3F1 Signals and Systems

(5) Discrete time systems as filters; final value theorem

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Recap

Where are we?

- What is a (discrete time) signal? What is a system/filter?
- Basic mathematical tools (z-transforms, inverse transform).
- Time/Frequency representations
  (difference equation, transfer function, convolution representation)
- System input/output behaviour: pulse response, FIR/IIR, basic properties (linearity, causality, stability).

In this lecture we will relate the internal description of a system/filter,

$$G(z)$$

to its **input-output** behaviour,

$$Y(z) = G(z)U(z)$$

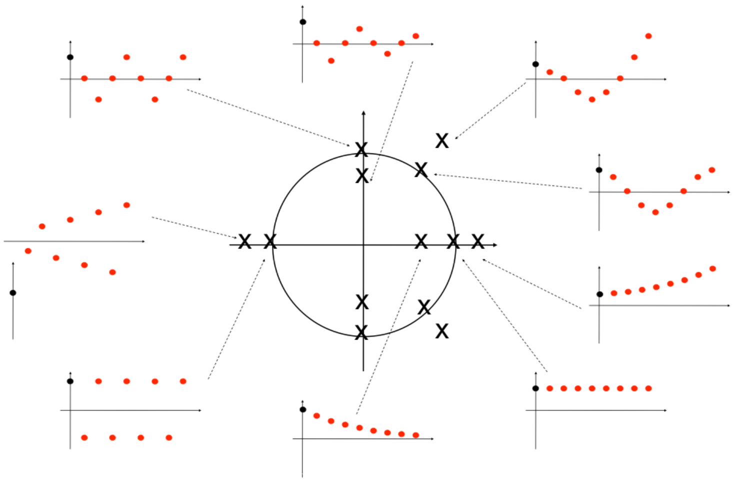
Consider a system $G(z)$.

Subjected to a pulse input: $u = (1, 0, 0, \dots)$. Then $U(z) = 1$.

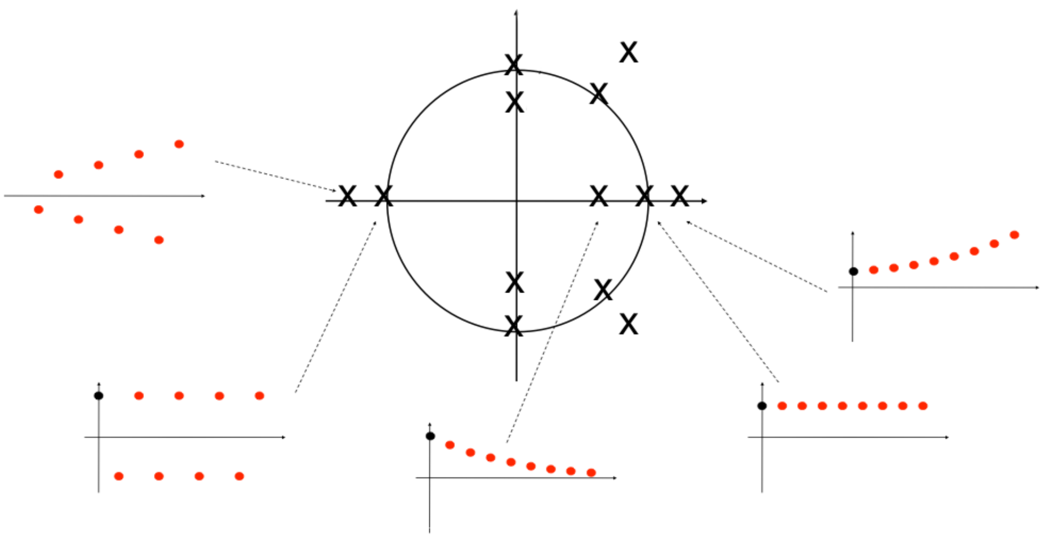
What is the output?

$$y = \mathcal{Z}^{-1}[Y(z)] = \mathcal{Z}^{-1}[G(z)U(z)] = \mathcal{Z}^{-1}[G(z)]$$

