Module 3F2: Systems and Control
EXAMPLES PAPER 3
OBSERVERS AND STATE-FEEDBACK

Observability and Controllability

1. Consider the satellite example of Section 3.7 of Lecture Notes 3.

   (a) Suppose the bias is on the star tracker (measurement of θ) rather than on the rate gyro (measurement of $\dot{\theta}$). Show that the technique shown in the Notes, based on appending the bias to the state vector, won't work in this case, because the system is not observable.

   (b) Suppose that there is a constant but unknown disturbance torque d acting on the satellite: $J\ddot{\theta} = u + d, \quad \dot{d} = 0$, and the star tracker measures θ correctly (without bias). Show that, if d is appended to the state vector then the system is observable (using the star tracker output only), and hence that d can be estimated.

2. (a) Derive state-space equations in standard form for the systems shown in Figures 1 and 2.

   (b) Investigate the controllability and observability of the systems obtained in (a), as the parameter α varies.

   (c) For values of α when either controllability or observability is lost determine the set of states that can be reached from the origin and the set of initial conditions that do not affect the output.

   (d) In each case that observability or controllability is lost, show that the corresponding eigenvalue does not appear as a pole of the transfer function, because of a pole-zero cancellation.

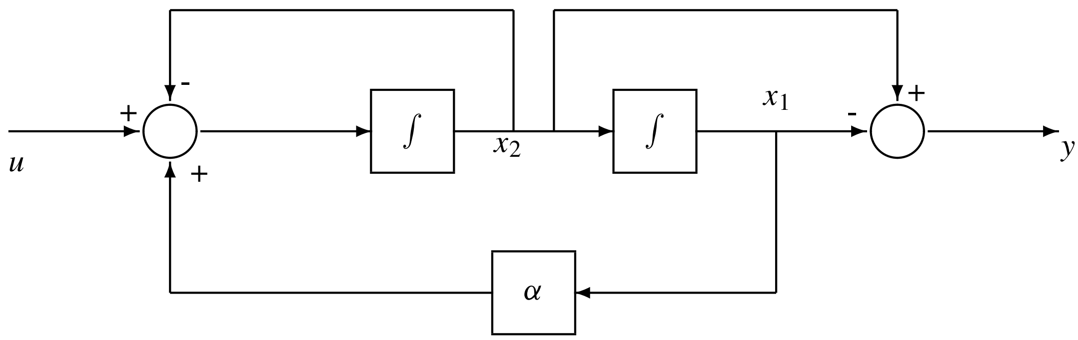

Figure 1:

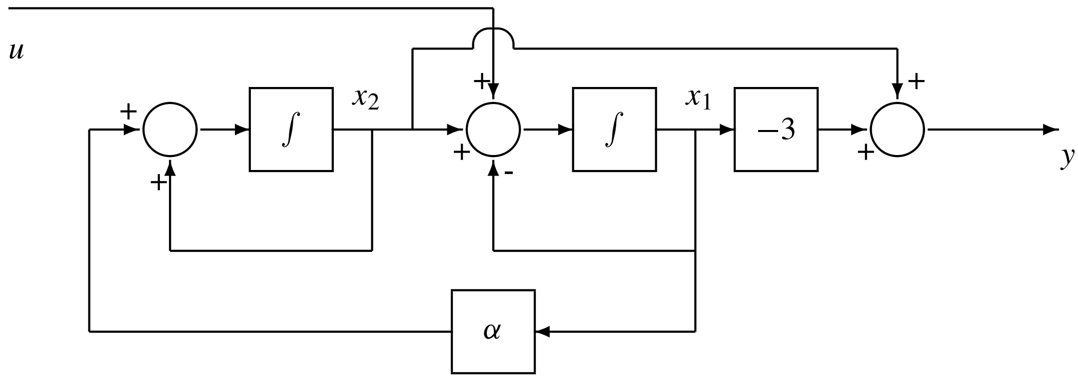

Figure 2:

3. The inverted pendulum laboratory experiment has the linearized equations:

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \omega_0^2 - \omega_1^2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \omega_0^2 & 0 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} u$$

   (a) Show that the system is controllable.

   (b) Will this system be observable if just one of the states is measured, i.e. if $y = x_i$ for each of the four cases $i = 1, 2, 3, 4$?

4. Consider the linearized two link manipulator of Question 7 of Examples Paper 1.

   (a) Is it controllable from

      (i) T_1 and T_2,

      (ii) T_1 alone ($T_2 = T_{2e}$),

      (iii) T_2 alone ($T_1 = T_{1e}$)?

