Module 3F1 – Signals and Systems
Examples Paper 3F1/1
SOLUTIONS

1. (i) $L(f(t)) = \dfrac{1}{s},\ f(t) = 1$ for $t \geq 0$

$f(kT) = 1$ for $k \geq 0$

$$Z\{f(kT)\} = \sum_{k\geq 0} 1 \times z^{-k} = \frac{1}{1-z^{-1}} \quad \text{for } |z| > 1 \text{ (geometric series)}$$

(ii) $L(f(t)) = \dfrac{1}{s+a},\ f(t) = e^{-at}$ for $t \geq 0$

$f(kT) = (e^{-aT})^k$

$$Z\{f(kT)\} = \sum_{k\geq 0} (e^{-aT}z^{-1})^k = \frac{1}{1-e^{-aT}z^{-1}}$$

(iii) $L(f(t)) = \dfrac{1}{(s+a)(s+b)} = \dfrac{1}{b-a}\left\{\dfrac{1}{s+a} - \dfrac{1}{s+b}\right\}$

$f(t) = \dfrac{1}{b-a}\left\{e^{-at} - e^{-bt}\right\}$

$$\begin{aligned} Z\{f(kT)\} &= \frac{1}{b-a}\left\{\frac{1}{1-e^{-aT}z^{-1}} - \frac{1}{1-e^{-bT}z^{-1}}\right\} \\ &= \frac{z^{-1}(e^{-aT}-e^{-bT})}{(b-a)(1-e^{-aT}z^{-1})(1-e^{-bT}z^{-1})} \end{aligned}$$

(iv) $L(f(t)) = \dfrac{s+a}{(s+a)^2+b^2}$

$f(t) = e^{-at}\cos(bt) = \dfrac{1}{2}e^{(-a+jb)t} + \dfrac{1}{2}e^{(-a-jb)t}$

$$\begin{aligned} Z\{f(kT)\} &= \frac{1}{2}\left(\frac{1}{1-e^{(-a+jb)T}z^{-1}}\right) + \frac{1}{2}\left(\frac{1}{1-e^{(-a-jb)T}z^{-1}}\right) \\ &= \frac{1-e^{-aT}\cos(bT)\,z^{-1}}{1-2e^{-aT}\cos(bT)\,z^{-1}+e^{-2aT}z^{-2}} \end{aligned}$$

2. Let $Y(z) = Z\{y_k\}$;
$U(z) = Z\{u_k\}_{k\geq 0} = 1 + z^{-1}$;
$Z\{u_{k-1}\}_{k\geq 0} = z^{-1} + z^{-2}$;
$Z\{u_{k-2}\}_{k\geq 0} = z^{-2} + z^{-3}$;
$Z\{y_{k-1}\} = z^{-1}Y(z) + y_{-1}$;
$Z\{y_{k-2}\} = z^{-2}Y(z) + z^{-1}y_{-1} + y_{-2}$;

(i) Take z-transform of $y_k = u_k + u_{k-1} + u_{k-2}$

$$\begin{aligned} Y(z) &= 1 + z^{-1} + z^{-1} + z^{-2} + z^{-2} + z^{-3} \\ &= 1 + 2z^{-1} + 2z^{-2} + z^{-3} \\ y_k &= 1, 2, 2, 1, 0, 0, \ldots \end{aligned}$$

(Alternatively solve the difference equation!)

(ii) z-transform of $y_k = 0.8y_{k-1} + 0.2u_k$ is

$$\begin{aligned} Y(z) &= 0.8(z^{-1}Y(z) + [y_{-1} = 0]) + 0.2(1 + z^{-1}) \\ &= \frac{0.2(1 + z^{-1})}{1 - 0.8z^{-1}} = 0.2 + \frac{0.36z^{-1}}{1 - 0.8z^{-1}} \\ y_k &= 0.2, 0.36, 0.36 \times 0.8, \ldots, 0.36 \times (0.8)^{k-1}, \ldots \end{aligned}$$

(iii) z-transform of $y_k = 0.98y_{k-1} - 0.9604y_{k-2} + u_k$ gives

$Y(z) - 0.98\left(z^{-1}Y(z) + y_{-1}\right) + 0.9604\left(z^{-2}Y(z) + z^{-1}y_{-1} + y_{-2}\right) = U(z) = 1 + z^{-1}$

$$Y(z) = \frac{0.0396 + z^{-1}}{1 - 0.98z^{-1} + 0.9604z^{-2}};$$

$$1 - 0.98z^{-1} + 0.9604z^{-2} = 1 - 2 \times 0.98\cos(\pi/3)z^{-1} + 0.98^2 z^{-2}$$

