

Module 3F1 – Signals and Systems

Examples Paper 3F1/1

The z -transform table in the Electrical and Information Data Book will be helpful.

1. Find from first principles the z -transforms of the sequences obtained by sampling, (with uniform sampling period T), the continuous-time waveforms whose Laplace transforms are

$$(i) \frac{1}{s} ; (ii) \frac{1}{(s+a)} ; (iii) \frac{1}{(s+a)(s+b)} ; (iv) \frac{s+a}{(s+a)^2 + b^2} .$$

Note that the first sample is taken at $t = 0^+$.

2. A sequence is given by $u_k = 1$ for $k = 0, 1$ and zero otherwise. Find the z -transform of this sequence and hence solve the following difference equations for y_k , using the z -transform technique.

(i) $y_k = u_k + u_{k-1} + u_{k-2}$, (FIR filter).

(ii) $y_k = 0.8 y_{k-1} + 0.2 u_k$, $y_{-1} = 0$. (exponential smoother)

(iii) $y_k = 0.98 y_{k-1} - 0.9604 y_{k-2} + u_k$, $y_{-1} = 0$, $y_{-2} = 1$. (IIR filter).

3. For the three difference equations of question 2, write down the z -plane transfer function relating $\{y_k\}$ to $\{u_k\}$.

(a) Evaluate their poles and zeros, and calculate and sketch the response to a unit step on u_k . (assume $y_k = 0$, $k < 0$).

(b) Assuming the time between samples is T , calculate and sketch the steady state response to $u_k = \cos \omega k T$ for $\omega T = 0, \pi/3, \pi$.

4. Supply and demand cycles. In agriculture a one season prediction of price is required by the farmer to determine how much product to produce. The price at harvesting will then depend on the supply via the demand curve. Assume that the demand at time k is given by

$$d_k = d_e - ap_k$$

where p_k is the price at time k . Now assume that the supply is given by

$$s_k = s_e + b\hat{p}_k ,$$

where \hat{p}_k is the predicted price at time k (note: $a, b > 0$, d_e and s_e are constants). The price then adjusts to equate supply and demand at time k (i.e. $s_k = d_k$). Let $c = b/a$ and determine conditions on c for the stability of this system (i.e. the stability of the difference equation determining p_k), for

- (i) $\hat{p}_k = p_{k-1}$ and
- (ii) $\hat{p}_k = 2p_{k-1} - p_{k-2}$ (i.e. a linear extrapolation through the last two prices).

5. Numerical solutions of differential equations

- (a) Euler's method for solving the differential equation

$$\frac{dx(t)}{dt} = f(x(t)) \tag{1}$$

is to make the approximation,

$$x((k+1)T) \cong x(kT) + Tf(x(kT)) \tag{2}$$

for $k = 0, 1, 2, \dots$, where T is the step length. Assuming $f(x) = ax$ for $a < 0$, what range of values of T in (2) will ensure that $x(kT) \rightarrow 0$ as $k \rightarrow \infty$?

- (b) Euler's method is inaccurate unless T is very small and an alternative is to consider higher-order extrapolation based on $x(kT)$ and $x((k-1)T)$ and $\dot{x}(kT) = f(x(kT))$. The function

$$g(t) = \{c - b + mT\} \frac{t^2}{T^2} + mt + b$$

is the quadratic function that satisfies

$$g(0) = b, \quad \dot{g}(0) = m, \quad g(-T) = c$$

Now if we let

$$b = x(kT), \quad m = f(x(kT)), \quad c = x((k-1)T)$$

then $x((k+1)T) = g(T) = x((k-1)T) + 2Tf(x(kT))$ is an extrapolation of the next value of x based on this quadratic approximation. If $f(x) = ax$ with $a < 0$ show that the method would be unstable for any $T > 0$ (!). What would be the nature of the instability?

