Module 3F1 – Signals and Systems
Examples Paper 3F1/3
SOLUTIONS

19. Method 1:

    Let X_k be the DFT of x_n. The FFT hardware applied to the conjugate sequence X_k^* produces the DFT

    $$Y_n = \sum_{k=0}^{N-1} X_k^* e^{-j\frac{2\pi}{N}kn} \; .$$

    Since $A^*B^* = (AB)^*$ we have that

    $$Y_n^* = \sum_{k=0}^{N-1} X_k e^{j\frac{2\pi}{N}kn} = N x_n \; .$$

    The inverse DFT is thus obtained by: input conjugation (pre-processing), FFT, output conjugation and division by N (post-processing). Matlab instructions: given the sequence `x` (vector of N elements), the DFT is computed by `X = fft(x)` and the inverse DFT is computed by `fft(X')'/N`.

    Method 2:

    Use the properties in Question 18. Let X_k be the DFT of x_n. Define a new sequence $y_n = X_n$ and denote by Y_k the DFT of y_n. Recall that $Y_k = Nx_{-k}$ (Question 1.d) so the FFT hardware applied to y_n produces the sequence Y_k which is "almost" the inverse DFT of X_k. Define $z_0 = \frac{1}{N}Y_0$ and $z_n = \frac{1}{N}Y_{N-n}$ for $1 \le n \le N-1$. Then z_n is the inverse DFT of X_k, that is, $z_n = x_n$.

20. For $M = 1$ and $N = 4$, the response of the filter is given by

    $$y_n \quad = \quad \underbrace{\sum_{k=0}^{M} h_k x_{n-k}}_{\text{standard convolution}} \quad = \quad \underbrace{\sum_{k=0}^{M} h_k x_{mod(n-k,N)}}_{\text{circular convolution}} \qquad M \le n < N$$

    since $mod(n-m, N) = n-m$ for $M \le n < N$. Thus, we can use the FFT hardware to perform circular convolution.

    We use vector notation for simplicity and we denote by FFT and IFFT respectively the fast Fourier transform and the inverse fast Fourier transform operations.

    – Define the first batch as $x_I = [0, x_0, x_1, x_2]$ and the second batch $x_{II} = [x_2, x_3, 0, 0]$. Define the extended filter $h = [h_0, h_1, 0, 0]$

- For $N = 4$, compute $X_I = \mathrm{FFT}(x_I)$, $X_{II} = \mathrm{FFT}(x_{II})$ and $H = \mathrm{FFT}(h)$. Define $Y_I = H \cdot X_I$ and $Y_{II} = H \cdot X_{II}$.
- Compute $y_I = \mathrm{IFFT}(X_I)$ and $y_{II} = \mathrm{IFFT}(X_{II})$.
- The sequence made by the last three samples of y_I followed by the second sample of y_{II} is the filtered response.

Example: let $h_0 = \frac{1}{2}, h_1 = \frac{1}{2}$ be the impulse response of the filter (average of the last two inputs). Consider an input signal given by a ramp of four elements $x_0 = 1, x_1 = 2, x_2 = 3, x_3 = 4$. Matlab code:

```
% (init)
M = 1;
N = 4;
h = [1/2;1/2;0;0]
x = [1;2;3;4]
% (pre processing)
xI = [0;x(1);x(2);x(3)];
xII = [x(3);x(4);0;0];
% (fft)
XI = fft(xI);
XII = fft(xII);
H = fft(h);
% (product)
YI = H.*XI;
YII = H.*XII;
% (inverse fft)
yI = fft(YI')'/N;
yII = fft(YII')'/N;
% (post processing)
y = [yI(2);yI(3);yI(4);yII(2)]
```

