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HEMH MASTER

3C5

Part IIA
Paper ~~G7~~ Dynamics and Vibrations

Ten Lectures on Rigid-Body Dynamics
Lectures ~~1-5~~: Theory
1-6

Dr H E M Hunt
Michaelmas 1997

1998
1999
2000
2001
2002
2003

2004 (* major update)
2005

NEW Master

3C5

CONTENTS - Lectures 1-5

1. INTRODUCTION

Mechanics can be divided broadly into Statics and Dynamics. Statics deals with geometry and equilibrium while dynamics deals with the time derivatives – kinematics and kinetics.

- Mechanics
  - Statics
    - geometry (displacement diagrams)
    - equilibrium (Force polygons)
  - Dynamics
    - kinematics (velocities)
    - kinetics (accelerations)

Done 2D stuff in IA & IB, now 3D!

Aims of the Dynamics course:

1. obtain equations of motion for bodies subject to known forces

Newton
d'Alembert
} used widely for planar problems. Not good for 3D

Euler
Lagrange
} Cope well with all problems in 3D though 3D problems are always complicated.

2. to solve these equations

general motion — Never a good idea – too hard

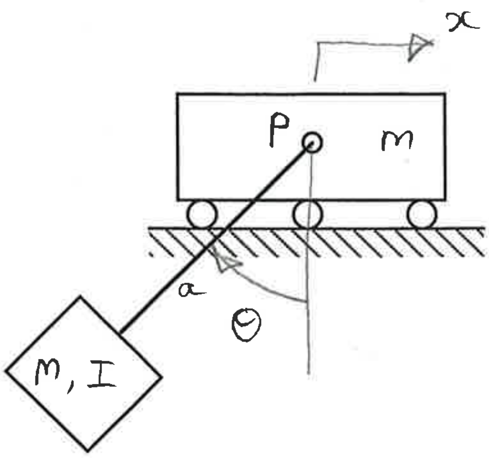
steady state — Best place to start

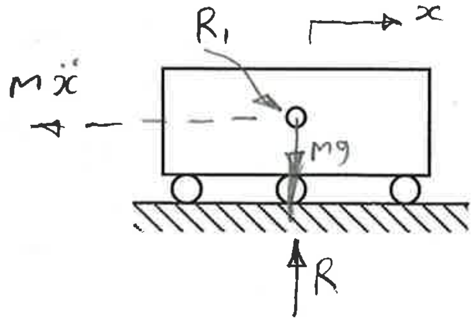
small vibrations & stability – deviations from Steady State

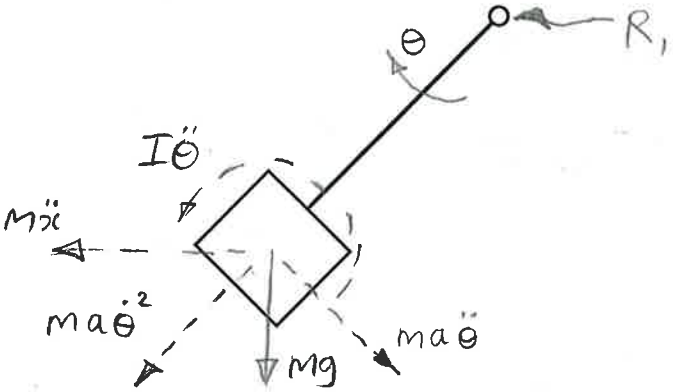
} we'll only do these

2. EQUATIONS OF MOTION IN THREE DIMENSIONS

2.1 Revision - plane (two-dimensional) motion of a rigid body







end L1 M07

End L1 M98

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moments about P for pendulum FBD ↻+

$$M\ddot{x}\, a\cos\theta - mga\sin\theta - ma^2\ddot{\theta} - I\ddot{\theta} = 0$$

whole system horizontal equilibrium ←

$$2M\ddot{x} + ma\dot{\theta}^2\sin\theta - ma\ddot{\theta}\cos\theta = 0$$

or $\frac{d}{dt}\left(2M\dot{x} - ma\dot{\theta}\cos\theta\right) = \cancel{\text{const}}\ 0$ (linear momentum) Same!

whole system vertical equilibrium ↑+

$$R = 2mg + ma\dot{\theta}^2\cos\theta + ma\ddot{\theta}\sin\theta$$

For plane motion in general we can safely say:

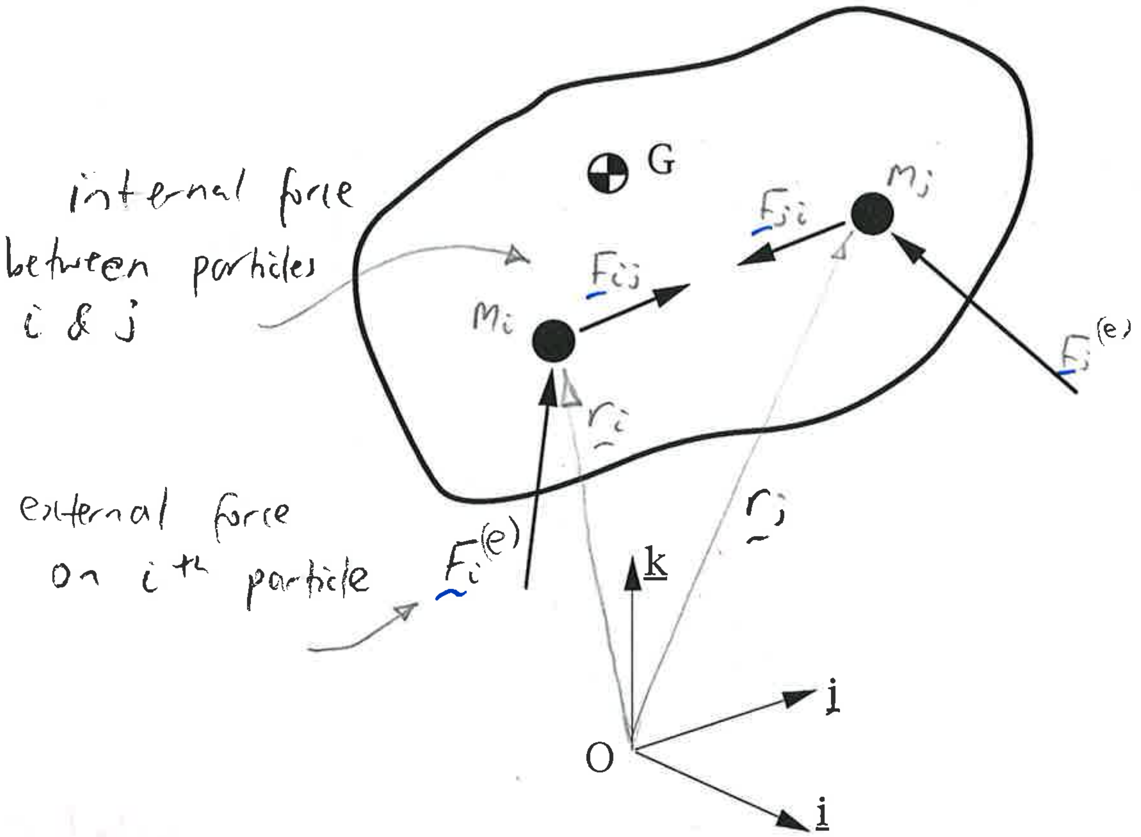
- $\Sigma\ \underline{F} = m\,\underline{a}_G \qquad = \dot{\underline{p}}_G$ Sum of forces = rate of change of lin mom
- $\Sigma\ \underline{M}_G = I_G\,\ddot{\underline{\theta}} \qquad = \dot{\underline{h}}_G$ sum of moments = rate of change of ang mom

Are these equations alone enough to describe motion in 3D ???

