Part IIA Module 3C6

Small vibration of discrete systems

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1. Revision of normal modes without damping

1.1 Key points from Part I

(a) In a $\left\{ \begin{array}{c} \text{vibration mode} \\ \text{normal mode} \\ \text{natural mode} \\ \text{mode} \end{array} \right\}$ all points of the system oscillate sinusoidally *in the same phase* (or 180° out of phase).

(b) An N degree-of-freedom (DOF) system has N modes, each with its own natural frequency.

(c) Each mode *on its own* behaves like a single DOF system (i.e. a harmonic oscillator).

(d) The total system response to driving (harmonic or transient) is just a superposition of the modal responses.

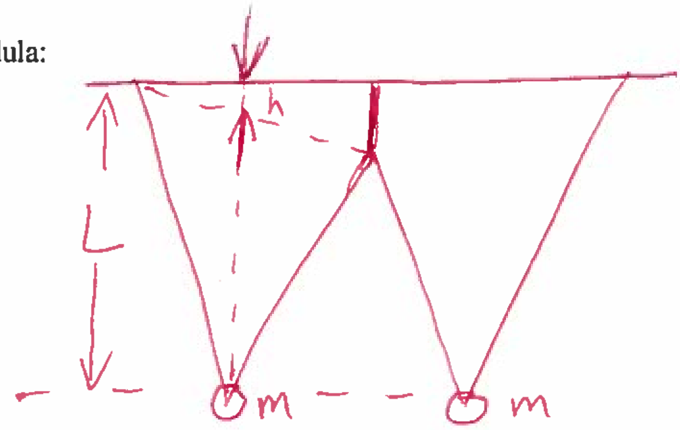
(e) A mode with *zero frequency* corresponds to a rigid-body displacement or rotation of the system.

(f) If the system is symmetric in some way, each mode is either symmetric or antisymmetric.

(g) Modes at lower frequencies involve motion with large length-scales. At higher frequencies, the modes become more intricate as their length-scales get shorter.

1.2. Example: symmetry and modal superposition

Two coupled pendula:



Observation: start one swinging, then gradually the other starts to move. After a while the first one stops and all the energy has transferred to the second. Then the process reverses, and repeats.

Using symmetry, the two modes are:

(a) Symmetric (swinging together) ie $[1, 1]$

Frequency $\omega_1 = \sqrt{g/L}$

(b) Antisymmetric $[1, -1]$

Frequency $\omega_2 = \sqrt{\frac{g}{L-h}} > \omega_1$

Start with equal amounts of both

$(1,1) + (1,-1) = (2,0)$ ie LH mass only

After half a beat period $\pi/(\omega_2 - \omega_1)$

have $(1,1) - (1,-1) = (0,2)$

i.e. only mass number 2 swinging. After another half-period, it is back to the original state with only mass number 1 swinging, and so on for ever in the absence of damping.

1.3 Modal calculations via Lagrange

Suppose we have an N-DOF system with generalised coordinates $q_1, q_2, \ldots, q_n$. The first stage in the Lagrangian procedure is to calculate the potential energy V and the kinetic energy T in terms of the q's and $\dot{q}$'s. To deal with *small* vibrations (about a position of stable equilibrium) we carry out series expansions of both V and T, keeping only the first interesting term which doesn't vanish:

$$V = V_0 \quad + \quad \sum_j V_j q_j \quad + \quad \frac{1}{2}\sum_j \sum_k V_{jk} q_j q_k \quad + \ldots\ldots$$

A constant, can be ignored

A vector of constant coefficients, which must vanish if $q = 0$ is equilibrium since $\frac{\partial V}{\partial q_j} = 0$

For convenience

A matrix of constant coefficients

i.e. $V \approx \frac{1}{2}\sum_j \sum_k V_{jk} q_j q_k$, a "quadratic form".

The matrix $[K]$ whose terms are V_{jk} is the *stiffness matrix*. We can always chose the values so as to make it *symmetric*.

The kinetic energy T consists of terms like "$\frac{1}{2}mv^2$", i.e. it is a quadratic expression

$$T = \frac{1}{2}\sum_j \sum_k T_{jk} \dot{q}_j \dot{q}_k$$

For convenience again

The coefficients T_{jk} may depend on the q's (things like "$\cos\theta$", for example) but for small vibrations we can approximate it by the constant values of T_{jk} at the equilibrium position $q = 0$. Including any higher terms in the Taylor series for T_{jk} would simply bring in terms of the third order and above in the q's and $\dot{q}$'s, since the expression is already second order.

Again, we can always choose to make the matrix $[M]$, whose terms are T_{jk}, *symmetric*. It is called the *mass matrix*.

Now we can use Lagrange's equations:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right)+\frac{\partial V}{\partial q_j}=\begin{cases}0 & \text{for free motion}\\ Q_j & \text{if externally driven}\end{cases}$$

i.e. $$\sum_k T_{jk}\ddot{q}_k+\sum_k V_{jk}q_k=\begin{cases}0\\ Q_j\end{cases}$$

These are linear equations, representing a set of coupled harmonic oscillators. Finding the vibration modes allows us to *uncouple* them. For a vibration mode, $q_j = u_j e^{i\omega t}$.

So we require

$$-\omega^2\sum_k T_{jk}u_k+\sum_k V_{jk}u_k=0$$

for free motion,

i.e. $$[K]\underline{u}=\omega^2[M]\underline{u}\ , \qquad (1)$$

which is (almost) a standard eigenvalue/eigenvector problem.

Solve it in the usual way:

(i) Solve $\det\left[K-\omega^2 M\right]=0$ for the natural frequencies $\omega=\omega_n$.

(ii) For each allowed ω_n, solve the simultaneous equations

$$[K]\underline{u}^{(n)}=\omega_n{}^2[M]\underline{u}^{(n)}$$

for the mode shape $\underline{u}^{(n)}$.

Since M and K are both symmetric, an extension of the results from the Part IA maths course shows that

(i) the values of $\omega_n{}^2$ are all *real*.

(ii) (proved at the end of this section) the eigenvectors are *orthogonal*, in the sense that

$$\begin{cases}\underline{u}^{(n)^t}M\underline{u}^{(m)}=0\\ \qquad\text{or}\\ \sum_j\sum_k T_{jk}u_j{}^{(n)}u_k{}^{(m)}=0\end{cases}\qquad \text{provided } n\neq m$$

It follows from (1) that $\begin{cases} \underline{u}^{(n)^t} K \underline{u}^{(m)} = 0 \\ \text{or} \\ \sum_j \sum_k V_{jk} u_j^{(n)} u_k^{(m)} = 0 \end{cases}$ provided $n \neq m$

The result (ii) makes it natural to *normalise* so that $\begin{cases} \underline{u}^{(n)^t} M \underline{u}^{(n)} = 1 \\ \sum_j \sum_k T_{jk} u_j^{(n)} u_k^{(n)} = 1 \end{cases}$

It then follows from equation (1) that $\begin{cases} \underline{u}^{(n)^t} K \underline{u}^{(n)} = \omega_n^2 \\ \sum_j \sum_k V_{jk} u_j^{(n)} u_k^{(n)} = \omega_n^2 \end{cases}$

These results allow us to see the significance of modes. First, express a general motion of the system as a linear combination of modal deformations:

$$\underline{q} = \alpha_1(t)\underline{u}^{(1)} + \alpha_2(t)\underline{u}^{(2)} + \ldots + \alpha_N(t)\underline{u}^{(N)}.$$

The α's are called "normal coordinates".

Now $\quad M\ddot{\underline{q}} + K\underline{q} = 0$,

i.e. $\quad \ddot{\alpha}_1 M\underline{u}^{(1)} + \ddot{\alpha}_2 M\underline{u}^{(2)} + \ldots + \alpha_1 K\underline{u}^{(1)} + \alpha_2 K\underline{u}^{(2)} + \ldots = 0$.

Now pre-multiply by $\underline{u}^{(1)t}$, and we find:

$$\ddot{\alpha}_1 + \omega_1^2 \alpha_1 = 0$$

(all other terms vanish by the orthogonality results)

Similarly, pre-multiply by $\underline{u}^{(2)t}$:

$$\ddot{\alpha}_2 + \omega_2^2 \alpha_2 = 0$$

and so on — each normal coordinate (i.e. modal amplitude) obeys a harmonic oscillator equation, independent of all others.

Proof of orthogonality

Suppose we have two modes, with corresponding natural frequencies, so that

$$\begin{cases} K\underline{u}^{(n)} = \omega_n{}^2 M\underline{u}^{(n)} \\ K\underline{u}^{(m)} = \omega_m{}^2 M\underline{u}^{(m)} \end{cases}$$

Then $$\begin{cases} \underline{u}^{(m)^t} K\underline{u}^{(n)} = \omega_m{}^2 \underline{u}^{(m)^t} M\underline{u}^{(n)} \\ \underline{u}^{(n)^t} K\underline{u}^{(m)} = \omega_m{}^2 \underline{u}^{(n)^t} M\underline{u}^{(m)} \end{cases}$$

But the two left-hand sides are equal because K is symmetric:

$$\underline{u}^{(m)^t} K\underline{u}^{(n)} = \left[\underline{u}^{(m)^t} K\underline{u}^{(n)}\right]^t \text{ as it is a scalar}$$

$$= \underline{u}^{(n)^t} K^t \underline{u}^{(m)}$$

$$= \underline{u}^{(n)^t} K\underline{u}^{(m)}$$

Similarly, $\underline{u}^{(m)^t} M\underline{u}^{(n)} = \underline{u}^{(n)^t} M\underline{u}^{(m)}$

by the same argument, as M is symmetric.