21. (a) PDF must integrate to 1:

$$\int_0^1 \int_0^1 kxy \,\mathrm{d}x\,\mathrm{d}y = k\left(\left.\frac{x^2}{2}\right|_0^1 \right) \times \left(\left.\frac{y^2}{2}\right|_0^1 \right) = \frac{k}{4} = 1$$

$$\therefore k = 4$$

(b)

$$\Pr\{X \leq 0.5, Y > 0.5\} = \int_{x=0}^{0.5} \int_{y=0.5}^{1} 4xy \,\mathrm{d}x\,\mathrm{d}y = \int_{x=0}^{0.5} \left.\frac{4xy^2}{2}\right|_{y=0.5}^{1} \mathrm{d}x$$

$$= \int_{x=0}^{0.5} \frac{4x}{2}\frac{3}{4}\,\mathrm{d}x = \frac{3}{2}\left.\frac{x^2}{2}\right|_0^{0.5} = \frac{3}{16}$$

(c)

$$f_X(x) = \int f_{XY}(x,y)\,dy = \int_0^1 4xy\,\mathrm{d}y \;=\; \left.\frac{4xy^2}{2}\right|_{y=0}^{1}$$

$$= 2x \text{ for } 0 \le x \le 1, \text{ 0 otherwise}$$

by symmetry,

$$f_Y(y) = 2y \text{ for } 0 \le y \le 1, \text{ 0 otherwise}$$

(d) $f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)} = 2y$

(c) $f_X(x) \times f_Y(y) = 2x \times 2y = 4xy = f_{XY}(x,y)$, therefore X and Y are independent.

22. At any time, t, $X(t)$ can take one of four values, $X(t,\alpha), \alpha = 1,2,3,4$, each with equal probability. Thus,

$$f_{X(t)}(x) = \frac{1}{4}[\delta(x-1) + \delta(x+2) + \delta(x - \sin(\pi t)) + \delta(x - \cos(\pi t))].$$

Therefore,

$$E[X(t)] = \int x f_{x(t)}(x)\,dx = \frac{1}{4}[1 - 2 + \sin(\pi t) + \cos(\pi t)] = \frac{\sin(\pi t) + \cos(\pi t) - 1}{4}.$$

Both $f_{X(t)}(x)$ and $E[X(t)]$ depend on t, so the process is not WSS, nor is it SSS.

23. (a) $X(t) = A$:

$$E[X(t)] = \int X(t,a) f_A(a)\,da = \int_0^1 a \times 1\,da = \left.\frac{a^2}{2}\right|_0^1 = 1/2$$

$$r_{XX}(t_1,t_2) = E[X(t_1)X(t_2)] = \int X(t_1,a)X(t_2,a)f_A(a)\,da$$

$$= \int_0^1 a^2 \times 1\,da = \frac{1}{3}$$

both independent of t, therefore process is WSS.

(b) $X(t) = \cos(2\pi ft + \Phi)$. Defining $\omega = 2\pi f$, we have:

$$
\begin{aligned}
E[X(t)] &= \int X(t,\phi) f_\Phi(\phi)\, d\phi = \int_0^{\phi_{max}} X(t,\phi)\frac{1}{\phi_{max}}\, d\phi = \int_0^{\phi_{max}} \frac{\cos(\omega t + \phi)}{\phi_{max}}\, d\phi \\
&= \left.\frac{\sin(\omega t + \phi)}{\phi_{max}}\right|_0^{\phi_{max}} = \frac{\sin(\omega t + \phi) - \sin(\omega t)}{\phi_{max}} \\
r_{XX}(t_1,t_2) &= E[X(t_1)X(t_2)] = \int X(t_1,\phi)X(t_2,\phi) f_\Phi(\phi)\, d\phi \\
&= \int_0^{\phi_{max}} \cos(\omega t_1 + \Phi)\cos(\omega t_2 + \Phi) f_\Phi(\phi)\, d\phi \\
&= \int_0^{\phi_{max}} \frac{1}{2}[\cos(\omega(t_1+t_2) + 2\phi) + \cos(\omega(t_1 - t_2))]\frac{1}{\phi_{max}}\, d\phi \\
&= \frac{1}{2\phi_{max}}\left[\frac{1}{2}\sin(\omega(t_1+t_2) + 2\phi)\right]\Bigg|_0^{\phi_{max}} + \frac{\phi_{max}}{2\phi_{max}}\cos(\omega(t_1 - t_2)) \\
&= \frac{\sin(\omega(t_1+t_2) + 2\phi_{max}) - \sin(\omega(t_1+t_2)) + 2\phi_{max}\cos(\omega(t_1-t_2))}{4\phi_{max}}
\end{aligned}
$$

