Cambridge University Engineering Dept. Third year

Module 3F2: Systems and Control

LECTURE NOTES 3a: OBSERVABILITY

Contents

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1 Solving Linear Equations

For convenience we will repeat some results and definitions from linear algebra.

Definition 1.1 *Let A be an $m \times n$ matrix then,*

(a) the set of all $\underline{x} \neq \underline{0}$ such that $A\underline{x} = \underline{0}$ is called the **Null Space** *of A ($\mathrm{null}(A)$).*

(b) the set of all $\underline{y}$ such that $\underline{y} = A\underline{x}$ for some $\underline{x}$ is called the **Range Space** *of A (or the range of A, $\mathrm{range}(A)$);*

(c) A is said to have full row rank if $\mathrm{range}(A) = \mathbb{R}^m$ (i.e. $\underline{z}^T A \neq \underline{0}$ for all $\underline{z} \neq \underline{0}$);

(d) A is said to have full column rank if $\mathrm{null}(A) = \emptyset$ (i.e. $A\underline{x} \neq \underline{0}$ for all $\underline{x} \neq \underline{0}$.)

Given an $m \times n$ matrix A and an $m \times 1$ vector $\underline{b}$, consider the equation:

$$A\underline{x} = \underline{b},$$

in the unknown $\underline{x}$ in $\mathbb{R}^n$. Two natural questions are:

(a) Does there exist a solution, $\underline{x}$?

(b) If so, is it unique?

Fact 1.2 *For the case $m = n$:*

(a) If $\det(A) \neq 0$ then for any $\underline{b}$ there exists a solution, $\underline{x}$, such that $A\underline{x} = \underline{b}$, and this solution is unique (Indeed it is given by $\underline{x} = A^{-1}\underline{b}$).

(b) If $\det(A) = 0$ then there exists $\underline{x} \neq \underline{0}$ such that $A\underline{x} = \underline{0}$.

Fact 1.3 *For any $m \times n$ matrix, M,*

$$M^T M\underline{x} = \underline{0} \quad \Leftrightarrow \quad M\underline{x} = \underline{0}.$$

Fact 1.4 *For the case $m \leq n$,*

(a) If $\det(AA^T) \neq 0$ *then* $\underline{x} = A^T \left(AA^T\right)^{-1} \underline{b}$*, solves* $A\underline{x} = \underline{b}$ *for any* $\underline{b}$*.*

(b) If $\det(AA^T) = 0$ *then there exists a* $\underline{b} \neq \underline{0}$ *such that* $\underline{b} \perp A\underline{x}$ *(i.e.* $\underline{b}^T A\underline{x} = \underline{0}$*) for all* $\underline{x}$*.*

For the case $m \geq n$,

(c) If $\det(A^T A) \neq 0$ *then there may not be a solution to* $A\underline{x} = \underline{b}$*, but if there is then it is unique.*

(d) If $\det(A^T A) = 0$ *then there exists* $\underline{x} \neq \underline{0}$ *such that* $A\underline{x} = \underline{0}$*.*

For hand calculations it is generally easiest to use the following observations:

(a) If you can find a set of n rows of A such that the determinant of the $n \times n$ submatrix given by these rows is nonzero, then A has full column rank.

(b) If you can find a nonzero vector, $\underline{x}$, such that $A\underline{x} = \underline{0}$ then clearly A does not have full column rank.

(c) If you can find a set of m columns of A such that the determinant of the $m \times m$ submatrix given by these columns is nonzero, then A has full row rank.

(d) If you can find a nonzero vector, $\underline{z}$, such that $\underline{z}^T A = \underline{0}$ then clearly A does not have full row rank.

2 Observability

A system:

$$\begin{aligned} \dot{\underline{x}} &= A\underline{x} + B\underline{u} \\ \underline{y} &= C\underline{x} \end{aligned}$$

is called **observable** if we can deduce the state, $\underline{x}(t)$, from measurements of $\underline{u}(\tau)$ and $\underline{y}(\tau)$ over some time interval.