$\Rightarrow$ poles of $G(z)$ define the response to a pulse



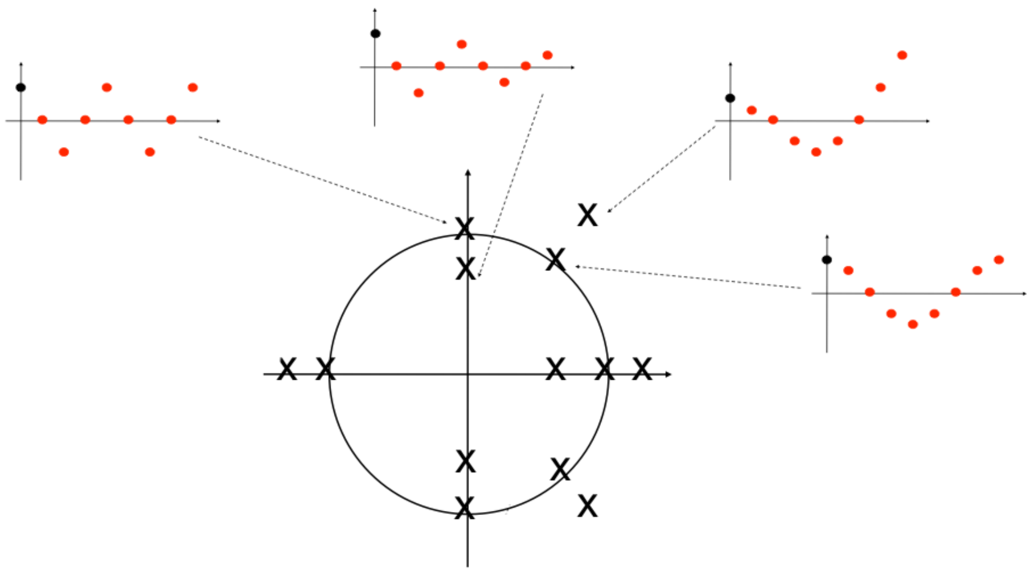
For real poles: $G(z) = \dfrac{1}{1-\lambda z^{-1}} \quad \xrightarrow{\mathcal{Z}^{-1}} \quad \lambda^k$



For complex poles: $G(z) = \dfrac{1}{1-(\lambda e^{j\theta})z^{-1}} + \dfrac{1}{1-(\lambda e^{-j\theta})z^{-1}}$

$$\xrightarrow{\mathcal{Z}^{-1}} \qquad \lambda^k \left(e^{j\theta k} + e^{-j\theta k}\right) = 2\lambda^k \cos(\theta k)$$

(real impulse response $\rightarrow$ pair of complex conjugate poles)



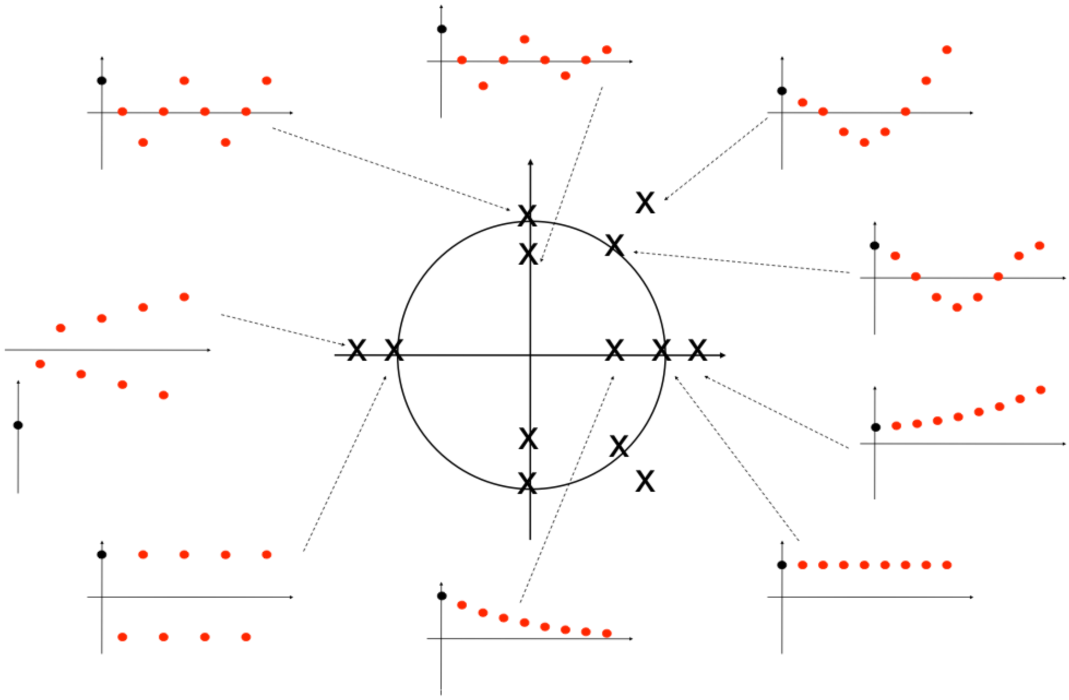
- what about repeated poles? $G(z) = \frac{1}{(1-pz^{-1})^2}$

