3F1 Signals and Systems

(1) Introduction and recap of continuous time systems and the Laplace transform

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Introduction

This course is about:

- **Signals**
- **Systems**

in **discrete** and **continuous** time. You have met continuous time signals and systems before. Why should we consider **discrete time systems**?

Motivation

Most applications in signal processing and control use digital systems (embedded or software). This involves sampling, so we need to be able to handle discrete time signals and systems. Examples include: digital audio, image processing, speech synthesis. Furthermore, many real-world processes are *discrete time in nature*.

Example: numerical solution of an ODE (Euler Method)

Consider an ODE:

$$\frac{dy}{dt} = f(y,t)$$

Define a fixed timestep, δt. Then (ignoring initial conditions), we can derive an approximate numerical solution of the ODE as follows:

$$y(t+\delta t) = y(t) + \delta t \times f(y(t),t)$$

Example: mortgage with repayment (compound interest with amortization)

$$\begin{aligned} \text{Let } y(k) &= \text{amount at beginning of year} k \\ r &= \text{interest rate} \\ b(k) &= \text{amount deposited or withdrawn at end of year } k. \end{aligned}$$

Then

$$\begin{aligned} y(k+1) &= y(k) + ry(k) + b(k) \\ y(k+1) &= (1+r)y(k) + b(k) \end{aligned}$$

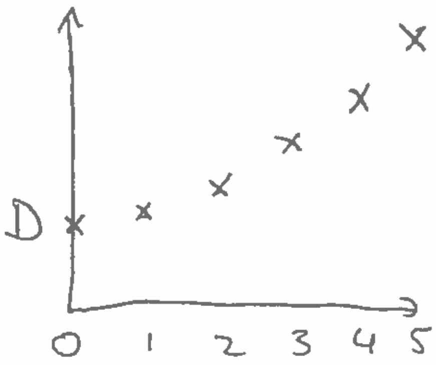
These are examples of **difference equations**. In this course we will develop methods to analyze and solve these equations, much like the methods we have for continuous time systems represented by ordinary differential equations.

Suppose $b(k) = b$ (constant) and let $y(0) = D$ (initial deposit/loan). Then we have:

$$\begin{aligned} y(1) &= (1+r)D + b \\ y(2) &= (1+r)[(1+r)D + b] + b = (1+r)^2 D + (1+r)b + b \\ y(3) &= (1+r)^3 D + (1+r)^2 b + (1+r)b + b \\ &\vdots \\ y(k) &= (1+r)^k D + \sum_{i=0}^{k-1} (1+r)^i b \end{aligned}$$

Behaviour over time:

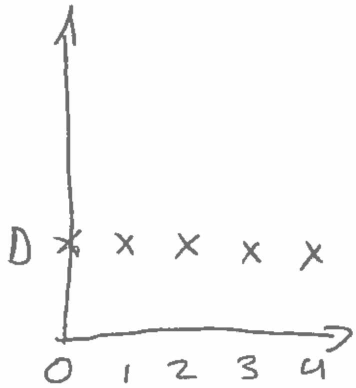
Case: $r > 0$



Unstable

$(1+r) > 1$

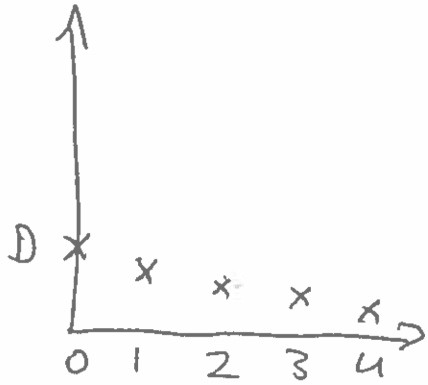
Case: $r = 0$



Marginally stable

$(1+r) = 1$

Case: $r < 0$



Stable

$(1+r) < 1$

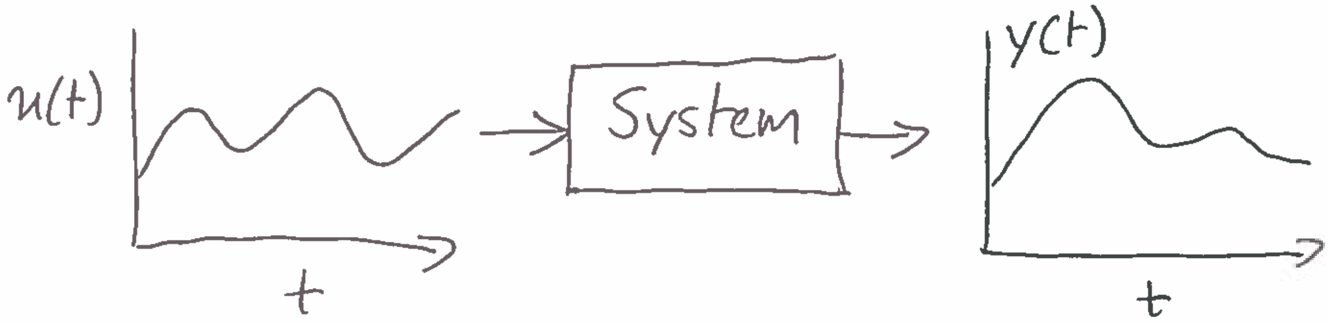
Question: is the Euler Method stable?