Comparing with the 7^{th} entry with z-transform table gives

$$\begin{aligned} y_k &= \frac{0.98^{k-1}}{\sin(\pi/3)}\{0.98 \times 0.0396\sin[(k+1)\pi/3] + \sin[k\pi/3]\} \\ &= 0.98^{k-1}\{0.0448\sin[(k+1)\pi/3] + 1.1547\sin[k\pi/3]\} \end{aligned}$$

Check: $y_0 = 0.0396 = \lim_{z\to\infty} Y(z) = u_0 - 0.9604y_{-1}$
$y_1 = 0.98y_0 - 0.9604y_{-1} + u_1 = 0.98 \times 0.0396 + 1$
$= 1.0388 = 0.0448\sin(2\pi/3) + 1.1547\sin(\pi/3)$
These checks are very easy and well worth doing especially for y_0

3. (i) $G(z) = 1 + z^{-1} + z^{-2}$; zeros $= \frac{-1}{2} \pm j\frac{\sqrt{3}}{2}$; poles $= 0, 0$ (twice)

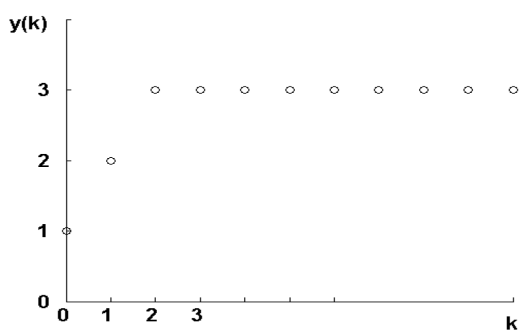
(ii) $G(z) = \dfrac{0.2}{1 - 0.8z^{-1}}$; pole $= 0.8$

(iii) $G(z) = \dfrac{1}{1 - 0.98z^{-1} + 0.9604z^{-2}}$; poles $= 0.98\, e^{\pm j\pi/3}$

(a) Step responses $U(z) = \dfrac{1}{1 - z^{-1}}$

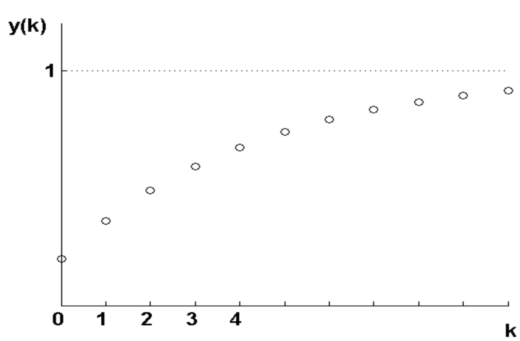
(i)

$$\begin{aligned} Y(z) &= G(z)U(z) \\ &= (1 + z^{-1} + z^{-2})(1 - z^{-1})^{-1} \\ &= (1 + z^{-1} + z^{-2})(1 + z^{-1} + z^{-2} + \ldots) \\ y(k) &= 1, 2, 3, 3, 3, \ldots \end{aligned}$$



(ii)

$$\begin{aligned} Y(z) &= \frac{1}{1 - z^{-1}} \times \frac{0.2}{1 - 0.8z^{-1}} \\ &= \frac{1}{1 - z^{-1}} - \frac{0.8}{1 - 0.8z^{-1}} \\ y(k) &= 1 - (0.8)^{k+1} \end{aligned}$$



(iii)

$$\begin{aligned} Y(z) &= \frac{1}{(1-z^{-1})(1-0.98z^{-1}+0.9604z^{-2})} \\ &= \left\{ \frac{A}{1-z^{-1}} + \frac{B+Cz^{-1}}{1-0.98z^{-1}+0.9604z^{-2}} \right\} \text{for some A, B, C} \end{aligned}$$

y_k could now be found as in Q.2(iii) but for variety we will calculate y_0, y_1, y_∞ and hence y_k

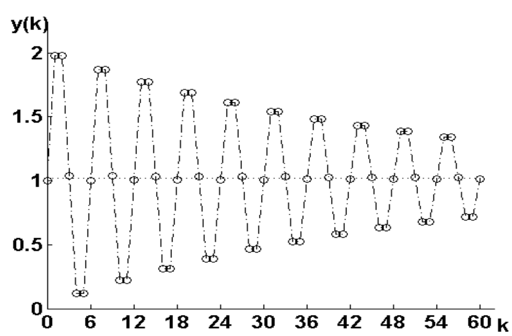
$$\left.\begin{aligned} y_0 &= u_0 = 1 \\ y_1 &= 0.98y_0 + u_1 = 1.98 \end{aligned}\right\} \text{From difference equation}$$