  rewrite as $G(z) = \frac{1}{1-pz^{-1}} + \frac{pz^{-1}}{(1-pz^{-1})^2}$ then $\xrightarrow{\mathcal{Z}^{-1}} p^k + kp^k$

- what about $G(z) = \frac{1}{1+2\xi\omega_n z^{-1}+\omega_n^2 z^{-2}}$ where $\xi < 1$?

  Show that $G(z)$ can be written as $G(z) = \frac{\alpha}{1+\beta z^{-1}} + \frac{\alpha^*}{1+\beta^* z^{-1}}$.
  Then $\xrightarrow{\mathcal{Z}^{-1}} \alpha\beta^k + \alpha^*(\beta^*)^k$. Why is this a real signal?

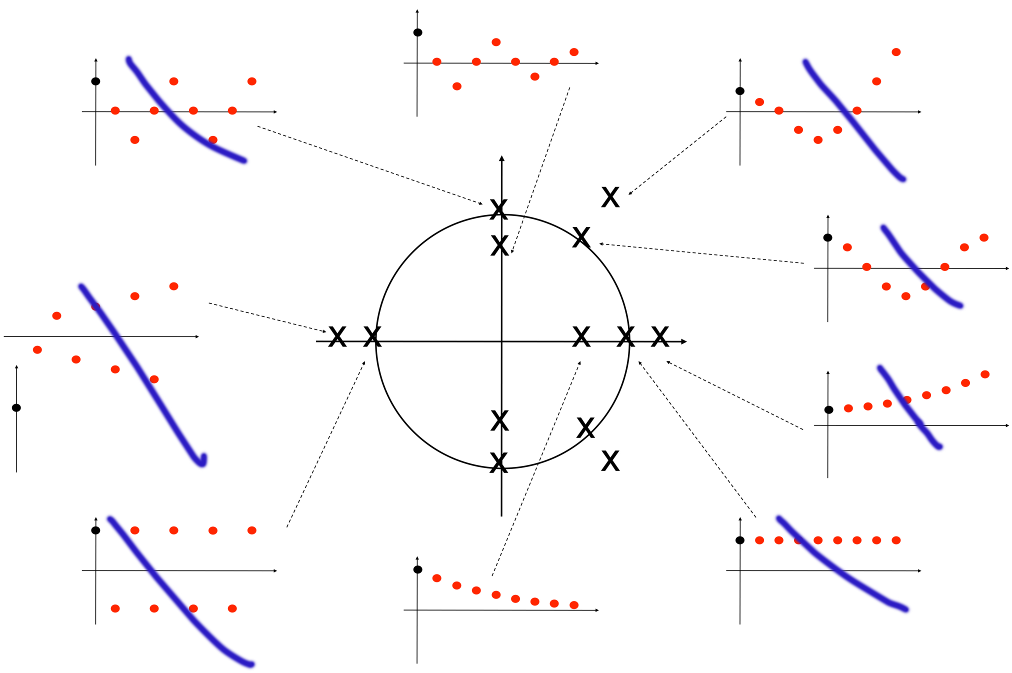
  Using polar coordinates $\alpha = |\alpha|e^{j\angle(\alpha)}$ $\beta = |\beta|e^{j\angle(\beta)}$ show that
  $\alpha\beta^k + \alpha^*(\beta^*)^k = 2|\alpha||\beta|^k \cos(\angle(\alpha) + \angle(\beta)k)$.



- distance of poles from origin is a measure of decay rate
- complex poles just inside unit circle give lightly damped oscillation
- oscillation is possible for real poles on negative real axis

Recall: stability of the system is equivalent to

- poles in the unit circle $|p_i| < 1$
- bounded impulse response: $\sum_{k=0}^{\infty} |g_k|$ finite.