Recall that

$$\underline{y}(t) = \underbrace{Ce^{At}\underline{x}(0)}_{\text{initial condition response}} \underbrace{+ D\underline{u}(t) + \int_0^t Ce^{A(t-\tau)}B\underline{u}(\tau)\,d\tau}_{\text{input response}}$$

and so if two initial states $\underline{x}_1 \neq \underline{x}_2$ give the same outputs then $\underline{0} = \underline{y}_2 - \underline{y}_1 = Ce^{At}(\underline{x}_2 - \underline{x}_1)$

A state $\underline{x}_o$ such that $Ce^{At}\underline{x}_o = \underline{0}$ for all t is called an **unobservable state**.

Define the **observability matrix** $Q = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$.

$$\begin{aligned} Q\underline{x}_o = \underline{0} \quad &\Rightarrow \quad CA^k\underline{x}_o = \underline{0} \text{ for } k = 0, \cdots n-1 \\ &\Rightarrow \quad CA^n\underline{x}_o = C\left(-\alpha_1 A^{n-1} \cdots - \alpha_{n-1}A - \alpha_n I\right)\underline{x}_o \text{ by Cayley-Hamilton Theorem} \\ &= \quad \underline{0} \\ &\Rightarrow \quad CA^k\underline{x}_o = \underline{0} \text{ for all } k \geq 0 \text{ by repeated use of Cayley-Hamilton theorem.} \\ &\Rightarrow \quad Ce^{At}\underline{x}_o = \underline{0} \text{ for all } t \text{ by the power series expansion of } e^{At} \end{aligned}$$

Hence $\underline{x}_0$ is an unobservable state if $Q\underline{x}_0 = \underline{0}$.

Conversely, $Ce^{At}\underline{x}_o = 0$ for all t implies $\frac{d^n}{dt^n}Ce^{At}\underline{x}_o = n!CA^ne^{At}\underline{x}_o = \underline{0}$ and so $Q\underline{x}_o = \underline{0}$.

Now consider differentiating $\underline{y}(t)$ to give

$$\underbrace{\begin{bmatrix} \underline{y}(t) \\ \dot{\underline{y}}(t) \\ \ddot{\underline{y}}(t) \\ \vdots \\ \underline{y}^{(n-1)}(t) \end{bmatrix}}_{\text{known}} = \underbrace{\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}}_{Q} \underbrace{\underline{x}(t)}_{?} + \underbrace{\begin{bmatrix} \underline{0} \\ CB\underline{u}(t) \\ CAB\underline{u}(t) + CB\dot{\underline{u}}(t) \\ \vdots \\ CA^{n-2}B\underline{u} + \cdots + CB\underline{u}^{(n-2)} \end{bmatrix}}_{\text{known}}$$

We can solve the above equation uniquely for $\underline{x}(t)$ if and only if rank $Q = n$, hence
Observability test:

$$\boxed{\text{The system is observable if and only if rank } Q = n}$$

2.1 Effect of Initial Condition on Output

Now consider the difference between two initial condition responses:

$$\underline{y}_o(t) = Ce^{At}\underline{x}_o \text{ and } \underline{y}(t) = Ce^{At}\left(\underline{x}_o + \underline{d}\right) \quad \text{so} \quad \underline{y}(t) - \underline{y}_o(t) = Ce^{At}\underline{d}$$

Can $\left(\underline{y}(t) - \underline{y}_o(t)\right)$ be small in spite of $\underline{d}$ being large? Measure the size of $\left(\underline{y}(t) - \underline{y}_o(t)\right)$ over the time interval $0 < t < t_1$ by

$$\begin{aligned} \int_0^{t_1} \left\| \underline{y}(t) - \underline{y}_o(t) \right\|^2 dt &= \int_0^{t_1} \left(\underline{y}(t) - \underline{y}_o(t)\right)^T \left(\underline{y}(t) - \underline{y}_o(t)\right) dt \\ &= \int_0^{t_1} \underline{d}^T\, e^{A^T t} C^T C e^{At}\, \underline{d}\, dt = \underline{d}^T\ W_o(t_1)\ \underline{d} \text{ where } W_o(t_1) = \int_0^{t_1} e^{A^T t} C^T C e^{At}\, dt \end{aligned}$$

Clearly this difference must be ≥ 0 so $W_o(t_1)$ is a positive semi-definite matrix. The system will be observable if $\underline{d}^T W_o(t_1)\underline{d} > 0$ for all $\underline{d} \neq \underline{0}$, i.e. if $W_o(t_1)$ is a positive definite matrix.