END L1 M03 End L1 M01

2.2 Newton's Laws in three dimensions - linear momentum

- Any "body" is a collection of particles



L1 M02

End L1 M97

Consider particle i of mass m_i at position $\underline{r}_i$

Apply Newton II:

external force   internal forces

$$m_i \ddot{\underset{\sim}{r}}_i = \underset{\sim}{F}_i^{(e)} + \sum_{j \neq i} \underset{\sim}{F}_{ij} \tag{2.1}$$

$\boxed{\text{End L1 M97}}$

Sum over all particles

$$\therefore \sum_i m_i \ddot{\underset{\sim}{r}}_i = \sum_i \underset{\sim}{F}_i^{(e)} + \sum_i \sum_{j \neq i} \underset{\sim}{F}_{ij} \tag{2.2}$$

Define: • $M = \sum_i m_i$ which is the total mass of the system

sum of mass of all particles

- $\underline{r}_G$ is the position of G (the centre of mass) so that

$$M \underline{r}_G = \sum_i m_i \underline{r}_i \qquad \therefore M \dot{\underset{\sim}{r}}_G = \sum_i m_i \dot{\underset{\sim}{r}}_i$$

$$M \ddot{\underset{\sim}{r}}_G = \sum m_i \ddot{\underset{\sim}{r}}_i \qquad \text{etc}$$

and • $\underline{F}^{(e)} = \sum_i \underline{F}_i^{(e)}$ which is the total external force.

sum of all external forces

Note that all internal forces cancel in pairs $\quad \sum_i \sum_{j \neq i} \underset{\sim}{F}_{ij} = 0$

because $\underset{\sim}{F}_{ij} = - \underset{\sim}{F}_{ji}$ by Newton 3

Equation (2.2) becomes:

$$M \ddot{\underline{r}}_G = \underline{F}^{(e)} \tag{2.3}$$

So it is <u>true</u> in 3D that "$\underline{F} = m \underline{a}$" works fine for a rigid body when applied to the Centre of gravity G. Newton knew this.

Denote the linear momentum of the body as $\underline{p} = M \dot{\underline{r}}_G$

$$\boxed{\dot{\underline{p}} = \underline{F}^{(e)}} \tag{2.4}$$

<u>On Data Sheet</u>

$\dot{\underline{p}}$ = rate of change of linear momentum

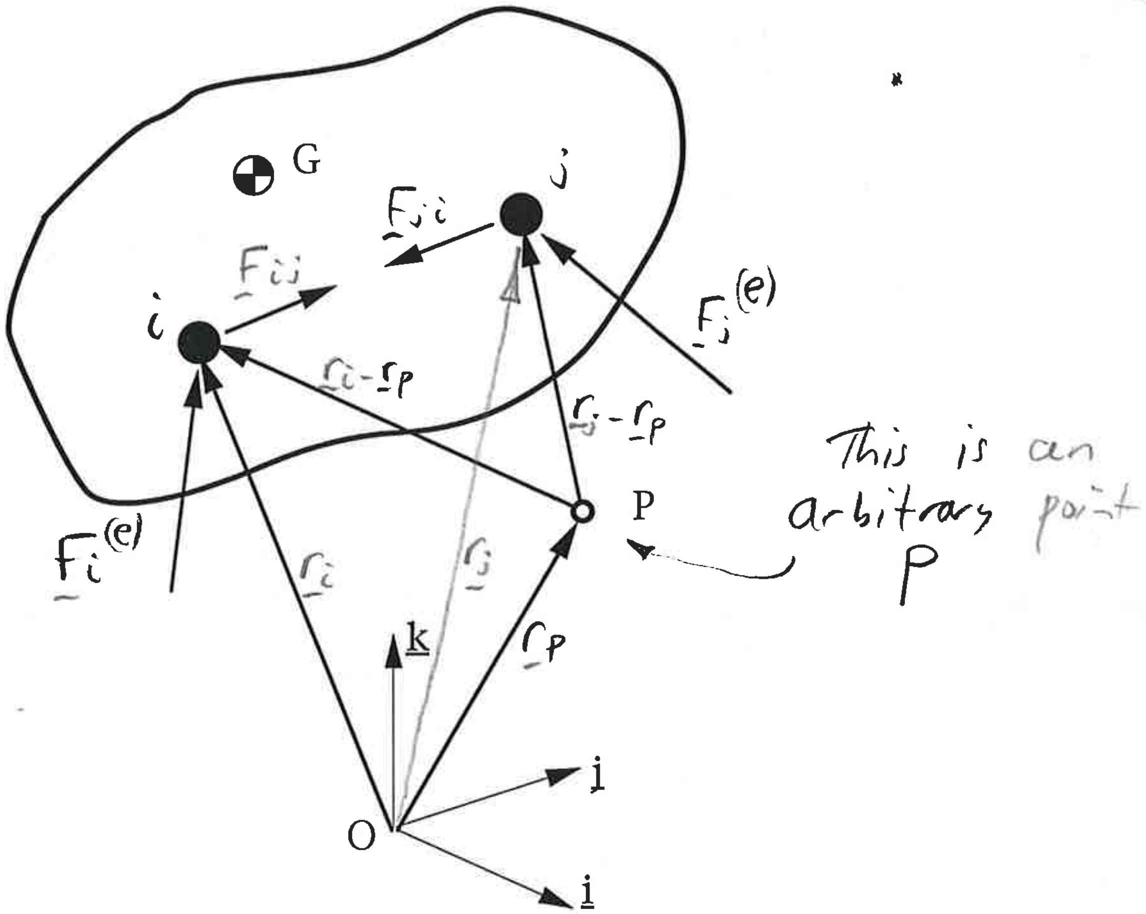
This seems like a trivial result

In fact, it's a beautiful result.

Good reason for focussing attention on "G" when ever possible.

Does 3D angular motion come out as simply as this?

2.3 Newton's Laws in three dimensions - moment of momentum



As before, apply Newton's 2nd law to particle i :

$$m_i \ddot{r}_i = F_i^{(e)} + \sum_{j \neq i} F_{ij} \qquad (2.1)$$

Take moments of (2.1) about an arbitrary (and not necessarily stationary) point P

$$(r_i - r_P) \times m_i \ddot{r}_i = (r_i - r_P) \times F_i^{(e)} + (r_i - r_P) \times \sum_{j \neq i} F_{ij}$$

and sum over all particles in the body

$$\sum_i (r_i - r_P) \times m_i \ddot{r}_i = \sum_i (r_i - r_P) \times F_i^{(e)} + \sum_i (r_i - r_P) \times \sum_{j \neq i} F_{ij}$$

$$= Q^{(e)} + 0 \qquad (2.5)$$

(moments of internal forces cancel in pairs)

where $Q^{(e)}$ is the total moment of external forces about P

Define the total moment of momentum about P as:

$$\boxed{\underline{h}_P = \sum_i (\underline{r}_i - \underline{r}_P) \times m_i \dot{\underline{r}}_i}$$

on data sheet (2.6)

Differentiate w r t time

$$\therefore \dot{\underline{h}}_P = \sum_i (\dot{\underline{r}}_i - \dot{\underline{r}}_P) \times m_i \dot{\underline{r}}_i + \sum_i (\underline{r}_i - \underline{r}_P) \times m_i \ddot{\underline{r}}_i$$

and from (2.5) subst $\underline{Q}^{(e)}$ so

$$\dot{\underline{h}}_P = -\dot{\underline{r}}_P \times \sum_i m_i \dot{\underline{r}}_i + \underline{Q}^{(e)}$$

but $\sum_i m_i \dot{\underline{r}}_i = M \dot{\underline{r}}_G = \underline{p}$ (the linear momentum)

(see section 2.2)

---

so $$\boxed{\underline{Q}^{(e)} = \dot{\underline{h}}_P + \dot{\underline{r}}_P \times \underline{p}}$$ (data sheet) (2.7)

where $\underline{Q}^{(e)}$ = total moment of external forces about P

$\dot{\underline{h}}_P$ = rate of change of total moment of momentum about P

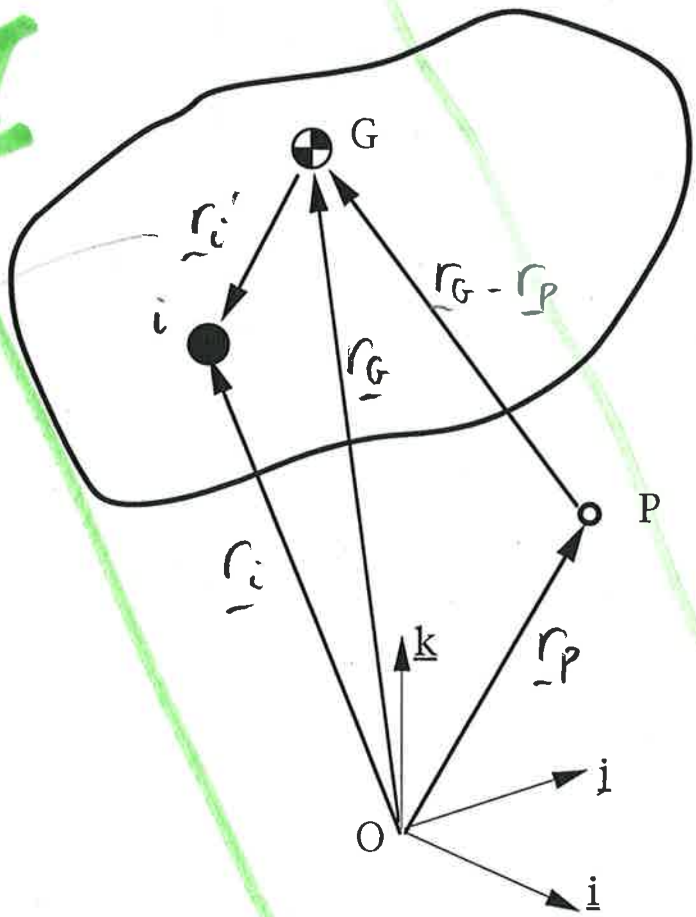
Don't forget it! $\dot{\underline{r}}_P \times \underline{p}$ accounts for the fact that point P may be moving

