3F4: Data Transmission (Lent 2019)
Examples Paper 2

1. *Orthonormal basis for QAM*: Recall that the QAM signal is

$$x(t) = \sum_{k \in \mathbb{Z}} \left[X_k^r f_k^r(t) \, + X_k^i f_k^i(t) \right],$$

where

$$f_k^r(t) = p(t - kT) \, \sqrt{2} \cos(2\pi f_c t),$$
$$f_k^i(t) = -p(t - kT) \, \sqrt{2} \sin(2\pi f_c t).$$

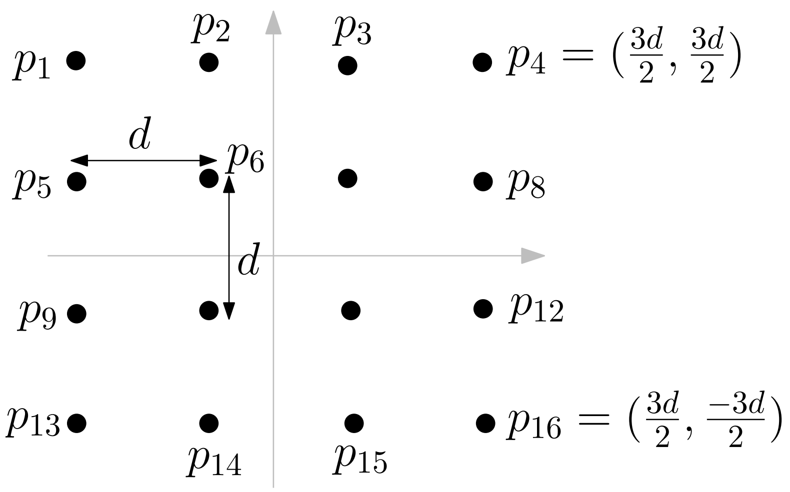
Assume that $p(t)$ is a baseband pulse with bandwidth $W \ll f_c$, and is chosen so that $p(t)$ is orthogonal to $p(t - nT)$ for all non-zero integers n.

Prove that $\left\{ f_k^r(t), \, f_k^i(t) \right\}, \ k \in \mathbb{Z}$ is an orthonormal set of functions. That is, you need to show that

$$\langle f_k^r(t), \, f_\ell^r(t) \rangle = \mathbf{1}\{k = \ell\}, \quad \langle f_k^i(t), \, f_\ell^i(t) \rangle = \mathbf{1}\{k = \ell\}, \quad \text{and} \quad \langle f_k^r(t), \, f_\ell^i(t) \rangle = 0 \text{ for all } k, \ell \in \mathbb{Z}.$$

Hint: To evaluate integrals such as $\int_{-\infty}^{\infty} p(t - kT) p(t - \ell T) \cos(4\pi f_c t) dt$, you can use Parseval's multiplication theorem, which says that for real-valued functions $x(t), y(t)$, the integral $\int x(t) y(t) dt = \int X(f) Y^*(f) df$.

2. *Quadrature Amplitude Modulation*: Consider the 16-QAM constellation shown in the figure below, with adjacent symbols in the vertical and horizontal directions spaced d apart.



This constellation is used for signalling (with uniform distribution on the symbols) over the AWGN channel

$$Y = X + N.$$

The noise N is a complex random variable, with real and imaginary parts being i.i.d. $\sim \mathcal{N}(0, N_0/2)$.

(a) Derive an upper bound for the probability of error when $X = p_1$ (or $X = p_4/p_{13}/p_{16}$, one of the corner points of the constellation).

(b) Derive an upper bound for the probability of error when $X = p_2$.

(c) Derive an upper bound for the probability of error when $X = p_6$.

(d) Using the union bound show that the average probability of error satisfies

$$P_e \leq 3Q\left(\frac{d}{\sqrt{2N_0}}\right) = 3Q\left(\sqrt{\frac{4E_b}{5N_0}}\right)$$

where E_b is the average energy per bit of the constellation.

3. *M-ary FSK*: After demodulation, an M-ary FSK receiver has the length-M vector $\mathbf{Y}$, given by

$$\mathbf{Y} = \mathbf{X} + \mathbf{N},$$

where $N_1, \ldots, N_M$ are i.i.d. Gaussian $\sim \mathcal{N}(0, N_0/2)$. If message i was transmitted, $\mathbf{X}$ has $\sqrt{E_s}$ in the ith entry and zeros elsewhere. Note that $E_s = E_b \log_2 M$ is the transmitted energy per symbol (message).

   (a) Derive the optimal detection rule for the M-ary FSK receiver, assuming all the messages are equally likely.

   (b) Show that the probability of detection error can be bounded as $P_e \leq \exp\left(-\frac{\log_2 M}{2}\left(\frac{E_b}{N_0} - 2\ln 2\right)\right)$. (The main steps are outlined in Handout 5. You also need to use the bound $\mathcal{Q}(x) < \frac{1}{2}e^{-x^2/2}$ for $x > 0$.)

