Handout 2

Classification of the equations

Differential equations are classified as Hyperbolic, Elliptic or Parabolic depending on their regions of influence. The type of equation has a great influence on the type of solution method used and on the boundary conditions to be applied. The exact mathematical derivation of the class is very complex (See Anderson Chapter 3) and we will only try to describe it on a physical basis.

Hyperbolic equations

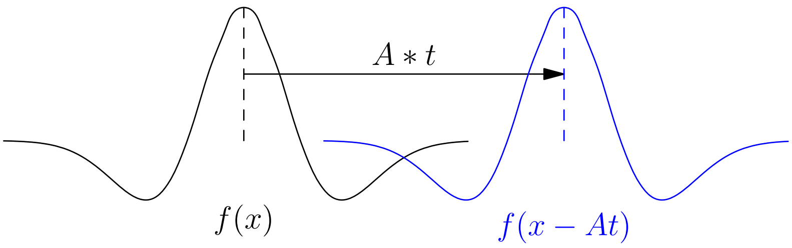
The simplest "hyperbolic" equation is the scalar convection equation

$$\frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} = 0$$

When A is constant, the general solution to this is

$$u = f(x - At)$$

where f is any function. A profile at time t is simply blown



forward (convected). To see this we note that

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial t}dt$$

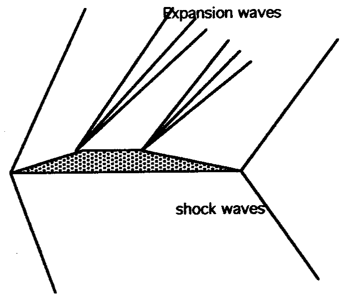
Thus $du = 0$ (i.e. $u =$ constant) on the line $\frac{dx}{dt} = A$. When A is a constant, this line is

$$x - At = constant$$

This curve is called the "characteristic curve" of the differential equation and A is the "characteristic velocity". In general, A varies with both time and space, giving curved characteristics.

Because the scalar convection equation has a characteristic curve with a real characteristic velocity it is said to be "hyperbolic". Associated with a hyperbolic equation is an "initial value problem" in which initial data is given for a range of x at an initial value of t, and for which solutions can be obtained in a region bounded by that range of x and by the characteristic. Information flows along characteristics in a wave-like manner and determines how points in the solution and boundary points can communicate. In particular, equations which are hyperbolic with respect to time (for example) are solved by marching forward in time.

You should be familiar with the behaviour of steady fully supersonic flow where if we introduce a disturbance, say a bump on an aerofoil, the disturbance only affects the flow along and downstream of the characteristic lines radiating from the disturbance. If we know the flow on the aerofoil surface we can solve for the flow anywhere between the characteristic emanating from the leading and trailing edges by marching in space along the characteristics. The flow quantities are conserved or *invariant* along the characteristics. Similarly, if we know the flow at exit from a supersonic nozzle we can calculate the flow in the jet as long as it remains supersonic by marching downstream using the method of characteristics.



Any unsteady inviscid flow, governed by the Euler equations, is hyperbolic in time since events at one time can only influence the flow at later time, i.e. we cannot change the past, and perturbations to the flow propagate in a wave-like manner (some things like vorticity or temperature convect, pressure propagates like sound, etc.). The unsteady Navier-Stokes equations on the other hand are a bit more problematical. There are many regions where the flow is effectively inviscid and they are then effectively hyperbolic in time. In regions (or Reynolds number ranges) where viscous effects are dominant, they are not hyperbolic.

The implications of hyperbolic equations are that if we specify the flow conditions at some boundary, e.g. on the aerofoil above, we can calculate the flow everywhere downstream of the characteristic lines radiating from the boundary. In unsteady flow if we specify the flow completely at some instant in time we can, in theory, calculate the flow at all subsequent intervals in time by marching forwards in time following the characteristic directions. This is the basis of the time-marching approach to solving the equations.