$\underline{p}$ = linear momentum $M \dot{\underline{r}}_G$

end L2 M03

2.4 Moment of momentum - special results for centre-of-mass

OMIT



$\underline{r}_i' =$ position vector wrt centre of mass

For the centre of mass G : $\sum m_i \underline{r}_i' = 0$ (2.8)

Recall (2.5) $\underline{Q}^{(e)} = \sum_i (\underline{r}_i - \underline{r}_P) \times m_i \ddot{\underline{r}}_i$

substitute $\underline{r}_i = \underline{r}_G + \underline{r}'_i$

$$\therefore \underline{Q}^{(e)} = \sum_i (\underline{r}_G + \underline{r}_i' - \underline{r}_P) \times m_i (\ddot{\underline{r}}_G + \ddot{\underline{r}}_i')$$

$$= (\underline{r}_G - \underline{r}_P) \times \left[\underbrace{\ddot{\underline{r}}_G \sum_i m_i}_{\dot{\underline{p}}} + \underbrace{\cancel{\sum m_i \ddot{\underline{r}}_i'}}_{0 \text{ from } 2.8} \right]$$

from (2.8) 0

$$+ \cancel{\sum_i m_i \underline{r}_i' \times \ddot{\underline{r}}_G} + \sum_i m_i \underline{r}_i' \times \ddot{\underline{r}}_i'$$

And $\underline{h}_G = \sum_i \underline{r}_i' \times m_i (\dot{\underline{r}}_G + \dot{\underline{r}}_i') = \sum_i m_i \underline{r}_i' \times \dot{\underline{r}}_i' + \underbrace{\cancel{\sum_i m_i \underline{r}_i' \times \dot{\underline{r}}_G}}_{0 \text{ from } 2.8}$

So $\dot{\underline{h}}_G = \sum_i m_i \underline{r}_i' \times \ddot{\underline{r}}_i' + \sum_i m_i \underbrace{\cancel{\dot{\underline{r}}_i' \times \dot{\underline{r}}_i'}}_{0}$

$$\therefore \boxed{\underline{Q}^{(e)} = (\underline{r}_G - \underline{r}_P) \times \dot{\underline{p}} + \dot{\underline{h}}_G} \quad (2.9)$$

where: $\underset{\sim}{Q}^{(e)}$ = total moment of external forces about P

$\underset{\sim}{\dot{p}}$ = rate of change of linear momentum

$\underset{\sim}{\dot{h}}_G$ = rate of change of moment of momentum about G

Note special results:

- From (2.7), if P is a fixed point $\underset{\sim}{r}_P = 0$

$$\underset{\sim}{Q}^{(e)} = \underset{\sim}{\dot{h}}_P \tag{2.10}$$

- From (2.9), if P is coincident with G $\underset{\sim}{r}_G \quad \underset{\sim}{r}_P = 0$

$$\underset{\sim}{p} = m\underset{\sim}{\dot{r}}_G = m\underset{\sim}{\dot{r}}_P$$

$$\text{So} \quad \underset{\sim}{\dot{r}}_P \times \underset{\sim}{p} = 0$$

$$\therefore \quad \underset{\sim}{Q}^{(e)} = \underset{\sim}{\dot{h}}_G \tag{2.11}$$

These are familiar – and elegant.

We have found that the familiar PLANAR results in 2D apply in 3D provided we use G or some other *fixed* point P as our reference for taking moment of momentum.

We often drop subscripts:

$$\underset{\sim}{F} = \underset{\sim}{\dot{p}}$$

$$\underset{\sim}{Q} = \underset{\sim}{\dot{h}} \quad \text{moments assumed about G or a fixed point P}$$

But why don't these describe 3D motion?

In section 4 we will discover why it is that these equations do not seem to describe 3D motion, but before doing so we will do an example and we will introduce some definitions of moments of inertia in 3D

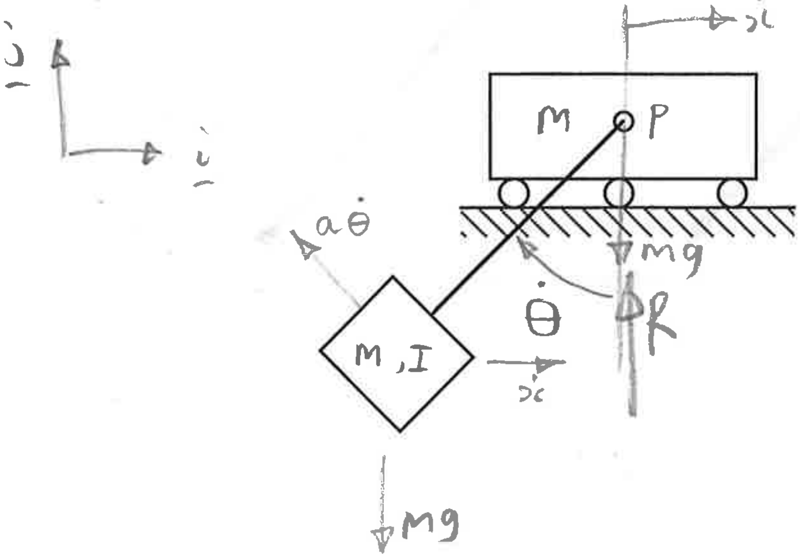
M00

end L2 M07 $\boxed{\text{End L2 Mic97}}$

Mic1 M99

2.5 Example - the momentum equations applied to plane motion

Consider the system from the example given in section 2.1:



We want to use equation 2.7 to check that it works

$$\underline{Q}^{(e)} = \dot{\underline{h}}_P + \dot{\underline{r}}_P \times \underline{p} \qquad (2.7)$$

From 2.6 we get $\underline{h}_P$ and we differentiate to get $\dot{\underline{h}}_P$

$$\underline{h}_P = \sum m_i (\underline{r}_i - \underline{r}_P) \times \dot{\underline{r}}_i = (-ma^2\dot\theta - I\dot\theta + ma\cos\theta\,\dot x)\,\underline{k}$$

$$\therefore \dot{\underline{h}}_P = (-ma^2\ddot\theta - I\ddot\theta + ma\cos\theta\,\ddot x - ma\dot\theta\sin\theta\,\dot x)\,\underline{k}$$

We get $\dot{\underline{r}}_P$ easily $\quad \dot{\underline{r}}_P = \dot x\,\underline{i} \quad$ velocity of point P

and $\underline{p}$ is the total linear momentum

$$\underline{p} = (M\dot x + m\dot x - ma\dot\theta\cos\theta)\,\underline{i} + ma\dot\theta\sin\theta\,\underline{j}$$

$$\& \ \underline{Q}^{(e)} = mga\sin\theta\,\underline{k}$$

Now use (2.7) to get

$$mga\sin\theta\,\underline{k} = (-ma^2\ddot\theta - I\ddot\theta + ma\cos\theta\,\ddot x - \cancel{ma\dot\theta\sin\theta\,\dot x})\,\underline{k} + \dot x\,\cancel{ma\dot\theta\sin\theta}\,\underline{k}$$

$$\therefore m\ddot x\,a\cos\theta - mga\sin\theta - ma^2\ddot\theta - I\ddot\theta = 0$$

just as in Section 2.1

End L2 M'05

We use 2.4 to get the other equations of motion: $\dot{\underline{p}} = \underline{F}^{(e)}$

~~horizontal:~~

$$\frac{d}{dt}\left(2m\dot{x} - ma\dot{\theta}\cos\theta\right)\underline{i} + \frac{d}{dt}\left(ma\dot{\theta}\sin\theta\right)\underline{j} = (R - 2mg)\underline{j}$$

~~vertical:~~

$$\therefore \quad 2m\dot{x} - ma\dot{\theta}\cos\theta = \text{const}$$

and $$R = ma\ddot{\theta}\sin\theta + ma\dot{\theta}^2\cos\theta + 2mg$$

The method may seem a bit involved but it is guaranteed to work in complicated 3D problems. It takes practice.

