ENGINEERING TRIPOS PART IIA - 2010/2011

Module 3A3 - Fluid Mechanics II

2D Compressible Flow

Dr J. P. JARRETT

Tafara Makuni

$M_1 = 1.86$

INTRODUCTION

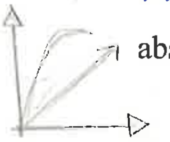
In this section of 3A3 we will take forward the basic 1D concepts already covered and extend into two dimensions. This will enable us to look at flows of significant relevance for the design of aircraft: where 1D may allow us to examine the flow through ducts / nozzles, 2D permits exploration of the flows around high speed airfoils and though complex jet engine intakes.

Our course starts with the advances in compressible aerodynamics made (in England) in the 1920s. In the wake of the First World War the recently formed RAF (created by merging the Royal Naval Air Service with the Army's Royal Flying Corps) re-equipped with the Sopwith *Snipe*: a strut-and-wire braced biplane with a maximum speed (at low level) of 12.5mph ($M=0.16$) ≈ incompressible.

With compressibility becoming significant above a Mach number of around $M=0.3$ (230mph) at sea level) more powerful aircraft soon started to experience compressibility effects on the tip sections of their propellers.

This problem rapidly extended to airframes through the 1920s as streamlining took hold in airplane design and speeds started to rise rapidly. This was felt most keenly in the design of racing seaplanes competing for the Schneider Trophy. The winning British aircraft of the 1927 race (Supermarine S5 whose chief designer, R.J. Mitchell, is famous for his later *Spitfire*) did so at a speed of 281mph $M=0.36$ On 12 March 1928 Flt Lieut S.M. Kinkead was killed in an *S.5* attempting to raise the absolute airspeed record above 300mph, $M=0.4$.

correction for compress'



The issue was that, though a vast amount of airfoil and design data had been accumulated over the preceeding 30 years from (often very accurate) wind tunnel testing, the data was essentially low-speed and thus only representative of incompressible flows. Thus the faster the designs flew as the 1920s progressed the greater the error w.r.t. the incompressible data.

option 1

The aeronautical community faced two options: scrap all the existing aerodynamic design data and start again (and add an order of magnitude to the problem by having to measure lift / drag against incidence *and* against Mach number) or deduce a compressibility correction such that the existing data might be modified to accurately predict the high speed behaviour.

option 2

This is the problem to which Hermann Glauert (an alumnus of Trinity College) applied himself in the early 1920s.

Glauert's Compressibility Correction

$C_L = f(M, Re)$ where $M = \dfrac{V}{\sqrt{\gamma R T}}$

Compressible Potential Flow

The equations of motion are:

$$\frac{C_p}{C_p} - \frac{C_v}{C_p} = \frac{R}{C_p} \qquad 1 - \frac{1}{\gamma} = \frac{R}{C_p} \qquad \frac{\gamma - 1}{\gamma} = \frac{R}{C_p}$$

$$C_p T = \frac{\gamma}{\gamma - 1} \cdot RT = \frac{\gamma R T}{\gamma - 1} = \underline{\underline{\frac{a^2}{\gamma - 1}}}$$

Mass: Continuity: $\nabla \cdot (\rho \underline{V}) = 0$ hence: $\rho \nabla \cdot \underline{V} + (\nabla \rho) \cdot \underline{V} = 0$ (A.1)

$$\Rightarrow \nabla \cdot \underline{V} + \frac{1}{\rho} \underline{V} \cdot \nabla \rho = 0$$

Euler (momentum): $-\frac{1}{\rho} \nabla p = \underbrace{(\underline{V} \circ \nabla)}_{\text{Scalar operator}} \underline{V}$ (A.2)

Isentropic: $p = k\,\rho^{\gamma} \;\Rightarrow\; dp = k\,\gamma \rho^{\gamma-1} d\rho \;\Rightarrow\; \nabla p = a^2 \nabla \rho$ } speed of sound (A.3)

Energy $\text{const} = h_0 = c_p T + \frac{1}{2}\left(u^2 + v^2\right) = \dfrac{a^2}{\gamma - 1} + \dfrac{1}{2}(u^2 + v^2)$ (A.4)

Equation for Velocity

Eliminating ∇p and $\nabla \rho$ from equation (A.2) using (A.1) and (A.3) leaves us with

$$\nabla . \mathbf{V} = -\frac{1}{\rho} \mathbf{V} . \overset{\text{speed of sound}}{\nabla \rho} = -\frac{1}{\rho a^2} \mathbf{V} . \underset{\text{euler}}{\nabla p} = \frac{1}{a^2} \left[\underline{V} \cdot (\underline{V} \cdot \nabla \underline{V}) \right]$$

Rearranging: $a^2 \nabla \cdot \underline{u} - \underline{V} \cdot (\underline{V} \circ \nabla \underline{V}) = 0$

or in terms of components $\mathbf{V} = (u, v)$

$$a^2 \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) - \left[u\,\mathbf{e}_x + v\,\mathbf{e}_y \right] . \left[\left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) \mathbf{e}_x + \left(u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) \mathbf{e}_y \right] = 0$$

$$a^2 \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) - u \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) - v \left(u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) = 0$$

$$\left(a^2 - u^2 \right) \frac{\partial u}{\partial x} - uv \frac{\partial u}{\partial y} - vu \frac{\partial v}{\partial x} + \left(a^2 - v^2 \right) \frac{\partial v}{\partial y} = 0$$

$$(a^2-u^2)\frac{\partial u}{\partial x} - uv\frac{\partial u}{\partial y} - vu\frac{\partial v}{\partial x} + (a^2-v^2)\frac{\partial v}{\partial y} = 0$$

Equation for Potential

The flow is assumed to be steady, two-dimensional and constant entropy (i.e. no significant shock or viscous losses). Under these conditions the flow is *irrotational*, and the velocity is given as the gradient of a potential (see 3A1).

no physical mechanism to cause spinning.

Irrotational : $\nabla \times \underline{u} = 0$

disturbance independent of x

$$\Rightarrow \underline{v} = \nabla\phi = \nabla(U_\infty x + \phi) = \begin{bmatrix} \frac{\partial}{\partial x}(U_\infty x + \phi) \\ \frac{\partial}{\partial y}(U_\infty x + \phi) \end{bmatrix} = \begin{matrix} U_\infty + \partial\phi/\partial x \\ \partial\phi/\partial y \end{matrix}$$

Substituting $u = U_\infty + \frac{\partial\phi}{\partial x}$ and $v = \frac{\partial\phi}{\partial y}$ and collecting terms together gives

$$\boxed{\left[a^2 - \left(\underbrace{U_\infty + \frac{\partial\phi}{\partial x}}_{u}\right)^2\right]\underbrace{\frac{\partial^2\phi}{\partial x^2}}_{\partial u/\partial x} - 2\left[\underbrace{U_\infty + \frac{\partial\phi}{\partial x}}_{u}\right]\underbrace{\frac{\partial\phi}{\partial y}}_{v}\underbrace{\frac{\partial^2\phi}{\partial x\partial y}}_{\frac{\partial u}{\partial y}=\frac{\partial v}{\partial x}} + \left[a^2 - \left(\underbrace{\frac{\partial\phi}{\partial y}}_{v}\right)^2\right]\underbrace{\frac{\partial^2\phi}{\partial y^2}}_{\partial v/\partial y} = 0} \quad \text{(A.5)}$$

which is supplemented by the energy equation

- ϕ small
- neglect 2nd order (aerofoils thin)

$$\frac{a^2}{(\gamma-1)} + \frac{1}{2}\left(\left(U_\infty + \frac{\partial\phi}{\partial x}\right)^{\cancel{2}} + \left(\frac{\partial\phi}{\partial y}\right)^{\cancel{2}}\right) = \boxed{\frac{a_\infty^2}{\gamma-1} + \frac{U_\infty^2}{2}} \quad \text{(A.6)}$$

Our interest here is in deriving what properties of the solution we can, and in particular how properties depend on Mach Number, etc., *without* the need to solve for ϕ explicitly.

Linearise Equation of Motion

Consider cases for which the flow is only *slightly* disturbed from a uniform one, as for thin aerofoils and linearise.

$$u = U_\infty + \frac{\partial\phi}{\partial x}, \quad v = \frac{\partial\phi}{\partial y} \quad \text{where } \frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y} \ll U_\infty \quad \text{(A.7)}$$

Equation (A.5) becomes

N.B. $M = \frac{V}{a}$

$$\left[a_\infty^2 - U_\infty^2\right]\frac{\partial^2\phi}{\partial x^2} + a_\infty^2\frac{\partial^2\phi}{\partial y^2} = 0 \qquad (1 - M_\infty^2)\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \quad \text{(A.8)}$$

Note that for $M_\infty < 1$ this equation is elliptic : "Laplace Like"

↓↓ transonic flow. ↓↓

and for $M_\infty > 1$ this equation is hyperbolic : "Wave Like"

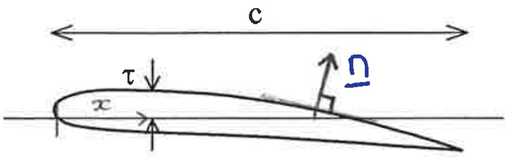
only some B.C's apply at any one time.

Unsteady, subsonic isentropic flow is hyperbolic w.r.t. time.

Linearise Boundary Conditions

where: τ is a measure of aerofoil thickness.
g is a shape Function.

For a typical thin compressible aerofoil,



$$\text{on} \quad y = \tau\, g_U\left(\frac{x}{c}\right) \quad \text{and} \quad \tau\, g_L\left(\frac{x}{c}\right)$$

$$\text{for} \quad 0 < x < c$$

$$\frac{dy}{dx} = \tau \cdot g_u' \cdot \frac{1}{c}$$

gradient $\frac{dy}{dx} = -\frac{\tau}{c} g'$

Since the normal is parallel to $\left[\overset{e_x}{-\frac{\tau}{c} g'}, \overset{e_y}{1} \right]$, this boundary condition is

$$\left[\overset{e_x}{-\frac{\tau}{c} g'}, \overset{e_y}{1} \right] \cdot \left[\overset{u}{U_\infty + \frac{\partial \phi}{\partial x}}, \overset{v}{\frac{\partial \phi}{\partial y}} \right] = \boxed{-\frac{\tau}{c} g' U_\infty \underbrace{\overset{\text{small}}{-\frac{\tau}{c} g'} \overset{\text{small}}{\frac{\partial \phi}{\partial x}}}_{\text{neglect}} + \frac{\partial \phi}{\partial y} = 0} \quad \text{on } y = \tau g$$

For a thin aerofoil $\frac{\tau}{c} \ll 1$ + Keep only linear terms.

$$\boxed{-\frac{\tau}{c} g' U_\infty + \frac{\partial \phi}{\partial y} = 0} \quad \text{on } y = \tau g$$

Further, since τ is small,

$$\frac{\tau}{c} g' U_\infty = \left.\frac{\partial \phi}{\partial y}\right|_{y=\tau g} = \left.\frac{\partial \phi}{\partial y}\right|_{y=0} + \underbrace{\tau g \left.\frac{\partial^2 \phi}{\partial y^2}\right|_{y=0}}_{\text{neglect}} + \ \ldots = \left.\frac{\partial \phi}{\partial y}\right|_{y=0} + \ldots$$

i.e. we can, to this order of approximation, apply the Boundary condition on $y=0$

The full problem is thus

$$(1 - M_\infty^2) \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

$$\boxed{\frac{\partial \phi}{\partial y} = \frac{\tau}{c} g' U_\infty} \quad \text{on } y = 0, \quad 0 < x < c \qquad \text{(A.9)}$$

M_∞, τ, g, c and U_∞ are all real and physical parameters.

Scaling

When faced with an equation like (A.9), which is still not that easy to solve, progress can often be made by looking for "similarity" solutions. This approach is summarised as:

"can we relate solutions corresponding to different data by suitably re-normalising (i.e. re-scaling) the dependent and independent variables.?

If such a relation *is* established, then, given a solution for one set of data (from numerical computation, experiment, etc), we can generate a family of solutions (called "similarity solutions") for a whole host of other sets of data. Sometimes a particular choice of scaling parameters can result in a simpler or "more studied" equation.

The parameters which appear in this problem are thus

$U_\infty, M_\infty, \tau, c$ and g (shapefunction).

The objective of the hunt for "similarity solutions" is to make substitutions using other scaled non-dimensional variables which reduces this number of parameters.

SCALING Length Scales:

An obvious choice for scaling x is $\boxed{\bar{x} = \frac{x}{c}}$ which means that the boundary conditions are applied on $0 < \tilde{x} < 1$.

It is not obvious how to scale y, so take $\boxed{\bar{y} = \beta \frac{y}{c}}$ with β still to be determined.

The blade boundary condition becomes

$$\frac{\partial \phi}{\partial \bar{y}} = \frac{c}{\beta} \cdot \frac{\partial \phi}{\partial y} = \frac{\tau}{\beta} U_\infty g'(\bar{x})$$

and clearly the way to scale ϕ is to take $\phi = \frac{\tau}{\beta} \cdot U_\infty \bar{\phi}$ so that this boundary condition becomes

$$\frac{\partial \bar{\phi}}{\partial \bar{y}} = \frac{\beta}{\tau} \cdot \frac{1}{U_\infty} \cdot \frac{\partial \phi}{\partial \bar{y}} = \frac{\cancel{\beta}}{\cancel{\tau}} \cdot \frac{1}{\cancel{U_\infty}} \left[\frac{\cancel{\tau}}{\cancel{\beta}} \cancel{U_\infty} \cdot g'(\bar{x}) \right]$$

Hence: $\frac{\partial \bar{\phi}}{\partial \bar{y}} = g'(\bar{x})$

$$\left\{ \begin{aligned} &(1-M_\infty^2)\frac{\partial^2\left[\frac{\tau U_\infty}{\beta}\bar{\phi}\right]}{\partial(\bar{x}c)^2} + \frac{\partial^2\left[\frac{\tau U_\infty}{\beta}\bar{\phi}\right]}{\partial\left(\frac{c}{\beta}\bar{y}\right)^2} = 0 \\ &(1-M_\infty^2)\frac{\partial^2\bar{\phi}}{\partial\bar{x}^2} + \beta^2\frac{\partial^2\bar{\phi}}{\partial\bar{y}^2} = 0 \\ &\frac{(1-M_\infty^2)}{\beta^2}\frac{\partial^2\bar{\phi}}{\partial\bar{x}^2} + \frac{\partial^2\bar{\phi}}{\partial\bar{y}^2} = 0 \end{aligned} \right\}$$

With this choice, equation (A.9) becomes

$$\frac{(1-M_\infty^2)}{\beta^2}\frac{\partial^2\bar{\phi}}{\partial\bar{x}^2} + \frac{\partial^2\bar{\phi}}{\partial\bar{y}^2} = 0$$

and the smart move is to take $\beta=\sqrt{1-M_\infty^2}$ (or $\sqrt{M_\infty^2-1}$ if the incoming flow is supersonic).

Recapping, if we make the following substitutions

$$\bar{\phi}=\frac{\sqrt{1-M_\infty^2}}{\tau U_\infty}\phi \; ; \quad \bar{x}=\frac{x}{c} \quad \text{and} \quad \bar{y}=\sqrt{1-M_\infty^2}\,\frac{y}{c} \qquad \text{(A.10)}$$

then equation (A.9) becomes

$$\frac{\partial^2\bar{\phi}}{\partial\bar{x}^2} + \frac{\partial^2\bar{\phi}}{\partial\bar{y}^2} = 0 \qquad \text{(A.11)}$$

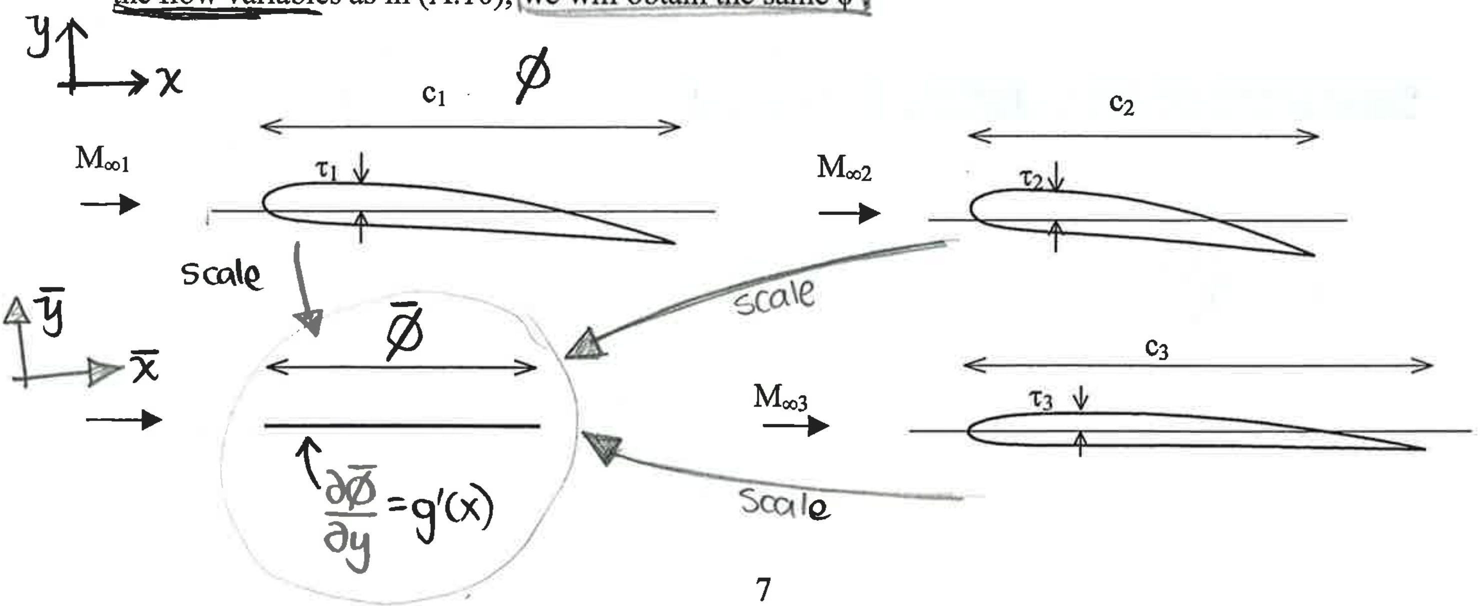
Boundary condition:

$$\frac{\partial\bar{\phi}}{\partial y} = g'(\bar{x}) \;\; \text{on} \;\; \bar{y}=0 \quad \text{where} \quad 0<\bar{x}<1 \qquad \text{(A.12)}$$

Thus

The problem for $\bar{\phi}$ is now completely independent of M_∞, c, τ or U_∞ and hence $\bar{\phi}$ is independent of these parameters.

This means that if we take a family of aerofoils with the same g_U and g_L, but with varying τ and c, (such a family of shapes is said to be *affinely related*), then, for many M_∞'s, if we normalise the flow variables as in (A.10), $\boxed{\text{we will obtain the same } \bar{\phi}}$



$$u = U_\infty + \frac{\partial \phi}{\partial x}$$

Significance $\frac{\partial u}{\partial x} = \frac{\partial^2 \phi}{\partial x^2}$

$$u\frac{\partial u}{\partial x} = U_\infty \frac{\partial^2 \phi}{\partial x^2} + \left[\frac{\partial^3 \phi}{\partial x^2}\right]$$ neglected?

This commonality of $\bar{\phi}$, has consequences for real flow quantities. Thus, for example, taking the linearised x-component of the (Euler) momentum equation

$$\frac{\partial p}{\partial x} = -\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) \rightarrow \frac{\partial p}{\partial x} = -\rho_\infty U_\infty \frac{\partial u}{\partial x} = -\rho_\infty U_\infty \frac{\partial^2 \phi}{\partial x^2}$$

$$\Rightarrow p = -\rho_\infty U_\infty \frac{\partial \phi}{\partial x} + f(y)$$

disturbance from the body

and far upstream $\frac{\partial \phi}{\partial x} = 0$, p is uniform, giving $p_\infty = f(y)$

$$-\frac{2}{U_\infty} \cdot \frac{\partial\left(\frac{\tau U_\infty}{\beta}\right)\bar{\phi}}{\partial(\bar{x} \cdot c)} = -\frac{2}{\cancel{U_\infty}} \cdot \frac{\tau \cancel{U_\infty}}{\beta c} \boxed{\frac{\partial \bar{\phi}}{\partial \bar{y}}}$$ k

$$\Rightarrow \quad p - p_\infty = -\rho_\infty U_\infty \frac{\partial \phi}{\partial x}$$

$$\Rightarrow \quad C_p = \frac{p - p_\infty}{\frac{1}{2}\rho_\infty U_\infty^2} = -\frac{\rho_\infty U_\infty}{\frac{1}{2}\rho_\infty U_\infty^2} \cdot \frac{\partial \phi}{\partial x} = -\frac{2}{U_\infty}\left[\frac{\partial \phi}{\partial x}\right] = \cancel{-\frac{2}{U_\infty}\left[\frac{\tau U_\infty}{\sqrt{1-M_\infty^2}} \cdot \frac{1}{c} \frac{\partial^2 \phi}{\partial x^2}\right]}$$

i.e. $$C_p = \frac{k\tau}{c\sqrt{1-M_\infty^2}} \qquad \text{(A.12)}$$

$$\frac{\tau}{\beta} \cdot \frac{U}{c} \cdot \frac{\partial^2 \phi}{\partial x^2} = \frac{\partial \phi}{\partial x}$$

where k depends only on g, the non-dimensional shape of the aerofoil, and on $\bar{x}$ the fraction of chord. Thus, if an aerofoil with thickness and chord τ_1 and c_1, is tested at M_1, and the value of C_p measured, then for other aerofoils in the same family (same value of g) with τ_2 and c_2, tested at M_2,

for aerofoils of same shape.

$$\left\{\frac{C_{p1} c_1 \sqrt{1-M_{\infty 1}^2}}{\tau_1} = \frac{C_{p2} c_2 \sqrt{1-M_{\infty 2}^2}}{\tau_2}\right\} \text{ hence: } C_{p2} = \frac{\tau_2}{\tau_1} \cdot \frac{c_1}{c_2} \cdot \frac{\sqrt{1-M_{\infty 1}^2}}{\sqrt{1-M_{\infty 2}^2}} \cdot C_{p1} \qquad \text{(A.13)}$$

(for the appropriate point on the aerofoil. i.e. at the same value of x/c).

Equation (A.12) indicates how low-speed data can be extended into the compressible range. If case (1) is taken as a low-speed test (i.e. zero Mach number), for an aerofoil having the same thickness-chord ratio and shape (i.e. being geometrically similar), then equation (A.13) becomes

$$\boxed{C_p = \frac{C_{p0}}{\sqrt{1-M_\infty^2}}} \xleftarrow{\text{hence}} \left\{ C_p = \frac{\cancel{\tau}}{\cancel{\tau}} \cdot \frac{\cancel{c}}{\cancel{c}} \cdot \frac{\sqrt{1-0}}{\sqrt{1-M_\infty^2}} \cdot C_{p0} \right.$$

which when integrated over the aerofoil also implies $C_L = \frac{C_{L0}}{\sqrt{1-M_\infty^2}}$ and $C_M = \frac{C_{LM}}{\sqrt{1-M_\infty^2}}$

where C_L and C_M are lift and moment coefficients.

The analysis presented is for *subsonic* flow, but clearly, a very similar one, with $\beta = \sqrt{1 - M_\infty^2}$ replaced by $\beta = \sqrt{M_\infty^2 - 1}$ and $\frac{\partial^2 \bar{\phi}}{\partial \bar{x}^2} - \frac{\partial^2 \bar{\phi}}{\partial \bar{y}^2} = 0$

SUPERSONIC (except we now get the wave equation).

for supersonic cases and, rather than getting Laplace's equation, we get the wave equation.