So subtracting,

$$\left(\omega_n{}^2 - \omega_m{}^2\right)\underline{u}^{(n)^t} M\underline{u}^{(m)} = 0$$

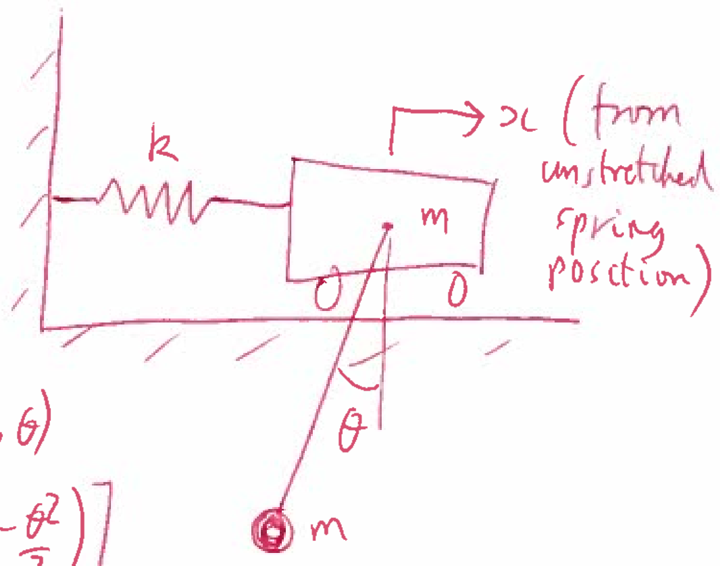
So if $\omega_n{}^2 \neq \omega_m{}^2$, we must have

$$\underline{u}^{(n)^t} M\underline{u}^{(m)} = 0$$

which is the required orthogonality result.

1.4. Example

(Recall 3C5 sheets 1/3)



Use generalised coordinates x, θ as shown.
$x = 0$, $\theta = 0$ at equilibrium.
Potential energy:

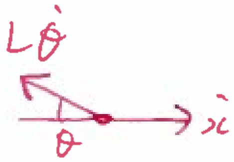
$$V = \tfrac{1}{2}kx^2 + mgL(1-\cos\theta)$$

$$\simeq \tfrac{1}{2}kx^2 + mgL\left[1-\left(1-\tfrac{\theta^2}{2}\right)\right]$$

$$= \frac{1}{2}\begin{bmatrix} x & \theta \end{bmatrix} \underbrace{\begin{bmatrix} k & 0 \\ 0 & mgL \end{bmatrix}}_{\text{Stiffness matrix } K} \begin{bmatrix} x \\ \theta \end{bmatrix}$$

Kinetic energy:

At pendulum mass:



$$T = \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}m\left[(\dot{x} - L\dot{\theta}\cos\theta)^2 + (L\dot{\theta}\sin\theta)^2\right]$$

$$= \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}m\left[\dot{x}^2 \underbrace{-2L\cos\theta\,\dot{x}\dot{\theta}}_{\text{Can take } \cos\theta = 1 \text{ here: } \dot{x}\dot{\theta} \text{ already small}} + L^2\dot{\theta}^2\right]$$

$$\therefore\ T \simeq m\dot{x}^2 - mL\dot{x}\dot{\theta} + \tfrac{1}{2}mL^2\dot{\theta}^2$$

$$= \frac{1}{2}\begin{bmatrix} \dot{x} & \dot{\theta} \end{bmatrix} \underbrace{\begin{bmatrix} 2m & -mL \\ -mL & mL^2 \end{bmatrix}}_{\text{Mass matrix } M} \begin{bmatrix} \dot{x} \\ \dot{\theta} \end{bmatrix}$$

To work out modes, choose the case $m = 1, k = 1, L = 1$ to save writing. Then

$$K = \begin{bmatrix} 1 & 0 \\ 0 & g \end{bmatrix}, \quad M = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

Natural frequencies from $\quad \det\left[K - \omega^2 M\right] = 0$

i.e. $$\begin{vmatrix} 1-2\omega^2 & \omega^2 \\ \omega^2 & g-\omega^2 \end{vmatrix} = 0$$

i.e. $$\left(1-2\omega^2\right)\left(g-\omega^2\right) - \omega^4 = 0$$

i.e. $$\omega^4 - (1+2g)\omega^2 + g = 0$$

i.e. $$\omega^2 = \frac{1}{2}\left[1+2g \pm \sqrt{(1+2g)^2 - 4g}\right]$$

$$= \frac{1}{2}\left[1+2g \pm \sqrt{1+4g^2}\right]$$

$$= \underset{(+)}{20.11}, \quad \underset{(-)}{0.487} \quad \text{with } g = 9.8$$

Mode vectors

$$K\underline{u} = \omega^2 M\underline{u}$$

i.e. $$\begin{cases} u_1 = \omega^2(2u_1 - u_2) & (1) \\ g u_2 = \omega^2(u_2 - u_1) \end{cases}$$

From (1) $\quad (2\omega^2 - 1)u_1 = \omega^2 u_2$

$$\therefore \frac{u_2}{u_1} = 2 - \frac{1}{\omega^2} = 2 - \frac{2}{1+2g \pm \sqrt{1+4g^2}}$$

$$= \underset{(+)}{1.95}, \quad \underset{(-)}{-0.052}$$

So the lower frequency mode has masses moving in the same direction, higher frequency has them moving in opposite directions.

Check: very easy using Matlab

All you need is:

```
K=[1 0;0 9.8];
M=[2 -1;-1 2];
[U,D]=eig(K,M)
```

Diagonal matrix of ω^2 values

Matrix of eigenvectors

or more generally

```
m=<whatever>;
k=<whatever>;
L=<whatever>;
g=9.8;
K=[k 0;0 m*g*L];
M=[2*m -m*L;-m*L 2*m*L^2];
[U,D]=eig(K,M)
```

Result of short program is

$$U = \begin{bmatrix} 0.9986 & 0.4563 \\ -0.0522 & 0.8898 \end{bmatrix}$$

$$D = \begin{bmatrix} 0.4873 & 0 \\ 0 & 20.1127 \end{bmatrix}$$

Agrees with calculation above

2. Vibration transfer functions

2.1 The response formula

The archetypal vibration measurement is to apply a sinusoidal force to one point on a structure and observe the response at another point. This transfer function can be expressed in terms of modes. Consider a harmonic force F at frequency ω, applied only to the jth generalised coordinate. Then response $\underline{q}$ is given by

$$M\ddot{\underline{q}} + K\underline{q} = \underline{Q}e^{i\omega t} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ F \\ 0 \\ \vdots \\ 0 \end{bmatrix} e^{i\omega t}$$

jth element

As before, write $\underline{q} = \sum_m \alpha_m \underline{u}^{(m)} e^{i\omega t}$.

Then $-\omega^2 \sum_m \alpha_m M\underline{u}^{(m)} + \sum_m \alpha_m K\underline{u}^{(m)} = \underline{Q}$

Pre-multiply by $\underline{u}^{(n)t}$ and use orthogonality:

$$-\omega^2\alpha_n + \alpha_n\omega_n^{\,2} = \underline{u}^{(n)t} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ F \\ 0 \\ \vdots \\ 0 \end{bmatrix} = F u_j^{(n)}$$

i.e. $\alpha_n = \dfrac{F u_j^{(n)}}{\omega_n^{\,2} - \omega^2}$.

So the response is given by

$$\underline{q} = \sum_n \alpha_n \underline{u}^{(n)} e^{i\omega t}$$

$$= F\sum_n \frac{u_j^{(n)}\underline{u}^{(n)}}{\omega_n^2 - \omega^2} e^{i\omega t}$$

Now the transfer function we want is the response at "point" k to drive at "point" j:

$$\boxed{G(j,k,\omega) = \frac{q_k}{F} = \sum_n \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2 - \omega^2}}$$

Most important result of the course

Many phenomena can be understood from this response formula.

2.2 Modal damping

Each term in the $\sum_n$ above is the response of a single undamped harmonic oscillator.

Before going any further we need to allow for some *damping* in the "modal oscillators".

A full discussion of damping is beyond this course. The mechanisms of damping are many, they are often not understood in detail, and they are often non-linear.

Examples:

* Internal (hysteric) dissipation within materials: very small for metals, but high for rubbers, some polymers etc.

* Friction at joints: hard to model, but often the *main* source of damping in engineering structures.

* Viscous damping in a surrounding fluid: usually small, unless deliberately induced (shock absorber, dashpot in door-closer)

* Radiation damping: energy loss into sound waves in air or water surrounding the structure, or sometimes structural waves going "off to infinity" (railway lines).

We will use an *ad hoc* approach which is good enough provided damping is *small*. We simply allow some "modal damping" for each modal oscillator, by analogy with the damped harmonic oscillator.

So we augment the response formula to

$$\boxed{G(j,k,\omega) = \sum_n \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2 + 2i\omega\omega_n c_n - \omega^2}}$$

where the c_n are dimensionless numbers (see Mechanics Data Book for 1-DOF case).

Some typical orders of magnitude:

Tuning fork	$c_n \approx 10^{-4}$
Guitar string	$c_n \approx 10^{-3}$
Built-up steel structure	$c_n \approx 10^{-2}$
Milk bottle "pop"	$c_n \approx 10^{-1}$

The last one sounds quite highly damped, but $c_n \sim 0.1$ is still a fairly small number for the purposes of our "small damping" assumption.

