Master (hemh)

Part IIA

Paper ~~G7~~ 3C5 Dynamics and Vibrations

Ten Lectures on Rigid-Body Dynamics

Lectures 6-10: Applications

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Michaelmas 1997

Major changes in 2002 :
pp 11-14 } no longer on syllabus
pp 21-24 }

CONTENTS - Lectures 6-10

For Rolling Balls see crib to EP2

portrait 100% / landscape 138%

5. GYROSCOPES

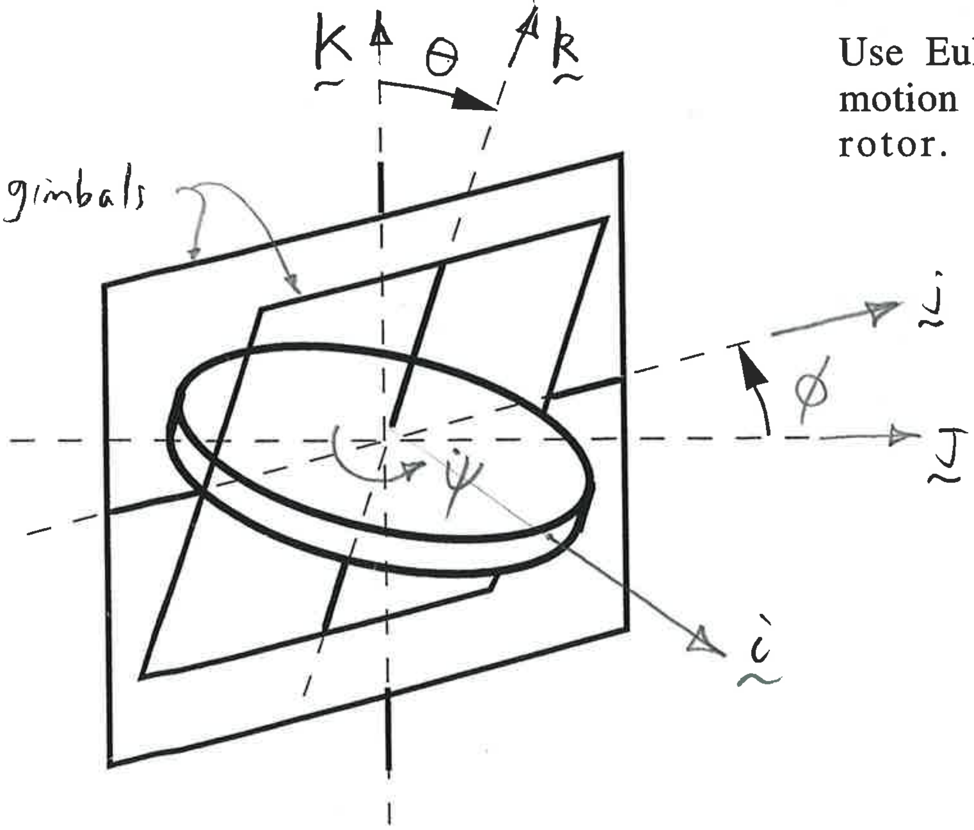
A Gyroscope (or "gyro") comprises a rotor spinning about its axis of symmetry in some kind of gimbals

Gimbals (or "Cardan's suspension") (pronounced "Gimbal") is the name given to the frame that allows the gyro rotor to assume any angular position G hired.

Gyros are widely used for navigation and stabilisation
ships, aircraft, spacecraft, satellites missiles

We will study
- precession — steady motion under applied couple
- nutation — wobbling, even with no couple.

The figure shows a rotor in a "Cardan Suspension" (gimbals)



Use Euler's angles to define motion of gimbals and of the rotor.

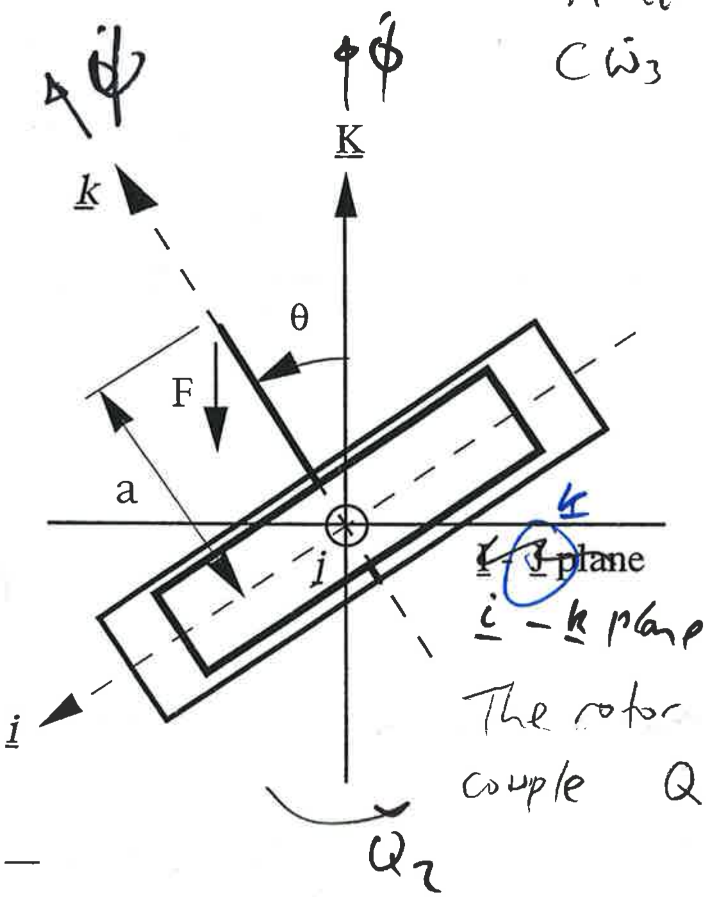
Rotor ~~"spin" is~~ ψ

end L6 M02

5.1 Steady Precession

Use the gyro equations:

$$A\dot{\Omega}_1 - (A\Omega_3 - C\omega_3)\Omega_2 = Q_1$$
$$A\dot{\Omega}_2 + (A\Omega_3 - C\omega_3)\Omega_1 = Q_2$$
$$C\dot{\omega}_3 = Q_3 \quad (4.8)$$



Remember the relations for Euler's angles:

$$\Omega_1 = -\dot{\phi}\sin\theta$$
$$\Omega_2 = \dot{\theta}$$
$$\Omega_3 = \dot{\phi}\cos\theta \quad (4.9)$$

$$\omega_3 = \Omega_3 + \dot{\psi}, \quad \dot{\psi} = \text{const}$$

The rotor "AAC" is subject to a couple $Q_2 = \cancel{Fa\sin\theta}$ ($\underline{j}$)

For steady state response, the moving frame has steady angular velocity

$$\therefore \dot{\Omega}_1 = \dot{\Omega}_2 = \dot{\Omega}_3 = 0 \quad \text{and also} \quad \dot{\theta} = 0, \quad \dot{\phi} = \text{const}$$

Applied couples are zero except Q_2 in the j direction

$$Q_1 = 0$$
$$Q_2 = \cancel{Fa\sin\theta}\ Q_2$$
$$Q_3 = 0$$

Substitute these into (4.8) and (4.9) to give

$$-(A\dot{\phi}\cos\theta - C\omega_3)\dot{\theta} = 0 \quad \underline{\text{true}} \quad (5.1a)$$

$$-(A\dot{\phi}\cos\theta - C\omega_3)\dot{\phi}\sin\theta = \cancel{Fa\sin\theta}\ Q_2 \quad (5.1b)$$

$$C\dot{\omega}_3 = 0 \quad \underline{\text{true}} \quad (5.1c)$$

equation "2" is the only interesting one

The second equation (5.1b) is a quadratic in $\dot{\phi}$,

$$\dot{\phi}^2 A\cos\theta \sin\theta - \dot{\phi}\, C\omega_3 \sin\theta + \cancel{Fa}\, Q_2 = 0$$

but for fast spin we assume $\dot{\phi} \ll \omega_3$ (ie slow precession)

(note we have divided through by $\sin\theta$ so take care with assumptions for $\theta \to 0$)

so that steady *precession* occurs at a constant rate

$$\dot{\phi} \approx \frac{\cancel{Fa}\, Q_2}{C\omega_3 \sin\theta} \quad \left(\begin{array}{l}\text{note independent of } \theta \\ \text{as in gyro lab}\end{array}\right) \tag{5.2}$$

This is the same as $Q = J\omega\Omega$ in Part IB but then you assumed $\theta = 90°$.

—

end L7 M03

If the spin is not "fast" then we should solve the quadratic in full.

$$\dot{\phi} = \frac{C\omega_3 \sin\theta \pm \sqrt{C^2\omega_3^2 \sin^2\theta - 4A\cos\theta \sin\theta\, \cancel{Fa}\, Q_2}}{2A\cos\theta\sin\theta}$$

There is another solution (which we don't call precession)

$$\dot{\phi} \approx \frac{C\omega_3}{A\cos\theta} \quad \text{Nutation.} \tag{5.3}$$

For thin discs, $C = 2A$ $\therefore$ $\dot{\phi} \approx 2\omega_3$ for $\theta \to 0$

This motion occurs at a very high rate (infinite when θ = 90°)

and it occurs without the need for an applied couple.

We call this motion *nutation* and for small amplitude (θ → 0°) we get the same frequency as computed below (eq 5.4)

Let's now calculate nutation frequency assuming it to be a small vibration superimposed on steady precession.

end L6 99, 01

Nutation is an oscillation superimposed on steady precession. It can also occur in the absence of any applied couple.