We derived these equations for 2D, but clearly they are also valid for 3D, provided the object under study produces relatively small variations in the flow:

$$(1 - M_\infty^2)\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

These relations do not hold for *mixed* supersonic and subsonic flow: the whole quality and nature of the flow changes when the subsonic and supersonic regions change size. This is the *transonic* region and is the subject of next year's Aerodynamics course (4A7).

Glauert's Compressibility Correction is often referred to as the "Prandtl-Glauert Similarity Rule" despite the fact that the two men did not collaborate on the work and, unlike Glauert, Prandtl published neither the result nor its derivation. Those interested by this historical quirk are encouraged to read Anderson's *"A History of Aerodynamics"*.

Thus Glauert demonstrated that, by his elegantly simple correction, low-speed (incompressible) test data could be reliably modified and thus used for airplane design at the appropriate Mach Number. His derivation was made public in the Proceedings of the Royal Society in the same year that Kinkead was killed attempting to exceed M=0.4.

The Schneider Trophy races continued. The rules stated that races would continue until one country had won three competitions in succession; the contest would then be judged to be over and the winner declared. A Supermarine *S.6* won the 1929 race, and a Supermarine *S.6B* in 1931: Britain had won the Trophy outright.

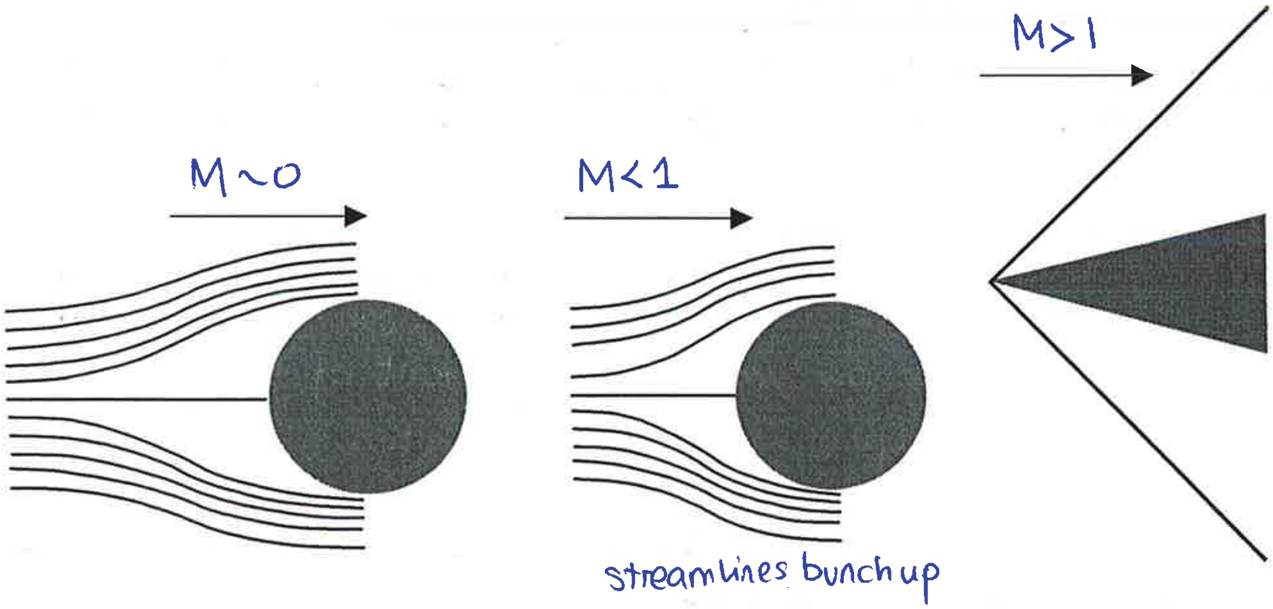
On 29 September 1931, a modified Supermarine *S.6B* set a new absolute airspeed record of 407.5mph – i.e at a Mach number of 0.53 (and over 40mph faster than a production Mk 1A *Spitfire* the first of which left the factory in June 1938).

INTRODUCTION TO 2D SUPERSONIC FLOW

The early research by Glauert and others into compressibility aided designers as airplane speeds continued to rise though the 1930s and particularly through the years of the Second World War: though the ground speed increases were themselves significant, much greater increases in Mach number occurred due to such speeds being attained at higher (therefore colder) altitudes (due the use of supercharging to increase the power available from aircraft piston engines in lower density air).

It was only a matter of time before humans made a serious attempt to fly supersonically...

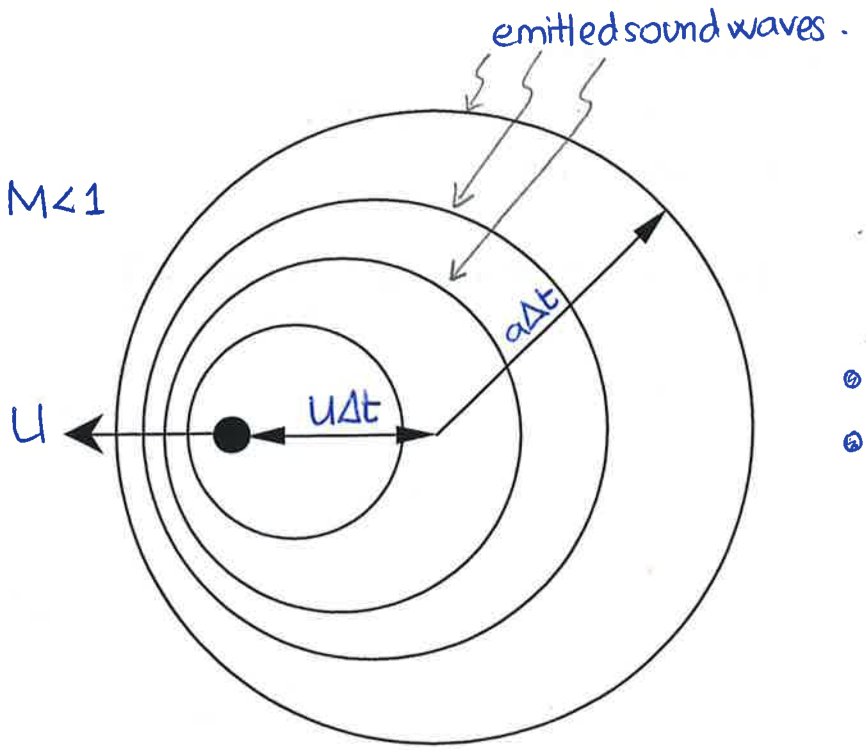
Terminology



Subsonic flow - information propagates upstream from body giving "advance warning". Streamtube areas adjust so that density changes corresponding to variations in velocity are small - flow is essentially incompressible. As M increases, there is less advance warning and the density changes become greater.

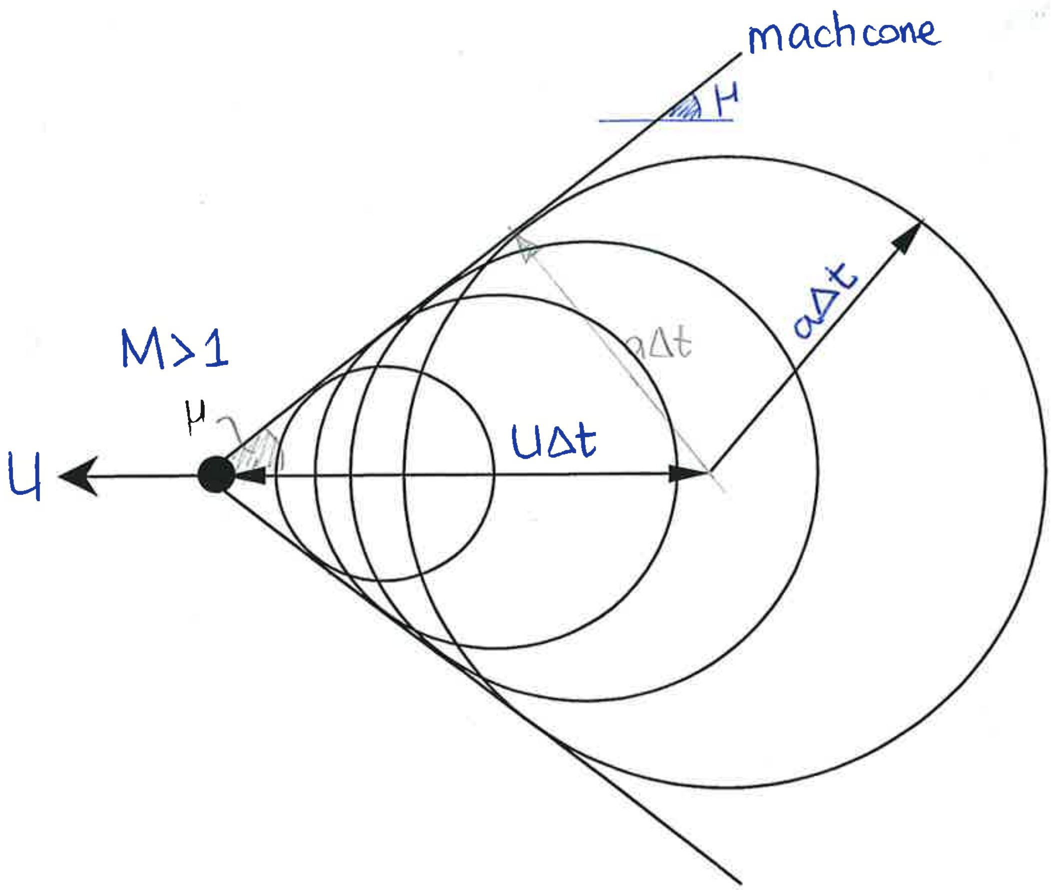
Supersonic flow - information cannot propagate upstream. The flow is uniform until it reaches the body, when it undergoes large changes in pressure, velocity and density.

Moving Sources of Sound



a = local speed of sound.

Sonic source:
- Waves influence entire domain
- Hence can hear source upstream.



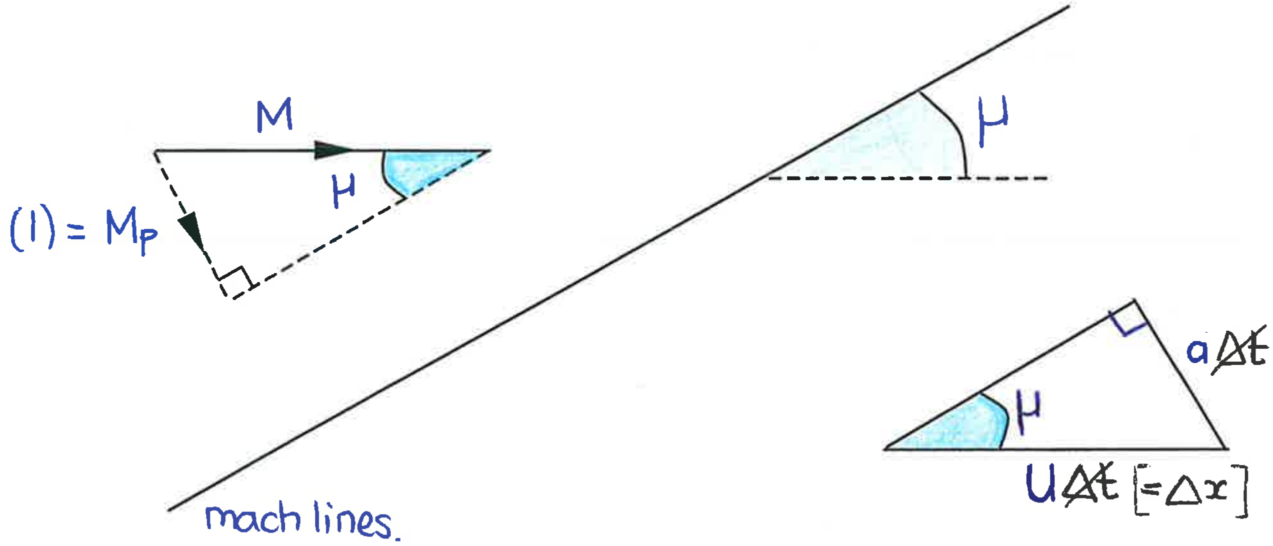
Supersonic Source
- waves coalesce to mach cone.
- No knowlege of source upstream.

Mach lines

Borders of Mach cone (or wedge in 2-D) are mach lines.

In frame with source stationary, Mach lines are at angle $\mu = sin^{-1}\left(\frac{a\Delta t}{\Delta x}\right) = sin^{-1}\left(\frac{1}{M}\right)$ to flow:

$$\mu = \sin^{-1}\left(\frac{a\cancel{\Delta t}}{U\cancel{\Delta t}}\right) = \sin^{-1}\left(\frac{a}{U}\right) = \sin^{-1}\left(\frac{1}{M}\right)$$



$\perp$ Mach line, flow Mach no $M_p = M \sin\mu = 1$, $V_p = a$

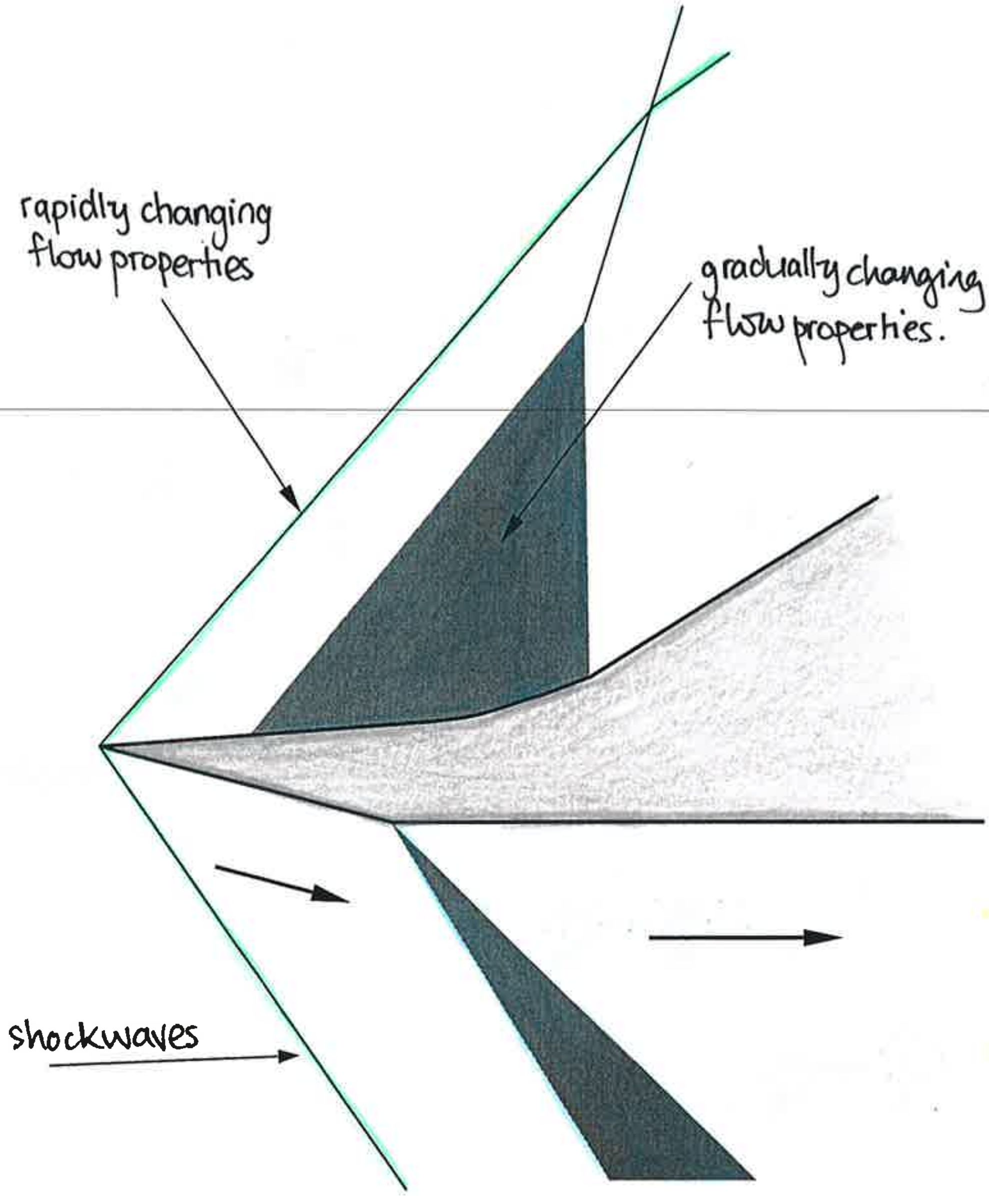
Sound wave stationary $\Leftrightarrow$ sound wave aligned wave on Mach line.

Typical supersonic flow features

Supersonic flowfields generally consist of

(i) large regions where properties change gradually, gradients are small

(ii) localised regions where properties change suddenly.



Analyse general flows by combining techniques for different regions.

Start by analysing flow in smooth regions - Method of characteristics

Equations of motion in intrinsic (streamline) co-ordinates

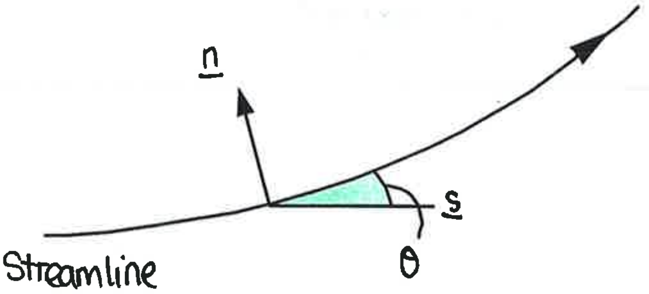
Assumptions: steady

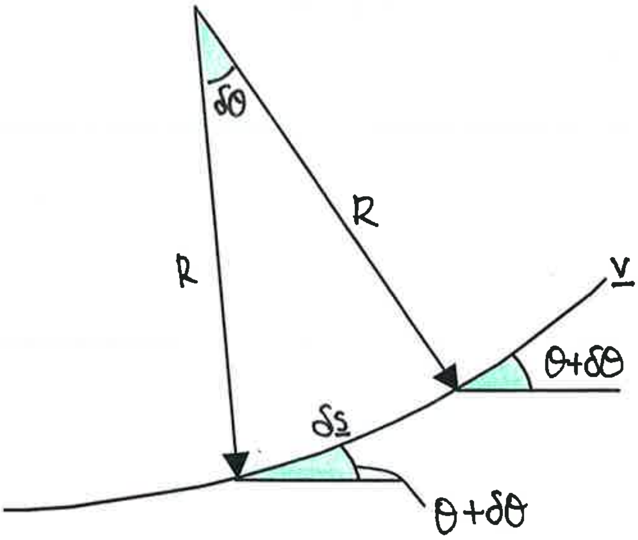
isentropic (= inviscid, non-heat conducting)

two dimensional

uniform conditions far upstream

Steady flow energy equation applied to a stream tube indicates that $h_0 = h + \dfrac{V^2}{2}$ = constant along the stream tube and hence uniform everywhere (since upstream conditions are uniform).





Momentum

- *s*-momentum (streamwise) $\boxed{\rho \cdot V \cdot \dfrac{\partial V}{\partial s} = -\dfrac{\partial p}{\partial s}}$

$\rho V \delta V + \delta p = 0$

$\rho V \dfrac{\delta V}{\delta s} + \dfrac{\delta p}{\delta s} = 0$ (1.1)

- *n*-momentum (ITDB) $\boxed{\rho \dfrac{V^2}{R} = -\dfrac{\partial p}{\partial n}}$

$\cancel{p} - (\cancel{p} + dp) = \rho V(\cancel{V} + dV - \cancel{V})$

$\Rightarrow dp + \rho V dV = 0$

$dp/ds + \rho V dV/ds = 0$

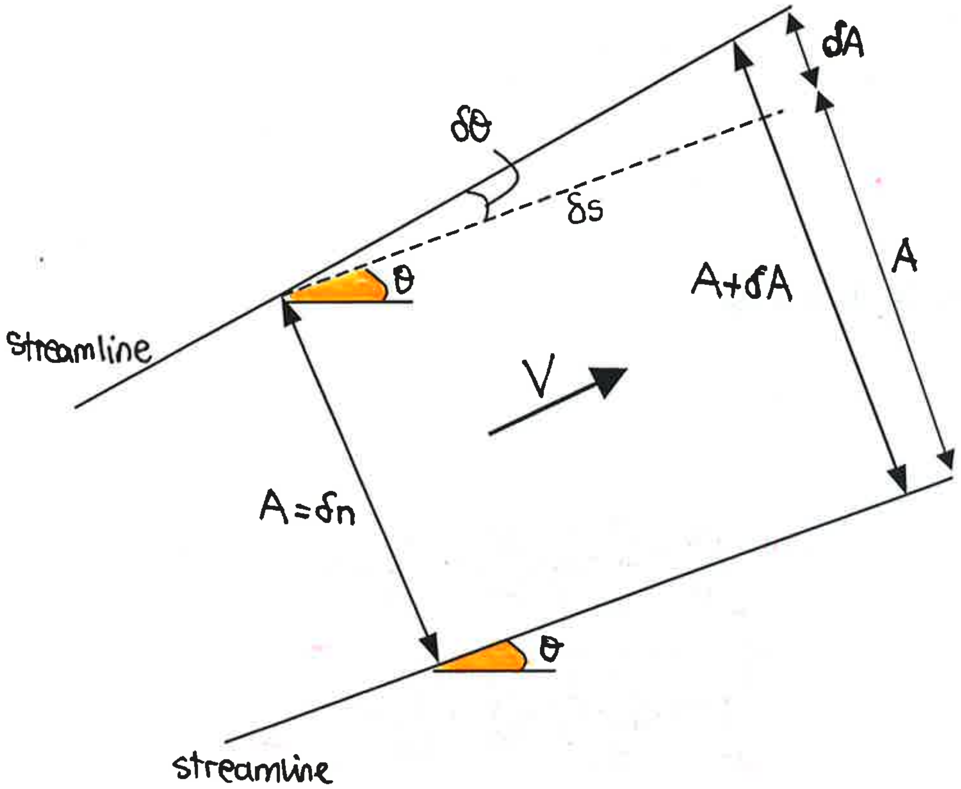
where R is the radius of curvature of a streamline.

Any streamline can be approximated locally by an arc with the appropriate radius of curvature. It is clear that arc length and angle change are related by

$\delta s = R\delta\theta \Rightarrow \boxed{\dfrac{1}{R} = \dfrac{\partial \theta}{\partial s}}$

So that the second equation becomes $\boxed{\rho V^2 \dfrac{\partial \theta}{\partial s} = -\dfrac{\partial p}{\partial n}}$ (1.2)

Conservation of mass (continuity)



Conservation of mass flow between the streamlines shown gives

no mass can cross streamlines:

$$\rho A V = \text{const} \Rightarrow \frac{\partial}{\partial s}(\rho A V) = 0 \Rightarrow \frac{\partial}{\partial s}(\rho A V) = A\frac{\partial(\rho V)}{\partial s} + \rho V \frac{\partial A}{\partial s} = 0$$

$$\Rightarrow \frac{\partial(\rho V)}{\partial s} + \frac{\rho V}{A}\frac{\partial A}{\partial s} = 0$$

$$\Rightarrow \boxed{\frac{\partial}{\partial s}(\rho V) + \rho V \cdot \frac{1}{A}\cdot\frac{\partial A}{\partial s} = 0}$$

For the streamlines shown

δ(normal)

$$A = \delta n \quad \text{and} \quad \delta A = \delta s\,\delta\theta$$

$$-\frac{1}{V}\frac{\partial V}{\partial s} - \frac{1}{\rho}\frac{\partial \rho}{\partial s}$$

Putting this all together gives

$$\boxed{\frac{1}{A}\cdot\frac{\partial A}{\partial s} = \frac{1}{\delta n}\cdot\frac{\cancel{\delta s}\,\delta\theta}{\cancel{\delta s}} = \frac{\partial\theta}{\partial n}}$$

$$\frac{\partial(\rho V)}{\partial s} = \rho\frac{\partial V}{\partial s} + s\frac{\partial\rho}{\partial s}$$

$$\Rightarrow \frac{1}{A}\frac{\partial A}{\partial s} = -\frac{1}{\rho V}\frac{\partial(\rho V)}{\partial s}$$

Further tidying up gives

(1.3)
$$\boxed{\frac{1}{\rho}\frac{\delta\rho}{\delta s} + \frac{1}{V}\frac{\partial V}{\partial s} + \frac{\partial\theta}{\partial n} = 0} \qquad (1.3)$$

$$\left\{\frac{1}{A}\frac{\partial A}{\partial s} = -\frac{1}{V}\frac{\partial V}{\partial s} - \frac{1}{\rho}\frac{\partial\rho}{\partial s}\right\}$$

Final Expression: $\frac{1}{\rho}\frac{\partial\rho}{\partial s} + \frac{1}{V}\frac{\partial V}{\partial s} + \frac{\partial\theta}{\partial n} = 0$

(1.1) $\rho V \dfrac{\partial V}{\partial s} = -\dfrac{\partial p}{\partial s}$ (1.3) $\dfrac{1}{\rho}\dfrac{\partial \rho}{\partial s} + \dfrac{1}{V}\dfrac{\partial V}{\partial s} + \dfrac{\partial \theta}{\partial n} = 0$

(1.1) $\rho V \dfrac{\partial V}{\partial s} = -a^2 \dfrac{\partial \rho}{\partial s}$ (1.3)* $-\dfrac{V}{a^2}\dfrac{\partial V}{\partial s} + \dfrac{1}{V}\dfrac{\partial V}{\partial s} + \dfrac{\partial \theta}{\partial n} = 0$

Entropy & Energy

(1.1) $\dfrac{1}{\rho}\dfrac{\partial \rho}{\partial s} = -\dfrac{V}{a^2}\dfrac{\partial V}{\partial s}$

Finally, we seek to complete the set by using the fact that the flow is isentropic. This helps in two ways. First it enables us to relate changes in pressure to those in density.