Using the Mechanics Data Book, we can write down immediately the impulse response of our system. Each separate "modal oscillator" will respond to an impulsive force according to the standard formula, and we simply combine the separate modal responses in the same linear combination as we found for the harmonic response:

$$g(j,k) \approx \sum_n \frac{u_j^{(n)} u_k^{(n)}}{\omega_n} \sin \omega_n t e^{-c_n \omega_n t}$$

"tap at j, observe at k"

(using $c_n << 1$)

2.3 Reciprocal theorem

Note that all these response formulae are symmetric in j and k. This leads to an important reciprocal property:

> The response (harmonic or transient) at one "point" to driving at another "point" is identical to that obtained when drive and response points are interchanged.

This very general result is not at all intuitive. (The lecturer can hear your voice exactly as clearly as you can hear his!) It is often useful in making measurements of transfer functions: you are free to measure the reciprocal of the quantity you really want, if that is easier.

E.g. to measure the transfer function from the combustion chamber of a rocket to somewhere outside, it may be much easier to make a noise outside and simply put a microphone into the combustion chamber.

2.4. Poles of the transfer function (cf IB control course)

One way to think about the response formula is in terms of *poles in the complex frequency plane*. One term in the transfer function is

$$\frac{a_n}{\omega_n{}^2 + 2i\,\omega\omega_n c_n - \omega^2} \quad \text{where } a_n = u_j{}^{(n)} u_k{}^{(n)}$$

This term contains poles at frequencies where

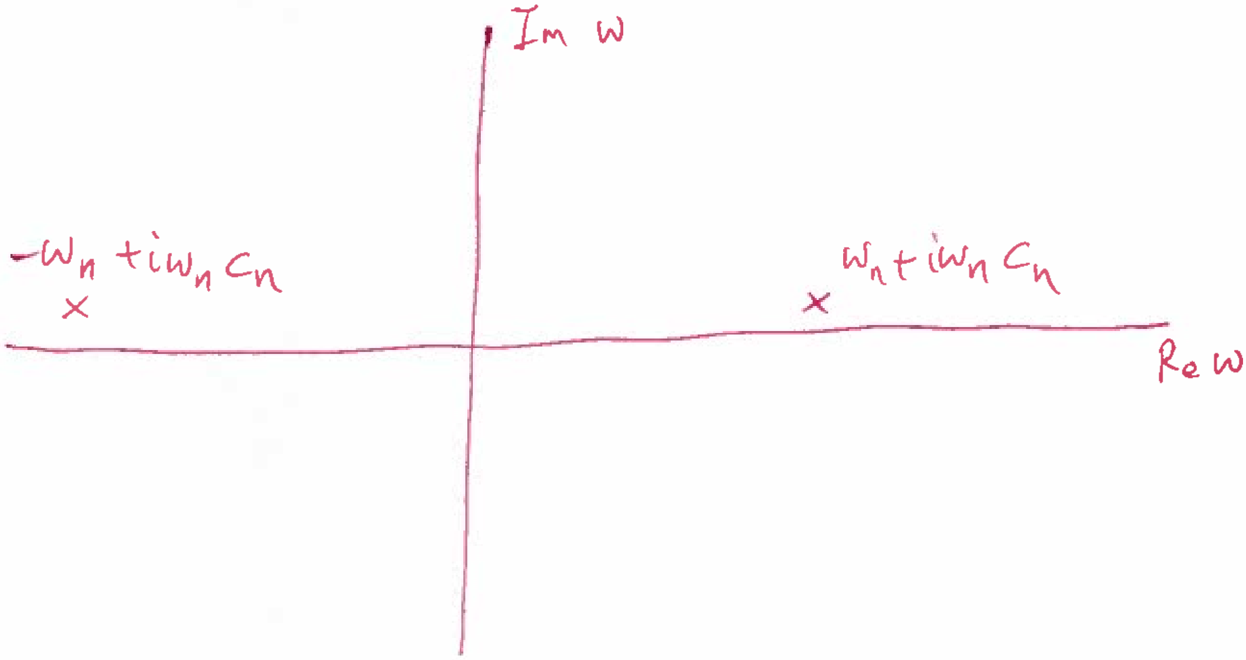
$$\omega^2 - 2i\omega\omega_n c_n - \omega_n{}^2 = 0$$

i.e.
$$\begin{aligned} \omega &= i\omega_n c_n \pm \sqrt{\omega_n{}^2 - c_n{}^2\omega_n{}^2} \\ &= \omega_n\left[\pm\sqrt{1-c_n^2} + ic_n\right] \\ &\approx \omega_n\left[\pm 1 + ic_n\right] \quad \text{when } c_n << 1 \end{aligned}$$

Now use partial fractions:

$$\frac{a_n}{\omega_n{}^2 + 2i\,\omega\omega_n c_n - \omega^2} \approx \frac{a_n}{2\omega_n}\left\{\frac{1}{\omega - \omega_n(-1+ic_n)} - \frac{1}{\omega - \omega_n(1+ic_n)}\right\}.$$

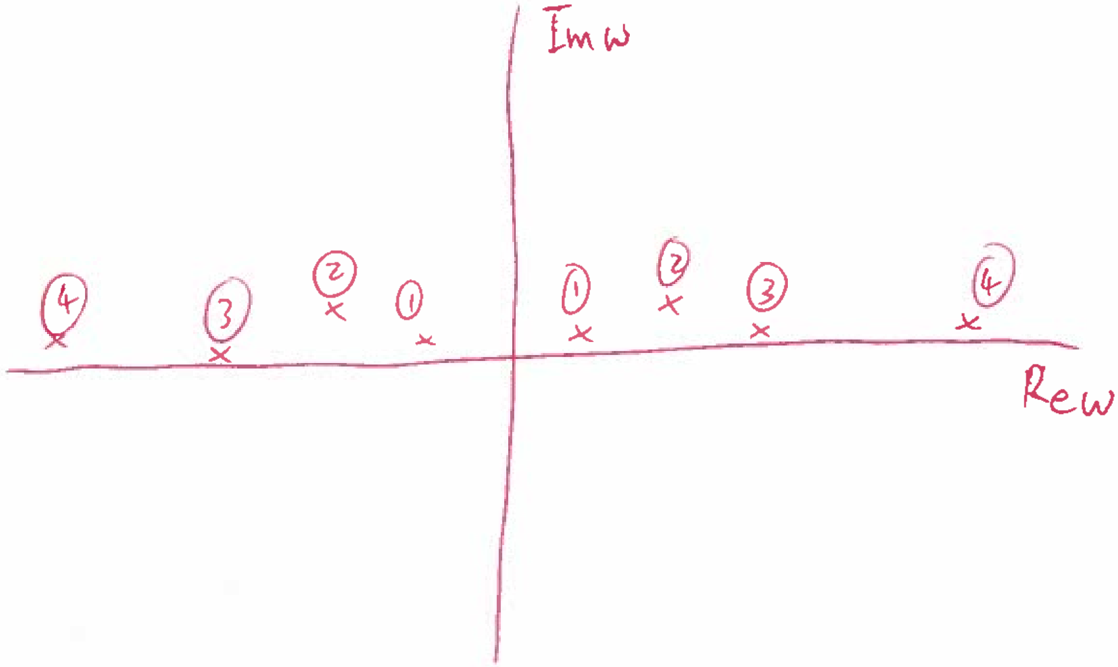
So each mode contributes a pair of poles to the transfer function:



"Small damping" means $c_n << 1$, so all poles are close to the real axis.

(This corresponds to the *imaginary* axis for Laplace transform plots — we are using a Fourier transform here.)

The pole plot for the complete transfer function will then look something like:



2.5. Behaviour near one pole

For a frequency ω close to one mode frequency ω_n, it may be reasonable to approximate the transfer function by just one pole term from the expansion:

$$G(j,k,\omega) \approx \frac{-a_n}{2\omega_n} \frac{1}{\omega - \omega_n(1 + ic_n)}$$

Call this

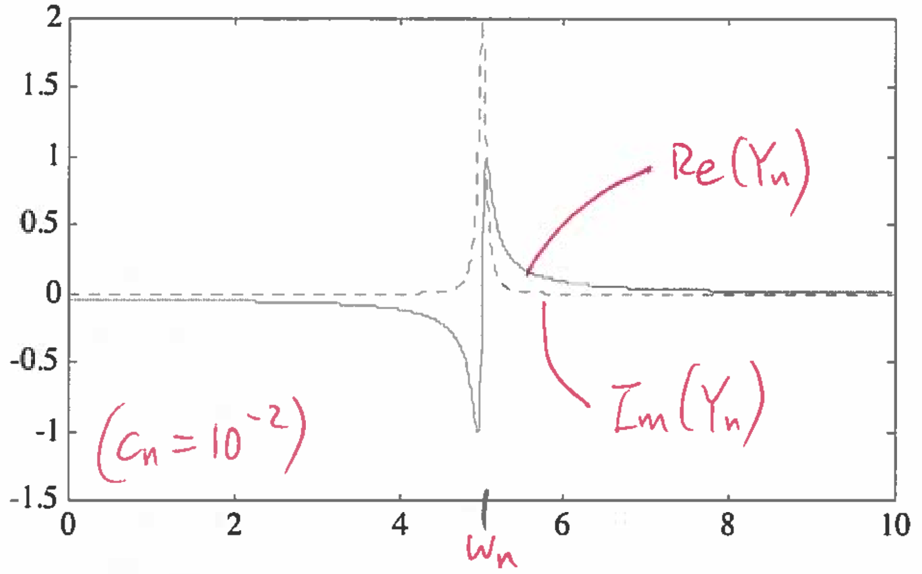
$$Y_n = \frac{b_n}{\omega - \omega_n(1 + ic_n)}$$

say, where $b_n = \dfrac{-a_n}{2\omega_n}$.

$$\text{Then} \begin{cases} \mathrm{Re}(Y_n) \approx \dfrac{b_n(\omega - \omega_n)}{(\omega - \omega_n)^2 + \omega_n{}^2 c_n{}^2} \\ \mathrm{Im}(Y_n) \approx \dfrac{b_n \omega_n c_n}{(\omega - \omega_n)^2 + \omega_n{}^2 c_n{}^2} \end{cases}$$

The peak in $|Y_n|$ occurs at $\omega \approx \omega_n$, and the peak height is $|Y_n| \approx \left| \dfrac{b_n}{\omega_n c_n} \right|$.

