3F1 Signals and Systems

(3) Discrete time systems

Timothy O'Leary

Michaelmas Term

Solution of linear difference equations

From data book: $\mathcal{Z}[\{y_{k+m}\}] = z^m \bar{y}(z) - z^m y_0 - \ldots - z y_{m-1}$

Example: a filter is implemented by the difference equation:

$$y_{k+2} = y_{k+1} - 0.25 y_k + u_k \qquad u_k = (1, 1, 1, \ldots)$$

with initial conditions $y_0 = 1, y_1 = 0$. Find the step response.

Taking Z transforms:

step.

$$z^2 \bar{y}(z) - z^2 = z \bar{y}(z) - z - 0.25 \bar{y}(z) + \frac{1}{1-z^{-1}}$$

$$(z^2 - z + 0.25)\bar{y}(z) = z^2 - z + \frac{1}{1-z^{-1}}$$

$$\bar{y}(z) = \frac{1 - 2z^{-1} + 2z^{-2}}{(1-\frac{1}{2}z^{-1})^2(1-z^{-1})}$$

$$= \frac{4}{1-z^{-1}} - \frac{5}{(1-\frac{1}{2}z^{-1})^2} + \frac{2}{1-\frac{1}{2}z^{-1}}$$

$4 \cdot \{1\}_{k \geq 0}$ $\qquad 2 \cdot \left(\frac{1}{2}\right)^k$

$$10z \frac{\frac{1}{2}z^{-1}}{(1-\frac{1}{2}z^{-1})^2}$$

advance

$\{k(\frac{1}{2})^k\}$

$$10 \underbrace{(k+1)\left(\frac{1}{2}\right)^{k+1}}_{kp^k}$$

$$\therefore \quad y_k = 4 - 10(k+1)\left(\frac{1}{2}\right)^{k+1} + 2\left(\frac{1}{2}\right)^k \qquad k \geq 0.$$

z-transfer function

A System described by linear difference equations

$$y_k + a_1 y_{k-1} + \dots + a_n y_{k-n} = b_0 u_k + \dots + b_m u_{k-m}$$

and subject to zero initial conditions

$$y_k = u_k = 0 \text{ for } k < 0$$

is **linear** and **time-invariant**, that is, is satisfies the principle of superposition and shifting the input to the right does the same to the output.

Such a system has a **z-transfer function**. Taking Z transforms:

$$Y(z) + a_1 z^{-1} Y(z) + \dots + a_n z^{-n} Y(z) = b_0 U(z) + \dots + b_m z^{-m} U(z)$$

Then define the **transfer function**:

$$G(z) = \frac{Y(z)}{U(z)} = \frac{b_0 + b_1 z^{-1} + \dots + b_m z^{-m}}{1 + a_1 z^{-1} + \dots + a_n z^{-n}}$$

Note: ratio is independent of $U(z)$.

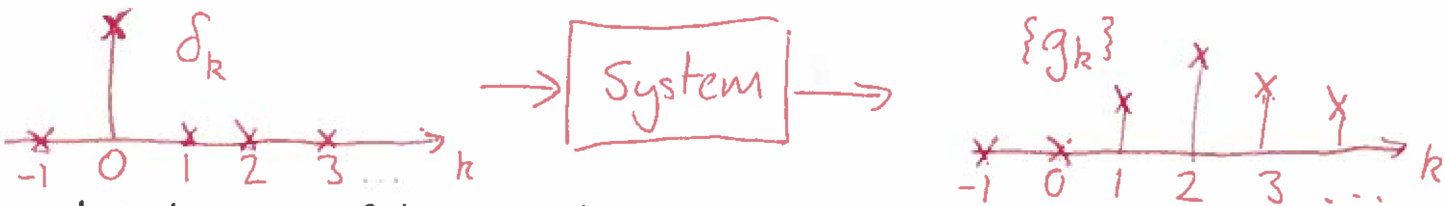
The unit pulse

By analogy with the Dirac delta function in continuous time, we define the **unit pulse** signal, δ_k in discrete time as:

$$\delta_k = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

Question: what is the Z transform of δ_k?

Consider the unit pulse input, $\{u_k\}_{k\geq 0} = \delta_k = (1, 0, 0, 0, \dots)$ to a system.



Let the output of the system be

$$\{g_k\}_{k\geq 0} = (g_0, g_1, g_2, g_3, \dots)$$

This is called the **pulse response** of the system.

