Engineering Tripos Part IIA THIRD YEAR

Module 3F1 – Signals and Systems

Examples Paper 3F1/3

19. Suppose that an FFT hardware unit is available for computing the DFT. Show how to use the same hardware to compute the inverse DFT (by suitable manipulations of input and output vectors).

20. Let h_n be the impulse response of a FIR filter of length 2. Suppose that an FFT hardware unit is available for computing 4-point DFT. Show how to use it to compute the first four samples of the response of the filter to a sequence x_n of length 4.

21. Two random variables X and Y have a joint Probability Density Function (PDF)

$$f_{XY}(x,y) = \begin{cases} kxy & 0 \le x \le 1, 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Determine:

(a) the value of k which makes f a valid PDF

(b) the probability of the event $X \le 1/2$ AND $Y > 1/2$

(c) the *marginal densities* $f_X(x)$ and $f_Y(y)$

(d) the *conditional density* $f_X(x)$ and $f_Y(y)$

(e) whether X and Y are independent.

22. A Random Process $\{X(t,\alpha)\}$ has an ensemble consisting of four sample functions:

$$X(t,1) = 1, \quad X(t,2) = -2, \quad X(t,3) = \sin(\pi t), \quad X(t,4) = \cos(\pi t)$$

where the random variable α takes the values $\{1,2,3,4\}$ with equal probability. Determine the first order PDF $f_{X(t)}(x)$ and the expected value of the process $E\{X(t)\}$. Is the process strict sense stationary? Is it wide sense stationary?

23. For each of the following Random Processes, sketch a few members of the ensemble, determine the expected values and autocorrelation functions of the process and state whether they are wide sense stationary (WSS):

    (a) $X(t) = A$, where A is uniformly distributed on $[0, 1]$

    (b) $X(t) = \cos(2\pi ft + \Phi)$, for fixed f and Φ uniformly distributed on $[0, \phi_{max}]$. Are there values of ϕ_{max} for which the process is WSS?

    (c) $X(t) = A\cos(2\pi ft + \Phi)$, where f is fixed, Φ uniformly distributed on $[0, 2\pi]$, A is uniformly distributed on $[0, 1]$ and ϕ and A are independent.

    Is the process in (a) Mean Ergodic?

24. Show the following results for a WSS, real-values random process $\{X(t)\}$ with autocorrelation function $r_{XX}(\tau)$ and power spectrum $\mathcal{S}_X(\omega)$:

    (a) $r_{XX}(-\tau) = r_{XX}(\tau)$

    (b) If $\{X(t)\}$ is the a random voltage across a 1Ω resistor, then the average power dissipated is $P_{av} = r_{XX}(0)$

    (c) $\mathcal{S}_X(-\omega) = \mathcal{S}_X(\omega)$

    (d) $\mathcal{S}_X(\omega)$ is real-valued

    (e) $P_{av} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{S}_X(\omega)d\omega$.

25. A white noise process, $\{X(t)\}$, is a Wide Sense Stationary, zero mean process with autocorrelation function:

    $$r_{XX}(\tau) = \sigma^2\delta(\tau)$$

    where $\delta(\tau)$ is the delta-function centred on $\tau = 0$ whose integral is unity and width is zero. Sketch the Power Spectrum for this process.

    A sample function, $X(t)$, from such a white noise process is applied as the input to a linear system whose impulse response is $h(t)$. The output is $Y(t)$.

    Derive expressions for the output autocorrelation function $r_{YY} = E[Y(t)Y(t+\tau)]$ and the input-output cross-correlation function, $r_{XY} = E[X(t)Y(t+\tau)]$. Hence obtain an expression for the output power spectrum $\mathcal{S}_Y(\omega)$ in terms of σ^2 and the frequency response, $\mathcal{H}(\omega)$, of the linear system.

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