Elliptic equations

The simplest "elliptic" equation is the scalar Laplacian:

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Elliptic equations are associated with values of unknowns (or their normal derivatives) given on a closed curve and require the solution of a "boundary value problem". The solution at some point in the interior depends on every boundary point and every other interior point. i.e. they have no characteristics.

Laplace's equation for *steady incompressible flow* or *steady heat conduction* is the classic example. For example, if we introduce a source into a steady subsonic flow then its influence will decay with distance from the source but it will never become completely zero (except at infinity). Similarly, if we introduce a source of heat at one point in a conducting solid the steady state temperature will be changed everywhere in the solid.

For elliptic equations information propagates in all directions. Hence, in a CFD solution every point influences every other point and we must apply boundary conditions at all points on the boundaries of the region we want to calculate. These boundary conditions may be: specified velocity, specified velocity potential or zero velocity normal to a surface. All steady subsonic flows are elliptic.

Parabolic equations

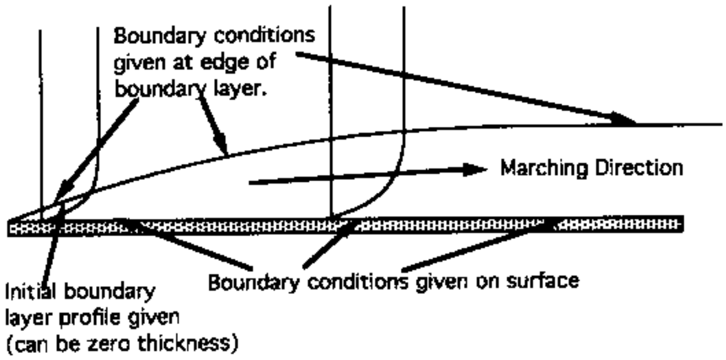
The simplest "parabolic equation" is the scalar equation for heat conduction

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0$$

This is a mixture of the two previous classifications and has a single characteristic direction, in the sense that solution may be obtained by marching in that direction from some specified initial condition, but information cannot really be said to travel along this characteristic, i.e. we do not have the equivalent of a conserved (invariant) quantity along a characteristic. If we change the temperature at some point in a solid at one instant in time, the disturbance spreads out in space with increasing time and we can find the solution at all subsequent times by marching the equations in time from the initial known temperature field.

Another example is the steady boundary layer approximation. If we specify the surface pressure distribution (or free-stream velocity distribution) everywhere along the surface, and make the usual boundary layer assumption that the pressure is constant through the boundary layer, then we can march the boundary layer equations downstream, starting at the leading edge of the body and obtain the boundary layer velocity distribution and hence displacement thickness, etc, at all downstream points. The pressure distribution on the surface, which is the same as that outside the boundary layer, must be obtained by some different method such as a solution of the inviscid flow about the body. This approach is quite different from solving the full N-S equations because we specify the pressure field rather than solve for it. If the surface pressure distribution used is accurate, and if the boundary layer approximations apply, it should give the same answer as the full N-S equations but with enormously less computational effort. We will look at a method of using this approach in some detail later.

In general we have to use different numerical techniques to solve each type of equation. Hence problems which are of mixed type are especially difficult The main example of this is mixed subsonic-supersonic flow. If we use say the method of characteristics in the



supersonic flow and a potential flow method in the subsonic part then we also need to know the location of the boundary between them and this is not easily obtainable. Hence the advantages of solving transonic flows by the time marching method which remains hyperbolic in both subsonic and supersonic flow and so enables a single numerical approach to be used without needing to know whether the flow is locally supersonic or subsonic.

Convection equation

Most aerodynamic flows are high Reynolds number and hence convection dominated. We therefore study the numerical simulation of the model 1-D convection equation

$$\frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} = 0$$

This equation is *hyperbolic* with characteristic curve

$$\left.\frac{dx}{dt}\right|_u = A$$

and is posed quite naturally as an initial value problem.