Also,

$$\begin{aligned} \underline{d} \text{ in Null Space of } W_o(t_1) \quad &\Leftrightarrow \quad W_o(t_1)\underline{d} = \underline{0} \Leftrightarrow \underline{d}^T W_o(t_1)\underline{d} = \underline{0} \Leftrightarrow Ce^{At}\underline{d} = \underline{0} \text{ for all } t < t_1 \\ &\Leftrightarrow \quad \underline{d} \text{ is an unobservable state.} \\ \Rightarrow \text{ Null Space of } W_o(t_1) \quad &= \quad \text{Null Space of } Q. \end{aligned}$$

Example

$$\underline{\dot{x}} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \underline{x}, \quad y = \begin{bmatrix} 1 & 1 \end{bmatrix} \underline{x} \;\Rightarrow Ce^{At} = \begin{bmatrix} e^{-t} & e^{-2t} \end{bmatrix}$$

$$W_o(t_1) \quad = \quad \int_0^{t_1} \begin{bmatrix} e^{-2t} & e^{-3t} \\ e^{-3t} & e^{-4t} \end{bmatrix} dt = \begin{bmatrix} \frac{1}{2}\left(1 - e^{-2t_1}\right) & \frac{1}{3}\left(1 - e^{-3t_1}\right) \\ \frac{1}{3}\left(1 - e^{-3t_1}\right) & \frac{1}{4}\left(1 - e^{-4t_1}\right) \end{bmatrix} \underset{\text{as } t_1 \to \infty}{\longrightarrow} \begin{bmatrix} \frac{1}{2} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{4} \end{bmatrix}$$

2.2 Change of State Coordinates when System is not Observable

If (A, C) is not observable then we can make a change of state coordinates to isolate the unobservable states as follows.

If the rank $Q = r < n$ then there exists a nonsingular $n \times n$ matrix T and a $pn \times r$ matrix $\tilde{Q}_1$ of rank r, such that *(Recall QR factorization)*

$$Q = \begin{bmatrix} \tilde{Q}_1 & 0 \end{bmatrix} T$$

Now change the state coordinates to $\underline{\tilde{x}} = T\underline{x}$:

$$\underline{\dot{\tilde{x}}} = \underbrace{TAT^{-1}}_{\tilde{A}} \underline{\tilde{x}} + \underbrace{TB}_{\tilde{B}} \underline{u}, \quad \underline{y} = \underbrace{CT^{-1}}_{\tilde{C}} \underline{\tilde{x}}.$$

Theorem 2.1 *In these coordinates if we partition the state,* $\underline{\tilde{x}} = \begin{bmatrix} \underline{\tilde{x}}_1 \\ \underline{\tilde{x}}_2 \end{bmatrix}$ *with* $\underline{\tilde{x}}_1$ *of dimension r, and compatibly partition:*

$$\tilde{A} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix}; \quad \tilde{B} = \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix}; \quad \tilde{C} = \begin{bmatrix} \tilde{C}_1 & \tilde{C}_2 \end{bmatrix}$$

then

$$\tilde{C}_2 = 0, \quad \tilde{A}_{12} = 0, \quad \textit{and } (\tilde{A}_{11}, \tilde{C}_1) \textit{ is observable}$$

Proof: Firstly $CA^kT^{-1} = CT^{-1}TA^kT^{-1} = \tilde{C}\tilde{A}^k$ so

$$\begin{bmatrix} \tilde{Q}_1 & 0 \end{bmatrix} = QT^{-1} = \begin{bmatrix} \tilde{C} \\ \tilde{C}\tilde{A} \\ \vdots \\ \tilde{C}\tilde{A}^{n-1} \end{bmatrix} \text{ and hence } \tilde{C}\tilde{A}^k \begin{bmatrix} 0 \\ I_{n-r} \end{bmatrix} = 0 \text{ for all } k.$$