$y_\infty = \lim_{z\to 1}(z-1)Y(z) = \dfrac{1}{1-0.98+0.9604} = \dfrac{1}{0.9804} = 1.020$ (Final Value Theorem)

From the form of $Y(z)$ we know

$$\begin{aligned} y_k &= A + (0.98)^k(B'\cos(k\pi/3) + C'\sin(k\pi/3)) \\ y_\infty &= \underline{A = 1.0200} \\ y_0 &= A + B' = 1; \underline{B' = -0.0200} \\ y_1 &= A + 0.98(B' \times 0.5 + C'\sqrt{3}/2) = 1.98 \Rightarrow \underline{C' = 1.1427} \end{aligned}$$

$$\left[\begin{array}{llll} \textit{check}: & & & \\ y_2 & = 0.98y_1 - 0.9604y_0 + u_2 & = & 1.98 \\ & = A + 0.98^2(B' \times (-0.5) + C'\sqrt{3}/2) & = & 1.98 \end{array}\right]$$



Damped oscillation with period $2\pi/(\pi/3) = 6$ steps and decaying at a rate $(0.98)^6 = 0.89$ per cycle towards steady value of 1.02

(b) Frequency response for $u_k = \cos(\omega kT)$
$y_k \to |G(e^{j\omega T})|\cos[\omega kT + \angle G(e^{j\omega T})]$

$$\underline{\omega T = 0}: \ \text{`}DC'\ gain = G(1) = \begin{cases} 3 & (i) \\ 1 & (ii) \\ 1.02 & (iii) \end{cases}$$

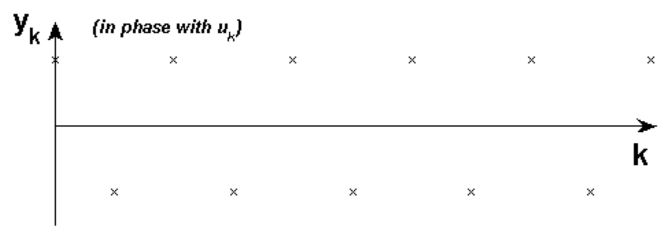
$$\underline{\omega T = \pi}: \qquad G(e^{j\omega T}) = G(-1)$$

$(maximum frequency)$

$$= \begin{cases} 1 & (i) \\ 1/9 & (ii) \\ 0.34 & (iii) \end{cases}$$

$hence$

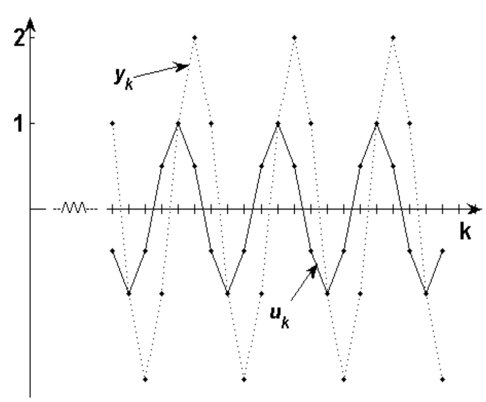
$$y_k \rightarrow \begin{cases} (-1)^k & (i) \\ (-1)^k/9 & (ii) \\ (-1)^k \times 0.34 & (iii) \end{cases}$$



$\underline{\omega T = \pi/3}:$

(i)

$$\begin{aligned} G(e^{j\pi/3}) &= 1 + \cos(-\pi/3) + \cos(-2\pi/3) + j\,(\sin(-\pi/3) + \sin(-2\pi/3)) \\ &= 1 - j\sqrt{3} = 2e^{-j\pi/3} \\ y_k &\rightarrow 2\cos[(k-1)\pi/3] \end{aligned}$$



$$
\begin{aligned}
(ii)\quad G(e^{j\pi/3}) &= \frac{0.2}{1-0.8e^{-j\pi/3}} = \frac{0.2}{[1-0.8\cos(\pi/3)+j0.8\sin(\pi/3)]} \\
&= \frac{0.2}{[0.6+j0.4\sqrt{3}]} = \frac{0.2}{0.9165e^{j0.857}} = 0.218e^{-j0.857} \\
y_k &\rightarrow 0.218\cos(k\pi/3-0.857) \\
y_{6l+n} &\rightarrow 0.218\cos(n\pi/3-0.857) = \begin{cases} 0.143 & n=0 \\ 0.214 & n=1 \\ 0.071 & n=2 \\ -0.143 & n=3 \\ -0.214 & n=4 \\ -0.071 & n=5 \end{cases}
\end{aligned}
$$

with similar sketch to the above.

