Engineering Pt IIa, Module 3F3 - Random Processes, Detection and Estimation
Examples paper

1. In this question you prove a number of key properties about random processes. You need many of these properties to answer later questions (and exam questions!).

   For jointly wide-sense stationary, discrete-time, real-valued random processes $\{X_n\}$ and $\{Y_n\}$:

   (a) + Show that the autocorrelation function is even:

   $$r_{XX}[k] = r_{XX}[-k]$$

   (b) + [Hence] show that power spectrum is always real-valued.

   (c) * Show the maximum of the autocorrelation function:

   $$\max |r_{XX}[k]| = r_{XX}[0]$$

   Hint: start with the certain relationship, which holds for any value of a:

   $$\mathbb{E}[(x_{n+k} - ax_n)^2] \geq 0$$

   Then simplify and choose particular value(s) of a which give the required result.

   (d) Show for the Cross-correlation function:

   $$r_{XY}[k] = r_{YX}[-k]$$

   (e) Show that power spectrum is even:

   $$\mathcal{S}_X(e^{j\omega}) = \mathcal{S}_X(e^{-j\omega})$$

   (f) + Show that power spectrum is periodic:

   $$\mathcal{S}_X(e^{j(\omega+2n\pi)}) = \mathcal{S}_X(e^{j\omega})$$

   (g) Show that the cross-power spectrum of two processes has the following symmetry:

   $$\mathcal{S}_{XY}(e^{j\omega}) = \mathcal{S}^*_{YX}(e^{j\omega})$$

   where * denotes the complex conjugate.

2. * $\{X_n\}$ is a wide sense stationary random process with finite mean. If $\{Y_n\}$ is the output of a linear time-invariant (LTI) system having impulse response $\{h_n\}$ and input $\{X_n\}$, then show that:

$$\mathbb{E}[Y_n] = \mathbb{E}[X_n] \sum_{p=-\infty}^{+\infty} h_p$$

and

$$r_{YY}[k] = \sum_{l=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} h_l \, h_i \, r_{XX}[l-i+k] = (\tilde{h} * h * r_{XX})[k]$$

where $\tilde{h}$ is the time-reversed sequence h. Show that $\{Y_n\}$ is guaranteed to be wide-sense stationary if:

$$\sum_{k=-\infty}^{+\infty} |h_k| < \infty$$

(i.e. the LTI system is strictly stable). To show this you will need to use the earlier property

$$\max_k(|r_{XX}[k]|) = r_{XX}[0]$$

3. The Bernoulli random process is defined such that each time point is independently assigned a value of +1 or -1 with probabilities p and $1-p$ respectively. A possible sample function from the process is, for example:

$$\{x_n\} = \{..., -1, +1, +1, -1, +1, +1, ...\}$$

Is the Bernoulli process a white noise process?

4. A stationary random process has autocorrelation function:

$$r_{XX}[0] = 2.8, \;\; r_{XX}[\pm 1] = 1$$

and zero values elsewhere. It is known that x_n is a noisy version of another signal d_n:

$$x_n = d_n + v_n$$

where v_n is a zero mean white noise signal with variance equal to 0.5 and uncorrelated with d_n. Determine the optimal first order (i.e. with two coefficients) FIR Wiener filter for estimation of d_n [Hint: first show that $r_{dd}[k] = r_{xx}[k] - r_{vv}[k]$].

Calculate the minimum mean-squared error for this filter.

How much (if at all) does the Wiener filter improve the mean-squared error compared with simply setting $\hat{d}_n = x_n$?

5. Determine the optimal frequency domain Wiener filter for the same signal as the previous question. Sketch the power spectrum of x_n, d_n, v_n and the frequency response of the filter. Comment on how reasonable the filter seems intuitively.

6. * The error signal for a signal estimation problem for estimating a signal d_n from some measurements x_n is:

$$\epsilon_n = d_n - \sum_{p=-\infty}^{\infty} h_p \, x_{n-p}$$

(a) Determine the autocorrelation function of $\{\epsilon_n\}$, i.e. $r_{\epsilon\epsilon}[k] = \mathbb{E}[\epsilon_n \epsilon_{n+k}]$

(b) Hence show that the power spectrum of the error is of the form

$$\mathcal{S}_\epsilon = \mathcal{S}_D - \mathcal{S}_{DX} H - S_{XD} H^* + |H|^2 \mathcal{S}_X$$

where the dependence of all $\mathcal{S}$ and H terms on $(e^{j\omega})$ has been omitted for clarity. $H(e^{j\omega})$ is the frequency response of the LTI filter.