Hence $\tilde{C}_2 = 0$. Furthermore

$$\tilde{Q}_1\tilde{A}_{12} = \begin{bmatrix} \tilde{Q}_1 & 0 \end{bmatrix} \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} \begin{bmatrix} 0 \\ I_{n-r} \end{bmatrix} = \begin{bmatrix} \tilde{C}\tilde{A} \\ \tilde{C}\tilde{A}^2 \\ \vdots \\ \tilde{C}\tilde{A}^n \end{bmatrix} \begin{bmatrix} 0 \\ I_{n-r} \end{bmatrix} = 0$$

which implies that $\tilde{A_{12}} = 0$ since $\tilde{Q}_1$ is full column rank.

Hence in these state coordinates we have,

$$\dot{\underline{\tilde{x}}}_1 = \tilde{A}_{11}\underline{\tilde{x}}_1 + \tilde{B}_1\underline{u}, \quad \underline{y} = \tilde{C}_1\underline{\tilde{x}}_1$$

and the input/output response (i.e. the transfer function) depends only on $\underline{\tilde{x}}_1$ and the states $\underline{\tilde{x}}_2$ are all unobservable.

3 Observers

3.1 Differentiating signals is a bad idea

Typically the state is not available for measurement,
but we can estimate $\underline{x}(t)$ from $\underline{y}$ and $\underline{u}$

In the section on observability we saw how to exactly deduce $\underline{x}(t)$ from

$$y, \dot{y}, .., y^{(n-1)}, u, \dot{u}, ..u^{(n-2)}$$

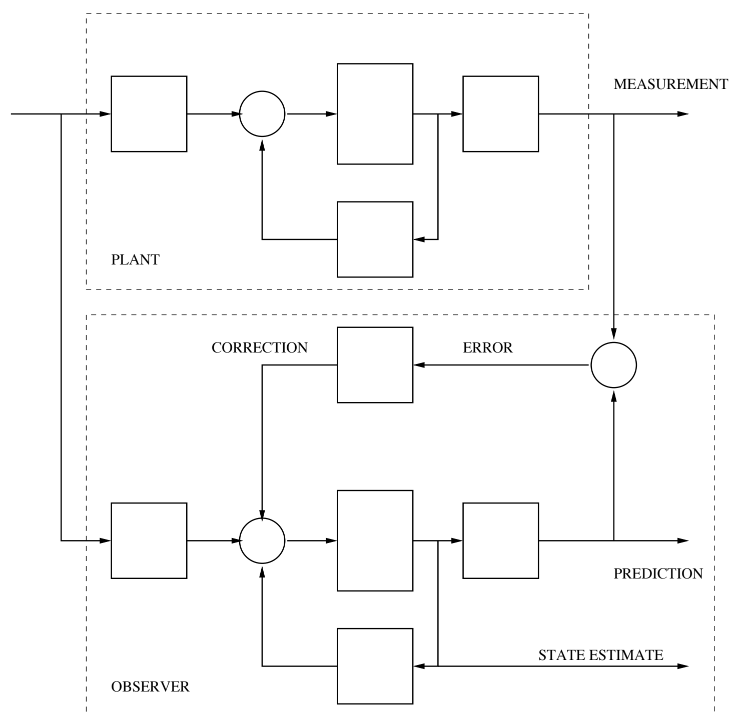
but differentiating signals has bad noise amplification problems:

$$\begin{array}{rclll} y(t) & = & \sin\omega t + \epsilon \sin\omega_n t & \text{S/N ratio} & = 1/\epsilon \\ \dot{y}(t) & = & \omega\cos\omega t + \epsilon\omega_n\cos\omega_n t & \text{S/N ratio} & = (\omega/\epsilon\omega_n) \\ \ddot{y}(t) & = & -\omega^2\sin\omega t - \epsilon\omega_n^2\sin\omega_n t & \text{S/N ratio} & = \frac{1}{\epsilon}\left(\frac{\omega}{\omega_n}\right)^2 \end{array}$$

3.2 Observer structure

Instead we will use a *state observer* (Luenberger Observer) which contains a dynamic model of the system and whose state, $\underline{\hat{x}}(t)$, approaches $\underline{x}(t)$ as $t \to \infty$.

