# 3F1, Signals and Systems

## PART IV

## Continuous/discrete interfaces

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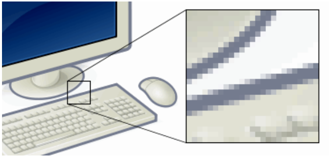
Goal of the lecture:

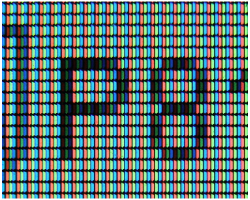
Digital to analog and analog to digital conversions

What sampling interval?

How many pixels do you need to “record” an image?
Example: a digital camera is a “sampler” (analog to digital)



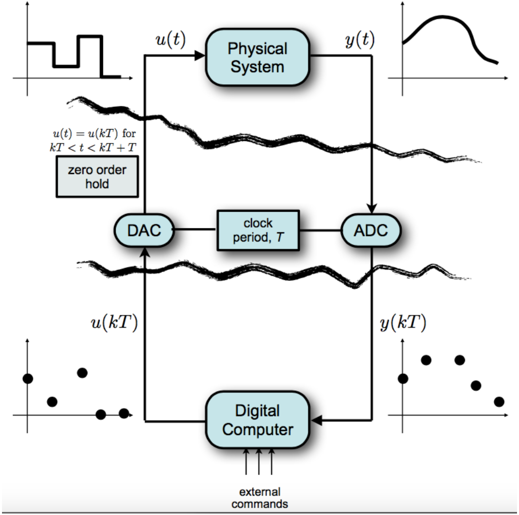
How many pixel do you need to “see” an image?
Example: a monitor is a “reconstructor” (digital to analog)



Do your eyes sample?
(about 100 mIllion rods and 6 millions cones on your retina...)

# Module A - Interfaces

Filtering: digital processing of continuous signals.
Control: closed loop of continuous process and digital controller.



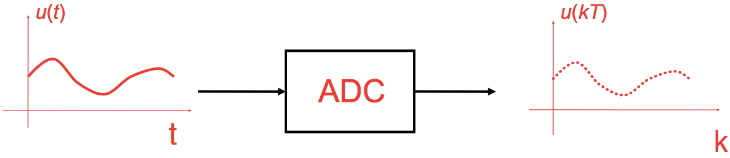

- What kind of ADC/DAC?
- What is the effect of the conversion?
- What sample time?

## Analog-to-digital converter (ADC)

Takes a continuous time signal $u(t)$, which is assumed to be continuous, and sample it to produce the number sequence $u(kT)$.

$T$ is the sampling time.

ADC also termed *sampler*

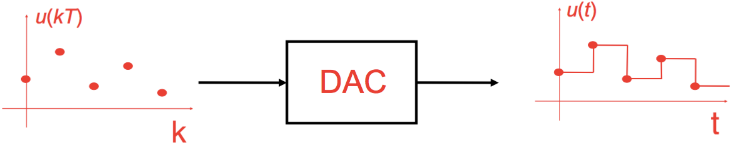


## Digital-to-analog converter (DAC)

Take the number sequence $u(kT)$ and produces a continuous time signal u(t).

*Zero order hold:*

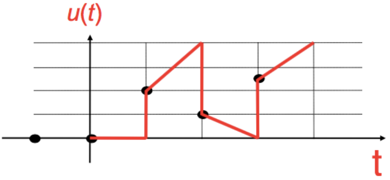
$$u(t) = u(kT) \qquad kT \leq t < (k+1)T$$



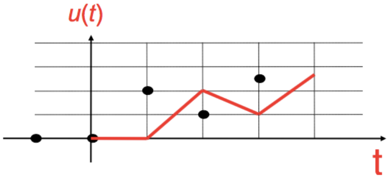
# Digital-to-analog converter (DAC)

