Module 3F2: Systems and Control
EXAMPLES PAPER 2 — ROOT-LOCUS

Solutions

1. (a) Equilibria at $x^2 = y^2 = 1/2$, so
$(x, y) = (-0.707, -0.707), (-0.707, 0.707), (0.707, -0.707), (0.707, 0.707)$.
Linearization

$$A = \begin{bmatrix} -2x & 2y \\ -2x & -2y \end{bmatrix}$$

evaluated at the equilibrium points.

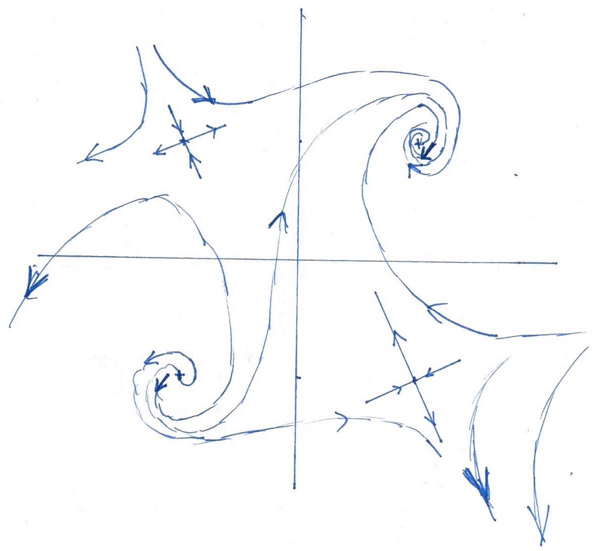
At $(0.707, 0.707)$, complex eigenvalues $-1.4142 \pm 1.4142i$, so spiralling in. Considering a point just to the right of the equilibrium (same y, larger x) we see that $\dot{x}$ and $\dot{y}$ are both negative, with the same magnitude, and so the trajectory is 45° down to the left (arrow shown). So, the spirals are clockwise.

At $(-0.707, -0.707)$, complex eigenvalues $1.4142 \pm 1.4142i$, so spiralling out. Considering a point to the left of the equilibrium, this time we see that the spirals are counter-clockwise.

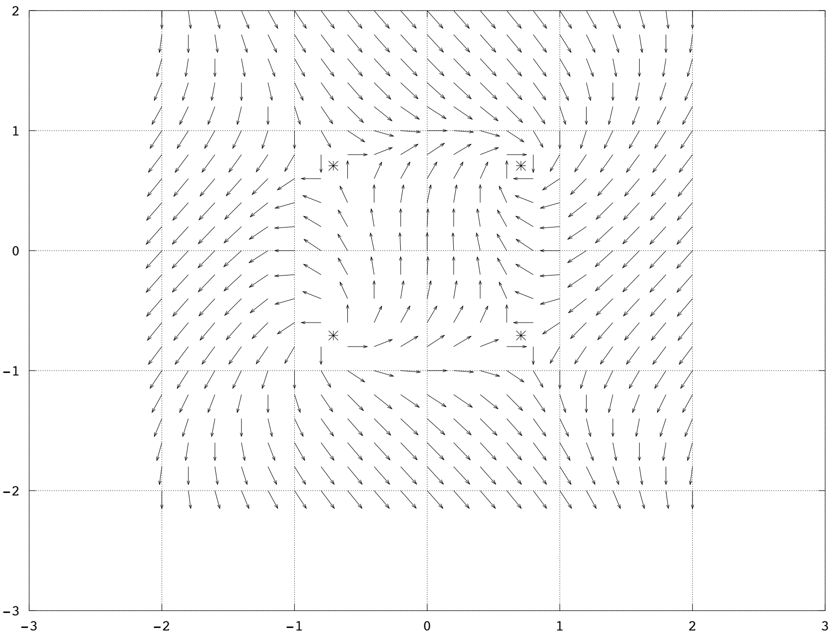
At $(0.707, -0.707)$, eigenvalues at -2, with eigenvector $[.92, .38]^T$ and 2 with eigenvector $[.38, -.92]^T$

At $(-0.707, 0.707)$, eigenvalues at 2, with eigenvector $[.92, .38]^T$ and -2 with eigenvector $[.38, -.92]^T$

Marking on the spirals and the stable and unstable manifolds at the equilibria, it is apparent that the state space trajectories must look something like:



For completeness, the actual vector field is shown here



(b) System will either end up at the only stable equilibrium, $(u/\sqrt{2}, u/\sqrt{2})$, or the trajectories will go to ∞.

