Module 3F2: Systems and Control
EXAMPLES PAPER 2 — STATE-SPACE TRAJECTORIES &ROOT-LOCUS

1. Consider the system

$$\frac{dx}{dt} = -x^2 + y^2$$
$$\frac{dy}{dt} = -x^2 - y^2 + u$$

   (a) Find all equilbria of this system, when $u = 1$, and find linearisations valid around each these equilibria.

   (b) Sketch the state space trajectories in the vicinity of each of these equilibria. What can you say about the behavior of the system in the rest of the state space (keep u at 1).

   (c) Now consider changing u. What steady state values of x and y will result for any given constant u?

2. (a) For the system

$$L(s) = \frac{1}{(s+a)(s+b)} \qquad (a, b \text{ both real})$$

   show that the root-locus diagram (for positive gains k) consists of the segment of the real axis between $-a$ and $-b$, and the perpendicular bisector of that segment.

   (b) Sketch the root-locus diagram for positive gains k for the system

$$L(s) = \frac{1}{s(s+1)^2}$$

   Find the (positive) value of k at which closed-loop stability is lost

   (i) from your diagram, and

   (ii) using the Routh-Hurwitz criterion.

   (c) Draw the root-locus diagram for positive gains k for the system

$$L(s) = \frac{s}{(s+0.5)(s+1)}$$

   and hence show that the closed-loop system is stable for all $k > 0$. Also sketch the root-locus diagram for negative gains, and find the value of k at which closed-loop stability is lost.

3. Consider again the position control system of Question 1 in Examples Paper 1. Suppose that the position feedback gain k_θ is fixed at 10 V/rad.
   (a) Sketch the variation of the closed-loop poles as the tacho feedback gain k_d varies
      (i) using root-locus construction rules,
      (ii) by finding an explicit expression for the closed-loop poles.
   (b) What is the damping factor of the closed loop as a function of k_d? Sketch the time response of the load angular position to a step change of 1 radian in desired angular position for values of $k_d = 0.6$ and $k_d = 1.2$.

4. A negative feedback system consists of a plant whose transfer function is that given in Question 2(b), and a controller which is just a positive gain k. The reference signal is a ramp $r(t) = 2t$. Suppose that k is set to that value which gives two coincident real closed-loop poles at $-1/3$. What is this value of k, and what is the steady-state error $e = r - y$ (where y is the output of the plant) obtained with this value?

   How should the gain be adjusted to reduce this error? What can be said about the locations of the closed-loop poles if this is done?

   (Use your results from Question 2(b).)

Answers

1. (a) — (b) $k = 2$. (c) $k = -3/2$.
2. (a) — (b) Damping factor: k_d.
3. $k = 4/27$. Steady-state error $= 27/2$.

Relevant questions on past 3F2 exam papers:
2008: Q.1. 2009: Q.2. 2010: Q.2. 2011: Q.2. 2012: Q.3. 2013: Q.2

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