3F1 Signals and Systems

(16) System response to random signals

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Real-world systems are often subjected to random fluctuations. For example:

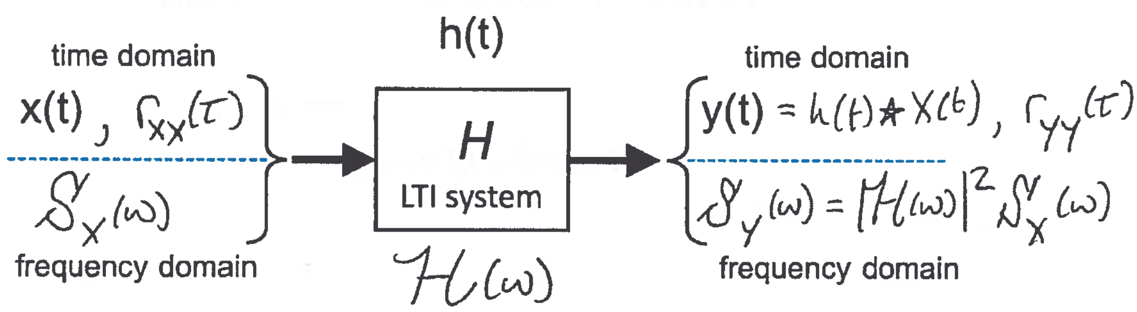
- power plants are subject to fluctuations in load
- large structures such as bridges and oil rigs are subject to wind gusts
- microorganisms are subject to thermal fluctuations and brownian collisions

This lecture is about understanding how systems (continuous time LTI systems) respond to random signals (noise).

Remarkable fact: it turns out that we can characterise the linear response of a system by simply observing its response to noise, subject to certain assumptions.

LTI system with WSS input

Let H be a continuous time LTI system (aka 'filter'!) with impulse response $h(t)$. For an input, $x(t)$, we can compute the output, $y(t)$:



and we can relate the autocorrelation functions and PSDs of the inputs and outputs, as follows.

Since the system is LTI, we can compute the output from the impulse response:

$$y(t) = h(t) \star x(t) = \int h(\beta)x(t-\beta)d\beta$$

Now suppose $x(t) = X(t)$, where $\{X(t)\}$ is a WSS random process. The output, $y(t)$, will also be a random process, $\{Y(t)\}$, with $Y(t) = h(t) \star X(t)$.

Will Y be WSS?

$$\begin{aligned}
E[Y(t)] &= E\left[\int h(\beta)x(t-\beta)d\beta\right] \\
&= \int h(\beta)E[x(t-\beta)]d\beta \\
&= \mu \int h(\beta)d\beta = \text{constant}
\end{aligned}$$

where μ is the mean value of X.

The autocorrelation function of Y is:

$$
\begin{aligned}
r_{YY}(t_1, t_2) &= E[Y(t_1)Y(t_2)] \\
&= E\Big[\int h(\beta_1)X(t_1-\beta_1)d\beta_1 \int h(\beta_2)X(t_2-\beta_2)d\beta_2\Big] \\
&= E\Big[\int\int h(\beta_1)h(\beta_2)X(t_1-\beta_1)X(t_2-\beta_2)d\beta_1 d\beta_2\Big] \\
&= \int\int h(\beta_1)h(\beta_2)E[X(t_1-\beta_1)X(t_2-\beta_2)]d\beta_1 d\beta_2 \\
&= \int\int h(\beta_1)h(\beta_2)r_{XX}(t_1-\beta_1, t_2-\beta_2)d\beta_1 d\beta_2
\end{aligned}
$$

Now, if X is WSS, then $r_{XX}(t_1, t_2) = r_{XX}(\tau)$. Thus, putting $\tau = t_2 - t_1, t = t_1$:

$$
\begin{aligned}
r_{YY}(t_1, t_2) &= E[Y(t)Y(t+\tau)] \\
&= \int\int h(\beta_1)h(\beta_2)r_{XX}(\tau+\beta_1-\beta_2)d\beta_1 d\beta_2 \\
&= r_{XX}(\tau) \star h(-\tau) \star h(\tau)
\end{aligned}
$$

