3F4: Data Transmission

Handout 3: Power spectral density of the PAM signal

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Recall the transmitted baseband PAM signal is $x(t) = \sum_k X_k p(t - kT)$.

- This is a *random signal*, so its power spectral density (PSD) describes the average power in any frequency band
- Also recall from 3F1 that integrating the PSD gives the average power of the signal
- We may want the PSD to be close zero outside the frequency band supported by the channel. Sometimes regulations also impose that the PSD lie below a given frequency-domain "mask"

To calculate the PSD, we will consider a slightly modified PAM signal:

$$x(t) = \sum_{k=-\infty}^{\infty} X_k \, p(t - kT - \Theta),$$

where Θ is a random dither (delay) uniformly distributed in $[0, T)$.

- The dither models the fact that an observer measuring the PSD has no information about when the transmitter's clock was switched on.
- Importantly, we will see that the dither makes $x(t)$ a wide-sense stationary (WSS) process, which greatly simplifies our PSD derivation.

Assumptions on the $\{X_k\}$ symbols

$$x(t) = \sum_{k=-\infty}^{\infty} X_k\, p(t - kT - \Theta),$$

- We assume that the X_k's are drawn from a constellation with zero mean, e.g., $\{-3A, -A, A, 3A\}$
- Furthermore, we also assume that the random process $\{X_k\}_{k=-\infty}^{k=\infty}$ is a WSS discrete-time process. What this means is that the autocovariance between X_j and X_{j+i} depends only on the shift i:

$$\mathbb{E}\left[(X_{j+i} - \mathbb{E}[X_{j+i}])\,(X_j - \mathbb{E}[X_j])\right] = \mathbb{E}[X_{j+i}X_j] =: R_X[i]$$

- In fact, if we assume that the input bits being transmitted are uniformly random, then the X_k's will also independent, i.e.,

$$R_X[i] = \begin{cases} \mathbb{E}X_k^2 \text{ (avg. energy of a constellation symbol)}, & i = 0 \\ 0, & i \neq 0 \end{cases}$$

- But our derivation will be for general $R_X[i]$. In Ex. paper 1, you will see a transmission scheme where X_k's are not independent

Autocovariance of $x(t)$

$$x(t) = \sum_{k=-\infty}^{\infty} X_k\, p(t - kT - \Theta),$$

Since $\{X_k\}$ have zero mean, so does $x(t)$. Therefore the *autocovariance function* of $x(t)$ is

$$\begin{aligned} R_x(t+\tau, t) &:= \mathbb{E}[x(t+\tau)x(t)] \\ &= \mathbb{E}\Big[\sum_{k=-\infty}^{\infty} X_k\, p(t+\tau-kT-\Theta) \sum_{\ell=-\infty}^{\infty} X_\ell\, p(t-\ell T-\Theta)\Big] \\ &= \sum_{k=-\infty}^{\infty}\sum_{\ell=-\infty}^{\infty} \mathbb{E}\Big[X_k X_\ell\, p(t+\tau-kT-\Theta)p(t-\ell T-\Theta)\Big] \end{aligned}$$

Note that the expectation is over (X_k, X_ℓ) as well as over $\Theta \sim \text{unif}[0, T)$. Since θ and the symbols $\{X_k\}$ are independent,

$$\begin{aligned} R_x(t+\tau, t) &= \sum_{k=-\infty}^{\infty}\sum_{\ell=-\infty}^{\infty} \mathbb{E}[X_k X_\ell]\,\mathbb{E}\big[p(t+\tau-kT-\Theta)p(t-\ell T-\Theta)\big] \\ &= \sum_{k=-\infty}^{\infty}\sum_{\ell=-\infty}^{\infty} R_X[k-\ell]\,\frac{1}{T}\int_0^T p(t+\tau-kT-\theta)p(t-\ell T-\theta)d\theta \end{aligned}$$

Making the change of variable $k - \ell = j$, we obtain

$$R_x(t+\tau, t) = \sum_{j=-\infty}^{\infty} R_X[j] \sum_{k=-\infty}^{\infty} \frac{1}{T}\int_0^T p(t+\tau-kT-\theta)p(t-kT+jT-\theta)d\theta$$
$$= \sum_{j=-\infty}^{\infty} R_X[j]\frac{1}{T}\int_{-\infty}^{\infty} p(t+\tau-\theta)p(t+jT-\theta)d\theta,$$

where the last equality above is obtained using the following fact: For any function $f(y)$ and any interval $[0, T]$,

$$\sum_{k=-\infty}^{\infty}\int_0^T f(y-kT)\,dy = \int_{-\infty}^{\infty} f(y)\,dy.$$