$$\begin{cases} \dot{\underline{\hat{x}}} &= A\underline{\hat{x}} + B\underline{u} + L(\underline{y} - \underline{\hat{y}}) \\ \underline{\hat{y}} &= C\underline{\hat{x}} \end{cases}$$



Consider the error $\underline{e}(t) = \underline{x}(t) - \underline{\hat{x}}(t)$

$$\begin{aligned}\dot{\underline{e}} &= \dot{\underline{x}} - \dot{\hat{\underline{x}}} = (A\underline{x} + B\underline{u}) - (A\underline{\hat{x}} + B\underline{u} + L(\underline{y} - \underline{\hat{y}})) \\ &= A(\underline{x} - \underline{\hat{x}}) - LC(\underline{x} - \underline{\hat{x}}) = (A - LC)\underline{e}\end{aligned}$$

$$\boxed{\dot{\underline{e}} = (A - LC)\underline{e}}$$

We want $e^{(A-LC)t} \to 0$ quickly as t increases.

This is achieved if the eigenvalues of $(A - LC)$ are large and negative, for example.

Can we assign the eigenvalues of $(A - LC)$ by choice of L?

Suppose (A, C) is **not** observable then in section 2.2 we found a change of coordinates, $\underline{\tilde{x}} = T\underline{x}$ such that,

$$\begin{bmatrix} \dot{\underline{\tilde{x}}}_1 \\ \dot{\underline{\tilde{x}}}_2 \end{bmatrix} = \begin{bmatrix} \tilde{A}_{11} & 0 \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} \begin{bmatrix} \underline{\tilde{x}}_1 \\ \underline{\tilde{x}}_2 \end{bmatrix} + \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix} \underline{u}, \quad \underline{y} = \begin{bmatrix} \tilde{C}_1 & 0 \end{bmatrix} \underline{x} + D\underline{u}$$

Hence

$$T(A - LC)T^{-1} = \tilde{A} - \tilde{L}\tilde{C} = \begin{bmatrix} \tilde{A}_{11} & 0 \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} - \begin{bmatrix} \tilde{L}_1 \\ \tilde{L}_2 \end{bmatrix} \begin{bmatrix} \tilde{C}_1 & 0 \end{bmatrix} = \begin{bmatrix} (\tilde{A}_{11} - \tilde{L}_1\tilde{C}_1) & 0 \\ (\tilde{A}_{21} - \tilde{L}_2\tilde{C}_1) & \tilde{A}_{22} \end{bmatrix},$$

and the eigenvalues of the observer,

$$\lambda_i(A - LC) = \lambda_i(\tilde{A} - \tilde{L}\tilde{C}) = \lambda_i(\tilde{A}_{11} - \tilde{L}_1\tilde{C}_1) \cup \lambda_i(\tilde{A}_{22}),$$

and $\lambda_i(\tilde{A}_{22})$ are not changed by $\tilde{L}$.

However *it can be shown that*

> We can arbitrarily assign the eigenvalues of $(A - LC)$ by choice of L if and only if the system is observable.

- We can thus make the error, $\underline{e}(t) \rightarrow 0$ arbitrarily quickly.

- But high gains might imply very large transient errors and noisy estimates.

