UNIVERSITY OF CAMBRIDGE

3F1, Signals and Systems

Part I: Frequency response and Bode diagrams

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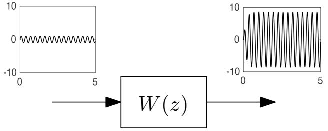
Intro - from math to engineering

Filters for signal processing

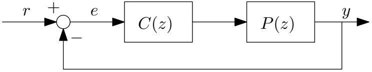
Design the filter $W(s)$ so that the filtered signal is...



Example: reduce high frequency noise from a signal, band-pass filter to select a specific radio station

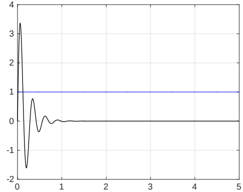
Filters for control

Design the controller $C(s)$ so that the closed loop $W_{r,e} = \frac{1}{1+PC} \ldots$



Example: shape the frequency response of the open loop PC to achieve performance/robustness of closed-loop tracking error $W_{r,e}$

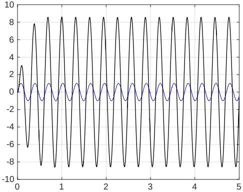
$$G(z) = \frac{z - 1}{z^2 - 1.85z + 0.9}$$

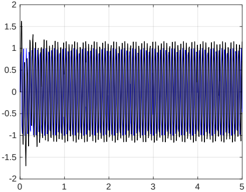


Every input is represented by a sum of sinusoids...

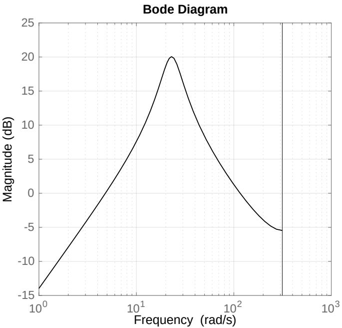
$\Rightarrow$

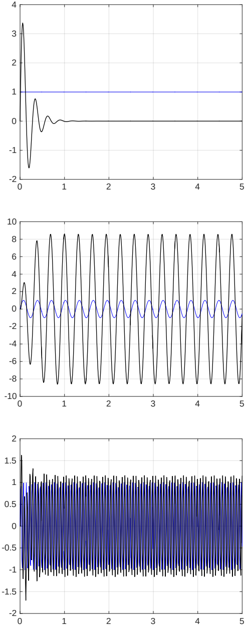
The filter behavior is completely characterized by its response to sinusoids.





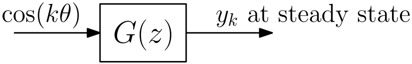
$$G(z) = \frac{z-1}{z^2 - 1.85z + 0.9}$$

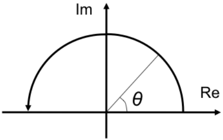




Module A

Frequency response



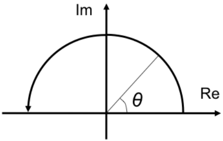


Analog signals:
Frequency: ω rad/s
Input signal: $\cos(\omega t)$
Output signal: $y(t)$ at steady state

Digital signals:
Sampling time: T
(Sampled) Input signal: $\cos(k\theta)$ for $\theta = \omega T$
Output signal: $y(k)$ at steady state

Note: $\theta \leq \pi$.
By Shannon sampling theorem the sampling frequency $f \geq 2\omega_{max}$, which gives a sampling time $T = \frac{2\pi}{f} \leq \frac{\pi}{\omega_{max}}$. Thus, $\theta = \omega T \leq \pi$.

$$\cos(k\theta) \longrightarrow \boxed{G(z)} \longrightarrow y_k \text{ at steady state}$$



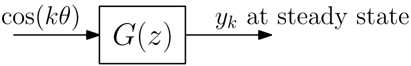
Easier to work with complex form: $u(k) = e^{j\theta k}$

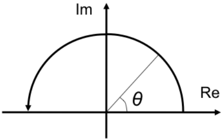
$\Rightarrow U(z) = \frac{1}{1-e^{j\theta}z^{-1}}$

$$\Rightarrow Y(z) = G(z)U(z) = \underbrace{\frac{G(e^{j\theta})}{1 - e^{j\theta}z^{-1}}}_{\text{input}} + \underbrace{\text{terms of the form } \frac{\beta_i}{1 - p_i z^{-1}}}_{\text{filter stable poles}}$$

Why is the input multiplied by $G(e^{j\theta})$? In general,
$Y(z) = \frac{\beta_0}{1-e^{j\theta}z^{-1}} + \text{terms of the form } \frac{\beta_i}{1-p_i z^{-1}}$ so

- $[(1 - e^{j\theta}z^{-1})Y(z)]_{z=e^{j\theta}} = \beta_0 + 0 \cdot \text{terms of the form } \frac{\beta_i}{1-p_i z^{-1}}$, but
- $[(1 - e^{j\theta}z^{-1})G(z)U(z)]_{z=e^{j\theta}} = G(e^{j\theta})$.