In steady state *precession* we have $\theta = \theta_0 = \text{const}$

$$Q_1 = Q_3 = 0 \quad , \quad \dot\Omega_1 = \dot\Omega_2 = \dot\Omega_3 = 0 \quad , \quad \dot\psi = \text{const}$$

and $$\dot\phi = \frac{Q_2}{C\omega_3 \sin\theta} \qquad \therefore \quad \Omega_1 = -\dot\phi \sin\theta = -\frac{Q_2}{C\omega_3}$$

Now perturb the motion by a small amount so that

$$\theta = \theta_0 + \alpha \qquad \alpha, \beta \text{ small}$$
$$\Omega_1 = -\frac{Q_2}{C\omega_3} + \beta$$

$$\therefore \quad \dot\theta = \dot\alpha = \Omega_2 \quad , \quad \ddot\theta = \ddot\alpha = \dot\Omega_2$$
$$\dot\Omega_1 = \dot\beta$$

Use the first two gyro equations (4.8): assume fast spin $C\omega_3 \gg A\Omega_3$

put Ω_2 ;

$$A\dot\beta + C\omega_3 \dot\alpha = 0 \tag{5.3a}$$
$$A\ddot\alpha - C\omega_3\left(-\frac{Q_2}{C\omega_3} + \beta\right) = Q_2 \tag{5.3b}$$

Note that the Q2 terms cancel in (5.3b) so that

$$A\ddot\alpha - C\omega_3\beta = 0 \tag{5.3c}$$

Now integrate (5.3a) $\;,\; \beta = -\dfrac{C\omega_3\alpha}{A} + \text{const}$

and substitute β into (5.3c) to give

$$\therefore \quad A\ddot\alpha - C\omega_3\left(-\frac{C\omega_3\alpha}{A} + \text{const}\right) = 0$$
$$\therefore \quad \ddot\alpha + \left(\frac{C}{A}\omega_3\right)^2 \alpha = \text{const}'$$

Hence the nutation frequency is $\dfrac{C\omega_3}{A}$ (5.4)

Do question 8 on G7/1 and question 1 on G7/2

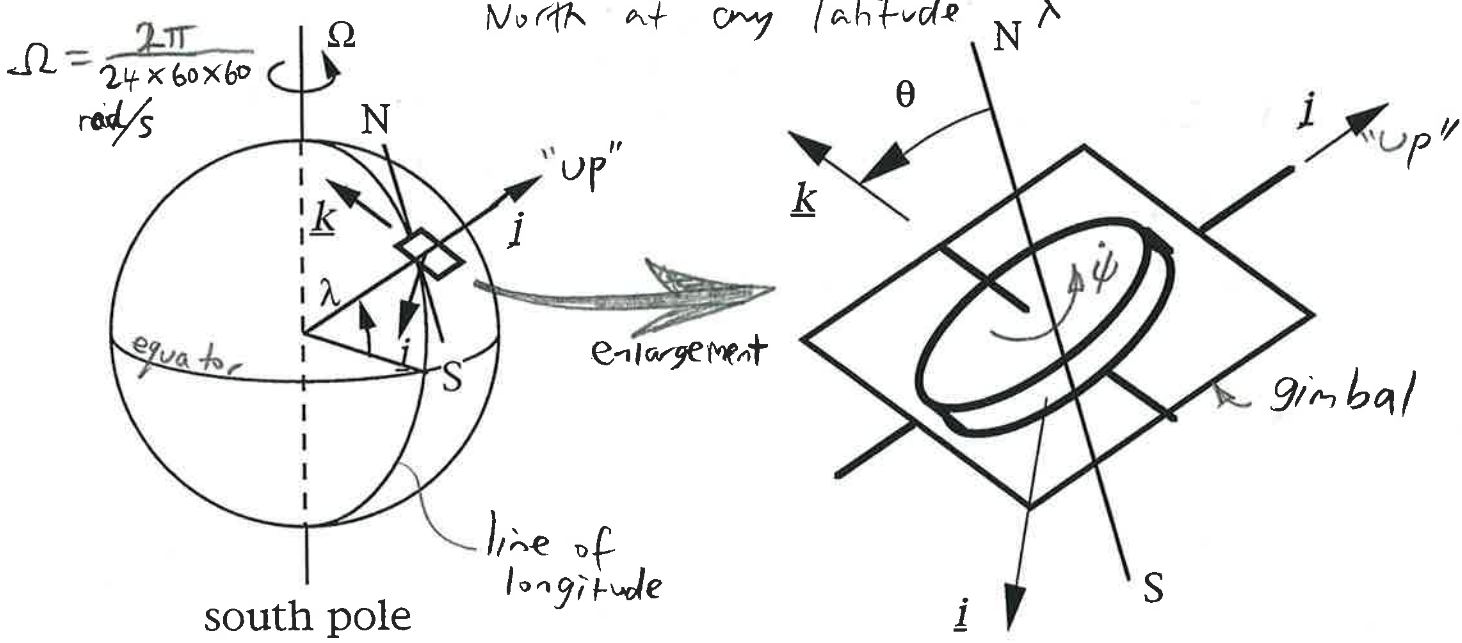
Note, same as (5.3) for $\theta \to 0$, ie small amplitude nutation

lend L600

High speed gyros serve as basic elements in many instruments for guidance and control of vehicles. Here are some examples.

6.1 Gyrocompass

– a rotor mounted in one gimbal. Points North at any latitude λ



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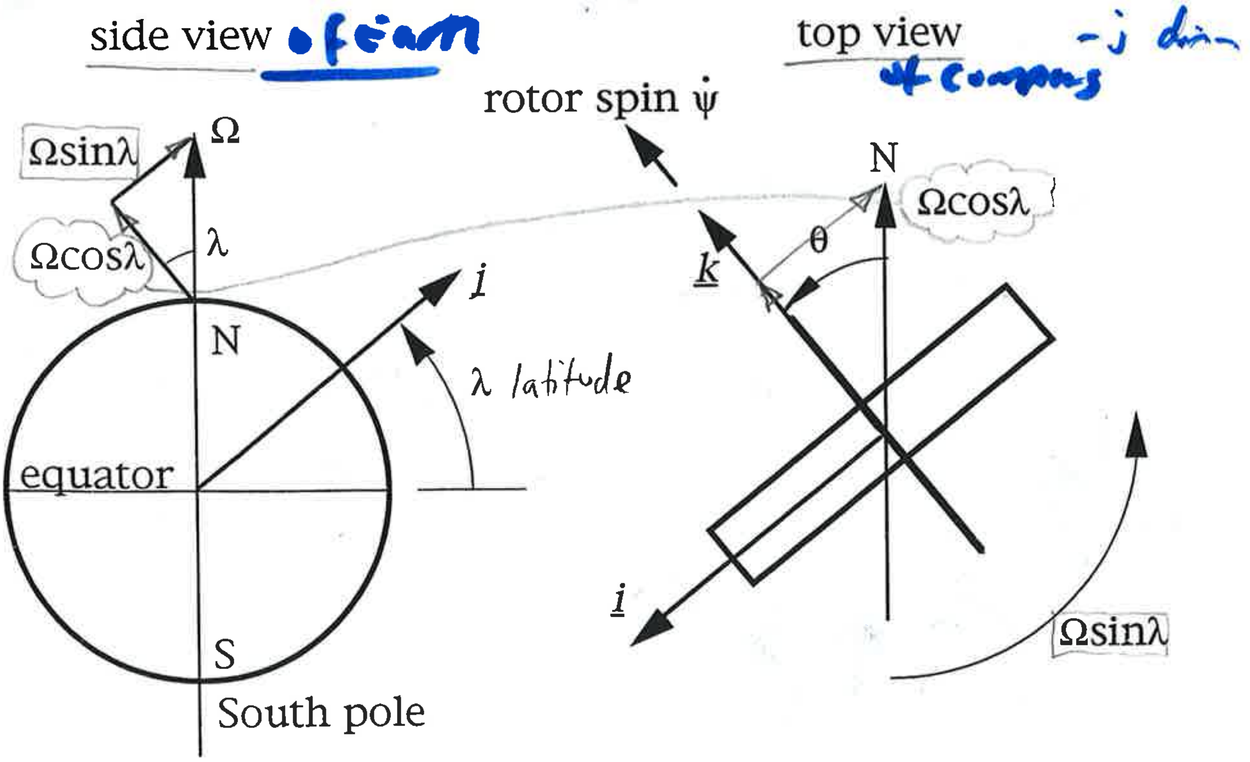
gimbal fixed

Find the angular velocity $\underset{\sim}{\Omega}$ of the non-body-fixed axes $\underline{i}$ $\underline{j}$ $\underline{k}$



$\underline{i}$: $\Omega_1 = -\Omega\cos\lambda\,\sin\theta$

($\underline{j}$: $\Omega_2 = \boxed{\Omega\sin\lambda} + \dot\theta$

$\underline{k}$: $\Omega_3 = \Omega\cos\lambda\,\cos\theta$) omit

Applied couples: ~~gyro~~ rotor is free to rotate about $\underline{j}$ and $\underline{k}$

$$\therefore \quad Q_1 \neq 0 \;, \quad Q_2 = 0 \;, \quad Q_3 = 0$$

Gyro equations (4.8)

$$A\dot{\Omega}_1 - (A\Omega_3 - C\omega_3)\Omega_2 = Q_1$$
$$A\dot{\Omega}_2 + (A\Omega_3 - C\omega_3)\Omega_1 = Q_2 = 0$$
$$C\dot{\omega}_3 = Q_3 = 0$$

Assume fast spin $\quad \omega_3 \gg \Omega_3 \quad$ hence 2nd gyro equation becomes

$$A\dot{\Omega}_2 - C\omega_3\Omega_1 = 0$$

should be $\simeq$ but I'm lazy!

