Handout 5 – Continuous vs Discrete Systems

The examples so far illustrate all the main features of continuous systems, and the kinds of calculations that can be made about them. Our toolbox now includes:

- modes and natural frequencies:
  - assume 'separation of variables' solution
  - find the general solution for the resulting ODE
  - apply the boundary conditions

- forced harmonic response and transfer functions:
  - 'Stitch' together free vibration solutions either side of applied point force
  - Use *continuity* and *force balance* to join solutions together

- transient response to particular initial conditions:
  - Use D'Alembert's method for 1D systems governed by the wave equation
  - OR express the initial condition shape as a combination of the free vibration mode shapes
  - OR find the inverse Fourier Transform of the Transfer Function
  - OR use a transmission line analogy

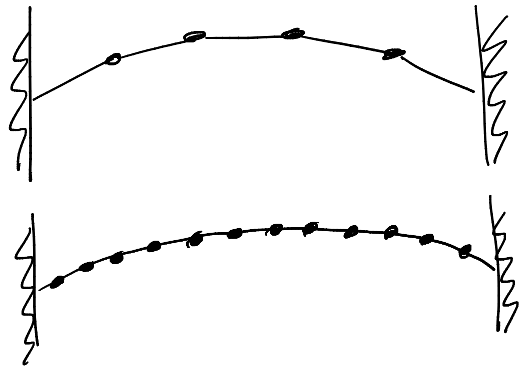
The same topics form part of the 'discrete' half of the course as well. The analogous concepts are as summarised below:

Discrete	**Continuous**
Vector of displacements, $\mathbf{u}$ (or generalised coordinates)	Continuous function of displacements, $u(x)$ (or generalised coordinates)
Matrix equations ('coupled oscillators') $\mathbf{M}\ddot{\mathbf{y}} + \mathbf{K}\mathbf{y} = \mathbf{F}$	Partial differential equations (PDE's) $\dfrac{\partial^2 y}{\partial t^2} + \mathcal{D}_x\left\{y\right\} = f(x,t)$
Finite number of modes	Infinite number of modes
Modes found by finding eigenvalues and eigenvectors, i.e. solve $\det\left\{\mathbf{K} - \omega^2\mathbf{M}\right\} = 0 \quad \text{and} \quad \mathbf{K}\mathbf{y} = \omega_n^2\mathbf{M}\mathbf{y}$	Modes found by solving unforced PDE together with boundary conditions: assume separation of variables

Continuous systems can always be thought of as limiting cases of discrete systems. This means that we can find useful discrete approximations of continuous systems as long as we include 'enough' degrees of freedom. Thinking about it the other way around: there are a variety of ways of 'discretising' continuous systems, the most common are summarised as follows:

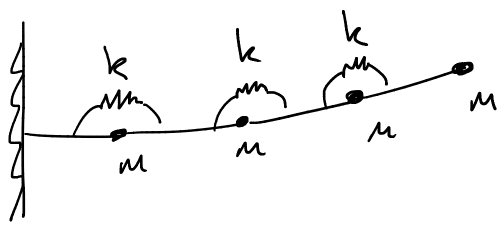
- 'Lumped mass' model: the structure is made up of point masses (or rigid bodies) connected by massless springs. This approximation is one of the simplest ways of approximating continuous systems. Although it is not the most accurate method, it is still widely used precisely because it is such a simple approach.

String:



Point masses joined by massless springs
See examples paper 1

Cantilever beam:



Rigid links joined by pins and torsion springs

- 'Modal reduction': the modes of a continuous structure can be identified, then the response can be approximated by only including a finite number of modes.

- 'Galerkin' approach: the deformation of the structure is assumed to be a linear sum of suitable functions (called basis / shape / test functions). The degrees of freedom are the magnitudes of each basis function.

- 'Finite Element' method: the structure is divided into small sections, the deformation of each section is assumed to be given by a suitable function (like the Galerkin approach) and the pieces are stitched together to give the overall deformation. The degrees of freedom are the states at each end of each section of the structure, which determine the underlying functions.

Transfer Functions

It is easier to prove general theorems in terms of discrete systems, and we will use the idea of approaching continuous systems as limits of discrete systems to justify applying these general results to continuous systems as well.