3.3 Application in Ball and Beam experiment

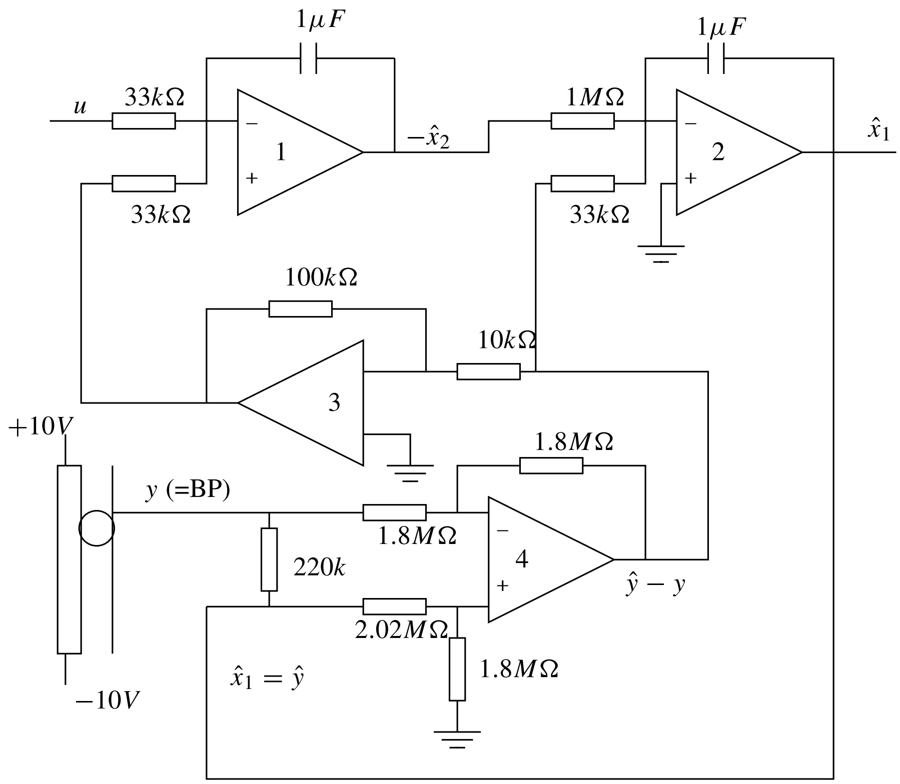
Ball Position (BP) is measured by discrete sensors.

We need Ball Velocity (BV). Differentiate? BP signal absent sometimes. **Use observer.**

Take $\underline{x} = [BP, BV]^T$ and $u = PP$ (Plank Position):

$$
\begin{aligned}
PLANT: \quad \dot{\underline{x}} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ \gamma \end{bmatrix} u, \quad y = [1, 0]\underline{x} \\
OBSERVER: \quad \dot{\hat{\underline{x}}} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \hat{\underline{x}} + \begin{bmatrix} 0 \\ \gamma \end{bmatrix} u + \begin{bmatrix} \ell_1 \\ \ell_2 \end{bmatrix} (y - \hat{y}) \\
ERROR: \quad \dot{\underline{e}} &= (A - LC)\underline{e} = \begin{bmatrix} -\ell_1 & 1 \\ -\ell_2 & 0 \end{bmatrix} \underline{e} \qquad \text{Eigenvalues: } s^2 + \ell_1 s + \ell_2 = 0
\end{aligned}
$$

Observable? $\begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. rank $= 2 \Rightarrow$ **Yes.**



Op. amps 1 and 2 implement $\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y})$. (2 integrators in series.)

Op. amp 3, together with the 33k resistor on its output, implements gain ℓ_2.

33k resistor on input of op.amp 2 implements ℓ_1.

ℓ_1 and ℓ_2 chosen to give eigenvalues $-15 \pm j8.6$

Op. amp 4 gives output = 0 if ball goes open-circuit — observer then predicts BP and BV without any data. (Clever circuit devised by Prof. Glover.)

3.4 Tracking disturbances, ignoring noise

Imagine tracking aircraft by radar (1-D). Aircraft position z is affected by random turbulence. Take $\underline{x} = [z, \dot{z}]^T$:

$$\dot{\underline{x}}(t) = A\underline{x}(t) + Bd(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} d(t)$$

The radar measurement is corrupted by noise:

$$y(t) = C\underline{x}(t) + n(t) = [1 \quad 0]\underline{x}(t) + n(t)$$

Observer: $\dot{\hat{\underline{x}}}(t) = A\hat{\underline{x}}(t) + L[y(t) - C\hat{\underline{x}}(t)]$ NB: $d(t)$ not known, so not used.

d large, n small: Believe the measurements. Use large L. *React quickly.*
d small, n large: Don't trust measurements, believe model. Use small L.
— *Smooth the measurements.*

3.5 Special case: Kalman Filter

Suppose that d and n are both 'white noise'.

Suppose we know their relative 'sizes': $var(d)$, $var(n)$.

How to design the observer gain L optimally —
minimise variance of tracking error $E\{\|\underline{x} - \hat{\underline{x}}\|^2\}$?