Flow isentropic $\Rightarrow \quad p = k\rho^\gamma \quad \Rightarrow \quad dp = \gamma k \rho^{\gamma-1} d\rho = \dfrac{\gamma \overbrace{k\rho^\gamma}^{p}}{\rho} d\rho = \dfrac{p\gamma}{\rho} d\rho$

$$\text{ie. } dp = \underbrace{\left[\frac{\gamma p}{\rho}\right]}_{\text{speed of sound.}} d\rho = a^2 d\rho \qquad (1.4)$$

Equation (1.4) is used to eliminate pressure in (1.1), and then $\dfrac{\partial \rho}{\partial s}$ is eliminated using (1.3). The **s-momentum equation becomes** $\left[\dfrac{V^2}{a^2} - 1\right] \cdot \dfrac{1}{V} \cdot \dfrac{\partial V}{\partial s} - \dfrac{\partial \theta}{\partial n} = 0$ (1.3)*

i.e.

$$\boxed{\left[M^2 - 1\right] \cdot \frac{1}{V} \cdot \frac{\partial V}{\partial s} - \frac{\partial \theta}{\partial n} = 0} \quad (1.5) \qquad (1.5)$$

Secondly since

$$Tds = dh - \frac{dp}{\rho} = dh_0 - VdV - \frac{dp}{\rho} \quad \text{and} \quad \textbf{s and } h_0 \textbf{ are uniform} \Rightarrow \boxed{VdV = -\frac{dp}{\rho}} \quad (\cancel{1.6})$$

(everywhere in the flowfield). Equation (1.2) becomes

$$\boxed{V\frac{\partial \theta}{\partial s} = \frac{\partial V}{\partial n}} \qquad \rho V^2 \frac{\partial \theta}{\partial s} = -\frac{\partial p}{\partial n} \leftarrow \partial p = -\rho V dV \qquad (1.6)$$

Equations (1.5) and (1.6) determine the flow, provided we can relate M to V (see later).

METHOD OF CHARACTERISTICS FOR SUPERSONIC FLOW

(non dimensionalising).

It turns out that equations (1.5) & (1.6) have an elegant geometric solution.

Equation (1.5) can be written

$$\sqrt{(M^2-1)}\cdot\frac{1}{V}\cdot\frac{\partial V}{\partial s}-\frac{1}{\sqrt{M^2-1}}\cdot\frac{\partial\theta}{\partial n}=0 \Rightarrow \frac{\sqrt{(M^2-1)}}{\sqrt{(M^2-1)}}\cdot\frac{1}{V}\cdot\frac{\partial V}{\partial s}-\frac{\sqrt{M^2-1}}{M^2-1}\cdot\frac{\partial\theta}{\partial n}=0$$

$$\Rightarrow (M^2-1)\cdot\frac{1}{V}\cdot\frac{\partial V}{\partial s}-\frac{\partial\theta}{\partial n}=0$$

replace with $\partial\nu$

We introduce the Prandtl-Meyer function

$$\nu=\int_1^M\sqrt{M^2-1}\,\frac{dV}{V} \tag{1.7}$$

so that $d\nu=\sqrt{M^2-1}\,\dfrac{dV}{V} \rightarrow \boxed{\dfrac{\partial\nu}{\partial s}-\dfrac{1}{\sqrt{M^2-1}}\cdot\dfrac{\partial\theta}{\partial n}=0}$

Equation (1.6) becomes

replace with $\frac{\partial\nu}{\sqrt{M^2-1}}$

$$\frac{\partial\theta}{\partial s}-\frac{\partial V}{V\partial n}=0$$

$$\boxed{\frac{\partial\theta}{\partial s}-\frac{1}{\sqrt{M^2-1}}\frac{\partial\nu}{\partial n}=0}$$

Adding and subtracting the two highlighted equations gives

$$\frac{\partial}{\partial s}(\nu+\theta)-\frac{1}{\sqrt{M^2-1}}\frac{\partial}{\partial n}(\nu+\theta)=0 \longrightarrow \text{A}$$

$$\frac{\partial}{\partial s}(\nu-\theta)+\frac{1}{\sqrt{M^2-1}}\frac{\partial}{\partial n}(\nu-\theta)=0 \longrightarrow \text{B}$$

Changes in $\nu-\theta$ throughout the flowfield satisfy

$$\boxed{d(\nu-\theta)=\frac{\partial}{\partial s}(\nu-\theta)ds+\frac{\partial}{\partial n}(\nu-\theta)dn}$$

If we choose to move, therefore, along a line which has direction given by

$$\text{gradient}=\frac{dn}{ds}=\frac{1}{\sqrt{M^2-1}}$$

relative to the flow direction, then *along this line*

$$d(\nu-\theta)=\frac{\partial}{\partial s}(\nu-\theta)ds+\frac{\partial}{\partial n}(\nu-\theta)dn$$

$$d(\nu-\theta)=\underbrace{\left[\frac{\partial}{\partial s}(\nu-\theta)+\frac{dn}{ds}\frac{\partial}{\partial n}(\nu-\theta)\right]}_{\text{B}\,(=0)}ds=0$$

$\frac{dn}{ds} = \frac{1}{\sqrt{M^2-1}}$

hence: $(\nu-\theta)=$ const. along streamline.

$\nu - \theta =$ const on a line which makes an angle $\frac{1}{\sqrt{M^2-1}}$ with the flow direction

$\nu + \theta =$ const on a line which makes an angle $-\frac{1}{\sqrt{M^2-1}}$ with the flow direction

To evaluate the Prandtl-Meyer function

it is necessary to express V as a function of M.

Now $T_0 = T\left[1 + \frac{\gamma-1}{2}M^2\right] \Rightarrow V^2 = a^2M^2 = \gamma R T M^2 = \dfrac{\gamma R T_0 M^2}{1 + \frac{\gamma-1}{2}M^2}$

stag. temp

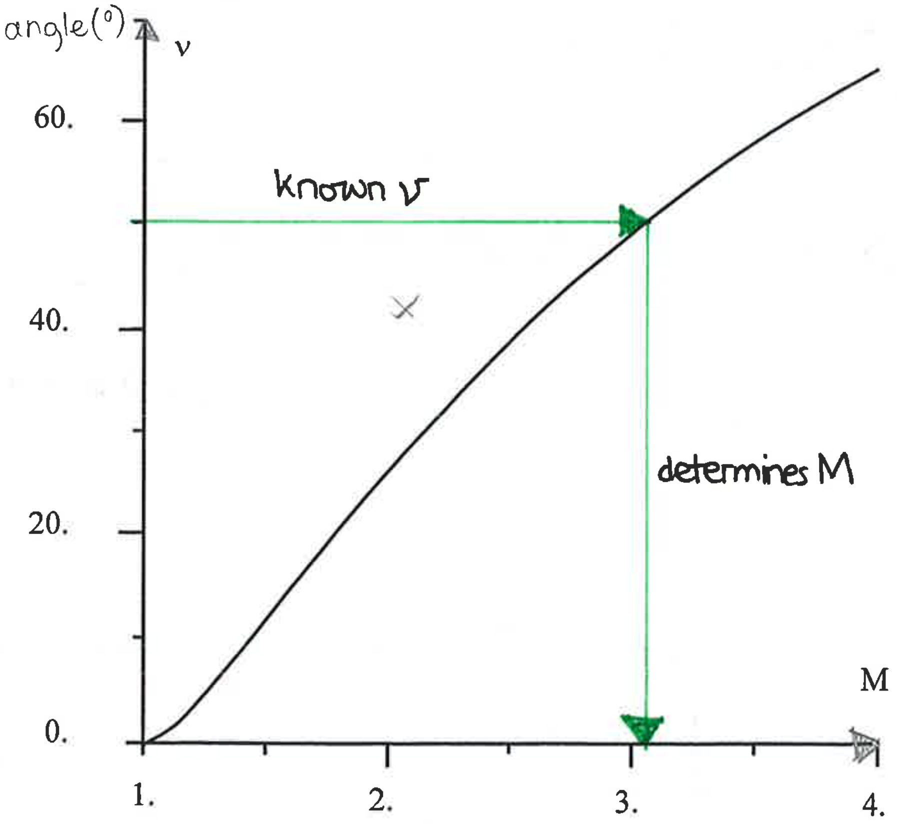
The integral can, in fact, be done analytically but you don't have to do it.

$$\boxed{\nu = \sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}\left(\frac{\gamma-1}{\gamma+1}\cdot(M^2-1)\right)^{1/2} - \tan^{-1}\left(\sqrt{M^2-1}\right)}$$

$$\nu = \int_1^M \sqrt{(M^2-1)}\cdot\frac{dV}{V} \qquad (1.9)$$

use temp to relate M to V.

This function is tabulated in Houghton and Brock and in the CUED tables, and a glance there shows that it increases monotonically with M. Note that it has units of angle, i.e. degrees or radians. CUED tables are in degrees.

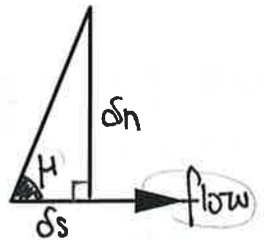


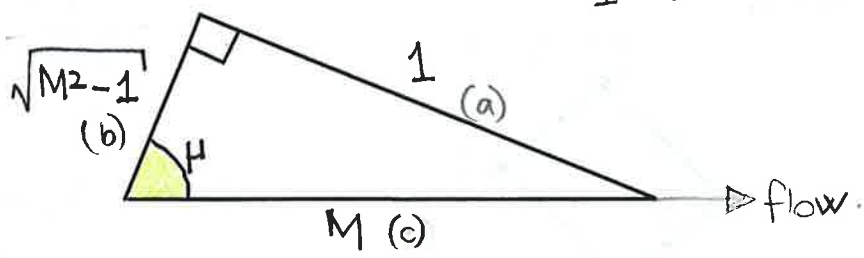
Finally, the line which is at angle $\left[\frac{dn}{ds} = \frac{1}{\sqrt{M^2-1}}\right]$ is actually at the *Mach angle* to the flow.

$c^2 - b^2 = a^2$

$M^2 - (M^2-1) = a^2$

$1 = a^2$





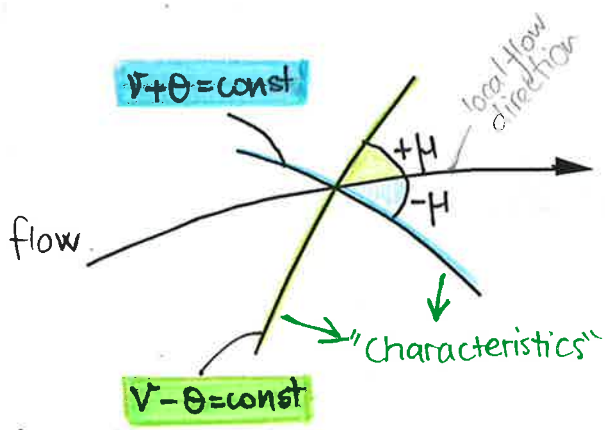
In summary, in inviscid, supersonic steady two dimensional flow, the general solution of the equations of motion are equivalent to the <u>algebraic</u> relationships

ie moves along line $\frac{dn}{ds} = \frac{1}{\sqrt{M^2-1}}$

$\nu - \theta$ = const on a line which makes an angle	$+\mu$	with the flow direction
$\nu + \theta$ = const on a line which makes an angle	$-\mu$	with the flow direction

Lines at $\pm\mu$ to the flow direction are called "Mach Lines" or "characteristics"

General Features of Supersonic Flow

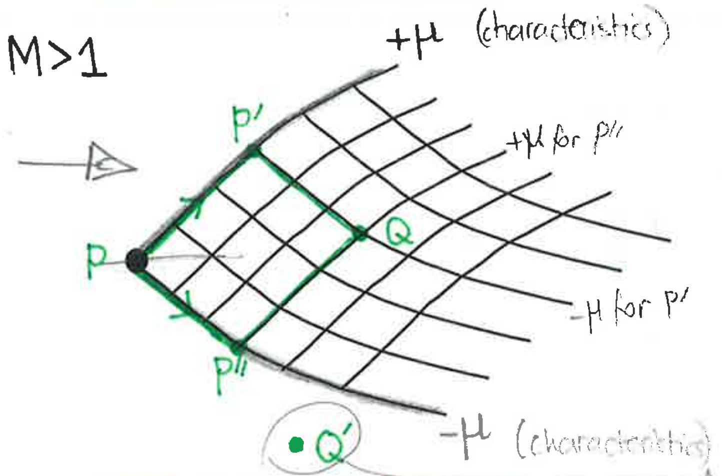


Through any point in a region of supersonic flow, there will be two characteristics at angles $\pm\mu$ to the local flow direction. The two relationships $\nu \pm \theta$ = const. are enough to determine ν and θ at the point. Since ν is a (single) monotomic function of M, then once ν is known so is M. The velocity follows immediately from the energy equation, and the other flow variables from simple compressible flow relationships. These other variables together with the known value of θ are the complete solution at this point.

$\int_{\text{limit 1}}^{\text{limit 2}}$ along this line → "riemann invariant"

Two important properties of hyperbolic equations dominate their method of solution.

a) Region of influence of P

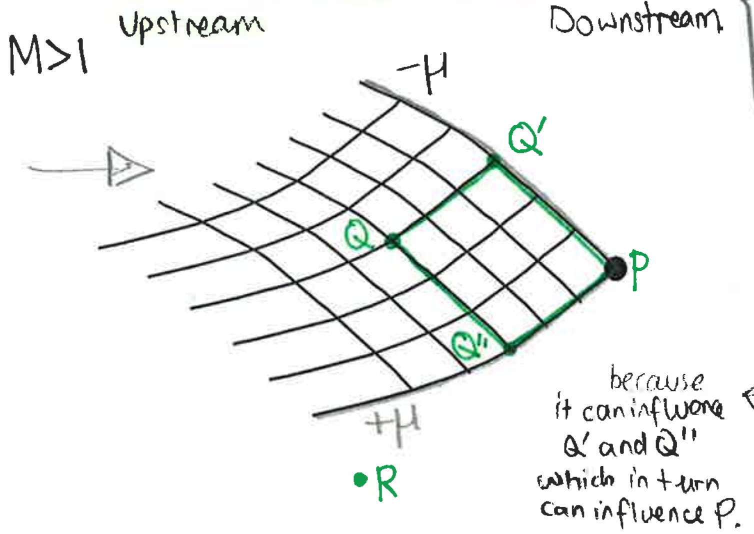


The solution for the flow at point P only influences those points downstream of the characteristics running through P.

∴ Flow change at P
↳ we know change at P' and P''
↳ we know change at Q.

N.B. Q' is not influenced if Q' lies outside the characteristic lines of P (as shown).

b) Domain of dependence of P

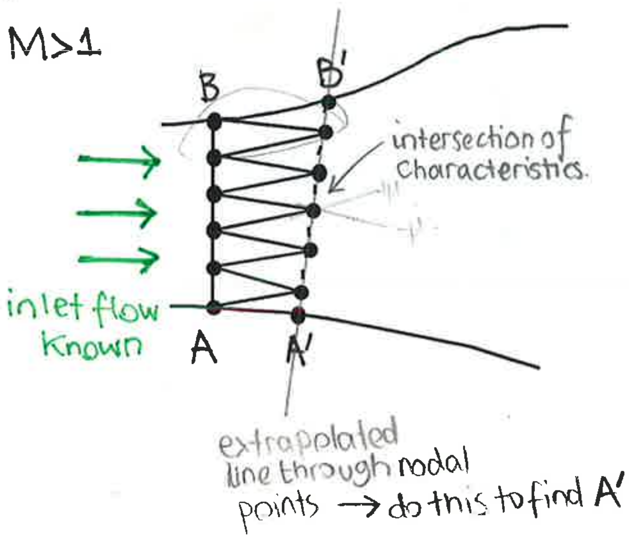


Conversely, only those points upstream of the characteristics running through P can influence/affect the solution at P (provided flow at P remains supersonic).

∴ Q can influence P.
P cannot influence Q.

Numerical solution by the method of characteristics exploits these properties embodied in the characteristic equations to numerically solve for the flow by sweeping downstream.

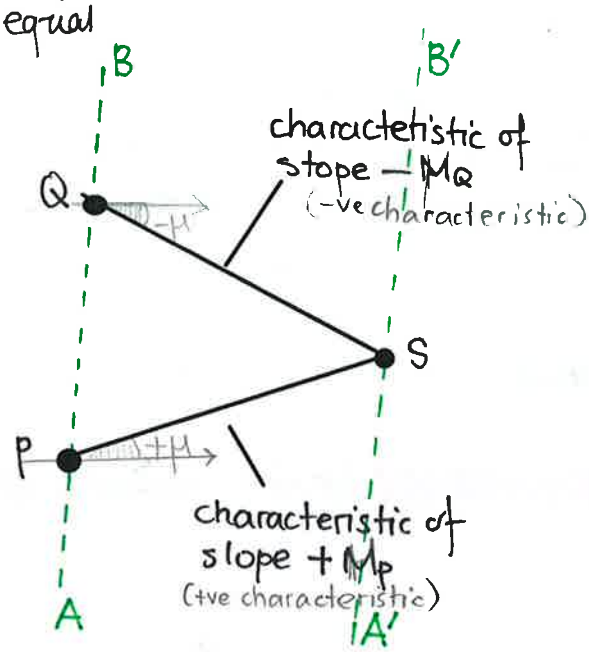
(i) Lattice method (like nodal analysis is part IA/B structures).



Having divided AB into a number of portions, characteristics can be drawn through all the nodal points. Where characteristics intersect defines the new nodal points at the downstream calculation curve A'B'.

(Close up of area circled on previous page)

Mach number at P does not necessarily equal mach number at Q.



$\nu_Q + \theta_Q = \nu_S + \theta_S \leftarrow (C) \leftarrow -\mu$

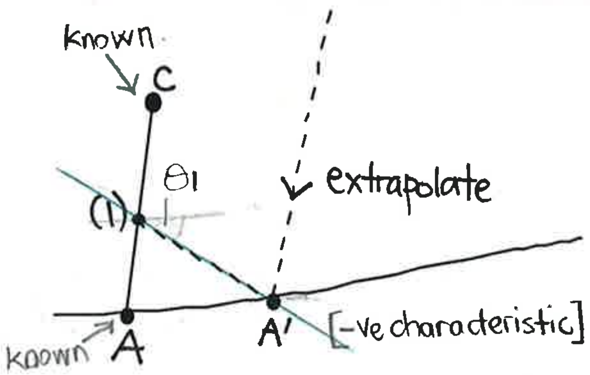
Hence

$\nu_P - \theta_P = \nu_S - \theta_S \leftarrow (D) \leftarrow +\mu$

$$\boxed{\nu_S = \frac{1}{2}(\nu_Q + \nu_P) + \frac{1}{2}(\theta_Q - \theta_P)} \leftarrow (C)+(D) \quad (1.11)$$

$$\boxed{\theta_S = \frac{1}{2}(\nu_Q - \nu_P) + \frac{1}{2}(\theta_Q + \theta_P)} \leftarrow (C)-(D) \quad (1.12)$$

To complete the calculation, at the downstream line A'B', boundary conditions must be applied at A' and B' themselves. There is a certain amount of ambiguity as to how this should be done. What follows is one way.



First extrapolate a curve through the known portion of A'B' to find the position of A'. One property is usually known at each of the points A'.

At a solid boundary $\theta_{A'}$ is known.

Point (1) is where the $-\mu$ characteristic would eminate from to ~~to~~ get to A'. ν_S exact position is unknown.

$\nu_A' + \theta_A' = \nu_{(1)} + \theta_{(1)}$

if (eg. edge of a jet), p_{const} therefore $M_{A'}$ known hence $\nu_{A'}$ is known

or if solid boundary; this is known.

If the boundary is one of constant pressure, as opposed to a solid boundary (e.g. edge of a jet), then p and hence M are known. i.e. $\nu_{A'}$ is known.

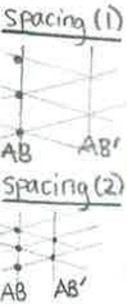
To find the other flow variable at A', it is necessary to estimate the characteristic emanating from somewhere along CA which passes through A'. Guess a point on CA and find where it meets the wall. Repeat till find correct one. The Riemann invariant along this line determines the remaining unknown at A'.

↳ "modern compressible flows" has a good on this.

The calculation continues to the next downstream line.

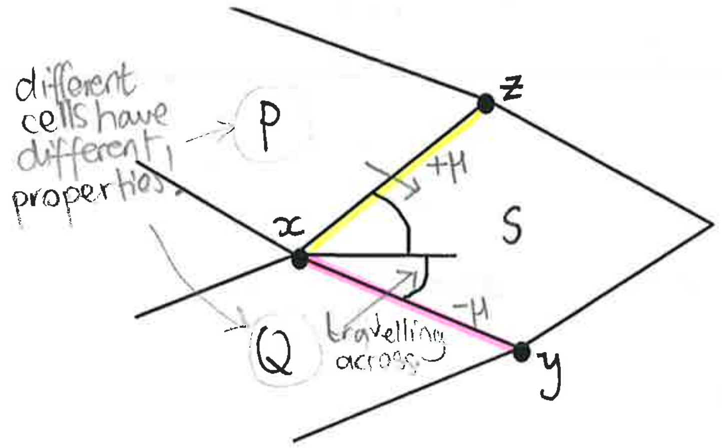
$\underbrace{\nu_{A'}}_{\text{Calculate}} + \theta_{A'} = \underbrace{\nu_1 + \theta_1}_{\text{known}}$ found from guessing a point on CA } repeat until $\nu_{A'} = (\nu_1 + \theta_1) - \theta_{A'}$

$$(1.11)\quad \nu_S = \frac{1}{2}(\nu_Q + \nu_P) + \frac{1}{2}(\theta_Q - \theta_P) \qquad (1.12)\quad \theta_S = \frac{1}{2}(\nu_Q - \nu_P) + \frac{1}{2}(\theta_Q + \theta_P)$$



Numerical packages based on characteristics use the lattice method. Since equations (1.11) and (1.12) are *algebraic* equations, they give can be solved *exactly*. The only error is in the local slopes of the characteristics i.e. the position of the lattice points or cells. If nodes get too far apart or cells become too large, one simply interpolates more points on AB and sweeps to A'B' again. There is no danger of numerical instability, false dissipation or dispersion, etc as there is with the solution of partial differential equations by finite difference or finite volume techniques. Phenomenal accuracy can, therefore, be obtained. In fact, solutions obtained by the method of characteristics are often treated as *exact* solutions against which to check those obtained by other methods.

