Module 3F2: Systems and Control
EXAMPLES PAPER 1 - STATE-SPACE MODELS

1. A feedback arrangement for control of the angular position of an inertial load is illustrated in Figure 1. (Note that this is not a block diagram in the control sense). Assume that:

   (i) The amplifier supplying the torque motor is voltage driven and produces an output current linearly proportional to input voltage.

   (ii) The torque motor produces an output torque linearly proportional to input current.

   (iii) The load, whose angular position is to be controlled, is a pure inertia and that all forms of friction may be neglected.

   (iv) The two angular position transducers are identical and produce output voltages linearly proportional to input angular position.

   (v) The tachogenerator produces an output voltage linearly proportional to torque motor shaft speed.

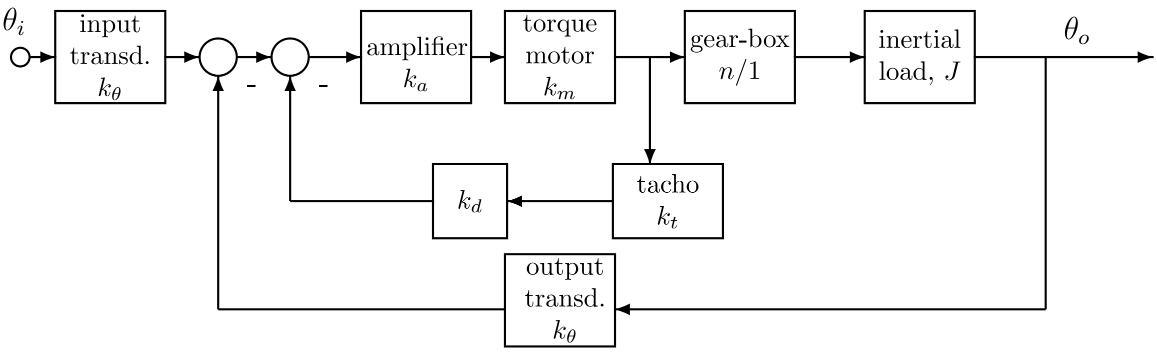

Figure 1:

The system has the following parameters.

Total inertia referred to output shaft, J	=	1 kg m^2
Amplifier conversion constant, k_a	=	0.125 A/V
Motor torque constant, k_m	=	2 N m/A
Gearbox speed reduction ratio, n	=	10/1 down
Position transducer constant, k_θ	=	10 V/radian
Tachogenerator constant, k_t	=	0.4 V/radian s^{-1}

The amount of tachogenerator feedback may be varied by adjustment of the velocity feedback constant k_d.

(a) Choosing θ_o and $\dot{\theta}_o$ as states, derive a state space model for this system. (Note that the gear-box amplifies torque by the factor n, and that the motor shaft rotates n times faster than the load.)

(b) Find the transfer function from θ_i to θ_o *from your state-space model.* Check your answer by finding the transfer function using methods from the second-year Linear Systems course.

(c) Calculate the poles of the closed-loop system as a function of k_d (i) directly from part (a), (ii) from part (b).

2. Control system compensators can be implemented using op-amp circuits. For each of the circuits in Figure 2,

   (a) Taking the capacitor voltages as internal states write down the state-space equations, except explaining why this is not possible for (ii).

   (b) Hence calculate the transfer functions, poles and zeros, and note the integral action controller, PI and PD controller circuits.