2.6 Summary

- 3D translational motion

$$\underline{F}^{(e)} = \dot{\underline{p}}$$

- 3D rotational motion

$$\underline{Q}_P^{(e)} = \dot{\underline{h}}_P + \dot{\underline{r}}_P \times \underline{p}$$

- special cases $\dot{r}_P = 0$ or P is at G

$$\underline{Q}_P^{(e)} = \dot{\underline{h}}_P$$

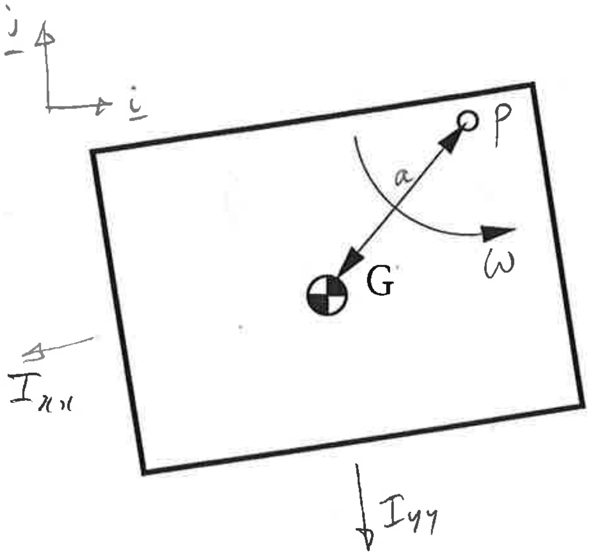
Calculation of $\underline{h}$ for rigid bodies is also a special case: see section 3

End L3 M03

Do questions 1 and 2 of examples paper G7/1

3. INERTIA OF A RIGID BODY

3.1 Revision - inertia of a lamina in plane motion



Moment of momentum about G

$$\underset{\sim}{h}_G = I_G \, \omega \, \underset{\sim}{k}$$

About P (assumed fixed, rarely stated !!)

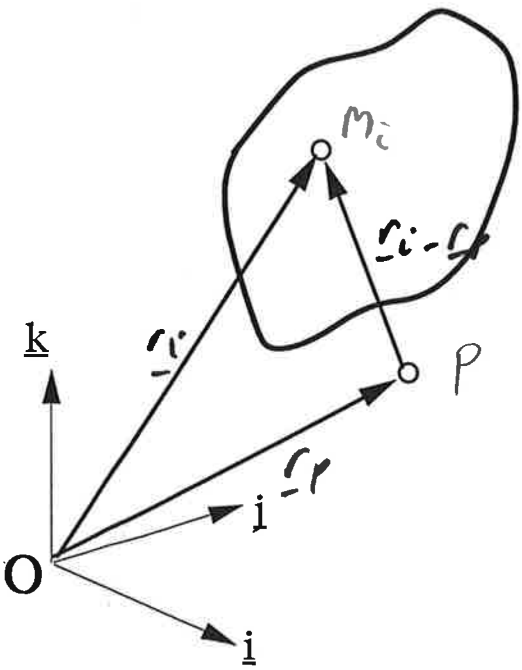
$$\underset{\sim}{h}_P = I_P \, \omega \, \underset{\sim}{k}$$

Parallel axis theorem : $I_P = I_G + m a^2$

Perpendicular axis theorem: $I_{xx} + I_{yy} = I_{zz}$

end L3 Mon

3.2 Inertia matrix



Moment of momentum about P for a collection of particles is (from 2.6)

$$\underset{\sim}{h}_P = \sum_i (\underset{\sim}{r}_i - \underset{\sim}{r}_P) \times m_i \dot{\underset{\sim}{r}}_i$$

Let point P be stationary and at the origin & fixed in the body so $\underline{r}_P = \underline{0}$

Angular velocity of body : $\underline{\omega} = \omega_1 \underline{i} + \omega_2 \underline{j} + \omega_3 \underline{k}$

(this is an arbitrary but usual notation)

velocity of particle i : $\dot{\underline{r}}_i = \underline{\omega} \times \underline{r}_i$ (part IA Mech)

so $\underline{h}_P = \sum_i m_i \underline{r}_i \times (\underline{\omega} \times \underline{r}_i)$ vector triple product

$= \sum_i m_i \left((\underline{r}_i \cdot \underline{r}_i)\underline{\omega} - (\underline{r}_i \cdot \underline{\omega})\underline{r}_i \right)$

and write $\underline{r}_i = x_i \underline{i} + y_i \underline{j} + z_i \underline{k}$

$$\text{so } \underline{h}_P = \sum_i m_i \left\{ (x_i^2 + y_i^2 + z_i^2) \begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{Bmatrix} - (x_i \omega_1 + y_i \omega_2 + z_i \omega_3) \begin{Bmatrix} x_i \\ y_i \\ z_i \end{Bmatrix} \right\}$$

shorthand for vector $\omega_1 \underline{i} + \omega_2 \underline{j} + \omega_3 \underline{k}$

---

or in matrix form,

$$\underline{h}_P = \begin{bmatrix} \sum m_i (y_i^2 + z_i^2) & -\sum m_i x_i y_i & -\sum m_i x_i z_i \\ -\sum m_i y_i x_i & \sum m_i (x_i^2 + z_i^2) & -\sum m_i y_i z_i \\ -\sum m_i z_i x_i & -\sum m_i z_i y_i & \sum m_i (x_i^2 + y_i^2) \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \begin{matrix} \underline{i} \\ \underline{j} \\ \underline{k} \end{matrix} \quad (3.1)$$

$\underline{h}_P = [I_P]\, \underline{\omega}$ is the preferred shorthand (3.1a)

$[I_P]$ is the *inertia matrix* or *inertia tensor*

- Always symmetric
- Only valid about chosen point P and chosen with axes
- Diagonal elements are called "Moments of inertia"
- Off-diagonal elements are called "Products of inertia"

end L 02

3.3 Principal moments of inertia

We have eq 3.1a $\underline{h}_P = [I_P]\ \underline{\omega}$

but we are used to problems where $\underline{h}$ is parallel to $\underline{\omega}$

eg $$\underset{\sim}{h}_P = \lambda\ \underset{\sim}{\omega} \qquad (\lambda \text{ is a scalar}) \tag{3.2}$$

which leads to a neat Eigenvalue problem

$$[I_P]\ \underset{\sim}{\omega} = \lambda\ \underset{\sim}{\omega} \tag{3.3}$$

The three eigenvalues (λ) of I_P and the three corresponding eigenvectors define the axes about which the body can rotate while maintaining $\underset{\sim}{h}$ parallel to $\underset{\sim}{\omega}$

- The three eigenvalues are the *principal moments of inertia*

We call them A B & C

- The three eigenvectors are the *principal axes of inertia*

They are orthogonal

- If we align our axes with the principle axes, then

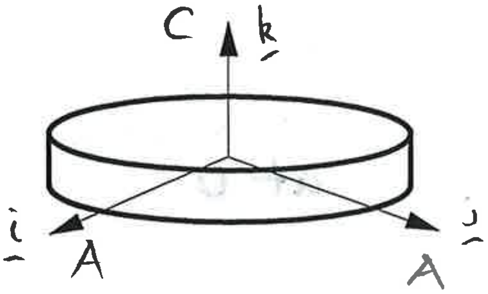
$$I_P = \begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{bmatrix}$$

Questions ~~5~~6d and ~~5~~6e on examples paper G7/1 involve calculating eigenvalues and eigenvectors of [I] to find the principal axes and moments of inertia.

end L3 M07

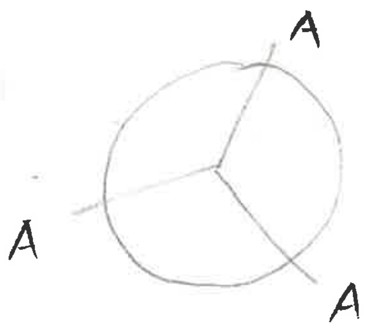
end L3 mich 97

Note: Generally $\underline{h}$ is not parallel to $\underline{\omega}$ except for rotation about a principal axis. There are special cases:



Cylinder, Disc or Square plate "AAC"

All axes in the $\underline{i}$ - $\underline{j}$ plane are principal as you find if you try to calculate eigenvectors



Sphere or cube "AAA"

All axes are principal

A cube is "equivalent" to a sphere!