The real and imaginary parts of Y_n may be plotted as follows:

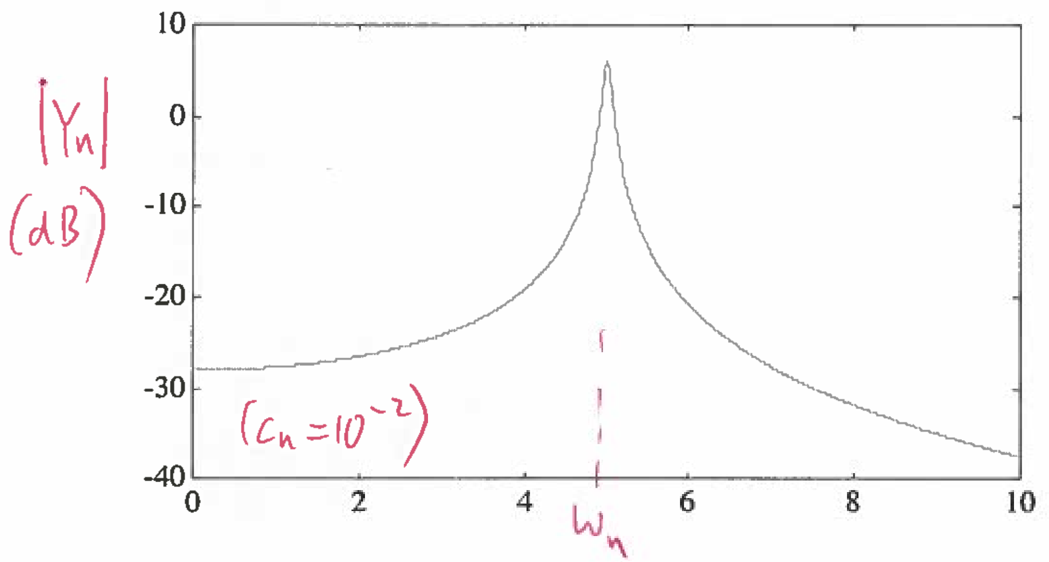


← numbers are arbitrary!

Far from the resonance, where $(\omega-\omega_n)^2 >> \omega_n{}^2 c_n{}^2$,

$$\begin{cases} \mathrm{Re}(Y_n) \approx \dfrac{b_n}{\omega-\omega_n} \\ \mathrm{Im}(Y_n) \approx \dfrac{b_n \omega_n c_n}{(\omega-\omega_n)^2} \end{cases}$$

So $\mathrm{Im}(Y_n)$ dies away much more rapidly than $\mathrm{Re}(Y_n)$, as the plot shows.
We are often concerned with plots of the amplitude of the transfer function, on a logarithmic scale: usually plot in *decibels* (dB), defined by $20\log_{10}|G|$. For our single pole, a decibel plot $20\log_{10}|Y_n|$ looks like:



The width of the peak is often characterised by the *half-power bandwidth*, which is $2\omega_n c_n$ (see example sheet 2, question 4).

Alternatively, for a system with many resonances, it may be useful to use the *modal overlap factor*, the ratio of the half-power bandwidth to the frequency spacing between adjacent modes, $|\omega_{n+1} - \omega_n|$.

If modal overlap is <<1:

* Modal peaks are well separated;

* Response near a resonance is well approximated by one pole;

* Response between two resonances may be well approximated by the two poles on either side.

If modal overlap is >>1:

* Modal peaks are overlapping in frequency;

* At any given frequency, several resonances may contribute to the response;

* Peaks in the total response are the result of *statistical* effects, as the various modes add constructively or destructively, depending on their relative phases.

If damping is small, high modal overlap can still arise in systems with high modal density, such as acoustic modes in a room. Typically, for a structural vibration problem, modal overlap will be low at low frequencies, but will increase as frequency increases, and may become high at high frequencies. Deterministic methods are useful for low modal overlap, but for high modal overlap statistical methods may be more appropriate. We consider only low modal overlap in this course.

2.6. The two-mode approximation (low modal overlap).

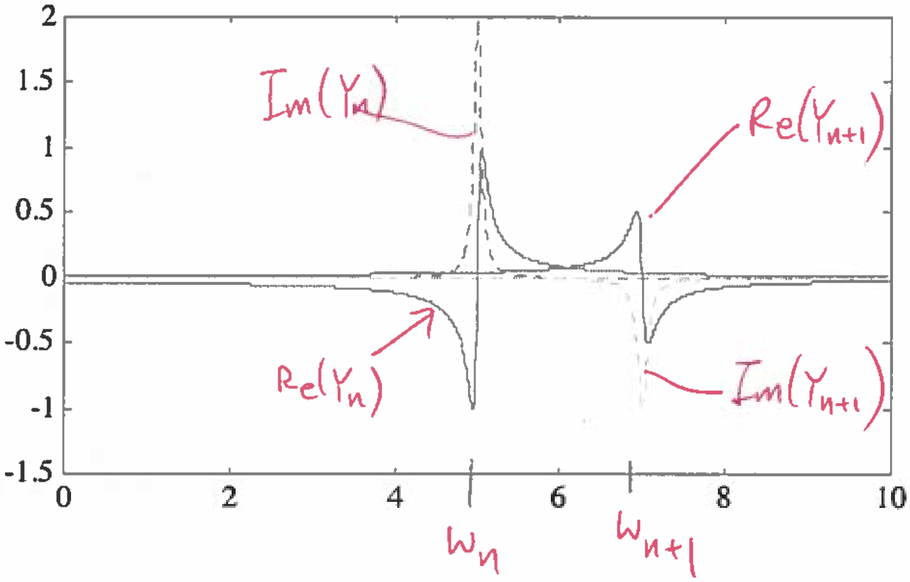
Consider two adjacent pole contributions:

$$Y_n + Y_{n+1} = \frac{b_n}{\omega - \omega_n(1 + ic_n)} + \frac{b_{n+1}}{\omega - \omega_{n+1}(1 + ic_{n+1})}.$$

The shape of the log response plot between the peaks depends critically on whether b_n and b_{n+1} have the *same sign* or *opposite signs*.

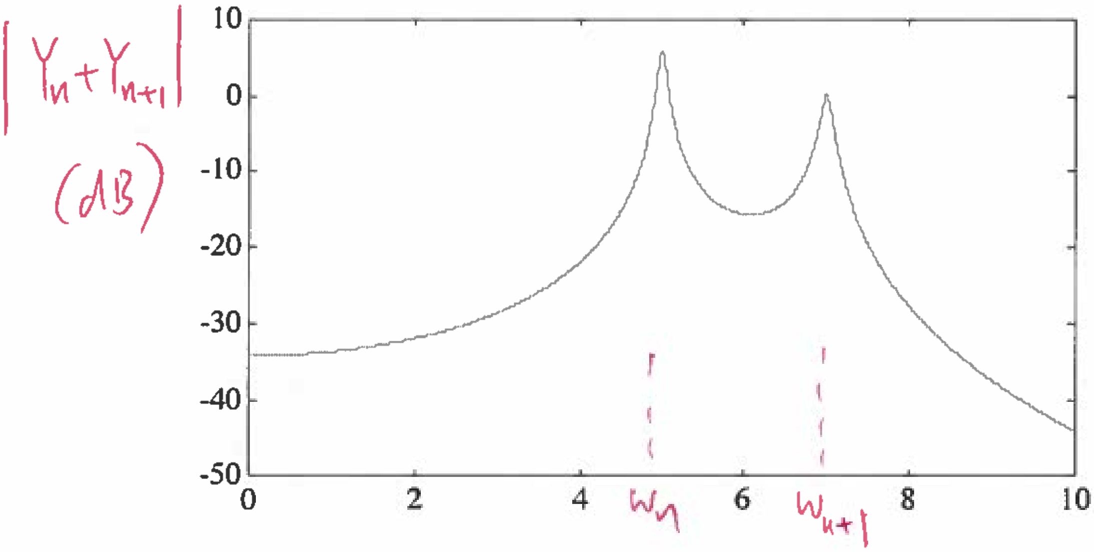
Case 1: b_n and b_{n+1} have opposite signs

The separate real and imaginary contributions look like this:



(Case plotted has $b_{n+1} = -b_n$ and $c_{n+1} = c_n = 10^{-2}$).