   (c) Compare the bandwidth efficiency (rate/bandwidth) of M-ary FSK with M-ary QAM assuming that the bandwidth of the QAM signal is $2W$ where $W = \frac{1}{T}$. Can you give an intuitive explanation for why QAM is more bandwidth efficient than FSK as M grows large?

   (d) How do the probabilities of detection error for the two modulation schemes (M-QAM and M-FSK) compare as M grows large? (*Hint*: For a square M-QAM constellation (like in the previous question), use a union bound to show that the probability of error for any symbol can be bounded by $4\mathcal{Q}(\frac{d}{\sqrt{2N_0}})$; then use the fact that $E_s = E_b \log_2 M = \frac{(M-1)d^2}{6}$.)

4. *Signalling using an arbitrary two-point constellation.*

   (a) Let $N = (N^r, N^i)$ be a complex Gaussian random variable, whose components N_r, N_i are i.i.d. $\sim \mathcal{N}(0, \frac{N_0}{2})$. Consider another complex random variable $\tilde{N} = (\tilde{N}^r, \tilde{N}^i)$ obtained by rotating N by an angle θ, i.e., $\tilde{N} = e^{j\theta}N$. Here, $\tilde{N}^r$ and $\tilde{N}^i$ denote the real and imaginary parts of $\tilde{N}$.
   Prove that $\tilde{N}^r, \tilde{N}^i$ are i.i.d. Gaussian $\sim \mathcal{N}(0, \frac{N_0}{2})$. Hence the distribution of N is invariant to rotation.

   (b) Consider an arbitrary two-point complex constellation $\{p_1, p_2\}$. This constellation is used for signalling over the AWGN channel
   $$Y = X + N,$$
   where N is complex Gaussian noise with the distribution specified in part (a).
   Using the result of part (a), prove that the probability of detection error is given by $\mathcal{Q}\left(\sqrt{\frac{d^2}{2N_0}}\right)$, where d is the distance between p_1 and p_2.

5. *Error probability of M-ary PSK.* Consider an M-ary PSK constellation consisting of M points uniformly spaced around a circle of radius A. This constellation is used for signalling over the AWGN channel

$$Y = X + N,$$

where N is complex Gaussian noise with real and imaginary parts being i.i.d. $\sim \mathcal{N}(0, N_0/2)$.
Prove that the probability of detection error P_e can be bounded as

$$P_e \leq 2\mathcal{Q}\left(\sqrt{\frac{2A^2}{N_0}}\sin(\pi/M)\right) = 2\mathcal{Q}\left(\sqrt{\frac{2E_b \log_2 M}{N_0}}\sin(\pi/M)\right).$$

Hint: Express the error event as the union of two events, apply the union bound, and then use the result of the previous problem to obtain the upper bound.

6. *Detection in exponential noise.* Consider a symbol X that takes value either s_1 or s_2 with equal probability. You observe $Y = X + N$, where the noise random variable $N > 0$ is *exponentially distributed* with parameter μ, that is the pdf of N is given by

$$f_N(x) = \mu e^{-\mu x}\mathbf{1}\{x \geq 0\}.$$

(Here $\mathbf{1}\{\mathcal{E}\}$ denotes the indicator function, which is 1 if the event $\mathcal{E}$ is true, and zero otherwise.)
Derive the optimal decision rule to detect X from Y. Does this rule reduce to the minimum-distance decision rule (as in the case of Gaussian noise)?

7. *Effect of imperfect synchronisation at the receiver.* Consider a PAM transmitted waveform

$$x(t) = \sum_{k=-\infty}^{\infty} X_k p(t - kT),$$

with symbols $X_k \in \{\sqrt{E}, -\sqrt{E}\}$ (equally likely), and a rectangular pulse $p(t) = \frac{1}{\sqrt{T}}\mathbf{1}\{0 \leq t < T\}$.

The received waveform is $r(t) = x(t) + n(t)$, where $n(t)$ is white Gaussian noise with power spectral density $N_0/2$.

Suppose that we are interested in detecting X_0. To do this, the ideal demodulator would observe $r(t)$ in the interval $[0, T)$, and compute its inner product with $p(t)$. However, due to imperfect synchronisation, the demodulator instead observes $r(t)$ in the interval $[\Delta, T + \Delta)$, where $0 < \Delta < T$ is the timing error.

(a) Write an expression for the demodulator output Y_0.

(b) Evaluate the exact probability of detection error as a function of $\frac{E}{N_0}$ and $\epsilon = \frac{\Delta}{T}$.

8. *Zero-forcing equalisers.* For a single pulse transmitted over a dispersive channel, assuming no noise at the receiver the demodulator output is: $g(0) = 1$, $g(T) = -0.4$, $g(2T) = -0.2$, and $g(nT) = 0$ for all other integers n. Here $g(t)$ is the overall filter (the combination of transmit filter, channel response, and the receive filter), and T is the symbol time of the pulse.

(a) If PAM symbols are transmitted over this channel with overall filter $g(t)$, write an expression for the (unequalised) demodulator output r_m, obtained at time $t = mT$. Assume that there is additive noise $n(t)$ affecting the channel output.

