Paper 3C6: VIBRATION

Examples paper 3 — Vibration response and Rayleigh's principle

*Tripos standard questions are marked **

Discrete systems

1 (a) Sketch the first five modes of vibration of a stretched string with fixed ends. Hence sketch the transfer function (as a decibel plot) when the string is driven with a harmonic force at the mid-point and the displacement is observed 1/5 of the way from one end.

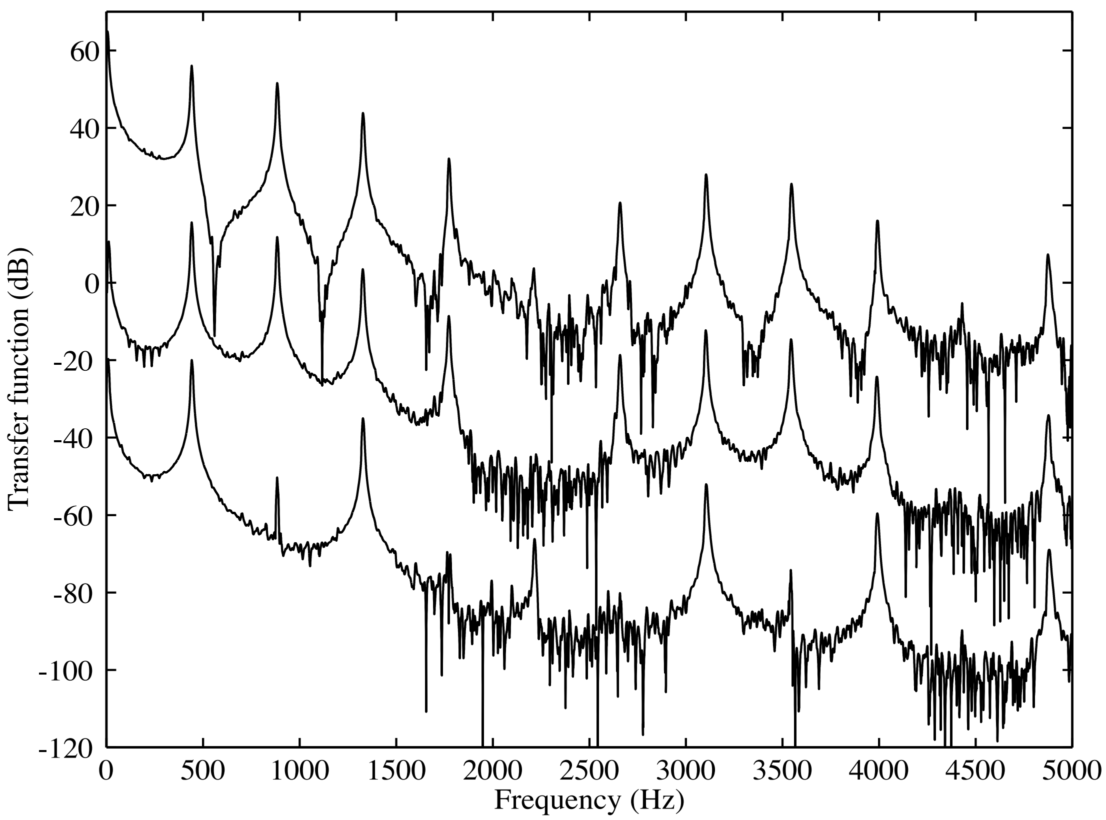
(b) Over the page are two sets of measured data, for two different vibrating systems. For each set, the observation point if fixed while the driving point is moved to three different positions. The curves are separated in the plots for clarity. For each set, try to answer the following questions:

(i) What kind of system might it be? (They are both systems which have been considered in the Continuous Systems lecture course.) How can you tell?

