Equations of motion in intrinsic (streamline) co-ordinates

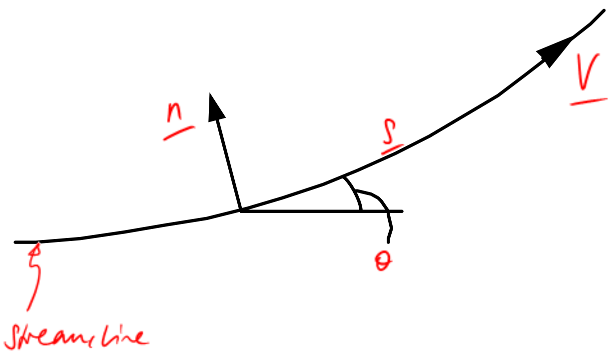
Assumptions: steady

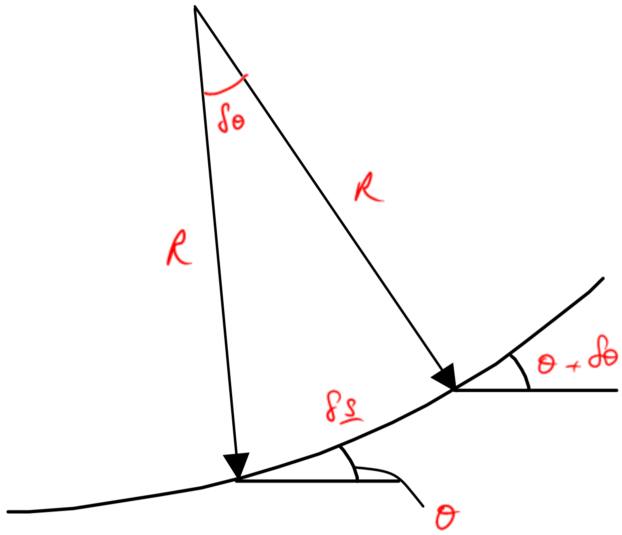
isentropic (= inviscid, non-heat conducting)

two dimensional

uniform conditions far upstream

Steady flow energy equation applied to a stream tube indicates that $h_0 = h + \frac{V^2}{2}$ = constant along the stream tube and hence uniform everywhere (since upstream conditions are uniform).





Momentum

s-momentum (streamwise)

$$\rho V \frac{dV}{ds} = -\frac{\partial p}{\partial s} \tag{1.1}$$

n-momentum

$$\frac{\rho V^2}{R} = -\frac{\partial p}{\partial n}$$

where R is the radius of curvature of a streamline.

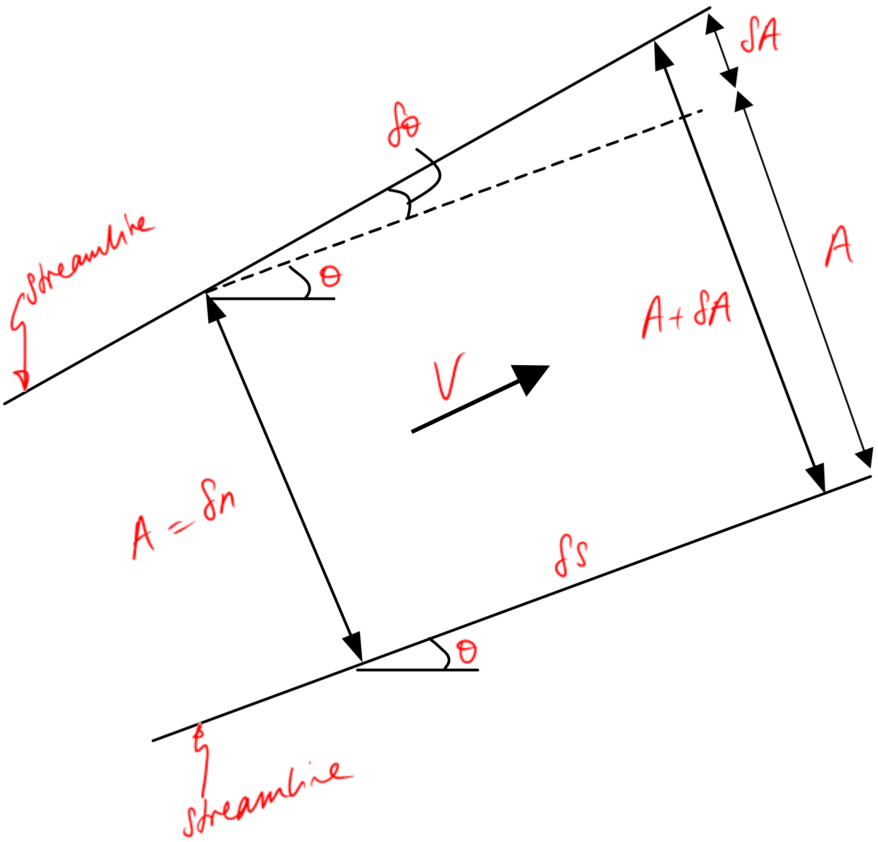
Any streamline can be approximated locally by an arc with the appropriate radius of curvature. It is clear that arc length and angle change are related by