(c) Use this result to show that the optimal Wiener Filter has a mean-squared error of

$$J_{\min} = \mathbb{E}[\epsilon_n^2] = r_{\epsilon\epsilon}[0] = \frac{1}{2\pi} \int_{-\pi}^{+\pi} \mathcal{S}_D(e^{j\omega}) - \mathcal{S}_{XD}^*(e^{j\omega}) H^{\text{opt}}(e^{j\omega}) d\omega$$

where H^{opt} is the optimal Wiener gain derived in the lecture notes.

(d) Hence show that in the standard scenario where $x_n = d_n + v_n$ and v_n is zero-mean and uncorrelated with d_n, we get

$$J_{\min} = \frac{1}{2\pi} \int_{-\pi}^{+\pi} \mathcal{S}_D(e^{j\omega})(1 - H^{\text{opt}}(e^{j\omega})) d\omega$$

Hint for whole question: you need several of the properties derived in q.1 to answer this question.

7. * A stationary AR(1) process can be written as:

$$d_n = a_1 d_{n-1} + e_n$$

where e_n is zero mean white noise with variance σ_e^2.

(a) Determine the power spectrum of the AR(1) process and its autocorrelation function. Sketch the power spectrum of the process for a few values of a_1 between 0 and 1, over the range $\omega = -4\pi, ..., +4\pi$.

(b) The AR process is now observed in zero mean white noise v_n with variance σ_v^2:

$$x_n = d_n + v_n$$

Assuming that v_n is uncorrelated with d_n, determine the frequency response $H(e^{j\omega})$ for the IIR Wiener filter which estimates d_n optimally from x_n.

(c) Hence, with $\sigma_v^2 = 0.25$, $\sigma_e^2 = 0.25$ and $a_1 = 0.5$, determine the Wiener filter impulse response. Is this filter causal or non-causal? Sketch the frequency response of this filter and explain why it is intuitively reasonable.

You may use the following DTFT pair:

$$\sum_{n=-\infty}^{\infty} \alpha^{|n|} e^{-jn\omega} = \frac{1-\alpha^2}{|1-\alpha e^{-j\omega}|^2}$$

8. It is desired to predict a future value of a WSS signal x_n m steps ahead of the current time, so that the desired signal is $d_n = x_{n+m}$. We are given the power spectrum of $\{x_n\}$ as $\mathcal{S}_x$.

   Show that the standard non-causal frequency domain Wiener filter formula is optimal for this problem:
$$H(e^{j\omega}) = \frac{\mathcal{S}_{xd}(e^{j\omega})}{\mathcal{S}_x(e^{j\omega})}$$
   Show that in this case,
$$\mathcal{S}_{xd}(e^{j\omega}) = \mathcal{S}_x(e^{j\omega})e^{jm\omega}$$
   and hence that the Wiener filter gain is:
$$H(e^{j\omega}) = e^{jm\omega}.$$
   Determine the impulse response of this filter and explain why this result could have been guessed without calculations.

9. * In an aircraft cockpit noise cancellation application a noisy signal is measured at a primary microphone:
$$x_n = d_n + v_{1,n}$$
   where d_n is the pilot's voice and $v_{1,n}$ is background noise. At a second microphone in the cockpit a measurement of pure noise $v_{2,n}$ is made. $v_{1,n}$ and $v_{2,n}$ are assumed jointly wide-sense stationary with cross-correlation function $r_{v_1v_2}[k]$, and both are uncorrelated with d_n.

   It is required to estimate the voice signal by FIR filtering the noise signal and subtracting from x_n:
$$\hat{d}_n = x_n - \sum_{p=0}^{P} h_p v_{2,n-p}$$
   (a) Using the mean-squared error criterion, show that the filter coefficients must satisfy
$$r_{v_2v_1}[k] = \sum_{q=0}^{P} h_q r_{v_2v_2}[k-q]$$
   for $k = 0, 1, 2, ..., P$. Hence show that the matrix form of the Wiener-Hopf equations for this case is:
$$\mathbf{R}_{v_2}\mathbf{h} = \mathbf{r}_{v_2v_1}$$
   where the matrices and vectors should be carefully defined.

   (b) Explain how, in a real environment, $r_{v_2v_2}[k]$ can be measured, assuming v_2 to be ergodic.

(c) Show that when d_n is stationary,

$$r_{v_2 v_1}[k] = r_{v_2 x}[k]$$

and hence explain how $r_{v_2 v_1}[k]$ could be estimated. Is the stationary assumption on d_n realistic in this application?

10. * In a digital communication system random bits are transmitted as a Bernoulli random process, that is, each time point is independently assigned a value of +1 or -1 with equal probabilities of 0.5 .

In the communications system the channel can be modelled by the following formula,

$$x_n = \sum_{i=0}^{1} c_i b_{n-i}$$

where $c_0 = 1$ and $c_1 = 0.1$.

