Handout 3 – Wave propagation perspective

We can obtain the response of a structure under three conditions using the following methods:

1. Transient free response to initial conditions: match the contribution of modes to the initial conditions
2. Steady-state response to sinusoidal excitation: find the transfer function
3. Transient response to arbitrary inputs: use the Fourier Transform and the transfer function

The results are based on modal analysis: a very general framework that applies to many kinds of systems and which gives a lot of insight into the behaviour of a structure. We have considered examples where analytic solutions are possible: in general the solutions can be computed numerically.

We now consider a different perspective: wave propagation. The two perspectives are wholly compatible: vibration modes are simply a combination of forward and backward travelling waves that cause a standing wave. But both perspectives are useful, and some problems are more readily solved using the perspective of travelling waves.

Transient response from D'Alembert's solution

The modal approach described above is very general and can be used to find the steady-state response (via the transfer function) or the transient response (via a modal summation) for any system. But for the particular case of 1D systems that satisfy the wave equation then there is another way to find the transient response for given initial conditions.

The wave equation has the form:

$$\frac{\partial^2 y}{\partial t^2} - c^2 \frac{\partial^2 y}{\partial x^2} = f(x,t)$$

and, without assuming sinusoidal time-variation, the general solution is given by:

$$y(x,t) = f(x-ct) + g(x+ct)$$

where f and g could be any function. It means that a given wave of shape $f(x)$ propagates forwards without changing shape, and similarly a given wave of shape $g(x)$ propagates backwards.

With some careful thought we can combine our knowledge of the initial conditions and the boundary conditions to decompose the initial conditions into forward and backward travelling components (f and g) for the plucked string.

At $t = 0$ (initial conditions), for all x in the range $0 < x < L$:

- $y = y_0(x)$, the initial trangular shape considered above. So

- $\dfrac{\partial y}{\partial t} = 0$, which corresponds to zero initial velocity:

integrating gives

combining and rearranging:

and because we are only interested in the total response $f + g$, then we can choose $K = 0$. Hence:

in the range $0 < x < L$ (that's the range over which y_0 is defined).

At $x = 0, L$ then $y = 0$ at all times (boundary conditions):

- at $x = 0$, $f(-ct) + g(ct) = 0 \implies f(\gamma) = -g(-\gamma)$

- at $x = L$, $f(L - ct) + g(L + ct) = 0$

combining gives:

which is true for all t. We get a similar expression for f, so both f and g must be periodic over distance $2L$.

To summarise:

- at $t = 0$: $f(x) = g(x) = y_0(x) / 2$
- at $t = 0$: $f(x) = -f(-x)$, it's initially an odd function
- for all t: f and g repeat every $2L$
- f travels forward and g travels backward at speed c.

Sketch for $t = 0$:

Sketch for $t = \mathrm{d}t$:

Gives the same result that we saw for the modal sum. See: `H3_Dalembert.m`

Transmission line analogy

1D structures that satisfy the wave equation are mathematically equivalent to electrical transmission lines. The wave equation has the form:

$$\frac{\partial^2 y}{\partial t^2} - c^2 \frac{\partial^2 y}{\partial x^2} = f(x,t)$$

and, without assuming sinusoidal time-variation, the general solution is given by:

$$y(x,t) = f(x - ct) + g(x + ct)$$

where f and g could be any function. It means that a given wave of shape $f(x)$ propagates forwards without changing shape, and similarly a given wave of shape $g(x)$ propagates backwards.

Recall the key results from Part IB transmission lines which also satisfies the wave equation:

We can apply the principles we already know in the electrical context to wave propagation in mechanical systems. We can choose the analogy: voltage to velocity, and current to force.

Example: torsional vibration of an oilwell drillstring

A drillstring is a long steel pipe used to drill for oil: it is usually several kilometers long, around 0.1m diameter, and is made up of hundreds of 10m long sections threaded together. This gives an aspect ratio of around $10^4 - 10^5$, hence the name. A motor at the surface rotates the whole drillstring and provides the cutting torque for the drill bit at the bottom.

Torsional vibration of the drillstring can be modelled using the 1D wave equation derived in Handout 1 for a uniform shaft:

$$\rho J \frac{\partial^2 \theta}{\partial t^2} - GJ \frac{\partial^2 \theta}{\partial x^2} = T(x,t)$$

We can choose the analogy: voltage to angular velocity, and current to torque:

A forward travelling torsional wave in a drillstring has the form $\theta_F(x - ct)$

Therefore:

$$\Omega_F = \dot{\theta}_F(x - ct) = -c\theta_F$$

$$T_F = -GJ\theta'_F(x - ct) = -GJ\theta_F$$

$$Z_0 \equiv \frac{\Omega_F}{T_F} = \frac{c}{GJ} = \frac{1}{J\sqrt{\rho G}}$$

recalling that $c = \sqrt{G/\rho}$

We can find the reflection coefficients downhole and at surface using the same expression as for the electrical case, noting that $Z_1 = 0$ and $Z_2 = \infty$

This lets us immediately determine the impulse or step response of the structure:

impulse response

step response

We would get the same answer if we calculated this via the transfer function: but it is very laborious to do so!