Centred space differences

The most obvious finite difference attempt at time marching is "forward in time" and "centred in space" :

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + A\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} = 0$$

$$u_i^{n+1} = u_i^n - \frac{c}{2}\left(u_{i+1}^n - u_{i-1}^n\right)$$

$$c = \frac{A\Delta t}{\Delta x}$$

Fig. 12 shows attempts to solve this equation using a centred method. It turns out to be *unstable for all values of c*.

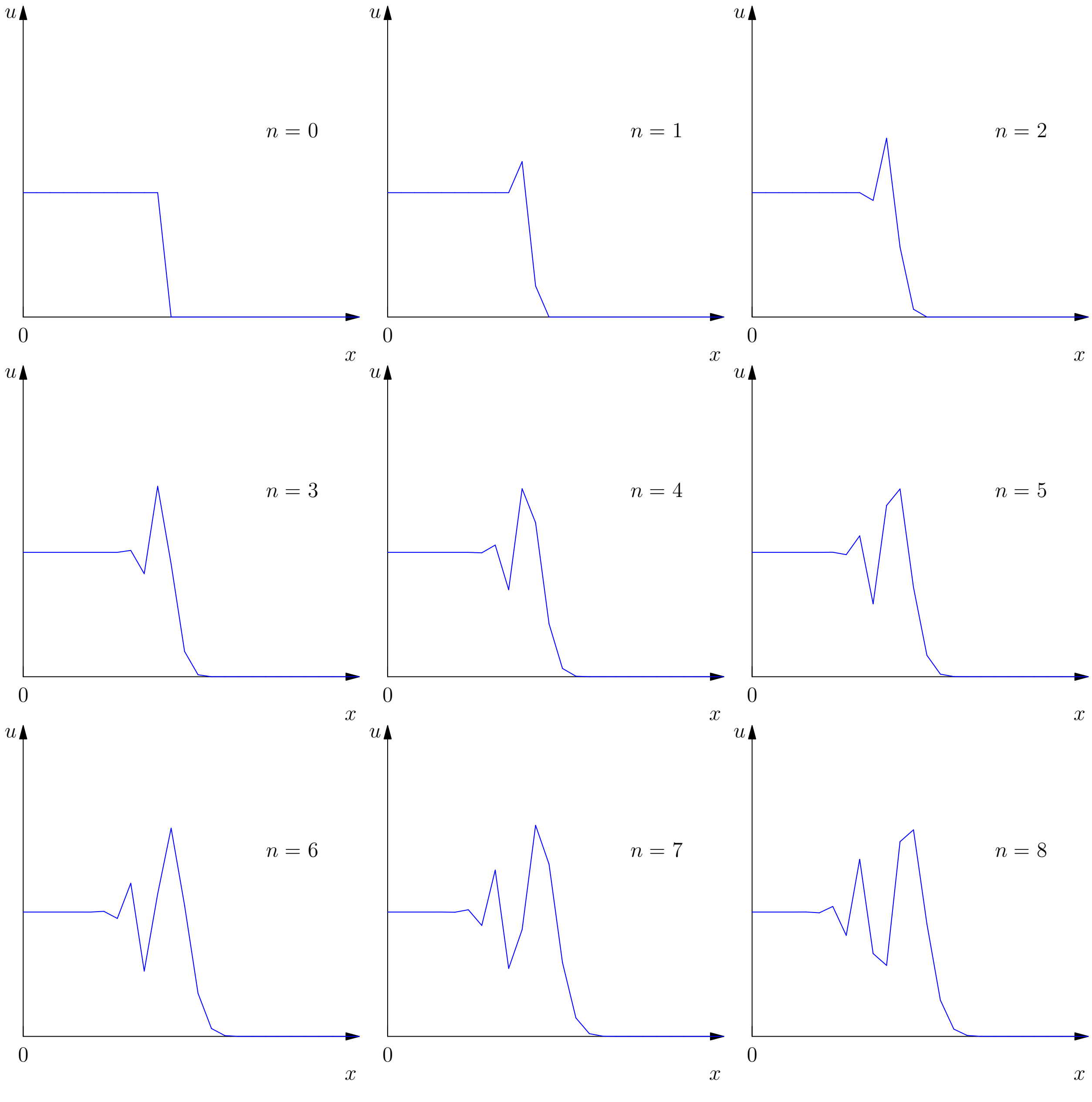
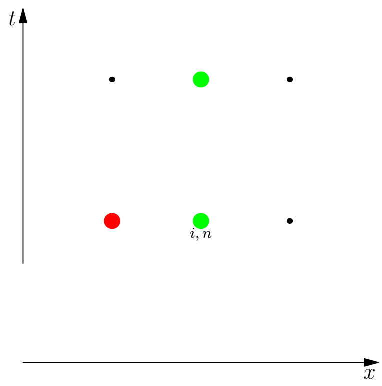