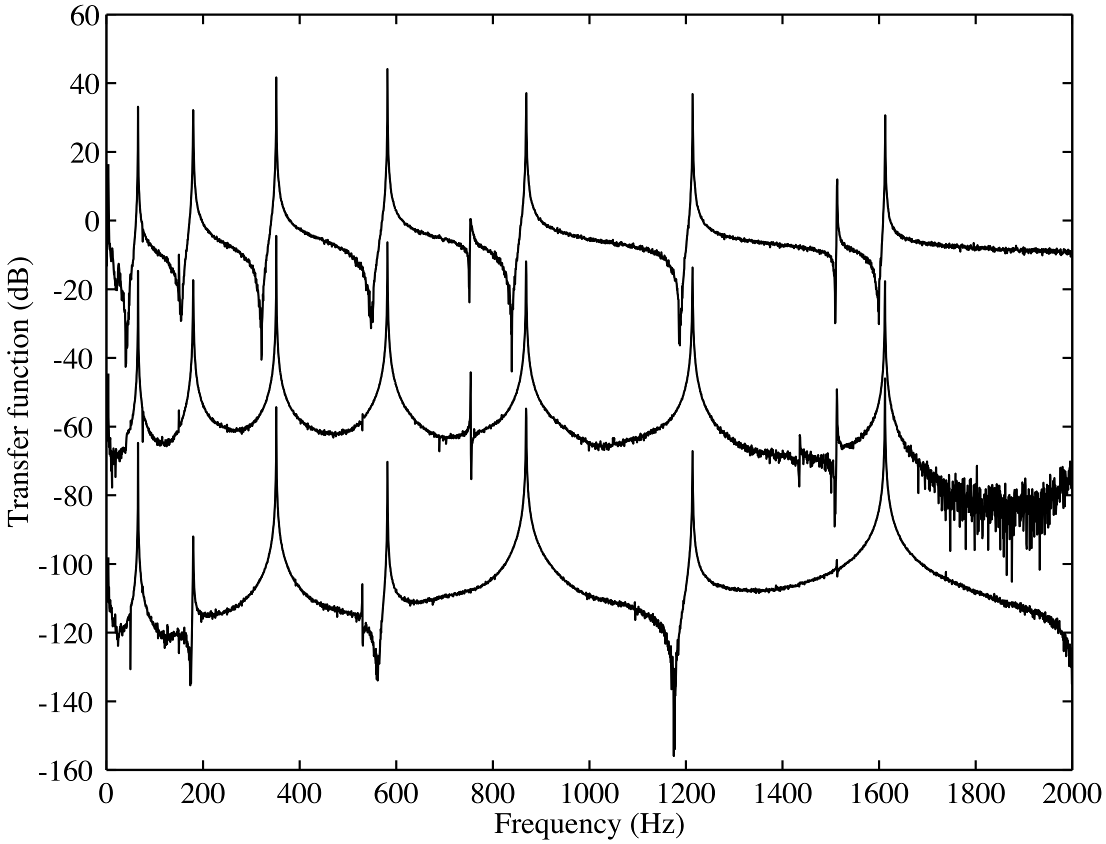
(ii) What can you say about the positions of the driving and observing points used?

(iii) What can you say about the possible boundary conditions of the systems?

(iv) Do the measurements show any unexpected features? How might these be accounted for?

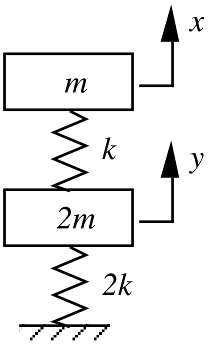


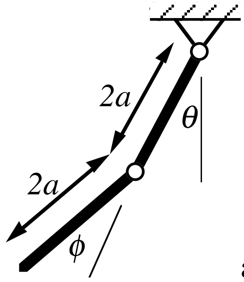
Transfer function (dB)
60
40
20
0
-20
-40
-60
-80
-100
-120
0 500 1000 1500 2000 2500 3000 3500 4000 4500 5000
Frequency (Hz)



Transfer function (dB)
60
40
20
0
-20
-40
-60
-80
-100
-120
-140
-160
0 200 400 600 800 1000 1200 1400 1600 1800 2000
Frequency (Hz)

2 For each of the two systems below, sketch the expected qualitative forms of the vibration modes. Find the mass and stiffness matrices, and use Rayleigh's principle to obtain approximations to all the mode frequencies: make *two different* plausible guesses for each mode shape, compare the frequency estimates you obtain, and say which is the more accurate.





as sheet 1 Q1(d)

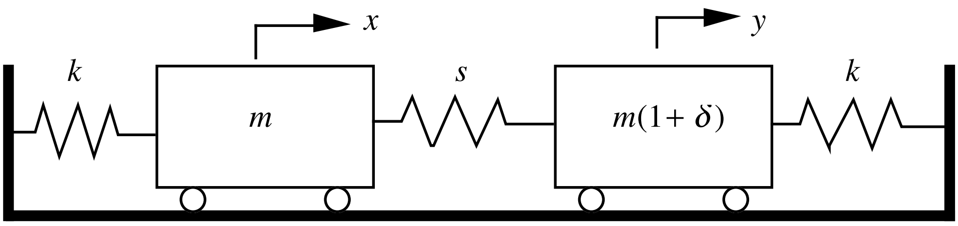
3* A light circular shaft carries three rotors of inertia I, $2I$ and $3I$ and rests in frictionless bearings at its ends. The torsional stiffness of the sections of shaft between each pair of rotors is k. Write down expressions for the kinetic and potential energies of the system in terms of the angles of rotation θ, ϕ and ψ of the three rotors. Deduce the mass and stiffness matrices.