End L3 M99

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3.4 Parallel axes theorem

We can compute $[I_P]$ at point P given $[I_G]$ at centre of mass

$$[I_P] = [I_G] + m \begin{bmatrix} y^2+z^2 & -xy & -xz \\ -yx & x^2+z^2 & -yz \\ -zx & -zy & x^2+y^2 \end{bmatrix}$$

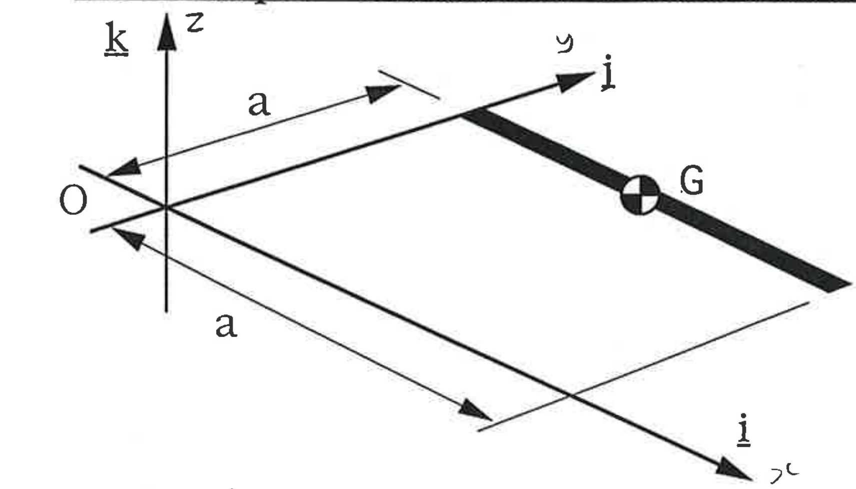
Where x, y, z are coordinates of point P relative to G

end L3 M05

A good place to end!

Prove this theorem yourself in question 4a of examples paper G7/1

3.5 Example - calculation of moments and products of inertia



Uniform rod
length a
mass m
find I_{zz} and I_{xy}
at O

From (3.1), $I_{zz} = \sum m_i (x_i^2 + y_i^2) = \int (x^2 + y^2)\, dm$

with $dm = \frac{m}{a} dx$ and $y = a$ (const)

$$\therefore \quad I_{zz} = \int_0^a (x^2 + a^2) \frac{m}{a}\, dx = \frac{m}{a}\left[\frac{x^3}{3} + a^2 x\right]_0^a = \frac{4}{3} m a^2$$

Check : $I_{zz} = \frac{ma^2}{12} + m\left(\left(\frac{a}{2}\right)^2 + a^2\right) = \underline{\frac{4}{3} m a^2}$ ✓

Data book — parallel axes theorem

$$I_{xy} = \cancel{A} \sum m_i x_i y_i \qquad \text{from (3.1)}$$

$$= \cancel{A} \int x y\, dm$$

$$= A \int_0^a x\, a \frac{m}{a}\, dx = A\left[\frac{x^2}{2} m\right]_0^a = \cancel{A} \frac{m a^2}{2}$$

Not easy to check ! (or even to interpret)

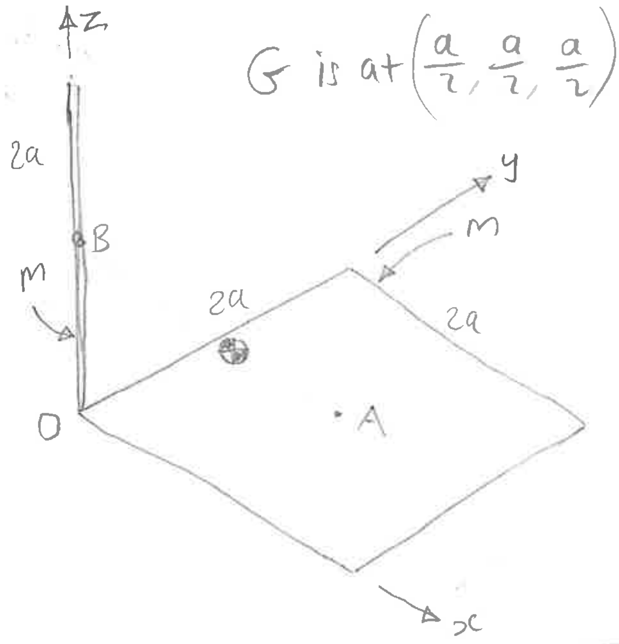
NEW EXAMPLE

3.6 Summary

Moment of momentum of a rigid body is given by

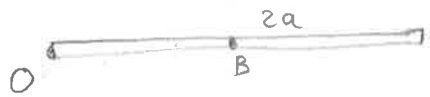
- $\underset{\sim}{h}_P = [I_P]\, \underset{\sim}{\omega}$ where $[I_P]$ is the 3×3 inertia matrix, **about axes through P**
- Principal axes & principal moments of inertia are the eigen vectors & values of $[I_P]$
- Two bodies with identical mass and principal moments of inertia are identical. (Sphere ≡ cube)

Do questions ~~3~~, ~~4~~ and ~~5~~ on examples paper G7/1
4 5 6/7



G is at $\left(\frac{a}{2}, \frac{a}{2}, \frac{a}{2}\right)$

- look first at rod

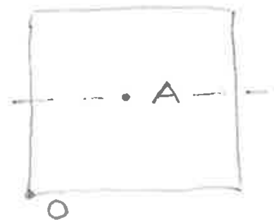


Data book $I_B = \frac{1}{12} m (2a)^2 = \frac{1}{3} ma^2$

$\parallel^l$ axis theorem $I_O = \frac{1}{3} ma^2 + ma^2$

$= \frac{4}{3} ma^2$

- then the plate



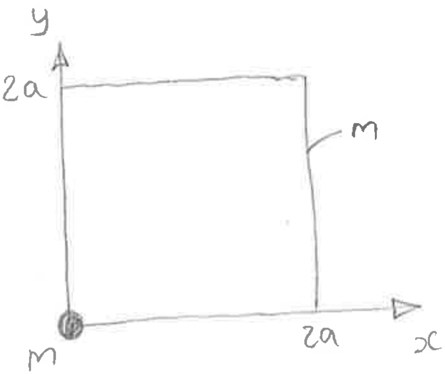
Data book $I_A = \frac{1}{12} m (2a)^2$

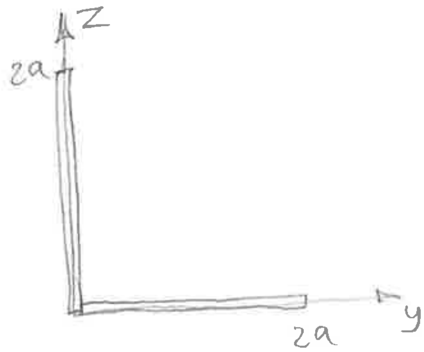
$= \frac{1}{3} ma^2$

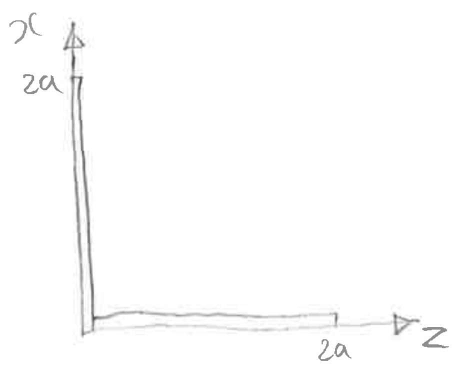
$\parallel^l$ axis : $I_O = \frac{4}{3} ma^2$

$\perp$ axis theorem $J_O = 2 \times \frac{4}{3} ma^2 = \frac{8}{3} ma^2$

View from each direction







$I_{zz} = \int (x^2+y^2)\, dm = \frac{8}{3} ma^2$

$I_{xy} = \int xy\, dm = \int_0^{2a}\int_0^{2a} xy\, dx\, dy \frac{m}{(2a)^2}$

$= \frac{1}{2}(2a)^2 \ \frac{1}{2}(2a)^2 \frac{m}{(2a)^2} = ma^2$

$I_{xx} = \int (y^2+z^2)\, dm = 2 \times \frac{4}{3} ma^2 = \frac{8}{3} ma^2$

$I_{yz} = \int yz\, dm = 0$

$I_{yy} = \frac{8}{3} ma^2$

$I_{zx} = 0$

$$I_O = \frac{ma^2}{3} \begin{bmatrix} 8 & -3 & 0 \\ -3 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

For I_A can use parallel axes theorm but need to use it <u>twice</u> : first $O \to G$ $\quad I_O = I_G + [\ \ ]$