Between the peaks, Y_n is dominated by its real part, as is Y_{n+1}. These two have the *same* sign (positive in the plot), so they add together to produce a smooth dip in $|Y_n + Y_{n+1}|$: the log plot looks like this:



The minimum value occurs somewhere near the middle of the interval, and

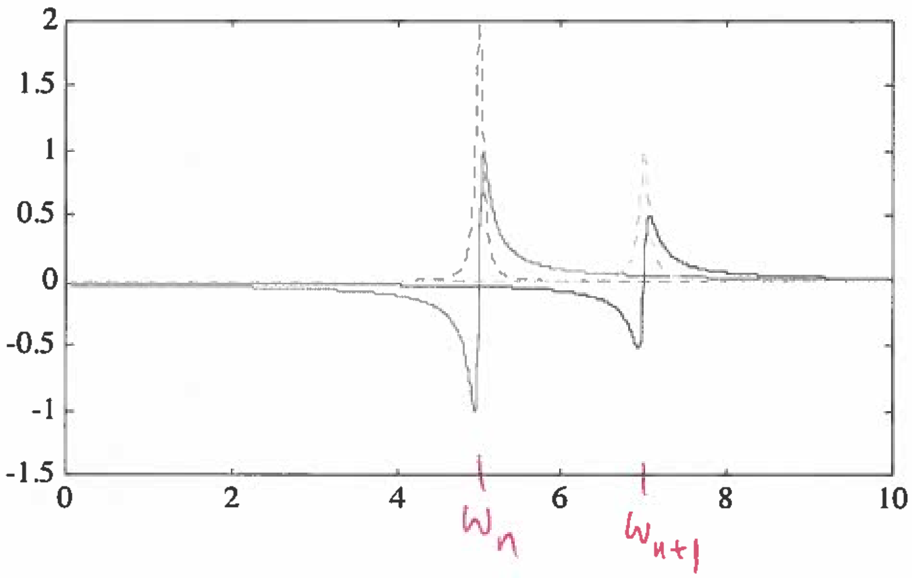
$$\min\left|Y_n + Y_{n+1}\right| \approx \mathrm{Re}(Y_n) + \mathrm{Re}(Y_{n+1}) \text{ at } \omega = \frac{\omega_n + \omega_{n+1}}{2}$$

$$\approx \frac{4|b_n|}{\omega_{n+1} - \omega_n} \text{ when } b_{n+1} = -b_n$$

Note that this is independent of the damping c_n

Case 2: b_n and b_{n+1} have same sign

Now the separate contributions are:



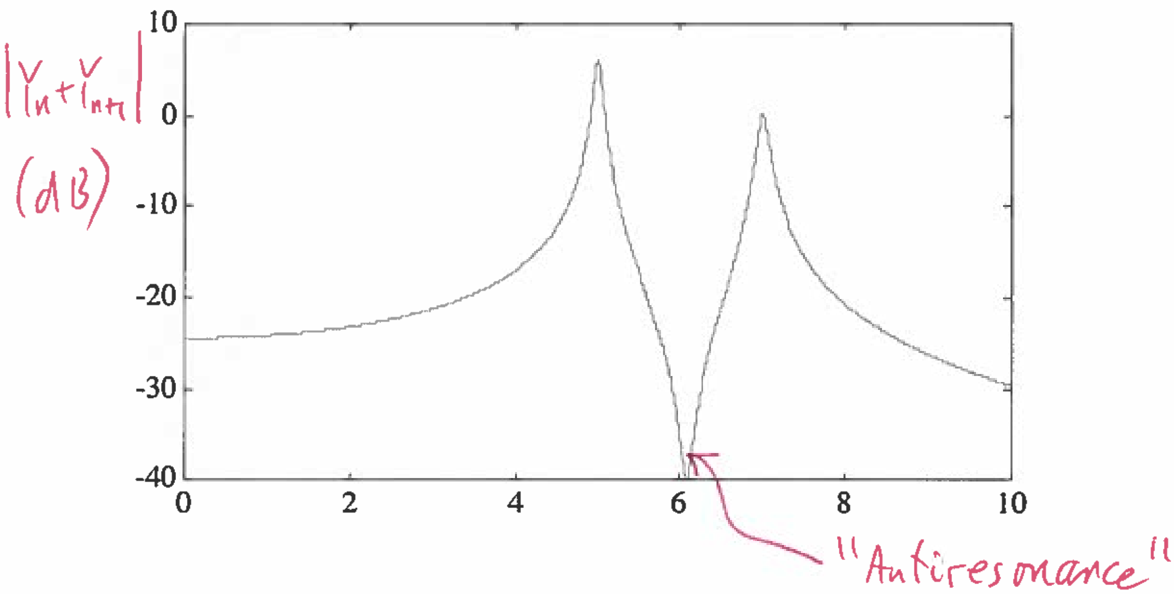
The dominant real parts of Y_n and Y_{n+1} now have opposite signs between the peaks, and somewhere they *cancel exactly*. This leaves only the much smaller imaginary parts, and at this *antiresonance* frequency the value falls to

$$|Y_n + Y_{n+1}| \approx |\mathrm{Im}(Y_n) + \mathrm{Im}(Y_{n+1})| \qquad \text{at } \omega \approx \frac{\omega_n + \omega_{n+1}}{2}$$

$$\approx \frac{8 b_n c_n \omega_n}{(\omega_{n+1} - \omega_n)^2} \text{ for } \begin{cases} c_{n+1} = c_n \\ b_{n+1} = b_n \\ \omega_n \text{ close to } \omega_{n+1} \end{cases}$$

This has a factor c_n in the numerator, so *the lower the damping, the deeper the antiresonances.*

The log plot of the total response now looks like:



So in summary, the peaks rise above the (logarithmic) typical level by a factor of order $1/c_n$, and the antiresonances fall below by a factor of order c_n.

i.e. the shapes of resonances and antiresonances in a log plot are rather similar — if the figure is plotted upside down, it looks somewhat similar, a mistake which can be made in practice when plotting transfer functions!

2.7 Interpreting response curves

Note first that all we have said about discrete systems carries over directly to continuous systems — in a finite frequency range there is always a finite number of modes, which could be matched by a discrete model.

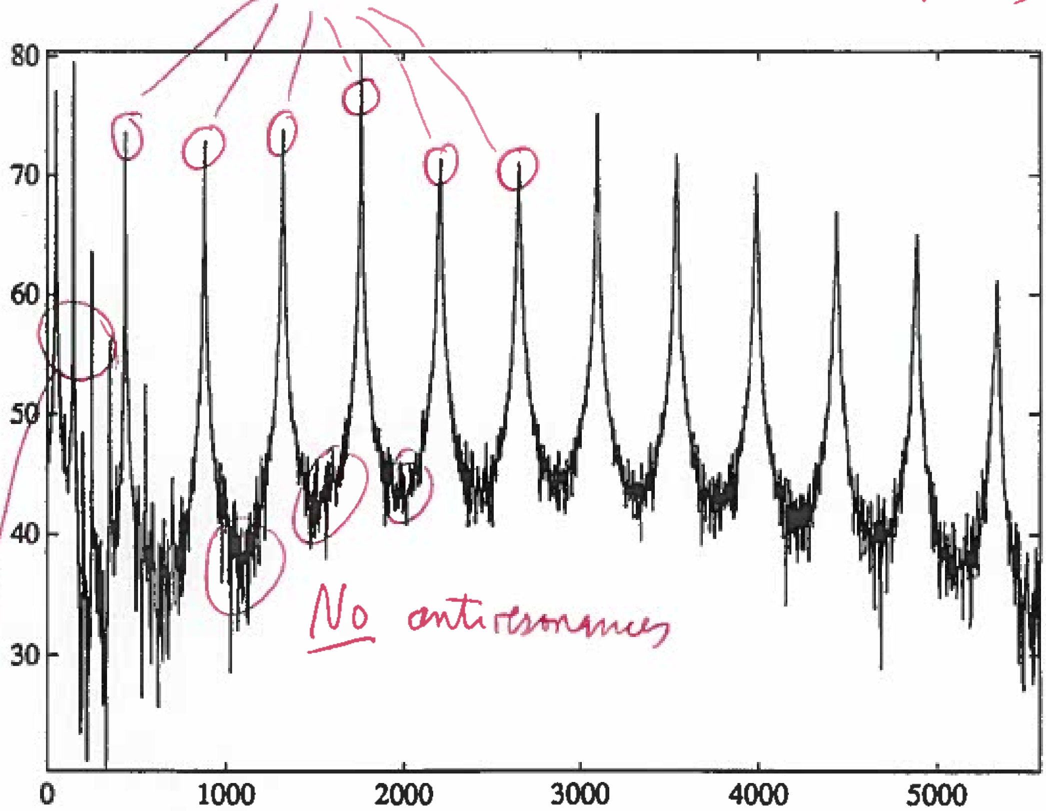
When looking at a response measurement, several questions should be asked:

* Are the peaks well separated? I.e. is the modal overlap low?
* How does the modal density behave?
* Are the peaks spaced regularly or irregularly?
* Is there any other "structure" in the pattern of peak spacings?

  (For example, do peaks occur in pairs or clusters?)
* What is the pattern of antiresonances?

