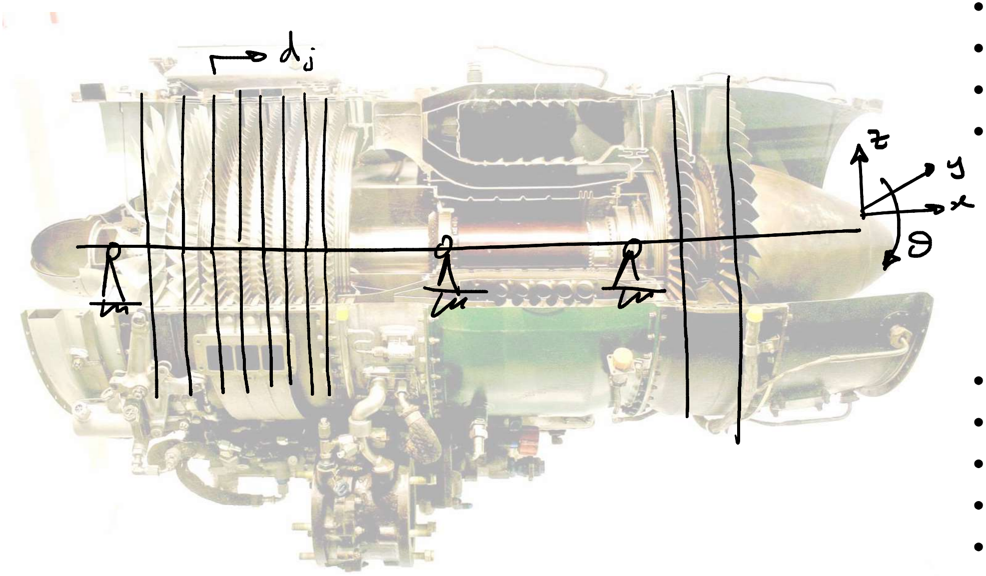
Handout 6 – Coupling and Rayleigh's Principle

We have looked briefly at the vibration behaviour of various simple systems. In engineering structures these can be regarded as typical components of more complicated systems.

e.g. turbojet engine



- shaft (torsion, axial, bending)
- pins / clamps (bending)
- free bearing for torsion
- cantilever beams (bending)

- Curved plates and panels
- Acoustic waves
- Friction dampers
- Spin
- Complex geometry

To study the behaviour of an assembled structure, we need to couple these components together in appropriate ways. We could use the Finite Element method: using a software package to define the geometry, create a mesh, compute the mass and stiffness matrices then find the eigenvalues (natural frequencies) and eigenvectors (mode shapes) of the model, or do some other analysis depending on what you are trying to find out. This is a very standard approach within industry which has many advantages:

- Mature software packages exist for creating and analysing models

- It is relatively straightforward to make models of structures with arbitrarily complicated geometries

- Finite Element modelling has a rigorous theoretical grounding that has been widely accepted.

But there are a few things to note:

- The Finite Element method essentially takes lots of smaller subcomponents and couples them together appropriately (so part of understanding the Finite Element method means we need to understand coupling of components);

- Finite Element models of industrial structures may have over 10^6 degrees of freedom: big models mean computationally expensive calculations that places significant demands on RAM and CPU;

- Large-scale Finite Element models may not give us much insight into what's going on: this is the problem of dealing with a large amount of data resulting from a computation;

- There can be a false-security in thinking that because you have a detailed model, then the answer must be 'right'.

Coupling of systems: using free vibration solutions

In Examples Paper 1 you have already tackled the simple problem of two joined stretched strings with different properties.

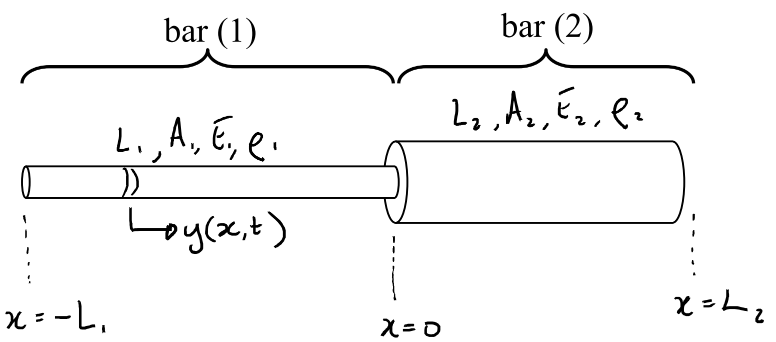
The method was as follows:

1. For each subsystem, write down or find the free vibration solution (using any boundary conditions you already know);

2. Apply continuity and balance of forces at the junction;

3. Solve to find the response of the coupled system.

Example: axial vibration of two bars



Boundary Conditions at ends : $x = -L_1$, $y'(-L_1,t) = 0$
$x = L_2$, $y'(L_2,t) = 0$

By separation of variables: $y(x,t) = U(x)e^{i\omega t}$

$$\begin{aligned} \text{for bar (1):} \quad U_1(x) &= \boxed{D_1 \cos\left[k_1\left(x+L_1\right)\right]} \quad \text{for} \quad x \le 0 \\ \text{for bar (2):} \quad U_2(x) &= \boxed{D_2 \cos\left[k_1\left(L_2-x\right)\right]} \quad \text{for} \quad x \ge 0 \end{aligned}$$

$k_1 = \omega/c_1 = \omega\sqrt{\rho_1/E_1}$

carefully chosen so BC's satisfied

At the junction between the two bars:

Continuity: $U_1(0) = U_2(0)$

$$D_1 \cos k_1 L_1 = D_2 \cos k_2 L_2$$

① $$D_1 \cos \tfrac{\omega L_1}{c_1} = D_2 \cos \tfrac{\omega L_2}{c_2}$$

Force balance: $E_1 A_1 U_1'(0) = E_2 A_2 U_2'(0)$

$$-D_1 E_1 A_1 k_1 \sin k_1 L_1 = D_2 E_2 A_2 k_2 \sin k_2 L_2$$

② $$-D_1 \frac{E_1 A_1 \cancel{\omega}}{c_1} \sin \frac{\omega L_1}{c_1} = D_2 \frac{E_2 A_2 \cancel{\omega}}{c_2} \sin \frac{\omega L_2}{c_2}$$

(ω cancels except for rigid body solution)

②/① $\Rightarrow$ $$-\frac{E_1 A_1}{c_1} \tan \frac{\omega L_1}{c_1} = \frac{E_2 A_2}{c_2} \tan \frac{\omega L_2}{c_2}$$

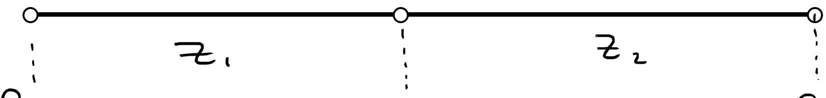
natural frequencies must satisfy this equation

Coupling of systems: transmission line analogy

We can think of the two-bar problem as being two transmission lines (or waveguides) each with different characteristic impedances:

Choose analogy:
voltage → velocity.
current → force

$$Z = \frac{V_F}{I_F} \longrightarrow Z = \frac{\dot{y}_F}{F_F}$$



free

$Z_A = \infty$

$$R = \frac{Z_A - Z_1}{Z_A + Z_1} = 1$$

$$R_F = \frac{Z_2 - Z_1}{Z_2 + Z_1}$$

$$T_F = \frac{2Z_2}{Z_2 + Z_1}$$

free

$Z_B = \infty$

$$R = 1$$

This analogy means that Z for the mechanical system is the characteristic *admittance*.

In general, this analogy does not immediately lend itself to finding the transfer function or natural frequencies but it can give an intuitive understanding of the early part of a transient response. If we applied an impulse to one end of bar 1 then:

- we can find the time T_1 that it would take for the pulse to travel the length of bar 1 $(T_1 = L_1/c_1)$
- and the time T_2 that it would take for the pulse to travel the length of bar 2 $(T_2 = L_2/c_2)$
- the reflection and transmission coefficients give the fraction of incoming to outgoing waves
- and you can build up an intuitive picture of the short-term response.

