Engineering Tripos Part IIA THIRD YEAR

Module 3A1: Fluid Mechanics I

APPLICATIONS TO EXTERNAL FLOWS

Examples paper 2—The flow around wings

3D panel methods

1. A vortex line element is shown in Figure 1. Points 1 and 2 are the start and end of the element, defined by the requirement that the vector from 1 to 2, $\mathbf{r}_v$, is in the same direction as the (vector) vorticity.

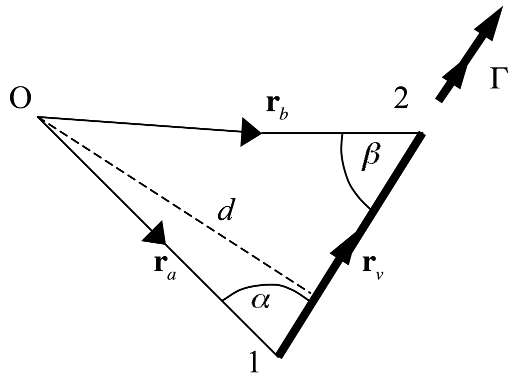

Fig. 1

(a) Show that:

(i) $\cos\beta = \dfrac{\mathbf{r}_b\,\mathbf{r}_v}{|\mathbf{r}_b||\mathbf{r}_v|}$

(ii) $\cos\alpha = -\dfrac{\mathbf{r}_a\,\mathbf{r}_v}{|\mathbf{r}_a||\mathbf{r}_v|}$

(iii) $d = \dfrac{|\mathbf{r}_a \times \mathbf{r}_b|}{|\mathbf{r}_v|}$

(b) Hence derive a vector form of the Biot-Savart result for the velocity at O.

(c) Write a Matlab function `v = linev(r1,r2,ro)` to calculate the velocity `v = [vx vy vz]` at `ro = [xo yo zo]` due to a vortex of unit circulation lying between `r1 = [x1 y1 z1]` and `r2 = [x2 y2 z2]`. (You will need the Matlab functions `dot`, `norm` and `cross`.)

Lifting-Line Theory

2. An untwisted wing with elliptical planform has a uniform aerofoil section shape, with zero-lift angle α_0, and aspect ratio A_R.

   (a) Show that:
   (i) the local section lift coefficient is equal to the wing lift coefficient;
   (ii) the downwash angle is proportional to the wing lift coefficient.

   (b) Hence find:
   (i) the wing lift coefficient as a function of angle of attack;
   (ii) the wing lift-curve slope.

3. An aircraft has an untwisted rectangular wing of aspect ratio A_R = 7.5. The chordwise section is uniform across the span, which runs from y = -s to +s. The circulation is given by

$$\Gamma(\theta) = Us\sum_{n=1}^{7} G_n \sin n\theta$$

   with $y = -s\cos\theta$, and $G_1 = 0.08100$, $G_3 = 0.01115$, $G_5 = 0.00262$, $G_7 = 0.00086$.

   (a) What is the lift coefficient of the wing?

   (b) What is the ratio of its induced drag coefficient to that of an elliptically-loaded wing with the same aspect ratio and lift coefficient?

   (c) Calculate the downwash angle at
   (i) the wing tips;
   (ii) the centre of the wing.

4. An aircraft with the tapered wing planform of Figure 2 is to be given an elliptic lift distribution via the use of twist, when flying at weight W and speed U in air of density ρ.

   (a) If the aerofoil section is symmetrical and the twist is defined to be zero at the wingtip, find
   (i) the aircraft incidence;
   (ii) the twist distribution.

   (b) Is it possible to maintain the elliptic lift distribution if the flight speed is changed? Justify your answer.

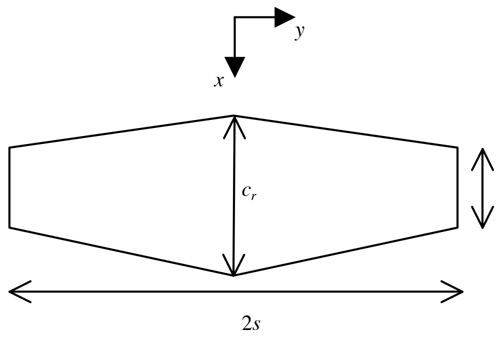

Fig. 2

5. A wing is to have lift distribution $\Gamma(y) = \Gamma_0(1 - y^2/s^2)^{3/2}$.

   (a) Find the coefficients G_n in the Fourier series representation

$$\Gamma(\theta) = Us \sum_{n \text{ odd}} G_n \sin n\theta$$

   with θ defined via $y = -s\cos\theta$. You may assume without proof that $\int_0^\pi \sin^3\theta \sin\theta \, d\theta = 3\pi/8$, $\int_0^\pi \sin^3\theta \sin 3\theta \, d\theta = -\pi/8$, and $\int_0^\pi \sin^3\theta \sin n\theta \, d\theta = 0$ for $n = 5,7,9....$

   (b) How would the induced drag of this wing compare with that of one with the same aspect ratio and lift coefficient, but with an elliptical lift distribution?