Pulse response of LTI systems

Now consider a linear, time-invariant system with pulse response $\{g_k\}_{k\geq 0}$. Take a general input,

$$\{u_k\}_{k\geq 0} = (u_0, u_1, u_2, u_3, \ldots)$$

What is the output, $\{y_k\}_{k\geq 0}$?

$$(1, 0, 0, \ldots) \to \boxed{G} \to (g_0, g_1, g_2, \ldots)$$

Also, by time-invariance,

$$(0, 1, 0, 0, \ldots) \to \boxed{G} \to (0, g_0, g_1, g_2, \ldots) \quad \text{etc}$$

Now represent $\{u_k\}_{k\geq 0}$ as a linear combination of δ-signals:

$$\begin{aligned}(u_0, u_1, u_2, \ldots) &= u_0(1, 0, 0, 0, \ldots) \\ &\quad + u_1(0, 1, 0, 0, \ldots) \\ &\quad + u_2(0, 0, 1, 0, \ldots) \quad \text{etc.}\end{aligned}$$

Then the output $\{y_k\}_{k\geq 0}$ for input $\{u_k\}_{k\geq 0}$ is:

$$\begin{aligned} y_k &= u_0(g_0, g_1, g_2, \ldots) \\ &\quad + u_1(0, g_0, g_1, \ldots) \qquad \text{(by linearity of } G\text{)} \\ &\quad + u_2(0, 0, g_0, g_1, \ldots) + \ldots \\ &= \sum_{i=0}^{k} u_i g_{k-i} \qquad \text{convolution!}\end{aligned}$$

Thus, for a LTI system

$$\{y_k\} = \{u_k\} * \{g_k\} = \{g_k\} * \{u_k\}.$$

Convolution representation of LTI systems

This shows that **any discrete-time LTI system can be represented as a convolution**, allowing us to compute the response to any input, $\{u_k\}_{k\geq 0}$ as:

$$y_k = \sum_{i=0}^{k} u_i g_{k-i} = \sum_{i=0}^{k} u_{k-i} g_i$$

that is,

$$\{y_k\} = \{g_k\} \star \{u_k\}$$

where $\{g_k\}_{k\geq 0}$ is the unit pulse response of the system.

Taking Z transforms gives:

$$\bar{y}(z) = \bar{g}(z)\bar{u}(z)$$

$$\text{(or} \quad Y(z) = G(z)U(z) \text{)}$$

In other words, **the transfer function equals the Z transform of the pulse response.**

Terminology: FIR, IIR and causality

Digital filters (i.e. "discrete time systems") whose pulse response terminates after a finite number of time steps:

$$\{g_k\} = (g_0, g_1, g_2, ...g_n, 0, 0, ..., 0)$$

are called **Finite Impulse Response (FIR)** filters. Otherwise, the system is called **Infinite Impulse Response (IIR)**.

FIR filters/systems have transfer functions with a special form:

$$G(z) = g_0 + g_1 z^{-1} + g_2 z^{-2} + \dots + g_n z^{-n} = \frac{z^n g_0 + \dots + g_n}{z^n}$$

This shows that **all of the poles of the transfer function of an FIR filter are at:** $z = 0$!

By contrast, IIR filters can have poles at arbitrary locations.

Aside: a feedback loop where the closed loop transfer function is FIR is called a *deadbeat* (or *finite settling time*) system. (Cannot happen in a continuous time linear system)

Discrete time systems whose pulse response is zero for negative time are called **causal**. The transfer functions of causal systems have the form:

$$G(z) = g_0 + g_1 z^{-1} + g_2 z^{-2} + \ldots$$

For causal systems, $G(z)$ is finite as $z \to \infty$ (and equals g_0 in fact). Furthermore, if

$$G(z) = \frac{b_0 z^m + b_1 z^{m-1} + \ldots + b_m}{z^n + a_1 z^{n-1} + \ldots + a_n}$$

then for the system to be causal, $\boxed{m \leq n.}$

eg

Causal :

$$(\ldots, 0, 0, \overset{k=0}{\downarrow \atop 1}, 0, 0, \ldots) \qquad (\ldots, 0, 0, \overset{k=0}{\downarrow \atop g_0}, g_1, g_2, \ldots)$$

$$\{\delta_k\} \longrightarrow \boxed{G} \longrightarrow \{g_k\}$$

Non-causal

$$(\ldots 0, 0, \overset{k=0}{1}, 0, 0, \ldots) \longrightarrow \boxed{G} \longrightarrow (\ldots, 0, g_{-1}, \overset{k=0}{\downarrow \atop g_0}, g_1, g_2, \ldots)$$

$g_{-1} \neq 0$ appears <u>before</u> the input!!