ENGINEERING TRIPOS PART IIA

Module 3A3 - Fluid Mechanics II

2D Compressible Flow

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INTRODUCTION

In this section of 3A3 we will take forward the basic 1D concepts already covered and extend into two dimensions. This will enable us to look at flows of significant relevance for the design of aircraft: where 1D may allow us to examine the flow through ducts / nozzles, 2D permits exploration of the flows around high speed airfoils and though complex jet engine intakes.

Our course starts with the advances in compressible aerodynamics made (in England) in the 1920s. In the wake of the First World War the recently formed RAF (created by merging the Royal Naval Air Service with the Army's Royal Flying Corps) re-equipped with the Sopwith *Snipe*: a strut-and-wire braced biplane with a maximum speed (at low level) of 125 mph ($M \sim 0.16$)

With compressibility becoming significant above a Mach number of around 0.3 ($\sim$230 mph at sea level) more powerful aircraft soon started to experience compressibility effects on the tip sections of their propellers.

This problem rapidly extended to airframes through the 1920s as streamlining took hold in airplane design and speeds started to rise rapidly. This was felt most keenly in the design of racing seaplanes competing for the Schneider Trophy. The winning British aircraft of the 1927 race (a Supermarine S.5 whose chief designer, R.J. Mitchell, is famous for his later *Spitfire*) did so at a speed of 281 mph ($M \simeq 0.36$). On 12 March 1928 Flt Lieut S.M. Kinkead was killed in an *S.5* attempting to raise the absolute airspeed record above 300 mph ($M \sim 0.4$)

The issue was that, though a vast amount of airfoil and design data had been accumulated over the preceeding 30 years from (often very accurate) wind tunnel testing, the data was essentially low-speed and thus only representative of incompressible flows. Thus the faster the designs flew as the 1920s progressed the greater the error w.r.t. the incompressible data.

The aeronautical community faced two options: scrap all the existing aerodynamic design data and start again (and add an order of magnitude to the problem by having to measure lift / drag against incidence *and* against Mach number) or deduce a compressibility correction such that the existing data might be modified to accurately predict the high speed behaviour.

This is the problem to which Hermann Glauert (an alumnus of Trinity College) applied himself in the early 1920s.

Glauert's Compressibility Correction

Compressible Potential Flow $C = f(M, Re) \quad M = V/a \quad Re = \frac{\rho V L}{\mu}$

The equations of motion are:

Mass: $\nabla\cdot(\rho V) = 0 \Rightarrow \nabla\cdot V + \frac{1}{\rho} V\cdot\nabla\rho = 0$ (A.1)

Euler (momentum): $-\frac{1}{\rho}\nabla p = V\cdot\nabla V$ (A.2)

Isentropic: $p = k\,\rho^{\gamma} \;\Rightarrow\; dp = k\,\gamma\rho^{\gamma-1}d\rho \;\Rightarrow\; \nabla p = a^2\nabla\rho$ (A.3)

Energy $\text{const} = h_0 = c_p T + \frac{1}{2}\left(u^2+v^2\right) = \frac{a^2}{\gamma-1} + \frac{1}{2}\left(u^2+v^2\right)$ (A.4)

Equation for Velocity

Eliminating ∇p and $\nabla\rho$ from equation (A.2) using (A.1) and (A.3) leaves us with

$$\nabla\cdot\mathbf{V} = -\frac{1}{\rho}\mathbf{V}.\nabla\rho = -\frac{1}{\rho a^2}\mathbf{V}.\nabla p = \frac{1}{a^2} V\cdot(V\cdot\nabla V)$$

rearranging: $a^2\nabla\cdot V - V\cdot(V\cdot\nabla V) = 0$

or in terms of components $\mathbf{V} = (u,v)$

$$a^2\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) - \left[u\,\mathbf{e}_x + v\,\mathbf{e}_y\right].\left[\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)\mathbf{e}_x + \left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)\mathbf{e}_y\right] = 0$$

$$a^2\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) - u\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right) - v\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right) = 0$$

$$\left(a^2-u^2\right)\frac{\partial u}{\partial x} - uv\frac{\partial u}{\partial y} - vu\frac{\partial v}{\partial x} + \left(a^2-v^2\right)\frac{\partial v}{\partial y} = 0$$

Equation for Potential

The flow is assumed to be steady, two-dimensional and constant entropy (i.e. no significant shock or viscous losses). Under these conditions the flow is *irrotational*, and the velocity is given as the gradient of a potential (see 3A1).