(ii) Field Method (easier method in practice).



Flow in cells P and Q is known.

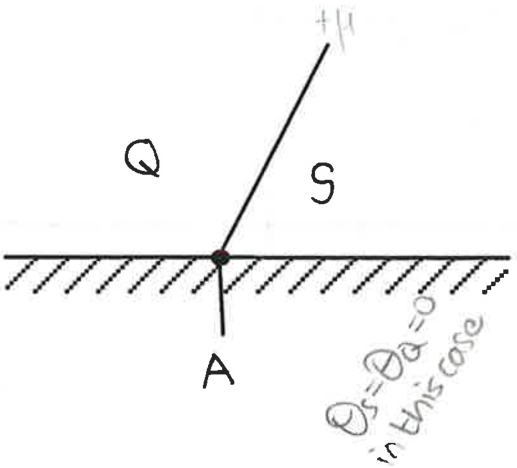
Instead of dealing with the flow properties at *nodes* or lattice points, it is equally plausible to split the flow domain into *cells* bounded by characteristics and to regard flow properties as being representative of conditions in a cell. Suppose flow conditions are known in cell P and cell Q. XY is a line at the Mach angle of the flow in Q (relative to the flow direction in Q), and XZ is at the Mach angle of the flow in P (relative to the flow direction in P).

Again:

$$\left.\begin{aligned} Q \Rightarrow S \quad & \{\, \nu_Q - \theta_Q = \nu_S - \theta_S \\ P \Rightarrow S \quad & \{\, \nu_P + \theta_P = \nu_S + \theta_S \end{aligned}\right\} \Rightarrow \text{solution for S.}$$

If you're going across characteristic: you keep the same sign as the characteristic.

leading once again to equations (1.11) and (1.12) for properties in S.

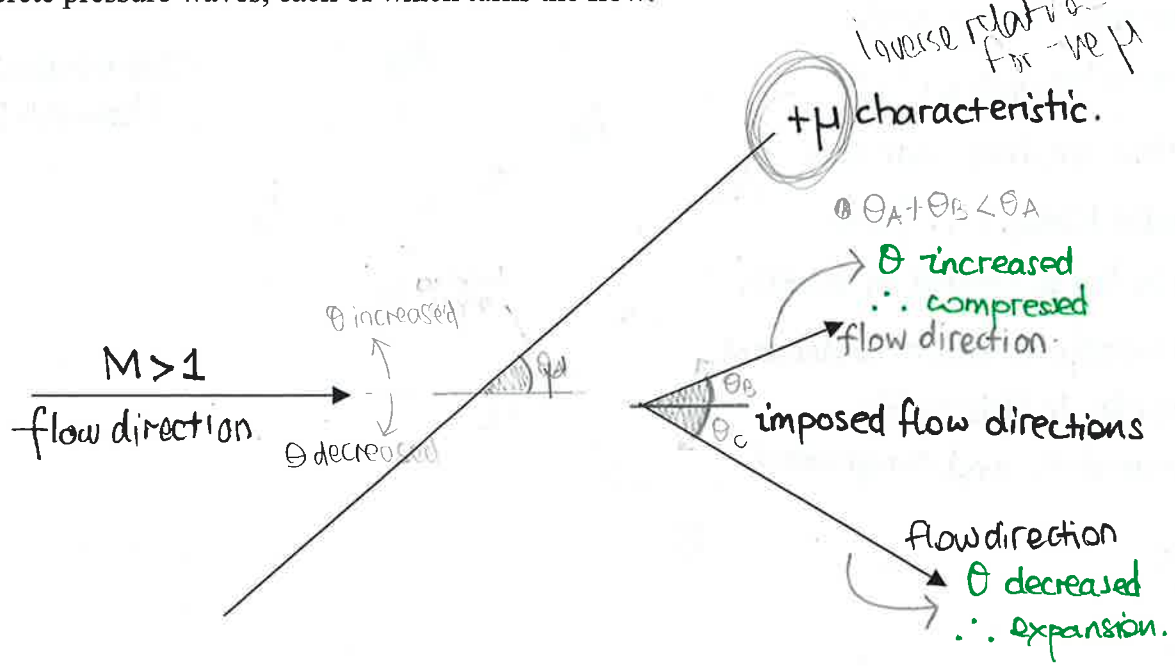


At a boundary, (for example the case of a solid wall), (value of the constants are the same)

$$\nu_Q + \theta_Q = \nu_S + \theta_S$$

and θ_S is determined by the slope of the wall

Clearly these two approaches are equivalent. The field method turns out to be much easier to use in hand computations. In the field method, the characteristics themselves are effectively being treated as discrete pressure waves, each of which turns the flow.



Use either:

i). A strict sign convention

ii.) Inspection → If θ is increased then
- ν must decrease (since $\nu + \theta = \text{const}$
- M decreases
  ↳ ∴ compression.

$\nu - \theta = \text{const}$ (becomes more -ve)

- Angle between characteristic and flow direction decreases for a compression.
  ↓ θ_A

flow turns into itself

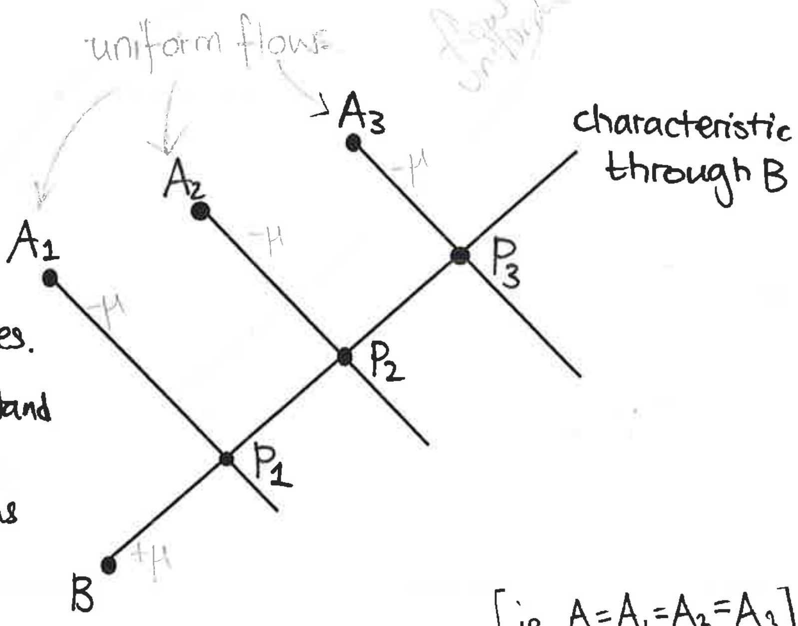
[i.e. $\theta_B < \theta_A$]

$$p + \frac{1}{2}V^2 = \text{const}$$

hence compression

SIMPLE EXPANSIONS AND COMPRESSIONS

Flow must be controlled by aerodynamics: jet engine intakes can only handle $M \leq 0.5$ $\therefore$ flow needs to be slowed in intakes. We therefore need to understand how to do this with expansions and compressions



[ie. $A = A_1 = A_2 = A_3$]

Suppose the points A_1, A_2 and A_3 lie in a region where the flow is uniform (denoted by subscript A). The equations determining the flow at P_1, P_2 and P_3 are each

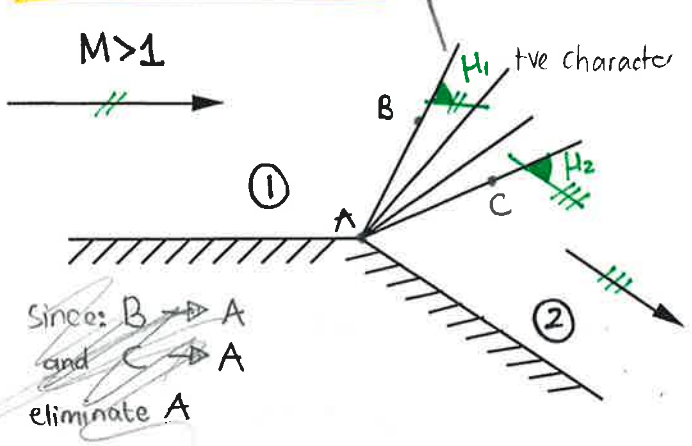
$$\left.\begin{aligned} V_P + \theta_P &= V_A + \theta_A \\ V_P - \theta_P &= V_B - \theta_B \end{aligned}\right\} \text{same expressions for all } p.$$

It follows that the solutions at P_1, P_2 and P_3 are *all the same*. $BP_1P_2P_3$ is thus a straight line and all flow properties are uniform along it.

Flow uniform along characteristic

• Subsonic : flow would seperate .
• Supersonic : what happens ?!?

Prandtl-Meyer Expansion



It is conventional not to draw characteristics which emanate from a region of uniform flow (in this case the $-\mu$ set), but to draw only those across which there is likely to be change in flow properties.

$$\{\nu_1 + \theta_1 = \nu_2 + \theta_2\}$$ travelling across characteristics ∴ same sign as characteristic

In this case $\nu_1 = f(M)\ ;\ \theta_1 = 0 \Rightarrow \nu_2 > \nu_1$ (since $\theta_2 < 0$)

~~$V_A - \theta_A = V_B - \theta_B$~~ ~~(travelling along characteristic.)~~
~~$V_A - \theta_A = V_c - \theta_c$~~
~~hence: $V_B - \theta_B = V_c - \theta_c$~~

$$p + \frac{\rho V^2}{2} = \text{const}$$

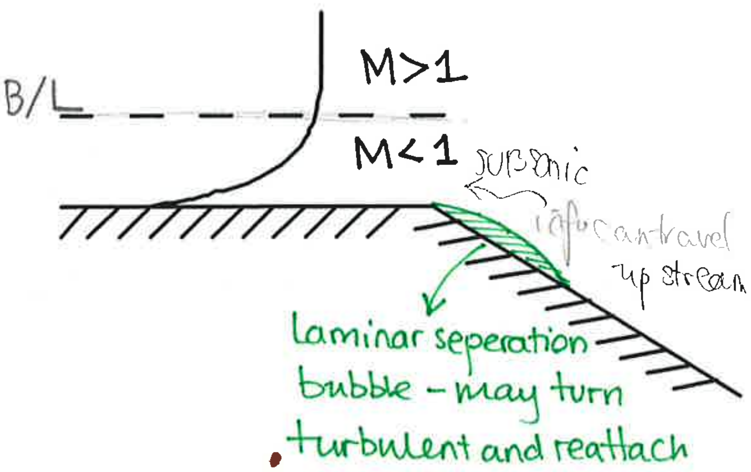
The flow thus speeds up , the pressure drops (hence expansion). The first ch'ic of the "fan" is at the Mach angle to the upstream flow and the last is at the Mach angle to the downstream flow.

$\boxed{\mu_2 > \mu_1}$ $\boxed{p_2 < p_1}$ ∴ we have an expansion. (causes favourable pressure gradient).

Note (i) "expansion" waves spread out.

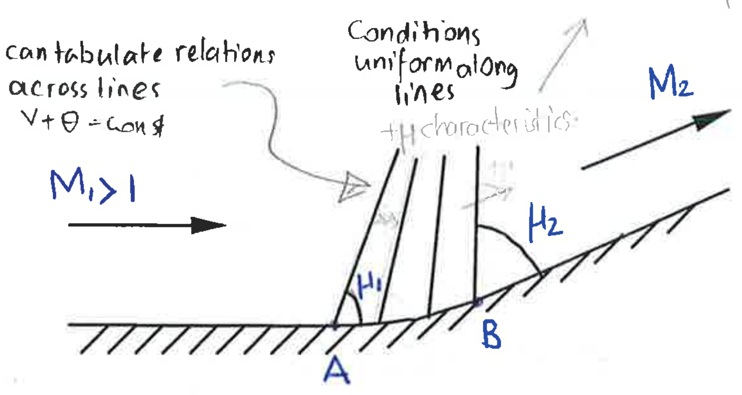
(ii) static pressure is lower in region 2 than in region 1. i.e. a favourable pressure gradient for the boundary layer. Often bulk of the the flow will turn a sharp corner. Flow in the boundary layer is mostly subsonic. Sometimes flow separates, sometimes separates locally then reattaches depending on the state of the boundary layer. Since the [flow near the wall is subsonic, pressure variations can feed upstream in this region.]

↳ buffetting and shock oscillation may occur.



Corner Compression

since we're travelling across them ⇒ keep same sign



The first characteristic from A, at the start of the curve is at the Mach angle of the M_1 flow, while the last, emanating from B, is at the Mach angle to the M_2 flow.

$\nu_1+\theta_1=\nu_2+\theta_2$ $\qquad \theta_2-\theta_1=\nu_1-\nu_2$

$\theta_2>\theta_1 \therefore \nu_2<\nu_1$ $\qquad$ +ve ⇒ $\nu_1>\nu_2$

$M_2<M_1$

In this case $\nu_1 = \nu_1(M_1)$, $\theta_1 = 0$, $\theta_2 > \theta_1 \Rightarrow \nu_2 < \nu_1$ i.e. $M_2 < M_1$

Flow slows down ⇒ (isentropic) compression.

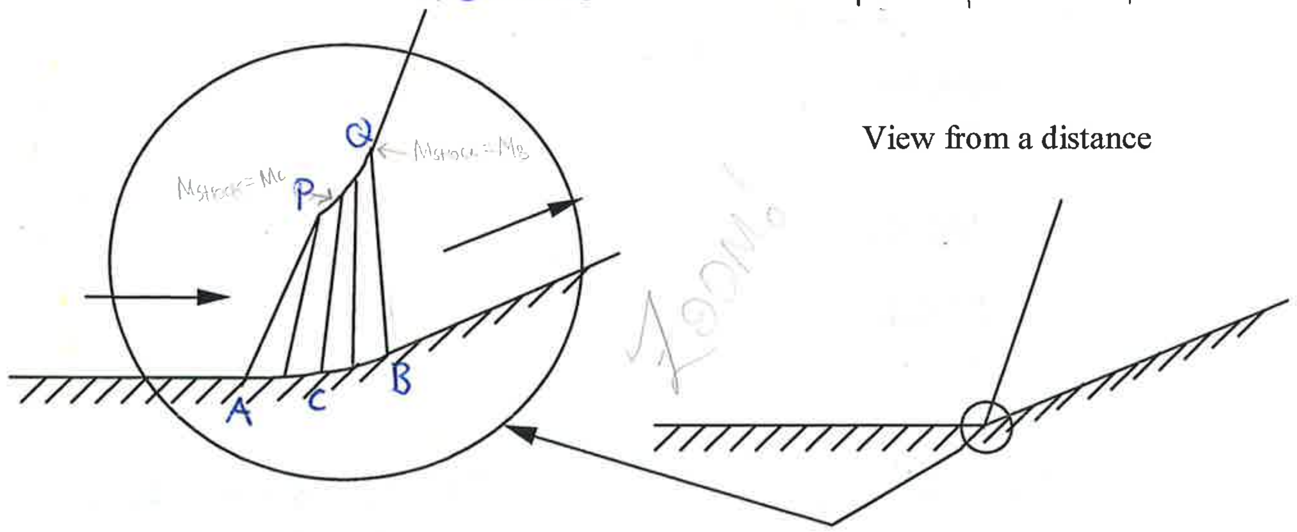
Characteristics converge.

Notes

(i) 'Compression' waves run together.

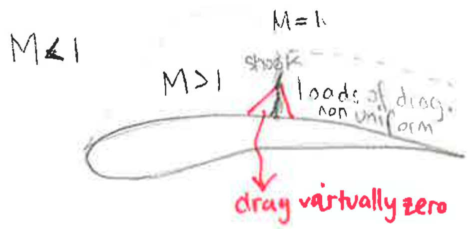
(ii) when two ch'ics of the same family meet ⇒ discontinuity i.e. shock

(Oblique shock) → weak (in terms of entropy but can generate significant pressure drops.
shockwave

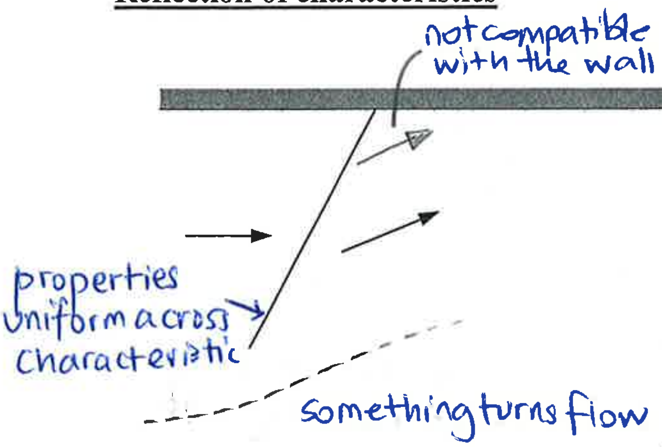


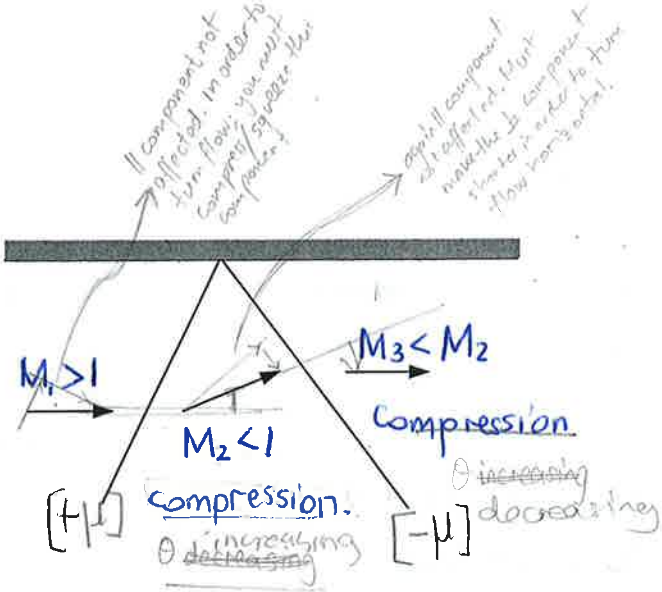
View from a distance

Shock starts at point where first ch'ics cross. Gets stronger as more ch'ics run into it (at P, $M_s = M_C$, and at Q, $M_s = M_B$)



Reflection of characteristics





Consider an incoming (compression) characteristic. This turns the flow and in order to satisfy the boundary condition at the wall, a second compression is needed.

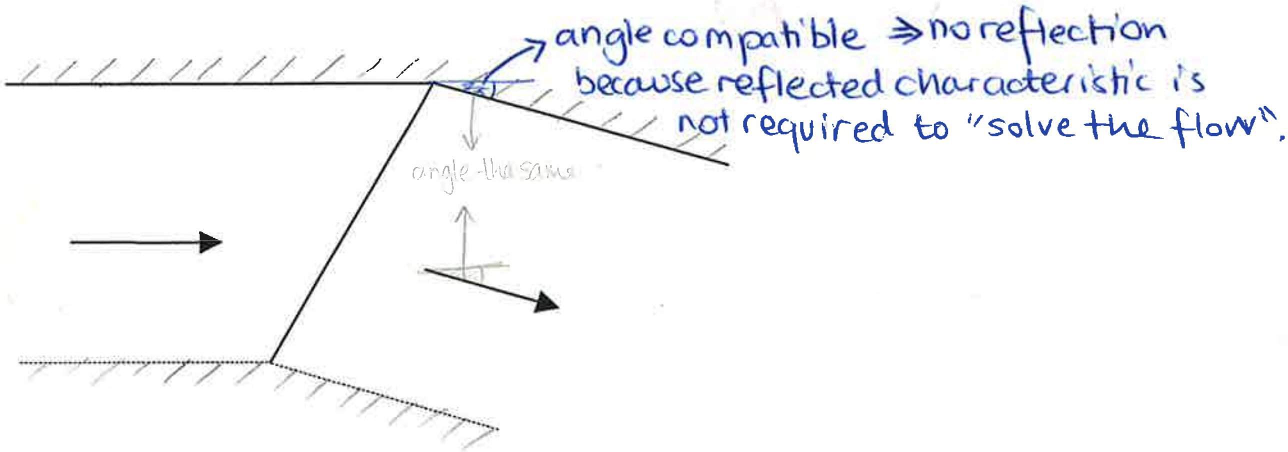
At solid walls

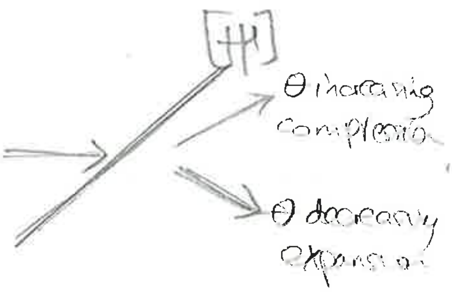
- expansion reflects as an expansion
- compression reflects as a compression

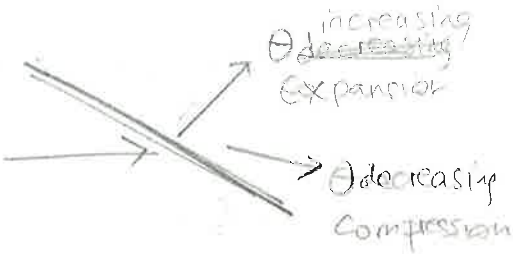
At constant pressure boundary

- expansion reflects as a compression
- compression reflects as an expansion

We can only have no reflection if the angle of the wall downstream of where the incoming characteristic meets it happens (or is designed) to have the same angle as the flow downstream of the characteristic. The characteristic is then said to have been cancelled.

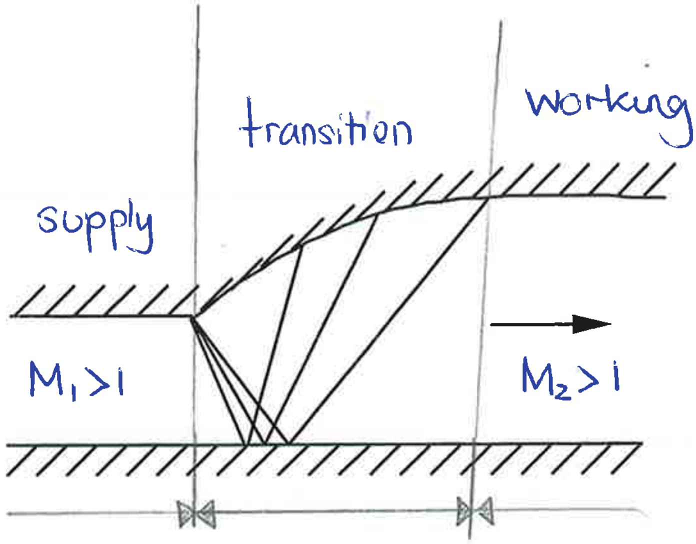






Example Wind Tunnel Expansion

For a supersonic wind tunnel try to profile end wall to avoid further reflection $\Rightarrow$ flow uniform in working section. This is a) difficult to do and b) each different M_2 needs a different shape.