(The LHS integrates the entire function by adding up the integrals over disjoint length T intervals.)
Making a change of variable in the integral $v = t+\tau-\theta$, we get

$$R_x(t+\tau, t) = \sum_{k=-\infty}^{\infty} R_X[j]\,\frac{1}{T}\int_{-\infty}^{\infty} p(v)p(v+jT-\tau)dv$$

Defining $R_p(\tau) = \int_{-\infty}^{\infty} p(v)p(v-\tau)dv$, we can summarize our result as

$$R_x(t+\tau, t) := \mathbb{E}[x(t+\tau)x(t)] = \frac{1}{T}\sum_{j=-\infty}^{\infty} R_X[j]R_p(\tau-jT)$$

PSD of $x(t)$

Observe that the RHS of $R_x(t+\tau, t)$ depends only on τ, not on t. Hence $x(t)$ is WSS, and we write

$$R_x(t+\tau, t) = R_x(\tau) = \frac{1}{T}\sum_{j=-\infty}^{\infty} R_X[k]R_p(\tau-kT) \qquad (1)$$

Fourier transform of $R_x(\tau)$ will give us power spectral density of $x(t)$.
Note:

- $R_p(\tau) = \int_{-\infty}^{\infty} p(v)p(v-\tau)dv = p(\tau) \star p(-\tau)$.
- Therefore, $\mathcal{F}[R_p(\tau)] = P(f)P(-f) = |P(f)|^2$ because $p(t)$ being real implies $P^*(f) = P(-f)$

Using these, we take Fourier transforms in (1) to get the PSD:

$$S_x(f) = \frac{1}{T}\sum_{k=-\infty}^{\infty} R_X[k]\mathcal{F}[R_p(\tau-kT)] = \frac{1}{T}\sum_{k=-\infty}^{\infty} R_X[k]\,|P(f)|^2 e^{-\mathrm{j}2\pi kfT}$$

Therefore the PSD of the transmitted PAM signal $x(t)$ is

$$S_x(f) = \frac{|P(f)|^2}{T}\sum_{k=-\infty}^{\infty} R_X[k]e^{-\mathrm{j}2\pi kfT}$$

When the $\{X_k\}$ are independent

In most (not all) applications, the $\{X_k\}$ symbols are independent as independent sets of bits are mapped to successive constellation symbols. In this case,

$$R_X[k] = \begin{cases} \mathbb{E}[X_j^2] = \mathcal{E}_s, & k = 0 \\ 0, & k \neq 0 \end{cases}$$

where $\mathcal{E}_s$ denotes the average energy per constellation symbol.
Then the formulas for autocovariance and PSD of $x(t)$ simplify to

$$R_x(\tau) = \frac{\mathcal{E}_s}{T} R_p(\tau), \qquad S_x(f) = \frac{\mathcal{E}_s}{T} |P(f)|^2$$

The average power of the PAM waveform is then calculated as

$$\frac{\mathcal{E}_s}{T} \int_{-\infty}^{\infty} |P(f)|^2 df = \frac{\mathcal{E}_s}{T} \int_{-\infty}^{\infty} |p(t)|^2 dt, \tag{2}$$

where we have used Parseval's theorem (see IB Signal & Data Analysis).

Thus, when the $\{X_k\}$ are independent:

- The expression in (2) is just what we would expect: in each symbol period of time T, we transmit a shift of pulse $p(t)$ modulated by a symbol whose average squared value is $\mathcal{E}_s$.
- If the pulse $p(t)$ has unit energy, then power of PAM signal is just $\mathcal{E}_s/T$.
- If the matched receive filter is chosen as $q(t) = p(-t)$, then the overall filter frequency response is

  $$G(f) = P(f)P(-f) = |P(f)|^2.$$

  Then the PSD on the previous slide can be written as

  $$S_x(f) = \frac{\mathcal{E}_s}{T} G(f).$$

When the $\{X_k\}$ are not independent, the PSD is given by the general expression on slide 7.