$$
\begin{aligned}
(iii)\quad G(e^{j\pi/3}) &= \frac{1}{[1-0.98\cos(\pi/3)+0.9604\cos(2\pi/3)+j(-0.98\sin(-\pi/3)+0.9604\sin(-2\pi/3))]} \\
&= \frac{1}{[0.0298+j0.01697]} \\
&= 29.16e^{j(-0.518)}
\end{aligned}
$$

This is the resonant frequency of this system.
$y_k \rightarrow 29.16\cos(k\pi/3-0.518)$

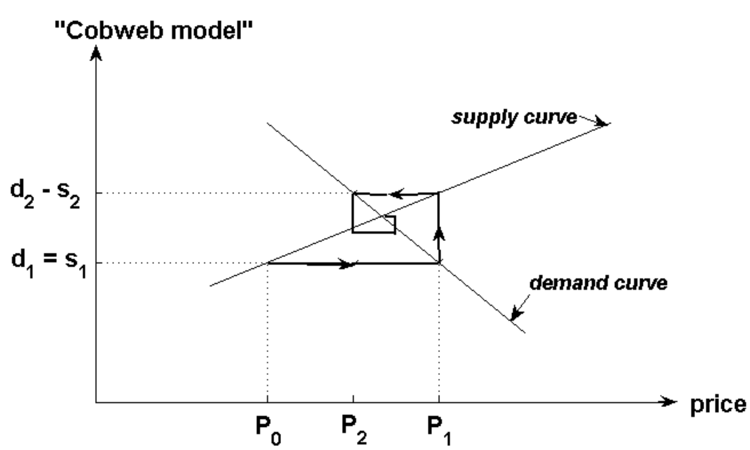
4. (i)

$$
\begin{aligned}
d_k &= d_0 - ap_k \\
s_k &= s_0 + bp_{k-1}
\end{aligned}
$$

equating supply and demand gives

$$
p_k = \frac{-b}{a}p_{k-1} + \frac{(-s_0+d_0)}{a}
$$

stable if $c = b/a < 1$



(ii)

$$\begin{aligned} d_k &= d_0 - ap_k \\ s_k &= s_0 + b(2p_{k-1} - p_{k-2}) \end{aligned}$$

equating supply and demand gives

$$p_k + 2cp_{k-1} - cp_{k-2} = \frac{d_0 - s_0}{a}$$

stable if roots of $z^2 + 2cz - c = 0$ are inside unit circle.
Roots are given by: $z = -c \pm \sqrt{c^2 + c}$. As c increases from zero, negative root hits the unit circle first at $z = -1$. Putting in $z = -1$ into the quadratic and solving for c gives $c = \frac{1}{3}$. Hence stable if $\boxed{c < \frac{1}{3}}$
Note that this price extrapolation tends to destabilise the system.

5. (a) For $f(x) = ax$ the difference equation is

$$\begin{aligned} x((k+1)T) &= x(kT) + Tax(kT) \\ &= (1 + aT)x(kT) \end{aligned}$$

For stability need $-1 < 1 + aT < 1$ *i.e.* $\boxed{-2 < aT < 0}$

(b) For $f(x) = ax$ the modified predictor of $x((k+1)T)$ is

$$\begin{aligned} x((k+1)T) = g(T) &= \\ &= \{x((k-1)T) - x(kT) + ax(kT)T\}1 + ax(kT)T + x(kT) \\ &= 2aTx(kT) + x((k-1)T) \end{aligned}$$

Stable method if roots of $z^2 - 2aTz - 1 = 0$ are inside unit circle.

$$z_{1,2} = aT \pm \sqrt{1 + a^2T^2}$$

For all $aT < 0 \quad z_2 < -1$, hence always unstable. This means that errors will grow as $(z_2)^k$ providing a hopeless method.

6. (a) The basic idea is to split $G(z)$ into partial fractions and note that each factor of the form $\dfrac{1}{1 - P_i z^{-1}}$ is the z-transform of $\{{P_i}^k\}_{k \geq 0}$. If $|P_i| < 1$ for each i then

$$\sum_{k=0}^{\infty} |g_k| \leq \sum_{k=0}^{\infty} |P_1|^k + \sum_{k=0}^{\infty} |P_2|^k + \ldots \sum_{k=0}^{\infty} |P_n|^k$$

and the right hand side is finite. The general argument which covers multiple poles goes as follows. A rational transfer function of the form

$$G(z) = \frac{b_0 + b_1 z^{-1} + \ldots + b_m z^{-m}}{(1 - P_1 z^{-1})^{n_1}(1 - P_2 z^{-1})^{n_2} \ldots (1 - P_r z^{-1})^{n_r}}$$