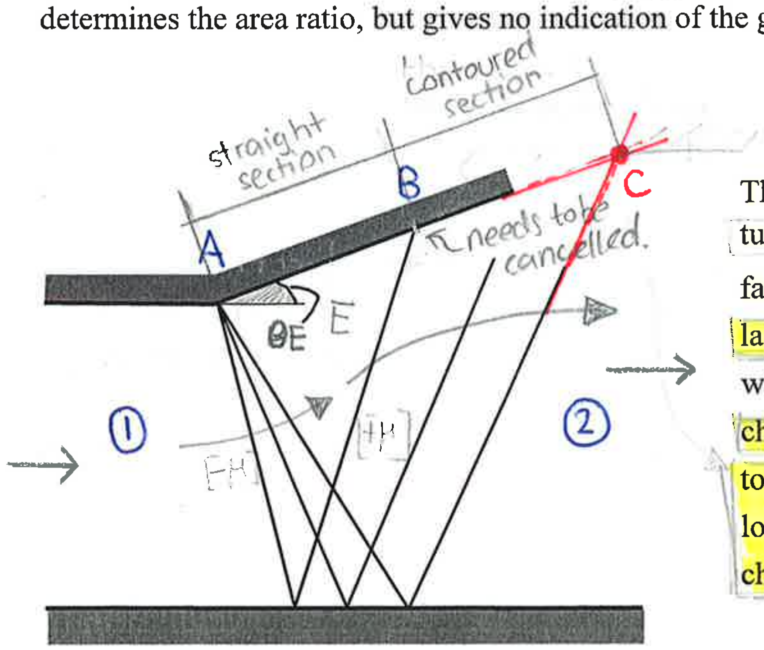
No heat added } $T_{02} = T_{01}$
No work done }
$\Rightarrow$ Isentropic $\Rightarrow$ $p_{02} = p_{01}$.

The equation for compressible non-dimensional mass flow.

$$f(M_2) = \frac{\dot{m}\sqrt{c_p T_{02}}}{p_{02} A_2} = \underbrace{\frac{\dot{m}\sqrt{c_p T_{01}}}{p_{01} A_1}}_{f(M_1)} \cdot \frac{A_1}{A_2} \tag{2.1}$$

determines area ratio of transition section.

determines the area ratio, but gives no indication of the geometry of the transitional piece AC.



The first section AB can be straight and the turning at A (θ_E say) will cause an expansion fan of characteristics to emanate from A. The last point on the straight portion of the upper wall, B, will be where the first of the reflected characteristics meets this upper wall. From B to C the end wall must be contoured to the local flow direction implied by these incoming characteristics.

Across the fan emanating from A,

$$\nu_A - \theta_A = \nu_E - \theta_E \qquad (\nu_A = \nu(M_1)\ ;\ \theta_1 = 0^\circ) \tag{2.2}$$

Across the reflected fan

$$\nu_E + \theta_E = \nu_C + \theta_C \qquad (\nu_C = \nu(M_2)\ ;\ \theta_C = \theta_2 = 0^\circ C) \tag{2.3}$$

Equations (2.2) and (2.3) determine θ_E. and ν_E, M_E.

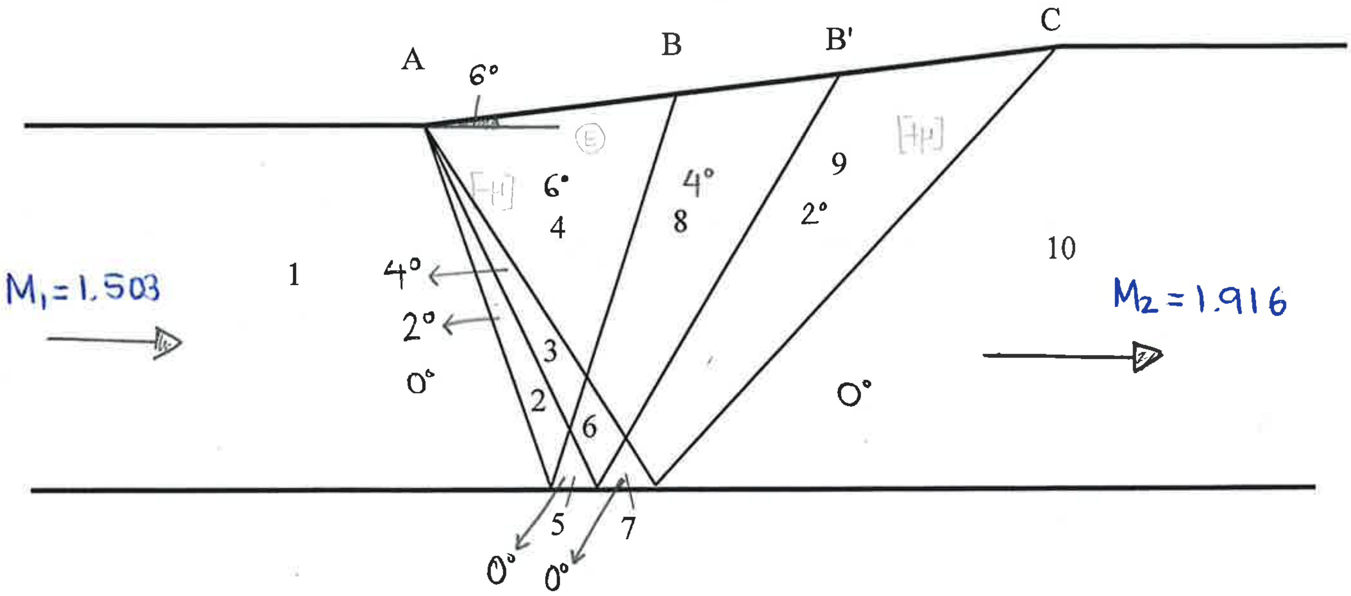
As a concrete example (chosen to make numbers easy !)

$M_1 = 1.503$ and $M_2 = 1.916$ $\nu_1 = 12^\circ$ $\nu_2 = 24^\circ$ (from tables)

Thus

$\nu_E - \theta_E = 12^\circ$ and $\nu_E + \theta_E = 24^\circ$ $\Rightarrow$ $\theta_E = 6^\circ$

$\nu_E - \theta_E = \nu_1 - \cancel{\theta_1}^{0} = 12$ $\nu_E + \theta_E = \nu_2 + \cancel{\theta_2}^{0} = 24$



To find the shape of the wall downstream of B, Use the field method with a discretisation

$\Delta\theta^\circ = 2^\circ = \Delta\nu$ Since $\theta \propto \nu$.

Prepare data from tables or calculator

$\boxed{\nu - \theta = \text{constant}}$

ν°	M	μ°
12	1·503	41·71
14	1·571	39·53
16	1·638	37·63
18	1·707	35·86
20	1·775	34·29
22	1·844	32·84
24	1·916	31·46

steps arbitrary data comes from tables

A) Expansion fan region 1 → 4

$\nu_1 - \theta_1 = \nu_2 - \theta_2$, $\theta_2 = 2$

$\nu_2 = 12 - 0 + 2 = 14$

Region	θ°	ν°
1	0	12
2	2	14
3	4	16
4	6	18

$\nu_1 = 12$

(by choice of angle increment)

$\nu_3 = \nu_1 - \theta_1 + \theta_3 = 12 - 0 + 4 = \underline{\underline{16}}$

$\nu_4 = \nu_1 - \theta_1 + \theta_4 = 12 - 0 + 6 = 18$

B) Region 5 (reflection from the lower wall).

$\underset{14}{\nu_2} + \underset{2}{\theta_2} = \nu_5 + \cancel{\theta_5}^{0} = 16^\circ$ and $\theta_5 = 0$ $\Rightarrow$ $\nu_5 = 16^\circ$ (convert to Mach number if you want).

C) Region 6

$$\left.\begin{aligned} \nu_3+\theta_3 &= \nu_6+\theta_6 = 20^\circ \\ \nu_5-\theta_5 &= \nu_6-\theta_6 = 16^\circ \end{aligned}\right\} \Rightarrow \quad \theta_6 = 2^\circ \quad \Rightarrow \quad \nu_6 = 18^\circ$$

D) Region 8

$$\left.\begin{aligned} \nu_4+\theta_4 &= \nu_8+\theta_8 = 24^\circ \\ \nu_6-\theta_6 &= \nu_8-\theta_8 = 16^\circ \end{aligned}\right\} \Rightarrow \quad \theta_8 = 4^\circ \quad \Rightarrow \quad \nu_8 = 20^\circ$$

E) Region 7

$$\nu_7 + \theta_7 = \nu_6 + \theta_6 = 20^\circ \quad \text{and} \quad \theta_7 = 0^\circ \quad \Rightarrow \quad \nu_7 = 20^\circ$$

F) Region 9

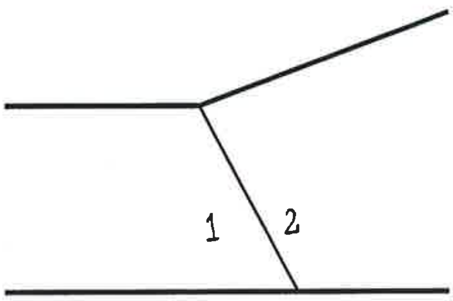
$$\left.\begin{aligned} \nu_8+\theta_8 &= \nu_9+\theta_9 = 24^\circ \\ \nu_7-\theta_7 &= \nu_9-\theta_9 = 20^\circ \end{aligned}\right\} \Rightarrow \quad \theta_9 = 2^\circ \Rightarrow \nu_9 = 22^\circ$$

G) Region 10

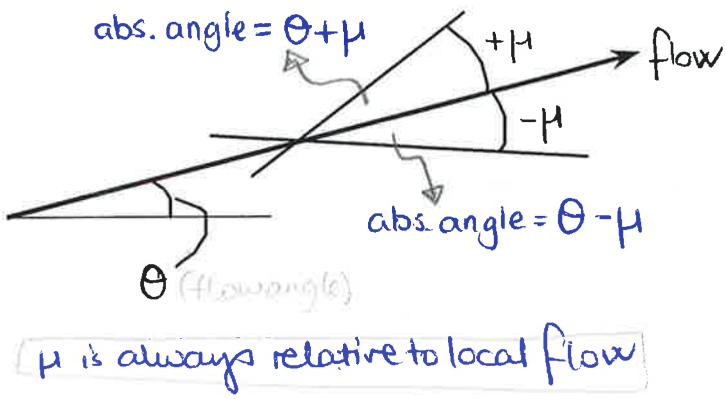
$$\nu_9 + \theta_9 = \nu_{10} + \theta_{10} = 24^\circ \quad \text{and} \quad \theta_{10} = 0 \quad \Rightarrow \quad \nu_{10} = 24^\circ$$

4° 2°

Finally, it remains only to find the co-ordinates of B, B' and C between which the local wall angle will be 4° and 2° respectively. These follow from the Mach angle of the various characteristics



General flow



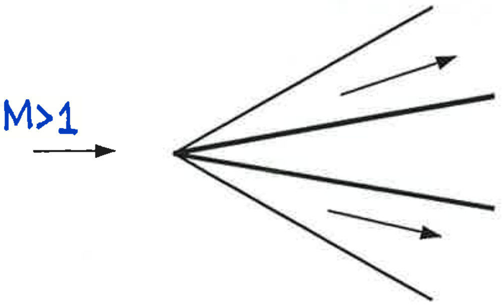
e.g. for the ch'ic separating regions 1 and 2, could take ch'ic at Mach angle to the upstream flow, but probably more accurate to use the average of conditions in regions 1 and 2. Remembering that μ is the angle of the ch'ic to the local <u>flow</u> direction,

Slope of ch'ic between ① and ②

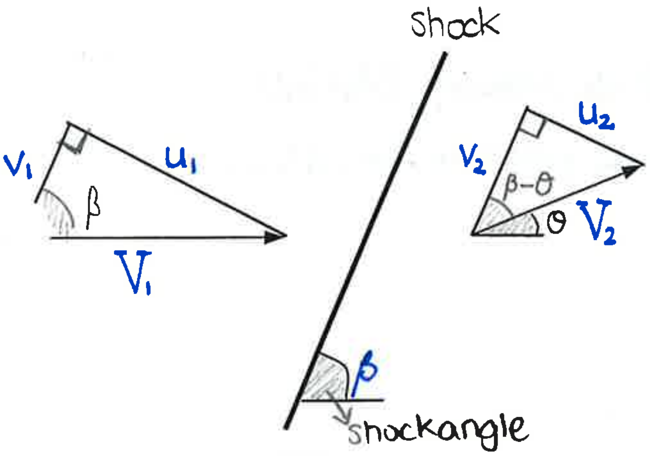
$$= \frac{1}{2}\left[(\theta-\mu)_2 + (\theta-\mu)_1\right]$$

$$= \frac{1}{2}\left[(\underset{\theta_2}{0}-\underset{\mu_2}{41.71}) + (\underset{\theta_1}{2}-\underset{\mu_1}{39.53})\right] = -39.6^\circ$$

OBLIQUE SHOCK WAVES



In most practical situations, shocks are not in fact normal to the flow direction and usually involve some turning of the flow.



Consider a shock wave at angle β to the incoming flow, with velocity V decomposed into components (u,v) normal and parallel to the shock.

A control volume analysis applied to the flow through the oblique shock shown exactly parallels that for a normal shock.

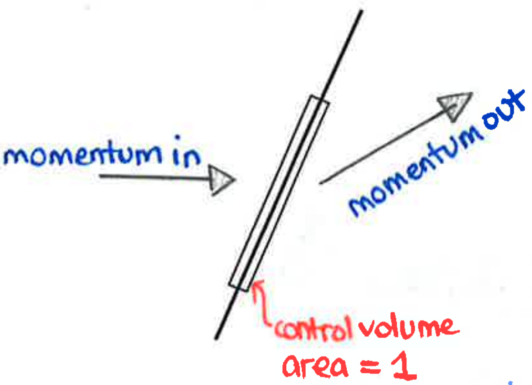
The *differences* are that

(i) now have *tangential* (i.e. parallel to the shock) momentum equation

$$\cancel{F}^{\,0} + \cancel{\sum pA}^{\,0} = \sum \dot{m} v$$

$$\boxed{\dot{m}_1 v_1 = \dot{m}_2 v_2}$$

tangential momentum.



Since $\dot{m}_1 = \dot{m}_2$

$\Rightarrow$ $v_1 = v_2$ (3.1)

(ii) the continuity and momentum equations only involve u_1 and u_2 (instead of V_1 and V_2)

$$\rho_1 u_1 = \rho_2 u_2 = \dot{m}$$

Use perfect gas relations,
+ trigonometry,
+ speed of sound.

γ cancels.

$\Rightarrow$ $$\frac{p_1^2}{\cancel{T_1}} \cdot M_1^2 \cdot \sin^2\beta = \frac{p_2^2}{\cancel{T_2}} \cdot M_2^2 \cdot \sin^2(\beta-\theta) \quad (3.2)$$

$\Rightarrow$ momentum $\perp$ shock: $$p_1 - p_2 = \dot{m}(u_2 - u_1) = \rho_2 u_2^2 - \rho_1 u_1^2 \quad (3.3)$$

$u_2^2 = V_2^2\sin^2(\beta-\theta)$, $u_1^2 = V_1^2\sin^2\beta$, $\rho_2 = \frac{\gamma p_2}{a_2^2}$, $\rho_1 = \frac{\gamma p_1}{a_1^2}$

$$p_1\left(1+\gamma M_1^2 \sin^2\beta\right) = p_2\left(1+\gamma M_2^2 \sin^2(\beta-\theta)\right)$$

(iii) the energy equation $c_pT_1 + \frac{1}{2}(u_1^2 + v_1^2) = c_pT_2 + \frac{1}{2}(u_2^2 + v_2^2)$

$$\Rightarrow T_1\left(1 + \frac{\gamma-1}{2}M_1^2\sin^2\beta\right) = T_2\left(1 + \frac{\gamma-1}{2}M_2^2\sin^2(\beta-\theta)\right) \qquad (3.4)$$

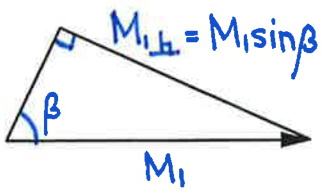
Equations (3.1) – (3.4) are solved in the same manner as the corresponding ones for a normal shock, energy × continuity ÷ momentum²

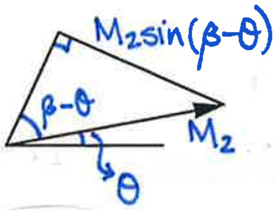
$$\frac{\left(1+\frac{\gamma-1}{2}M_1^2 \sin^2\beta\right)M_1^2 \sin^2\beta}{\left(1+\gamma M_1^2 \sin^2\beta\right)^2} = \frac{\left(1+\frac{\gamma-1}{2}M_2^2 \sin^2(\beta-\theta)\right)M_2^2 \sin^2(\beta-\theta)}{\left(1+\gamma M_2^2 \sin^2(\beta-\theta)\right)^2} \qquad (3.5)$$

leading to a quadratic equation for $M_2^2 \sin^2(\beta-\theta)$ in terms of $M_1 \sin\theta$

⇒ Implies two correct answers... problems for CFD...

Another View





Cannot do this for STAGNATION quantities!

Imagine an observer moving parallel to the shock at speed v (the component of V parallel to the shock). Since $v_1 = v_2$, the observer will see the flow as a normal shock with upstream Mach number $M_1 \sin\beta$.

The moving observer sees the same value for every **static** quantity as does a stationary one. Thus all relationships between **static** flow quantities (but *not* stagnation ones) are identical with those for a normal shock if we replace

M_1 by $M_{\perp 1} = M_1 \sin\beta$ and M_2 by $M_{\perp 2} = M_2 \sin(\beta-\theta)$

Thus, for example,

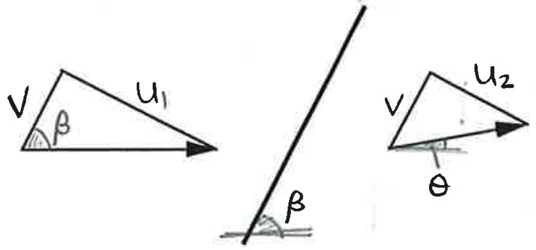
$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma+1}\left(M_1^2\sin^2\beta - 1\right) \qquad (3.6)$$

$$M_2\sin(\beta-\theta) = \left[\frac{1 + \frac{\gamma-1}{2}M_1^2\sin\beta}{\gamma M_1^2\sin^2\beta - \frac{\gamma-1}{2}}\right]^{1/2} \qquad (3.7)$$

Remember that the moving observer will see *different* values of stagnation quantities to an observer at rest (if a fluid particle is brought to rest relative to the moving observer it will have speed v relative to an observer at rest). Care is thus necessary in framing relationships between upstream and downstream stagnation quantities.

Flow Turning

The final quantity of interest is the angle θ through which the flow has been turned.



Continuity gives

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{v\tan\beta}{v\tan(\beta-\theta)} \tag{3.8}$$

and the density ratio across a shock is given by ~~equation~~ the normal shock relationship with M_1 replaced by $M_1 \sin\beta$

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1)M_1^2\sin^2\beta}{2\left[1+\frac{\gamma-1}{2}M_1^2\sin^2\beta\right]} \tag{3.9}$$

Equation (3.8) can be solved for $\tan(\beta-\theta)$ which is then substituted into the identity

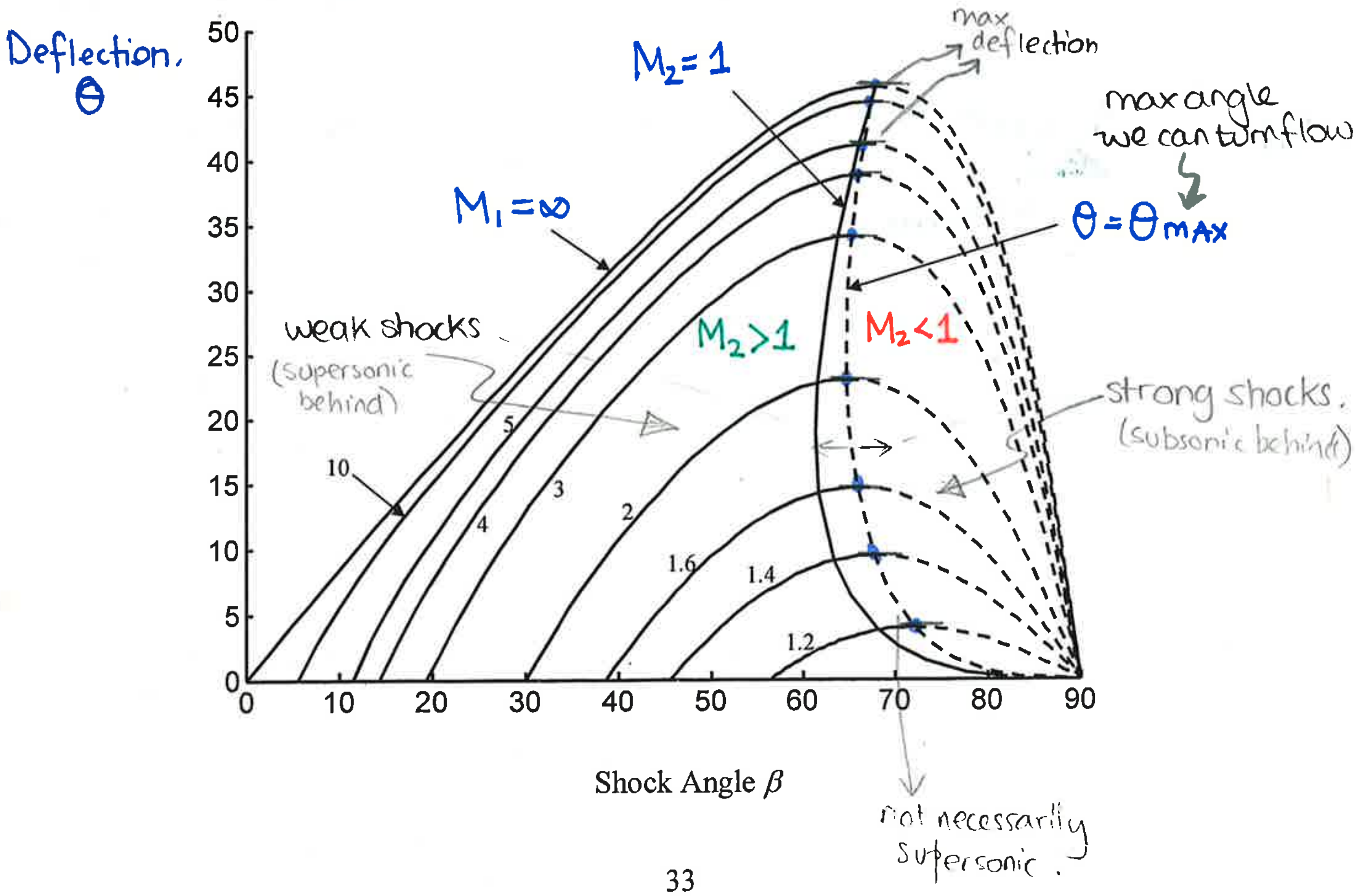
$$\tan\theta = \tan(\beta-(\beta-\theta)) = \frac{\tan\beta - \tan(\beta-\theta)}{1+\tan\beta\tan(\beta-\theta)}$$

to give

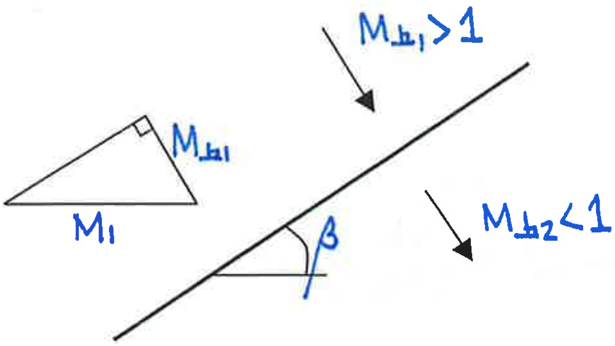
$$\tan\theta = \frac{2\cot\beta\cdot(M_1^2\sin^2\beta-1)}{(\gamma+1)M_1^2 - 2(M_1^2\sin^2\beta-1)} = fn(M_1,\beta) \tag{3.10}$$

} oblique shocks are a function of 2 variables.

The figure shows a plot of the deflection against the shock wave angle for various values of M_1.



Shock angle range: $\boxed{\mu < \beta < 90^\circ}$



Consequences

(1) For each M_1, β must lie between $\boxed{90^\circ}$ (normal shock) and $\boxed{\beta} = \sin^{-1}\dfrac{1}{M_1}$, corresponding to $M_{\perp 1} = M_1 \sin\beta = 1$. (i.e. is the $\boxed{\text{Mach Angle } \mu}$).