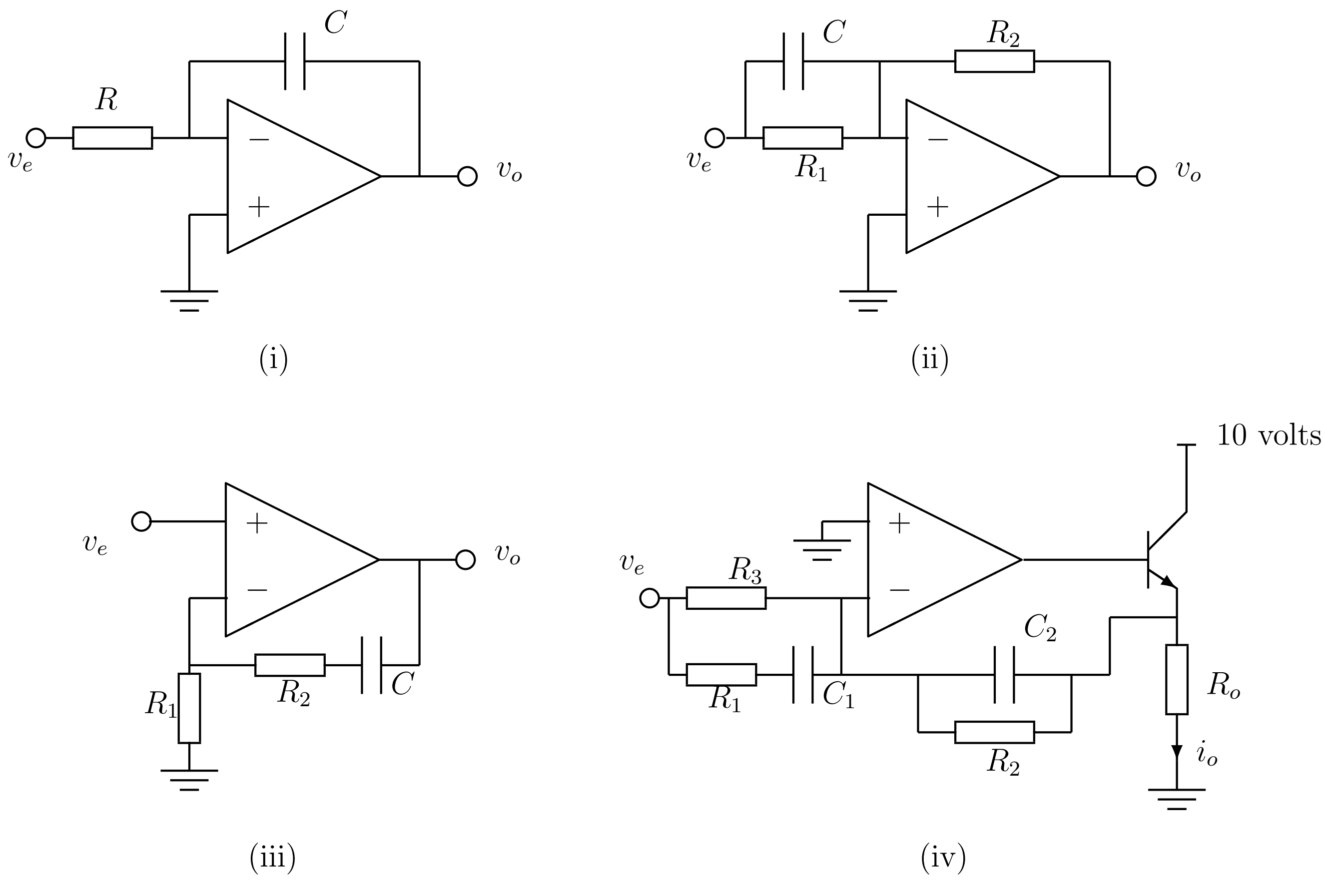

Figure 2:

3. For the linearisation worked example given in lectures in section 3.3.1 verify the (1,2) element of the matrix P.

4. The following are matrices for state-space models in the form $\dot{x} = Ax + Bu$, $y = Cx + Du$. ('$0_{p,m}$' denotes the $p \times m$ zero matrix.) In each case determine (i) how many inputs, states and outputs there are, (ii) the *dimensions* of the transfer function matrix, and (iii) the transfer function matrix:

   (a)
   $$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 4 & 5 & 6 \end{bmatrix}, \quad D = 0$$

   (b)
   $$A = \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

   (c)
   $$A = -2, \qquad B = \begin{bmatrix} 1 & 2 \end{bmatrix}, \qquad C = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \quad D = 0_{3,2}$$

5. Consider the state-space equation,
   $$\dot{\underline{x}}(t) = A\underline{x}(t), \quad \underline{x}(0) = \underline{x}_0.$$

   (a) Verify that, if $\underline{x}_0$ is an eigenvector of A, then $\underline{x}(t) = e^{\lambda t}\underline{x}_0$ will satisfy this state-space equation if λ is the corresponding eigenvalue.

   (b) If
   $$A = \begin{bmatrix} 0 & 1 \\ -k & -2 \end{bmatrix}$$
   calculate the state transition matrix for $k = -3, 0, 1$ and 5, and verify that part (a) holds for all eigenvectors of A. Are there any non-zero equilibrium states?

   (c) For the circuit of question 2(iv) determine intial states, $\underline{x}_0$, such that the resulting responses with $v_e(t) = 0$ will be $\underline{x}(t) = e^{-t/R_2C_2}\underline{x}_0$ and $\underline{x}(t) = e^{-t/R_1C_1}\underline{x}_0$.