*First order hold:*

Linear extrapolation through the last two discrete inputs

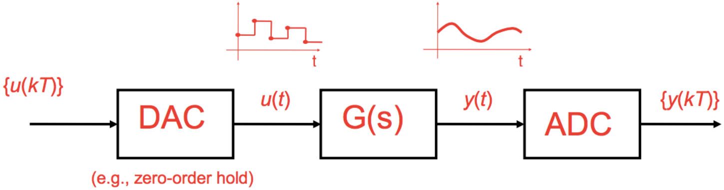



Linear interpolation of last two discrete inputs



## Transfer function analysis of DAC and ADC interfaces

Hybrid: G(s) linear continuous system,
discrete input/output - before DAC/after ADC




- What is the transfer function $G(z)$ from $u$ to $y$?
  (does the transfer function exist?)

## Transfer function analysis of DAC and ADC interfaces


It is **linear** and **time-invariant** thus it has a $z$-transfer function.

How to find it? Take any input, find the output, take the ratio of the $z$ transform. Take $u(kT) = 1$ for all $k \geq 0$.

$$\begin{aligned} &\Rightarrow \quad u(t) = 1 \qquad \forall t \geq 0 \\ &\Rightarrow \quad Y(s) = G(s)\frac{1}{s} \\ &\Rightarrow \quad y(kT) = \mathcal{L}^{-1}\left(\frac{G(s)}{s}\right)_{t=kT\geq 0} \end{aligned}$$

Since $\mathcal{Z}(u(kT)) = \frac{1}{1-z^{-1}}$ we get

$$G(z) = \frac{z-1}{z}\mathcal{Z}\left(\mathcal{L}^{-1}\left(\frac{G(s)}{s}\right)_{t=kT\geq 0}\right)$$

(like step invariant approximation of $G(s)$...)

## Transfer function analysis of DAC and ADC interfaces


$$G(s) = \frac{1}{s+1} \quad \text{(example)}$$

take any input

$$u(t) = 1 \;\Rightarrow\; U(s) = \frac{1}{s}$$

find the output

$$Y(s) = \frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1} \quad \Rightarrow \quad y(kT) = \left(1 - e^{-t}\right)_{t=kT\geq 0}$$

take the ratio of $z$ transforms

$$G(z) = Y(z)/U(z) = \frac{z-1}{z}\mathcal{Z}\left(1 - e^{-kT}\right)$$

$$= \frac{z-1}{z}\left(\frac{z}{z-1} - \frac{z}{z-e^{-T}}\right) = 1 - \frac{z-1}{z-e^{-T}}$$

# Module B - Sampling: frequency analysis

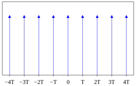
(Recall from IB Paper 6)

Impulse response

$$g(t)$$

Sampling

$$g_s(t) = g(t) \underbrace{\sum_{n=-\infty}^{\infty} \delta(t - nT)}_{\delta_p(t)} = g(t) \underbrace{\frac{1}{T} \sum_{n=-\infty}^{\infty} e^{jn\omega_0 t}}_{\text{Fourier series of } \delta_p} \qquad \omega_0 = \frac{2\pi}{T}$$




For instance,

$$\delta_p(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T}$$

$$c_n = \frac{1}{T} \int_{-T/2}^{T/2} \delta_p(t) e^{-jn\omega_0 t} dt = \frac{1}{T} \quad \text{for all } n$$

(Recall from IB Paper 6)

Impulse response

$$g(t)$$

Fourier transform

$$G(j\omega) = \int_{-\infty}^{\infty} g(t) e^{-j\omega t}$$

Sampling

$$g_s(t) = g(t)\ \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{jn\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T}$$

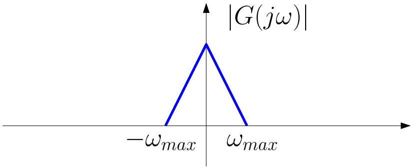
Fourier transform

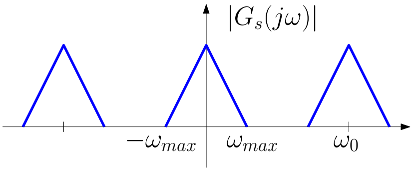
$$G_s(j\omega) = \int_{-\infty}^{\infty} g_s(t) e^{-j\omega t} = \int_{-\infty}^{\infty} g(t) \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{jn\omega_0 t} e^{-j\omega t}$$

$$= \frac{1}{T} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} g(t) e^{-j(\omega - n\omega_0)t} = \frac{1}{T} \sum_{n=-\infty}^{\infty} G(j(\omega - n\omega_0))$$