Now look at some measurements taken on contrasting systems:

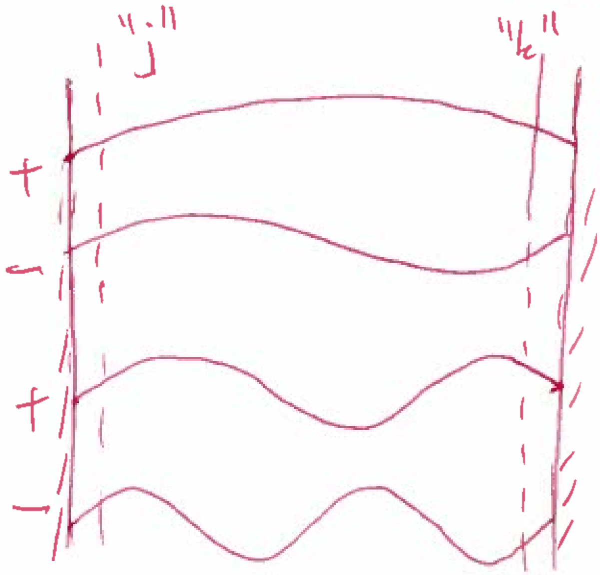
Well separated peaks, regular constant spacing



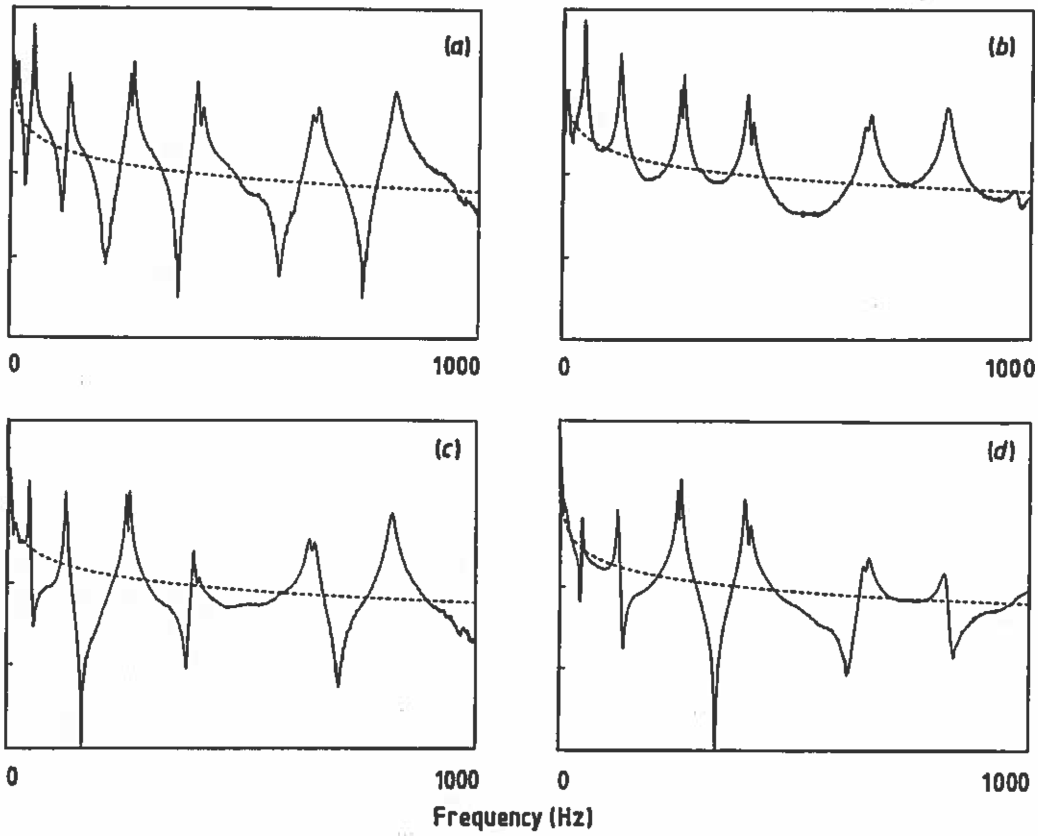
Extra peaks at low frequency: multiples of 50 Hz, electrical interference

This is a transfer function between points near the ends of a violin string (A, 440 Hz). Mode frequencies approximately harmonic.

Modes alternately symmetric and antisymmetric, so $u_j^{(n)} u_k^{(n)}$ changes sign every time, hence no antiresonances



4 measurements on the same system:



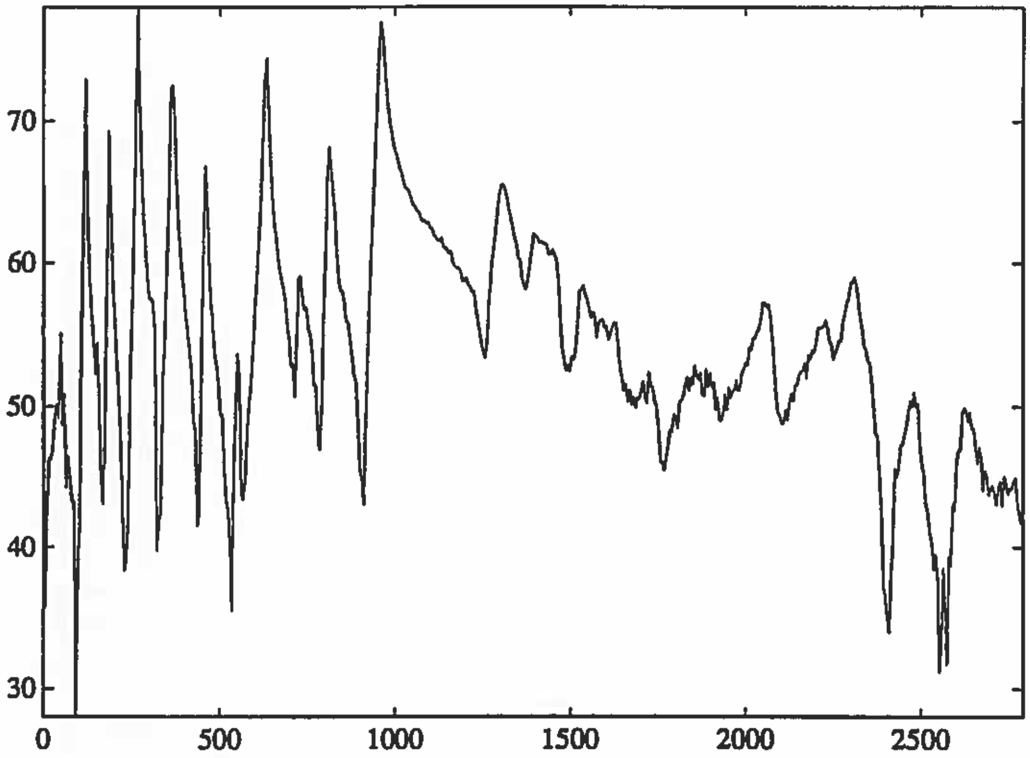
(a) shows well separated peaks, regular spacing which gets progressively wider. Some peaks are split in 2. Antiresonances between every pair of peaks.

It is driving-point response of a free-free bending beam, measured near one end. $u_j^{(n)} u_k^{(n)} = \left(u_j^{(n)}\right)^2$ so always positive, so always antiresonances.

Actually a piece of 50×50 mm wood: peaks split because of different stiffness in two directions.

(b) shows end-to-end response: no antiresonances as previous

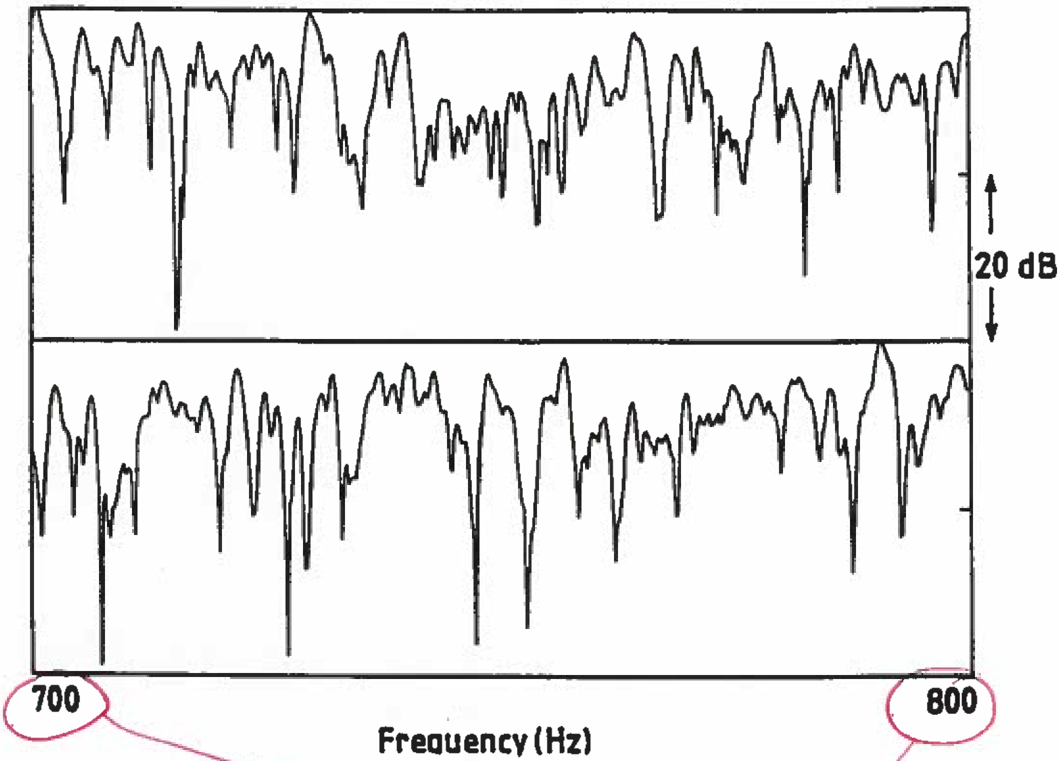
(c) & (d) show other choices of drive & observation: peaks always the same, but heights and antiresonances vary



Low modal overlap at low frequencies

high modal overlap at higher frequencies

A typical structure (actually the back plate of a violin). Frequently see this pattern.



Note narrow frequency range.

Two tests on the same system: acoustic response in Clare College chapel.

High modal overlap.

Rounded peaks, not the same in both cases

Occasional sharp dips caused by statistical cancellation of different modal contributions

3. Rayleigh's principle

3.1 The Rayleigh quotient

Recall: potential energy $V = \frac{1}{2}\underline{q}^t K \underline{q}$

kinetic energy $T = \frac{1}{2}\dot{\underline{q}}^t M \dot{\underline{q}}$

$$= -\omega^2 \cdot \frac{1}{2}\underline{q}^t M \underline{q}$$

$$= -\omega^2 \tilde{T} \text{ say.}$$

We now prove an important result about the "Rayleigh quotient" $\dfrac{V}{\tilde{T}} = \dfrac{\underline{q}^t K \underline{q}}{\underline{q}^t M \underline{q}}$.