2. (a) **For the system**

$$L(s) = \frac{1}{(s+a)(s+b)} \qquad (a, b \textbf{ both real})$$

show that the root-locus diagram (for positive gains k) consists of the segment of the real axis between $-a$ and $-b$, and the perpendicular bisector of that segment.

Using *Rule 3* (see Lecture Notes 2) every point on the real axis between the two poles, namely every point on the segment between $-a$ and $-b$, is on the root-locus, since every such point is to the left of one pole.

Using *Rule 5* there are 2 asymptotes, perpendicular to the real axis (angles $(2\ell+1)\pi/2, \ell = 0, 1$). These asymptotes emanate from the point $(-a-b)/2$, namely the mid-point of the segment between $-a$ and $-b$.

We are asked to show that every point on these asymptotes is actually on the root-locus itself. Consider a point s_0 on one of these asymptotes. From Figure 1 it is clear that $\angle(s_0+a) + \angle(s_0+b) = \pi$ (geometry of isosceles triangles). Hence every such point satsifies the *Angle Condition* and so is on the root-locus.

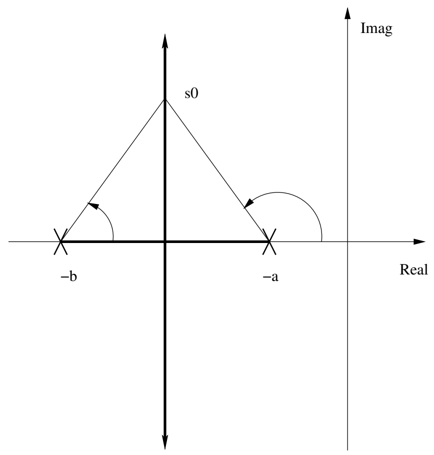


Figure 1:

Note that this question can also be answered by considering the roots of the quadratic equation $1 + kL(s) = 0$, or

$$s^2 + (a+b)s + (ab+k) = 0$$

and noting that the sum of the two roots is always $-(a+b)$.

(b) **Sketch the root-locus diagram for positive gains k for the system**

$$L(s) = \frac{1}{s(s+1)^2}$$

From *Rule 3* every point on the negative real axis is on the root-locus — because for $-1 < s_0 < 0$, s_0 is to the left of one pole, and for $s_0 < -1$ it is to the left of 3 poles.
There are 3 poles and no zeros ($n = 3, m = 0$), so by *Rule 5* there are 3 asymptotes, making angles $(2\ell+1)\pi/3, \ell = 0, 1, 2$ with the positive real axis, and the asymptotes emanate from $(-1-1+0)/3 = -2/3$.
Look for breakaway points, using *Rule 4*:

$$\begin{aligned}
\frac{d}{ds}L(s) &= \frac{d}{ds}\{s^{-1}(s+1)^{-2}\} \\
&= -s^{-2}(s+1)^{-2} - 2s^{-1}(s+1)^{-3} \\
&= \frac{-(s+1)-2s}{s^2(s+1)^2} = \frac{-3s-1}{s^2(s+1)^2} \\
&= 0 \quad \text{if } s = -\frac{1}{3}
\end{aligned}$$

This leads to the sketch shown in Figure 2. *Note: The breakaway point need not be located accurately, since the question says 'sketch', so the use of Rule 4 is optional here.*

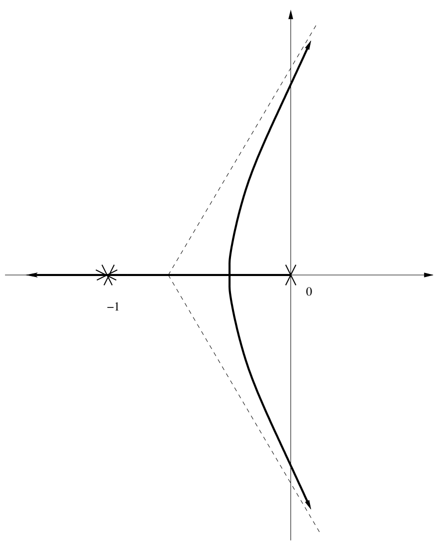


Figure 2:

Find the (positive) value of k at which closed-loop stability is lost

(i) from your diagram, and

(ii) using the Routh-Hurwitz criterion.