—

As usual, substitute for Ω_1, Ω_2 and $\omega_3 \approx \dot{\psi}$ (fast spin)

$$\therefore \quad A\ddot{\theta} + C\dot{\psi}\Omega\cos\lambda\sin\theta = 0$$

get a 2nd order de in θ

For steady state

$$\ddot{\theta} = 0 \quad \text{so}$$

$$C\dot{\psi}\Omega\cos\lambda\sin\theta = 0$$

$$\therefore \quad \theta = 0 \quad \text{or} \quad \pi$$

(but unstable about $\theta = \pi$ since $\sin(\pi + x) = -\sin x$)

Oscillations about steady state $\theta = 0$ = rotor pointing north

$$\ddot{\theta} + \left[\frac{C}{A}\dot{\psi}\Omega\cos\lambda\right]\theta \approx 0 \qquad \text{SHM}$$

Bearing friction

(Need damping to give steady reading)

Causes errors (see examples paper 2, Q2)

Reduced accuracy near poles

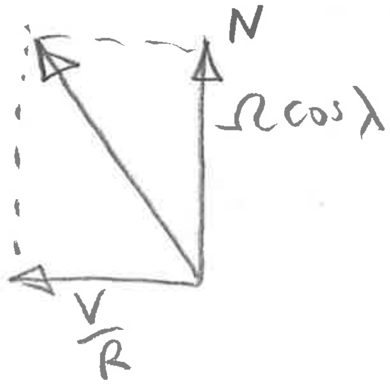
$\cos\lambda \to 0$ as $\lambda \to 90°$. Oscillation frequency $\to 0$

$\therefore$ settling time is very long.

Errors due to vehicle motion:

Suppose vehicle (ship, aircraft) is travelling due north at speed v.

This causes an angular velocity $\frac{V}{R}$ in westerly dirn.



The gyrocompass then points in the direction of the resultant angular velocity

The error is small for ships but large errors occur in fast planes.

$V = 600\text{mph} \equiv 300\text{m/s}$, $R = 6000\text{km}$ $\Big\} \frac{V}{R} = 5\times10^{-5}$, $\Omega = 7\times10^{-5}$ for $\lambda = 0$ (equator)

No error for travel in E-W direction.

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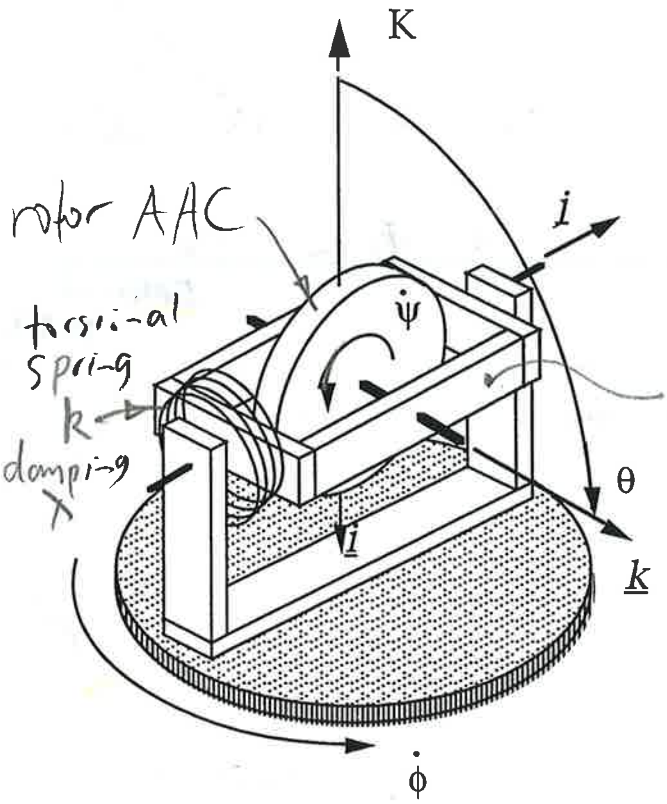
Do question 2 on examples paper G7/2

By the way, cribs for examples papers will be issued by supervisors. 4th year students who did G7 may already have passed on their cribs to you. Don't kid yourself by copying out cribs. You're not fooling anyone.

end L7 01

end L7 99



The rate gyro measures absolute angular velocity. Many applications:

- aircraft navigation
- car active suspension control
- missile guidance

Gimbal inertia I_G about $\underline{j}$ axis

The idea is that θ is a measure of $\dot\phi$ (rate)

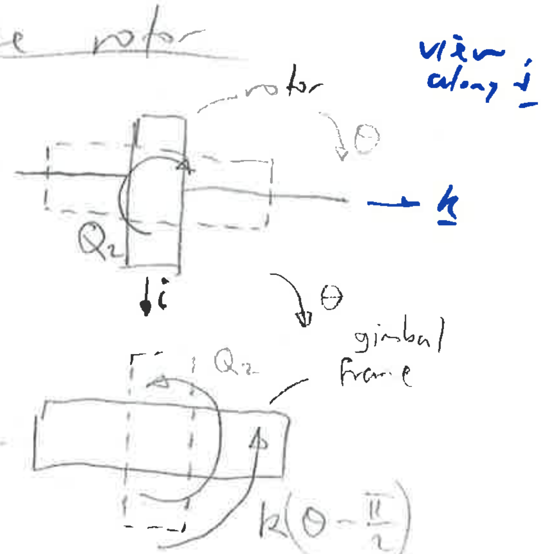
Gyroscope equations (4.8):

$$\left.\begin{aligned} A\dot\Omega_1 - (A\Omega_3 - C\omega_3)\Omega_2 &= Q_1 \\ A\dot\Omega_2 + (A\Omega_3 - C\omega_3)\Omega_1 &= Q_2 \\ C\dot\omega_3 &= Q_3 \end{aligned}\right\}\ \text{as usual}$$

Euler angles (4.9):

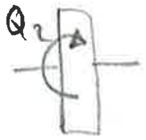
$$\left.\begin{aligned} \Omega_1 &= -\dot\phi\sin\theta \\ \Omega_2 &= \dot\theta \\ \Omega_3 &= \dot\phi\cos\theta \\ \omega_3 &= \Omega_3 + \dot\psi \end{aligned}\right\}$$

Couples applied to the rotor



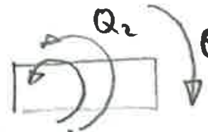
$k\left(\theta - \frac{\pi}{2}\right) + \lambda\dot\theta + I_G\ddot\theta$

Couples applied to the rotor:



$$Q_2 = -k\left(\theta-\tfrac{\pi}{2}\right) - \lambda\dot\theta - I_G\ddot\theta$$

spring (includes damping) gimbal d'Alembert inertia



$k\left(\theta-\frac{\pi}{2}\right)+\lambda\dot\theta + I_G\ddot\theta$

2nd gyro equation for fast spin $\omega_3 \gg \Omega_3$

END L8 M02

$$A\ddot\theta + C\dot\psi\dot\phi\sin\theta = -k\left(\theta-\tfrac{\pi}{2}\right) - \lambda\dot\theta - I_G\ddot\theta$$

$$\therefore\ (A+I_G)\ddot\theta + \lambda\dot\theta + k\left(\theta-\tfrac{\pi}{2}\right) = -C\dot\psi\dot\phi\sin\theta$$

put $\theta = \frac{\pi}{2} - \alpha$, $\dot\theta = -\dot\alpha$, $\ddot\theta = -\ddot\alpha$

$$\therefore\ (A+I_G)\ddot\alpha + \lambda\dot\alpha + k\alpha = C\dot\psi\dot\phi \qquad (\text{small } \alpha,\ \cos\alpha \approx 1)$$

—

In steady state, the angle α is proportional to the platform turning rate $\dot\phi$

$$k\alpha = C\dot\psi\dot\phi \qquad \text{with } \dot\alpha, \ddot\alpha = 0$$

$$\therefore\ \alpha = \frac{C\dot\psi}{k}\dot\phi$$

Damping λ needed to reach steady state.