Express $\underline{q}$ in terms of modes, as in §2.1:

$$\underline{q} = \sum_n \alpha_n \underline{u}^{(n)}.$$

Then

$$\frac{V}{\tilde{T}} = \frac{\left(\sum \alpha_n \underline{u}^{(n)t}\right) K \left(\sum \alpha_m \underline{u}^{(m)}\right)}{\left(\sum \alpha_n \underline{u}^{(n)t}\right) M \left(\sum \alpha_m \underline{u}^{(m)}\right)}$$

$$= \frac{\alpha_1{}^2 \omega_1{}^2 + \alpha_2{}^2 \omega_2{}^2 + \ldots\ldots + \alpha_N{}^2 \omega_N{}^2}{\alpha_1{}^2 + \alpha_2{}^2 + \ldots\ldots + \alpha_N{}^2} \qquad (1)$$

(using the orthogonality results from §1.3).

Now suppose that the vector $\underline{q}$ is an approximation to one of the eigenvectors, say $\underline{u}^{(k)}$. That means we can take $\alpha_k = 1$, and all the other α_n, are small:

$$|\alpha_n| << 1, \quad n \neq k.$$

Then

$$\frac{V}{\tilde{T}} = \frac{\alpha_1{}^2 \omega_1{}^2 + \ldots + \omega_k{}^2 + \ldots + \alpha_N{}^2 \omega_N{}^2}{\alpha_1{}^2 + \ldots + 1 + \ldots + \alpha_N{}^2}$$

It is obvious that $\dfrac{V}{\tilde{T}} \approx \omega_k{}^2$.

But the result is stronger than that: $\underline{q}$ differs from the exact mode shape by terms of order $\alpha_1, \alpha_2 \ldots$, but $\dfrac{V}{\tilde{T}}$ only differs from $\omega_k{}^2$ by terms of order $\alpha_1{}^2, \alpha_2{}^2$ etc.

Roughly, if $\underline{q}$ has say 10% errors, the Rayleigh quotient will approximate $\omega_k{}^2$ with errors only around 1%.

We can say more: suppose in eq. (1) we replace all the terms $\omega_2{}^2, \omega_3{}^2$ etc. by $\omega_1{}^2$ (where they are in order $\omega_1{}^2 \le \omega_2{}^2 \le \omega_3{}^2 \le ... \le \omega_n{}^2$).

Then we obviously *reduce* the value of the Rayleigh quotient, whatever the values of the α's:

$$\text{so} \qquad \frac{V}{\bar{T}} \ge \frac{\alpha_1{}^2\omega_1{}^2 + \alpha_2{}^2\omega_1{}^2 + ... + \alpha_N{}^2\omega_1{}^2}{\alpha_1{}^2 + \alpha_2{}^2 + ... + \alpha_N{}^2}$$

$$= \omega_1{}^2.$$

Similarly, if we replace all the $\omega_j{}^2$ by $\omega_N{}^2$, we can deduce

$$\frac{V}{\bar{T}} \le \omega_N{}^2.$$

Gathering the results up:

If the quantity $\dfrac{V}{\bar{T}} = \dfrac{\underline{q}^t K \underline{q}}{\underline{q}^t M \underline{q}}$ is evaluated with *any* vector $\underline{q}$, the result will be

(1) $\ge$ the smallest squared frequency

(2) $\le$ the largest squared frequency

(3) a surprisingly good approximation to $\omega_k{}^2$ if $\underline{q}$ is an approximation to $\underline{u}^{(k)}$

(Formally, $\dfrac{V}{\bar{T}}$ is *stationary* near each mode.)

This result can be used

(a) to find general theorems about vibration behaviour;

(b) to estimate mode frequencies, using rather crude guesses about mode shapes;

(c) to find the effect of *small changes* to a system on the vibration resonance frequencies.

3.2 Two general theorems

We can use Rayleigh's principle to give a clear proof of two results which will seem immediately intuitively plausible:

(1) If the inertia of any part of a system is increased without changing the stiffness, then all vibration frequencies will go down (or in special cases, some may remain unchanged).

(2) If the stiffness of any part of a system is increased without changing the inertia, all natural frequencies will go up.

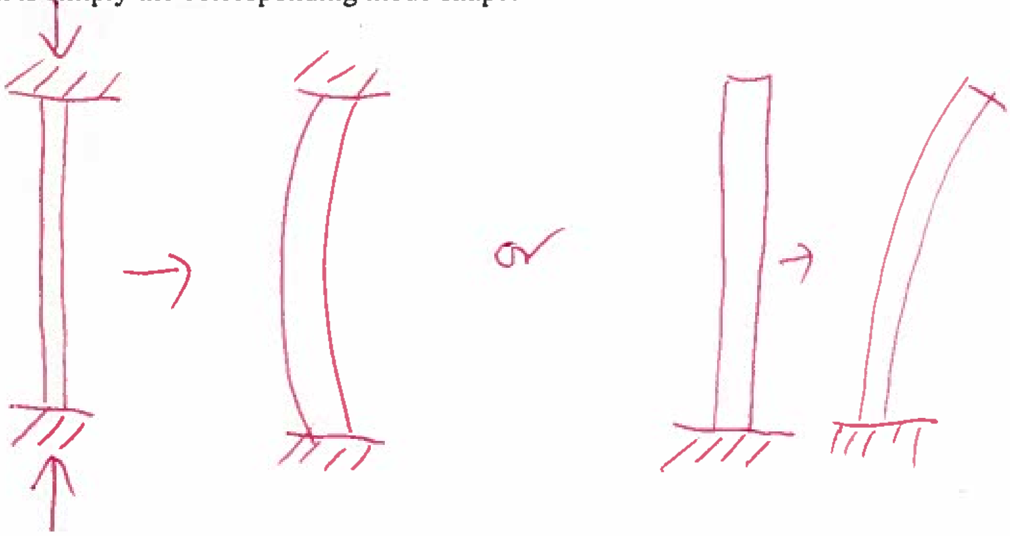
Proof of (1):

Think of the inertia increase made up in infinitesimal increments. For each increment, we can estimate the new frequencies using Rayleigh's principle, with the mode shapes before the new increment is added. The potential energy will be unchanged, and the kinetic energy will increase (unless the mode has a nodal point where the new mass is added). Thus this estimate of the frequency will have reduced. There would be a small correction because of the change in mode shape, but this will only be of second order, and can be neglected for a sufficiently small change. Thus each actual frequency is reduced.

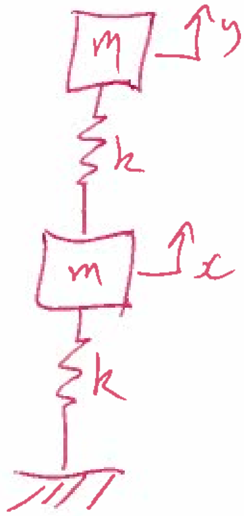
Proof of (2) is essentially identical.

An aside on buckling:

If there is a contribution to potential energy which is *negative* (e.g. compression in a strut, or self-weight of a column) then all vibration frequencies will be *reduced*. If the lowest frequency is pushed right down to zero, the system will buckle, in a shape which is simply the corresponding mode shape.



3.2. A simple example



$$V = \tfrac{1}{2}kx^2 + \tfrac{1}{2}k(x-y)^2,$$

$$T = \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}m\dot{y}^2$$

so

$$K = k\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}, \quad M = m\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Try to guess the lowest mode shape — let's simply try $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

So estimate $V = \frac{k}{2}\begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = k$

$$\tilde{T} = \frac{m}{2}\begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{5}{2}m$$

so $\omega^2 \simeq \frac{V}{\tilde{T}} = \frac{2k}{5m}$, so $\omega \simeq 0.633\sqrt{\frac{k}{m}}$

A good guess for "Rayleigh" estimates is often a suitable *static* deflection pattern.

The self-weight deflections are: $x = \dfrac{-2mg}{k}, y = \dfrac{-3mg}{k}$ (check them...)

So try $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ in Rayleigh quotient:

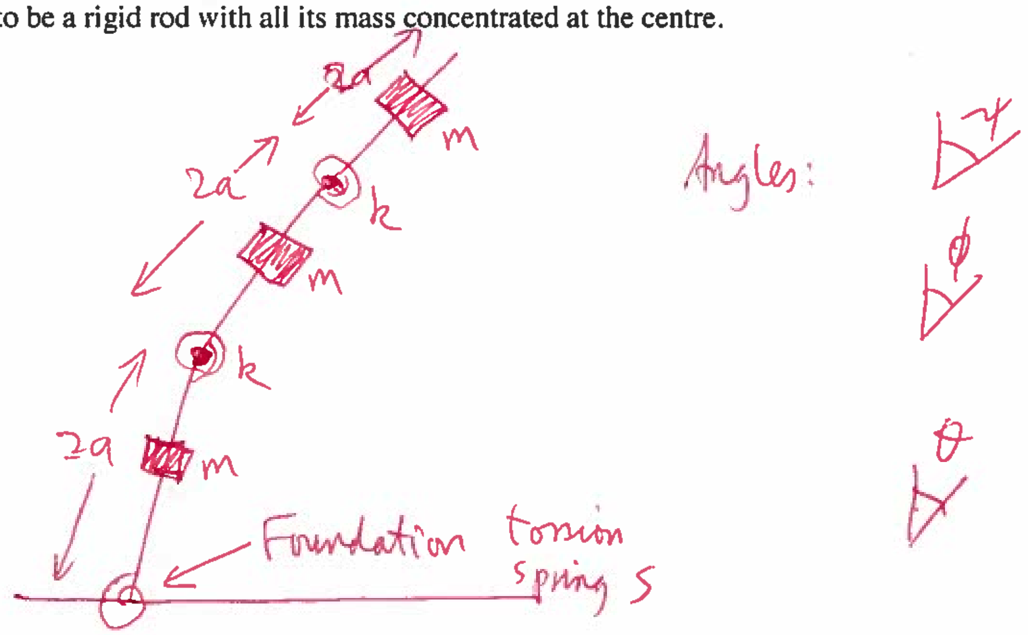
$$V = \frac{1}{2}\begin{bmatrix} 2 & 3 \end{bmatrix}\begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \frac{5k}{2}, \qquad \tilde{T} = \frac{1}{2}m\left(2^2 + 3^2\right) = \frac{13}{2}m$$