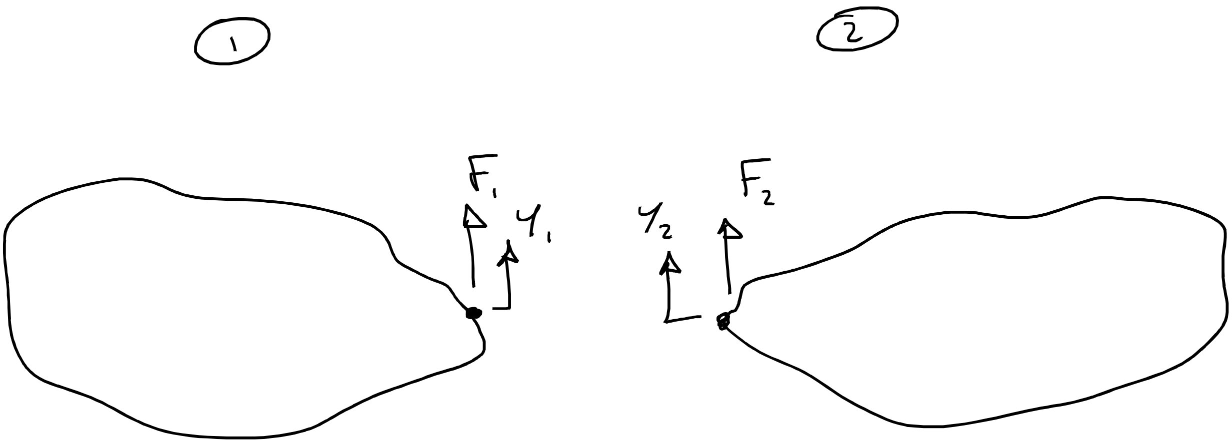
The full response involves a complicated sequence caused by multiple internal and end reflections.

When there is a significant change of impedance then the reflection coefficient at the boundary will be close to $\rho = \pm 1$. This means that the two structures will behave approximately independently, and the coupled modes will be similar to the modes of the separate structures. For example: at the ends of a guitar string the reflection coefficient is approximately $R = -1$, so that the string modes follow what we expect for a string fixed at each end. The fact that the transmission coefficient is not exactly $T = 0$ is what allows us to hear the guitar.

Coupling of systems: using Transfer Functions

A particular case of coupling that is simple to treat occurs when two systems are connected together at a point.

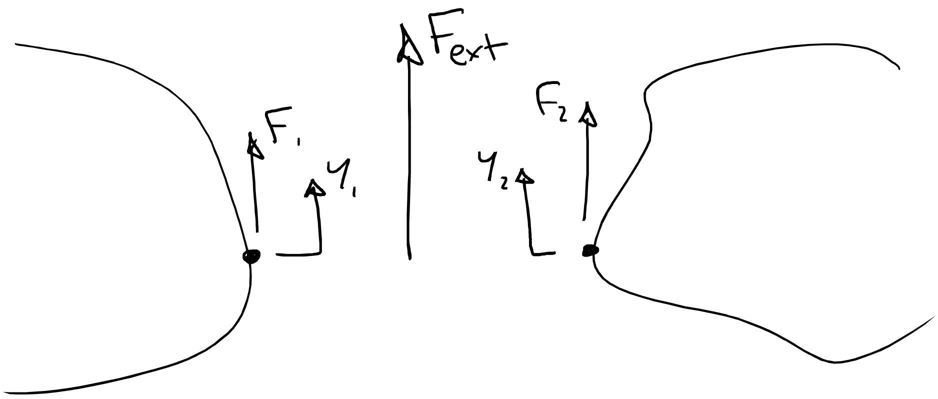
Consider two arbitrary systems that we will couple:



We could measure or calculate the driving point transfer functions for each subsystem when they are uncoupled (you do this in the laboratory experiment for the two beams).

$$G_1 \equiv \frac{Y_1(\omega)}{F_1(\omega)}, \quad G_2 \equiv \frac{Y_2(\omega)}{F_2(\omega)}$$

If we want to know the new transfer function of the coupled system then we can imagine applying an external force at the same point as where they are coupled:



Then we can apply *continuity* and *force balance* at the junction (being careful with signs!):

$$y_1 = y_2 = y, \qquad F_1 + F_2 = F_{ext}$$

Note that the external force F_{ext} is the resultant of the internal forces F_1 and F_2.

Combining these expressions at the junction with the subsystem Transfer Function definitions gives:

$$y_1/G_1 + y_2/G_2 = F_{ext}$$

$$y\left(\frac{1}{G_1} + \frac{1}{G_2}\right) = F_{ext}$$

$$G_c = \frac{y}{F_{ext}} = \left(\frac{1}{G_1} + \frac{1}{G_2}\right)^{-1} = \frac{G_1 G_2}{G_1 + G_2}$$

which is analogous to the result obtained for parallel connection of resistors in electrical circuits.

Although this result is specific to the driving point response at the position of the connection, the natural frequencies that are predicted by this expression are nevertheless intrinsic properties of the coupled system and are given by $G_1 + G_2 = 0$.

What happens if $|G_1| \ll |G_2|$? $G_c \longrightarrow \frac{G_1 \cancel{G_2}}{\cancel{G_2}} = G_1$

What happens if $G_1 = G_2$? $G_c = \frac{G_1^{\cancel{2}}}{2\cancel{G_1}} = G_1/2$?