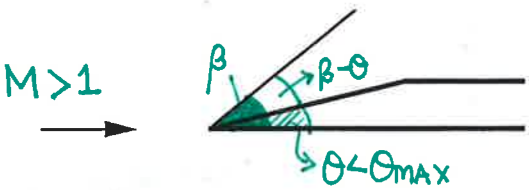
(2) For each M_1 $\boxed{\text{there is a } \textit{maximum possible deflection } \theta = \theta_{max}}$, (which increases as M_1 increases.)

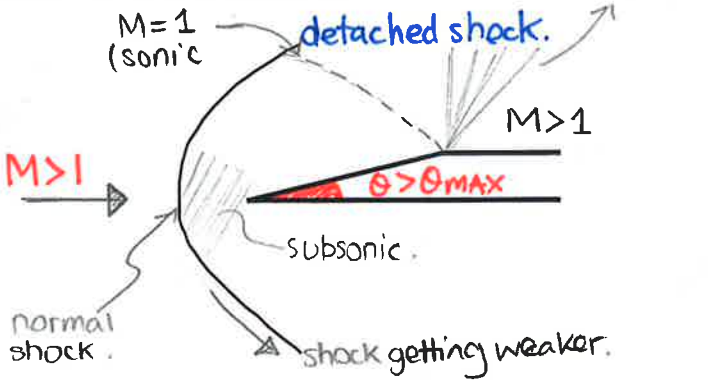
(3) For each value of $\boxed{\theta < \theta_{max} \text{ there are } \textit{two} \text{ possible values of the shock angle } \beta.}$ In practice, it is very rare to find the stronger branch (strong oblique) (higher value of $M_\perp = M_1 \sin\beta$). Nearly always find the part of the curve shown solid (other than, of course, normal shocks).

(4) The Mach number for the $\boxed{\text{perpendicular velocity downstream of the shock is } \textit{always subsonic} \text{ i.e. } M_2 \sin(\beta - \theta) < 1.}$ But M_2 is nearly always greater than unity (apart from a small region near $\theta = \theta_{max}$)

(5) Shocks which correspond to the part of the curve to the left of $\theta = \theta_{max}$ are referred to as 'weak' shocks and those corresponding to the right as 'strong' shocks.

(6) The wedge angle determines the $\boxed{\text{flow turning, } \theta,}$ which then $\boxed{\text{determines } \beta.}$
If $\theta > \theta_{max}$ for this value of M_1, then the shock becomes normal and 'stands off'





(7) It can be shown that the entropy rise across a normal shock is given by

$$\frac{\Delta s}{R} = \ln\frac{p_{01}}{p_{02}} \quad \text{and} \quad \frac{p_{01}}{p_{02}} = 1 + O(M_1 - 1)^3 \quad \text{as } M_1 \to 1.$$

For oblique shocks, we have

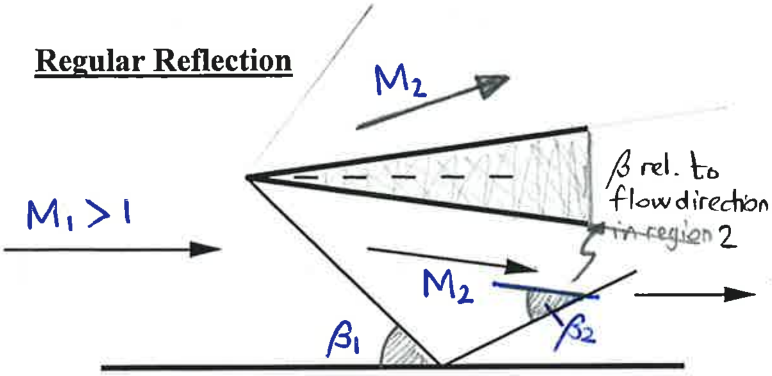
$$\frac{\Delta s}{R} = O(M_1 \sin\beta - 1)^3 \quad \text{as } M_1 \sin\beta \to 1.$$

E.g. with $M_1 = 2$ and $\beta = 35^\circ$ $(M_1 \sin\beta - 1)^3 = 3\times10^{-3}$ (tiny).

Thus oblique shocks are often nearly isentropic even for relatively high Mach numbers.

SHOCK WAVE REFLECTION

Regular Reflection



When an oblique shock meets a solid boundary a reflected wave forms, exactly as for characteristics, since downstream of the shock the flow is aligned with the wedge direction, and it must be turned back to the original direction at the wall.

The turning of the flow back to the original direction requires another shock.

E.g. $\boxed{M_1 = 2.5}$ and $\boxed{\theta = 8°}$

A set of tables (Houghton and Brock)

TABLE V

FLOW OF DRY AIR THROUGH A PLANE OBLIQUE SHOCK WAVE

↑ weak
↓ strong

M_1	θ ~~δ~~	β	p_2/p_1	ρ_2/ρ_1	T_2/T_1	M_2	$\Delta S/c_v$
2.50	2.000	25.052	1.1408	1.0984	1.0386	2.4151	0.0001
	4.000	26.613	1.2968	1.2029	1.0781	2.3320	0.0007
	6.000	28.266	1.4690	1.3132	1.1187	2.2496	0.0023
→	8.000	30.015	1.6585	1.4288	1.1608	2.1673	0.0052
	10.000	31.863	1.8661	1.5490	1.2047	2.0844	0.0099
	12.000	33.818	2.0930	1.6733	1.2508	2.0005	0.0163
	14.000	35.887	2.3401	1.8010	1.2993	1.9149	0.0247
	16.000	38.083	2.6088	1.9315	1.3507	1.8272	0.0353
	18.000	40.422	2.9007	2.0642	1.4052	1.7368	0.0481

$\Rightarrow M_2 = 2.1673$ (and $\beta = 30°$ ↗ for M_1)

Thus for second shock $\boxed{M = 2.1673}$ and $\boxed{\theta = 8°}$ ← turning back to original direction.

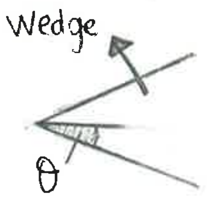
2.15	2.000	29.295	1.1246	1.0873	1.0344	2.0746	0.0001
	4.000	30.964	1.2611	1.1794	1.0693	2.0003	0.0005
	6.000	32.732	1.4104	1.2762	1.1051	1.9263	0.0017
→	8.000	34.606	1.5733	1.3776	1.1421	1.8520	0.0038
2.20	2.000	28.594	1.1268	1.0888	1.0349	2.1234	0.0001
	4.000	30.243	1.2660	1.1826	1.0705	2.0480	0.0005
	6.000	31.988	1.4183	1.2813	1.1070	1.9730	0.0017
→	8.000	33.837	1.5847	1.3845	1.1446	1.8977	0.0040

$\Rightarrow$ 2nd shock angle *relative to the flow direction*, $\boxed{\beta_2 = 34.3°}$ and $\boxed{M_3 = 1.87}$ } interpolate

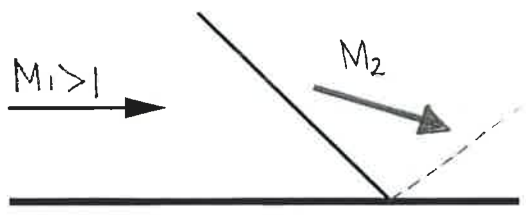
i.e. $M_3 = 1.87$ and the angle of the shock *relative to the wall* = $26.3°$

↖ $34.3 - 8 = \underline{\underline{26.3°}}$

Mach Reflection



What would have happened in the example above if the turning required by the second shock, the wedge half-angle = θ had turned out to be greater than the maximum possible turning, θ_{max}, for this value of M_2 ?

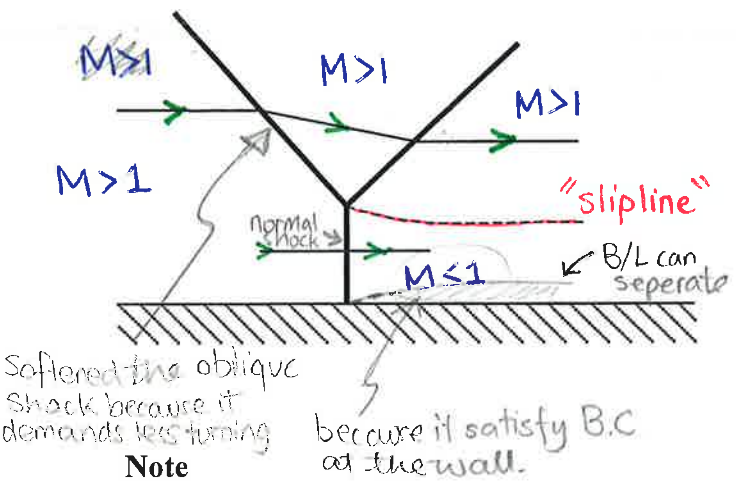


E.g. $M_1 = 2.5$ and $\theta = 22°$

Tables/Chart $\Rightarrow$ $\beta_1 = 45.7°$ and $M_2 = 1.54$

and θ_{max} for $M = 1.54$ is about $13.3°$

look at the graph.



Get *Mach Reflection*

In the downstream region there is a 'slip line', which in inviscid flow is a discontinuity of velocity, but in practice is a thin shear layer.

The two regions either side of the slip line have the same static pressure, but different levels of velocity.

Note

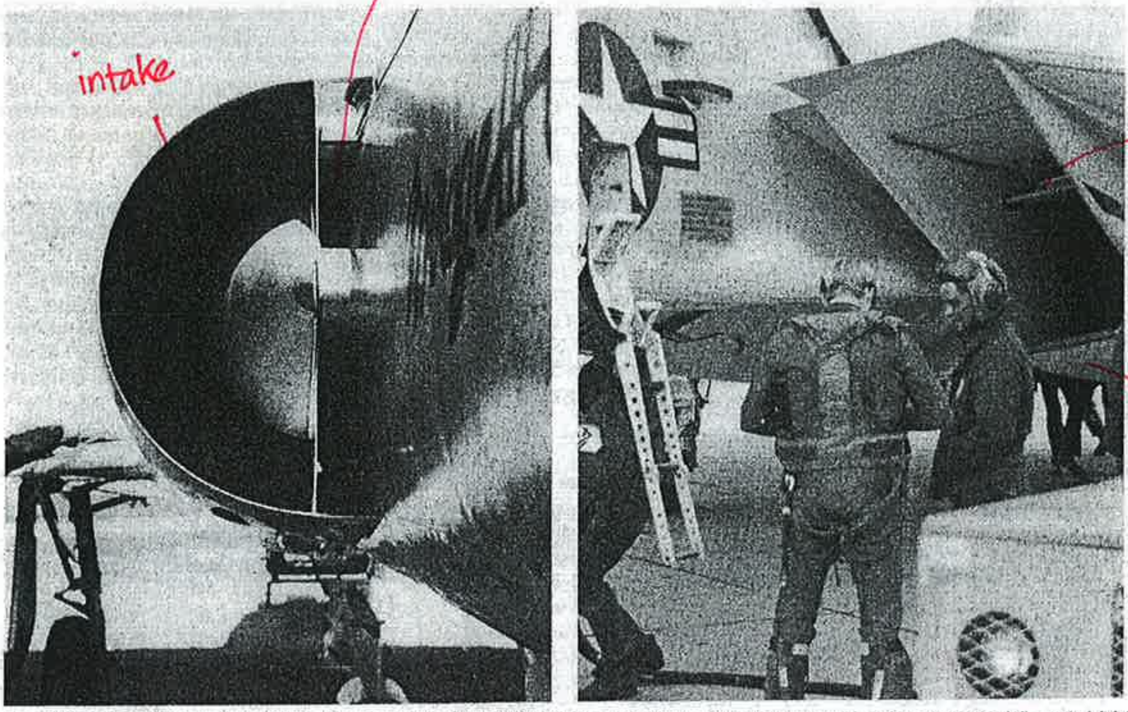
These types of reflections assume inviscid flow near the wall. In practice, there is a boundary layer there. The incident shock is a sudden pressure rise and may or may not separate the boundary layer depending on the shape and state of the boundary layer flow. This is a highly complex situation and an area of active research.

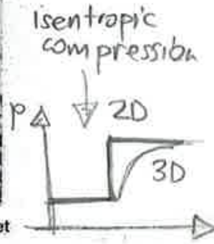
Supersonic Air Intakes

The flow through an aero-engine is always subsonic (typically M = 0.5) and this poses quite a design problem for intakes for supersonic aircraft. The air intakes of the Mirage and F-14 aircraft show two common approaches to this problem. Note the sharp angles, which are intended to induce only oblique shocks, wherever possible.

The flow through typical conical centre-body type intakes is shown for design speed, along with the improvement in intake performance obtained by using oblique shock compression for the design of a particular intake. Oblique shocks form a very efficient compression of the air immediately prior to the intake. This slows the flow down considerably such that the final normal shock, after which the flow is subsonic is at a modest Mach number and hence has relatively small entropy rise.

to avoid B/L injection into intake.

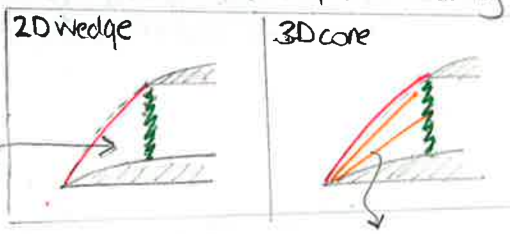


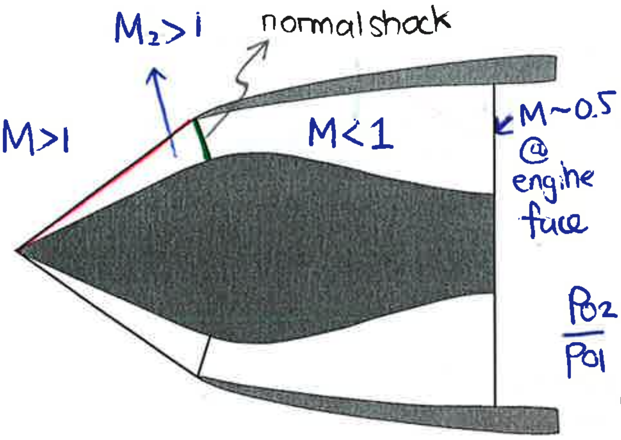


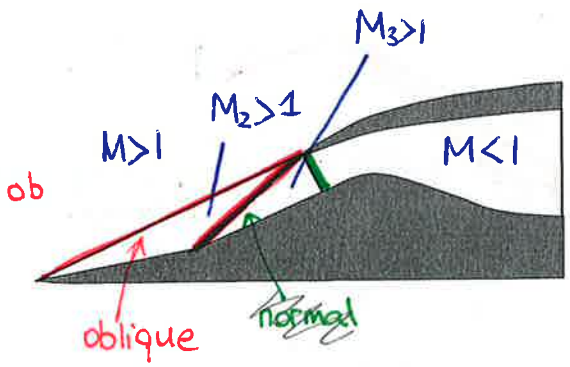
7-3 Inlets for modern combat aircraft. Above: normal shock diffusers for a maximum of double the speed of sound; on the left, ventral inlet (F-16), on the right side-mounted D-shape inlet (Viggen).
Below: multiple-shock diffusers, suitable for flight speeds of more than twice the speed of sound; on the left: variable-geometry axi-symmetric inlet (Mirage); on the right: variable-geometry two-dimensional inlet (F-14).

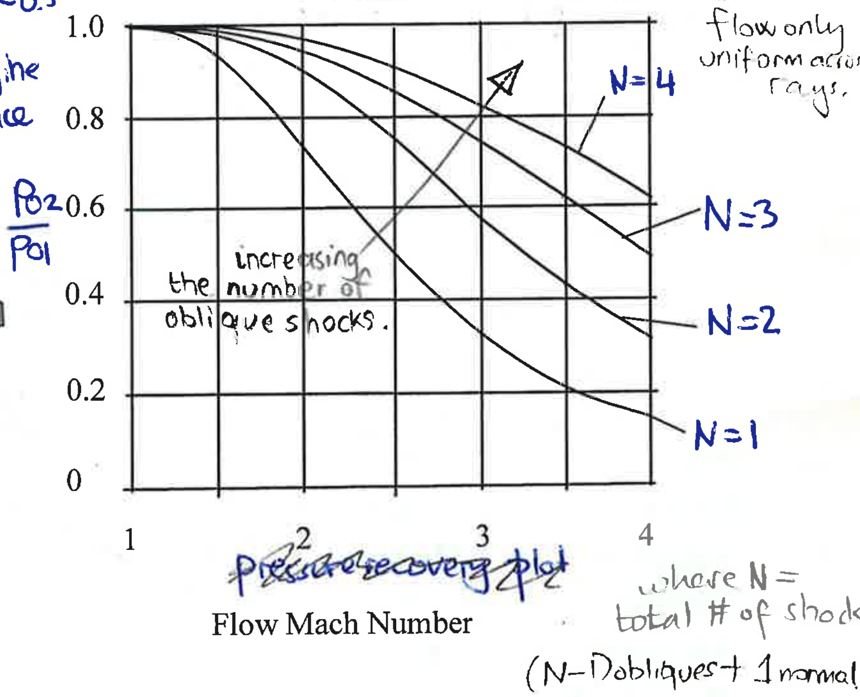
We're formally only considering 2D wedge systems.

taken from Klaus Huenecke - Modern Combat Aircraft Design





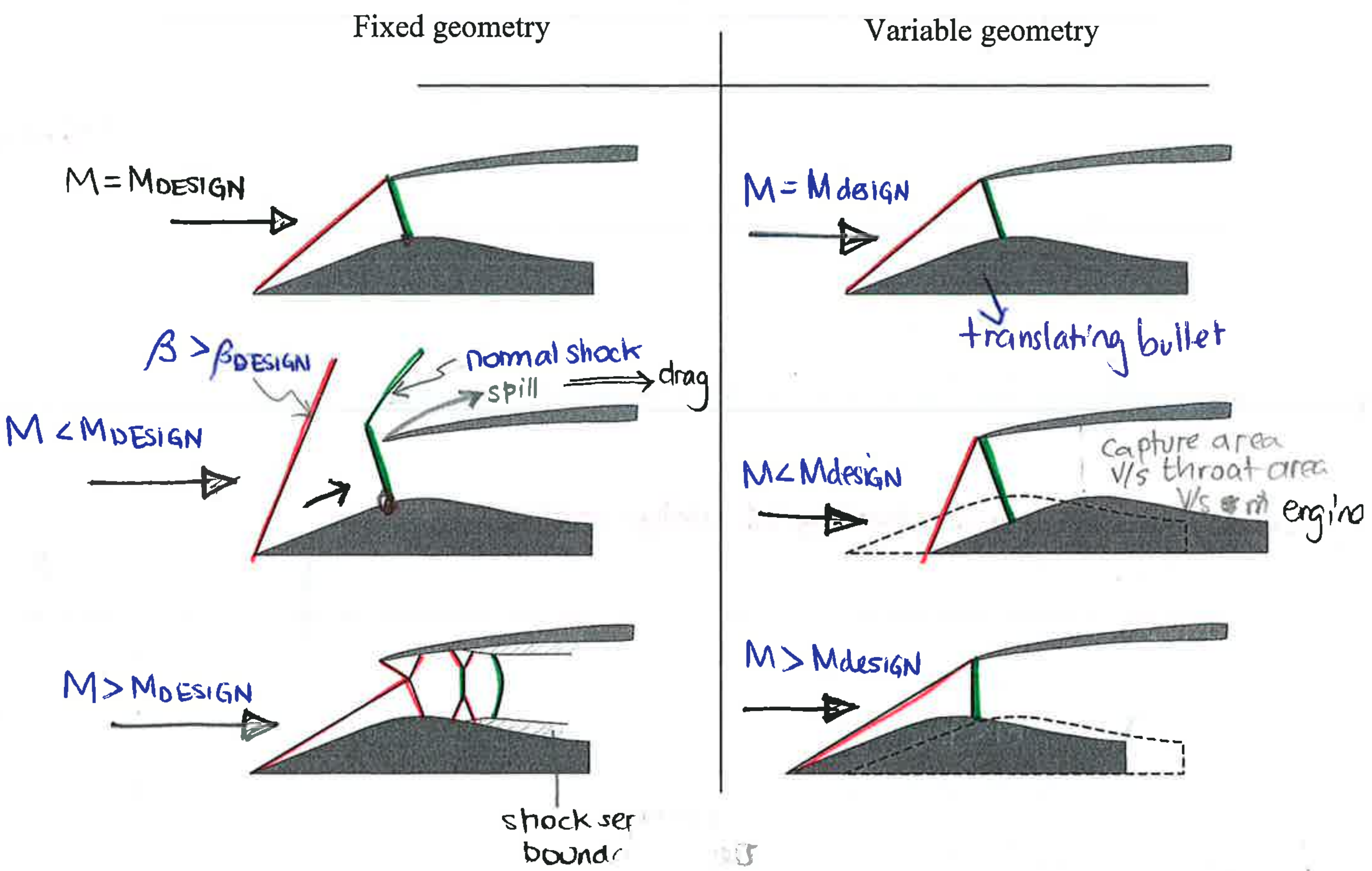




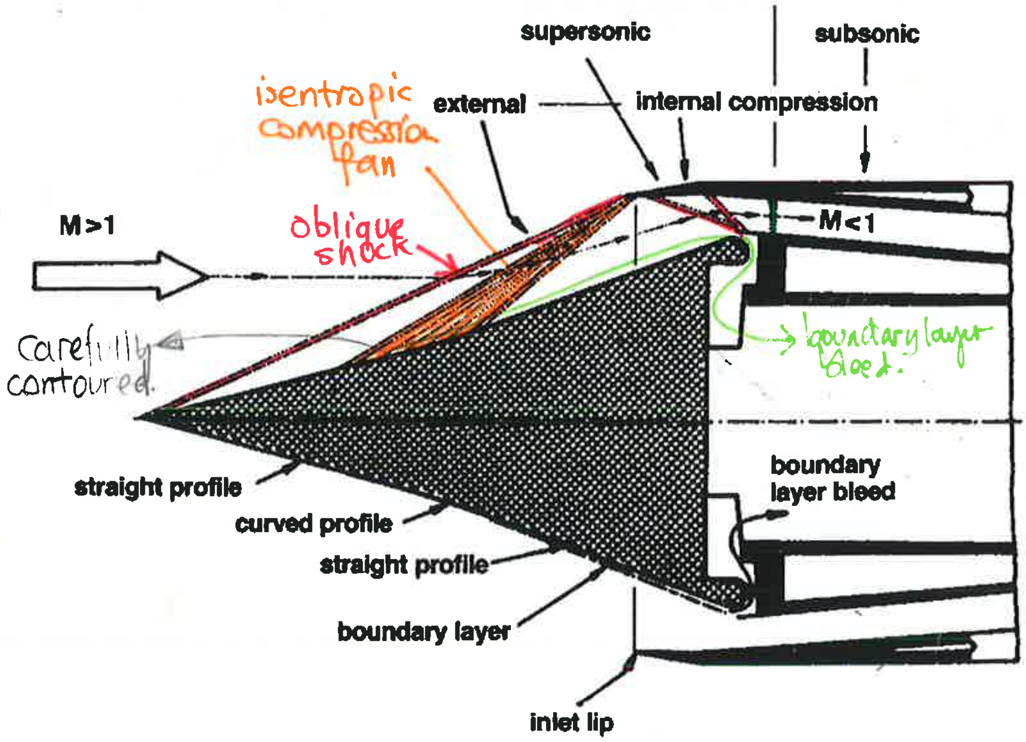
For the oblique (i.e. weak) shocks, the loss of stagnation pressure across them is relatively small and the main loss is across the normal shock. As the number of shocks N increases, the more the flow can be slowed down by the oblique shocks and the lower the Mach Number ahead of the normal shock, and hence the smaller the loss across it.

More shocks = more complexity = greater cost + lower reliability.