Figure 12: Centred-difference solution to the convection equation. For the correct solution the initial disturbance should convect downstream (unchanged) as time (n) increases. However, the solution is clearly unstable.

Upwind space differences



We recognise the physics revealed by the presence of a characteristic curve and mimic this numerically with an "upwind" difference.

Thus the difference equation is

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + A\frac{u_i^n - u_{i-1}^n}{\Delta x} = 0$$

$$u_i^{n+1} = u_i^n - c\left(u_i^n - u_{i-1}^n\right)$$

$$c = \frac{A\Delta t}{\Delta x}$$

The small perturbation method using a sawtooth (try the single point disturbance as an exercise) gives

$$u_i^{n+1} = \epsilon - c(\epsilon + \epsilon)$$

$$\frac{u_i^{n+1}}{\epsilon} = 1 - 2c$$

Thus for stability, $-1 \leq 1 - 2c \leq 1$, which implies $c \leq 1$. The parameter c is called the Courant number or Courant-Freidrichs-Lewy (CFL) number.

The stability criterion, CFL ≤ 1, has a sound basis in the physics of the initial value problem:

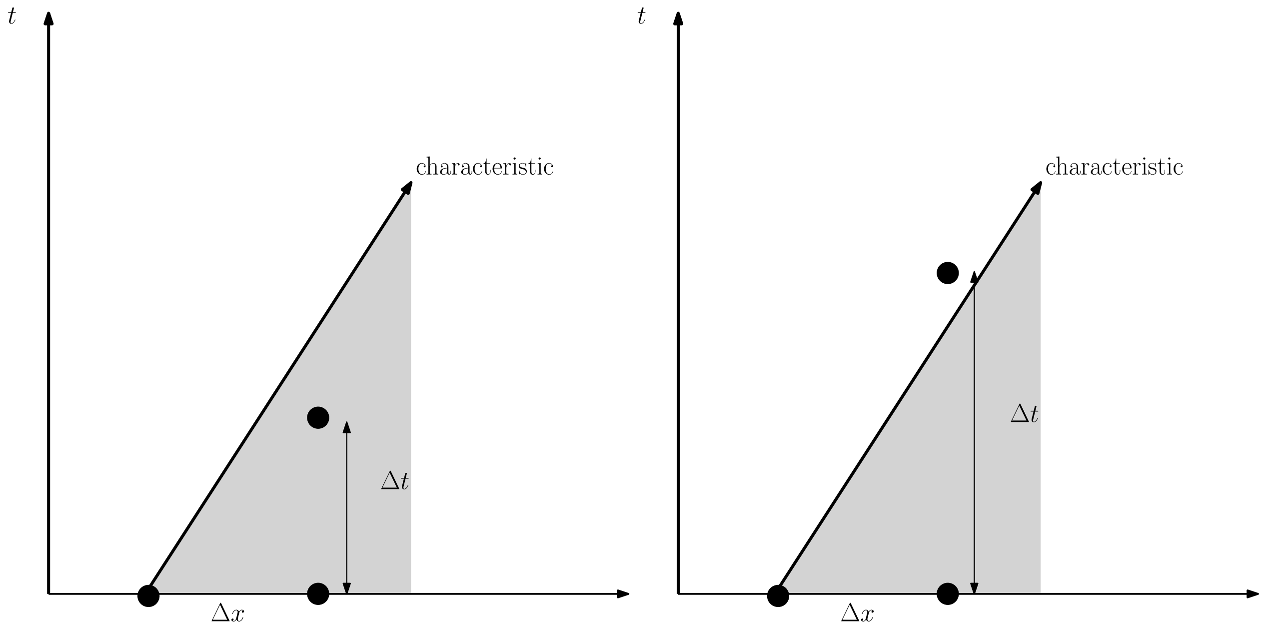


Figure 13: Numerical domain of dependence must be contained within the physical one for stability.