From the 'discrete' half of the course, we know that the transfer function from an input force at node j to the response at node k is given by:

$$G(j,k,\omega) = \frac{y_k}{f_j} = \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2 - \omega^2}$$

By regarding a continuous system as a limit of discrete systems the proof you have already seen carries over directly, so that the transfer function from an input force at position x to the response at position y is given by:

input position

measurement position

$$G(x,y,\omega) = \frac{U}{F} = \sum_{n=1}^{N} \frac{u_n(x) u_n(y)}{\omega_n^2 - \omega^2}$$

modes, n

BUT: the result relies on orthogonality of modes, and the mode shapes must be suitably normalised before the formula is used.

Discrete **Continuous**

orthogonality

$$\underline{u}_a^T M \underline{u}_b = \begin{cases} 0 & \text{if } a \neq b \\ 1 & \text{if } a = b \end{cases} = \int u_a(x) u_b(x) \underbrace{dm}_{\text{eg } \rho A dx \text{ for beam}}$$

normalisation

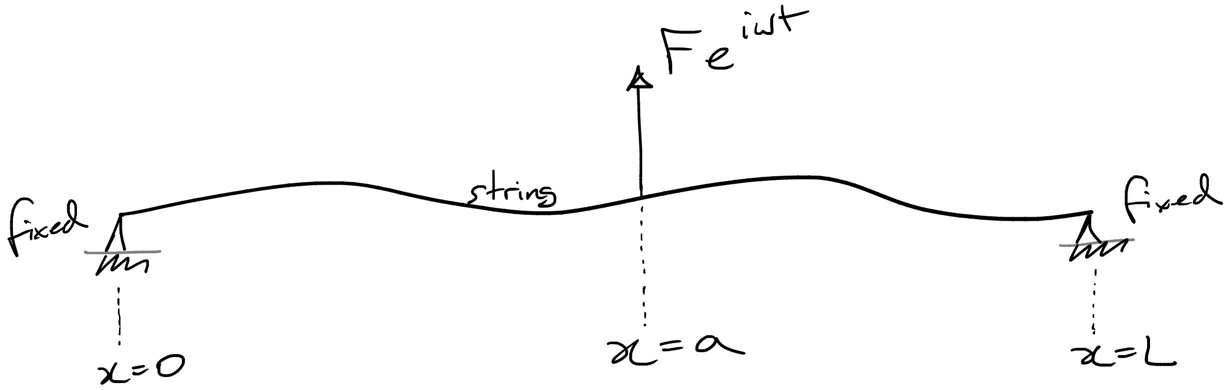
eg $\rho A dx$

for beam

Recall that when we found the transfer function for a continuous system, we had to stitch together free solutions of the system such that there was *continuity* and *force balance* at the site of the input force (see Handout 2).

This time, we have found the same Transfer Function directly from the vibration modes of the system, *without directly considering the input force*. The expressions that we obtain are equivalent to each other, but notice that the modal approach is the summation of a series which converges to the exact solution as the number of modes included tends to infinity, while the direct approach gives the exact expression without the summation.

The stretched string example again:



The transfer function found in Handout 2 was:

$$G(a,x,\omega) = \frac{U(x)}{F(a)} = \begin{cases} \dfrac{c}{\omega P}\dfrac{\sin\left[\omega(L-a)/c\right]}{\sin(\omega L/c)}\sin\left[\omega x/L\right] & \text{for} \quad 0 < x < a \\ \\ \dfrac{c}{\omega P}\dfrac{\sin\left[\omega a/c\right]}{\sin(\omega L/c)}\sin\left[\omega(L-x)/L\right] & \text{for} \quad a < x < L \end{cases}$$

OR applying the modal formula:

$$G(a,x,\omega) = \frac{U(x)}{F(a)} = \sum_{n=1}^{N}\frac{u_n(a)u_n(x)}{\omega_n^2-\omega^2}$$

For the stretched string we know that:

$$u_n(x) = C_n \sin\frac{n\pi x}{L}$$

but we need to find C_n in order to normalise the mode shape:

$$\begin{aligned}\int_0^M u_n^2(x)\mathrm{d}m &= \int_0^L C_n^2 \sin^2\left(\frac{n\pi x}{L}\right) m\mathrm{d}x \\ &= \frac{mC_n^2}{2}\int_0^L \left(1 - \cos\left(\frac{2n\pi x}{L}\right)\right)\mathrm{d}x \\ &= \frac{mLC_n^2}{2} \\ &= 1\end{aligned}$$

giving:

$$C_n = \sqrt{\frac{2}{mL}}$$

Substituting into the modal expression for the Transfer Function gives:

$$G(a,x,\omega) = \frac{2}{mL}\sum_n \left[\frac{\sin\left(\frac{n\pi a}{L}\right)\sin\left(\frac{n\pi x}{L}\right)}{\omega_n^2 - \omega^2}\right]$$

where $\omega_n = \frac{n\pi c}{L}$, $c = \sqrt{P/m}$

For convenience, put the two expressions next to each other to compare:

DIRECT: $$G(a,x,\omega) = \frac{U(x)}{F(a)} = \begin{cases} \dfrac{c}{\omega P}\dfrac{\sin\left[\omega(L-a)/c\right]}{\sin(\omega L/c)}\sin\left[\omega x/L\right] & \text{for} \quad 0 < x < a \\ \dfrac{c}{\omega P}\dfrac{\sin\left[\omega a/c\right]}{\sin(\omega L/c)}\sin\left[\omega(L-x)/L\right] & \text{for} \quad a < x < L \end{cases}$$

MODAL: $$G(a,x,\omega) = \frac{U(x)}{F(a)} = \frac{2}{mL}\sum_n \frac{\sin\left(\frac{n\pi a}{L}\right)\sin\left(\frac{n\pi x}{L}\right)}{\omega_n^2 - \omega^2}$$

The two expressions look entirely different, but are completely equivalent. We can demonstrate it numerically: see `H5_transfer_function_equivalence.m`

Notice reciprocity again: it is particularly clear in the modal expression that *a* and *x* can be swapped. Also note that in general the coefficients C_n would normally depend on *n*: this example which gives constant C_n is a special case.

With a modal expression for the transfer function, we can readily find the impulse response as the inverse Fourier (or Laplace) Transform of each term of the summation can easily be found using the databook.

The modal expression for the transfer function is:

$$G(x,y,\omega) = \sum_n \frac{u_n(x)\,u_n(y)}{\omega_n^2 - \omega^2}$$

from the Electrical databook and using $s = i\omega$ (i.e. $s^2 = -\omega^2$), the inverse Laplace Transform gives the impulse response:

$$g(x,y,t) = \sum_n \frac{u_n(x)\,u_n(y)}{\omega_n} \sin(\omega_n t)$$

Damping

So far we have considered the vibration of structures without damping. In reality there is always some source of energy loss, such that resonant peaks are finite and antiresonances are small but not zero. Just as in the discrete case, we can make the transfer function a little more realistic by including some damping.

Damping mechanisms are not usually very well understood in most practical problems (go to 4C6 next year for more on this), but if damping is small then we can use an *ad hoc* approximation, simply including a 'modal damping factor' for each mode. The transfer function then becomes:

$$G(x,y,\omega) \simeq \sum_n \frac{u_n(x)\,u_n(y)}{\omega_n^2 + 2i\zeta_n\omega_n\omega - \omega^2}$$