(i) *From the diagram:* At any point s_0 on the root-locus, the *Angle Condition* is satisfied: $\angle(s_0) + 2\angle(s_0+1) = \pi$ (since one pole is at 0, and two are at -1). Let's find exactly where the root-locus crosses the imaginary axis — that's the point at which closed-loop stability is lost. If s_0 is on the imaginary axis then $\angle s_0 = \pi/2$. Hence the angle condition reduces to $2\angle(s_0+1) = \pi/2$, or $\angle(s_0+1) = \pi/4$. Hence we must have $s_0 = j1$ — see Figure 3. Now find the corresponding gain

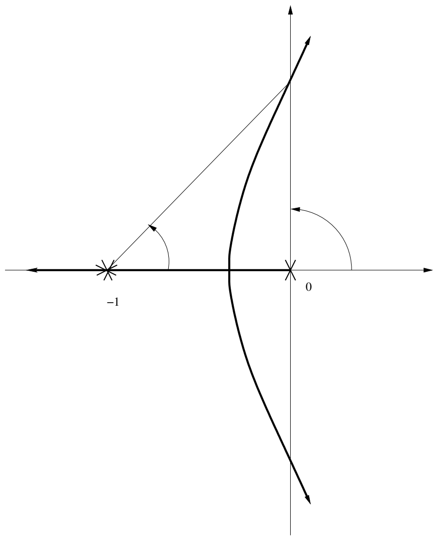


Figure 3:

from (see section 2.3 of Lecture Notes 2):

$$k = \frac{1}{|L(s_0)|} = |s_0| \times |s_0 + 1| \times |s_0 + 1| = 1 \times \sqrt{2} \times \sqrt{2} = 2.$$

(ii) *From Routh-Hurwitz criterion:* The closed-loop poles are the solutions of $1 + kL(s) = 0$, namely:

$$\begin{aligned} 1 + k\frac{1}{s(s+1)^2} &= 0 \\ \Rightarrow s(s+1)^2 + k &= 0 \\ \Rightarrow s^3 + 2s^2 + s + k &= 0 \end{aligned}$$

Now the Routh-Hurwitz criterion for this to have all solutions with negative real parts is (see section 3 of Lecture Notes 2, with $n = 3$):

$$2 \times 1 > 1 \times k$$

which is just violated when $k = 2$.

(c) **Draw the root-locus diagram for positive gains k for the system**

$$L(s) = \frac{s}{(s + 0.5)(s + 1)}$$

and hence show that the closed-loop system is stable for all $k > 0$. Also sketch the root-locus diagram for negative gains, and find the value of k at which closed-loop stability is lost.

$k > 0$: Using *Rule 3*, one branch of the root-locus is the real axis between -0.5 and 0, and the other branch is the real axis to the left of -1. And that is the whole of the root-locus. So both roots are real and negative for all $k > 0$, and hence the closed-loop is (asymptotically) stable.

$k < 0$: Using *Rule 3* — modified for $k < 0$ — every point on the real axis to the right of 0 is on the root-locus, as is every point on the real axis between -1 and -0.5. There is a breakaway point somewhere between -1 and -0.5 (could be calculated using *Rule 4* but not important for rough sketch) and another one somewhere to the right of 0 — see Figure 4. The negative gain at which stability is just lost can be calculated in (at least) two ways:

1. Analytically, since only second-order in this case:
   $1 + k\frac{s}{(s+1)(s+0.5)} = 0$
   $\Rightarrow (s + 1)(s + 0.5) + ks = 0$
   $\Rightarrow s^2 + (1.5 + k)s + 0.5 = 0$
   $\Rightarrow$ stability just lost when $1.5 + k = 0$ or $k = -1.5$.