6. A system's dynamical behaviour is defined by the state-space equation set

$$\frac{d\underline{x}}{dt} = \begin{bmatrix} 0 & 1 & 3 \\ -1 & 0 & 1 \\ 0 & 0 & 3 \end{bmatrix} \underline{x} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u, \quad y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \underline{x}.$$

   (a) Find a change of state variables described by

   $$\underline{z} = T^{-1}\underline{x}$$

   where T is a complex nonsingular matrix such that the state equations for $\underline{z}$ are in diagonal form, and find the appropriately transformed state equations.

   (b) Determine the system's state transition matrix.

   (c) If $\underline{x}(0) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and the input $u(t) = 1$ for $t \geq 0$, find the resulting output $y(t)$ for $t \geq 0$.

   (d) Repeat parts (b) and (c) using Laplace transforms.

7. Figure 3 represents a two-link manipulator in a vertical plane, to be controlled by the two motors at the joints producing torques T_1 and T_2 as shown. Ignoring frictional and damping terms this particular system satisfies the following differential equations (where the dots over symbols denote differentiation with respect to time):

$$\begin{aligned} T_1 &= -(14.25 + 4\cos\theta_2)\ddot{\theta}_1 - (1.5 + 2\cos\theta_2)\ddot{\theta}_2 \\ &\quad +120\sin\theta_1 + 20\sin(\theta_1 + \theta_2) + 2\dot{\theta}_2(2\dot{\theta}_1 + \dot{\theta}_2)\sin\theta_2 \\ T_2 &= -(1.5 + 2\cos\theta_2)\ddot{\theta}_1 - \ddot{\theta}_2 + 20\sin(\theta_1 + \theta_2) - 2\dot{\theta}_1^2\sin\theta_2 \end{aligned}$$

   [T_1 and T_2 in Nm, θ_1 in radians and time in seconds].

   (a) Calculate the torques T_{1e}, T_{2e} required to maintain the system in equilibrium at $\theta_1 = \pi/6$ and $\theta_2 = \pi/3$.
   The linearised equations about this equilibrium point can be shown to be given by:

   $$\underline{\dot{x}} \cong A\underline{x} + B\underline{u}$$

   where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \alpha^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\beta & 0 & 0 & 0 \end{bmatrix}, \underline{x} = \begin{bmatrix} \theta_1 - \pi/6 \\ \dot{\theta}_1 \\ \theta_2 - \pi/3 \\ \dot{\theta}_2 \end{bmatrix}, \ B = \begin{bmatrix} 0 & 0 \\ -0.1 & 0.25 \\ 0 & 0 \\ 0.25 & -1.625 \end{bmatrix} \ \underline{u} = \begin{bmatrix} T_1 - T_{1e} \\ T_2 - T_{2e} \end{bmatrix}$$

and $\alpha^2 = 6\sqrt{3}$ and $\beta = 15\sqrt{3} = 5\alpha^2/2$.

(b) What are the open-loop poles of this linearized system? Determine an initial condition such that $\underline{x}(t) \to \underline{0}$ as $t \to \infty$.

(c) Calculate e^{At}. (Note that $\begin{bmatrix} X & 0 \\ Y & Z \end{bmatrix}^{-1} = \begin{bmatrix} X^{-1} & 0 \\ -Z^{-1}YX^{-1} & Z^{-1} \end{bmatrix}$ when the inverses exist, and $\mathcal{L}(\sinh(\alpha t) - \alpha t) = \alpha^3/s^2(s^2 - \alpha^2)$.)

(d) If you can release the system from an initial condition and measure the states how could you measure the $(3, 2)$ element of the state transition matrix?

(e) Explain the physical reasons for the difference in response when the system is released from a small displacement in θ_1 and θ_2 .

(f) Calculate the transfer function from u_2 to x_3 , (Hint: this only depends on the $(3, 2)$ and $(3, 4)$ elements of $(sI - A)^{-1}$ and the second column of B), and hence deduce the response of x_3 due to a step input on u_2.

(g) $*$ Suppose that this system is connected to a digital computer. The ADC measures x_1 and x_3 and the DAC controls u_1 and u_2 and synchronised with sampling period T . Assuming that $u_1(kT) = 0$ for all k, calculate the state-space difference equation relating the number sequences $u_2(kT)$ output by the computer and the sequence $\underline{x}(kT)$.