6. A discrete time system has impulse response $\{g_k\}$ and transfer function $G(z)$.
- Show that $\sum_{k=0}^{\infty} |g_k|$ is finite if all the poles of $G(z)$ lie strictly inside the unit circle.
 - Suppose $\sum_{k=0}^{\infty} |g_k|$ is infinite. Explain how the output can be made arbitrarily large at some time instant, using an input of plus and minus ones.
 - Let $G(z) = 1/(z^2 - \sqrt{2}z + 1)$. Find a bounded input which gives an unbounded output.
7. For a sampling period of 35 milliseconds the Bode plots of the following transfer functions were plotted (see figure 1). As usual, however, it was forgotten to label the graphs. Can you help?

$$(1) \frac{z+2}{z-1}$$

$$(2) \frac{2z+1}{z-1}$$

$$(3) \frac{1}{z^2 - 0.5z + 0.9}$$

$$(4) \frac{1}{z(z^2 - 0.5z + 0.9)}$$

$$(5) \frac{(z+1)^2}{4(z+3)(z+0.5)}$$

$$(6) \frac{3z+1}{4z+2}$$

8. An economic indicator is measured once every quarter and it is desired to estimate the underlying trend in the face of seasonal fluctuations. Assume that the indicator at the k^{th} quarter, v_k , is related to the underlying trend, u_k , by

$$v_k = u_k + w_k, \quad k \geq 0,$$

where $u_k = a + bk$, and the seasonal variation, w_k , is a periodic function with period 4 and zero mean value, (i.e. $w_{4k+i} = w_i$ and $w_0 + w_1 + w_2 + w_3 = 0$).

- (a) Show that

$$\begin{aligned} W(z) = \mathcal{Z}(\{w_k\}) &= \frac{w_0 + w_1 z^{-1} + w_2 z^{-2} + w_3 z^{-3}}{1 - z^{-4}} \\ &= \frac{w_0 + (w_0 + w_1)z^{-1} - w_3 z^{-2}}{(1 + z^{-1})(1 + z^{-2})} \end{aligned}$$

- (b) The underlying trend is estimated by passing $\{v_k\}$ through a FIR filter giving

$$y_k = \frac{1}{8} \{v_k + 2v_{k-1} + 2v_{k-2} + 2v_{k-3} + v_{k-4}\}$$

with error given by

$$e_k = y_k - u_{k-2}$$

(i.e. y_k is supposed to be an estimate of u_{k-2}).

- (i) Find the transfer function, $G(z)$, of this filter, and its zeros, and show that the poles of $W(z)$ are cancelled by zeros of $G(z)$.
- (ii) Show that $(G(z) - z^{-2}) = \frac{1}{8}(1 - z^{-1})^2(1 + 4z^{-1} + z^{-2})$, and hence that its zeros cancel the poles of $U(z)$.
- (iii) Hence show that the error, e_k , will be zero for $k \geq 4$, and hence this filter accurately extracts the trend under these assumptions.
- (iv) Show that

$$G(e^{j\theta}) = \cos \theta \cos^2(\theta/2) e^{-j2\theta}$$

and sketch the Bode diagram. Comment on the frequencies at which $G(e^{j\theta}) = 0$.