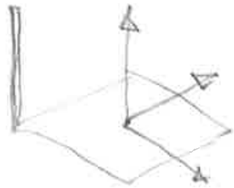
then $G \to A$ $\quad I_A = I_G + [\ \ ]$

So for I_G use $(x, y, z) = \left(\frac{-a}{2}, \frac{-a}{2}, \frac{-a}{2}\right)$ = position of O w.r.t G

& for I_A then use $(x, y, z) = \left(\frac{a}{2}, \frac{a}{2}, \frac{-a}{2}\right)$ A w.r.t G

∴

This gives $\quad I_A = \frac{ma^2}{3}\begin{bmatrix} 8 & -3 & 3 \\ -3 & 8 & 3 \\ 3 & 3 & 8 \end{bmatrix}$



For principal axes find eigenvalues & vectors

$$\begin{vmatrix} 8-\lambda & -3 & 3 \\ -3 & 8-\lambda & 3 \\ 3 & 3 & 8-\lambda \end{vmatrix} = 0$$

which gives $(\lambda - 2)(\lambda - 11)^2 = 0$

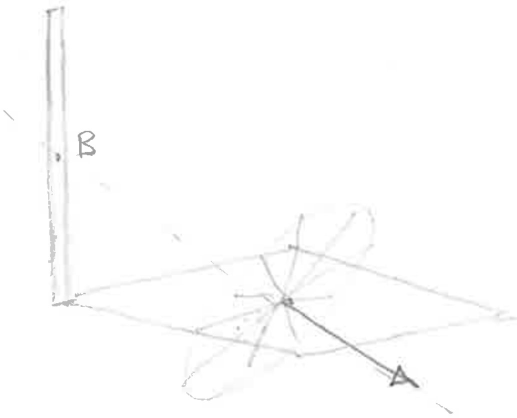
So $I_1 = \frac{11ma^2}{3}$

$I_2 = \frac{11ma^2}{3}$

$I_3 = \frac{2ma^2}{3}$ $\quad (1, 1, -1)$

eigenvalues are principal moments of inertia

eigenvectors are principal axes



Note repeated eigenvalues

∴ this is an AAC body

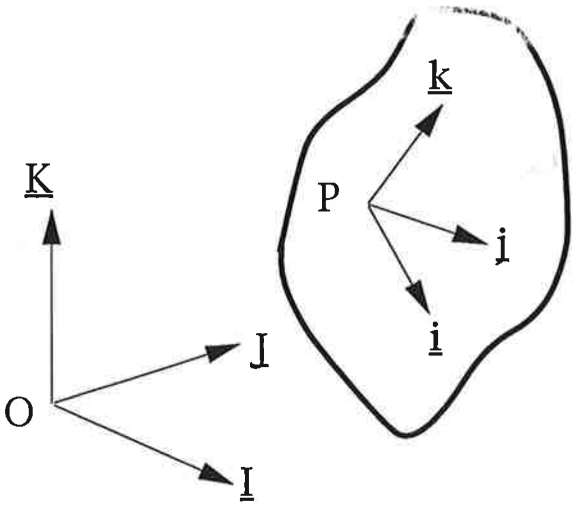
4. Euler's equations

4.1 Body-fixed reference frame

$\underset{\sim}{h}_P = [I_P]\,\underset{\sim}{\omega}$ is fine, but as the body rotates the elements of $[I_P]$ change.

Either • find $[I_P]$ as a function of time (hard)

or • fix a set of axes in the body aligned with the principal axes for ultimate simplicity.



The $\underline{\mathbf{i}}$ $\underline{\mathbf{j}}$ $\underline{\mathbf{k}}$ axes are body-fixed axes

The $\underline{\mathbf{I}}$ $\underline{\mathbf{J}}$ $\underline{\mathbf{K}}$ axes are ground-fixed axes ("inertial")

- Align $\underline{\mathbf{i}}$ $\underline{\mathbf{j}}$ $\underline{\mathbf{k}}$ with principal axes

    Easy by inspection for most bodies

- Define instantaneous angular velocity $\underline{\boldsymbol{\omega}}$ in body-fixed axes $\underset{\sim}{i}$ $\underset{\sim}{j}$ $\underset{\sim}{k}$

$$\underset{\sim}{\omega} = \omega_1 \underset{\sim}{i} + \omega_2 \underset{\sim}{j} + \omega_3 \underset{\sim}{k}$$

We'll account for motion of $\underset{\sim}{i}$ $\underset{\sim}{j}$ & $\underset{\sim}{k}$ just like we did with $\dot{\underset{\sim}{e}} = \omega \underset{\sim}{e}^*$ last year.

We can then write the moment of momentum of the body about P as:

$$\underset{\sim}{h}_P = \begin{bmatrix} A & & 0 \\ & B & \\ 0 & & C \end{bmatrix} \underset{\sim}{\omega} = A\omega_1 \underset{\sim}{i} + B\omega_2 \underset{\sim}{j} + C\omega_3 \underset{\sim}{k} \tag{4.1}$$

and since $\underline{i}$ $\underline{j}$ $\underline{k}$ move with the body this equation *always* holds.

Recall : $\underset{\sim}{r} = r \, \underset{\sim}{e}_r$

unit vector

scalar

$\therefore \; \dot{\underset{\sim}{r}} = \dot{r} \, \underset{\sim}{e}_r + r \, \dot{\underset{\sim}{e}}_r$

$= \dot{r} \, \underset{\sim}{e}_r + r \, \underset{\sim}{\omega} \times \underset{\sim}{e}_r$

$= \dot{r} \, \underset{\sim}{e}_r + \underset{\sim}{\omega} \times \underset{\sim}{r}$

$\dot{\underset{\sim}{r}}\big|_F = \dot{\underset{\sim}{r}}\big|_R + \underset{\sim}{\omega} \times \underset{\sim}{r}$ (See Mech Data Book 1.3.2)

Differentiating equation 4.1, in the same way

$$\dot{\underset{\sim}{h}}_P\big|_F = \dot{\underset{\sim}{h}}_P\big|_R + \underset{\sim}{\omega} \times \underset{\sim}{h}_P \tag{4.2}$$

true value of $\dot{\underline{h}}_P$ in fixed frame

value of $\dot{\underline{h}}_P$ calculated in rotating frame as if $\underline{i}$ $\underline{j}$ $\underline{k}$ were fixed

Correction due to rotation of $\underline{i}$ $\underline{j}$ $\underline{k}$ frame

We will consider only cases where either P is stationary or P is at G (why?) Special result $\dot{\underline{h}} = \underline{Q}$ applies

so use (2.10) or (2.11)

$$\dot{\underline{h}}_P = \underline{Q}^{(e)}$$

Now find expressions for the $\underline{i}\,\underline{j}\,\underline{k}$ components of both sides

$$\underline{Q}^{(e)} = Q_1\,\underline{i} + Q_2\,\underline{j} + Q_3\,\underline{k} \qquad \text{(say)}$$

and $$\dot{\underline{h}}_P = \dot{\underline{h}}_P\big|_R + \underline{\omega} \times \underline{h}_P \qquad (4.2)$$

where $$\underline{h}_P = A\omega_1\,\underline{i} + B\omega_2\,\underline{j} + C\omega_3\,\underline{k} \qquad (4.1)$$

so $$\dot{\underline{h}}_P\big|_R = A\dot{\omega}_1\,\underline{i} + B\dot{\omega}_2\,\underline{j} + C\dot{\omega}_3\,\underline{k}$$

and $$\underline{\omega} = \omega_1\,\underline{i} + \omega_2\,\underline{j} + \omega_3\,\underline{k}$$

Next, equate $\underline{i}\,\underline{j}\,\underline{k}$ components

noting $\underline{\omega} \times \underline{h}_P = (\omega_2 C\omega_3 - \omega_3 B\omega_2)\,\underline{i}$ + etc.