Since the mean and r_{XX} depend on t the process is not in general WSS However, if $\phi_{max} = 2n\pi$ (n integer) the mean is zero, hence independent of t. Furthermore, if $\phi_{max} = n\pi, r_{XX} = \frac{1}{2}\cos(\omega(t_1 - t_2))$ which depends only on $t_1 - t_2$. If both of these conditions are satisfied then the process is WSS.

(c) $X(t) = A\cos(2\pi ft + \Phi)$. The ensemble is composed of functions from (a) times functions from (b), so sample space consists of multivariate pairs $\{A, \Phi\}$, with A and Φ independent. Thus the (joint) PDF for the process is:

$$f_{A\Phi}(a,\phi) = f_A(a) f_\Phi(\phi)$$

Then, defining $\omega = 2\pi f$,

$$
\begin{aligned}
E[X(t)] &= \int\int X(t,a,\phi) f_{A\Phi}(a,\phi)\, da\, d\phi \\
&= \int\int a\cos(\omega t + \phi) f_A(a) f_\Phi(\phi)\, da\, d\phi \\
&= \int_0^1 a\, da \times \frac{1}{2\pi}\int_0^{2\pi} \cos(\omega t + \phi) d\phi \\
\text{from (a) and (b):} &= \frac{1}{4\pi}[\sin(\omega t + 2\pi) - \sin(\omega t)] \\
&= 0.
\end{aligned}
$$

Similarly,

$$\begin{aligned}
r_{XX}(t_1, t_2) &= E[X(t_1)X(t_2)] \\
&= \int\int a^2 \cos(\omega t_1 + \phi)\cos(\omega t_2 + \phi) f_A(a) f_\Phi(\phi)\, da\, d\phi \\
&= \int_0^1 a^2\, da \times \frac{1}{2\pi}\int_0^{2\pi} \cos(\omega t_1 + \phi)\cos(\omega t_2 + \phi) d\phi \\
\text{from (a) and (b):} &= \frac{1}{3} \times \frac{1}{2}\cos(\omega(t_1 - t_2)) \\
&= \frac{1}{6}\cos(\omega(t_1 - t_2))
\end{aligned}$$

Hence the process is WSS.

Is the process in (a) Mean Ergodic?
The time average is the mean over t for a single realisation, a:

$$E_t[X(t,a)] = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} X(t,a)\, dt = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} a\, dt = a$$

The ensemble average is the mean over A for a single instant t:

$$E_A[X(t,a)] = \int X(t,a) f_A(a)\, da = \frac{1}{2}$$

These averages are not the same, therefore the process in (a) is not ergodic.

24. (a) Let $t' = t + \tau$. Then

$$\begin{aligned}
r_{XX}(\tau) &= E[X(t)X(t+\tau)] \\
&= E[X(t+\tau)X(t)] \\
&= E[X(t')X(t'-\tau)] \\
&= r_{XX}(-\tau)
\end{aligned}$$

(b) Instantaneous power $= V^2/R = X^2(t)/1$.
Average power, $P_{av} = E[X^2(t)] = r_{XX}(0)$.

(c) Let $\tau' = -\tau$. Then

$$\begin{aligned}
\mathcal{S}_X(\omega) &= \int_{-\infty}^{\infty} r_{XX}(\tau) e^{-j\omega\tau}\, d\tau \\
&= \int_{-\infty}^{\infty} r_{XX}(-\tau') e^{j\omega\tau'}\, d\tau' \\
&= \int_{-\infty}^{\infty} r_{XX}(\tau') e^{j\omega\tau'}\, d\tau' \quad \text{(from (a))} \\
&= \mathcal{S}_X(-\omega)
\end{aligned}$$