Estimate forms for the two vibration modes with non-zero frequencies (hint: likely modes are orthogonal to the rigid-body-rotation mode). Hence use Rayleigh's principle to obtain approximations to the natural frequencies of these two modes.

Calculate the exact expressions for these natural frequencies, and compare them with the Rayleigh estimates.

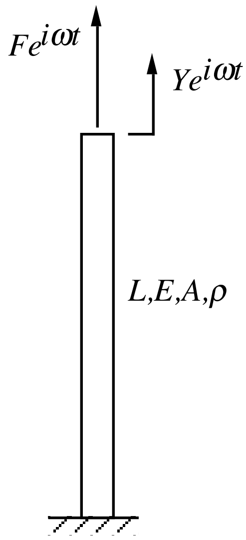
4 Two oscillators are coupled through a spring of stiffness s. The oscillators have masses m and $m(1+\delta)$, each supported by a spring of stiffness k. Obtain the mass and stiffness matrices for the system, in terms of the displacements x and y of the two masses.

For the case $\delta = 0$ write down the mode shapes by inspection, and hence obtain the natural frequencies. Now use these mode shapes in Rayleigh's principle to estimate the modified natural frequencies when δ is small but non-zero. Finally, use these modified frequencies in the eigenvector equation to obtain approximations (valid to first order in δ) for the modified mode shapes. Verify that these modified mode shapes are orthogonal (to the same order of approximation).



Continuous systems

5* (i) The undamped elastic column shown is of length L, density ρ and cross-sectional area A. It is subjected to a harmonic force $Fe^{i\omega t}$ at its free end and the displacement response here is $Ye^{i\omega t}$. Show that the frequency-response function is



$$H(L,L,\omega)=\frac{2}{\rho AL}\sum_{j=1}^{\infty}\frac{1}{-\omega^2+\omega_j^2}$$

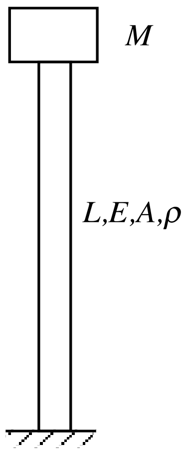
where ω_j is the j^{th} natural frequency of longitudinal vibration.

(ii) Recall that in Q6 of Examples Sheet 1, $H(L,L,\omega)$ was also shown to be given by

$$H(L,L,\omega)=\frac{c}{EA\omega}\tan\frac{\omega L}{c} \qquad \text{where} \quad c=\sqrt{\frac{E}{\rho}} \ .$$

Sketch a graph of $\frac{EA}{L}H(L,L,\omega)$ against $\frac{\omega L}{c}$ and verify *in broad terms* that the two results for $H(L,L,\omega)$ produce similar graphs.
(It can be shown by employing mathematical results from the summation of series that they are exactly the same, as they must be.)

(iii) A rigid mass M is fixed to the top of the column as shown. Show that the natural frequencies of the resulting system are given by the solutions of



$$\frac{-1}{M\omega^2}+H(L,L,\omega)=0 \ .$$

(iv) Devise a graphical construction using the graph from part (iii) to obtain approximate solutions for the natural frequencies of the mass-column combination. Hence check that when M is small the low natural frequencies are approximately those of the column alone, and when M is large the higher natural frequencies are those for a column fixed at both ends. Confirm that adding M always reduces the natural frequencies.

6. The frequency-response function (FRF) obtained from modal analysis for forced vibration of an undamped bending beam is

$$H_{YF}(x,y,\omega) = \sum_{j=1}^{\infty} \frac{u_j(x)u_j(y)}{\left(-\omega^2 + \omega_j^{\,2}\right)}$$

where the response Y is measured at x due to a transverse force F applied at y.

(i) By considering the response due to two very-closely-spaced forces F and $-F$ show that the FRF for the displacement Y due to a moment M applied at y is

$$H_{YM}(x,y,\omega) = \sum_{j=1}^{\infty} \frac{u_j(x)u_j'(y)}{\left(-\omega^2 + \omega_j^{\,2}\right)}$$

and show by differentiation that $$H_{\theta M}(x,y,\omega) = \sum_{j=1}^{\infty} \frac{u_j'(x)u_j'(y)}{\left(-\omega^2 + \omega_j^{\,2}\right)}$$

and that $$H_{\theta F}(x,y,\omega) = H_{YM}(y,x,\omega)$$
where θ is the rotation of the beam at x.