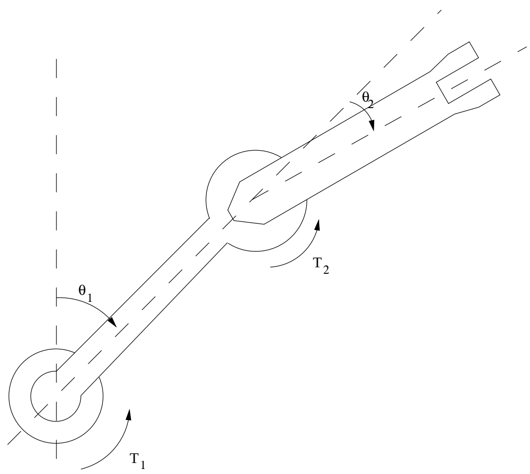

Figure 3: Robot arm

8. (a) A feedback system is given by

$$\underline{Y}(s) = G(s)\left(\underline{R}(s) - \underline{Y}(s)\right)$$

where $G(s) = C\,(sI - A)^{-1}\,B$. By considering the corresponding state-equations (i.e. $\underline{\dot{x}} = A\underline{x} + B(\underline{r} - \underline{y})$, $\underline{y} = C\underline{x}$) deduce that $\underline{Y}(s) = H_{CL}(s)\underline{R}(s)$ where

$$H_{CL}(s) = (I + G(s))^{-1}\,G(s) = C\,(sI - A + BC)^{-1}\,B$$

Optional extra: Verify this identity using the *matrix inversion lemma* — namely: if the indicated inverses exist, then

$$(W + XY^{-1}Z)^{-1} = W^{-1} - W^{-1}X(Y + ZW^{-1}X)^{-1}ZW^{-1}$$

(b) In the equations

$$\begin{aligned} \underline{\dot{x}}(t) &= A\underline{x}(t) + B\underline{u}(t) \\ \underline{y}(t) &= C\underline{x}(t) + D\underline{u}(t) \end{aligned}$$

when D is $p \times p$ and invertible, substitute $\underline{u}(t) = D^{-1}\left(\underline{y}(t) - C\underline{x}(t)\right)$ and hence show that the transfer function of the inverse system, $\underline{U}(s) = (G(s))^{-1}\,\underline{Y}(s)$, is given by,

$$(G(s))^{-1} = \left(D + C\,(sI - A)^{-1}\,B\right)^{-1} = D^{-1} - D^{-1}C\left(sI - A + BD^{-1}C\right)^{-1}BD^{-1}$$

Optional extra: Verify this by using the matrix inversion lemma.

Answers

1. (a)

$$\begin{aligned} \underline{\dot{x}} &= \begin{bmatrix} 0 & 1 \\ -25 & -10k_d \end{bmatrix}\underline{x} + \begin{bmatrix} 0 \\ 25 \end{bmatrix}\theta_i \\ \theta_0 &= [1\ \ 0]\,\underline{x} \end{aligned}$$

(b) $\frac{25}{s^2+10k_d s+25}$

(c) Poles at $-5\left(k_d \pm j\sqrt{1-k_d^2}\right)$.

2. (a) (i) $A = 0,\ B = -1/CR,\ C = 1,\ D = 0.$
   (ii) There is no standard state space form because of the derivative action.
   (iii) $A = 0,\ B = 1/CR_1,\ C = 1,\ D = 1 + R_2/R_1.$
   (iv) $A = \begin{bmatrix} -1/C_1R_1 & 0 \\ 1/C_2R_1 & -1/C_2R_2 \end{bmatrix},\ B = \begin{bmatrix} 1/C_1R_1 \\ -1/R_1C_2 - 1/R_3C_2 \end{bmatrix},$
   $C = \begin{bmatrix} 0 & 1/R_o \end{bmatrix},\ D = 0.$

   (b)

$$\begin{aligned} G_1(s) &= -1/sCR \\ G_2(s) &= -R_2/R_1 - sCR_2 \\ G_3(s) &= 1 + R_2/R_1 + 1/sCR_1 \\ G_4(s) &= \frac{-R_2[1 + sC_1(R_1 + R_3)]}{R_0R_3(1 + sC_1R_1)(1 + sC_2R_2)} \end{aligned}$$

3. Note that the (1,2) element of the matrix P should be $\partial\ddot{\theta}/\partial z$, evaluated at the equilibrium.