(d) For $z = a + jb$, define the complex conjugate, $z^* = a - jb$. Then

$$\begin{aligned}
\mathcal{S}_X(\omega) &= \int_{-\infty}^{\infty} r_{XX}(\tau) e^{-j\omega\tau}\, d\tau \\
&= \int_{-\infty}^{\infty} r_{XX}(\tau)(\cos(-\omega\tau) + j\sin(-\omega\tau))\, d\tau \\
&= \int_{-\infty}^{\infty} r_{XX}(\tau)(\cos(\omega\tau) - j\sin(\omega\tau))\, d\tau \\
&= \int_{-\infty}^{\infty} r_{XX}(-\tau)(\cos(\omega\tau) - j\sin(\omega\tau))\, d\tau \quad \text{(from (a))} \\
&= \mathcal{S}_X(-\omega)^* \\
&= \mathcal{S}_X(\omega)^* \quad \text{(from (c))}
\end{aligned}$$

Therefore $\mathcal{S}_X(\omega)$ is real.

(e) From (b):

$$P_{av} = r_{XX}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{S}_X(\omega) e^{j\omega 0}\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{S}_X(\omega)\, d\omega$$

25. The power spectrum, $\mathcal{S}_X(\omega)$ for the white noise process is:

$$\mathcal{S}_X(\omega) = \int_{-\infty}^{\infty} r_{XX}(\tau) e^{-j\omega\tau}\, d\tau = \int_{-\infty}^{\infty} \sigma^2 \delta(\tau) e^{-j\omega\tau}\, d\tau = \sigma^2 e^{-j\omega 0} = \sigma^2$$

i.e. constant over all frequencies, ω.

For the linear system:

$$Y(t) = h(t) * X(t) = \int_{-\infty}^{\infty} h(\beta) X(t-\beta)\, d\beta$$

Autocorrelation function

$$
\begin{aligned}
r_{YY}(\tau) &= E[Y(t)Y(t+\tau)] \\
&= E\Big[\int h(\beta_1)X(t-\beta_1)\,d\beta_1 \int h(\beta_2)X(t+\tau-\beta_2)\,d\beta_2\Big] \\
&= E\Big[\int\int h(\beta_1)h(\beta_2)X(t-\beta_1)X(t+\tau-\beta_2)\,d\beta_1\,d\beta_2\Big] \\
&= \int\int h(\beta_1)h(\beta_2)E[X(t-\beta_1)X(t+\tau-\beta_2)]\,d\beta_1\,d\beta_2 \\
&= \int\int h(\beta_1)h(\beta_2)r_{XX}(\tau+\beta_1-\beta_2)\,d\beta_1\,d\beta_2 \\
&= \int\int h(\beta_1)h(\beta_2)\sigma^2\delta(\tau+\beta_1-\beta_2)\,d\beta_1\,d\beta_2 \\
&= \sigma^2\int h(\beta_2-\tau)h(\beta_2)\,d\beta_2 \\
&= \sigma^2 h(-\tau) * h(\tau)
\end{aligned}
$$

Cross-correlation function

$$
\begin{aligned}
r_{XY}(\tau) &= E[X(t)Y(t+\tau)] \\
&= E\Big[\,X(t)\int h(\beta)X(t+\tau-\beta)\,d\beta\Big] \\
&= E\Big[\int h(\beta)X(t)X(t+\tau-\beta)\,d\beta\Big] \\
&= \int h(\beta)E[X(t)X(t+\tau-\beta)]\,d\beta \\
&= \int h(\beta)r_{XX}(\tau-\beta)\,d\beta \\
&= \int h(\beta)\sigma^2\delta(\tau-\beta)\,d\beta \\
&= \sigma^2 h(\tau)
\end{aligned}
$$

Taking the Fourier transform of the expression for $r_{YY}(\tau)$:

$$
\mathcal{S}_Y(\omega) = \sigma^2\mathcal{H}^*(\omega)\mathcal{H}(\omega) = \sigma^2|\mathcal{H}(\omega)|^2
$$

since $\mathcal{H}^*(\omega)$ is the transform of $h(-\tau)$.

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