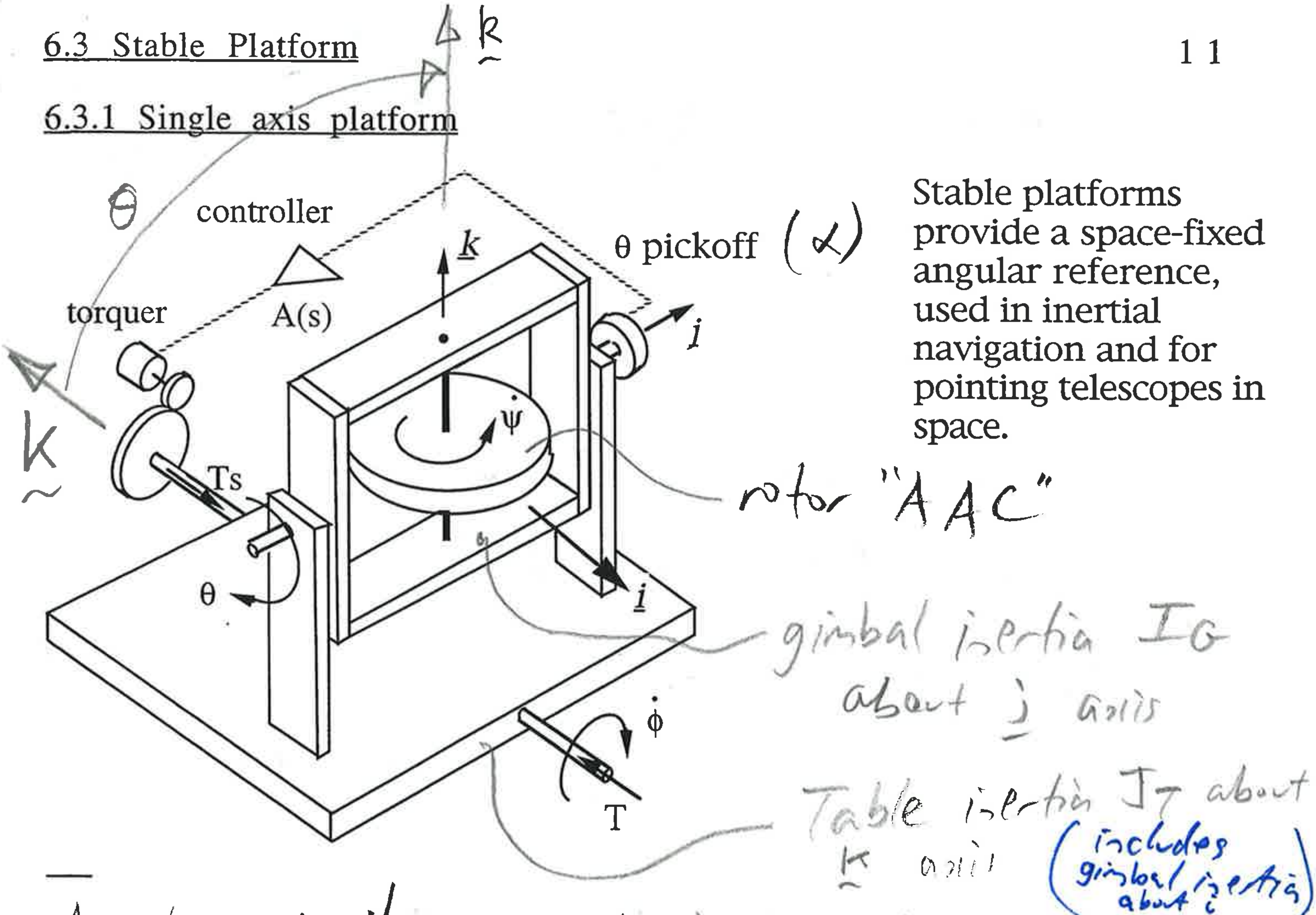
Damped SHM just like Case (a) in Mechanics data book

Do question 3 on G7/2 (but do part (c) later)

end L7 M00

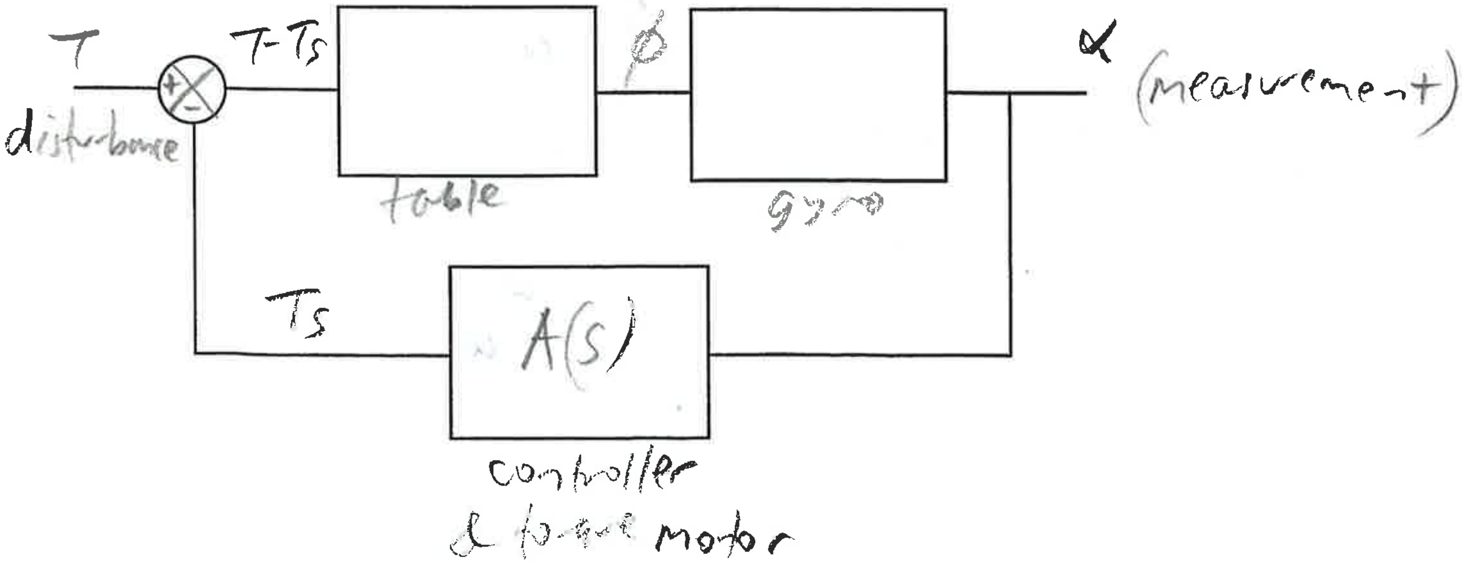
6.3 Stable Platform

6.3.1 Single axis platform



Stable platforms provide a space-fixed angular reference, used in inertial navigation and for pointing telescopes in space.

Angular motion ϕ of the table is detected as a change in angle $\alpha = \frac{\pi}{2} - \theta$. A torque motor acts to keep the table horizontal by generating a servo torque T_s to oppose any disturbance torque T



Gyroscope equations (as usual)

$$\left.\begin{aligned} Q_1 &= A\dot{\Omega}_1 - (A\Omega_3 - C\omega_3)\Omega_2 \\ Q_2 &= A\dot{\Omega}_2 + (A\Omega_3 - C\omega_3)\Omega_1 \\ Q_3 &= C\dot{\omega}_3 \end{aligned}\right\} \quad (4.8)$$

Euler's angles (as usual) with $\theta = \frac{\pi}{2} - \alpha$

$$\Omega_1 = -\dot{\phi}\sin\theta = -\dot{\phi}\cos\alpha \simeq -\dot{\phi}$$

$$\Omega_2 = \dot{\theta} = -\dot{\alpha}$$

$$\Omega_3 = \dot{\phi}\cos\theta = \dot{\phi}\sin\alpha \simeq \dot{\phi}\alpha$$

$$\omega_3 = \Omega_3 + \underset{\omega}{\dot{\psi}} = \underset{\omega}{\dot{\psi}} \quad \text{for fast spin}$$

—

Applied moments

$$Q_1 = T_s - T + J_T\ddot{\phi} \quad \text{— d'Alembert} \quad (\text{assume } \cos\alpha \simeq 1)$$

$$Q_2 = -I_G\ddot{\theta} = I_G\ddot{\alpha} \quad (\text{d'Alembert again})$$

Substitute into the gyro equations and ignore small quantities

1st gyro equation:

$$T_s - T + J_T\ddot{\phi} = -A\ddot{\phi} - C\overset{\omega}{\dot{\psi}}\dot{\alpha}$$

$$\therefore \quad T - T_s = (A + J_T)\ddot{\phi} + C\underset{\omega}{\dot{\psi}}\dot{\alpha}$$

and slip into Laplace transform notation à la control theory

$$T - T_s = (A + J_T)s^2\phi + C\overset{\omega}{\dot{\psi}}\, s\alpha$$

2nd gyro equation:

$$I_G \ddot{\alpha} = -A\ddot{\alpha} + C\overset{\omega}{\dot{\psi}}\dot{\phi}$$

Laplacify: $$\therefore \quad \alpha = \frac{C\overset{\omega}{\dot{\psi}}\phi}{(A+I_G)s}$$

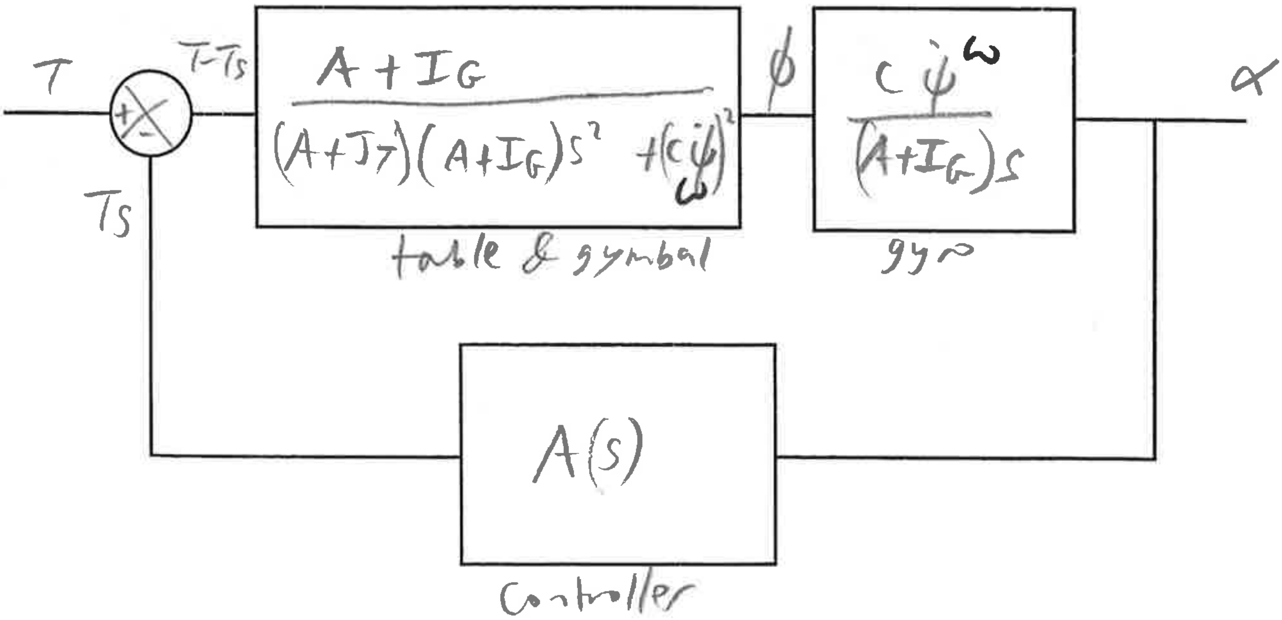
note we divided by s $\therefore$ $s \to 0$ requires care