(with $n_1 + n_2 + \ldots + n_r = n > m$) can be split into partial fractions of the form

$$G(z) = \sum_{i=1}^{r} \sum_{l=1}^{n_r} \frac{A_{il}}{(1 - P_i z^{-1})^l}$$

Thus $\{g_k\}$ is a sum of the number sequences which are the inverse z-transform of $A_{il}/(1 - P_i z^{-1})^l$. To show that $\sum_{k=0}^{\infty} |g_k|$ is finite it is sufficient to show the same for each of these sequences. Note (from z-transform table) that

$$Z^{-1}\left(\frac{1}{(1 - P_i z^{-1})^l}\right) = \left\{\frac{(k+l-1)!\, P_i{}^k}{k!\,(l-1)!}\right\}_{k \geq 0}$$

Note further that the sum below is finite and

$$\sum_{k=0}^{\infty} \frac{(k+l-1)!}{k!\,(l-1)!} |P_i|^k = \frac{1}{(1 - |P_i|)^l}$$

since $|P_i| < 1$. Thus

$$\sum_{k=0}^{\infty} |g_k| \leq \sum_{i=1}^{r} \sum_{l=1}^{n_i} \frac{|A_{il}|}{(1 - |P_i|)^l}$$

which is finite.

(b) Since $\sum_{k=0}^{\infty} |g_k|$ is infinite then, for any M (no matter how large), we can find an N so that

$$\sum_{k=0}^{N} |g_k| > M$$

At time $k = N$ we have

$$y_N = \sum_{k=0}^{N} g_k\, u_{N-k}$$

Thus, if we set $u_{N-k} = sign(g_k)$ for $k = 0, \ldots, N$ we get

$$y_N = \sum_{k=0}^{N} |g_k|$$

So we have made the output bigger than an arbitrarily large M at the time instant $k = N$.

(c) The basic observation is that $G(z)$ has a pair of poles on the unit circle, so the system naturally oscillates. To make the output grow without bound we need to excite at the same frequency, *i.e.* put in another pair of poles at the same locations. (Recall case in lectures where the output of the "summer" is unbounded for a step input). To check things formally we can proceed as follows: (The factors in the numerator don't affect the reasoning - they can be chosen to facilitate the calculation of inverse z-transforms.) First note that

$$Z^{-1}\left(\frac{z^2}{z^2-\sqrt{2}z+1}\right)=\left\{\sqrt{2}\sin\left(\frac{\pi}{4}(k+1)\right)\right\}_{k\geq 0}$$

i.e.

$$\frac{1}{z^2-\sqrt{2}z+1}=\sum_{k=0}^{\infty}\sqrt{2}\sin\left(\frac{\pi}{4}(k+1)\right)z^{-k-2} \qquad (*)$$

Differentiating both sides of (*) with respect to z gives

$$\frac{-(2z-\sqrt{2})}{(z^2-\sqrt{2}z+1)^2}=-\sum_{k=0}^{\infty}(k+2)\sqrt{2}\sin\left(\frac{\pi}{4}(k+1)\right)z^{-k-3}$$

We now set $U(z)=\dfrac{z^2-\frac{1}{\sqrt{2}}z}{z^2-\sqrt{2}z+1}$. This is a bounded input:

$$u_k=\sqrt{2}\left[\sin\left(\frac{\pi}{4}(k+1)\right)-\frac{1}{\sqrt{2}}\sin\left(\frac{\pi}{4}k\right)\right]$$

At the output we have:

$$\begin{aligned} Y(z) &= \sum_{k=0}^{\infty}(k+2)\frac{1}{\sqrt{2}}\sin\left(\frac{\pi}{4}(k+1)\right)z^{-k-2} \\ &= \sum_{k=2}^{\infty}k\frac{1}{\sqrt{2}}\sin\left(\frac{\pi}{4}(k-1)\right)z^{-k} \end{aligned}$$

The corresponding output sequence is:

$$y_k=\begin{cases} 0 & k=0,1 \\ \frac{k}{\sqrt{2}}\sin\left(\frac{\pi}{4}(k-1)\right) & k\geq 2 \end{cases}$$