There are pros and cons to using discrete time methods in applications. Some of these include:

advantages	disadvantages
-Algorithms can be implemented on any hardware -Flexible, easy to modify	-Reliability can be difficult to guarantee! -Need ADC/DAC hardware -Finite sampling rate

Revision of continuous time signals and systems

A **signal** is real-valued function $x(t)$, $0 \leq t < \infty$. A **system** turns input signals, $u(t)$ into outputs, $y(t)$.



Signals may be smooth and/or continuous. One way to represent a (smooth) system is as a linear ODE.

Example:

$$\ddot{y}(t) + a_1\dot{y}(t) + a_2 y(t) = b_0\dot{u}(t) + b_1 u(t)$$

Recall the **Laplace transform** for a continuous time signal $x(t)$

$$\bar{x}(s) = \int_{0^+}^{\infty} x(t)e^{-st}\,\mathrm{d}t$$

Applying the Laplace Transform to the system above (assuming zero initial conditions) gives:

$$s^2\bar{y}(s) + a_1 s\bar{y}(s) + a_2\bar{y}(s) = b_0 s\bar{u}(s) + b_1\bar{u}(s)$$

Rearranging:

$$\frac{\bar{y}(s)}{\bar{u}(s)} = \frac{b_0 s + b_1}{s^2 + a_1 s + a_2} := G(s)$$

$G(s)$ is called the **Transfer Function** of the system.

We can also represent the system using a **convolution representation**, using the system's **impulse response**, $g(t)$:

$$y(t) = \int_0^{\infty} g(t-\tau)u(\tau)\,\mathrm{d}\tau$$

Taking Laplace transforms gives:

$$\bar{y}(s) = \bar{g}(s)\bar{u}(s)$$

Thus, $G(s) = \bar{g}(s)$. [Note: we are assuming $u(t), g(t) \equiv 0$ for $t < 0$]

A system, L, is **Linear time-invariant** (LTI), iff the following conditions both hold:

Linear: if $L(u_1(t)) = y_1(t)$ and $L(u_2(t)) = y_2(t)$, then for any scalars α_1, α_2:

$$L(\alpha_1 u_1(t) + \alpha_2 u_2(t)) = \alpha_1 y_1(t) + \alpha_2 y_2(t))$$

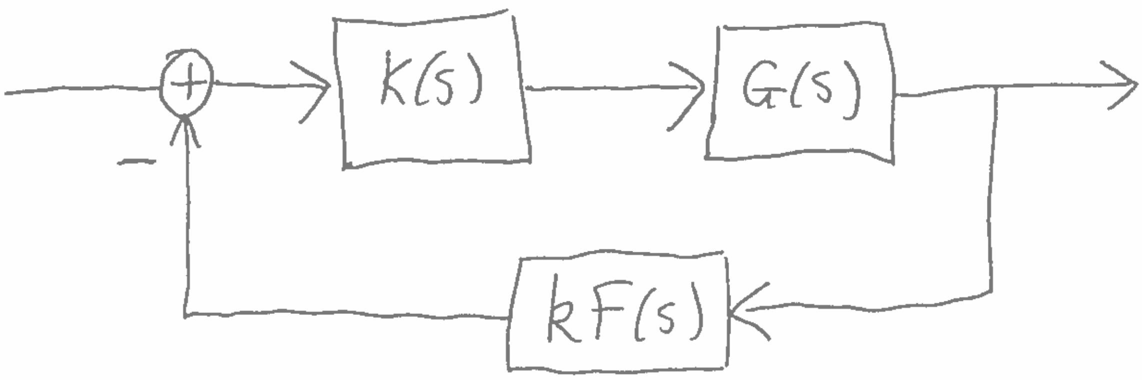
Time-invariant: if $L(u(t)) = y(t)$ then $L(u(t+T)) = y(t+T)$ for any time interval, T. In other words, if the system's response properties do not change over time.