so $$\omega^2 \approx \frac{5}{13}\frac{k}{m} \to \omega \approx 0.620\sqrt{\frac{k}{m}}, \text{ cf exact result } 0.618\sqrt{\frac{k}{m}}.$$

smaller than previous estimate
so better estimate

3.3 A flexible chimney on a resilient foundation

As a more realistic application of Rayleigh's principle, consider a simple discrete model of a tall, flexible chimney on a resilient foundation. Model the chimney by three rigid sections joined by torsion springs. Each section is assumed, for simplicity, to be a rigid rod with all its mass concentrated at the centre.

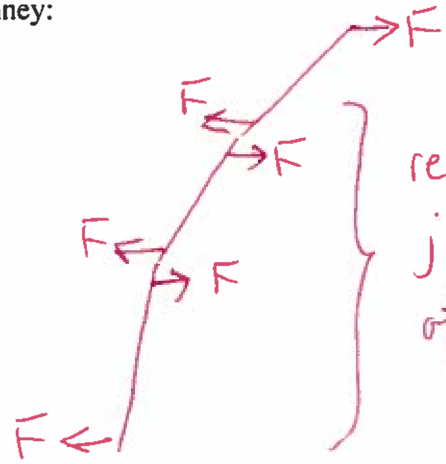


So $$V=\frac{1}{2}s\theta^2+\frac{1}{2}k(\theta-\phi)^2+\frac{1}{2}k(\psi-\phi)^2$$
$$\left.\begin{aligned}&-mga(1-\cos\theta)-mga[2-2\cos\theta+1-\cos\phi]\\&-mga[2-2\cos\theta+2-2\cos\phi+1-\cos\psi]\end{aligned}\right\}\ \text{"inverted pendulum" terms}$$
$$\approx\frac{1}{2}s\theta^2+\frac{1}{2}k(\theta-\phi)^2+\frac{1}{2}k(\psi-\phi)^2$$
$$-mga\left[\frac{\theta^2}{2}+\frac{2\theta^2}{2}+\frac{\phi^2}{2}+\frac{2\theta^2}{2}+\frac{2\phi^2}{2}+\frac{\psi^2}{2}\right]$$
$$=\frac{1}{2}s\theta^2+\frac{1}{2}k(\theta-\phi)^2+\frac{1}{2}k(\psi-\phi)^2-\frac{mga}{2}\left(5\theta^2+3\phi^2+\psi^2\right)$$

$$T=\frac{1}{2}m\left(a\dot\theta\right)^2+\frac{1}{2}m\left(2a\dot\theta+a\dot\phi\right)^2+\frac{1}{2}m\left(2a\dot\theta+2a\dot\phi+a\dot\psi\right)^2$$
$$=\frac{1}{2}ma^2\left[9\dot\theta^2+5\dot\phi^2+\dot\psi^2+12\dot\theta\dot\phi+4\dot\theta\dot\psi+4\dot\phi\dot\psi\right]$$

Now try to estimate the lowest frequency of vibration using Rayleigh. This would be the most important for wind-excited vibrations, a hazard for tall chimneys.

To obtain a sensible approximation to the mode shape, it is often a good idea to solve a simple problem of static deflection. In this case, a reasonable guess for the lowest mode shape might be the response to a horizontal load applied to the top of the chimney:



reaction shear forces at the joints, by force equilibrium of each section

Now impose moment balance for each section:

$$\begin{cases} 2aF = k(\psi-\phi) \\ 2aF = k(\phi-\psi)+k(\phi-\theta) \\ 2aF = s\theta + k(\theta-\phi) \end{cases}$$

Solving these gives $\theta = \dfrac{6aF}{s}, \phi = \dfrac{6aF}{s} + \dfrac{4aF}{k}, \psi = \dfrac{6aF}{s} + \dfrac{6aF}{k}$

So for a guessed mode shape, try

$$\theta = 1, \quad \phi = 1 + \frac{2}{3}\lambda, \quad \psi = 1+\lambda$$

where $\lambda = s/k$.

Then

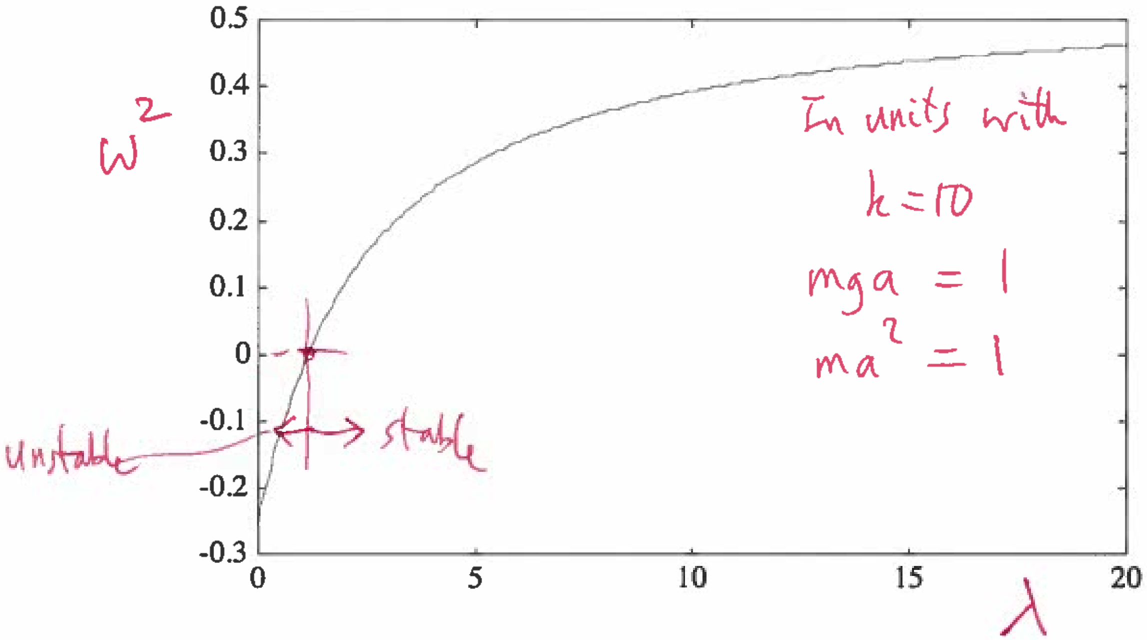
$$V = \frac{1}{2}s + \frac{1}{2}k\left(\frac{2}{3}\lambda\right)^2 + \frac{1}{2}k\left(\frac{\lambda}{3}\right)^2 - \frac{mga}{2}\left[5 + 3\left(1+\frac{2}{3}\lambda\right)^2 + (1+\lambda)^2\right]$$

$$= \frac{1}{2}k\lambda + \frac{1}{2}k\lambda^2\frac{5}{9} - \frac{mga}{2}\left(9 + 6\lambda + \frac{7}{3}\lambda^2\right)$$

$$\tilde{T} = \frac{1}{2}ma^2\left[9 + 5\left(1+\frac{2}{3}\lambda\right)^2 + (1+\lambda)^2 + 12\left(1+\frac{2}{3}\lambda\right) + 4(1+\lambda) + 4\left(1+\frac{2}{3}\lambda\right)(1+\lambda)\right]$$

Now $\frac{V}{\tilde{T}}$ gives our approximation to ω^2.

Plotted as a function of λ , it gives:



With these parameters, $\omega^2 > 0$ for $\lambda > 1.2$, but for $\lambda < 1.2$, $\omega^2 < 0$ so that *buckling* occurs: the spring at the base is not strong enough to stop the chimney toppling under its own weight.

We can look at some limiting cases:

As $\quad \lambda \to 0, V \to -\frac{9}{2}mga$, so always unstable (since $\tilde{T} > 0$ for all λ).

As $\quad \lambda \to \infty, \begin{cases} V \to \left(\frac{5}{18}k - \frac{7}{6}mga\right)\lambda^2 \\ \tilde{T} \to \frac{1}{2}ma^2\lambda^2\left[\frac{20}{9} + 1 + \frac{8}{3}\right] \end{cases}$

This is a case of a column *clamped* at the base: it can still buckle if $\frac{7}{6}mga > \frac{5}{18}k$,

i.e. if $k < \frac{21}{5}mga$.

For the case plotted above, k is greater than this limit, so that there is a limiting value of ω^2 as $\lambda \to \infty$:

$$\omega^2 \to \frac{\left(\frac{5}{18}k - \frac{7}{6}mga\right)}{\frac{53}{18}ma^2}$$

Notice that it would have been hard to deduce all this information without Rayleigh's method. It is easy to compute eigenvalues and eigenvectors for *particular* values of the system parameters, but to keep the parameter dependence visible requires an analytic method. Analytic solution of a 3×3 matrix problem is quite tricky (although not impossible).