Important fact:

a stable filter "forgets" the initial conditions.

In the z-domain any linear filter can be written as

$$A(z)Y(z) = B(z)U(z) + C(z, y_i)$$

$C(z, y_i)$ takes into account the *initial conditions* of the filter

- For example, the filter

$$y(k+2) + \frac{1}{2}y(k) = u(k)$$

  in the z-domain has coefficients

$$A(z) = z^2 + \frac{1}{2}, \; B(z) = 1, \; C(z, y_i) = z^2 y_0 + z y_1.$$

  Usually $C(z, 0) = 0$ and $G(z) = B(z)/A(z)$.

If we write an arbitrary filter as:

$$A(z)Y(z) = B(z)U(z) + C(z, y_i)$$

then

$$\begin{aligned} \lim_{k\to\infty} y(k) &= \lim_{k\to\infty} \mathcal{Z}^{-1}\left[\frac{B(z)}{A(z)}U(z) + \frac{C(z, y_i)}{A(z)}\right] \\ &= \lim_{k\to\infty} \mathcal{Z}^{-1}\left[\frac{B(z)}{A(z)}U(z)\right]. \end{aligned}$$

- Stable roots in $A(z)$ enforce the exponential decay of $\mathcal{Z}^{-1}[\frac{C(z,y_i)}{A(z)}]$.
- $\frac{B(z)}{A(z)}U(z)$ may not decay exponentially for $U(z) \neq 1$.

Final Value Theorem: Suppose that all the poles of $(z-1)Y(z)$ lie strictly inside the unit circle. Then

$$\lim_{k\to\infty} y(k) = \lim_{z\to 1}(z-1)Y(z)$$

Proof: All poles of $Y(z)$ are in the unit circle except possibly a (non-repeated) pole at $z=1$. Assume distinct poles, we can rewrite

$$Y(z) = \frac{\alpha_0}{1-z^{-1}} + \sum_i \frac{\alpha_i}{1-p_i z^{-1}}$$

where $|p_i| < 1$. By inverse z-transform:

$$\lim_{k\to\infty} y(k) = \lim_{k\to\infty}\left(\alpha_0 + \sum_i \alpha_i p_i^k\right) = \alpha_0 \ .$$

$$\lim_{z\to 1}(z-1)Y(z) = \lim_{z\to 1}\left(z\alpha_0 + (z-1)\sum_i \frac{\alpha_i}{1-p_i z^{-1}}\right) = \alpha_0 \ .$$

If poles are not distinct, the right-hand side of the $\sum$ is more convoluted but limits still converge to zero as $k\to\infty$ and $z\to 1$.

Application of theorem: step response

$$u(k) = 1 \text{ for } k \geq 0 \quad \stackrel{\mathcal{Z}}{\longrightarrow} \quad U(z) = \frac{1}{1-z^{-1}} = \frac{z}{z-1}$$

$$\begin{aligned}
\lim_{k\to\infty} y(k) &= \lim_{z\to 1}(z-1)Y(z) \\
&= \lim_{z\to 1}(z-1)G(z)U(z) \\
&= \lim_{z\to 1} zG(z) \\
&= G(1)
\end{aligned}$$

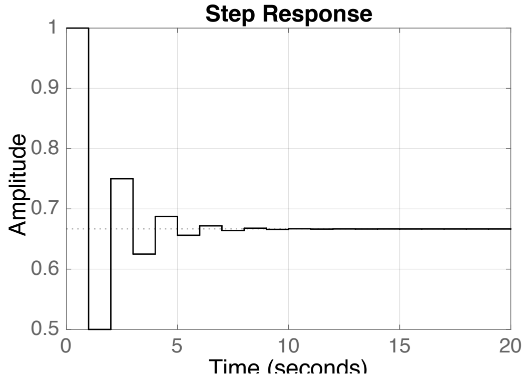
Example: consider the filter

$$y(k+1) + \frac{1}{2}y(k) = u(k+1)$$

z-transform with initial $y_0 = 0$

$$\left(z + \frac{1}{2}\right) Y(z) = zU(z)$$

$$\Rightarrow \quad Y(z) = \underbrace{\frac{2z}{2z+1}}_{G(z)} U(z)$$



Steady-state response to a unit step input $U(z) = \frac{z}{z-1}$

$$\lim_{k\to\infty} y(k) = G(1) = \frac{2}{3}$$