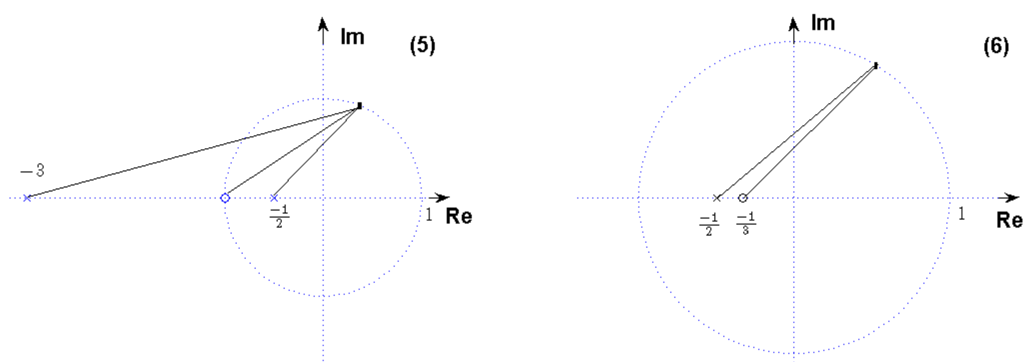
which is unbounded.

7. Both (1) and (2) have infinite gain at frequency 0 ($z=e^{j0}=1$), but there is only one gain plot that is not finite at low frequencies, so suspect that they both have the same gain at all frequencies. (This is true, but not that easy to show unless you spot that $|e^{j\theta}+2|=|2e^{j\theta}+1|$, or know Apollonius' Circle from geometry.) Check high frequencies ($z=e^{j\pi}=-1$): $G_1(-1)=-1/2$ and $G_2(-1)=+1/2$, so the gains are the same, but the phases should be 180° different at high frequencies. Both (1) and (2) have phase $-90°$ at low frequencies because of the pole at 1. It

seems that there is only one plot starting at $-90°$, but it separates into two as frequency increases, ending at 0 and 180°, respectively, consistently with the analysis above. (*Further things that could be spotted:* In (1) the zero is *outside* the unit circle, which gives no net change in phase as θ goes from 0 to π, whereas in (2) the zero is *inside* the unit circle which gives +180°of phase change over the frequency range.)

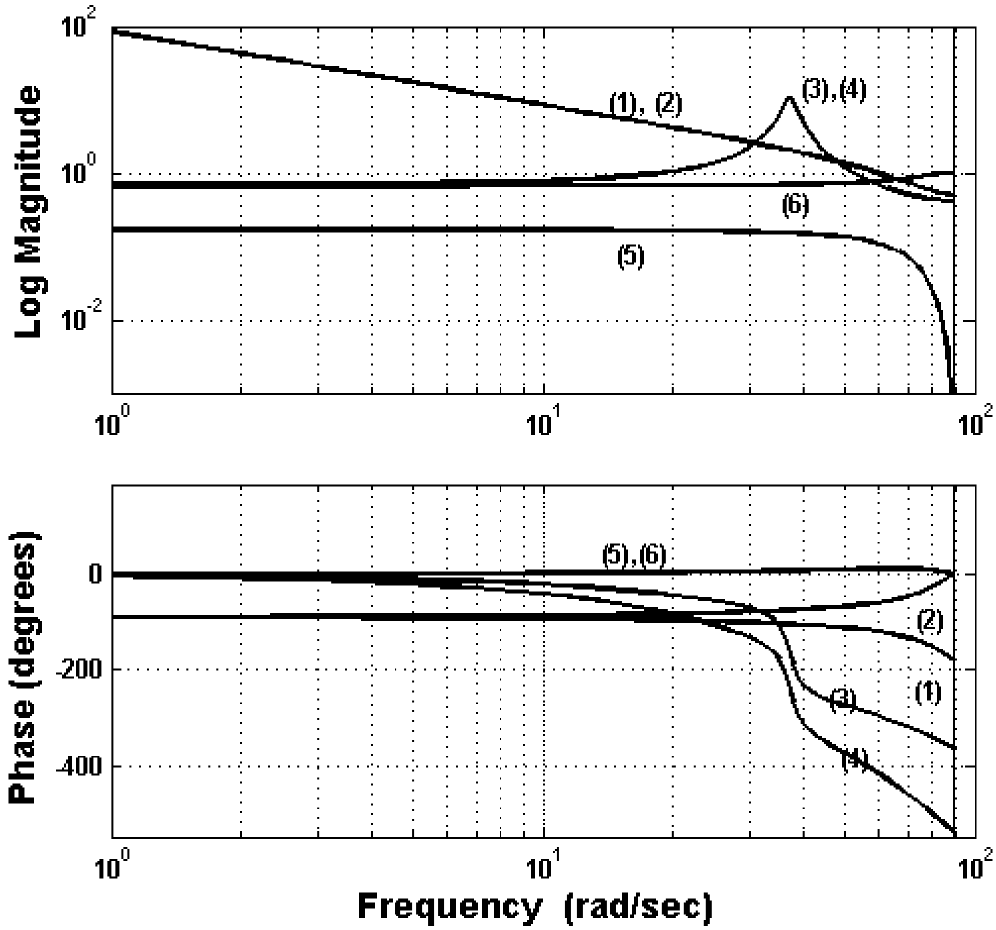
Both (3) and (4) have the same magnitudes, since they differ only by the factor z, and $|z| = 1$ when $z = e^{j\theta}$. Both have poles at $z = re^{\pm j\phi}$, with $r^2 = 0.9$, so $r = 0.95$, ie very near the unit circle. Therefore the gain will have a sharp peak near frequency ϕ. Both phases decrease rapidly near this resonant frequency ϕ — consider the vector $e^{j\theta} - re^{j\phi}$: it flips by nearly 180°as θ changes from just below ϕ to just above. The extra factor of z^{-1} in (4) gives an extra phase lag of 180°at $\theta = \pi$. Estimating ϕ (just as a check): $(z - re^{j\phi})(z - re^{-j\phi}) = z^2 - 2r\cos\phi + r^2$, so $2\cos\phi = 0.5/0.95$, hence $\phi = 1.3$ rad, but $\phi = \omega T$ and $T = 0.035$ sec, so $\omega = 1.3/.035 = 37.3$ rad/sec, which agrees with the location of the resonant peak in Fig.1.