See `H6_coupling.m`

A simple application of the coupling formula arises if we ask what happens to the natural frequencies of a system if we attach a point mass, *M*, to a point *x*. Suppose the original system has normalised mode shapes u_n and natural frequencies ω_n.

The transfer function (without the mass) is given by the standard expression:

$$G_1 = G(x, x, \omega) = \sum_n \frac{u_n^2(x)}{\omega_n^2 - \omega^2}$$

and the transfer function of the mass on its own (free mass) is:

$$G_2 = -1/M\omega^2$$

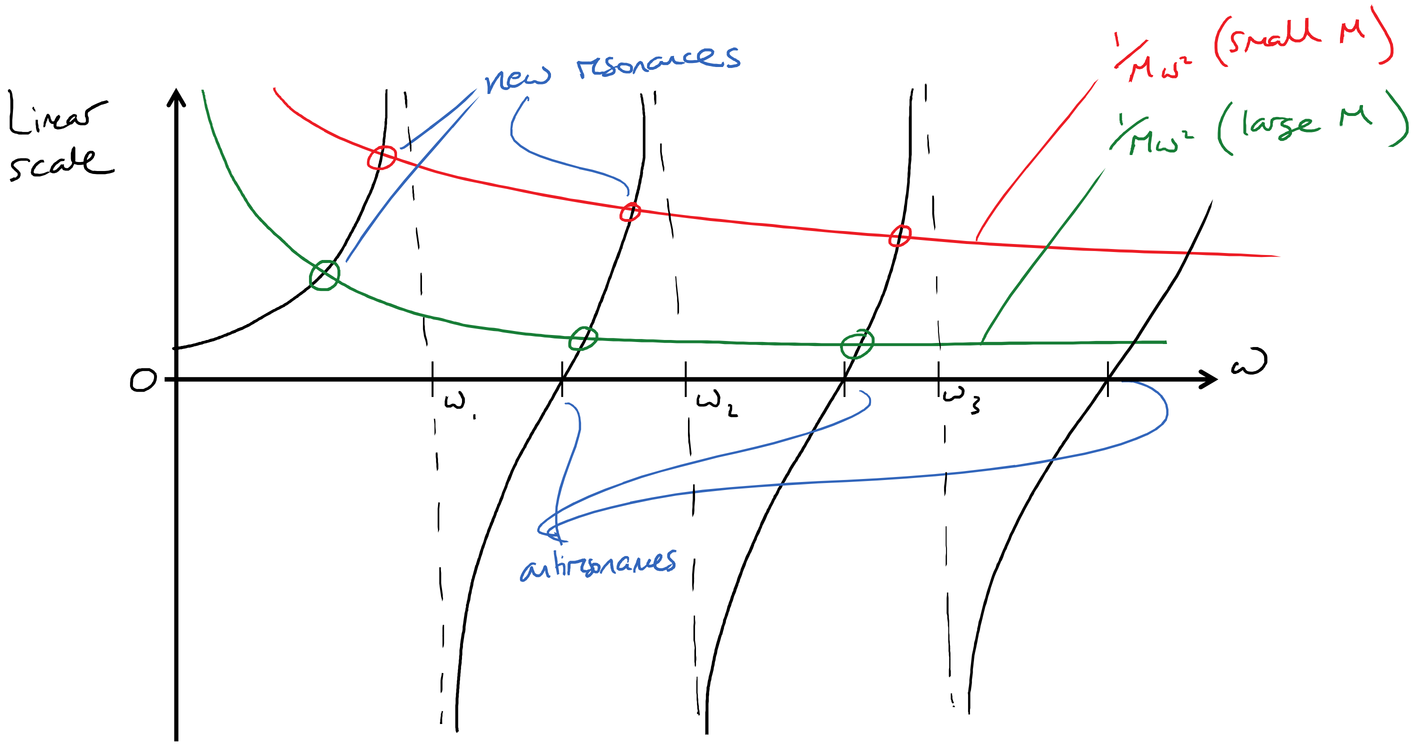
The coupled transfer function G_C could be found using:

$$\frac{1}{G_C} = \frac{1}{G_1} + \frac{1}{G_2}$$
$$= \frac{1}{G_1} - M\omega^2$$

but we just want to know what happens to the natural frequencies. The natural frequencies of the coupled system will be given by the poles of the G_C or the zeros of $1/G_C$, i.e. when

$$\frac{1}{G_1} - M\omega^2 = 0, \quad \text{or} \quad G_1 = \frac{1}{M\omega^2}$$

We can easily see what happens from a graphical argument, by plotting G_1 and $1/M\omega^2$ and finding the intersections:



In summary:

- for small masses and at low frequency: the natural frequencies are only reduced a little

- for large masses and at high frequency: the natural frequencies are reduced significantly. The mass constrains the system and creates a nodal point at the position of the mass.

