3A3: The equations of fluid flow & their numerical solution

Examples Paper 1

February 2016

1 The differential equation

$$\frac{dy}{dt} = -y$$

with initial conditions $y(0) = 1$, is to be solved approximately by a finite-difference method. Let $y_n = y(n\Delta t)$ denote the approximation for y at time $t = n\Delta t$. By using a first order accurate forward difference approximation for dy/dt, show that $y_n = (1-\Delta t)^n$, and hence that, for small values of Δt, $y_n \approx e^{-t}$. (You may find it helpful to use one of the limit formulae in section 2 of the Maths Data book).

2 (a) By considering the points at $i-2$, $i-1$, i, $i+1$ and $i+2$, derive a central difference approximation to $\partial^3 F/\partial x^3$ on a uniform grid with spacing Δx.

(b) By considering points at $i = 1$, 2 and 3 on a uniform grid with spacing Δx, obtain approximations to $\partial F/\partial x$ and $\partial^2 F/\partial x^2$ at the boundary point $i = 1$. What is the order of accuracy of these approximations?

(c) By considering points at $i = 1$, 2, 3 and 4 on a uniform grid with spacing Δx, obtain a second-order accurate approximation to $\partial^2 F/\partial x^2$ at the boundary point $i = 1$.

3 A non-uniform grid has spacing Δx_+ between points i and $i+1$ and spacing Δx_- between $i-1$ and i. Use a Taylor series to obtain approximations to $\partial F/\partial x$ and $\partial^2 F/\partial x^2$ at the central grid point i. Show that these expressions are second order accurate only if $(\Delta x_+ - \Delta x_-)$ is small, i.e. or order Δx^2.

4 Show that a downwind difference scheme for the one dimensional convection equation

$$\frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x}$$

of the form

$$u_i^{n+1} = u_i^n - c\left(u_{i+1}^n - u_i^n\right)$$

is unstable to a disturbance of the form of a "sawtooth" wiggle for all values of c.

5 Show that the Lax-Wendroff scheme applied to the one-dimensional convection equation is like solving the equivalent PDE

$$\frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} = -\frac{A\Delta x^2}{6}\left(1 - \frac{A^2\Delta t^2}{\Delta x^2}\right)\frac{\partial^3 u}{\partial x^3}$$

6 Consider the convection-diffusion equation

$$\frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}$$

(a) By using a first-order forward difference formula for $\partial u/\partial t$, first-order upwind in space for $\partial u/\partial x$ and a second-order central difference for $\partial^2 u/\partial x^2$ show that the finite-difference equation is given by

$$u_i^{n+1} = u_i^n - c\left(u_i^n - u_{i-1}^n\right) + \alpha\left(u_{i-1}^n - 2u_i^n + u_{i+1}^n\right)$$

where c is the CFL and $\alpha = \nu\Delta t/\Delta x^2$.

(b) Modify the `convection.m` code from Handout 2, to solve this finite difference equation for a "ramp" initial condition with $c = 0.5$ and $\alpha = 1/6$. Compare the result (plot u versus x on the same graph) after 10 time steps with that obtained for $\alpha = 0$.

(c) Give a physical explanation for why the ramp has a lower slope when $\alpha \neq 0$.