   (b) What implication can you draw on the achievable closed-loop behaviour for this system for these three cases? Include a discussion of the achievable steady-state conditions.

5. (a) In lectures we defined the *controllability Gramian* $W_c(t)$ as

$$W_c(t) = \int_0^t e^{A\tau} B B^T e^{A^T \tau} d\tau.$$

Assume that the system is asymptotically stable; then $W_c(\infty) = \lim_{t\to\infty} W_c(t)$ exists. By considering

$$\frac{d}{d\tau}\left\{e^{A\tau} B B^T e^{A^T \tau}\right\}$$

show that

$$A W_c(\infty) + W_c(\infty) A^T = -B B^T$$

(b) For the system shown in Figure 1, with $\alpha = -1$, find $W_c(\infty)$. (Note that $W_c(\infty)$ is symmetric.)

Observers, State Feedback

6. Design an observer for Question 1(b), such that the state estimation error decays with a time constant of 1 sec.

7. Design a state-feedback controller for the system of Figure 1, which places both closed-loop poles at -10 sec^{-1}. (Assume that both state variables are available for feedback.)

Minimum energy control

8. A "pirate ship"[1] at a fairground is controlled by a suitable electric motor. An idealised and simplified model of the system is given by:

$$I\ddot{\theta} = -MgL\theta + \tau$$

where θ is the angle (of the ship's mast to the vertical), $I = L^2 M$, $g \simeq 10\text{ms}^{-2}$, $L = 10\text{m}$, and τ (Nm) is the geared motor torque. It is desired to minimise the losses in the motor which are assumed to be proportional to

$$J = \int_0^{t_1} \tau^2(t) dt.$$

Show that the minimum possible value of J when moving the ship from rest to $\theta(t_1) = \pi/4$, $\dot{\theta}(t_1) = 0$, with $t_1 = n\pi$ sec ($n = 1, 2, \cdots$) is given by,

$$J_{\min} = \frac{\pi I^2}{8n}$$

[1] For those who do not frequent fairgrounds this ride looks like a ship swinging from a pivot above it.

when the optimal input is

$$\tau_{\text{opt}}(t) = \frac{(-1)^{n+1} I \sin(t)}{2n}.$$

Hint: Define the input to be the scaled torque: $u = \tau/I$.

State feedback, Observers, Everything

9. A system satisfies the state equation

$$\begin{aligned} \dot{\underline{x}} &= A\underline{x} + Bu \\ y &= C\underline{x} + Du \end{aligned}$$

where

$$A = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}, \; B = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 1 \end{bmatrix}, \; D = 0$$

and it is desired to design a feedback controller so that $y(t)$ tracks an external reference signal $r(t)$ as closely as possible with satisfactory stability margins.

(a) Verify that the transfer function is

$$G(s) = \frac{(1-s)}{s(s+1)}$$

with impulse response $1 - 2e^{-t}$. (Note the zero at $s = +1$ makes the response initially go in the 'wrong' direction).

(b) Design a state feedback controller

$$u = -K\underline{x} + Mr$$

so that the closed-loop poles are both at $-\beta$ and $y(t) \to r$ when r is a step. Calculate the resulting response of $y(t)$ to a step change in r.

(c) Design a state observer with gain matrix L so that the poles of the observer are both at $-\alpha$. Calculate the state estimation error if $\underline{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and the observer state, $\hat{\underline{x}}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

(d) Show that if estimated state feedback is now used then the controller equation will be

$$\begin{aligned} \dot{\hat{\underline{x}}} &= (A - BK - LC)\,\hat{\underline{x}} + Ly + BMr \\ u &= -K\hat{\underline{x}} + Mr \end{aligned}$$

and observe that the step response from r to y will be the same as in (b) if $\underline{x}(0) = \hat{\underline{x}}(0) = \underline{0}$.

(e) Now suppose that the system actually satisfies the state equations

$$\begin{aligned} \dot{\underline{x}} &= A_a \underline{x} + B_a u \\ y &= C_a \underline{x} \end{aligned}$$

and the measurement is $(y + v)$ where v is observation noise.