The solution is given by *Kalman Filter* theory —
optimal trade-off between tracking d and rejecting n. Guarantees stable observer.

Generalises to arbitrary dimension state vectors, multi-input, multi-output,
and to arbitrary disturbance/noise spectra. *Matlab*: `kalman, dkalman, estim` etc.

Very widely used *Navigation & guidance, Telecomms, Control, Finance, ...*
Especially in discrete time — software implementation.

3.6 Application to sensor fusion

Satellite, 1 axis of rotation: $J\ddot{\theta} = u + d$ (u = control torque, d = disturbance torque).

Two noisy sensors: Star sensor: $y_1 = \theta + n_\theta$, Rate gyro: $y_2 = \dot{\theta} + n_\omega$

Let $\underline{x} = [\theta, \dot{\theta}]^T$. State-space model:

$$
\begin{aligned}
\underline{\dot{x}} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 & 0 \\ 1/J & 1/J \end{bmatrix} \begin{bmatrix} u \\ d \end{bmatrix} \\
\underline{y} &= \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} n_\theta \\ n_\omega \end{bmatrix} = I\underline{x} + \begin{bmatrix} n_\theta \\ n_\omega \end{bmatrix}
\end{aligned}
$$

Observable? Yes. ($C = I$, so rank $C = 2$, so rank $\begin{bmatrix} C \\ CA \end{bmatrix} = 2$.)

Observer:

$$
\begin{aligned}
\underline{\dot{\hat{x}}} &= A\underline{\hat{x}} + B \begin{bmatrix} u \\ 0 \end{bmatrix} + L(\underline{y} - C\underline{\hat{x}}) \qquad (d \text{ not known}) \\
&= (A - LC)\underline{\hat{x}} + \begin{bmatrix} 0 \\ 1/J \end{bmatrix} u + L\underline{y} \qquad \text{but } C = I \text{ so:} \\
&= \begin{bmatrix} -\ell_{11} & 1 - \ell_{12} \\ -\ell_{21} & -\ell_{22} \end{bmatrix} \underline{\hat{x}} + \begin{bmatrix} 0 \\ 1/J \end{bmatrix} u + \begin{bmatrix} \ell_{11} & \ell_{12} \\ \ell_{21} & \ell_{22} \end{bmatrix} \underline{y}
\end{aligned}
$$

Place both eigenvalues at -10 (say): Using trace$(A - LC) = \sum_i \lambda_i$ and det$(A - LC) = \prod_i \lambda_i$: $-\ell_{11} - \ell_{22} = -20$ and $\ell_{11}\ell_{22} + \ell_{21}(1 - \ell_{12}) = 100$. This leaves some design freedom.

$n_\theta \ll n_\omega$: Make $\ell_{11} \gg \ell_{12}$ and $\ell_{21} \gg \ell_{22}$.

$n_\theta \gg n_\omega$: Make $\ell_{11} \ll \ell_{12}$ and $\ell_{21} \ll \ell_{22}$.

Optimal trade-off: *Kalman Filter* again.

3.7 Application to sensor bias estimation

Satellite, as before: $J\ddot{\theta} = u$

Sensors: Star tracker measures angular position: $y_1 = \theta$
Rate gyro measures angular velocity with bias: $y_2 = \dot{\theta} + b_\omega$.

Augment state vector: $\underline{x} = [\theta, \dot{\theta}, b_\omega]^T$, and assume bias is constant: $\dot{b}_\omega = 0$.

State-space model:

$$\begin{aligned} \dot{\underline{x}} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 1/J \\ 0 \end{bmatrix} u \\ \underline{y} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \underline{x} \end{aligned}$$

Is the state observable?

$$\begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

First 3 rows are linearly independent (Or: All three columns are linearly independent).

So rank = 3. Hence: **Observable**. So can use observer to estimate $\underline{x}$:

$$\dot{\hat{\underline{x}}} = A\hat{\underline{x}} + Bu + L(y - C\hat{\underline{x}})$$

$A - LC$ stable $\Rightarrow \hat{x}_3 \to b_\omega$ as $t \to \infty$. Rate of convergence depends on eigenvalues of $A - LC$.