(ii) Two identical free-free beams of length L, A and B, are initially separate. Let H_{YF}, H_{YM} , $H_{\theta F}$ and $H_{\theta M}$ be the FRFs relating the end rotation and displacement of each beam subject to a force and moment at the same end, so that

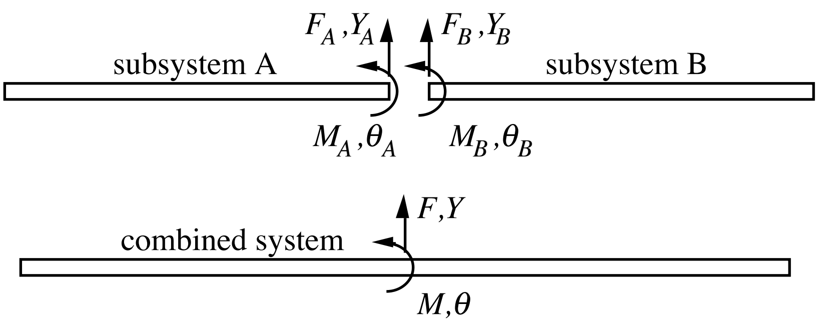
$$\begin{bmatrix} Y_A \\ \theta_A \end{bmatrix} = \begin{bmatrix} H_{YF} & H_{YM} \\ H_{\theta F} & H_{\theta M} \end{bmatrix} \begin{bmatrix} F_A \\ M_A \end{bmatrix}$$

and a similar equation for beam B. The two beams are now rigidly joined together as shown below. By considering what constraints are imposed by this joining, in terms of the matrix representation above, demonstrate that the two non-zero FRFs $\overline{H}_{YF}$ and $\overline{H}_{\theta M}$ at the centre of the combined free-free beam of length $2L$ are given by

$$\overline{H}_{YF} = \frac{\Delta}{2H_{\theta M}} \quad \text{and} \quad \overline{H}_{\theta M} = \frac{\Delta}{2H_{YF}}$$

where $\Delta = H_{YF}H_{\theta M} - H_{YM}H_{\theta F}$.

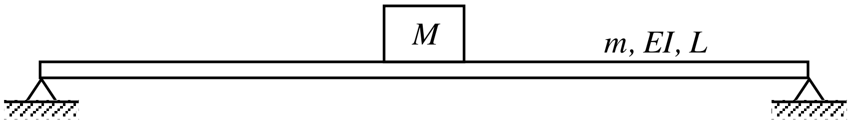
Explain why you expect $\overline{H}_{YM} = \overline{H}_{\theta F} = 0$.



7. A mass M is resting at the centre of a simply-supported beam of length L, flexural rigidity EI and mass m per unit length.

(i) Use Rayleigh's method with a suitable assumed mode shape to estimate the lowest natural frequency.

(ii) For the case of $m = 0$ use the Part I Structures Data Book to obtain an expression for the natural frequency of the beam with mass M at its centre. How does this compare with the answer obtained in (i) above?



8. An undamped string of length L and mass per unit length m has tension P. Its deflection in its first mode of vibration is described by $y(x,t) = u(x)\, e^{i\omega t}$ where $u(x)$ is the (first) mode function and ω is the (lowest) natural frequency.

(i) Show that the increase in length of an element dx of the string when at maximum deflection is

$$\frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2 dx \, .$$

(ii) Hence show that the potential energy of the string is

$$V = \frac{P}{2}\int_0^L \left(\frac{\partial y}{\partial x}\right)^2 dx \quad .$$

(iii) Assuming that $u(x) = \sin(\pi x / L)$, use Rayleigh's Principle to show that

$$\omega = \frac{\pi}{L}\sqrt{\frac{P}{m}} \quad .$$

Compare this result with the exact answer found by solving the equation of motion. Why do the results agree exactly?

(iv) Now assume that $u = (x / L)(1 - x / L)$ and repeat the calculation to find ω. Check that this result is approximately 0.7% too big.

Answers

3. Exact frequencies $\omega^2 = \dfrac{k}{6I}\left(7 \pm \sqrt{13}\right)$.

4. $\omega^2 \approx \dfrac{2k}{m(2+\delta)}, \dfrac{2k+4s}{m(2+\delta)};\quad \dfrac{y}{x} \approx 1 + \dfrac{k\delta}{2s}, -1 + \dfrac{\delta}{2s}(k+2s)$.

7. (i) using a sinusoidal presumed mode, $\omega = \pi^2 \sqrt{\dfrac{EI}{(m + \frac{2M}{L})\, L^4}}$

(ii) $\omega = \sqrt{\dfrac{48EI}{M\,L^3}}$ which compares very well with $\sqrt{\dfrac{48.7EI}{M\,L^3}}$ above when $m = 0$