To test the difference scheme we attempt to convect a ramp profile forward, as shown in Fig. 14. The results show that although the initial profile is indeed convected physically, the slope of the ramp reduces monotonically. This implies the presence of diffusion – *False Diffusion*.

The evident diffusion is false because it must arise from the

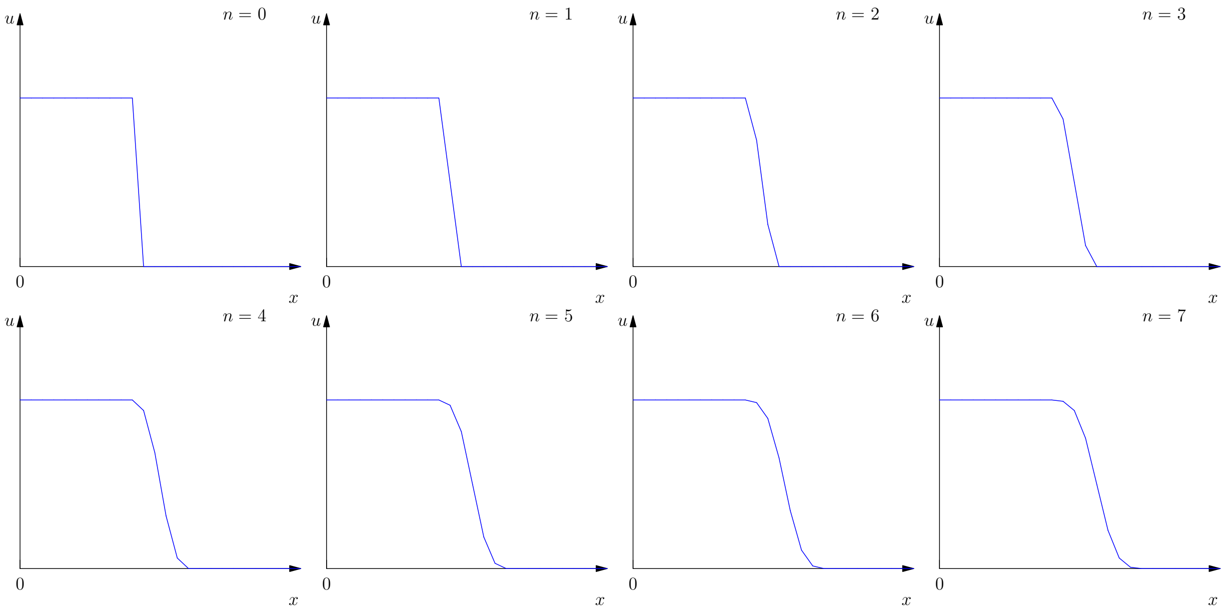


Figure 14: $c = 0.5$

difference equation itself (it is not present in the original differential equation). We can see how this false diffusion arises by analysing the form of the truncation error using Taylor expansions:

$$u_i^{n+1} = u_i^n + \Delta t \frac{\partial u}{\partial t}\bigg|_{i,n} + \frac{\Delta t^2}{2!}\frac{\partial^2 u}{\partial t^2}\bigg|_{i,n} + \cdots$$

$$u_{i-1}^n = u_i^n - \Delta x \frac{\partial u}{\partial x}\bigg|_{i,n} + \frac{\Delta x^2}{2!}\frac{\partial^2 u}{\partial x^2}\bigg|_{i,n} - \cdots$$