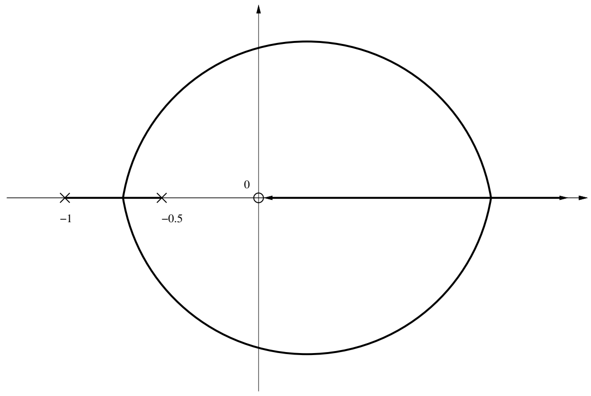


Figure 4:

2. Using root-locus methods: First find the value of ω_0, where $s_0 = j\omega_0$ is the point at which the root-locus crosses the imaginary axis.
The *angle condition* for $k < 0$ is: $\angle(j\omega_0 + 1) + \angle(j\omega_0 + 0.5) - \angle(j\omega_0) = 0$
$\Rightarrow \alpha + \beta - \frac{\pi}{2} = 0$, so $\alpha + \beta = \frac{\pi}{2}$, where $\alpha = \angle(j\omega_0 + 1)$ and $\beta = \angle(j\omega_0 + 0.5)$.
Now $\tan\alpha = \omega_0$, $\tan\beta = 2\omega_0$, and $\tan(\alpha + \beta) = \infty$.
But $\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$, so $\tan\alpha\tan\beta = 1$,
hence $2\omega_0^2 = 1$ or $\omega_0 = 1/\sqrt{2}$. (This could also be estimated graphically by trial and error — in more complicated cases an analytical solution would not be possible.)
Now the corresponding gain must be found, using the method shown in section 2.3 of Lecture Notes 2:
$|k| = \frac{|j/\sqrt{2}+1| \times |j/\sqrt{2}+0.5|}{1/\sqrt{2}} = \frac{\sqrt{3/2} \times \sqrt{3/4}}{1/\sqrt{2}} = \frac{3}{2}$.
Hence $k = -3/2$.

3. (a) **Sketch the variation of the closed-loop poles as the tacho feedback gain k_d varies (i) using root-locus construction rules**

The closed-loop poles are the solutions of
$s^2 + 10k_d s + 25 = 0$
$\Rightarrow (s^2 + 25) + 10k_d s = 0$
$\Rightarrow 1 + k_d \frac{10s}{s^2+25} = 0$ which is in the root-locus form.
So the root-locus diagram has one zero at 0 and two poles at $\pm 5j$. By *Rule 3* every point on the negative real axis is on the root-locus. By *Rule 5* there is one asymptote, along the negative real axis, and by *Rule 2* the zero attracts one branch of the root-locus. So there must be one breakaway point where the two complex roots become real, and this can *(optionally)* be calculated using *Rule 4*:

$$\begin{aligned} \frac{d}{ds}\left(\frac{s}{s^2+25}\right) &= 0 \\ \Rightarrow \frac{1(s^2+25) - s(2s)}{(s^2+25)^2} &= 0 \\ \Rightarrow -s^2 + 25 = 0 \quad \Rightarrow & \quad s = \pm 5 \end{aligned}$$

so the breakaway point is at -5. ($+5$ is not on the locus.)
The locus traced by the two complex branches is in fact a semi-circle, but this is not easy to show by this method. See Figure 5.

Sketch the variation of the closed-loop poles as the tacho feedback gain k_d varies (ii) by finding an explicit expression for the closed-loop poles.

The closed-loop poles are the solutions of $s^2 + 10k_d s + 25 = 0$, namely:

$$s = \frac{-10k_d \pm \sqrt{(10k_d)^2 - 4 \times 25}}{2} = -5\left(k_d \pm \sqrt{k_d^2 - 1}\right)$$

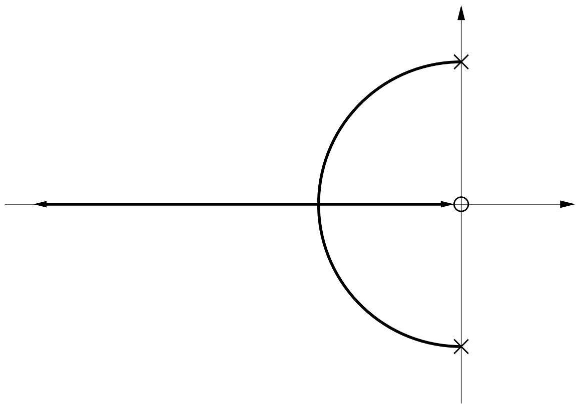


Figure 5:

When $k_d < 1$ these are complex conjugates, $-5\left(k_d \pm j\sqrt{1-k_d^2}\right)$, with modulus $5\sqrt{k_d^2 + (1-k_d^2)} = 5$, so the locus for $k_d < 1$ lies on a circle of radius 5. (This can also be seen by noting that $\omega_n = 5$ independently of k_d.)
When $k_d > 1$ these are two real roots, $s_1 < -5$, $-5 < s_2 < 0$ (since $\sqrt{k_d^2 - 1} < k_d$).