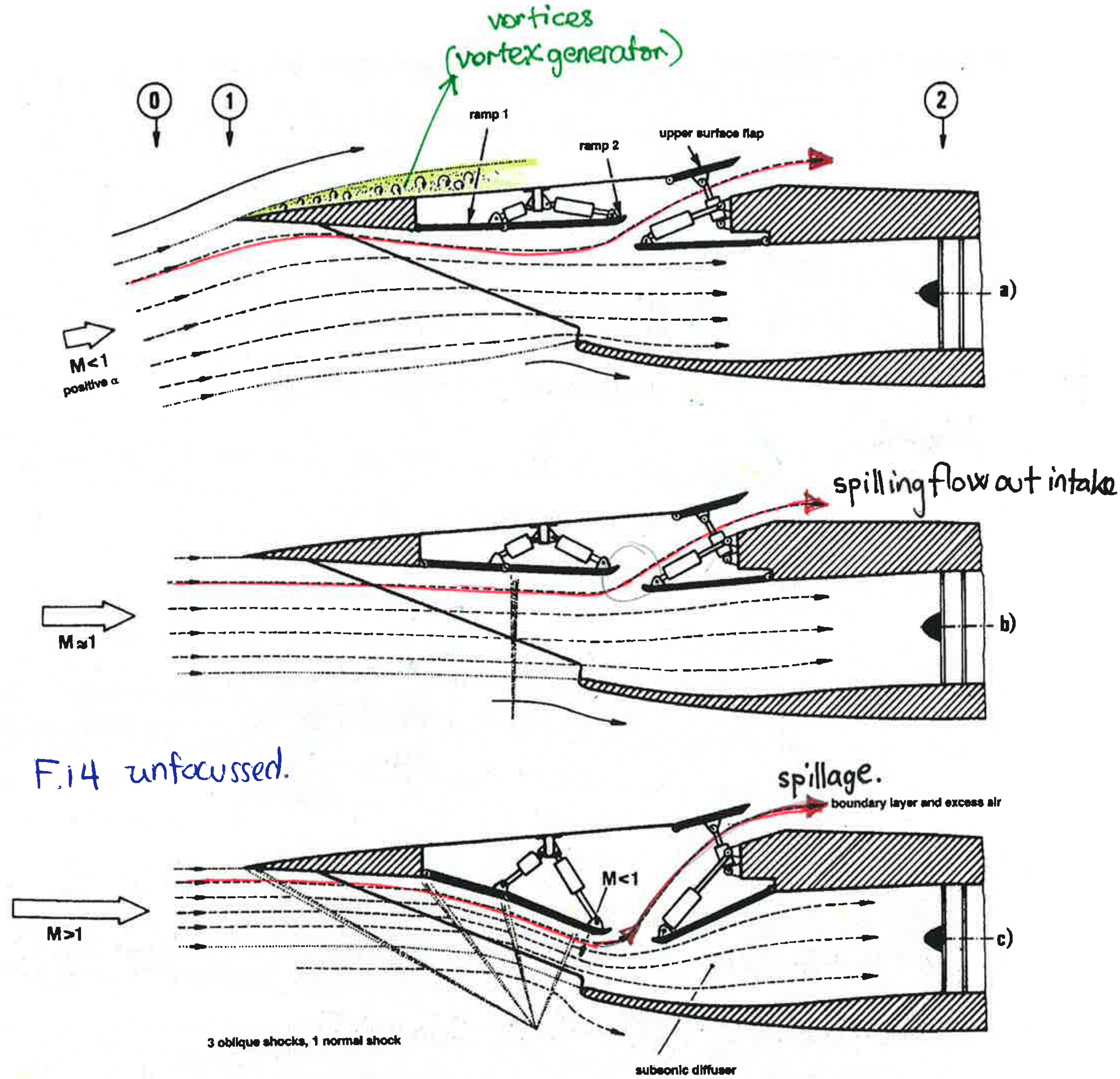
As the mach number of the aircraft increases or as the proportion of flight time spent in supersonic flight increases, the inlets become more complex. There are two things to balance: the aircraft speed and the engine flow demand. Good intake recovery (low total pressure loss) is obtained by using variable geometry intakes with spill doors for excess air.



The most efficient conical intake is for the SR-71 (Mach 3.2) and square intake is for Concorde (Mach 2.05). The SR-71 intake operating at design Mach number has multiple external oblique shocks all designed to intersect just inside the intake on the outer casing, with reflection as a single oblique shock.

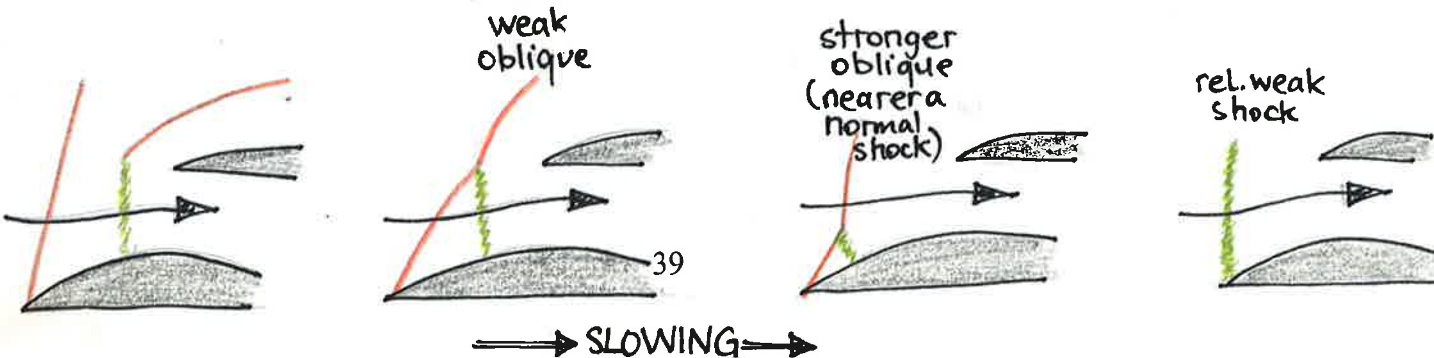


Axisymmetric inlet with mixed compression (Lockheed SR-71).



7-26 **Inlet of the F-14**
Ramp positions and flow at various flight conditions: a) subsonic speed and high angle-of-attack (typical manoeuvre case); b) transonic flow; c) supersonic flow.

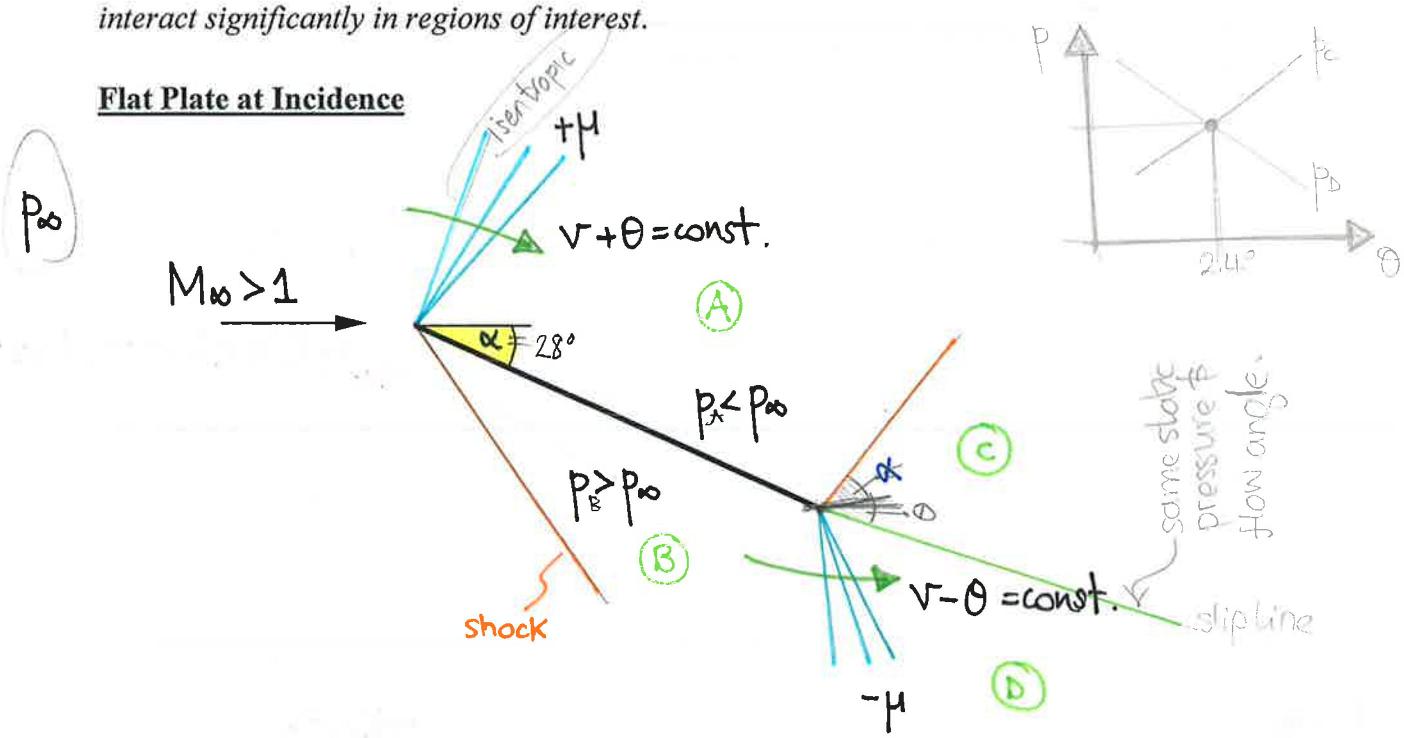
This picture is also taken from Klaus Huenecke - Modern Combat Aircraft Design. It illustrates the design ingenuity needed to balance the competing demands of good intake recovery (low total pressure loss) over a wide range of flow speeds, coupled with varying engine flow demands and geometric simplicity.



FURTHER EXAMPLES (SHOCK-EXPANSION THEORY)

The supersonic flow around a variety of shapes can be calculated by patching together various combinations of shocks and isentropic expansions, *provided that the individual elements do not interact significantly in regions of interest.*

Flat Plate at Incidence



Kutta condition at trailing edge doesn't normally hold.

Transition $\infty \to A$ $\quad M_\infty = 2.41 \quad \alpha = 28°$

This is an *expansion* across a $+\mu$ ch'ic. Thus

$\nu_A + \theta_A = \nu_\infty + \theta_\infty \qquad \nu_\infty = 36.89 \quad \theta_\infty = 0° \qquad \theta_A = -28°$

$\Rightarrow \nu_A = 36.89 + 0 - (-28) = 64.89° \Rightarrow M_A = 3.95$

We will see later that ,to complete the solution, we need to calculate the static pressure here. This is done by remembering that the expansion is isentropic, so that there is no loss of stagnation pressure. This gives

$p_{0A} = p_{0\infty} \qquad \Rightarrow \boxed{p_A = 0.1052\, p_\infty} \qquad \dfrac{p_A}{p_0} = 0.1052$

$\uparrow$ $p_{0\infty}$

Transition $\infty \to B$

This is a shock, with upstream Mach number 2.41 and a deflection of ______ H & B gives

M_1	$\frac{p_2}{p_1} = \frac{p_B}{p_\infty}$	$M_2 = M_B$
2.40	4.8648	1.0705
2.45	4.8671	1.1325
2.41	4.8653	1.0829

interpolate

$\boxed{\nu_B = 1.01}$

Transitions A → C and B → D

To match the flow at the trailing edge, the flow in region A must be compressed (by a shock) and that in region B expanded. The shocks at the leading and trailing edge will, however, have different strengths in general and the drop of total pressure across each of them will thus be different. Downstream of the trailing edge there will thus be two streams. These must have the same static pressure and flow angle and but different velocities and Mach numbers. There is no simple way of determining what value of flow angle is appropriate and it is necessary to iterate.

looking at shock on top surface

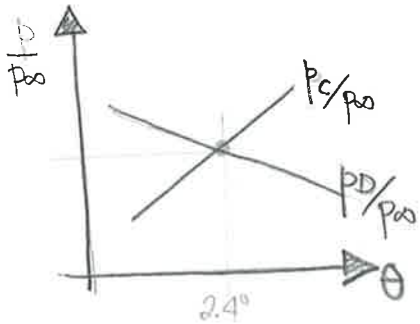
where $p_A = 0.1052 p_\infty$

θ	δ_{AC} ($\theta+\alpha$)	$\frac{p_2}{p_1}\left(=\frac{p_C}{p_A}\right)$	$\frac{p_C}{p_\infty}$	$\Delta\theta_{BD}$	ν_D	M_D	$\frac{p_D}{p_\infty}$
0	28	8.1982	.8625	28	29.01	2.099	1.1148
2	30	9.1018	.9575	30	31.01	2.175	.9900
4	32	10.0742	1.0598	32	33.01	2.214	.9315

oblique shock — expansion wave

Interpolating between the last two values
$\theta = x\,4 + (1-x)\,2$ and equal static pressures in the two streams means that

ie. $\theta = 2.4°$



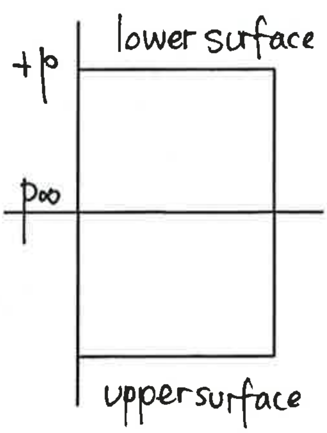
Notes

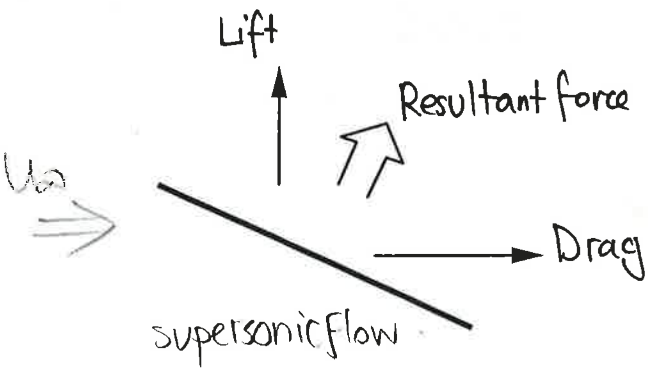
(i) There is *no singularity* at the leading edge. Hence no suction force parallel to plate. → no induced drag

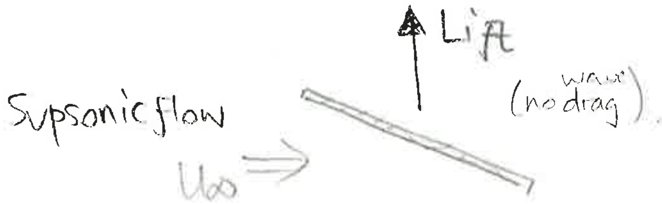
i.e. force on plate ⊥ plate.

The drag on the plate is referred to as 'wave drag' since it is caused by the momentum carried away by the wave system. "WAVE RIDER"

Contrast this behaviour with incompressible (or for that matter subsonic) flow. For that case, there is a leading edge singularity and lift but no drag.

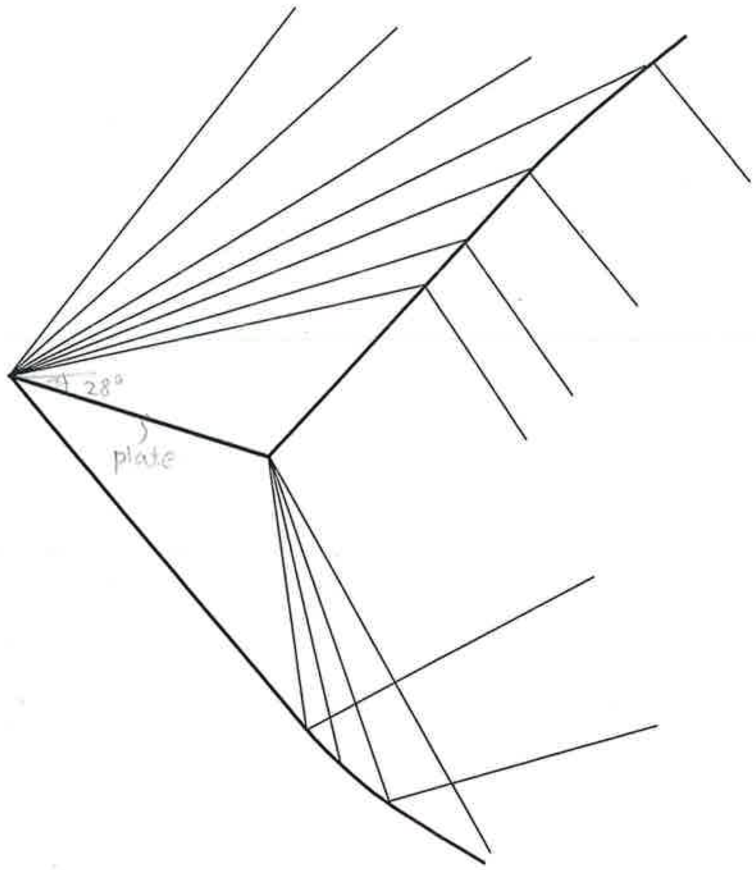




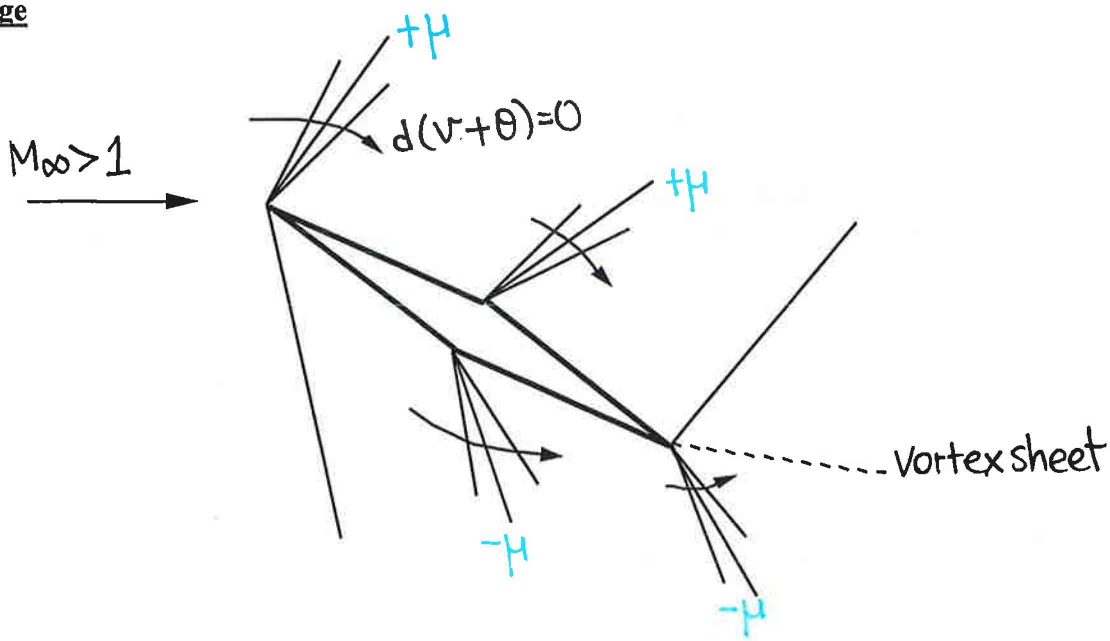


(ii) The pressures on the upper and lower surfaces do not equalise at the trailing edge (i.e. no Kutta condition as for the subsonic case). There is turning of the flow downstream of the trailing edge (called 'supersonic deviation'). $p_A \neq p_B$ @ trailing edge.

(iii) The solution we have obtained assumes that the various shocks, expansions and slip line do not interact (i.e. intersect). At some distance from the plate they will do so

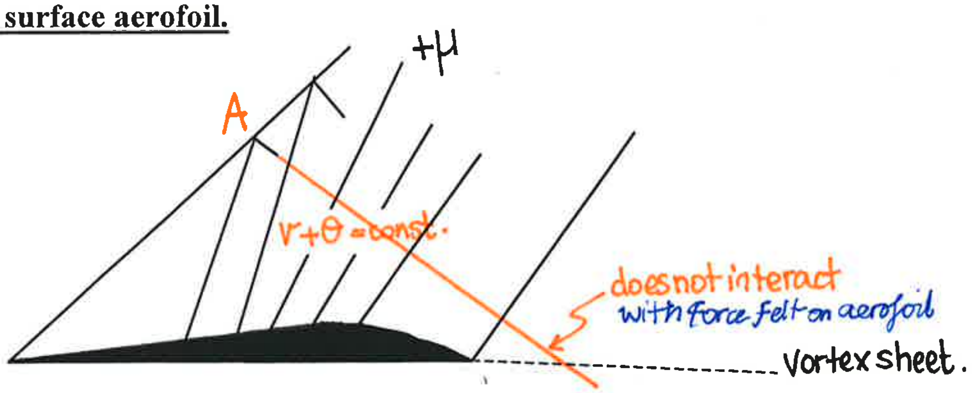


Double Wedge



In this case, there is not only lift and drag, there is also a pitching moment.

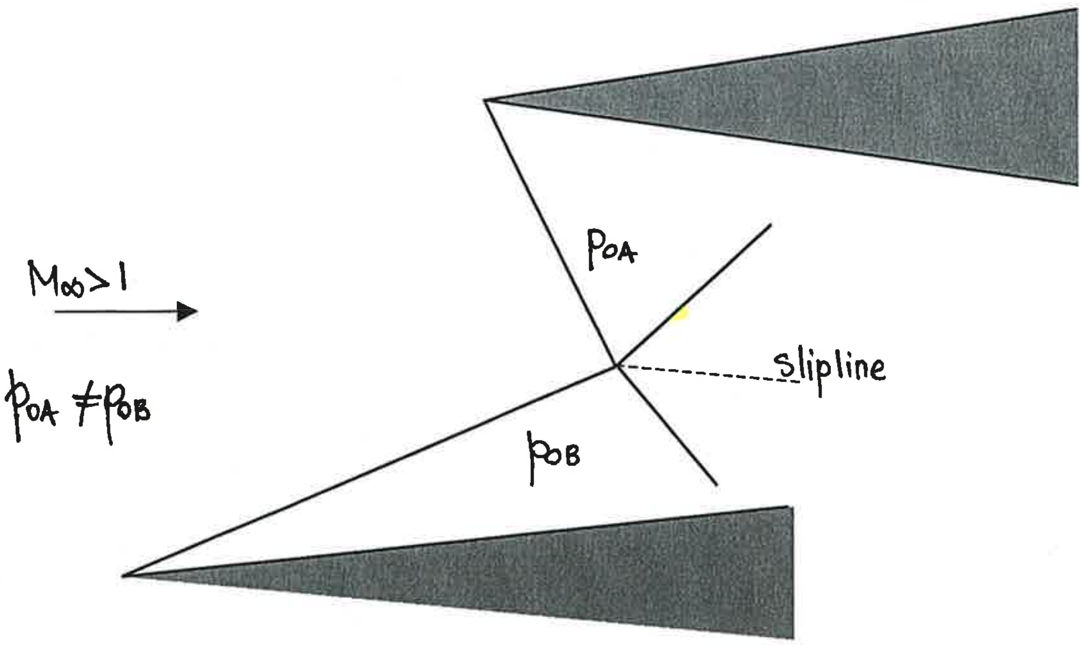
Cambered upper surface aerofoil.



When the first characteristic from the aerofoil intersects the shock, then the shock starts to curve and its strength is not constant ⇒ flow downstream does not have uniform entropy ⇒ simple characteristics only approximately valid. If μ– ch'ic from A does not intersect the aerofoil surface, then this fact does not influence consideration of force on the aerofoil. In addition, the characteristic waves which are reflected from the shock are *very* weak, and the approximation which ignores them (i.e. shock-expansion theory) is a good one.

Intersecting Shocks.