Now combine the two gyro equations

$$T - T_s = \left[(A+J_T)s^2 + \frac{(C\overset{\omega}{\dot{\psi}})^2}{A+I_G}\right]\phi$$

—

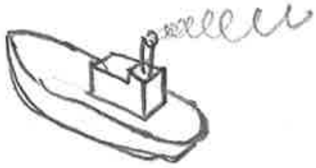
Hence we construct a block diagram

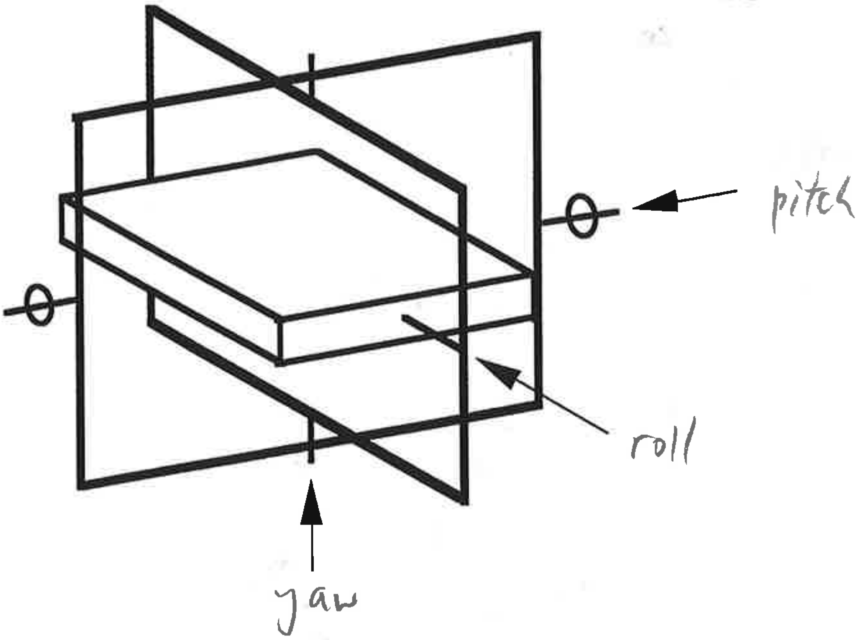


Do question 4 on examples paper G7/2

end L8 99 – rushed!

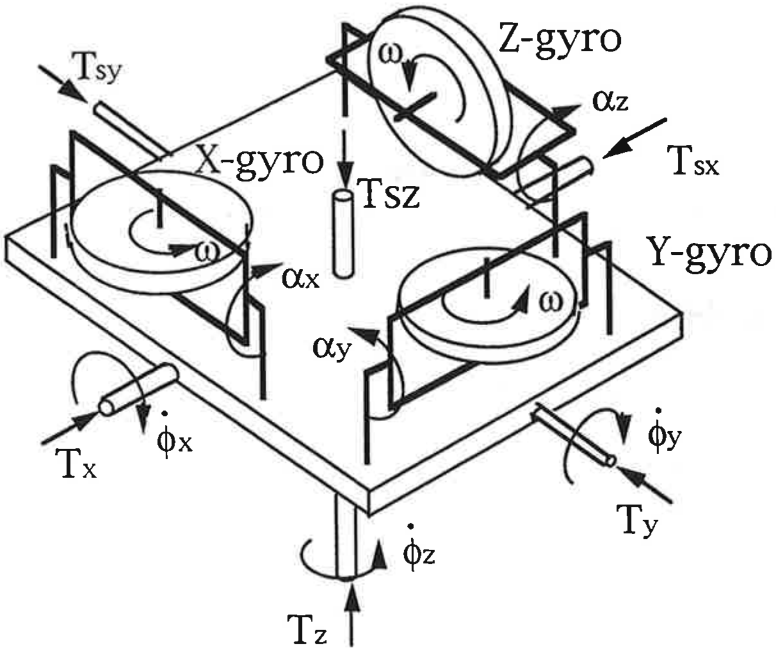
6.3.2 Three-axis platform





The platform is mounted in a Cardan's suspension giving it all three degrees of rotational freedom

The platform supports three gyros that respond to rotation on three mutually perpendicular axes (see diagram below)



Assume gimbal axes parallel to platform axes.

The gyro pickoffs measure gyro gimbal rotation δ *relative to the platform*. We need to convert these into absolute rotations α.

eg X-gyro

$\alpha_x = \delta_x + \phi_y$

Use $T_S = H(s)\ \alpha$ in three separate single-axis-platform block diagrams.

Do question 5 on examples paper G7/2

end L8 M01

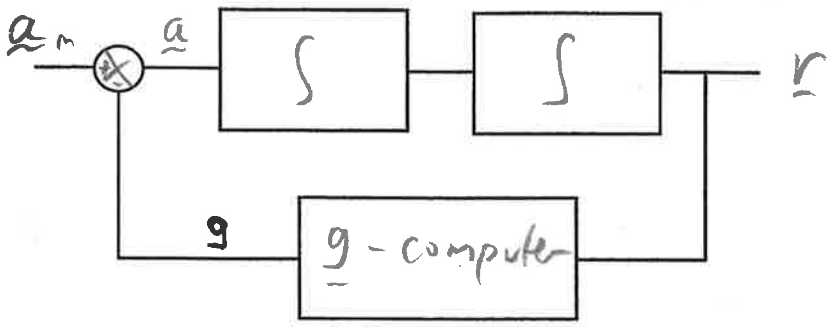
6.4 Inertial Navigation

[ref: Britting KR, 'Inertial Navigation Systems Analysis' CUED NY23]

In space, use 3 rate gyros & integrate to get rotational motion + 3 accelerometers & double integrate to get translational motion.

Gravity Correction

An accelerometer can't distinguish between $\ddot{z}$ & g. Need to know the value of $\underline{g}$ (mag & dirn) all round the globe



$\underline{a}_m$ measured is corrected using the value of $\underline{g}$ for the known position & altitude

1. Strapdown system (chapter 8)

Mount 3 accelerometers & 3 gyros directly on vehicle ∴ lots of computation, resolving component vectors in 3D. Large errors can result if there is vibration & noise. Mechanically simple.

2. Space stabilised platform (chapter 6)

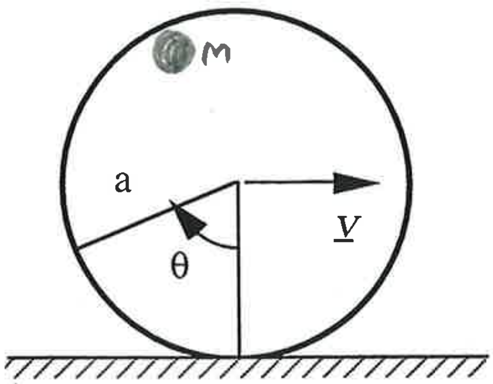
Mount 3 accelerometers on a 3-axis stable platform. No tricky computations required (except $\underline{g}$ computer) since accelerometers are always aligned with $\underline{i}$ $\underline{j}$ $\underline{k}$.

3. Locally level platform (chapter 7)

Contrive a platform which is always normal to $\underline{g}$. For surface navigation don't need $\underline{g}$ correction at all.

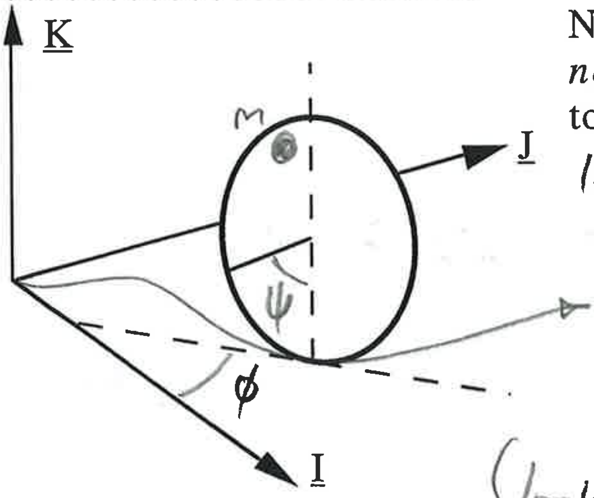
Do question 3(c) on paper G7/2

End L8 MOO



Holonomic system: Number of d.o.f. is equal to number of co-ordinates needed to define system completely.