A system is **stable** if "bounded inputs give bounded outputs." For a system with a rational transfer function, $G(s)$, this means G has *no poles in the right-half complex plane or imaginary axis.*

For a stable LTI system, $G(s)$, the **steady-state response** for an input $u(t) = \sin(\omega t)$ is:

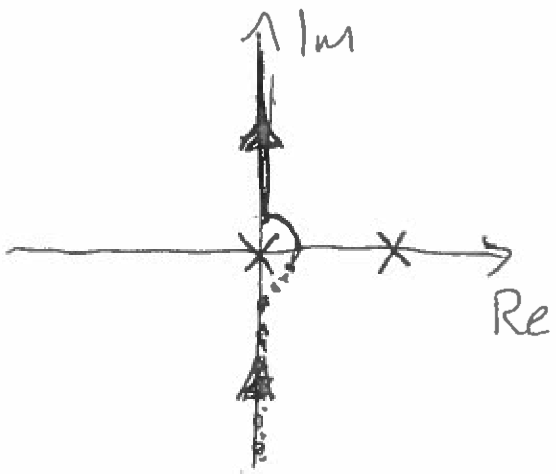
$$y_{ss}(t) = |G(j\omega)| \sin(\omega t + \angle(G(j\omega)))$$

The **Nyquist Stability Criterion** allows us to determine the stability of a closed loop system by analysing its open loop properties:

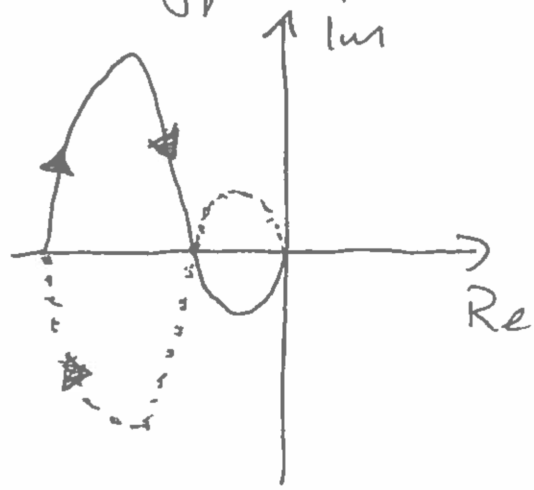


- Plot locus of $FGK(s)$ as s moves upwards along the imaginary axis, that is, the locus of the quantity $F(j\omega)G(j\omega)K(j\omega)$, $-\infty < \omega < \infty$.
- Let N = number of anti-clockwise encirclements of the point $-1/k$
- Then the closed loop system is stable if and only if: N = # of RHP poles of $F(s)G(s)K(s)$.

Example $\quad G(s) = \dfrac{s+1}{s(s-1)}$, $K(s) = K$, $F(s) = 1$



Nyquist plot



- There is one RHP pole at 1 (open loop)
- There is one anticlockwise encirclement of $-1/k$ for $1 < k < \infty$
- Thus the closed loop system is stable $\Longleftrightarrow \quad 1 < k < \infty$.

Next steps: discrete time signals and systems

Our goal is to take the theory and machinery developed for continuous time systems and develop discrete-time analogs.

A **discrete time signal** is a number sequence, indexed from zero:

$$(x_0, x_1, x_2, \ldots) \text{ or } (x(0), x(T), x(2T), \ldots)$$

where T is the **sampling period**. We normally assume the sequence is identically zero or undefined for index values less than zero. A standard notation is:

$$\{x_k\}_{k\geq 0} \text{ or } \{x(kT)\}_{k\geq 0}$$

A **discrete time system** takes discrete time signal inputs and produces discrete time signal outputs (as in the case for continuous time systems):

For continuous time signals, $x(t)$, the Laplace transform is defined as:

$$\mathcal{L}[x(t)] = \bar{x}(s) = \int_{0^+}^{\infty} x(t)e^{-st}\,\mathrm{d}t$$

By analogy we can find a discrete-time equivalent.

Let $\delta(t)$ be the Dirac delta function (recall definition!) For a discrete time signal $\{x(kT)\}_{k\geq 0}$, heuristically define a continuous time equivalent:

$$x^*(t) = \sum_{k=0}^{\infty} x(k)\delta(t - kT)$$

Now take the Laplace transform of x^*:

$$\begin{aligned}
\mathcal{L}[x^*(t)] &= \int_0^{\infty} \sum_{k=0}^{\infty} x(k)\delta(t - kT)e^{-st}\,\mathrm{d}t \\
&= \sum_{k=0}^{\infty} x(kT)e^{-skT} = \sum_{k=0}^{\infty} x(k)z^{-k} = \mathcal{Z}\{x_k\}
\end{aligned}$$

where $z = e^{sT}$. $\mathcal{Z}\{x_k\}$ is called the **z-transform**.