(Recall from IB Paper 6)

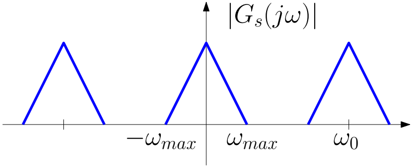
$$G_s(j\omega) = \frac{1}{T} \sum_{n=-\infty}^{\infty} G(j(\omega - n\omega_0))$$



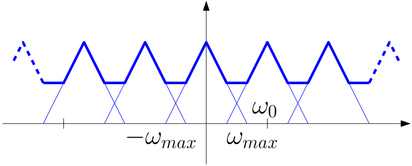




periodicity of the sampled signal spectrum

# Module C - Aliasing

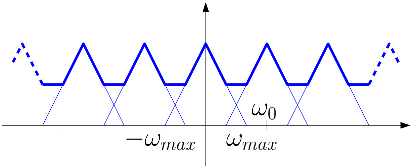



$$\omega_{max} < \frac{\omega_0}{2} = \frac{\pi}{T} \text{ , otherwise aliasing}$$

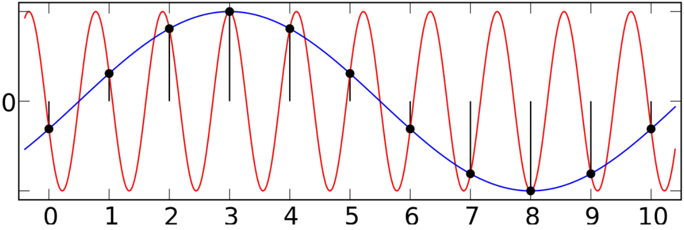



- Can reconstruct an analogue signal perfectly from its digital samples provided its bandwidth is less than $\frac{\pi}{T}$.
- For signals with bandwidth $\omega_{max} > \frac{\pi}{T}$ we must pre-filter (ideal lowpass filter, max cut-off $\frac{\pi}{T}$).
- Take the sampling time $T < \frac{\pi}{\omega_{\max}}$ ($\frac{1}{\omega_{\max}}$ if Hz) for signals with finite bandwidth $\omega_{max}$.

Aliasing in frequency




Aliasing in time




Identical sampling of the two sinusoids for $T = 1$

Examples:

▸ Video aliasing: airplane propeller and digital cameras

▸ Audio aliasing: effects of undersampling

▸ Using aliasing to analyze fast motion: spring modes

▸ Using aliasing to analyze fast motion: fan

▸ Using aliasing to analyze fast motion: pulses of water

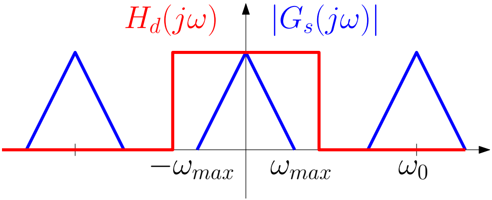
# Module D - Reconstruction

## Shannon theorem

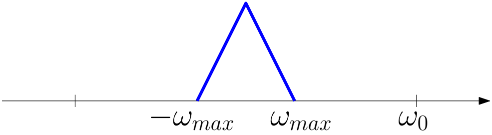
A continuous-time signal $g(t)$ width bandwidth $\omega_{max}$ can be reconstructed exactly from its sample version $g_s(t)$ if the sampling time satisfies $\omega_{max} < \omega_0/2 = \pi/T$.