In 2D rolling, θ uniquely defines disc position



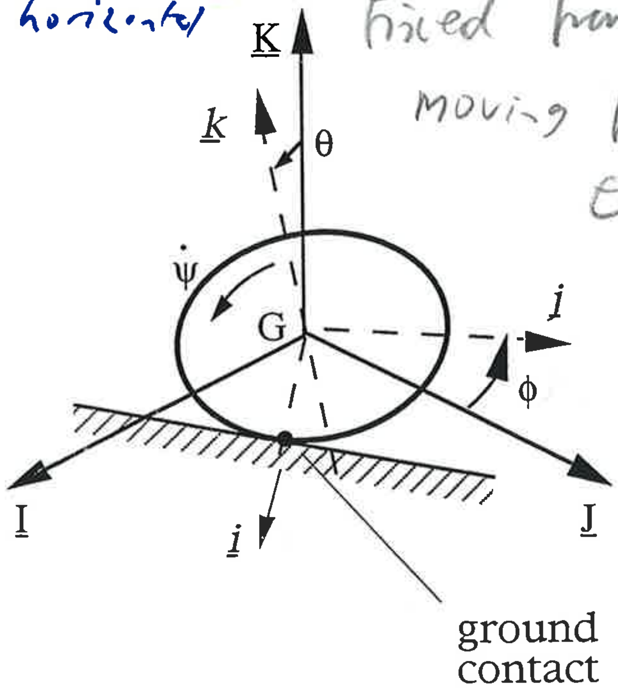
Non-holonomic system: Number of d.o.f. is *not* equal to number of co-ordinates needed to define system completely.

In 3D rolling, need θ, ϕ, ψ, x & y for position yet disc has only 3 degrees of freedom θ, ϕ, ψ

(look at mark made on car tyre)

We will consider a disc of radius a and principal moments of inertia A, A, C rolling on a plane. This is Non-Holonomic. ∴ Don't use Lagrange

horizontal



Fixed frame $\underline{I}, \underline{J}, \underline{K}$

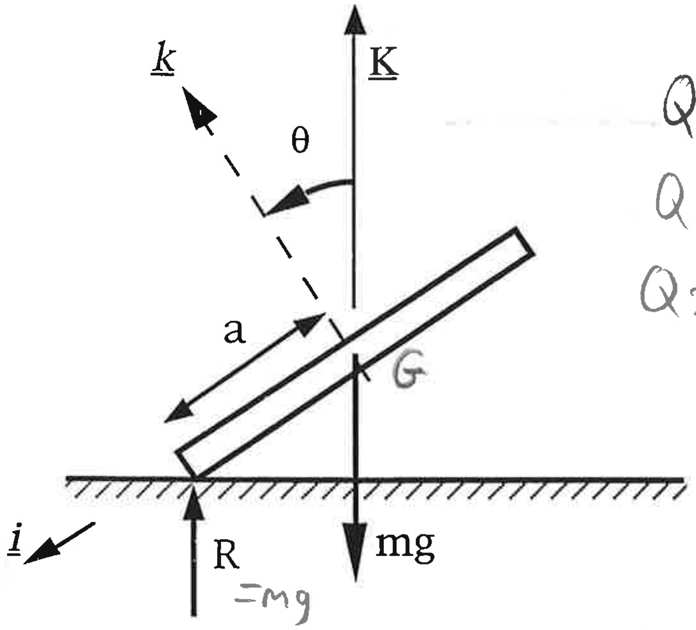
moving frame $\underline{i}, \underline{j}, \underline{k}$ with Euler's angles. $\underline{k}$ points along disc axis

$\underline{j}$ is always horizontal

$\underline{i}$ always points to point of contact with table

Use the gyro equations

This is the "coin wobbling on the counter in the College Bar" problem



Gyro equations:

$$Q_1 = A\dot{\Omega}_1 - (A\Omega_3 - C\omega_3)\Omega_2$$

$$Q_2 = A\dot{\Omega}_2 + (A\Omega_3 - C\omega_3)\Omega_1$$

$$Q_3 = C\dot{\omega}_3$$

Euler's angles:

$$\Omega_1 = -\dot{\phi}\sin\theta$$

$$\Omega_2 = \dot{\theta}$$

$$\Omega_3 = \dot{\phi}\cos\theta$$

$$\omega_3 = \dot{\psi} + \Omega_3$$

Steady state - assume G is stationary. $\therefore R = mg$

Moments: $Q_1 = 0$

$$Q_2 = -mga\cos\theta$$

$$Q_3 = 0$$

Angles $\left.\begin{array}{l} \theta = \text{const} \\ \dot{\phi} = \text{const} \end{array}\right\}$ in steady state

$= $ "wobble rate"

No-slip condition at point of contact with ground:

$$\underline{v} = \underline{\omega} \times a\underline{i} = \underline{0}$$

$$\therefore (\omega_1 \underline{i} + \omega_2 \underline{j} + \omega_3 \underline{k}) \times a\underline{i} = \underline{0}$$

$$\therefore \omega_3 a \underline{j} - \omega_2 a \underline{k} = \underline{0}$$

$$\therefore \omega_2 = 0 \quad \& \quad \omega_3 = 0$$

end L9 M99

The second gyro equation gives

$$-mga\cos\theta = -(A\dot{\phi}\cos\theta - C\omega_3)\dot{\phi}\sin\theta$$

and we have $\omega_3 = 0$ for no slip

Near horizontal rolling means θ is small , $\cancel{\cos\theta \simeq 1}$, $\sin\theta \simeq \theta$

So $mga \simeq A\dot{\phi}^2\theta$

$$\therefore \dot{\phi} \simeq \sqrt{\frac{mga}{A\theta}}$$

For a thin disc $A = \frac{1}{4}ma^2$

$$\therefore \dot{\phi} \simeq 2\sqrt{\frac{g}{a\theta}}$$

This is the "wobble rate" $\to \infty$ as $\theta \to 0$ as observed.

Now compute the observed rate $\dot{\beta}$ of turning of the Queen's head on the coin. Do this in Q 6(b) to get $\dot{\beta} \simeq \sqrt{\frac{g\theta^3}{a}}$

ie $\dot{\beta} \to 0$ as $\theta \to 0$

Conundrum Disc angular velocity $\underset{\sim}{\omega} = \omega_1 \underset{\sim}{i} = \Omega_1 \underset{\sim}{i}$

and $\Omega_1 = -\dot{\phi}\sin\theta$

Resolve $\underset{\sim}{\omega}$ into $\underset{\sim}{k}$ direction $\therefore \dot{\beta} = \dot{\phi}\sin^2\theta$

$$= 2\sqrt{\frac{g}{a\theta}}\,\theta^2$$

$$= 2\sqrt{\frac{g\theta^3}{a}}$$

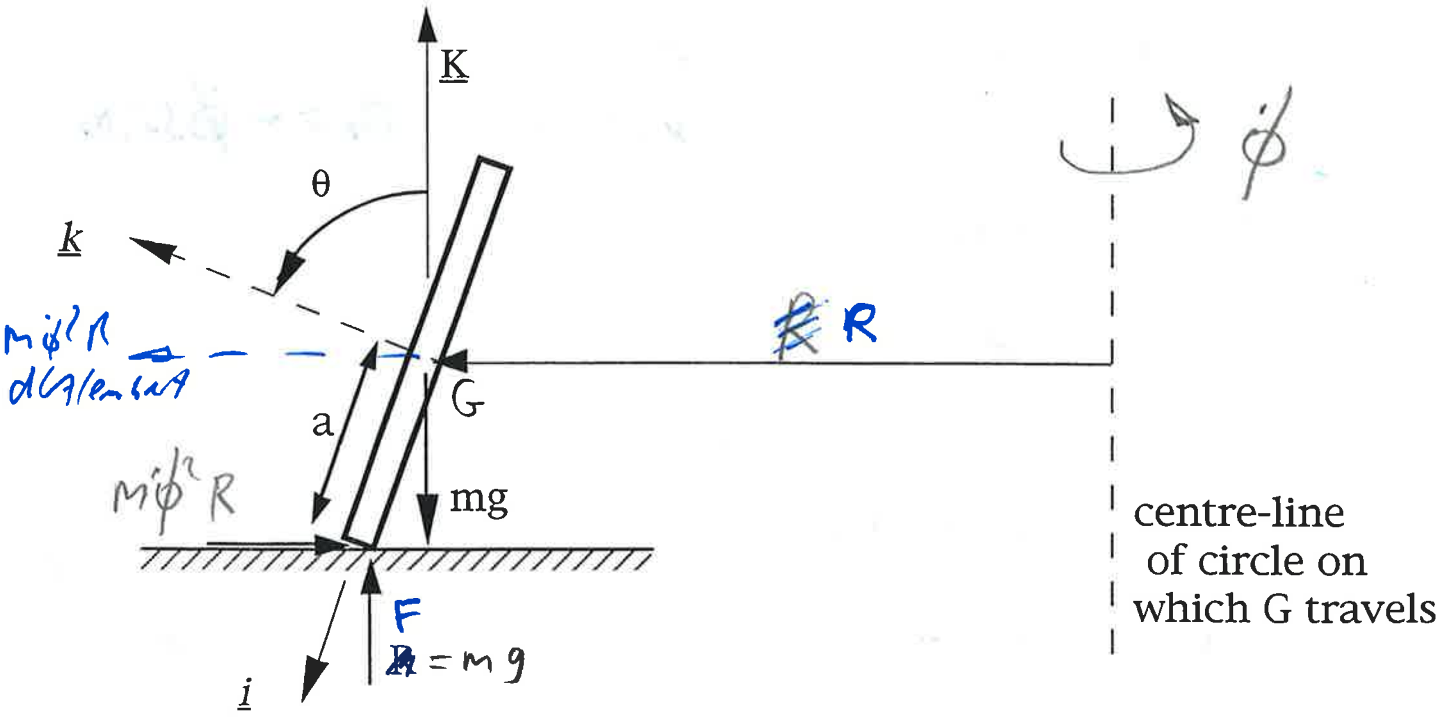
Not as above. Why not??

This is question 6(b) on examples paper G7/2

end L9 M01

7.2 Near Vertical Rolling

This is what happens if you roll a Frisbee along the ground.



Couple $Q_2 = -mga\cos\theta + m\dot\phi^2 Ra\sin\theta$

Steady rolling, assume G moves on circle radius R.