4. (a) (i) 1,3,1. (ii) 1×1 (iii) $\frac{12}{s+1} + \frac{10}{s+2} + \frac{6}{s+3}$ or $\frac{28s^2+118s+114}{(s+1)(s+2)(s+3)}$

   (b) (i) 1,2,2. (ii) 2×1 (iii) $\dfrac{\begin{bmatrix} s^2 + 11s + 16 \\ s^2 + 6s + 7 \end{bmatrix}}{(s+1)(s+2)}$

   (c) (i) 2,1,3. (ii) 3×2 (iii) $\dfrac{\begin{bmatrix} 3 & 6 \\ 0 & 0 \\ 1 & 2 \end{bmatrix}}{s+2}$

5. (b)

$$\begin{aligned} k = -3: &\quad \frac{1}{4}\begin{bmatrix} e^{-3t} + 3e^{t}, & -e^{-3t} + e^{t} \\ -3e^{-3t} + 3e^{t}, & 3e^{-3t} + e^{t} \end{bmatrix}, \\ k = 0: &\quad \begin{bmatrix} 1 & \frac{1}{2}(1 - e^{-2t}) \\ 0 & e^{-2t} \end{bmatrix} \\ k = 1: &\quad e^{-t}\begin{bmatrix} 1 + t & t \\ -t & 1 - t \end{bmatrix} \\ k = 5: &\quad e^{-t}\begin{bmatrix} \cos 2t + \frac{1}{2}\sin 2t, & \frac{1}{2}\sin 2t \\ \frac{-5}{2}\sin 2t, & \cos 2t - \frac{1}{2}\sin 2t \end{bmatrix}. \end{aligned}$$

   With $k = 0$, $\underline{x}_e = \begin{bmatrix} \alpha \\ 0 \end{bmatrix}$ is an equilibrium state for any α.

   (c) $\underline{x}_0 = \begin{bmatrix} 1/C_2R_2 - 1/C_1R_1 \\ 1/C_2R_1 \end{bmatrix}$, and $\underline{x}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

6. (b) $\exp(At) = \begin{bmatrix} \cos t & \sin t & (-\cos t + e^{3t}) \\ -\sin t & \cos t & \sin t \\ 0 & 0 & e^{3t} \end{bmatrix}$

   (c) $y(t) = e^{3t} - \cos t + \sin t$.

7. (a) $T_{1e} = 80\text{Nm},\ T_{2e} = 20\text{Nm}$.

   (b) Poles at $0, 0, \pm\alpha$. $\underline{x}_0 = \begin{bmatrix} -2 \\ 2\alpha \\ 5 \\ -5\alpha \end{bmatrix}$.

   (c) $e^{At} = \begin{bmatrix} \cosh \alpha t & \alpha^{-1} \sinh \alpha t & 0 & 0 \\ \alpha \sinh \alpha t & \cosh \alpha t & 0 & 0 \\ -\beta\alpha^{-2}(\cosh \alpha t - 1) & -\beta\alpha^{-3}(\sinh \alpha t - \alpha t) & 1 & t \\ -\beta\alpha^{-1} \sinh \alpha t & -\beta\alpha^{-2}(\cosh \alpha t - 1) & 0 & 1 \end{bmatrix}$

   (f)
$$\begin{aligned} G_{32}(s) &= \frac{-1}{s^2} - \frac{0.625}{s^2 - \alpha^2} \\ x_2(t) &= -\frac{1}{2}t^2 - \frac{5}{48\sqrt{3}}[\cosh \alpha t - 1] \end{aligned}$$

   (g)
$$\begin{aligned} \underline{x}((k+1)T) &= \Phi \underline{x}(kT) + \Gamma u_2(kT) \\ \Phi &= \exp(AT) \text{ as in 6(c).} \\ \Gamma &= \tfrac{1}{8} \begin{bmatrix} 2C \\ 2S \\ -5C - 4T^2 \\ -5S - 8T \end{bmatrix}, \qquad \begin{aligned} C &= (\cosh \alpha T - 1)/\alpha^2 \\ S &= \sinh(\alpha T)/\alpha \end{aligned} \end{aligned}$$

8. —

Suitable questions from past 3F2 Tripos papers:
2013: 1, 2012: 1, 2011: 1, 3(a), 3(b)(i). 2010: 1(a)–(d). 2009: 1, 3(a),(b). 2008: 2(b),(d), 3(a). 2007: 1.

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