3F1 Signals and Systems

(5) Stability

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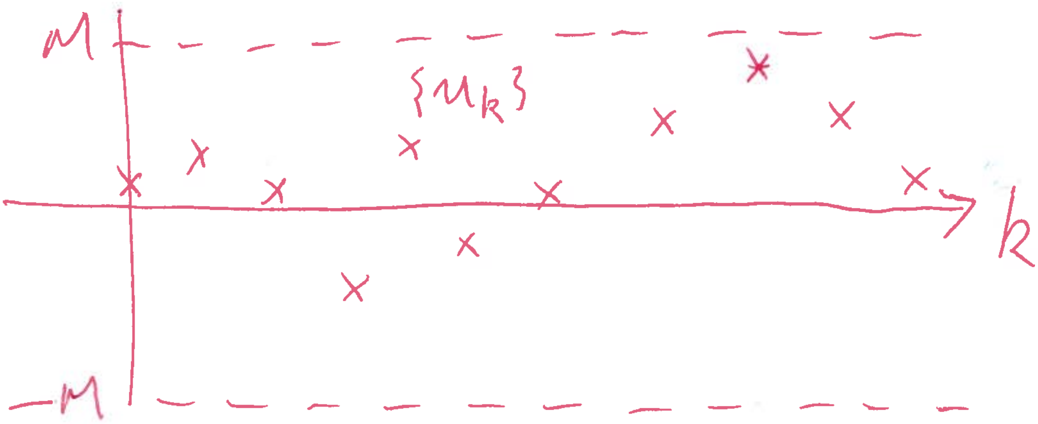
BIBO stability

We say that a signal $\{u_k\}$ is **bounded** if there exists a positive constant M such that $|u_k| < M$ for all k.



A discrete time system is **stable** if
bounded inputs give bounded outputs (BIBO stability).

$$\underset{\text{bounded}}{\{u_k\}} \longrightarrow \boxed{\text{System}} \longrightarrow \underset{\underline{\text{bounded}}}{\{y_k\}}$$

Theorem

Let G be a discrete time system with a rational transfer function,

$$G(z) = \frac{n(z)}{d(z)} = \frac{b_0 z^m + \ldots + b_m}{z^n + a_1 z^{n-1} + \ldots + a_n}$$

with $m \leq n$, and *no common factors between $n(z)$ and $d(z)$. Let the pulse response of G be $\{g_k\}_{k \geq 0}$. Then the following are equivalent:*

1. *G is stable*
2. *All of the roots, p_i, of $d(z)$ satisfy $|p_i| < 1$*
3. *$\sum_{k=0}^{\infty} |g_k|$ is finite*

Logical sequence of proof:

$$(1) \xrightarrow{A} (2) \xrightarrow{\text{Ex. sheet 6a}} (3) \xrightarrow{B} (1)$$

(Moreover, Example sheet 6b shows $1 \implies 3$.)

Proof (sketch of main ideas)

A: $(1) \implies (2)$ (using contrapositive: we prove "not" $\neg(2) \Rightarrow \neg(1)$)

(in part) Suppose $p_1, p_2, p_3, \ldots$ are distinct. Then we can decompose G using partial fractions:

$$G(z) = \frac{\alpha_1}{1 - p_1 z^{-1}} + \ldots + \frac{\alpha_n}{1 - p_n z^{-1}}$$

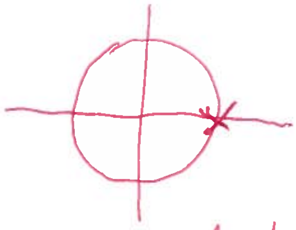
Then

$$g_k = \alpha_1 p_1^k + \alpha_2 p_2^k + \ldots + \alpha_n p_n^k$$

Now suppose $|p_i| > 1$ for some i. Then g_k is unbounded. Therefore a pulse input (bounded) gives an unbounded output and G is not stable.

(Contrapositive: $P \Rightarrow Q \equiv (\neg Q) \Rightarrow (\neg P)$)

Example: pole on unit circle. Let



$$G(z) = \frac{1}{1 - z^{-1}}$$

Then G has a pole at $z = 1$ and $g_k = 1$ for all $k \geq 0$.

Now consider an input $\{u_k\} = (0, 1, 1, 1, ...)$

Accumulator

$$U(z) = \frac{z^{-1}}{1 - z^{-1}}$$

$$Y(z) = G(z)U(z) = \frac{z^{-1}}{(1 - z^{-1})^2}$$

Therefore, $y_k = k \quad k \geq 0$

Aside: look at system back in time domain

Have $Y(z) = G(z)U(z)$

$= \frac{1}{1-z^{-1}} U(z)$

$\Rightarrow (1-z^{-1})Y(z) = U(z)$

$\Rightarrow y_k - y_{k-1} = u_k$

Adder/accumulator!

Therefore a bounded input gives an unbounded output and G is not stable in this example. We will not consider the general case of poles on unit circle in proof.

B: (3) $\implies$ (1)

Let $\{u_k\}$ be a bounded input, i.e. $|u_k| < M$ for $k \geq 0$. Then the output, $\{y_k\}$ is given by:

$$y_k = \sum_{i=0}^{k} g_i u_{k-i}$$

Then $$|y_k| = \left| \sum_{i=0}^{k} g_i u_{k-i} \right|$$

$$\leq \sum_{i=0}^{k} |g_i||u_{k-i}|$$

$|u_k| < M \ \forall k.$

$$\leq \sum_{i=0}^{k} |g_i| M = M \sum_{i=0}^{k} |g_i|$$

$$\leq M \sum_{i=0}^{\infty} |g_i|$$

So $\sum_{i=0}^{\infty} |g_i|$ finite $\Rightarrow \{y_k\}$ bounded $\therefore$ system is stable $\square$