$$\delta s = R\,\delta\theta \quad \Rightarrow \quad \frac{1}{R} = \frac{\partial \theta}{\partial s}$$

So that the second equation becomes

$$\rho V^2 \frac{\partial \theta}{\partial s} = -\frac{\partial p}{\partial n} \tag{1.2}$$

Conservation of mass (continuity)



Conservation of mass flow between the streamlines shown gives

$$\rho V A = \text{const.} \quad \Rightarrow \frac{\partial}{\partial s}(\rho V A) = 0$$

$$\Rightarrow \frac{\partial}{\partial s}(\rho V) + \rho V \frac{1}{A}\frac{\partial A}{\partial s} = 0$$

For the streamlines shown

$$A = \delta n \qquad \text{and} \qquad \delta A = \delta s \, \delta\theta$$

Putting this all together gives

$$\frac{1}{A}\frac{\partial A}{\partial s} = \frac{1}{\delta n} \cdot \frac{\partial s \, \delta\theta}{\partial s} = \frac{\partial\theta}{\partial n} \Rightarrow \rho \frac{\partial V}{\partial s} + V \frac{\partial \rho}{\partial s} + \rho V \frac{\partial\theta}{\partial n} = 0$$

Further tidying up gives

$$\frac{1}{\rho}\frac{\partial\rho}{\partial s} + \frac{1}{V}\frac{\partial V}{\partial s} + \frac{\partial\theta}{\partial n} = 0 \qquad (1.3)$$

Entropy & Energy

Finally, we seek to complete the set by using the fact that the flow is isentropic. This helps in two ways. First it enables us to relate changes in pressure to those in density.

Flow isentropic $\Rightarrow \quad p = k\rho^{\gamma} \quad \Rightarrow \quad dp = \gamma k\rho^{\gamma-1}d\rho$

$$\text{i.e. } dp = \frac{\gamma p}{\rho}\,d\rho = a^2 d\rho \tag{1.4}$$

Equation (1.4) is used to eliminate pressure in (1.1), and then $\frac{\partial \rho}{\partial s}$ is eliminated using (1.3). The s-momentum equation becomes $\left[\frac{V^2}{a^2} - 1\right]\frac{1}{V}\frac{\partial V}{\partial s} - \frac{\partial \theta}{\partial n} = 0$

i.e.

$$\boxed{\left[M^2 - 1\right]\frac{1}{V}\frac{\partial V}{\partial s} - \frac{\partial \theta}{\partial n} = 0} \tag{1.5}$$

Secondly since

$$Tds = dh - \frac{dp}{\rho} = dh_0 - VdV - \frac{dp}{\rho} \quad \text{and} \quad s \text{ and } h_0 \text{ are uniform} \Rightarrow \boxed{VdV = -\frac{dp}{\rho}}$$

(everywhere in the flowfield). Equation (1.2) becomes

$$\boxed{V\frac{\partial \theta}{\partial s} = \frac{\partial V}{\partial n}} \tag{1.6}$$

Equations (1.5) and (1.6) determine the flow, provided we can relate M to V (see later).

Method of Characteristics for Supersonic Flow

It turns out that equations (1.5) & (1.6) have an elegant geometric solution.