(a) Determine the cross-correlation function between $\{b_n\}$ and $\{x_n\}$, and also the autocorrelation function of $\{x_n\}$.

(b) It is desired to optimally estimate the bit sequence $\{b_n\}$ from the channel data $\{x_n\}$. Design the first order FIR Wiener filter for this task, i.e. form an estimate of the type:

$$\hat{b}_n = \sum_{i=0}^{1} h_i x_{n-i}$$

where h_0 and h_1 are to be determined according to the Wiener criterion.

11. An FIR filter is applied to the problem of detecting a signal $s_{1,n}$:

$$s_{1,n} = \begin{cases} 1 & 0 \leq n < D/2 \\ -1 & D/2 \leq n < D \\ 0 & \text{Otherwise} \end{cases}$$

For simplicity, D is a multiple of 4.

$s_{1,n}$ is observed at an unknown time location n_0, and in additive, zero mean white noise of variance σ_v^2:

$$x_n = s_{1,n-n_0} + v_n, \quad n = 0, 1, ..., N-1, \quad N >> D$$

x_n is filtered with a FIR filter $h_{1,p}$ to give y_n:

$$y_n = \sum_{p=0}^{P} h_{1,p} x_{n-p}$$

(a) Using such a filter, explain how to perform optimal detection of the time n_0 where s_1 occurs. What are the filter coefficients? Sketch s_1, the optimal filter h_1 and the output of the optimal filter in response to x_n. This part is most easily done in Matlab. The following generates the signal s_1 for $D = 12$ with the usual one sample offset for Matlab vectors:

```
D=12
s_1=ones(D,1);
s_1(D/2+1:D)=-1;
```

(b) Determine the maximum expected Signal-to-Noise Ratio (SNR) for the optimal filter at time $n_0 + D - 1$ in terms of D and compare this with the maximum SNR obtainable from the unfiltered x_n. Suggest another significant advantage of the matched filter over this simple scheme for this choice of signal $s_{1,n}$.

(c) Using the optimal filter, express the expected signal-to-noise ratio (SNR) at the output of the filter for times $n \neq n_0 + D - 1$. Hence show the output from the filter at times $n \neq n_0 + D - 1$ always has SNR less than that at n_0.

(d) True or false (and why?): this result guarantees correct detection of the time n_0 at which the signal occurs.

(e) A second signal $s_{2,n}$ may also be present in the data:

$$s_{2,n} = \begin{cases} -1 & 0 \leq n < D/4 \\ +1 & D/4 \leq n < 3D/4 \\ -1 & 3D/4 \leq n < D \\ 0 & \text{Otherwise} \end{cases}$$

Determine and sketch the output of filter $h_{1,p}$ applied to $s_{2,n}$. Again, most easily done in Matlab. The following generates s_2:

```
s_2=ones(D,1);
s_2(1:D/4)=-1;
s_2(3*D/4+1:D)=-1;
```

Hence suggest how simple matched filters can be used to detect the potential presence of *either or both* s_1 and s_2 in the data.

Answers

1.

2.

3. Yes

4.

$$h_0 = 0.795 \ h_1 = 0.073, \ J_{min} = 0.398$$

5.

6. (a)

$$r_{dd}[k] - h_p * r_{dx}[k] - h_{-p} * r_{xd}[k] + h_p * h_{-p} * r_{xx}[k]$$

7. (a) Power spectrum is

$$\frac{\sigma_e^2}{|(1 - a_1 e^{-j\omega})|^2}$$

Autocorrelation function is:

$$\frac{\sigma_e^2}{1 - a_1^2} a_1^{|n|}$$

(b)

$$H(e^{j\omega}) = \frac{\sigma_e^2}{\sigma_e^2 + \sigma_v^2 (1 - a_1 e^{-j\omega})(1 - a_1 e^{j\omega})}$$

(c) $h_p = 0.496(0.234)^{|p|}$, non-causal.

8.

9. Impulse response is:

$$h_p = \delta[p + m]$$

10. (a)

$$r_{bx}[m] \begin{cases} c_0, & m = 0 \\ c_1, & m = 1 \\ 0, & \text{otherwise} \end{cases}$$

$$r_{xx}[0] = 1.01, \ r_{xx}[\pm 1] = 0.1, \text{ and zero otherwise}$$

(b)

$$\mathbf{h} = \begin{bmatrix} 1 \\ -0.1 \end{bmatrix}$$

11. (d) False

Suitable past tripos questions from 3F3:
2010 Q2(c)
2011 Q3
2012 Q3
2014 Q3
2015 Q3
2016 Q3 (mainly for lectures 5-8)
2017 Q2 (d) and (e)

Simon Godsill