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$\Rightarrow U(z) = \frac{1}{1-e^{j\theta}z^{-1}}$

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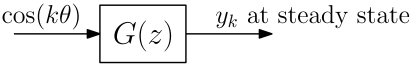
$\Rightarrow y(k) = G(e^{j\theta})e^{j\theta k} + \text{terms decaying to 0 as } k \to \infty.$

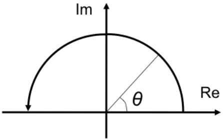
$$y_{ss}(k) = G(e^{j\theta})e^{j\theta k} = |G(e^{j\theta})|e^{j(\theta k + \angle G(e^{j\theta}))}$$

Alternative derivation: frequency response by convolution

$$u_k = e^{j\theta k}$$

$$\begin{aligned}
y_k &= \sum_{i=0}^{k} g_{k-i} u_i = \sum_{i=0}^{k} g_i u_{k-i} = \sum_{i=0}^{k} g_i e^{j\theta(k-i)} = e^{j\theta k} \sum_{i=0}^{k} g_i e^{-j\theta i} \\
&= e^{j\theta k} \left(\sum_{i=0}^{\infty} g_i e^{-j\theta i} - \sum_{i=k+1}^{\infty} g_i e^{-j\theta i} \right) \\
&= e^{j\theta k} \left(\underbrace{\sum_{i=0}^{\infty} g_i e^{-j\theta i}}_{G(e^{j\theta})} - \sum_{i=k+1}^{\infty} g_i e^{-j\theta i} \right) \\
&\leq G(e^{j\theta}) e^{j\theta k} + \underbrace{\sum_{i=k+1}^{\infty} |g_i|}_{\to 0 \text{ as } k \to \infty} = |G(e^{j\theta})| e^{j(\theta k + \angle G(e^{j\theta}))} + \underbrace{\sum_{i=k+1}^{\infty} |g_i|}_{\to 0 \text{ as } k \to \infty}
\end{aligned}$$



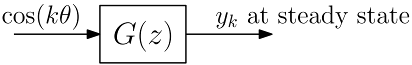


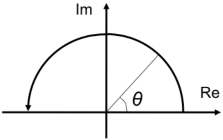
Use linearity

$$u(k) = \cos(\theta k) = \frac{1}{2}\left(e^{j\theta k} + e^{-j\theta k}\right)$$

$$\begin{aligned} y_{ss}(k) &= \frac{|G(e^{j\theta})|}{2}\left(e^{j(\theta k + \angle G(e^{j\theta}))} + e^{-j(\theta k + \angle G(e^{j\theta}))}\right) \\ &= |G(e^{j\theta})| \cos(\theta k + \angle G(e^{j\theta})) \end{aligned}$$

The filter amplifies the sinusoidal input $\cos(\theta k)$ by a factor $|G(e^{j\theta})|$ and shifts its phase by a factor $\angle G(e^{j\theta})$.

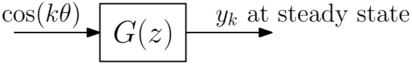


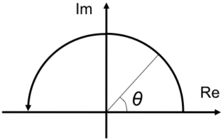


$$u(k) = \cos(\theta k)$$

$$y_{ss}(k) = |G(e^{j\theta})| \cos(\theta k + \angle G(e^{j\theta}))$$

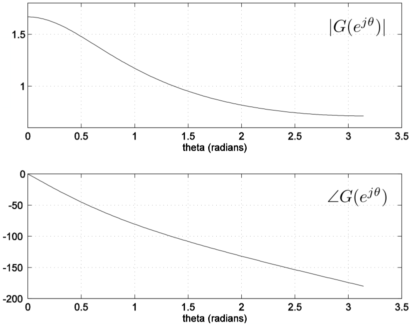
- Filter amplification and phase shift can be characterized at each frequency $\rightarrow$ Bode diagrams.
- Bode diagrams provide a good representation of the filter behavior since every signal at the input can be decomposed into sum of sinusoids.
- Note that $\cos(\omega Tk) = \cos((\omega + \frac{2\pi}{T})Tk)$ so going more than a complete revolution is redundant.





