Paper 3C6: VIBRATION

Examples paper 1 — Vibration modes

*Tripos standard questions are marked **

Discrete systems

1. For each of the four systems shown in Fig. 1, express the potential and kinetic energy functions in terms of the given generalised coordinates, for small motions of the system. Hence obtain the mass and stiffness matrices in each case. Notice that you need to include the effect of gravity in (c), but not in (b): why is this?

(a) A rigid, uniform rectangular panel of dimensions $2a$x$2b$ and mass m, supported at its corners by eight identical springs of stiffness k, attached to a rigid frame. Motion is in the plane of the panel only, described by generalised coordinates x and y (coordinates of the centre of mass) and θ (rotation about the centre of mass).

(b) Bounce, pitch and roll motion of a car body. A rigid body is supported above rigid horizontal ground by four identical springs of stiffness k, positioned symmetrically as shown. The total mass is m, the centre of mass lies in the centre plane of the car, and the moments of inertia of the car body about the centre of mass are I and J, as shown. Generalised coordinates are the vertical displacement z of the centre of mass, and rotations through angles θ and ϕ about the centre of mass.

(c) Planar motion of a compound pendulum, consisting of two identical, uniform, rigid rods of length $2a$ and mass m. Both pivots are frictionless. Generalised coordinates are the angle θ of the upper rod to the vertical, and the angle ϕ between the two rods.

(d) Torsional motion of three rotors on a light shaft which can turn freely in bearings at each end. The rotors have moments of inertia I, J and I as shown, and each section of shaft has torsional stiffness k Nm/rad. Generalised coordinates are the rotations θ, ϕ and ψ of the rotors.

2. For the four systems in question 1, which of the modes and/or natural frequencies can be deduced without further calculation? Think about *symmetry* and its consequences. What other feature of one of these systems reveals a mode and natural frequency by inspection?

Find equations which determine all the modes and their corresponding natural frequencies, for all four systems. (Messy algebra need not be followed through to solve these equations in detail.)