There are only 5 phase plots in Fig.1, but 6 transfer functions. So two of them must have the same phase characteristic. The reasoning above rules out (1)–(4), which have already been identified, so that leaves (5) and (6). We have $G_5(1) = 1/6$ and $G_6(1) = 2/3$, so they have the same phase (0) at frequency 0 — this doesn't prove that they are the same at all frequencies, but is easy to check. We also have that $G_5(-1) = 0$, so the gain of (5) goes down to $-\infty$ on the log scale at high frequencies. This identifies which gain plots belong to (5) and (6), but as an additional check note that $G_3(1) = G_4(1) \approx G_6(1)$. The magnitude and phase characteristics can be understood in these and the other cases from a pole-zero diagram:



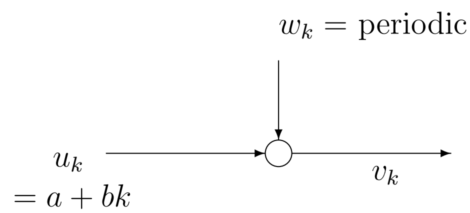
(To see analytically that the phases of (5) and (6) are the same, note that the quotient of the two transfer functions is:

$$\frac{(e^{j\theta}+1)^2}{(e^{j\theta}+3)(3e^{j\theta}+1)} = \frac{e^{2j\theta}+2e^{j\theta}+1}{3e^{2j\theta}+10e^{j\theta}+3} = \frac{\cos(\theta)+1}{3\cos(\theta)+5} \qquad \text{which is real.)}$$



Students should be encouraged to generate Bode plots in this and other examples using Matlab Control Toolbox functions `bode` *or* `ltiview` *to check results, so long as they can deploy arguments such as those above.*

8.



(a)

$$
\begin{aligned}
W(z) &= (w_0 + w_1 z^{-1} + w_2 z^{-2} + w_3 z^{-3}) \\
&\quad + z^{-4}(w_0 + w_1 z^{-1} + w_2 z^{-2} + w_3 z^{-3}) \\
&\quad \vdots \\
&\quad + z^{-4k}(w_0 + w_1 z^{-1} + w_2 z^{-2} + w_3 z^{-3}) \\
&\quad \vdots \\
&= (1 + z^{-4} + z^{-8} + \ldots)(w_0 + w_1 z^{-1} + w_2 z^{-2} + w_3 z^{-3}) \\
&= \frac{1}{1 - z^{-4}}(w_0 + w_1 z^{-1} + w_2 z^{-2} + w_3 z^{-3})
\end{aligned}
$$

