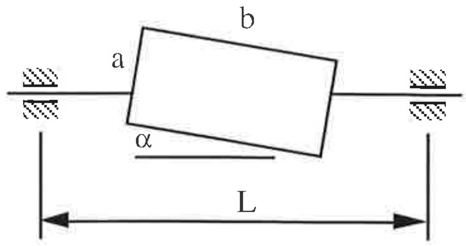


Fig. 2

(a) Determine the components of angular velocity along each of the principal axes of the plate.

(b) Use Euler's dynamical equations to evaluate the magnitude of the rotating bearing reaction forces. Why are these zero for the case of a square plate (ie. when a = b)?

8. A dinner plate, or a pencil, or a wooden spoon can all be considered approximately as cylinders of various aspect ratios. Consider a general cylinder with principal moments of inertia A, A and C with its centre of mass moving freely under the influence of no external couples. (Notation: the components of the cylinder's angular velocity ω_1, ω_2 and ω_3 in a body-fixed reference frame are aligned with the principal axes A, A, C respectively).

(a) Use Euler's equations (i) to show that ω_3 is constant and (ii) to obtain a pair of coupled differential equations involving ω_1 and ω_2. Show then that the general motion is oscillatory (SHM) and find expressions for the variation of ω_1 and ω_2 with time.

(b) Describe the solution in geometric terms for the case of a thin circular disc (eg a dinner plate) and for a slender rod (eg a pencil). How do these solutions differ? Experiment by tossing pencils and dinner plates in the air!

Answers

1.(c) (i) $\dot{\mathbf{h}}_G = \mathbf{Q}^{(e)}$ and (ii) $\dot{\mathbf{h}}_P = \mathbf{Q}^{(e)}$

2. (a) $\mathbf{h}_P = (-ma^2\dot\theta + ma\cos\theta\,\dot x)\mathbf{k}$, $\mathbf{r}_P = x\,\mathbf{i}$ $\mathbf{p} = (2m\dot x - ma\dot\theta\cos\theta)\mathbf{i} + ma\dot\theta\sin\theta\,\mathbf{j}$

(b) $R = ma\ddot\theta\sin\theta + ma\dot\theta^2\cos\theta + 2mg$

3.(a) $\dfrac{3ma^2}{16}, \dfrac{3ma^2}{16}, \dfrac{ma^2}{6}$; identical to a cylinder because it has two equal moments of inertia

(b) all moments of inertia are now $\dfrac{5ma^2}{32}$, indistinguishable from a sphere

4.(a) Typical diagonal element: $A_P = A_G + (Y^2 + Z^2)M$;
Typical off-diagonal element $-D_P = -D_G - YZM$

(b) at G: $1/6\ ma^2$, $1/6\ ma^2$, $1/6\ ma^2$; at a vertex: $11/12\ ma^2$, $11/12\ ma^2$, $1/6\ ma^2$.

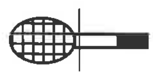
(c) Initial angular acceleration $12g/11a\sqrt{2}$.

5. (a): $I_o = ma^2\begin{bmatrix} 8/3 & 1 & -1 \\ 1 & 11/3 & 0 \\ -1 & 0 & 11/3 \end{bmatrix}$ (b): $[\,0 \;\; a/3 \;\; a/3\,]$ (c): $I_G = ma^2\begin{bmatrix} 2 & 1 & -1 \\ 1 & 10/3 & 1/3 \\ -1 & 1/3 & 10/3 \end{bmatrix}$

(d) & (e): ma^2 in direction (2, -1, 1); $(11/3)ma^2$ in direction (0, 1, 1); $4ma^2$ in direction (1, 1, -1)

6. (a)

$$A\,\dot\omega_1 - (B - C)\,\omega_2\,\omega_3 = 0$$
$$B\,\dot\omega_2 - (C - A)\,\omega_3\,\omega_1 = 0$$
$$C\,\dot\omega_3 - (A - B)\,\omega_1\,\omega_2 = 0$$

(d) 

7.(a) $\Omega\sin\alpha$ (parallel to short side), $\Omega\cos\alpha$ (parallel to long side), 0 (perpendicular to plate)

(b) $\dfrac{m\Omega^2(b^2 - a^2)}{12L}\cos\alpha\,\sin\alpha$

8.(a) $\omega_1 = A_o\sin\lambda t$ $\omega_2 = A_o\cos\lambda t$ where $\lambda = \dfrac{A-C}{A}\,\omega_3$
and where A_o is the oscillatory amplitude.

(b) A constant spin ω_3 about the symmetry axis plus a 'wobble' around the direction of **h**. A disc wobbles twice per revolution (for small-amplitude oscillations); a rod does not wobble at all (This is not easy to visualize – toss a coin and a pencil into the air for confirmation. The wobble is called 'nutation' and is the same as the faster of the two precession rates found in Q8).

Suitable Tripos Questions

2009 Q1	2008 Q1abc	2007 Q2	2006 Q2	2005 Q1, Q2	2004 Q3
2003, Q1	2002, Q1	2001, Q2	2000, Q1	1999, Q1	1998, Q1, Q2, Q3
1997, Q1	1996, Q4	1995, Q1			

Part II, Paper G7 (Old Tripos)

1994, Q2, Q3	1993, Q2	1992, Q3	1991, Q2, Q3	1990, Q2

HEMH Oct 2011