Dispersion equation

So far the structures we have discussed follow the 1D wave equation, where the wavenumber and frequency are related by $\omega = ck$ (where c is constant). This is not always true, and to see what happens we need to look more closely at how a group of waves propagate. For a thorough treatment, see M. J. Lighthill "Waves in Fluids", CUP 1978, library reference TA277.

Imagine causing a local disturbance to a structure, such that a group of waves start to propagate:

we can represent the motion $y(x,t)$ as a slowly varying envelope function $Y_0(x,t)$ multiplied by a fast-varying harmonic function of time and space:

Now take a snapshot at $x = x_0$ and $t = t_0$ and use a Taylor approximation of α for small changes in x and t:

$$\alpha(x,t) \approx \alpha(x_0,t_0) + \left.\frac{\partial\alpha}{\partial x}\right|_{x_0}(x - x_0) + \left.\frac{\partial\alpha}{\partial t}\right|_{t_0}(t - t_0)$$

So if we imagine a group of sinusoidal waves propagating

$$y(x,t) = Y_0(x,t)\mathrm{e}^{i(ct-kx)}$$

then comparing expressions gives

$$k = -\frac{\partial\alpha}{\partial x}, \quad \omega = \frac{\partial\alpha}{\partial t}$$

Differentiating k with respect to t and ω with respect to x gives:

$$\frac{\partial k}{\partial t} + \frac{\partial\omega}{\partial x} = 0$$

and we know that k and ω are related by some function $\omega(k)$, so

$$\frac{\partial k}{\partial t} + c_g\frac{\partial k}{\partial x} = 0$$

where c_g is called the group velocity and is given by

The main result is that k, and hence ω, are constant along lines of $(x\text{-}c_g t)$ = constant. In other words the group velocity tells us the speed at which a group of waves travels: not just how fast a particular crest of a wave moves, but how fast the overall cluster or packet of waves travels.

For the stretched string:

$$m\frac{\partial^2 y}{\partial t^2} - P\frac{\partial^2 y}{\partial x^2} = f(x,t)$$

The forward travelling component of the free vibration solution is:

$$y_F(x,t) = A\mathrm{e}^{i(\omega t - kx)}$$

substituting into the governing PDE gives:

So the wave speed c is the same as the group velocity c_g: equivalently the group velocity is constant and does not depend on frequency. This means that there is no dispersion: all frequencies travel at the same speed and this is why arbitrarily shaped disturbances can propagate without changing shape (dispersion) along structures that satisfy the wave equation.

This is not true for all structures, as we will see next.

Vibration of a bending beam

The next system to consider is important in many engineering applications: bending vibration of a slender beam, which we will treat in the simplest approximation often called the 'Euler beam' or 'Euler-Bernoulli beam'.

We assume that the beam is made of material with Young's modulus E and density ρ, and that it has a uniform cross-section with area A, and flexural rigidity EI. It vibrates with small transverse displacement $y(x,t)$ in response to applied transverse force $f(x,t)$ per unit length. To derive the PDE governing its behaviour we can follow the same steps as we have seen before.

1. Draw a big diagram of a small section of the structure

2. label the coordinates: the start and length of the small section
3. label the displacements from equilibrium: nominal values on the left, small changes on the right
4. identify any distributed external forces
5. work out expressions for the internal forces on both sides of the small section
6. apply $F = ma$ and take limits

Work out expressions for the internal forces:

so

Apply $F = ma$:

Take limits:

The values of I for cross-sections of common shapes can be found in the mechanics databook.

Now assume that a wave is propagating forward along the beam of wavenumber k and frequency ω:

$$y(x,t) = A\mathrm{e}^{i(\omega t - kx)}$$

Substitute into the governing PDE to get the dispersion equation for a bending beam:

The group velocity is given by

which tells us that high frequencies travel faster than slow. So if we apply an impulse to a beam, the different frequency components of the impulse will spread as the group of waves travel. By the time they reflect, the effect is sometimes audibly noticeable.

Summary

Wave propagation through 1D structures that satisfy the wave equation are analagous to electrical transmission lines. We can use the concepts of *characteristic impedance* and *reflection coefficients* at boundaries to calculate the transient response.

We can find the transient response for 1D systems governed by the wave equation using D'Alembert's solution: decomposing the response into forward and backward travelling waves.

The relationship between frequency ω and wavenumber k gives the *dispersion equation* for wave propagation in a structure. The group velocity is given by:

$$c_g = \frac{\partial \omega}{\partial k}$$

The group velocity is constant for structures satisfying the wave equation, i.e. there is no dispersion and waves propagate without changing shape. When the group velocity depends on frequency, then wave shapes disperse over time.

Vibration of a bending beam is an example of a non-constant group velocity that is important in many engineering structures.