$\therefore$ need centripetal force $m\dot\phi^2 \cancel{R}R$

We also need the no-slip condition at the contact with the ground:

$$\underline{V}_P = \underline{V}_G + \underline{\omega} \times a\underline{i} = \underline{0}$$

$$\therefore\ V_1\underline{i} + V_2\underline{j} + V_3\underline{k} + a\omega_3\underline{j} - a\omega_2\underline{k} = \underline{0}$$

$$\therefore\ V_1 = 0$$

$$V_2 = -a\omega_3 = R\dot\phi \quad (\text{forward rolling speed})$$

and $V_3 = a\omega_2$ but $\omega_2 = \dot\theta = 0$ in S.S.

Second gyro equation (it's always the 2nd equation that is useful)

$$Q_2 = A\dot{\Omega}_2 + (A\Omega_3 - C\omega_3)\Omega_1$$

Steady state : $\dot{\Omega}_2 = 0$

Eulers angles $\Omega_3 = \dot{\phi}\cos\theta$ $\Omega_1 = -\dot{\phi}\sin\theta$

$$\therefore -mga\cos\theta + m\dot{\phi}^2 R a \sin\theta = -(A\dot{\phi}\cos\theta - C\omega_3)\dot{\phi}\sin\theta$$

Let $\theta = \frac{\pi}{2} - \alpha$ and $R\dot{\phi} = -a\omega_3$ means $\omega_3 \gg \dot{\phi}$ for $R \gg a$

$$\therefore -mga\sin\alpha - ma^2\omega_3\dot{\phi}\cos\alpha = C\omega_3\dot{\phi}\cos\alpha$$

and for small α

$$\dot{\phi} \approx \frac{-mga\alpha}{(C+ma^2)\omega_3}$$

but $\omega_3 = -\frac{v_2}{a}$ and $C = \frac{1}{2}ma^2$

$$\therefore \boxed{\dot{\phi} = \frac{2}{3}\frac{g\alpha}{v_2}} = \text{turning rate}$$

$$\& \quad R = \frac{v_2}{\dot{\phi}} = \frac{3}{2}\frac{v_2^2}{g\alpha} = \text{Cornering radius}$$

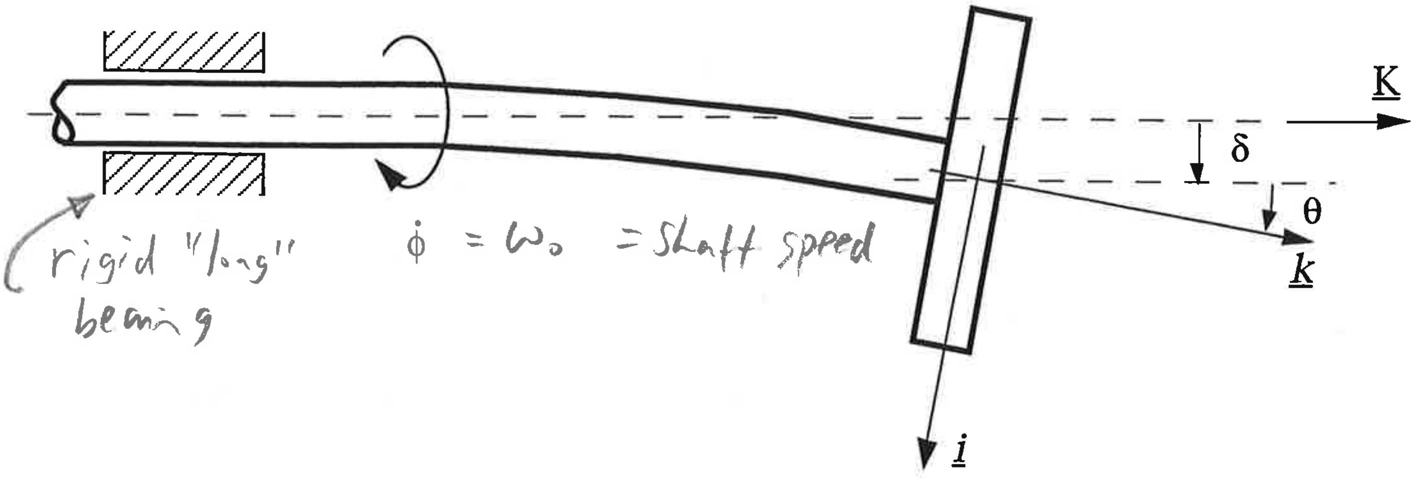
Note as $\alpha \to 0$ then $R \to \infty$

$\therefore$ straight line rolling

Do question 6 on examples paper G7/2

8 ROTOR WHIRL

How do gyroscopic effects affect vibrating shafts with spinning rotors?



Whirl can be "synchronous" or "non-synchronous".

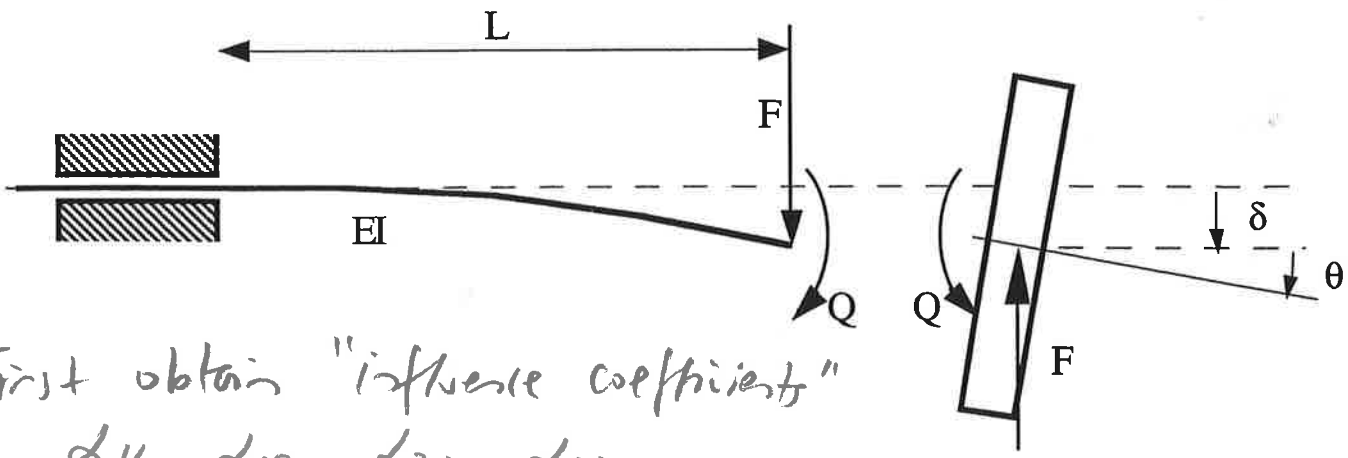
- Synchronous whirl: plane of bent shaft (i-k plane) rotates at the same speed as the shaft itself. Most common. Excited by out of balance forces

- Non synchronous whirl: plane of bent shaft rotates at a speed different to ω_0. The complete Non synchronous analysis is not hard but off the G7 syllabus

Some sources of excitation:

- Out of balance (Synchronous)
- Asymmetric horizontal shaft (keyways etc) gravity provides excitation at twice shaft speed
- Oil film whirl (a hydrodynamic effect à la G6) gives excitation at half shaft speed
- dry bearing whirl due to clearance between shaft & hole (at any frequency)
- Nearby machinery vibrating at any frequency

Consider forces on shaft and rotor:



First obtain "influence coefficients"
$\alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22}$

Use the Structures Data Book to get deflections and rotations for given applied forces and couples

$$\delta = \frac{FL^3}{3EI} + \frac{QL^2}{2EI} \qquad \theta = \frac{FL^2}{2EI} + \frac{QL}{EI}$$

$$\therefore \begin{Bmatrix} \delta \\ \theta \end{Bmatrix} = \frac{L}{6EI}\begin{bmatrix} 2L^2 & 3L \\ 3L & 6 \end{bmatrix}\begin{Bmatrix} F \\ Q \end{Bmatrix} = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}\begin{Bmatrix} F \\ Q \end{Bmatrix} \quad (8.1)$$

Use the gyro equations but note that $\underline{\omega} = \underline{\Omega}$ for synchronous whirl

2nd gyro equation $\quad A\dot{\omega}_2 + (A-C)\,\omega_1\omega_3 = Q_2$

Steady state so $\quad \dot{\omega}_2 = 0 \quad , \quad \dot{\phi} = \omega_0$

& Moment $\quad Q_2 = -Q$

Euler angles $\quad \omega_1 = -\dot{\phi}\sin\theta \simeq -\omega_0\,\theta$

$\qquad \omega_3 = \dot{\phi}\cos\theta = \omega_0$

$$\therefore \quad Q \simeq (A-C)\,\omega_0^2\,\theta \qquad (8.2)$$

Newton's 2nd law for the circular motion:

$$F = M\omega_0^2\,\delta \tag{8.3}$$

and substitute (8.2) and (8.3) into (8.1)

$$\delta = \alpha_{11} M\omega_0^2\,\delta + \alpha_{12}(A-C)\,\omega_0^2\,\theta$$
$$\theta = \alpha_{21} M\omega_0^2\,\delta + \alpha_{22}(A-C)\,\omega_0^2\,\theta$$

where α_{11}, α_{21}, α_{12}, α_{22} are influence coefficients

or in matrix form

$$\begin{bmatrix} 1-\alpha_{11}M\omega_0^2 & \alpha_{12}(C-A)\omega_0^2 \\ -\alpha_{21}M\omega_0^2 & 1+\alpha_{22}(C-A)\omega_0^2 \end{bmatrix} \begin{Bmatrix} \delta \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{8.4}$$