Substituting into our finite-difference scheme,

$$u_i^{n+1} = u_i^n - c\left(u_i^n - u_{i-1}^n\right)$$

shows

$$\frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} = \frac{A\Delta x}{2}\frac{\partial^2 u}{\partial x^2} - \frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2} + \cdots$$

Now the true differential equation implies that

$$\frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} = 0 \implies \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}\left(-A\frac{\partial u}{\partial x}\right) = -A\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right) = -A\frac{\partial}{\partial x}\left(-A\frac{\partial u}{\partial x}\right) = A^2\frac{\partial^2 u}{\partial x^2}$$

so that the difference equation is really more like the equivalent PDE :

$$\frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} = \left\{\frac{A\Delta x}{2}\left(1 - \frac{A\Delta t}{\Delta x}\right)\right\}\frac{\partial^2 u}{\partial x^2}$$

Clearly if $c = A\Delta t/\Delta x = 1$, then there is no false diffusion. This can be seen in the solution in Fig. 15. In practical problems, however, Δx and A vary in space and it is not possible to achieve c exactly equal to 1, implying the use of smaller time steps. If we use

$c = 0.5$, the effective viscosity is

$$\nu_{\text{eff}} = \left\{ \frac{A\Delta x}{2} \left(1 - \frac{A\Delta t}{\Delta x} \right) \right\} = \frac{A\Delta x}{4}$$

This can be very restrictive; for example, if we have an aerofoil with a $\Delta x = \text{chord}/100$ then

$$Re_{\text{eff}} = \frac{A\,\text{chord}}{\nu_{\text{eff}}} = \frac{4A\text{chord}}{A\Delta x} = 400!$$

Typical flight Reynolds numbers are of the order of 10^6.