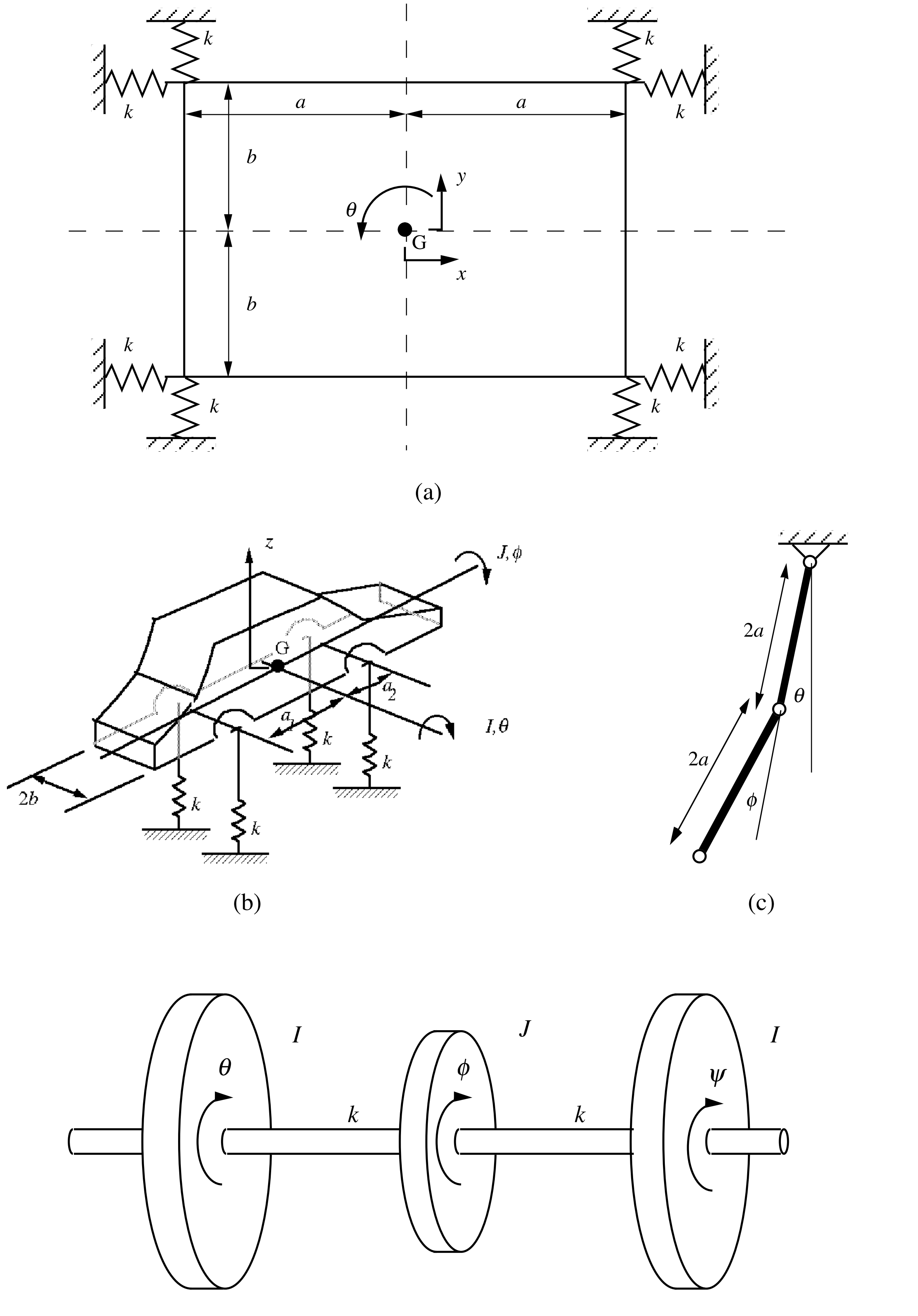

Figure 1

3. A simple discrete model of a stretched string can be constructed as follows: three identical masses m are attached to a light string with tension T. The spaces between each pair of masses and between the end masses and the fixed end points of the string are all equal to a. The transverse displacements of the three masses are denoted x, y and z.

Calculate the kinetic and potential energy for small displacements, and hence find the mass and stiffness matrices.

[Hint: to calculate the potential energy, consider the work done by the (small) extension of each section of string against the tension T.]

Determine the vibration modes and their natural frequencies. How accurately does this model predict the behaviour of the first three modes of a stretched string with continuous mass distribution?

*4. Two equal masses m are attached to a third mass M by two equal leaf springs of stiffness k, as shown in Fig. 2. For small motions in the transverse direction, characterised by generalised coordinates x, y and z as shown, determine the mass and stiffness matrices and hence the vibration modes and their natural frequencies.

Regarding this as a simple model for a tuning fork, which of the modes corresponds to 'the' frequency of the fork as normally understood, and why?

The sound of a tuning fork is often amplified by placing the 'stalk' (i.e. the mass M) on a table so the fork is normal to the table. How are vibrations transmitted to the table by doing this?

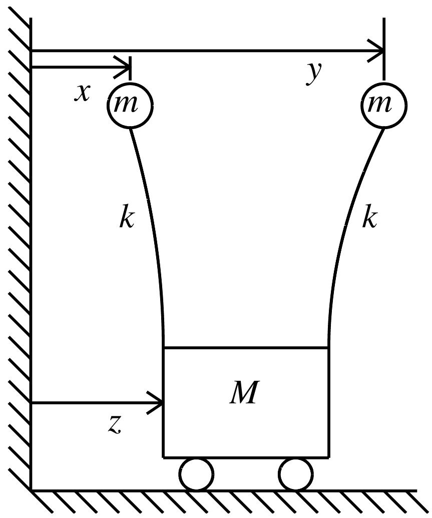

Figure 2

Continuous systems

5 A column of length L with uniform cross section of area A is rigidly fixed at its base, and the top is free. It undergoes axial vibration with small amplitude. The Young's modulus is E and the density is ρ. What boundary conditions must be satisfied at the two ends of the column? Find the modes and natural frequencies of the column (ignoring any effects of gravity). Sketch the first three mode shapes.