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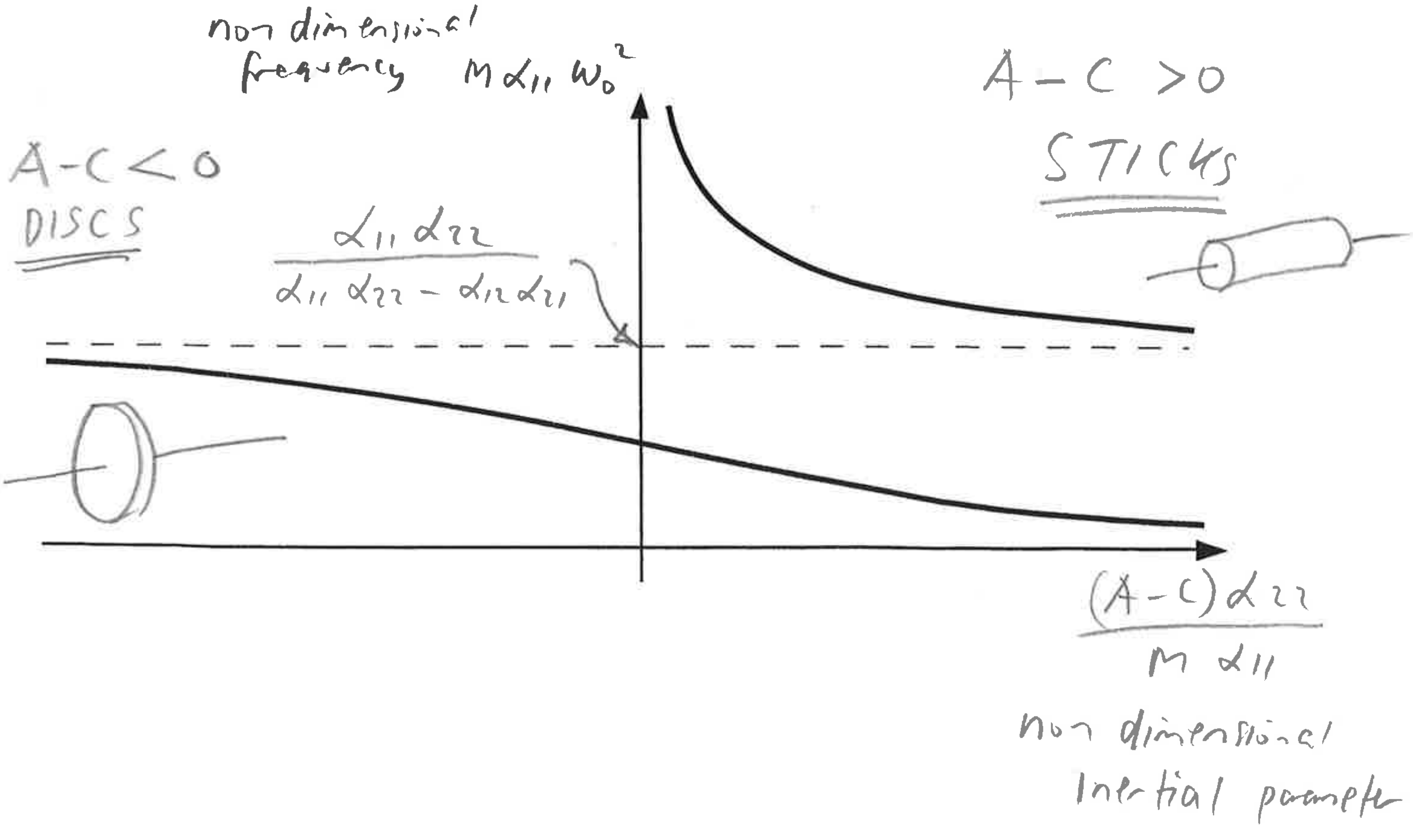
The determinant of the matrix give the two critical speeds for synchronous whirl as the solution to a quadratic in ω_0^2. These can be shown graphically:



Newton's 2nd law for the circular motion:

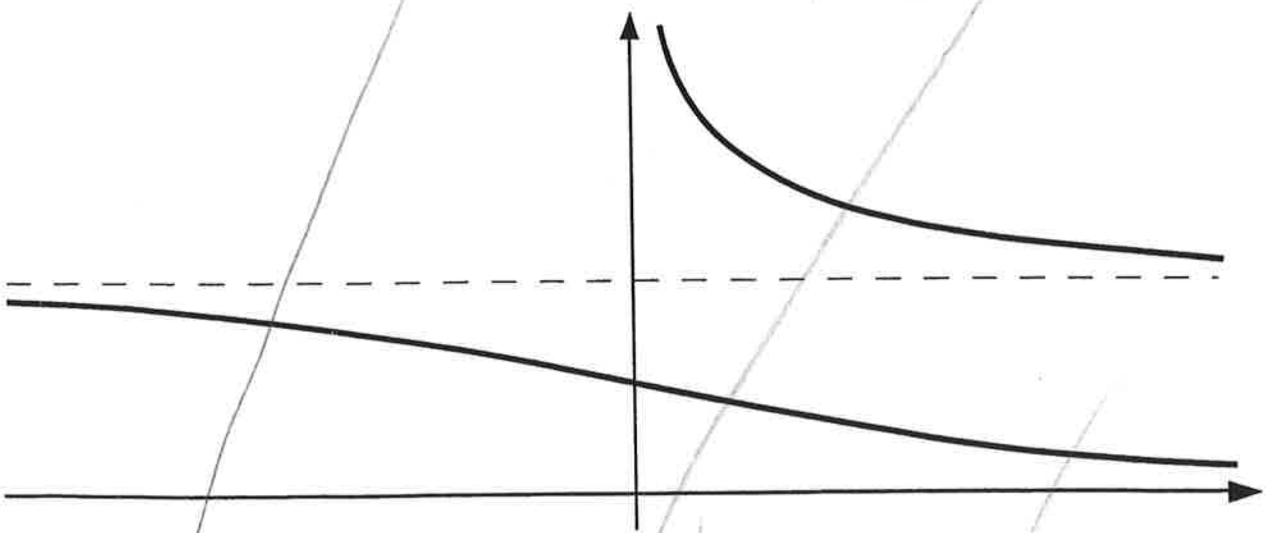
(8.3)

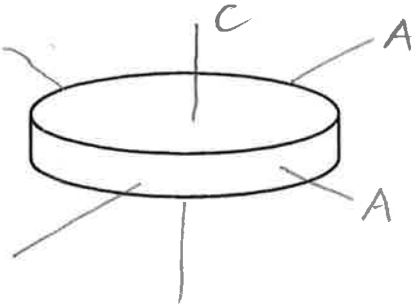
and substitute (8.2) and (8.3) into (8.1)

or in matrix form

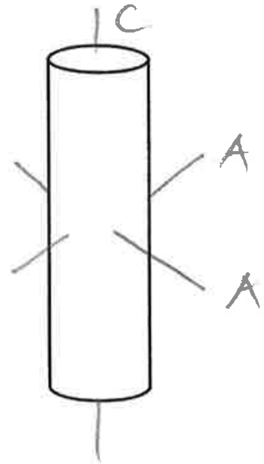
(8.4)

The determinant of the matrix give the two critical speeds for synchronous whirl as the solution to a quadratic in ω_o^2. These can be shown graphically:





When $A - C < 0$ there is only one critical speed because the shaft is gyroscopically stabilized. DISCS



When $A - C > 0$ there are two critical speeds as usual for a rigid body. STICKS

Many machines operate above their first critical speed. It is necessary to accelerate rapidly through the resonance to avoid damage.

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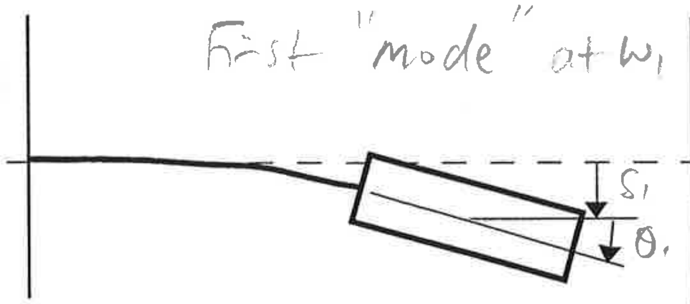
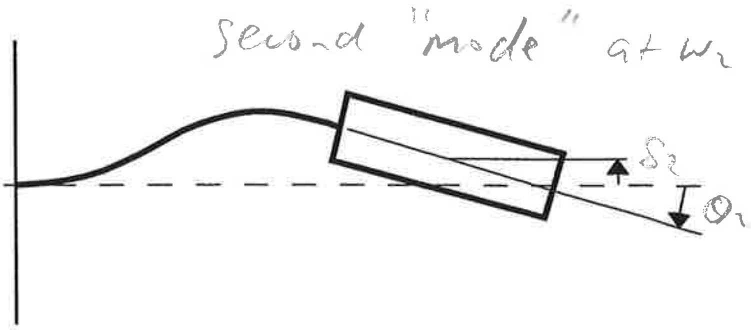
For sticks, the deflected shape is different for the two critical speeds. Determine these "mode shapes" in the usual way by substituting the values of ω_0 back into the equations and determine the ratio δ/θ .





$\begin{Bmatrix} \delta_1 \\ \theta_1 \end{Bmatrix}$ satisfies (8.4) with $\omega_0 = \omega_1$

$\begin{Bmatrix} \delta_2 \\ \theta_2 \end{Bmatrix}$ satisfies (8.4) with $\omega_0 = \omega_2$

Do questions 7 and 8 on examples paper G7/2