Taking Fourier transforms allows us to relate the PSDs of the input and output of the system, H.

$$
\begin{aligned}
\mathcal{S}_Y(\omega) &= FT\{r_{YY}(\tau)\} \\
&= \int\Big(\int\int h(\beta_1)h(\beta_2)r_{XX}(\tau+\beta_1-\beta_2)d\beta_1 d\beta_2\Big)e^{-j\omega\tau}d\tau \\
&= \int\int h(\beta_1)h(\beta_2)\Big(\int r_{XX}(\tau+\beta_1-\beta_2)e^{-j\omega\tau}d\tau\Big)d\beta_1 d\beta_2 \\
&= \int\int h(\beta_1)h(\beta_2)\Big(\int r_{XX}(\lambda)e^{-j\omega(\lambda-\beta_1+\beta_2)}d\lambda\Big)d\beta_1 d\beta_2 \\
&= \int h(\beta_1)e^{j\omega\beta_1}d\beta_1 \int h(\beta_2)e^{-j\omega\beta_2}d\beta_2 \int r_{XX}(\lambda)e^{-j\omega\lambda}d\lambda \\
&= \mathcal{H}^*(\omega)\mathcal{H}(\omega)\mathcal{S}_X(\omega)
\end{aligned}
$$

where $\mathcal{H}(\omega) = FT\{h(t)\}$.
More compactly,

$$\mathcal{S}_Y(\omega) = |\mathcal{H}(\omega)|^2\mathcal{S}_Y(\omega)$$

So the PSD of Y is the PSD of X times the power gain of the system at frequency ω.

Thus if a system is subject to WSS fluctuations we can measure $r_{XX}(\tau)$ and $r_{YY}(\tau)$, transform these to the PSDs and obtain:

$$|\mathcal{H}(\omega)| = \sqrt{\frac{\mathcal{S}_Y(\omega)}{\mathcal{S}_X(\omega)}}$$

Thus if an important system (e.g. a power plant) is subject to measurable random fluctuations in input (e.g. fluctuating demand), we can measure the system frequency response **without taking the plant offline**. In general, we may be able to exploit natural fluctuations in systems to measure their (linear) properties, e.g. response of microscopic systems to thermal fluctuations.

However, this does not give any information about the phase of the system.

To get the phase of $\mathcal{H}(\omega)$, we need to measure the cross-correlation function (CCF) between input and output:

$$\begin{aligned}
r_{XY}(t_1, t_2) &= E[X(t_1)Y(t_2)] \\
&= E\left[X(t_1)\int h(\beta)X(t_2-\beta)d\beta\right] \\
&= E\left[\int h(\beta)X(t_1)X(t_2-\beta)d\beta\right] \\
&= \int h(\beta)E[X(t_1)X(t_2-\beta)]d\beta \\
&= \int h(\beta)r_{XX}(t_1, t_2-\beta)d\beta
\end{aligned}$$

Thus if X (and hence Y) are WSS:

$$r_{XY}(\tau) = E[X(t)Y(t+\tau)] = \int h(\beta)r_{XX}(\tau-\beta)d\beta = h(\tau)\star r_{XX}(\tau)$$

Taking Fourier transforms gives:

$$\mathcal{S}_{XY}(\omega) = FT\{r_{XY}(\tau)\} = \mathcal{H}(\omega)\mathcal{S}_X(\omega)$$

$\mathcal{S}_{XY}(\omega)$ is the **Cross Spectral Density** between X and Y. The above equation gives:

$$\mathcal{H}(\omega) = \frac{\mathcal{S}_{XY}(\omega)}{\mathcal{S}_X(\omega)}$$

This gives both the amplitude and phase of $\mathcal{H}(\omega)$. Again, this can be achieved by passively monitoring a system/plant without taking it offline.

