4F7-STATISTICAL SIGNAL ANALYSIS

Examples Paper

Question 1: Let $Z_1, \ldots, Z_N$, be independent Gaussian random variables with mean 0 and variance 1. Find the probability density functions that are being approximated by the sample average estimates

$$\frac{1}{N}\sum_{j=1}^{N}\frac{1}{\sqrt{2\pi}}\exp\left[-0.5(y-Z_j)^2\right]h(Z_j)$$

and

$$\frac{\sum_{j=1}^{N}\exp\left[-0.5(y-Z_j)^2\right]h(Z_j)}{\sum_{j=1}^{N}\exp\left[-0.5(y-Z_j)^2\right]}$$

where $h:\mathbb{R}\to\mathbb{R}$ is some function of interest. Which, if any of these approximations, are unbiased?

Question 2: Let $J_1, \ldots, J_N$ be discrete valued random variables, $J_i \in \{1, \ldots, N\}$, with joint conditional probability mass function

$$\Pr\left(J_1 = j_1, \ldots, J_N = j_N \mid Z_1 = z_1, \ldots, Z_N = z_N\right)$$

$$= \Pr\left(J_1 = j_1 \mid Z_1 = z_1, \ldots, Z_N = z_N\right)\cdots\Pr\left(J_N = j_N \mid Z_1 = z_1, \ldots, Z_N = z_N\right).$$

That is, given the values of $Z_1, \ldots, Z_N$, the random variables $J_1, \ldots, J_N$ are independent. Furthermore, let

$$\Pr\left(J_i = j \mid Z_1 = z_1, \ldots, Z_N = z_N\right) = \frac{\exp\left[-0.5(y - z_j)^2\right]}{\sum_{i=1}^{N} \exp\left[-0.5(y - z_i)^2\right]}.$$

(a) The variables $J_1, \ldots, J_N$ are the outputs of a multinomial resampling algorithm. Give the input weighted samples that are being resampled.

(b) Find $\mathbb{E}\left\{h(Z_{J_1}) \mid Z_1 = z_1, \ldots, Z_N = z_N\right\}$.

(c) Find $\mathbb{E}\left\{h(Z_{J_1})\frac{1}{N}\left(\sum_{i=1}^{N} \frac{1}{\sqrt{2\pi}} \exp\left[0.5(y - Z_i)^2\right]\right)\right\}$.

You may use the following fact about the conditional expectation: for any pair of random variables X and Y and a real valued function $g(x, y)$,

$$\mathbb{E}(g(X, Y)) = \mathbb{E}\left[\mathbb{E}(g(X, Y) \mid Y)\right].$$

Furthermore, if X is discrete valued but Y is continuous then

$$\mathbb{E}\left[\mathbb{E}(g(X, Y) \mid Y)\right] = \int \left(\sum_{x} g(x, y) p_{X|Y}(x \mid y)\right) p_Y(y) dy.$$

Let $p(x_0, \ldots, x_n \mid y_0, \ldots, y_n)$ be the conditional probability density function of a hidden Markov model with state transition probability density function $f(x_k, x_{k+1})$ and observation probability density function $g(x_k, y_k)$. Assume $X_0 \sim p(x_0)$.

Let $\pi_n(x_{0:n}) = p(x_{0:n}, y_{0:n})$. Let $X_{0:n}^i \sim q_n(x_0, \ldots, x_n)$, $i = 1, \ldots, N$, be independent samples from a proposal probability density function $q_n(x_{0:n})$ and let $w_n^i = \pi_n(X_{0:n}^i)/q_n(X_{0:n}^i)$.

Question 3: Write down the multinomial resampling algorithm for the weighted samples $\{(X_{0:n}^i, w_n^i)\}_{i=1}^N$.

Question 4: Let J denote a particle index produced by the multinomial resampling algorithm. Show that $\mathbb{E}\left\{h_n(X_{0:n}^J)W_n/N\right\} = \int h_n(x_{0:n})\pi_n(x_{0:n})dx_{0:n}$ where $W_n = \sum_{j=1}^N w_n^j$.

Question 5: Write down the particle filter algorithm when the proposal probability density function $q_n(x_{0:n})$ is

$$q_n(x_{0:n}) = p(x_0)f(x_0, x_1)\cdots f(x_{n-1}, x_n).$$

Give the particle filter estimate for $p(y_{0:n})$ and $\int h_n(x_{0:n})p_n(x_{0:n} \mid y_{0:n})dx_{0:n}$ where $h_n(x_{0:n})$ is a real valued function.

Question 6: Explain why the particle filter produces an unbiased estimate of the integral $\int h_k(x_{0:k})\pi_k(x_{0:k})dx_{0:k}$ for any time k and any function $h_k(x_{0:k})$. Hence show that the particle filter's estimate of $p(y_{0:n})$ is unbiased.

Consider the following hidden Markov model. Let

$$X_k = aX_{k-1} + \sqrt{b}W_k, \quad k = 0, 1, \ldots$$

where W_k are independent and identically distributed $\mathcal{N}(0,1)$. Let $X_{-1} = x_{-1} = 0$. The observation process Y_k, $k = 0, 1, \ldots$ is integer valued, $Y_k \in \{0, 1, \ldots\}$ and follows a Poisson distribution with rate $c\exp(X_k)$,

$$\Pr\left(Y_k = y \mid X_k = x_k\right) = \frac{e^{-c\exp(x_k)}\left(c\exp(x_k)\right)^y}{y!}.$$

Let the probability mass function for Y_k given $X_k = x_k$ be $g(x_k, y_k)$, i.e. $g(x_k, y_k) = \Pr(Y_k = y_k \mid X_k = x_k)$.

Question 7: Write down $\log f(x_{k-1}, x_k)$ and $\log g(x_k, y_k)$and show that this hidden Markov model belongs to the exponential family.

Question 8: Assume constants a and b are known and only c is to be learnt from the data record $y_0, \ldots, y_n$. Write down the intermediate function

$$Q_n(c, c') = \int \log p_{c'}(x_{0:n}, y_{0:n}) p_c(x_{0:n} \mid y_{0:n}) dx_{0:n}$$

of the Expectation-maximisation algorithm.

Question 9: Find the value c' that maximises $Q_n(c, c')$.

Question 10: Find the gradient $d \log p_c(y_{0:n})/dc$. Write down the gradient ascent algorithm for maximising $\log p_c(y_{0:n})$ and explain how a particle filter may be used to implement it.

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