   (c) The wing is to be untwisted, with a symmetrical aerofoil section. The angle of attack, α, is specified by $\alpha = 3\Gamma_0/2Us$.
   (i) Find an expression for the ratio of chord to semi-span, $c(\theta)/s$.
   (ii) Sketch this ratio as a function of spanwise distance.
   (iii) Make a rough estimate of the wing aspect ratio that has resulted from this choice of the constant of proportionality between α and Γ_0.

   (d) (i) Find an expression for the local lift coefficient, $c_l(\theta)$.
   (ii) Sketch this result as a function of spanwise distance, and briefly comment on the implications for the wing's stall behaviour.

6. *Challenge*: In an attempt to take account of structural, as well as aerodynamic, considerations, Prandtl proposed that the weight of an aircraft would be partly dependent on the average wing bending moment, which he showed to be given (in the usual notation) by

$$\overline{M} = \frac{\rho U}{2s} \int_{-s}^{s} \Gamma(y) y^2 dy$$

   (a) By expanding the circulation in the usual Fourier series, show that the dimensionless average bending moment, $\tilde{\overline{M}} = \overline{M}/\tfrac{1}{2}\rho U^2 s^3$, is given by

$$\tilde{\overline{M}} = \frac{\pi}{8}(G_1 + G_3)$$

(b) Now assume that the weight, W, of the aircraft, is given in dimensionless form by

$$\tilde{W} = \tilde{W}_0 + 8\kappa\tilde{\tilde{M}}$$

where $\tilde{W} = W/\frac{1}{2}\rho U^2 s^2$. Use the fact that the aircraft lift must equal its weight in equilibrium flight to obtain an equation linking the coefficients G_1 and G_3, and plot this equation, taking $\kappa = 1/2, \tilde{W}_0 = 0.1$.

(c) Plot the dimensionless induced drag, $\tilde{D}_i = D_i/\frac{1}{2}\rho U^2 s^2$, as a function of G_3 only, assuming that all higher Fourier coefficients are zero. Discuss the implications of your result for the optimum design.

(d) Sketch the approximate form of the lift distribution for the optimum design, and indicate how the corresponding (untwisted) wing planform will differ from elliptical.

The Horseshoe Vortex Model

Warning! Manual calculation of horseshoe vortex velocities for the numerical examples can be time-consuming and tedious, particularly for qu12. If qu1(c) didn't seem worth the effort at the time, you may wish to reconsider it now. You might also find it useful to write a Matlab function that calls your 'linev' routine 3 times to calculate the velocity due to a horseshoe vortex.

7. An aeroplane flying at height h above the ground is represented by a horseshoe vortex with circulation Γ and span $2s'$.

(a) Show that the average upwash experienced by the bound vortex due to its ground image is

$$\frac{\Gamma}{8\pi s'} \log\left[1 + \frac{s'^2}{h^2}\right]$$

(b) Estimate the change in drag coefficient of the aircraft due to the image upwash.

(c) *Discussion*. Comment on the importance of this, 3D, ground effect in comparison with the 2D phenomenon discussed in the lectures.

8. Three aircraft of equal weight are flying in formation. Their horseshoe vortex representation is shown in Figure 3.

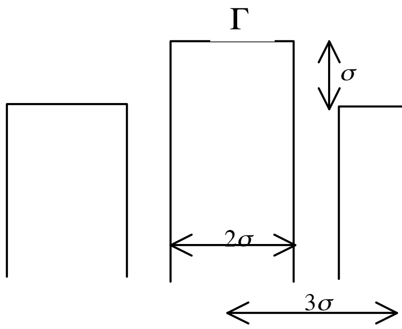

Fig. 3

(a) Find the downwash on the bound vortex of one of the aircraft flying alone, assuming that the lift distribution is elliptical, and that the effective horseshoe vortex span is unchanged.

(b) Calculate the downwash at the centre of each aircraft's wing, as a percentage of your answer in (a). Comment on your results.

9. An aircraft is flying with speed U at an altitude h. The effective span of its horseshoe vortex model is 2σ.

(a) Calculate the associated streamwise velocity distribution on the ground underneath as a function of streamwise (x) and spanwise (y) coordinates, assuming that $h \gg \sigma$. Take the coordinate origin to be fixed at the centre of the bound leg of the vortex.

(b) Use Bernoulli's equation to estimate the ground plane pressure. You may assume that the velocities found in (a) are small in comparison to the free-stream.

(c) Show that the weight of the aircraft is carried by the ground.

Viscous Effects and Stall

10. An untwisted, tapered wing has taper ratio (ie tip chord/root chord) λ and semi-span s.

(a) Estimate the spanwise location of the maximum local lift coefficient. You may assume that the lift distribution can be approximated as elliptical.

(b) What can you conclude about the stall behaviour of highly tapered wings with constant cross-section geometry?