$\therefore$

$$\boxed{\begin{aligned} A\dot{\omega}_1 - (B-C)\,\omega_2\omega_3 &= Q_1 \\ B\dot{\omega}_2 - (C-A)\,\omega_3\omega_1 &= Q_2 \\ C\dot{\omega}_3 - (A-B)\,\omega_1\omega_2 &= Q_3 \end{aligned}} \qquad (4.3)$$

on data sheet

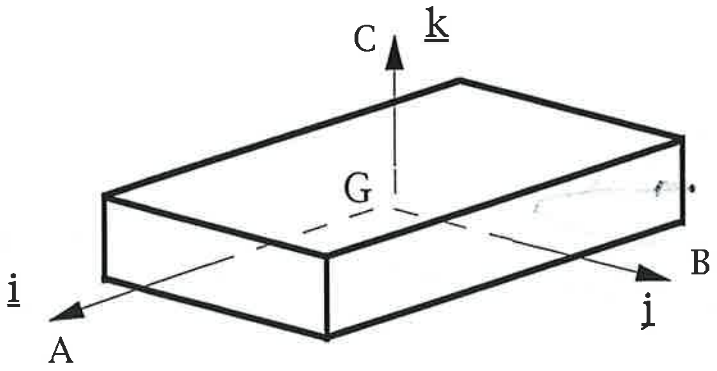
Note cyclic symmetry

These are *Euler's equations* in a body-fixed axis frame. They determine the angular motion of a body subject to an external couple $\underline{Q}$ about a point P which MUST either be a FIXED POINT or at G .

end L4 M01

$Q = 0$

4.2 Example - stability of free rotation about a principal axis



Choose body-fixed axes aligned with principal axes

$$I_G = \begin{bmatrix} A & & 0 \\ & B & \\ 0 & & C \end{bmatrix}$$

Spin the body about it's $\underline{i}$ axis with angular velocity Ω.
Write down the angular velocity vector and perturb the motion by a very small amount.

Steady state : $\underset{\sim}{\omega} = \Omega \underset{\sim}{i} \quad = \text{const}$

perturbation : $\underset{\sim}{\omega} = (\Omega + \omega_1') \underset{\sim}{i} + \omega_2' \underset{\sim}{j} + \omega_3' \underset{\sim}{k}$

small

Substitute the angular velocity expressions into Euler's equations

$Q = 0$

$$A\dot{\omega}_1' - (B - C)\,\omega_2'\,\omega_3' = 0$$

$$B\dot{\omega}_2' - (C - A)\,\omega_3'(\Omega + \omega_1') = 0$$

$$C\dot{\omega}_3' - (A - B)(\Omega + \omega_1')\,\omega_2' = 0$$

and ignore 2nd order terms

$$A\dot{\omega}_1' \simeq 0 \qquad (4.4a)$$

$$B\dot{\omega}_2' - (C - A)\,\omega_3'\,\Omega \simeq 0 \qquad (4.4b)$$

$$C\dot{\omega}_3' - (A - B)\,\omega_2'\,\Omega \simeq 0 \qquad (4.4c)$$

Ed.

$(4.4a) \longrightarrow \omega_1' \simeq \text{constant}$

Follow this exactly

Differentiate (4.4b)

$$B\ddot{\omega}_2' - (C-A)\Omega\dot{\omega}_3' = 0 \qquad (4.5)$$

and substitute $\dot{\omega}_3'$ from ~~into~~ (4.4c)

$$\therefore \quad \ddot{\omega}_2' + \frac{(A-C)(A-B)}{BC}\Omega^2 \omega_2' = 0 \qquad (4.6)$$

$$\text{or} \quad \ddot{\omega}_2' + \lambda^2 \omega_2' = 0$$

And if $\lambda^2 > 0 \rightarrow$ SHM ie stable

$\lambda^2 < 0 \rightarrow e^{\lambda t}$ ie unstable

At last we can explain why the spinning motion of a rigid body is unstable about it's intermediate moment of inertia.

$\lambda^2 > 0$ when $A-C$ and $A-B$ have the same sign ie $A>C$ & $A>B$

or $A<C$ & $A<B$

So for STABLE motion A must be the largest or smallest principal inertia

$\lambda^2 < 0 \quad \therefore \quad A$ is the middle principal inertia $\rightarrow$ unstable motion

exactly as observed.

End L4 M00

This wonderfully simple result can be demonstrated with a book, a tennis racquet, an oval antique bone-china dinner plate . . .

End L5 M05

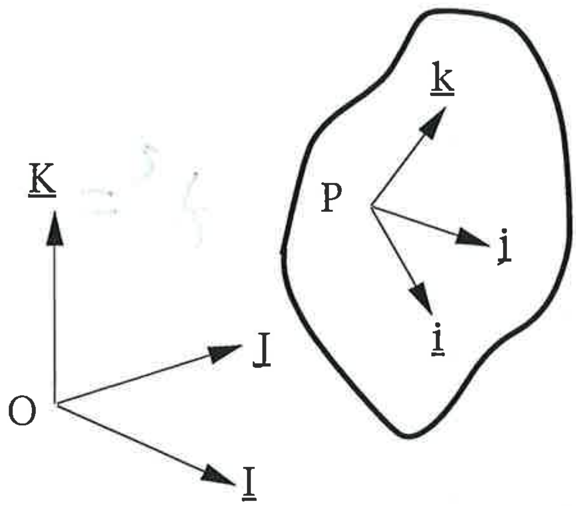
Now Do EP1 Q6

END L5 M07 with example

4.3 Non-body-fixed reference frame

In section 4.1 Euler's equations were derived using body-fixed axes so as to maintain alignment with $\underline{i}\ \underline{j}\ \underline{k}$. For bodies with axi-symmetry (cylinders, discs, ellipsoids, spheres) we can use non-body-fixed axes.

We align an axis with a principal axis.



Lets be general first.

As before, body angular velocity

$$\underline{\omega} = \omega_1 \underline{i} + \omega_2 \underline{j} + \omega_3 \underline{k}$$

The reference frame $\underline{i}\ \underline{j}\ \underline{k}$ is *not* fixed in the body. The reference frame moves with angular velocity $\underline{\Omega}$ while the body itself has angular velocity $\underline{\omega}$. $\underline{\Omega} \neq \underline{\omega}$!

$$\underline{\Omega} = \Omega_1 \underline{i} + \Omega_2 \underline{j} + \Omega_3 \underline{k}$$

Use $\dot{\underline{h}}_P = \underline{Q}^{(e)}$ as usual about fixed P or G

Consider only bodies which move so as to keep $\underline{i}\ \underline{j}\ \underline{k}$ aligned with principal axes (must have axisymmetry). We can still use (4.1)

$$\underline{h}_P = A\omega_1 \underline{i} + B\omega_2 \underline{j} + C\omega_3 \underline{k} \quad (4.1)$$

Differentiating (4.1) gives (noting that angular velocity of reference frame is now $\underline{\Omega}$ and *not* $\underline{\omega}$)

$$\dot{\underline{h}}_P = \dot{\underline{h}}_P\big|_R + \underline{\Omega} \times \underline{h}_P = \underline{Q}^{(e)}$$

we had $\underline{\omega}$ before

$$\therefore \quad \boxed{\begin{aligned} A\dot{\omega}_1 - (B\,\omega_2\Omega_3 - C\,\omega_3\Omega_2) &= Q_1 \\ B\dot{\omega}_2 - (C\,\omega_3\Omega_1 - A\,\omega_1\Omega_3) &= Q_2 \\ C\dot{\omega}_3 - (A\,\omega_1\Omega_2 - B\,\omega_2\Omega_1) &= Q_3 \end{aligned}} \qquad (4.7)$$

Check this for yourselves.

- (4.7) are *Euler's equations* in a *non*-body-fixed axis frame.

  ie $\underline{\Omega} \neq \underline{\omega}$

- If $\underline{\Omega} = \underline{\omega}$ then the simpler Euler equations (4.3) are obtained.

  ie body fixed axes $\underline{i}\ \underline{j}\ \underline{k}$

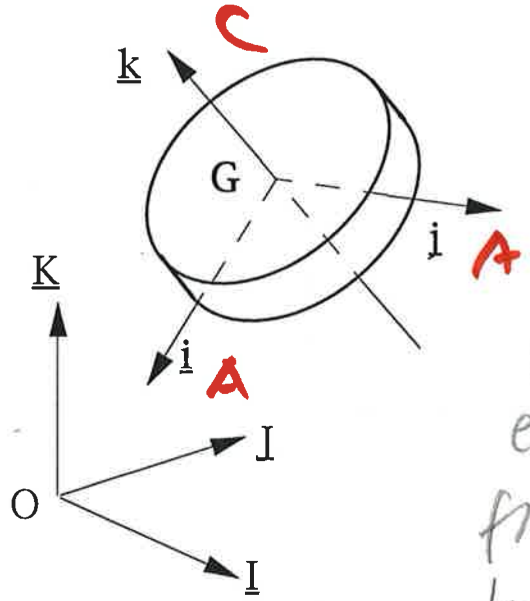
- As usual, for equations (4.7) to hold, point P MUST either be a FIXED POINT or at G .

- (4.7) can only really work with axisymmetry and simplifications result as follows in section 4.4.

we haven't incorporated the axisymmetry yet.

