Module 3F2: Systems and Control
EXAMPLES PAPER 2 — TRANSFER FUNCTIONS &ROOT-LOCUS

1. (a) A feedback system is given by

$$\underline{Y}(s) = G(s)\left(\underline{R}(s) - \underline{Y}(s)\right)$$

where $G(s) = C\,(sI - A)^{-1}\,B$. By considering the corresponding state-equations (i.e. $\dot{\underline{x}} = A\underline{x} + B(\underline{r} - \underline{y})$, $\underline{y} = C\underline{x}$) deduce that $\underline{Y}(s) = H_{CL}(s)\underline{R}(s)$ where

$$H_{CL}(s) = (I + G(s))^{-1}\,G(s) = C\,(sI - A + BC)^{-1}\,B$$

Verify this identity using the *matrix inversion lemma* — namely: if the indicated inverses exist, then

$$(W + XY^{-1}Z)^{-1} = W^{-1} - W^{-1}X(Y + ZW^{-1}X)^{-1}ZW^{-1}$$

(b) In the equations

$$\begin{aligned}\dot{\underline{x}}(t) &= A\underline{x}(t) + B\underline{u}(t)\\ \underline{y}(t) &= C\underline{x}(t) + D\underline{u}(t)\end{aligned}$$

when D is $p \times p$ and invertible, substitute $\underline{u}(t) = D^{-1}\left(\underline{y}(t) - C\underline{x}(t)\right)$ and hence show that the transfer function of the inverse system, $\underline{U}(s) = (G(s))^{-1}\,\underline{Y}(s)$, is given by,

$$(G(s))^{-1} = \left(D + C\,(sI - A)^{-1}\,B\right)^{-1} = D^{-1} - D^{-1}C\left(sI - A + BD^{-1}C\right)^{-1}BD^{-1}$$

Verify this by using the matrix inversion lemma.

2. (a) For the system

$$L(s) = \frac{1}{(s+a)(s+b)} \qquad (a, b \text{ both real})$$

show that the root-locus diagram (for positive gains k) consists of the segment of the real axis between $-a$ and $-b$, and the perpendicular bisector of that segment.

(b) Sketch the root-locus diagram for positive gains k for the system

$$L(s) = \frac{1}{s(s+1)^2}$$

Find the (positive) value of k at which closed-loop stability is lost

(i) from your diagram, and

(ii) using the Routh-Hurwitz criterion.

(c) Draw the root-locus diagram for positive gains k for the system

$$L(s) = \frac{s}{(s+0.5)(s+1)}$$

and hence show that the closed-loop system is stable for all $k > 0$. Also sketch the root-locus diagram for negative gains, and find the value of k at which closed-loop stability is lost.

3. Consider again the position control system of Question 1 in Examples Paper 1. Suppose that the position feedback gain k_θ is fixed at 10 V/rad.
   (a) Sketch the variation of the closed-loop poles as the tacho feedback gain k_d varies
      (i) using root-locus construction rules,
      (ii) by finding an explicit expression for the closed-loop poles.
   (b) What is the damping factor of the closed loop as a function of k_d? Sketch the time response of the load angular position to a step change of 1 radian in desired angular position for values of $k_d = 0.6$ and $k_d = 1.2$.

4. A negative feedback system consists of a plant whose transfer function is that given in Question 2(b), and a controller which is just a positive gain k. The reference signal is a ramp $r(t) = 2t$. Suppose that k is set to that value which gives two coincident real closed-loop poles at $-1/3$. What is this value of k, and what is the steady-state error $e = r - y$ (where y is the output of the plant) obtained with this value?

   How should the gain be adjusted to reduce this error? What can be said about the locations of the closed-loop poles if this is done?

   (Use your results from Question 2(b).)

Answers

2. (a) — (b) $k = 2$. (c) $k = -3/2$.
3. (a) — (b) Damping factor: k_d.
4. $k = 4/27$. Steady-state error = 27/2.

Relevant questions on past 3F2 exam papers:
2008: Q.1. 2009: Q.2. 2010: Q.2. 2011: Q.2. 2012: Q.3. 2013: Q.2. 2014: Q.2
2015: Q.4. 2016: Q.4. 2017: Q.2

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