Show that the resulting closed loop equation will be

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} \underline{x} \\ \hat{\underline{x}} \end{bmatrix} &= \begin{bmatrix} A_a & -B_a K \\ LC_a & A - BK - LC \end{bmatrix}\begin{bmatrix} \underline{x} \\ \hat{\underline{x}} \end{bmatrix} + \begin{bmatrix} B_a M & 0 \\ BM & L \end{bmatrix}\begin{bmatrix} r \\ v \end{bmatrix} \\ \begin{bmatrix} y \\ u \end{bmatrix} &= \begin{bmatrix} C_a & 0 \\ 0 & -K \end{bmatrix}\begin{bmatrix} \underline{x} \\ \hat{\underline{x}} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ M & 0 \end{bmatrix}\begin{bmatrix} r \\ v \end{bmatrix} \end{aligned}$$

(f) Show that the return-ratio (open-loop transfer function) of the system with the loop 'broken' at the plant output will be,

$$H(s) = C_a(sI - A_a)^{-1}B_a \; K(sI - A + BK + LC)^{-1}L$$

and note that this transfer function will determine the stability margins with respect to gain and phase uncertainty in the plant model.

(g) Investigate the appropriate choices for α and β by considering the speed of the step responses from r to y, the amplitude of the required input, u, and the gain and phase margins. A MATLAB .m file has been written to perform these calculations. It can be invoked from MATLAB as follows (for example):

```
>> alpha = 1
>> beta = 1
>> fact = 1
>> Q83F2
```

The gain of $G(s)$ is multiplied by the term `fact`.

The resulting plots and printout then give the Bode diagram for $G(j\omega)$; the eigenvalues of the closed loop A-matrix as in part (e); the step response from r to y in the presence of approximately white noise on v (Normally distributed sequence of standard deviation 0.01, and sampled at 0.01 s); and the Nyquist diagram for the loop gain $H(j\omega)$ as in part (f).

Answers

1. —

2. (a) Figure 1: $\underline{\dot{x}} = \begin{bmatrix} 0 & 1 \\ \alpha & -1 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$, $y = [-1, \quad 1]\underline{x} + 0u$.

   Figure 2: $\underline{\dot{x}} = \begin{bmatrix} -1 & 1 \\ \alpha & 1 \end{bmatrix} \underline{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$, $y = [-3, \quad 1]\underline{x} + 0u$.

   (b) and (c): Figure 1 — Always controllable. Not observable if $\alpha = 2$, in which case states of the form $\underline{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rho$ are unobservable, for any scalar ρ.

   Figure 2 — Not controllable if $\alpha = 0$, in which case the reachable states are of the form $\underline{x} = \begin{bmatrix} \rho \\ 0 \end{bmatrix}$. Not observable if $\alpha = 3$, in which case states of the form $\underline{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \rho$ are unobservable.

   (d) Figure 1: When $\alpha = 2$ the eigenvalues are $1, -2$, but the transfer function is $1/(s+2)$.

   Figure 2: When $\alpha = 0$ the eigenvalues are $1, -1$ but the transfer function is $-3/(s+1)$. When $\alpha = 3$ the eigenvalues are $2, -2$ but the transfer function is $-3/(s+2)$.

3. (b) Observable from x_1 only.

4. (a) Controllable from (u_1 and u_2) or u_2 alone, but not from u_1 alone.

   (b) Can place poles using u_2 alone, but cannot maintain a steady state value with $x_1 \neq 0$.

5. (b) $W_c(\infty) = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$

6. Observer gain: $L = [\ell_1, \ell_2, \ell_3]^T$, where $\ell_1 = 3$, $\ell_2 = 3$, $\ell_3 = J$. (This places all three poles at -1 sec^{-1}, which is not the only possible solution.)

7. $u = -K\underline{x} = -[k_1, k_2]\underline{x}$, where $k_1 = 100 + \alpha$ and $k_2 = 19$.

8. —

9. (a) $g(t) = 1 - 2e^{-t}$

   (b) $K = \left[\beta^2, \frac{1}{2}(1-\beta)^2\right], \quad M = \beta^2, \quad y(t) = 1 - e^{-\beta t}(1 + \beta(1+\beta)t).$

   (c) $L = \begin{bmatrix} \alpha^2 \\ -(1-\alpha)^2 \end{bmatrix}, \quad \underline{e}(t) = \exp(-\alpha t) \begin{bmatrix} 1 + \alpha(1-2\alpha)t \\ 1 + (1-2\alpha)(1-\alpha)t \end{bmatrix}.$

   (g) Note the large noise amplification and poor stability margins when α or β are chosen too large.