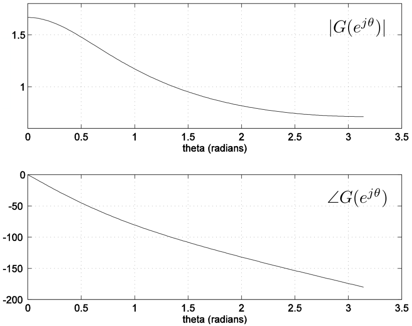
$$u(k) = \cos(\theta k)$$

$$y_{ss}(k) = |G(e^{j\theta})| \cos(\theta k + \angle G(e^{j\theta}))$$

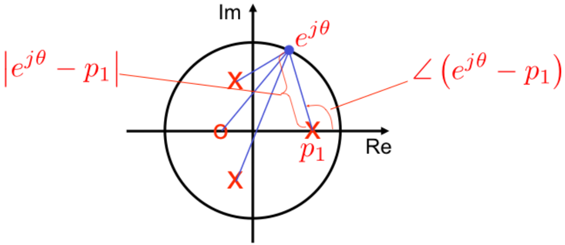


Module B

Gain-Phase plots / Bode diagrams



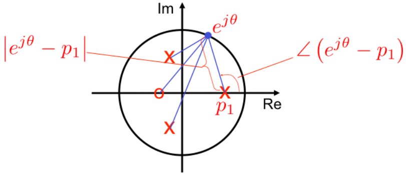
- $G(z) = c\frac{\prod_{k=1}^{m}(z-z_k)}{\prod_{k=1}^{n}(z-p_k)}$



- $$|G(e^{j\theta})| = |c|\frac{\prod_{k=1}^{m}|e^{j\theta} - z_k|}{\prod_{k=1}^{n}|e^{j\theta} - p_k|}$$

  $$\begin{aligned}|G(e^{j\theta})|_{dB} &= 20\log(|G(e^{j\theta})|) \\ &= 20\Big(\log|c| + \sum_{k=1}^{m}\log|e^{j\theta} - z_k| - \sum_{k=1}^{n}\log|e^{j\theta} - p_k|\Big)\end{aligned}$$

- $\angle G(e^{j\theta}) = \angle(c) + \sum_{k=1}^{m}\angle(e^{j\theta} - z_k) - \sum_{k=1}^{n}\angle(e^{j\theta} - p_k)$



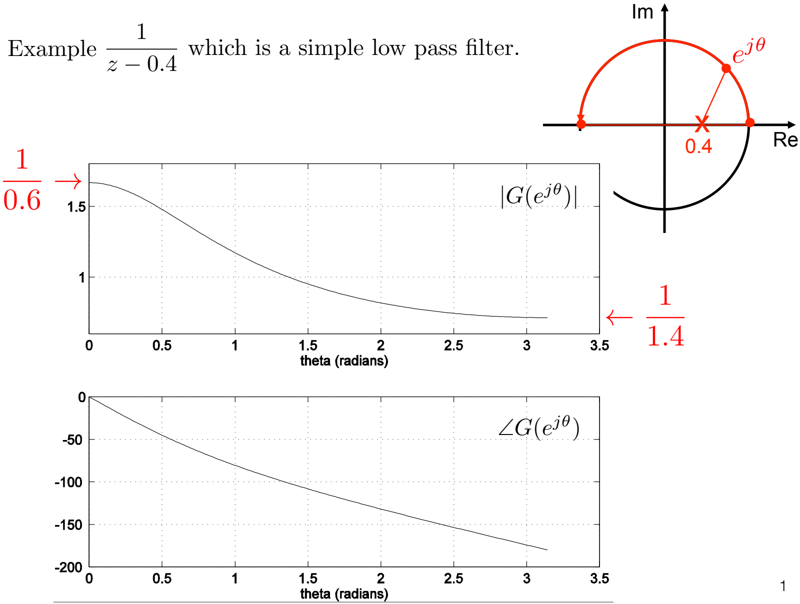
Hints on Bode diagrams

- The magnitude of the frequency response is given by the product of the distances from the zeros to $e^{j\theta}$ divided by the product of the distances from the poles to $e^{j\theta}$

  In dB products and divisions become sums and subtractions.

- The phase response is given by the sum of the angles from the zeros to $e^{j\theta}$ minus the sum of the angles from the poles to $e^{j\theta}$.
- Thus when $e^{j\theta}$ "is close to" a pole, the magnitude of the response rises (resonance). When $e^{j\theta}$ "is close to" a zero, the magnitude falls (a null).
- The behavior of the phase response is less intuitive but similar principles apply.
- Differently from the continuous case, there are no simple rules for drawing Bode diagrams of digital filters. Numerical tools help!