4.4 Non-body-fixed reference frame for axisymmetric bodies (The gyroscope equations)

Axisymmetric bodies are "A A C" instead of "A B C" ie A = B



The k axis is aligned with the symmetry axis

i j k are always principal. This works for cylinders, discs, ellipsoids, tops, prisms, flat square plates, beer bottles, ... Anything with AAC

So put A = B into 4.7 and note that $\omega_1 = \Omega_1$ & $\omega_2 = \Omega_2$ since k is always moving with the body

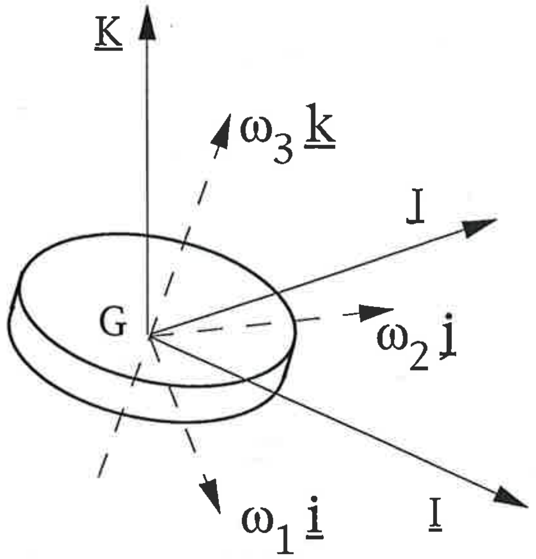
$$\therefore \boxed{\begin{aligned} A\dot{\Omega}_1 - (A\,\Omega_3 - C\,\omega_3)\,\Omega_2 &= Q_1 \\ A\dot{\Omega}_2 + (A\,\Omega_3 - C\,\omega_3)\,\Omega_1 &= Q_2 \\ C\,\dot{\omega}_3 &= Q_3 \end{aligned}} \qquad (4.8)$$

in data sheet

These are the Gyroscope Equations. Applications will be given in sections 5 to 8 later in this course.

(end L603 way behind!)

4.5 Euler's angles



How do we solve ~~Euler's~~ the Gyro equations?

We can't just integrate them because the reference frame $\underline{i}\ \underline{j}\ \underline{k}$ moves.

Try an example for yourself rotating a book by 90° about one body fixed axis & then the other. The order matters!

- We want to find the variation with time of the body's orientation and we need a set of coordinates that relate to the fixed frame $\underline{I}\ \underline{J}\ \underline{K}$

- Direct integration doesn't work because the reference frame is always moving

we'd have to do $\theta_1(t) = \int_0^T \omega_1(t)\, \underline{i}(t)\, dt$

which is not easy

- Let's invent *unique* angular coordinates - Euler's angles.

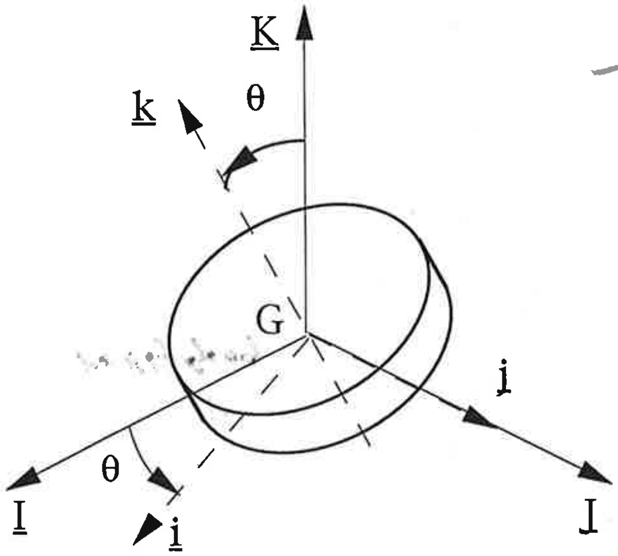
$\theta\ ,\ \phi\ ,\ \psi$

I J K is the absolute reference frame.

K is "vertical" in most problems

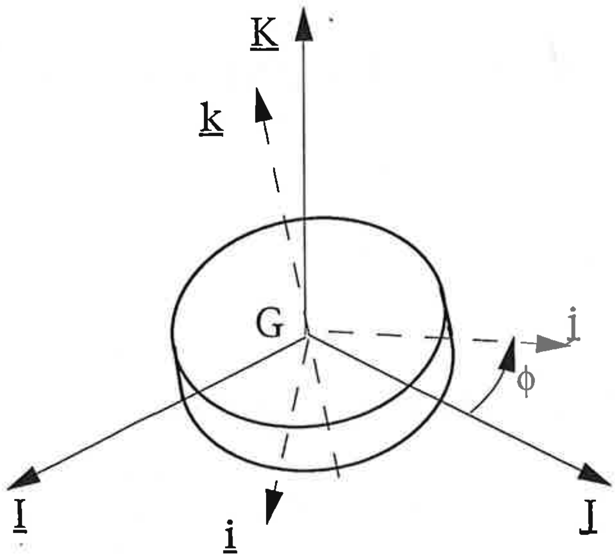
i j k is the body-fixed reference frame, initially aligned with I J K

First tilt by θ then turn by ϕ then spin by ψ



Step 1

Tilt the i j k axis frame by θ about the j axis



Step 2

Turn the i j k frame by ϕ about K axis

Note that j is always horizontal.

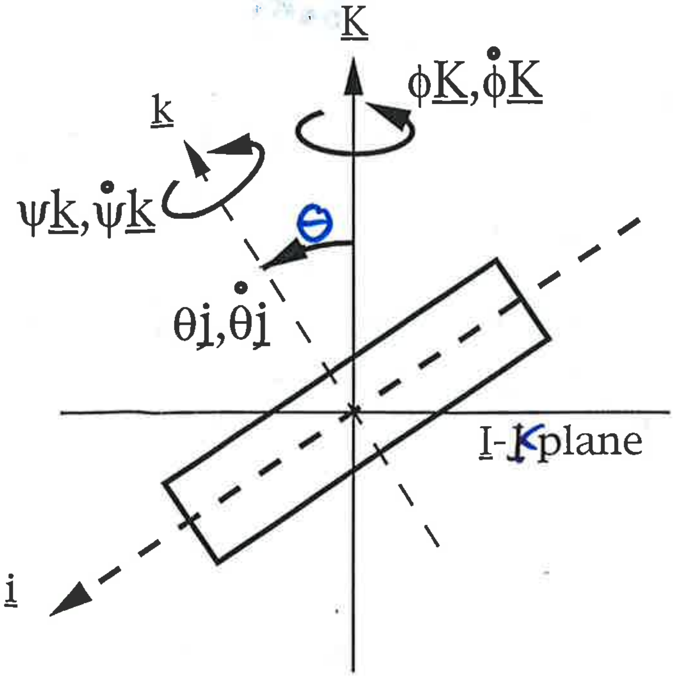
Step 3

Spin the body by ψ about the k axis

The Euler's Angles θ ϕ ψ completely specify the angular position of the body. They are independently integrable.

Obtaining Ω_1, Ω_2 Ω_3 from θ, ϕ, ψ

View along the $\underline{j}$ axis and resolve $\dot{\theta}\,\underline{j}$ & $\dot{\phi}\,\underline{K}$ into $\Omega_1 \underline{i} + \Omega_2 \underline{j} + \Omega_3 \underline{k}$



Not on Data Sheet

x.s

$$\therefore \boxed{\begin{aligned} \Omega_1 &= -\dot{\phi}\sin\theta \\ \Omega_2 &= \dot{\theta} \\ \Omega_3 &= \dot{\phi}\cos\theta \end{aligned}} \quad \cancel{(4.9)}$$

Body angular velocity $\underline{\omega}$:

$\omega_1 = \Omega_1 \qquad \omega_2 = \Omega_2$

$\omega_3 = \Omega_3 + \dot{\psi}$

$\dot{\psi}$ is often called "the spin" of a rotor. It is the relative angular velocity (about $\underline{k}$) between the rotor & the reference frame.

end LS M01 (fast!)

4.6 Summary

- Euler's equations in body-fixed reference frame

Equations (4.3) relate $\underline{\omega}$ to external couple $\underline{Q}$ about body fixed point P or G.

- Gyroscope equations for axisymmetric bodies

Equations 4.8 relate $\underline{\Omega}$ & ω_3 to the external couple.

Very wide range of applications, especially gyroscopes, satellites, rolling bodies

---

- Euler's angles θ ϕ ψ

define the motion of an axisymmetric body without ambiguity

[not strictly true - if $\theta = 0$ we can't distinguish between ϕ & ψ. Try to avoid this!]

Do question ~~6~~ 7 on examples paper G7/1