6 Axial vibration of the column of Q5 is excited by a harmonic force $Fe^{i\omega t}$ applied at the top, in a vertical direction. Find the appropriate boundary condition at the top of the column to take account of the applied force, and hence find the amplitude of forced displacement at a given point x in the column. Show that the displacement at the top of the column is

$$u = \frac{Fc}{EA\omega}\tan\frac{\omega L}{c} \quad \text{where} \quad c = \sqrt{E/\rho}\ .$$

Verify that the behaviour is consistent with the natural frequencies found in Q5.

7* A piano string is excited by the impulsive rebound of a felt-covered hammer. The vibration may be modelled approximately as follows. An ideally flexible stretched string of tension P and mass m per unit length is fixed at its ends $x = 0$, $x = L$. At time $t = 0$ the string has no lateral displacement $w(x,t,) = 0$, but the impact of the hammer gives an initial velocity distribution

$$\frac{\partial w}{\partial t} = \begin{cases} 0 & 0 \le x < a \\ V & a \le x \le b \\ 0 & b < x \le L \end{cases}$$

Express the subsequent motion of the string as a modal sum, find the modal amplitudes, and hence find an expression for the vibration of the string.

In what ways is this model unrealistic, for the behaviour of real piano strings?

8 Explain how d'Alembert's solution of the wave equation can be used to express the transient motion of a plucked string. For the case of a pluck at the mid-point of the string, sketch the graphical construction for a few "movie frames" distributed through the first period of the vibration.

9* A stretched string with tension P is fixed at points $x = -L_1$ and $x = L_2$. For $-L_1 \le x \le 0$ the mass per unit length is m_1 and for $0 \le x \le L_2$ the mass per unit length is m_2.
(a) What boundary conditions must be satisfied at the point $x = 0$?
(b) Show that the natural frequencies ω_n are the solutions of the equation

$$c_1 \tan\left(\frac{\omega L_1}{c_1}\right) + c_2 \tan\left(\frac{\omega L_2}{c_2}\right) = 0$$

where c_1 and c_2 are the wave speeds in the two sections of the string.
(c) Verify that when the two sections of string have the same mass per unit length, this equation predicts the usual natural frequencies of a uniform string of length $L_1 + L_2$.

Answers

1 (a) $K = 4k\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a^2+b^2 \end{bmatrix}$, $M = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m\left(a^2+b^2\right)/3 \end{bmatrix}$

(b) $K = 2k\begin{bmatrix} 2 & a_1-a_2 & 0 \\ a_1-a_2 & a_1^2+a_2^2 & 0 \\ 0 & 0 & 2b^2 \end{bmatrix}, M = \begin{bmatrix} m & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & J \end{bmatrix}$

(c) $K = mga\begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix}, M = \dfrac{ma^2}{3}\begin{bmatrix} 32 & 10 \\ 10 & 4 \end{bmatrix}$.

(d) $K = k\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}, M = \begin{bmatrix} I & 0 & 0 \\ 0 & J & 0 \\ 0 & 0 & I \end{bmatrix}$

2 (a) All can be deduced trivially; (b) one can be deduced, the other two frequencies are roots of a quadratic equation; (c) the two frequencies require solution of a quadratic equation; (d) all three can be calculated explicitly.

3 $\begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}, \lambda = 2-\sqrt{2}; \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \lambda = 2; \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix}, \lambda = 2+\sqrt{2};$ where $\lambda = \omega^2 ma/T$.

4 $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \lambda = 0; \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \lambda = 1; \begin{bmatrix} 1 \\ 1 \\ -2m/M \end{bmatrix}, \lambda = (2m+M)/M;$ where $\lambda = \omega^2 m/k$.

5 $w(0) = 0; \dfrac{\partial w(L)}{\partial x} = 0$.

$\omega_n = (n-1/2)\pi c/L; u_n = \sin[(n-1/2)\pi x/L]$.

6 $\dfrac{\partial w(L)}{\partial x} = \dfrac{F}{EA}e^{i\omega t}; \qquad u(x) = \dfrac{Fc}{EA\omega}\dfrac{\sin(\omega x/c)}{\cos(\omega L/c)}$

7 Amplitude $b_n = \dfrac{2V}{n^2\pi\Omega}\left(\cos\dfrac{n\pi b}{L} - \cos\dfrac{n\pi a}{L}\right)$

9 (a) w and $\dfrac{\partial w}{\partial x}$ both continuous.