(b) **What is the damping factor of the closed loop as a function of k_d?**

The closed-loop characteristic polynomial is $s^2 + 10k_d s + 25$. Comparing this to the standard second-order form $s^2 + 2\zeta\omega_n s + \omega_n^2$ (or $\frac{s^2}{\omega_n^2} + \frac{2\zeta}{\omega_n} + 1$) gives $\omega_n = 5$ and hence $10\zeta = 10k_d$, hence $\zeta = k_d$.

Sketch the time response of the load angular position to a step change of 1 radian in desired angular position for values of $k_d = 0.6$ and $k_d = 1.2$.

This is a second-order system with $\zeta = k_d$, and $\omega_n = 5$. From Examples Paper 1, Q.1(b) we know that the closed-loop transfer function from θ_d to θ is $G(s) = 25/(s^2 + 10k_d s + 25)$, so the steady-state gain is $G(0) = 1$, hence the final value of θ will be 1 rad. Now get the step response sketches from the Mechanics Data Book, paying attention to the correct calibration of the time axis (the Data Book shows $\omega_n t$), and showing the final value correctly. See Figure 6.

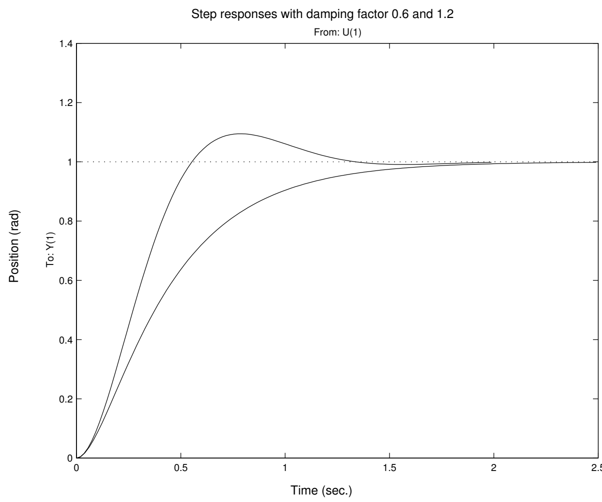


Figure 6:

4. **A negative feedback system consists of a plant whose transfer function is that given in Question 1(b), and a controller which is just a positive gain k. The reference signal is a ramp $r(t) = 2t$. Suppose that k is set to that value which gives two coincident real closed-loop poles at $-1/3$. What is this value of k, and what is the steady-state error $e = r - y$ (where y is the output of the plant) obtained with this value?**

   Use the root-locus diagram obtained for Question 1(b).
   The question tells you that the breakaway point on the root-locus diagram is at $-1/3$ (in case it has not been worked out in Question 1(b)). Working out the corresponding gain, using the method described in section 2.3 of Lecture Notes 2 gives:

$$k = \frac{1}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{4}{27}$$

   The transfer function from the reference input to the error is (from notes or work out from first-principles):

$$\frac{\bar{e}(s)}{\bar{r}(s)} = \frac{1}{1 + kL(s)}$$

   and recall that $\bar{r}(s) = \frac{2}{s^2}$. Apply the Final-Value Theorem to get the steady-state value:

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} s\bar{e}(s) = \lim_{s\to 0} s \times \frac{1}{1 + kL(s)} \times \frac{2}{s^2} = \lim_{s\to 0} \frac{2}{s\left(1 + \frac{4/27}{s(s+1)^2}\right)} = \frac{54}{4} = \frac{27}{2}$$

   How should the gain be adjusted to reduce this error? What can be said about the locations of the closed-loop poles if this is done?

   From the working above it can be seen that the steady-state error is, in general, $2/k$. So the gain should be increased to reduce the steady-state error. From the root-locus diagram the effect of this will be to make two of the closed-loop poles complex. If the gain is increased too far, these will become very underdamped or even unstable. [From the answer to Question 1(b) it is seen that stability is lost for $k > 2$. So the smallest steady-state error achievable is 1, but in practice rather greater than that in order to have reasonable closed-loop damping (stability margins).]

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