$$\nabla \times V = 0 \quad \Rightarrow V = \nabla \Phi = \nabla(U_\infty x + \phi)$$

Substituting $V = \left(U_\infty + \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}\right)$ and collecting terms together gives

$$\boxed{\left[a^2 - \left(U_\infty + \frac{\partial \phi}{\partial x}\right)^2\right]\frac{\partial^2 \phi}{\partial x^2} - 2\left[U_\infty + \frac{\partial \phi}{\partial x}\right]\frac{\partial \phi}{\partial y}\frac{\partial^2 \phi}{\partial x \partial y} + \left[a^2 - \left(\frac{\partial \phi}{\partial y}\right)^2\right]\frac{\partial^2 \phi}{\partial y^2} = 0} \qquad \text{(A.5)}$$

which is supplemented by the energy equation neglect 2nd order

$$\frac{a^2}{(\gamma - 1)} + \frac{1}{2}\left(\left(U_\infty + \cancel{\frac{\partial \phi}{\partial x}}\right)^2 + \cancel{\left(\frac{\partial \phi}{\partial y}\right)^2}\right) = \frac{a_\infty^2}{(\gamma-1)} + \frac{U_\infty^2}{2} \qquad \text{(A.6)}$$

Our interest here is in deriving what properties of the solution we can, and in particular how properties depend on Mach Number, etc., *without* the need to solve for ϕ explicitly.

Linearise Equation of Motion

Consider cases for which the flow is only *slightly* disturbed from a uniform one, as for thin aerofoils and linearise.

$$u = U_\infty + \frac{\partial \phi}{\partial x}, \quad v = \frac{\partial \phi}{\partial y} \quad \text{where } \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y} \ll U_\infty \qquad \text{(A.7)}$$

Equation (A.5) becomes

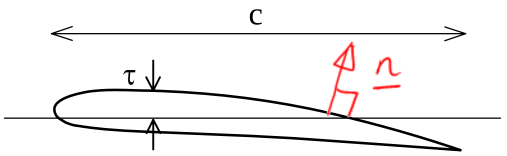
$$\left[a_\infty^2 - U_\infty^2\right]\frac{\partial^2 \phi}{\partial x^2} + a_\infty^2 \frac{\partial^2 \phi}{\partial y^2} = 0 \qquad \boxed{(1 - M_\infty^2)\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0} \qquad \text{(A.8)}$$

Note that for $M_\infty < 1$ this equation is elliptic (Laplace-like)

and for $M_\infty > 1$ this equation is hyperbolic (wave-like)

Linearise Boundary Conditions

For a typical thin compressible aerofoil,



$\underline{n}\cdot\underline{V}=0$ on $y = \tau\, g_U\left(\frac{x}{c}\right)$ and $\tau\, g_L\left(\frac{x}{c}\right)$

for $0 < x < c$

where τ is a measure of airfoil thickness & g is a shape function

Since the normal is parallel to $\left[-\frac{\tau}{c}g',1\right]$, this boundary condition is

$$\left[-\frac{\tau}{c}g',1\right]\cdot\left[U_\infty+\frac{\partial\phi}{\partial x},\frac{\partial\phi}{\partial y}\right] = -\frac{\tau}{c}g'U_\infty \underbrace{\cancel{-\frac{\tau}{c}g'\frac{\partial\phi}{\partial x}}}_{\text{small}} + \frac{\partial\phi}{\partial y} = 0 \quad \text{on } y=\tau g$$

For a thin airfoil, $\frac{\tau}{c} \ll 1$ and keeping only linear terms:

$$-\frac{\tau}{c}g'U_\infty + \frac{\partial\phi}{\partial y} = 0$$

Further, since τ is small,

$$\frac{\tau}{c}g'U_\infty = \left.\frac{\partial\phi}{\partial y}\right|_{y=\tau g} = \left.\frac{\partial\phi}{\partial y}\right|_{y=0} \underbrace{\cancel{+\tau g\left.\frac{\partial^2\phi}{\partial y^2}\right|_{y=0}}}_{\text{neglect}} + \ldots = \left.\frac{\partial\phi}{\partial y}\right|_{y=0} + \ldots$$

i.e. we can, to this order of approximation, apply the boundary condition on $y=0$.

The full problem is thus

$$\boxed{\begin{aligned} &1-M_\infty^2\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0 \\ &\frac{\partial\phi}{\partial y}=\frac{\tau}{c}g'\left(\frac{x}{c}\right)U_\infty \text{ on } y=0,\ 0<x<c \end{aligned}} \qquad \text{(A.9)}$$