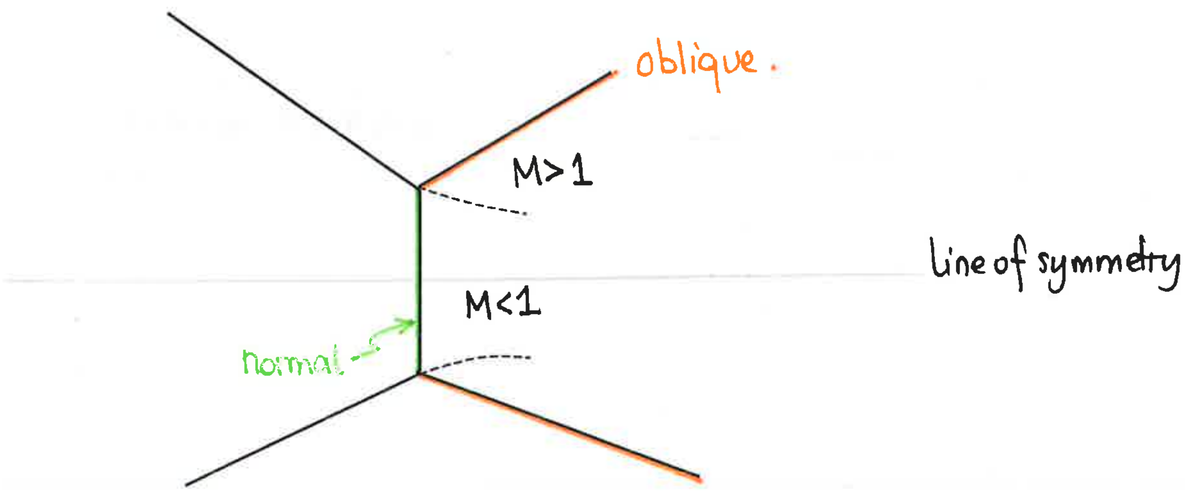
The intersection of two shocks is another case where it is usually necessary to iterate to find the downstream flow angle and static pressure.



First two shocks will be known, provided the wedge angles are known. The second two shocks must be iterated in the same way as for the flat plate aerofoil. i.e. a guess made of the downstream flow angle and the calculations performed across the two shocks. Compatibility will be obtained when the downstream static pressures are equal. In general there will be a slip line/vortex sheet.

The special case of two equal intersecting shocks is greatly simplified by symmetry considerations which fix the downstream angle at the same as the far upstream one.

A similar phenomenon to Mach Reflection at boundaries is possible with intersecting shocks. If the two downstream shocks can not be made to deliver compatible flow, then something like the following pattern is observed.



Nozzle Flows

The one dimensional flow in nozzles of slowly varying area exhausting into a plenum shows the full range of shock and expansion behaviour. The next two figures are taken from Liepmann & Roshko.

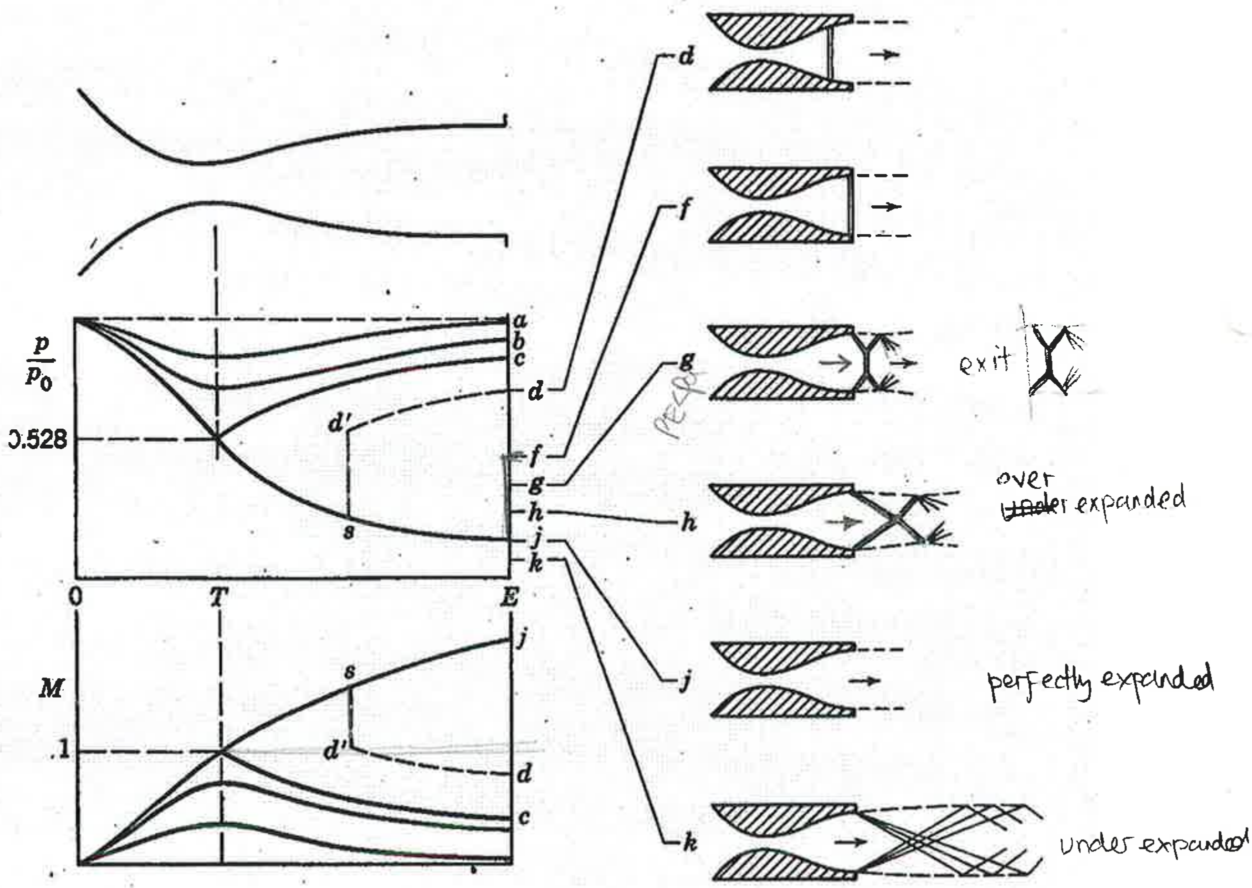
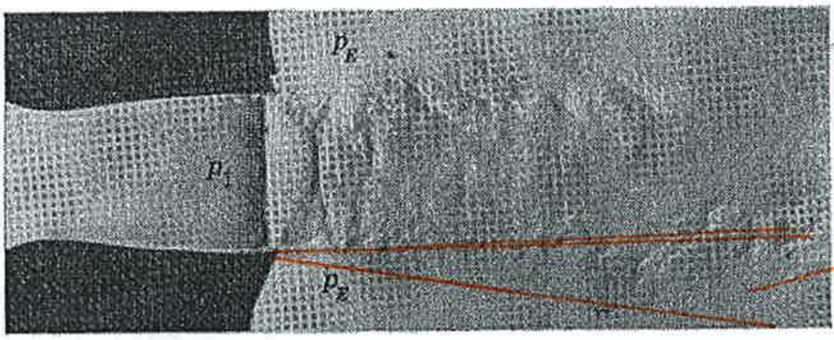


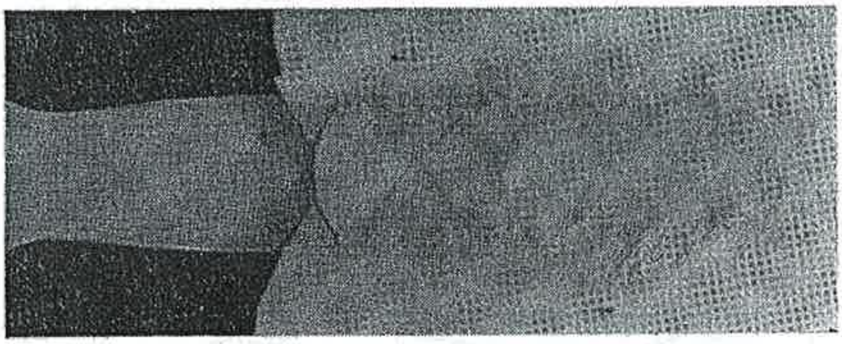
FIG. 5·3 Effect of pressure ratio on flow in a Laval nozzle.

$\frac{p_1}{p_E} < 0.4$ f

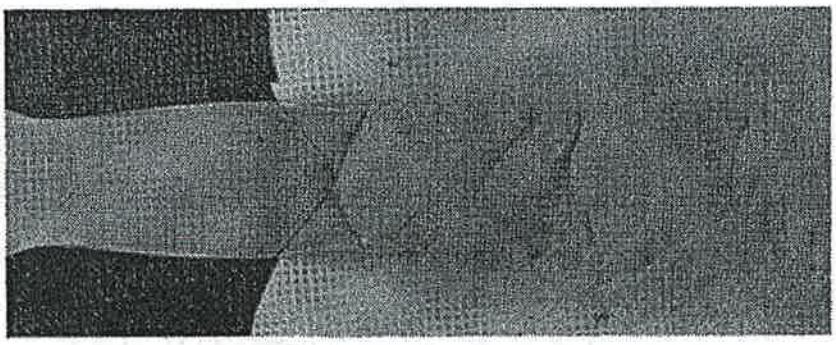


shear layer

$\frac{p_1}{p_E} = 0.66$ g

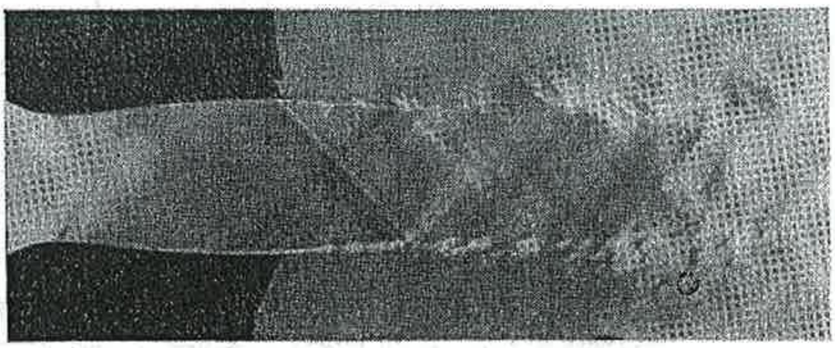


$\frac{p_1}{p_E} = 0.85$ h



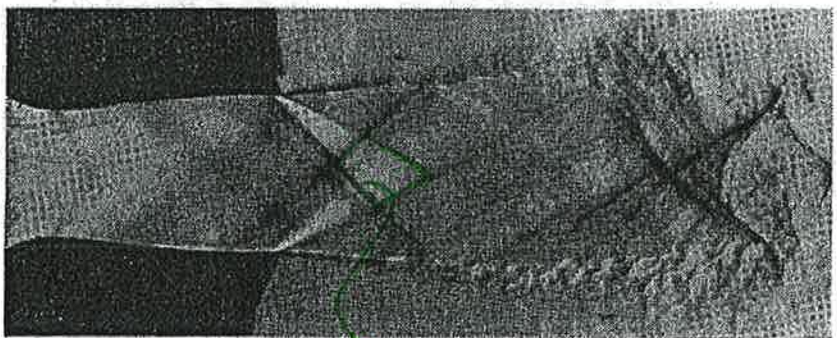
under expanded

$\frac{p_1}{p_E} = 1.00$ j



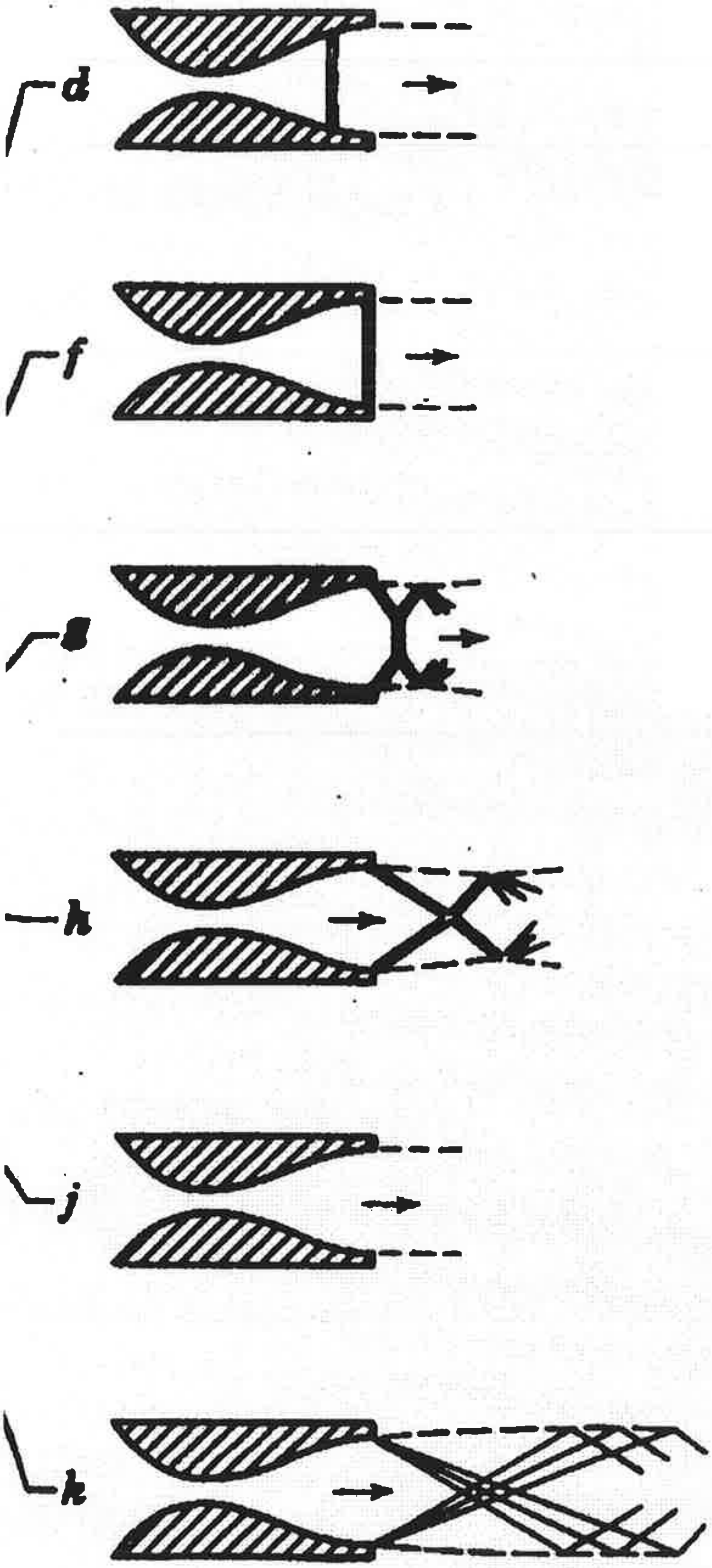
fully expanded

$\frac{p_1}{p_E} = 1.50$ k



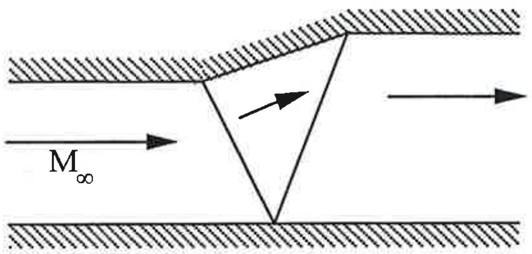
over expanded.

diamond patterns.


d
f
g
h
j
k

LINEARISED SHOCK EXPANSION THEORY (LINEARISED METHOD OF CHARACTERISTICS)

Calculations using shock expansion theory give good accuracy but involve considerable effort in anything other than the simplest cases. An inspection of the flat plate calculation indicates, however, that the values used for that example were rather extreme. While such values are appropriate for such applications as re-entry vehicles, they are much larger than those likely to be met in the study of flow around aircraft, turbomachinery blades, etc. For these cases, the flow changes are much more modest. A useful first approximation is to treat all changes as small.



If the changes in the flow angle from the upstream flow are small, then we can linearise the Prandtl-Meyer relationships.

For isentropic compressions and expansions

a) Neglect spread of the fan of characteristics and treat as a single 'wave'

b) Linearise the Prandtl-Meyer relationships $\nu \pm \theta = \text{const.}$

Recall that $\nu = \int^M \sqrt{M^2-1}\,\frac{dV}{V} \quad \Rightarrow \quad d\nu = \sqrt{M^2-1}\cdot\frac{dV}{V} = \pm d\theta$

i.e. $$\frac{dV}{V} = \mp \frac{d\theta}{\sqrt{M^2-1}} \tag{6.1}$$

c.) In addition, recall that the entropy rise in an oblique shock wave is proportional to

$(M_{\perp 1} - 1)^3$ where $M_\perp$ = mach $\perp$ to shock.

and to this level of approximation (all changes of flow angle small), oblique shocks can be treated as reversible expansions. $\boxed{\text{Equations (6.1)}}$ thus hold across them.

Changes in static pressure follow from the these change in velocity using the streamwise momentum equation (equation (1.2)

$$\rho V \frac{\partial V}{\partial s} = -\frac{\partial p}{\partial s}$$

$\Rightarrow$ Along a streamline $\rho V\,dV = -dp$

$$\Rightarrow \frac{dp}{p} = \mp \frac{\rho V^2}{p}\cdot\frac{d\theta}{\sqrt{M^2-1}} = \mp \frac{\gamma M^2 d\theta}{\sqrt{M^2-1}} \tag{6.2}$$

log derivative

A similar formula for the change in Mach number comes from

$$M = \frac{V}{c} \Rightarrow \frac{dM}{M} = \frac{dV}{V} - \frac{dc}{c} \quad \text{and} \quad c^2 = \frac{\gamma p}{\rho} = k\,p^{\frac{\gamma-1}{\gamma}} \quad \text{in isentropic flow,}$$

$$\Rightarrow 2\frac{dc}{c} = \frac{\gamma-1}{\gamma}\frac{dp}{p}$$

$$\frac{dM}{M} = \pm\frac{d\theta}{\sqrt{M^2-1}} \pm \frac{\gamma-1}{2\gamma}\frac{M^2\gamma\, d\theta}{\sqrt{M^2-1}} \Rightarrow \frac{dM}{M} = \pm\boxed{\frac{1+\frac{\gamma-1}{2}M^2}{\sqrt{M^2-1}}}\, d\theta \qquad (6.3)$$

For a *succession* of changes in the flow, as long as the flow angle never becomes large

$$\theta = d\theta_1 + d\theta_2 + d\theta_3 + \ldots = \frac{\sqrt{M_\infty^2-1}}{V}\left(\pm dV_1 \pm dV_2 \pm dV_3 + \ldots\right)$$

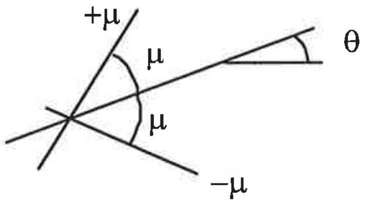
$\pm d\theta = \sqrt{M^2-1}\cdot\frac{dV}{V}$

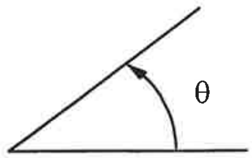
Notes on signs

(a) The relationships

$\nu + \theta = \text{const}$ across a $+\mu$ ch'ic

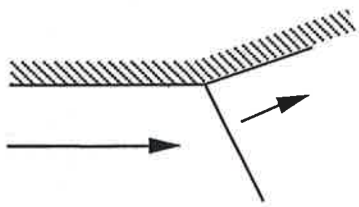
$\nu - \theta = \text{const}$ across a $-\mu$ ch'ic





together with those for the various small deflection shock relations are only valid if θ is measured positive anticlockwise.

(b) For linearised cases it may be easier to work out the sign of changes according to whether or not the flow has passed through a compression or an expansion.



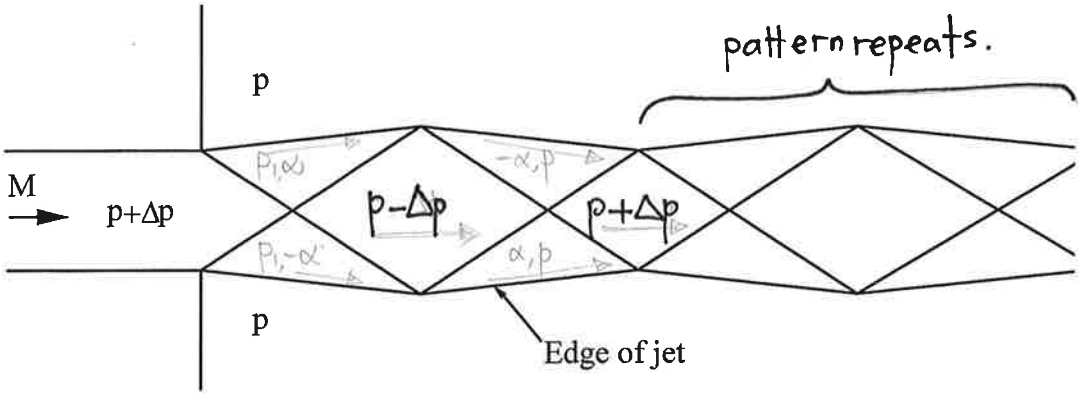
e.g. Expansion $\Rightarrow dM > 0,\ dp < 0,\ dV > 0$ (and $d\nu > 0$)

Linearised Theory Applied to 'Shock Cells'

Recall, linearised theory = take all characteristics and shocks at the Mach angle of the upstream flow, ignore change of characteristics angles at characteristics crossings, treat shocks as isentropic, and use equations (6.1) & (6.2) for the changes in flow properties

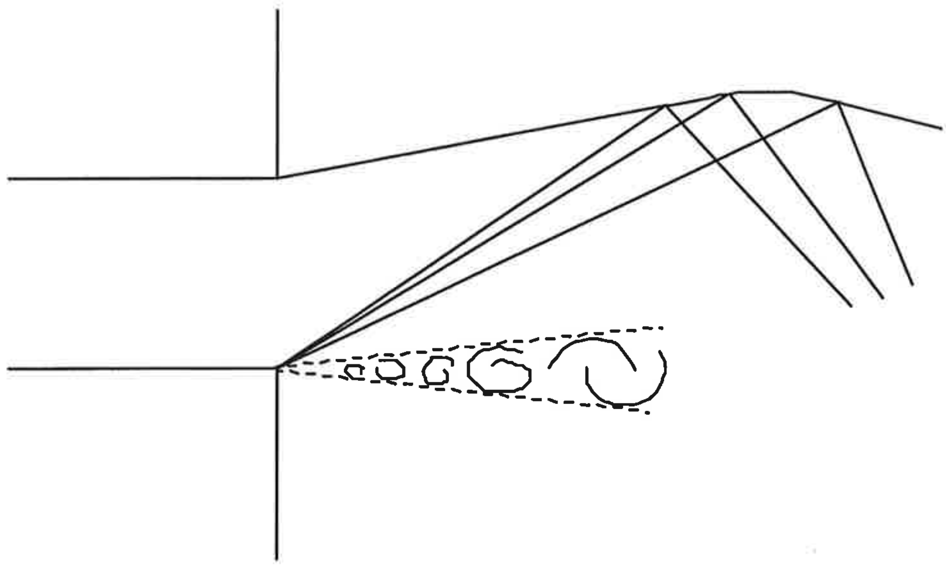
$$\frac{\Delta p}{p} = \pm\frac{\gamma M^2}{\sqrt{M^2-1}}\Delta\theta \qquad\qquad \frac{\Delta M}{M} = \pm\frac{1+\frac{\gamma-1}{2}M^2}{\sqrt{M^2-1}}\Delta\theta\ .$$

p's are the same.



This repetition of shock and expansion patterns downstream of an over-expanded jet is referred to as a set of "shock cells".

N.B. For full non-linear theory the expansion around the initial corner "spreads out", implying that shock cells 'weaken' in the downstream direction.



An added complication is the fact that the edge of the jet is unstable, and becomes a turbulent shear layer!

cookman.