(b) Determine the ideal zero-forcing equaliser, and draw a block diagram showing how you would implement it.

(c) Design a 4-tap FIR zero-forcing equaliser for this channel.

(d) Write an expression for the output of the FIR equaliser obtained in part(c), indicating the residual interference and the additive noise.

(e) What is the noise enhancement factor of the FIR equaliser, assuming that the noise random variables in the (unequalised) demodulator output are i.i.d. Gaussian?

9. *MMSE Equaliser.* The discrete-time sequence at demodulator output of a dispersive channel is given by

$$r_m = \sum_{\ell=0}^{L} g_\ell X_{m-\ell} + n_m, \qquad \text{for } m = 0, 1, \ldots. \tag{1}$$

A $(K+1)$-tap MMSE equaliser $\underline{c} = [c_0, \ldots, c_K]$ estimates the mth information symbol as $\hat{X}_m = \underline{c}^T \underline{r}$, where

$$\underline{c} = \begin{bmatrix} c_0 \\ \vdots \\ c_K \end{bmatrix} \quad \text{and } \underline{r} = \begin{bmatrix} r_m \\ \vdots \\ r_{m+K} \end{bmatrix}.$$

It was shown in Handout 7 that the MMSE equaliser is given by $\underline{c} = \mathbf{R}^{-1}\mathbf{p}$, where

$$\mathbf{R} = \mathbb{E}[\underline{r}\underline{r}^T], \quad \mathbf{p} = \mathbb{E}[\underline{r}\, X_m]. \tag{2}$$

(a) Considering Eq. (1) for $m, m+1, \ldots, m+K$, show that $\underline{r}$ can be expressed as

$$\underline{r} = \mathbf{U} \begin{bmatrix} X_{m-L} \\ X_{m-L+1} \\ \vdots \\ X_{m+K}, \end{bmatrix} + \underline{n},$$

where $\mathbf{U}$ is a $(K+1) \times (L+K+1)$ matrix with entries determined by $g_0, \ldots, g_L$, and $\underline{n} = [n_m, \ldots, n_{m+K}]^T$.

(b) Assume that the information symbols $\{X_k\}_{k\in\mathbb{Z}}$ are chosen i.i.d. from a PAM constellation with zero mean and average symbol energy $\mathcal{E}$. Also assume that the noise variables $\{n_k\}_{k\in\mathbb{Z}}$ are i.i.d. $\mathcal{N}(0, N_0/2)$. Then show that the matrix $\mathbf{R}$ and the vector $\mathbf{p}$ in (2) can be computed as

$$\mathbf{R} = \mathcal{E}\mathbf{U}\mathbf{U}^T + \frac{N_0}{2}\mathbf{I}, \quad \text{and} \quad \mathbf{p} = \mathcal{E}\,\underline{u}_L,$$

where $\mathbf{I}$ is the $(K+1)\times(K+1)$ identity matrix, and the columns of $\mathbf{U}$ are denoted by $\underline{u}_0, \ldots, \underline{u}_{L+K}$.

10. *OFDM.*

(a) A Digital Audio Broadcast (DAB) system uses coded QPSK modulation on an orthogonal frequency division multiplexed set of carriers. The system has the following parameters.

- Channel bandwidth: 2.4 MHz
- Frequency spacing between sub-carriers: 1000 Hz
- Symbol rate on each sub-carrier: 750 symbol/s.
- Rate of error-correcting code: $\frac{1}{2}$.

Calculate the maximum bit rate available to the user, and the duration (in secs.) of the guard interval used.

(b) A digital TV system employs coded OFDM with 64-QAM as the underlying modulation method.

The signal bandwidth is 9 MHz, the spacing between sub-carriers is 5 kHz, and the duration of the guard band is 10 μs. A rate $\frac{2}{3}$ code is for error correction. 10% of the sub-carriers are reserved for pilot tones for carrier phase and amplitude recovery (i.e., no user information is transmitted on these sub-carriers).

Calculate the user data rate, and hence the number of TV channels that could be accommodated within this COFDM signal. (Assume that one TV channel requires a data rate of 4Mb/s.)

(c) Compute the bandwidth efficiency (bits/s of user data rate per Hz of bandwidth) of the digital TV system in part (b) and the DAB system in part(a).

Answers to Selected Questions

7 (b). $P_e = \frac{1}{2}\left(\mathcal{Q}\left(\sqrt{\frac{2E}{N_0}}\right) + \mathcal{Q}\left(\sqrt{\frac{2E(1-2\epsilon)^2}{N_0}}\right)\right)$.

8. (c) 4-tap FIR ZF equaliser: $[h_0,\, h_1,\, h_2,\, h_3] = [1,\, 0.4,\, 0.36,\, 0.224]$. (e) The noise variance at the output of the FIR equaliser is 1.3398 times the noise variance of the unequalised output.

10. a) Maximum user data rate = 1.8 Mb/s, duration of guard interval = 0.333 ms. b) User data rate = 30.86 Mb/s, max. number of TV channels = 7. c) Bandwidth efficiency for TV signal = 3.43 bits/s per Hz, for DAB signal =0.75 bits/s per Hz.