Equation (1.5) can be written

$$\sqrt{M^2-1}\,\frac{1}{V}\frac{\partial V}{\partial s} - \frac{1}{\sqrt{M^2-1}}\frac{\partial \theta}{\partial n} = 0$$

We introduce the Prandtl-Meyer function

$$\nu = \int_1^M \sqrt{M^2-1}\,\frac{dV}{V} \qquad (1.7)$$

so that $d\nu = \sqrt{M^2-1} \Rightarrow \boxed{\dfrac{\partial \nu}{\partial s} - \dfrac{1}{\sqrt{M^2-1}}\dfrac{\partial \theta}{\partial n} = 0}$

Equation (1.6) becomes

$$\boxed{\frac{\partial \theta}{\partial s} - \frac{1}{\sqrt{M^2-1}}\frac{\partial \nu}{\partial n} = 0}$$

Adding and subtracting the two highlighted equations gives

$$\frac{\partial}{\partial s}(\nu+\theta) - \frac{1}{\sqrt{M^2-1}}\frac{\partial}{\partial n}(\nu+\theta) = 0$$

$$\frac{\partial}{\partial s}(\nu-\theta) + \frac{1}{\sqrt{M^2-1}}\frac{\partial}{\partial n}(\nu-\theta) = 0$$

Changes in $\nu-\theta$ throughout the flowfield satisfy

$$d(\nu-\theta) = \frac{\partial}{\partial s}(\nu-\theta)\,ds + \frac{\partial}{\partial n}(\nu-\theta)\,dn$$

If we choose to move, therefore, along a line which has direction given by $\dfrac{dn}{ds} = \dfrac{1}{\sqrt{M^2-1}}$

relative to the flow direction, then *along this line*

$$d(\nu-\theta) = \frac{\partial}{\partial s}(\nu-\theta)\,ds + \frac{\partial}{\partial n}(\nu-\theta)\,dn$$

$$= \left[\frac{\partial}{\partial s}(\nu-\theta) + \frac{dn}{ds}\frac{\partial}{\partial n}(\nu-\theta)\right]ds = 0$$

i.e. $(\nu-\theta) = $ constant

$\nu - \theta =$ const on a line which makes an angle $\dfrac{1}{\sqrt{M^2-1}}$ with the flow direction
$\nu + \theta =$ const on a line which makes an angle $-\dfrac{1}{\sqrt{M^2-1}}$ with the flow direction

To evaluate the Prandtl-Meyer function : $\nu = \displaystyle\int_1^M \sqrt{M^2-1}\,\frac{dV}{V}$

it is necessary to express V as a function of M .

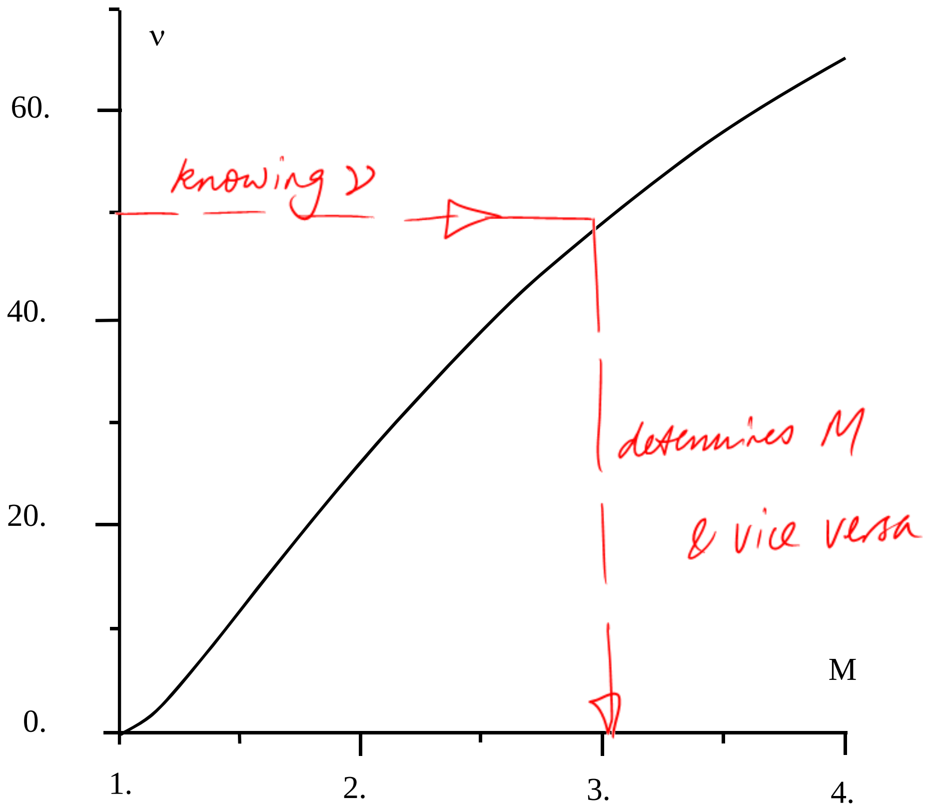
Now

$$T_0 = T\left[1+\frac{\gamma-1}{2}M^2\right] \Rightarrow V^2 = a^2M^2 = \gamma R T M^2 = \frac{\gamma R T_0 M^2}{1+\frac{\gamma-1}{2}M^2}$$