Rayleigh's Principle

Rayleigh's principle has been covered in the Discrete Systems part of the lecture course. The result can be carried over directly to continuous systems, just as seen for the results for transfer functions and transient response.

The starting point is the Rayleigh quotient, the ratio of the potential energy to the "kinetic energy with the time derivatives ignored". The result is that if the Rayleigh quotient is evaluated with an approximation to a mode shape $u_n(x)$, then the value will be a better approximation to the squared natural frequency ω_n^2. If the mode in question is the lowest mode of a system, then the Rayleigh quotient will always give an overestimate of the squared natural frequency.

As usual, continuous systems differ from discrete systems in the mathematical details. For a discrete system, the Rayleigh quotient is expressed in terms of quadratic algebraic expressions, based on the stiffness and mass matrices. For a continuous system, the potential and kinetic energies, and thus the Rayleigh quotient, involve integrals. The kinetic energy is usually quite simple, being just the integral over the whole system of "1/2 $m\, v^2$".

For example, for vibration of a stretched string the kinetic energy is

$$T = \frac{1}{2}m \int \left(\frac{\partial y}{\partial t}\right)^2 \mathrm{d}x$$

So the quantity that would be used in the Rayleigh quotient is

$$\tilde{T} = \frac{1}{2}m \int y^2 \mathrm{d}x$$

The potential energy of the vibrating string is not quite so obvious, but was already worked out in the course of question 3 of Examples Sheet 1. By calculating the work done in stretching each small element of the string as it moves away from the original straight position, the energy can be shown to be:

$$V = \frac{1}{2}P \int \left(\frac{\partial y}{\partial x}\right)^2 \mathrm{d}x$$

Potential and kinetic energies for the other systems considered in this course are listed on the 3C6 datasheet.

Example: stretched string

Let us guess the lowest mode shape of a stretched string, and use Rayleigh's principle to estimate the frequency. We need a guessed shape which satisfies the fixed boundary conditions at the two ends of the string, and the simplest expression which does that is:

$$u(x) = x(L-x)$$

So $$\tilde{T} = \frac{1}{2} m \int_0^L [x(L-x)]^2 \, dx$$

$$= \frac{1}{2} m \int_0^L (x^4 - 2x^3 L + x^2 L^2) \mathrm{d}x$$

$$= \frac{1}{2} m \left[\frac{L^5}{5} - \frac{2L^5}{4} + \frac{L^5}{3} \right]$$

$$= \frac{mL^5}{60}$$

$$V = \frac{1}{2} P \int_0^L [u'(x)]^2 \, dx$$

$$= \frac{1}{2} P \int_0^L (L - 2x)^2 \, dx$$

$$= \frac{1}{2} P \int_0^L (L^2 - 4xL + 4x^2) \mathrm{d}x$$

$$= \frac{1}{2} P \left[L^3 - \frac{4L^3}{2} + \frac{4L^3}{3} \right]$$

$$= \frac{PL^3}{6}$$

$$\omega_1^2 \simeq \frac{V}{\tilde{T}} = \frac{10P}{mL^2}$$ giving $$\omega_1 \simeq \frac{\sqrt{10}}{L} \sqrt{P/m}$$

The exact result is:

$$\omega_1 = \frac{\pi}{L}\sqrt{\frac{P}{m}}$$

so our estimate is extremely close:

$$\sqrt{10} = 3.162$$
$$\pi = 3.142$$
$$\text{error} < 1\%$$

Summary

We can analyse coupled systems by applying continuity and force balance at the connection point and either:

- join the free vibration solution for each of the subsystems
- OR use a Transfer Function approach

For 1D systems, we can gain some insight into the short-term transient response by considering the junction in terms of reflection and transmission coefficients in a waveguide.

Adding mass always lowers the natural frequencies, having a larger effect at higher frequencies

Rayleigh's principle applies to continuous systems in the same way as for discrete systems, but the 'derivative-free kinetic energy' and potential energy involve integrals: see the 3C6 Datasheet!

We can guess mode shapes and use Rayleigh's quotient to estimate natural frequencies

We can estimate the effect of small structural changes by assuming that the mode shapes haven't changed and finding the new Rayleigh quotient.