Example $\dfrac{1}{z-0.4}$ which is a simple low pass filter.



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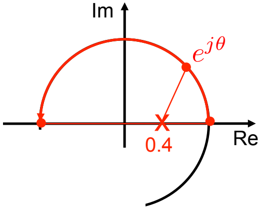
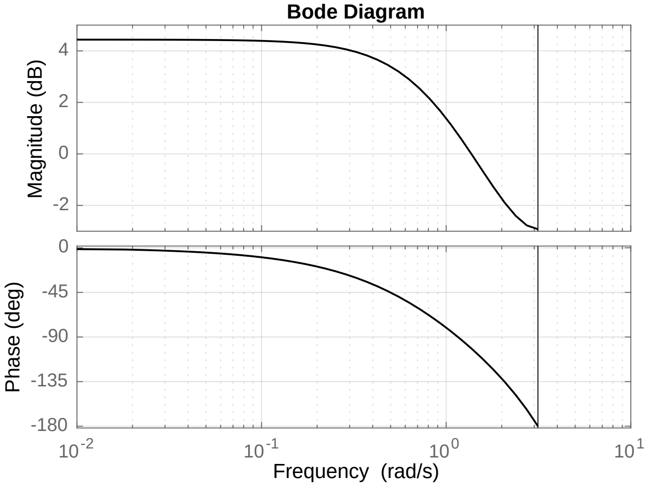
Example $\dfrac{1}{z-0.4}$ which is a simple low pass filter.





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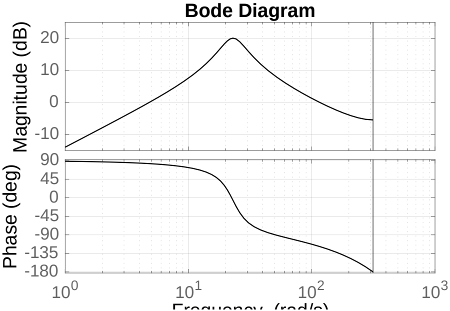
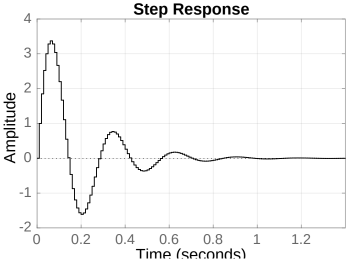
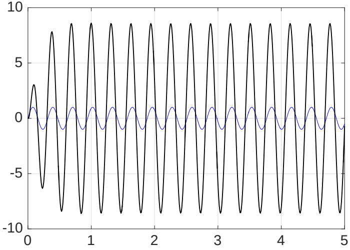
Matlab code for Bode diagrams and sinusoidal input/output

```
%Definition of the filter
num = [ 1 -1 ];  %filter numerator
den = [ 1 -1.85 0.9 ]  %filter denominator
T = 0.01  %sampling time
MaxFreqHz = 1/(2*T)  %max signals frequency in Hz
MaxFreqRads = 2*pi/(2*T)  %max signals frequency in rad/s
G = tf(num,den,T)  %transfer function

%Bode diagram on [0,pi]
bode(G)

%Unit step response
step(G)

%Filtering a 20 rad/s sinusoid for 5 seconds
omega = 20; %20 rad/s, that is, 3.18 Hz
t = 0:T:5; %sampling times
u = sin(omega*t); %digital input (sampled)
y = lsim(G,u,t); %filter output
plot(t,u,'-b'); hold on; plot(t,y,'-k','linewidth',1) % plot signals
```


Bode Diagram
Magnitude (dB)
20
10
0
-10
Phase (deg)
90
45
0
-45
-90
-135
-180
10^0
10^1
10^2
10^3


Step Response
Amplitude
4
3
2
1
0
-1
-2
0
0.2
0.4
0.6
0.8
1
1.2


10
5
0
-5
-10
0
1
2
3
4
5

Module D

From analysis to design: a first example (with Matlab code)

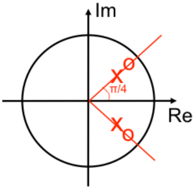
Digital notch filter

Design a causal digital notch filter to attenuate $50Hz$ noise. Assume a sampling period of $T = 2.5$ milliseconds.

We need to arrange a dip in the magnitude of the filter at frequency $\omega_c = 50 \cdot 2\pi rad/s$, that is at the normalized frequency $\theta_c = \omega_c T = 50 \cdot 2\pi \cdot 0.0025 = \frac{\pi}{4} rad$.