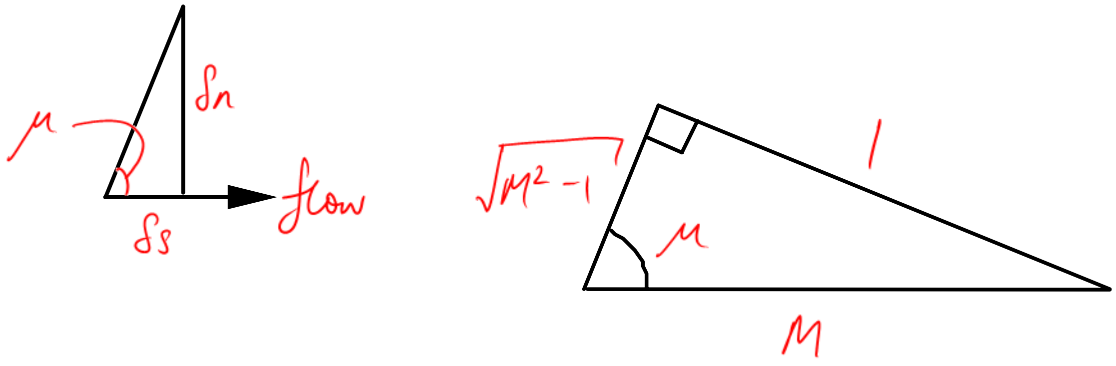
The integral can, in fact, be done analytically

$$\nu = \sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}\left[\frac{\gamma-1}{\gamma+1}\left(M^2-1\right)\right]^{\frac{1}{2}} - \tan^{-1}\sqrt{M^2-1} \qquad (1.9)$$

This function is tabulated in Houghton and Brock and in the CUED tables, and a glance there shows that it increases monotonically with M . Note that it has units of angle. i.e. degrees or radians. CUED tables are in degrees.



Finally, the line which is at angle $\frac{dn}{ds} = \frac{1}{\sqrt{M^2-1}}$ is actually at the *Mach angle* to the flow.



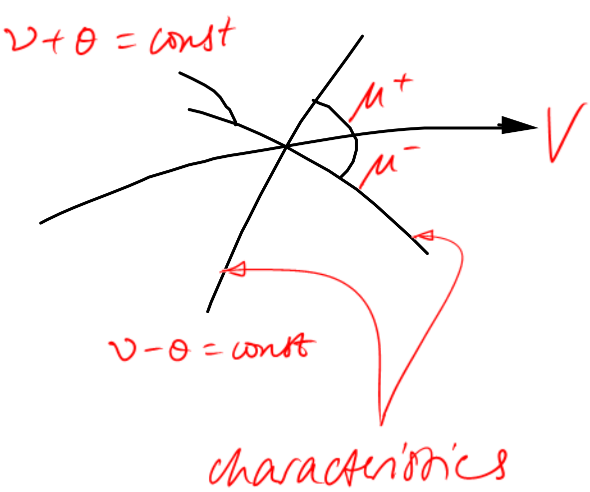
In summary, in inviscid, supersonic steady two dimensional flow, the general solution of the equations of motion are equivalent to the algebraic relationships

$\nu - \theta =$ const on a line which makes an angle $+\mu$ with the flow direction

$\nu + \theta =$ const on a line which makes an angle $-\mu$ with the flow direction

Lines at $\pm\mu$ to the flow direction are called "Mach lines" or "CHARACTERISTICS"

General Features of Supersonic Flow



Through any point in a region of supersonic flow, there will be two characteristics at angles $\pm\mu$ to the local flow direction. The two relationships $\nu \pm \theta =$ const. are enough to determine ν and θ at the point. Since ν is a monotomic function of M, then once ν is known so is M. The velocity follows immediately from the energy equation, and the other flow variables from simple compressible flow relationships. These other variables together with the known value of θ are the complete solution at this point.