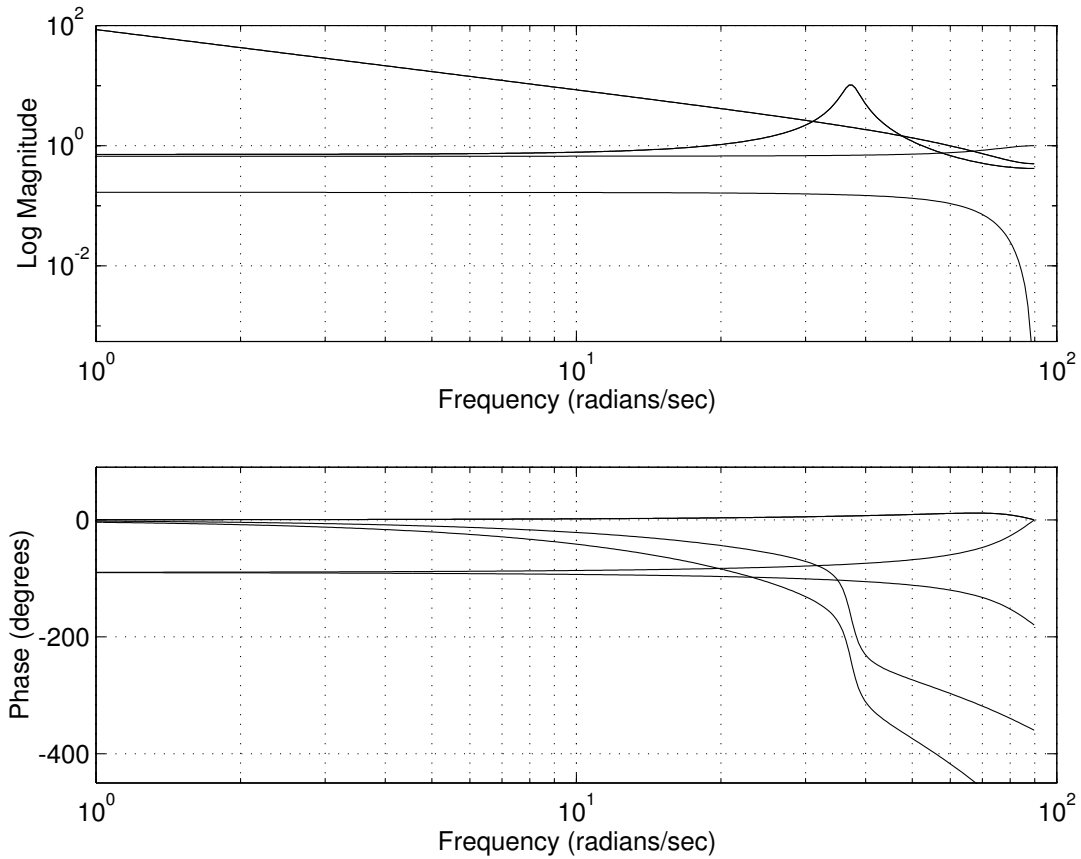


Figure 1: Bode plots for Question 7

Answers:

1. $1/(1 - z^{-1}), 1/(1 - e^{-aT}z^{-1}),$

$$\frac{z^{-1}(e^{-aT} - e^{-bT})}{(b - a)(1 - e^{-aT}z^{-1})(1 - e^{-bT}z^{-1})}, \frac{(1 - e^{-aT} \cos(bT)z^{-1})}{(1 - 2e^{-aT} \cos(bT).z^{-1} + e^{-2aT}z^{-2})}$$

2. $\mathcal{Z}\{u_k\} = 1 + z^{-1}$. (i) $1, 2, 2, 1, 0, 0, \dots$

(ii) $0.2, 0.36, \dots, 0.36(0.8)^{k-1}$

(iii) $(0.98)^{k-1}\{0.0448 \sin[(k+1)\pi/3] + 1.1547 \sin[k\pi/3]\}$.

3. (a) Poles and zeros:

	Poles	Zeros
(i)	0, 0	$-\frac{1}{2} \pm j\sqrt{3}/2$
(ii)	0.8	0
(iii)	$0.98e^{\pm j\pi/3}$	0, 0

Step responses:

(i) $1, 2, 3, 3, 3, \dots$

(ii) $1 - (0.8)^{k+1}$

(iii) $1.02 + (0.98)^k\{-0.02 \cos(k\pi/3) + 1.1427 \sin(k\pi/3)\}$

(b)

$\omega T = 0$: (i) 3; (ii) 1; (iii) 1.02

$\omega T = \pi$: $A(-1)^k$ for (i) $A = 1$, (ii) $A = 1/9$, (iii) $A = 0.34$

$\omega T = \pi/3$: $A \cos(k\pi/3 + \theta)$ for (i) $(A, \theta) = (2, -\pi/3)$,
(ii) $(A, \theta) = (0.218, -0.857)$, (iii) $(A, \theta) = (29.16, -0.518)$.

4. $c < 1, c < 1/3$.

8. (b)(i) $G(z) = \frac{1}{8}(1 + z^{-1})^2(1 + z^{-2})$. Zeros at $-1, \pm j$.