modal damping factor, dimensionless

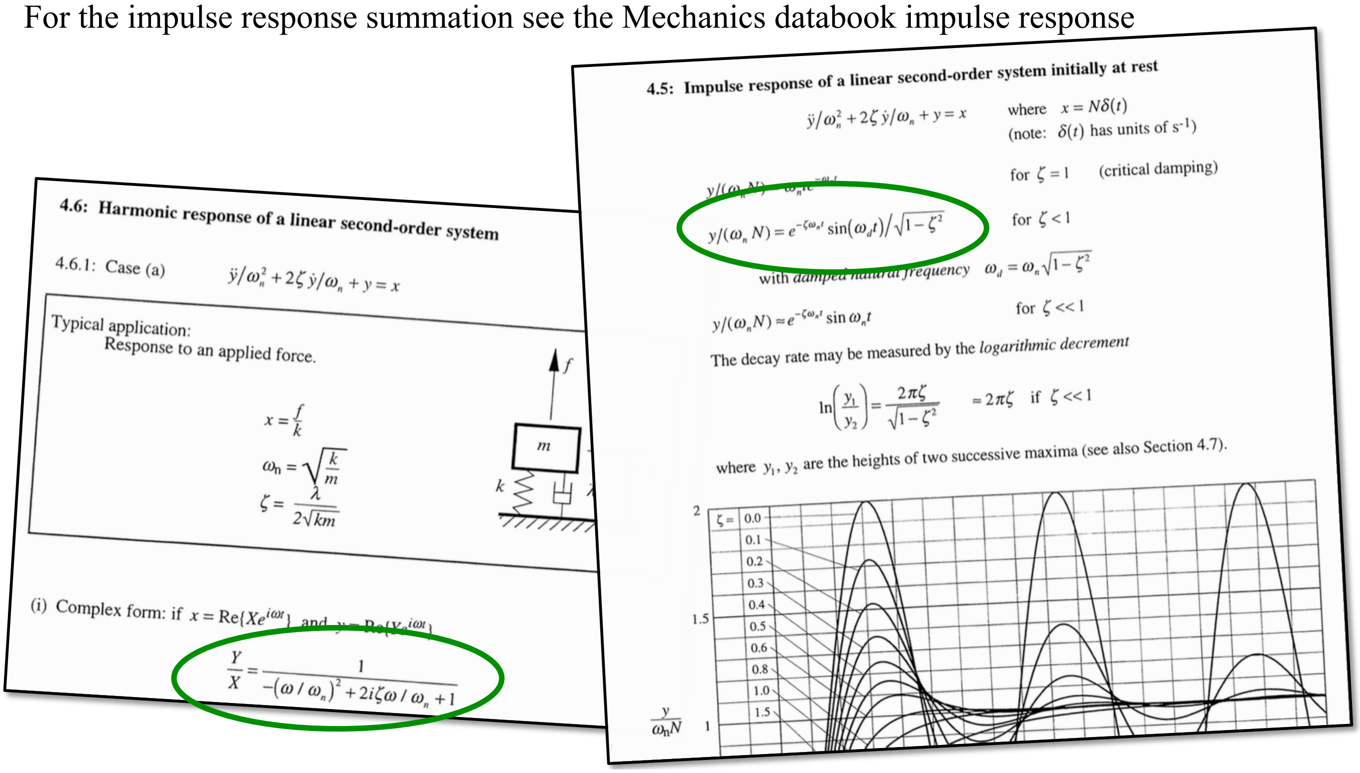
and the corresponding impulse response is given by:

$$g(x,y,t) \simeq \sum_n \frac{u_n(x)\,u_n(y)}{\omega_n} e^{-\zeta_n\omega_n t} \cdot \sin\omega_n t$$

Similarly the step response can also be found very readily. These expressions can all be found on the 3C6 datasheet.

Each term of the sum is simply the response of a damped mass-on-spring oscillator. The mode shape factors in the numerator tell us how strongly that particular mode contributes to the response.

- For the transfer function summation see the Mechanics databook harmonic case (a)
- For the impulse response summation see the Mechanics databook impulse response



So the Transfer Function for the <u>stretched string</u> example with light damping becomes:

$$G(a,x,\omega) = \frac{U(x)}{F(a)} = \frac{2}{mL}\sum_n \frac{\sin\left(\frac{n\pi a}{L}\right)\sin\left(\frac{n\pi x}{L}\right)}{\omega_n^2 + 2i\zeta_n\omega_n\omega - \omega^2}$$

We can carry out the same approach for any kind of continuous structure (as long as the behaviour is reasonably approximated as linear).

The <u>pinned-pinned beam</u> gives is an interesting case: recall that the mode shapes are identical to a stretched string and only the natural frequencies are different (see Handout 4). So the Transfer Function expression is identical to the one above, but with:

$$\omega_n = \left(\frac{n\pi}{L}\right)^2\sqrt{\frac{EI}{\rho A}}$$

From the differential equations it is not at all obvious that the Transfer Function expressions for a pinned-pinned beam and a stretched string should be identical, but just with new values for the natural frequencies.

For a beam with other boundary conditions we can use the same modal summation formula, but

- there is no explicit formula for the natural frequencies;
- it is hard to normalise the mode shapes except numerically.

Summary

- Continuous systems can be thought of as a limiting case of discrete systems
- Similarly: continuous systems can be approximated using a discrete representation: the 'lumped mass' approach is one of the simplest ways of doing this
- Results from discrete systems can be applied to continuous systems
- The modal summation formula for discrete systems can be applied to continuous systems (but remember to normalise the mode-shapes).
- All Transfer Functions can be expressed as:

$$G(x,y,\omega) \approx \sum_{n=1}^{N} \frac{u_n(x)u_n(y)}{\omega_n^2 + 2i\zeta_n\omega_n\omega - \omega^2}$$

- with corresponding impulse responses

$$g(x,y,t) \approx \sum_{n=1}^{N} \frac{u_n(x)u_n(y)}{\omega_n} \mathrm{e}^{-\zeta_n\omega_n t} \sin\omega_n t$$

- and assuming light damping, with dimensionless modal damping factor ζ_n