11. An aircraft has an untwisted, elliptic wing of aspect ratio 9 with a symmetrical aerofoil section. Its zero-lift drag coefficient (due to viscous effects) is 0.01.

(a) Ignoring fuselage and tail contributions, and assuming that the lift-dependent drag is entirely due to inviscid effects, calculate:
(i) the maximum lift-to-drag ratio of the aircraft; and
(ii) the incidence at which it is achieved.

(b) Two-dimensional measurements on the aerofoil section show that the additional viscous drag due to lift increases the (2D) drag coefficient by $0.004c_l^2$, where c_l is the (2D) lift coefficient.
(i) Assuming that the 2D results apply to local lift and drag coefficients on the wing, recalculate the maximum lift-to-drag ratio of the aircraft.
(ii) Comment on the accuracy of your previous calculation, and discuss how this would be affected by changes in wing aspect ratio and flight Reynolds number.

Sweep

12. Figure 4 shows a crude extended lifting-line discretisation of a swept wing of uniform chord and cross-section. The latter is symmetrical.

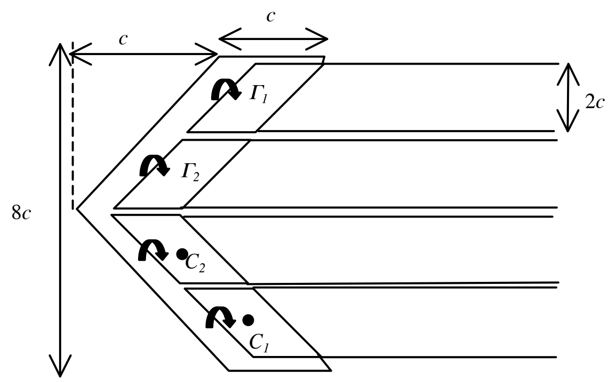

Fig. 4

(a) Find the vortex-induced downwash at the collocation points C_1 and C_2.

(b) If the free-stream velocity is U, at incidence α, calculate the vortex circulations.

(c) *Discussion.* Explain qualitatively how the downwashes and corresponding circulations will alter if the wing is unswept, and comment.

Answers

1 b) $\frac{\Gamma}{4\pi}\frac{\mathbf{r}_a\times\mathbf{r}_b}{\left|\mathbf{r}_a\times\mathbf{r}_b\right|^2}\mathbf{r}_v\cdot\left(\frac{\mathbf{r}_b}{\left|\mathbf{r}_b\right|}-\frac{\mathbf{r}_a}{\left|\mathbf{r}_a\right|}\right)$

2 b) (i) $C_L=\frac{2\pi}{1+2/A_R}\left(\alpha-\alpha_0\right)$ (ii) $\frac{dC_L}{d\alpha}=\frac{2\pi}{1+2/A_R}$

3 a) 0.477 b) 1.063 c) (i) 0.0722 rad (ii) 0.0137 rad

4 a) (i) $\frac{W}{2\pi\rho U^2 s^2}$ (ii) $\frac{2W}{\pi^2\rho U^2\left[c_r s-\left(c_r-c_t\right)y\right]}\sqrt{1-\frac{y^2}{s^2}}$ b) No

5 a) $G_1=\frac{3}{4}\frac{\Gamma_0}{Us}$; $G_3=-\frac{1}{4}\frac{\Gamma_0}{Us}$ b) 33% higher

c) (i) $\frac{c(\theta)}{s}=\frac{2}{3\pi}\frac{\sin^3\theta}{1+\frac{1}{4}\cos 2\theta}$ d) (i) $c_l(\theta)=3\pi\frac{\Gamma_0}{Us}\left[1+\frac{1}{4}\cos 2\theta\right]$

6 b) $G_1=\frac{1}{1-\kappa}\left[\frac{\widetilde{W}_0}{\pi}+\kappa G_3\right]$

7 b) $\frac{\Gamma^2}{2\pi U^2 S}\log\left[1+\left(\frac{s'}{h}\right)^2\right]$, where S is the wing area.

8 a) $\frac{\pi}{16}\frac{\Gamma}{\sigma}$ b) 86.8% leader, 84.0% followers

9 a) $U-\frac{\Gamma}{\pi}\frac{\sigma h}{\left(x^2+y^2+h^2\right)^{3/2}}$ b) $p-p_\infty=\frac{\rho U\Gamma}{\pi}\frac{\sigma h}{\left(x^2+y^2+h^2\right)^{3/2}}$

10 a) $y=(1-\gamma)s$

11 a) (i) 26.6 (ii) 5.93° b) (i) 25.2

12 a) $w_{d1}=6.50\frac{\Gamma_1}{4\pi c}-1.17\frac{\Gamma_2}{4\pi c}$, $w_{d2}=-0.86\frac{\Gamma_1}{4\pi c}+5.65\frac{\Gamma_2}{4\pi c}$

b) $\frac{\Gamma_1}{4\pi cU\alpha}=0.191$, $\frac{\Gamma_2}{4\pi cU\alpha}=0.206$

W N Dawes, February 2014