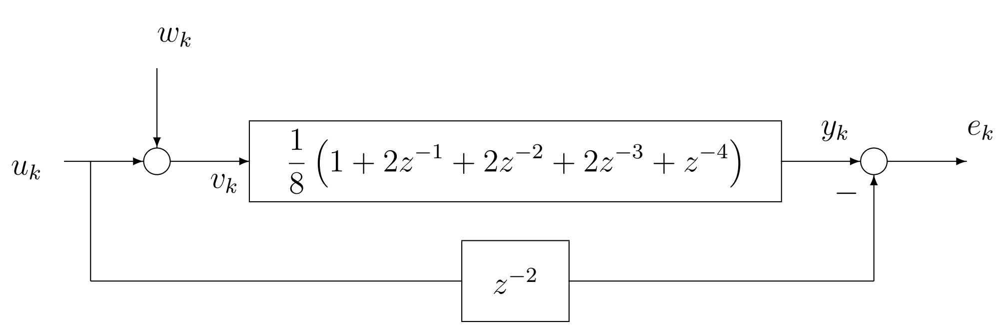
But $w_0 + w_1 + w_2 + w_3 = 0$ so

$$w_0 + w_1 z^{-1} + w_2 z^{-2} + w_3 z^{-3} = (1 - z^{-1})(w_0 + (w_0 + w_1)z^{-1} - w_3 z^{-2})$$

and $(1 - z^{-4}) = (1 - z^{-1})(1 + z^{-1})(1 + z^{-2})$
hence

$$W(z) = \frac{w_0 + (w_0 + w_1)z^{-1} - w_3 z^{-2}}{(1 + z^{-1})(1 + z^{-2})}$$

(b)



(i)

$$
\begin{aligned}
G(z) &= \frac{1}{8}\left(1+2z^{-1}+2z^{-2}+2z^{-3}+z^{-4}\right)\\
&= \frac{1}{8}\left(1+z^{-1}\right)\left(1+z^{-1}+z^{-2}+z^{-3}\right)\\
&= \frac{1}{8}\left(1+z^{-1}\right)^2\left(1+z^{-2}\right) \text{ which cancel poles of } W(z)
\end{aligned}
$$

(ii)

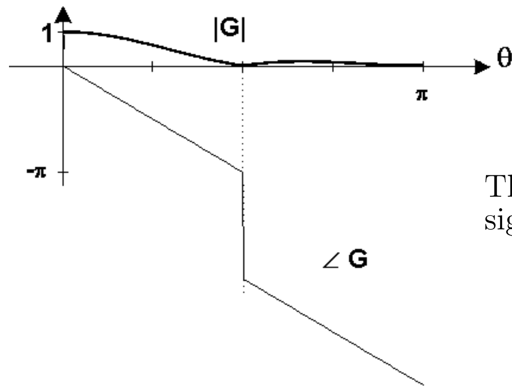
$$
\begin{aligned}
G(z)-z^{-2} &= \frac{1}{8}\left(1+2z^{-1}-6z^{-2}+2z^{-3}+z^{-4}\right)\\
&= \frac{1}{8}\left(1-z^{-1}\right)\left(1+3z^{-1}-3z^{-2}-z^{-3}\right)\\
&= \frac{1}{8}\left(1-z^{-1}\right)^2\left(1+4z^{-1}+z^{-2}\right)\\
U(z) &= Z(a+bk) = \frac{a}{1-z^{-1}}+\frac{bz^{-1}}{(1-z^{-1})^2}\\
& \quad \text{and hence zeros of } \left(G(z)-z^{-2}\right) \text{ cancel poles of } U(z).
\end{aligned}
$$

(iii)

$$
\begin{aligned}
E(z) &= -z^{-2}U(z)+G(z)\left(W(z)+U(z)\right)\\
&= \left(G(z)-z^{-2}\right)U(z)+G(z)W(z)\\
&= \frac{1}{8}\left(1+4z^{-1}+z^{-2}\right)\left(a(1-z^{-1})+bz^{-1}\right)\\
& \quad +\frac{1}{8}\left(1+z^{-1}\right)\left(w_0+(w_0+w_1)z^{-1}-w_3z^{-2}\right)\\
& \quad \text{which is a polynomial in } z^{-1} \text{ of degree 3}\\
& \quad \text{hence } e_k=0 \text{ for } k\geq 4\\
&\Rightarrow y_k=e_k+u_{k-2}=u_{k-2} \text{ for } k\geq 4\\
& \quad \text{and } y_k \text{ extracts the trend with a delay of 2 samples.}
\end{aligned}
$$

(iv)

$$
\begin{aligned}
G(e^{j\theta}) &= \frac{1}{8}\left(1+e^{-j\theta}\right)^2\left(1+e^{-2j\theta}\right)\\
&= \frac{1}{8}e^{-2j\theta}\left(e^{j\theta/2}+e^{-j\theta/2}\right)^2\left(e^{j\theta}+e^{-j\theta}\right)\\
&= \cos^2(\theta/2)\cos\theta e^{-2j\theta}\\
&= 0 \text{ for } \theta=\pi/2 \text{ and } \theta=\pi.
\end{aligned}
$$



The zeros reject the period-4 periodic signal $\{w_k\}$ and its 2nd harmonic.

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