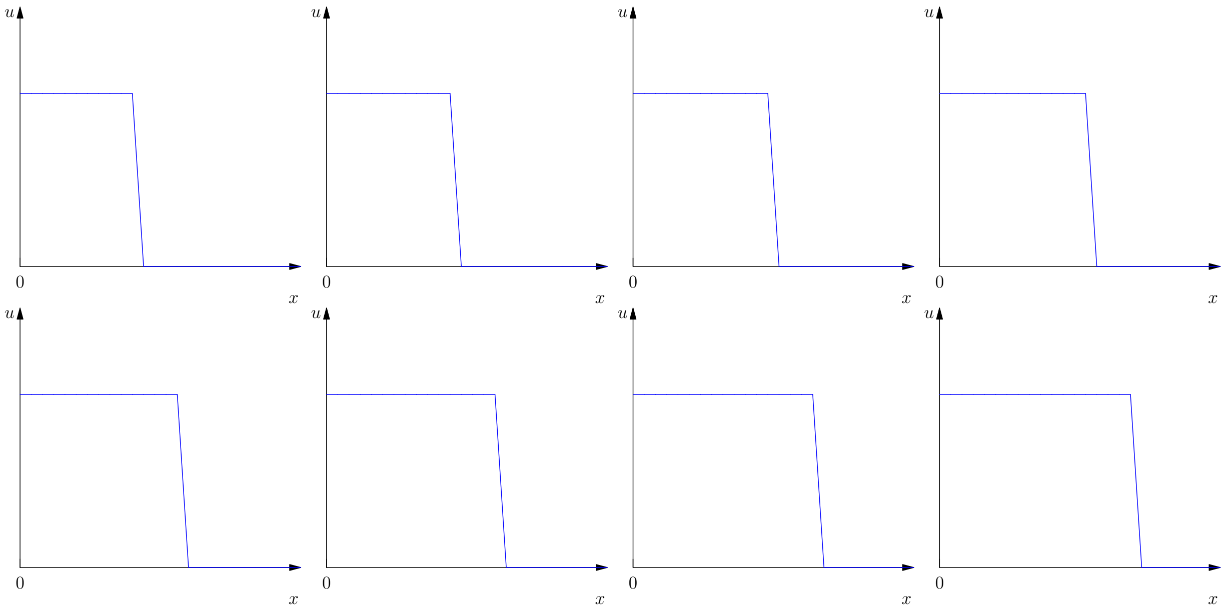


Figure 15: $c = 1.0$

The Matlab code for solving the convection equation is give below:

```
clear;
nx=201;
for i=1:nx
    x(i) = (i-1)*1./(nx-1);
    if i < nx/20.
        u(i) = 1.;
    else
        u(i) = 0.;
    end
%   u(i) = exp(-((x(i)-0.2)/0.05)^2);
end
newplot;
hold on;
plot(x,u,'r','LineWidth',1);
c = 1.005;
offset = 0;
nt = 100;
```

```
nplot=10;
for n=1:nt
    for i=2:nx-1
        un(i) = u(i)-c*(u(i)-u(i-1));
    end
    un(1) = 1;
    un(nx) = 0;
    if rem(n,nplot)==0
        offset = offset + .02;
        for i=1:nx
            up(i) = un(i)+offset;
        end
        plot(x,up,'b','Linewidth',1.5);
    end
    u = un;
end
hold off;
```

Lax-Wendroff differencing

This is a classic attempt to produce a stable solution procedure free from false diffusion. The basic idea is to expand the time difference to be accurate to second order. We start with the convection equation

$$\frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} = 0$$

and the Taylor series

$$u_i^{n+1} = u_i^n + \Delta t \left.\frac{\partial u}{\partial t}\right|_{i,n} + \frac{\Delta t^2}{2!}\frac{\partial^2 u}{\partial t^2}_{i,n} + \cdots$$

The convection equation can be rearranged and differentiated to give

$$\frac{\partial u}{\partial t} = -A\frac{\partial u}{\partial x} \text{and} \frac{\partial^2 u}{\partial t^2} = A^2\frac{\partial^2 u}{\partial x^2}$$

Substituting this into the Taylor series above, we get

$$u_i^{n+1} = u_i^n + \Delta t \left(-A\left.\frac{\partial u}{\partial x}\right|_{i,n}\right) + \frac{\Delta t^2}{2!}\left(A^2\left.\frac{\partial^2 u}{\partial x^2}\right|_{i,n}\right) + \dots$$

If the space derivatives are now replaced by second-order central differences, we obtain the Lax-Wendroff scheme:

$$u_i^{n+1} = u_i^n - \frac{c}{2}\left(u_{i+1}^n - u_{i-1}^n\right) + \frac{c^2}{2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)$$

with c as before = CFL = $A\Delta t/\Delta x$. We can examine the stability of this scheme by considering discrete perturbations. The "sawtooth wiggle", for example, gives:

$$u_i^{n+1} = \epsilon - \frac{c}{2}\left(-\epsilon - (-\epsilon)\right) + \frac{c^2}{2}\left((-\epsilon) - 2\epsilon + (-\epsilon)\right)$$

Thus

$$\frac{u_i^{n+1}}{\epsilon} = 1 - 2c^2$$

and for stability,

$$\left|\frac{u_i^{n+1}}{\epsilon}\right| \leq 1 \implies -1 \leq 1 - 2c^2 \leq 1$$

i.e.

$$c \leq 1$$

Again we find that a scheme for a convection equation is only stable provided that the numerical domain of dependence is contained within the physical one. Applying the Lax-Wendroff scheme to our model problem, Fig. 16, shows results in complete contrast to those obtained using the upwind scheme. The ramp profile convects with little change in slope (suggesting there is little false diffusion) but the profile becomes very "wiggly". What is happening here is that for the differential equation, all wavelengths travel at the convection speed A. The Lax-Wendroff difference scheme, however, has the short wavelengths travelling at different speeds and they spread out. This phenomena is referred to as *dispersion* and, if it leads to upstream propagation, *false convection*.