Note: for WSS processes, $r_{XY}(\tau) = r_{YX}(-\tau)$, but unlike ACFs, CCFs need not be symmetric about 0: $r_{XY}(\tau) \neq r_{XY}(-\tau)$. Thus, $\mathcal{S}_{XY}(\omega)$ may not be purely real and can contain phase information.

Aside: how can we measure the ACF/CCF of a signal, practically?

Another aside: evolution developed simple correlator circuits a long time ago! Here's a neural circuit for motion detection:

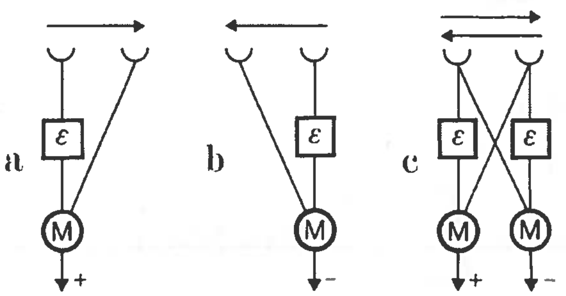
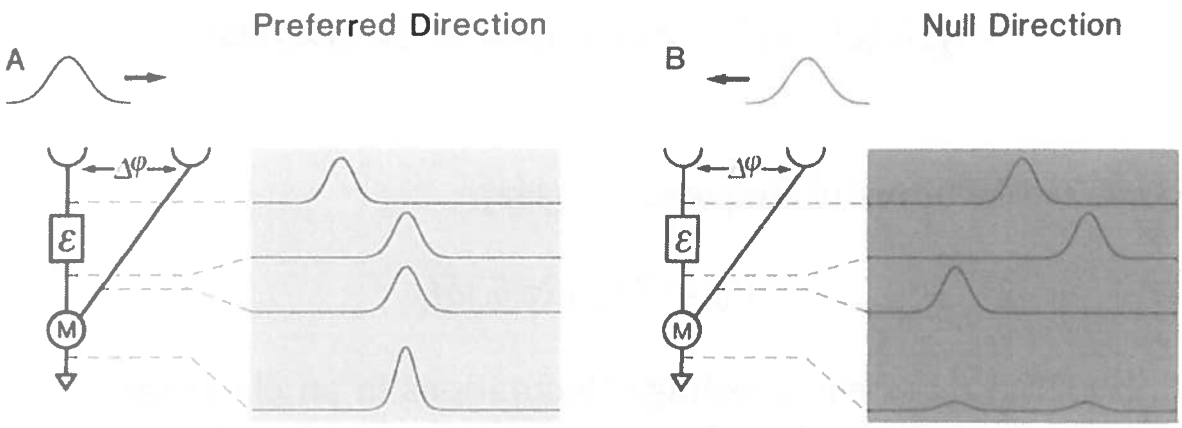
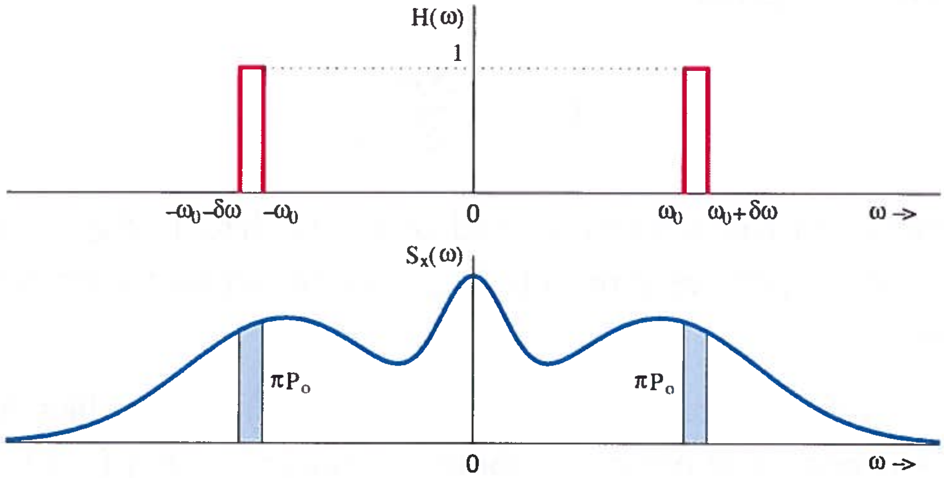


Fig. 4a–c. Representation of the operations of movement detectors. Undirectional (a, b) and bidirectional movement detectors (c) consisting of two light receptors, signal delay ε in one or two channels and elementary signal multiplication stages M. As in Fig. 3 the preferred direction of movement is shown by the arrows



After Werner Reichardt, J Comp Physiol A (1987).