The continuous-time signal can be computed from the sampled signal by the interpolation formula

$$g(t) = \sum_{n=-\infty}^{\infty} g(nT) \frac{\sin \frac{\pi}{T}(t - nT)}{\frac{\pi}{T}(t - nT)}$$




$$G(j\omega) = H_d(j\omega)G_s(j\omega)$$



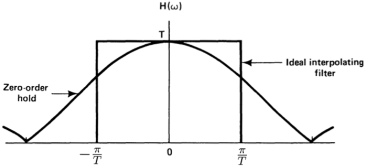

$$g(t) = g_s(t) * \frac{\sin(\frac{\pi}{T}t)}{\frac{\pi}{T}t}$$

Shannon/ideal reconstruction

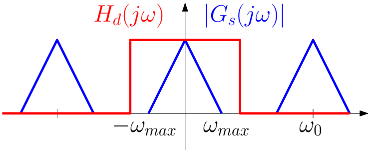
$$
\begin{aligned}
g(t) &= g_s(t) * \frac{\sin(\frac{\pi}{T}t)}{\frac{\pi}{T}t} \\
&= \int_{-\infty}^{\infty} g(\tau) \sum_{n=-\infty}^{\infty} \delta(\tau - nT) \frac{\sin(\frac{\pi}{T}(t-\tau))}{\frac{\pi}{T}(t-\tau)} \\
&= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} g(\tau)\delta(\tau - nT) \frac{\sin(\frac{\pi}{T}(t-\tau))}{\frac{\pi}{T}(t-\tau)} \\
&= \sum_{n=-\infty}^{\infty} g(nT) \frac{\sin\frac{\pi}{T}(t-nT)}{\frac{\pi}{T}(t-nT)}
\end{aligned}
$$

non causal ...

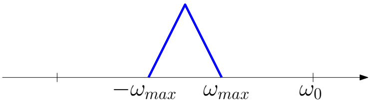
## Ideal and ZOH reconstruction




$$ZOH(s) = \frac{1 - e^{-sT}}{s}$$



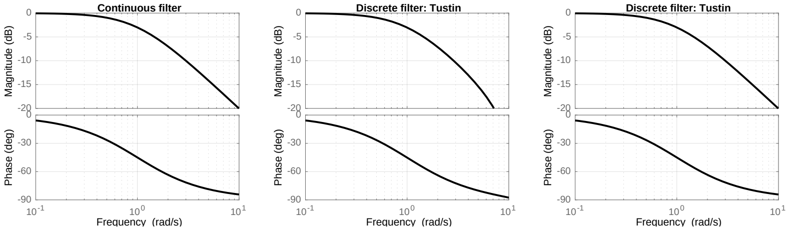

$$G(j\omega) = H_d(j\omega)G_s(j\omega)$$



# Module E - Choice of the sampling period

Frequency distortion of discretized filters.

- Continuous
- Sampling $T = 0.25$ ($\omega_{max} = \frac{\pi}{T} = 4\pi$ rad/s).
- Sampling $T = 0.01$ ($\omega_{max} = \frac{\pi}{T} = 100\pi$ rad/s).

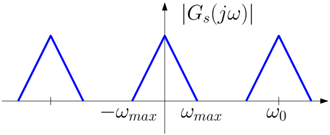



Distortion reduces as $T \to 0$.

Rule of thumb: $\frac{\pi}{T} > 10$(process frequency)

## Example: audio signal

We can hear sounds with frequency components between 20 Hz and 20 kHz. What sampling period?




rad/s:

$$2\omega_{max} < \omega_0 = 2\frac{\pi}{T} \quad \Rightarrow \quad \omega_{max} < \frac{\pi}{T}$$

Hz:

$$2\omega_{max} < \omega_0 = \frac{1}{T} \quad \Rightarrow \quad 2\omega_{max} < \frac{1}{T}$$

Audio: 20 KHz thus $\omega_0 > 2\omega_{max} = 40kHz$, that is,

$$T = \frac{1}{2\omega_{max}} = \frac{1}{40kHz} = 0.000025s = 25\mu s$$

Min sampling rate music (cd, streaming): 44.1 kHz (transition band...)