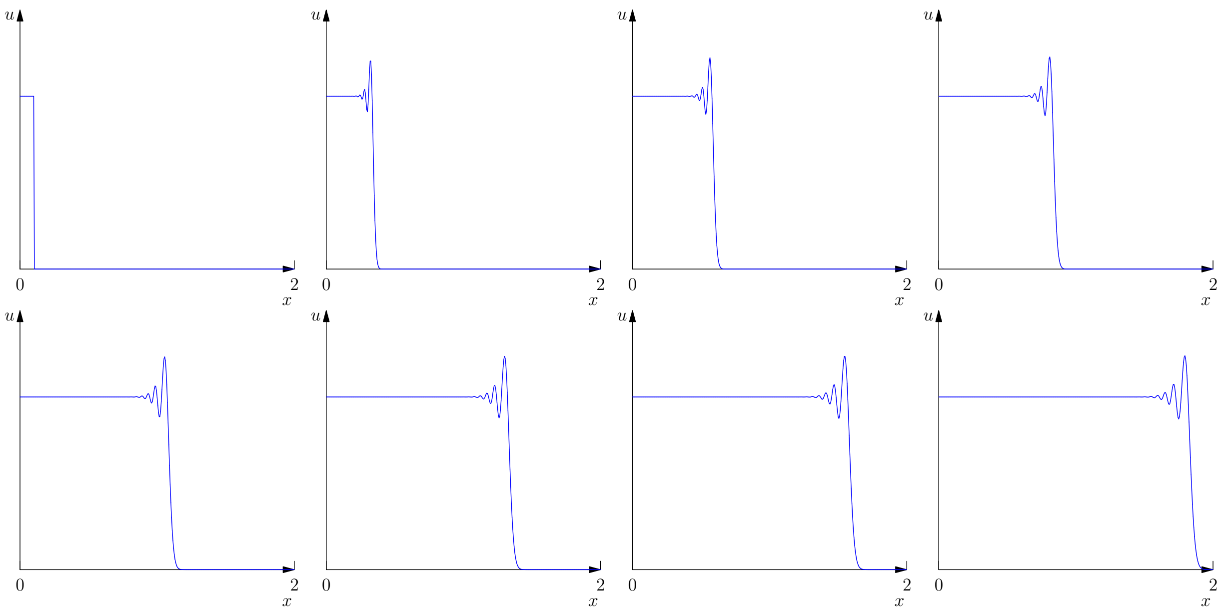


Figure 16: Lax Wendroff soltuion. $c = 0.5$.

In general, numerical diffusion and/or dispersion are present in all numerical schemes to some extent. False convection is a common problem with centred schemes.

The sharp step exaggerates the unwanted properties of this type of numerical scheme. For more "gentle" functions, the effects are still present but less noticeable (and will cause problems if you fail to notice them!). The examples paper has some exercises to examine less extreme profiles and shows the extra accuracy of the second order Lax-Wendroff scheme over the first order upwind case.

MacCormack differencing

The MacCormack scheme is widely used for solving fluid flow equations. The scheme is implemented in a two-step process: predictor and corrector. For the convection equation, the two steps are given by

$$\text{Predictor:} \qquad \overline{u_i^{n+1}} = u_i^n - A\frac{\Delta t}{\Delta x}(u_{i+1}^n - u_i^n)$$

$$\text{Corrector:} \qquad u_i^{n+1} = \frac{1}{2}\left[u_i^n + \overline{u_i^{n+1}} - A\frac{\Delta t}{\Delta x}(\overline{u_i^{n+1}} - \overline{u_{i-1}^{n+1}})\right]$$

The term $\overline{u_i^{n+1}}$ is a temporary "predicted" value of u that is corrected by the "corrector" step to get u at time level $n+1$. Note that in the predictor step a forward difference is used, whereas in the corrector step a backward difference is used. This order of differencing can be reversed.

For the linear convection problem, it can be shown that the MacCormack differencing is identical to the Lax-Wendroff differencing. Hence, like the Lax-Wendroff differencing it is second order accurate in space and time and shares the same stability characteristics.

The MacCormack scheme is more easy to implement than the Lax-Wendroff method for nonlinear problems. Thus it provides an easy way to solve a nonlinear problem, such as fluid flow equations that is free from numerical dissipation (up to second order).