Physical interpretation of Power Spectral Density



Consider a random signal X passed through a narrow-band filter of bandwidth $\delta\omega = 2\pi\delta f$, as shown in the figure.

$$\mathcal{H}(\omega) = \begin{cases} 1 & \text{for } \omega_0 < |\omega| \leq \omega_0 + \delta\omega \\ 0 & \text{otherwise} \end{cases}$$

To find the average power output from the filter (shaded area),

$$\begin{aligned} P_0 = r_{YY}(0) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{S}_Y(\omega)d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{S}_X(\omega)|\mathcal{H}(\omega)|^2 d\omega \\ &= \frac{1}{2\pi}\left(\int_{-(\omega_0+\delta\omega)}^{-\omega_0} \mathcal{S}_X(\omega)d\omega + \int_{\omega_0}^{\omega_0+\delta\omega} \mathcal{S}_X(\omega)d\omega\right) \\ &\approx \frac{1}{2\pi}(\mathcal{S}_X(-\omega_0) + \mathcal{S}_X(\omega_0))\delta\omega = \frac{1}{\pi}\mathcal{S}_X(\omega_0)\delta\omega \end{aligned}$$

Expressed in terms of frequency, f (Hz),

$$P_0 \approx 2\mathcal{S}_X(2\pi f_0)\delta f$$

If, say, X(t) represented voltage fluctuations in an electrical system, then we see that $\mathcal{S}_X$ is indeed a **power density** with units V^2/Hz (assuming unit impedance).

White and coloured noise

A **White Noise** process is a zero-mean WSS process, $X(t)$ with autocorrelation function:

$$r_{XX}(\tau) = P_X \delta(\tau)$$

where P_X is a constant.

The PSD of such a process is then:

$$\mathcal{S}_X(\omega) = \int P_X \delta(\tau) e^{-j\omega\tau} d\tau = P_X e^{-j\omega 0} = P_X$$

so the power is constant across all frequencies, as is (approximately) true for white light. However,

$$\text{Power of } X = \tfrac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{S}_X(\omega) d\omega = \infty$$

This means pure white noise is unrealisable in practice. It is, however, a very useful conceptual entity and a good approximation of 'nearly white' processes for which $r_{XX}(\tau)$ is a very narrow pulse (that is, noise processes whose fluctuations are rapid in time).

If a white noise process is passed through a filter with frequency response $\mathcal{H}(\omega)$ we obtain a noise process with PSD:

$$\mathcal{S}_Y = \mathcal{S}_X |\mathcal{H}(\omega)|^2 = P_X |\mathcal{H}(\omega)|^2$$

We can therefore shape the PSD of the noise process.

Many natural phenomena exhibit characteristic power spectra. Consider a 'Brownian particle' (e.g. pollen grain in water). We can model the impacts of water molecules as a white noise process. Newton's second law (in 1D) then gives:

$$m\dot{v} + \gamma v = X(t)$$

where m = particle mass, γ = viscous drag, v = velocity. The frequency response of this system is $\frac{1}{j\omega + \gamma/m}$, yielding a PSD:

$$\mathcal{S}_Y(\omega) = |\mathcal{H}(\omega)|^2 P_X = \frac{1}{\omega^2 + (\gamma/m)^2}$$

Correspondingly, noise processes with power spectrum $\propto 1/\omega^2$ are called **brown noise** processes. Other 'colours' of noise exist, including pink ($\propto 1/\omega$) noise.