Approach: place a pair of zeros of $H(z)$ at $\lambda e^{\pm j\ \theta_c}$. We need to put in a pair of poles to make it causal. Let's place the poles at $\mu e^{\pm j\ \theta_c}$ where $\mu < \lambda$. This will result in the frequency response being close to one for frequencies away from θ_c.

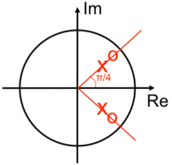
$$
\begin{aligned}
G(z) &= c\frac{(z - \lambda e^{j\theta_c})(z - \lambda e^{-j\theta_c})}{(z - \mu e^{j\theta_c})(z - \mu e^{-j\theta_c})} \\
&= c\frac{z^2 + \lambda\sqrt{2}z + \lambda^2}{z^2 + \mu\sqrt{2}z + \mu^2}
\end{aligned}
$$



choose c so that $G(1) = 1$ (unity d.c. gain).

Digital notch filter

$$G(z) = c\,\frac{z^2 + \lambda\sqrt{2}z + \lambda^2}{z^2 + \mu\sqrt{2}z + \mu^2}$$

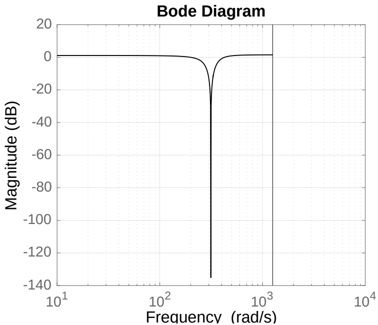


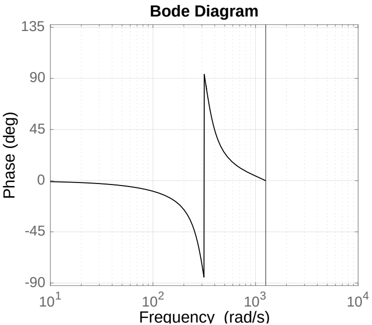
Simple description Y(z) = G(z)U(z) given by

$$(z^2 + \mu\sqrt{2}z + \mu^2)Y(z) = c(z^2 + \lambda\sqrt{2}z + \lambda^2)U(z)$$

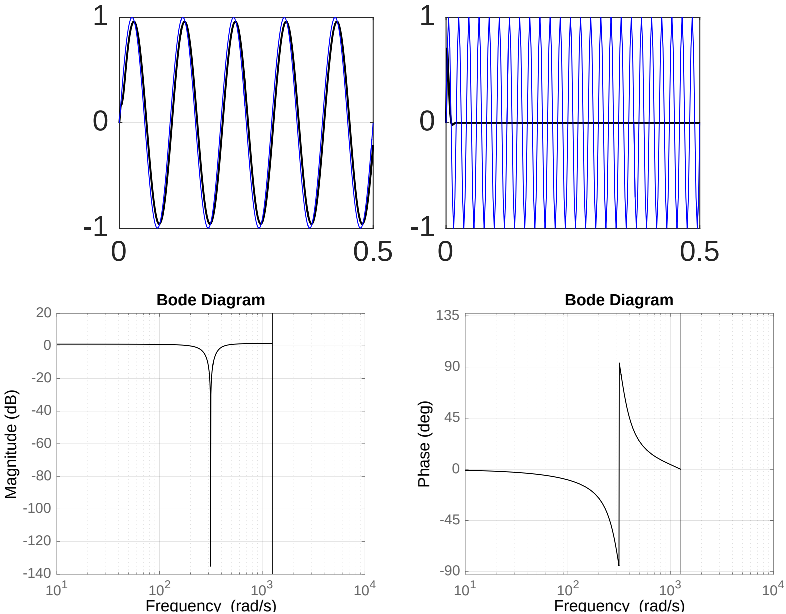
and simple implementation, by the difference equation

$$y_k = -\mu\sqrt{2}y_{k-1} - \mu^2 y_{k-2} + cu_k + c\lambda\sqrt{2}u_{k-1} + c\lambda^2 u_{k-2}$$





Digital notch filter



Matlab code for notch filter

```
%Filter definition with cut frequency at 50Hz
phase = 2*pi*50*0.0025
z = [exp(j*phase),exp(-j*phase)]  %zeros
p = (1/1.2)*[exp(j*phase),exp(-j*phase)]  %poles
notchF= zpk(z,p,1,0.0025)  %(zeros, poles, dc gain, sample time)

%Bode diagram
bode(notchF)

%Simulation with input at frequency omega
t = 0:0.0025:0.5
omega = 50
u = sin(omega*2*pi*t)
y = lsim(notchF,u,t)
plot(t,y,'-k','linewidth',1)
hold on
plot(t,u,'-b')
```