Handout 8 – Case Study

The 3C6 Toolbox: Continuous Systems

Handout 1: Equations of motion: deriving PDEs for 1D structures
Vibration modes and natural frequencies

Handout 2: Transient response to initial conditions: Matching mode shapes to initial conditions
Harmonic forced response: analytic Transfer Functions

Handout 3: D'Alembert's solution (1D wave equation)
Transmission line analogy
Harmonic travelling wave solution and the dispersion equation

Handout 4: Euler beams: boundary conditions and vibration modes

Handout 5: Discretisation of continuous systems: lumped mass model / generalised coordinates
Transfer Functions from modal summation, and modal damping
Impulse / step response using modal summation

Handout 6: Coupling subsystems
Rayleigh approximation

Handout 7: Experimental methods

Case study: Turbine Blade Vibration

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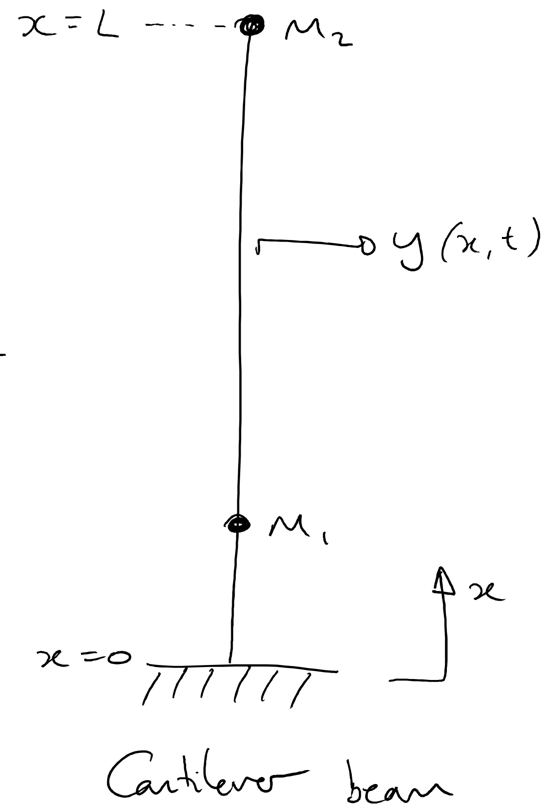
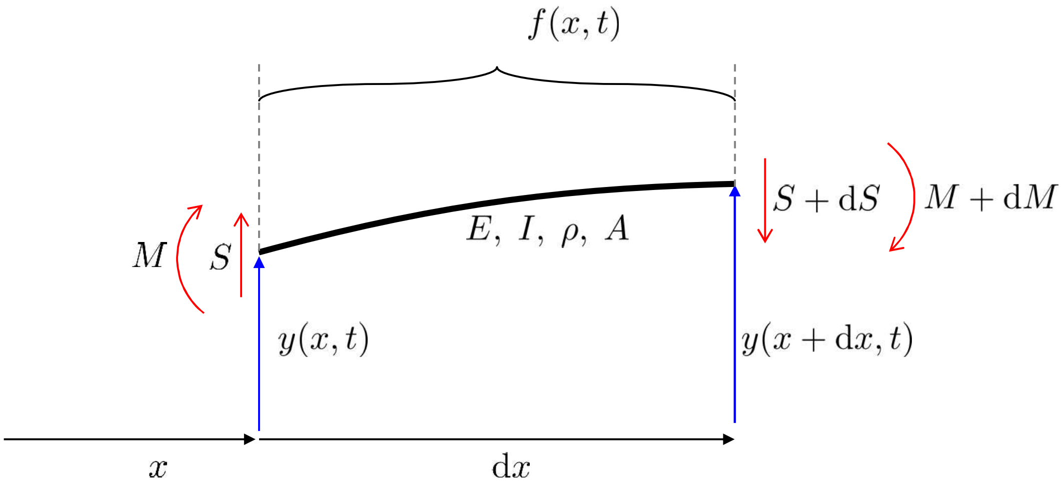


Beam: equations of motion



Shear / Moment / Curvature: $S = \frac{\mathrm{d}M}{\mathrm{d}x}, \quad M = EI\mathrm{d}\kappa, \quad \mathrm{d}\kappa = \frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$

Change in shear: $dS = \frac{dS}{dx} dx = EI \frac{\partial^4 y}{\partial x^4} dx$

$F = ma$: $-\mathrm{d}S + f(x,t)\mathrm{d}x = \rho A \mathrm{d}x \frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$

Giving the equation of motion:

$$\rho A \frac{\partial^2 y}{\partial t^2} + EI \frac{\partial^4 y}{\partial x^4} = f(x,t)$$

Forward travelling sinusoidal waves satisfy: $y(x,t) = Y_F \mathrm{e}^{i(kx-\omega t)}$

Allowing the dispersion relation to be found:

$$\left(-\rho A \omega^2 + k^4 EI\right) Y_F = 0 \implies \omega = k^2 \sqrt{\frac{EI}{\rho A}}$$

This gives the phase velocity c_p and group velocity c_g:

$$c_p = \frac{\omega}{k} = k\sqrt{\frac{EI}{\rho A}} \quad \text{and} \quad c_g = \frac{\partial \omega}{\partial k} = 2k\sqrt{\frac{EI}{\rho A}}$$

Free vibration modes take the form

$$y(x,t) = u(x) e^{i\omega t}$$

Giving the fourth order ODE for free vibration modes:

$$EI u^{iv} - \rho A \omega^2 u = 0$$

The solutions can be written in the form:

$$U(x) = D_1 \cos kx + D_2 \sin kx + D_3 \cosh kx + D_4 \sinh kx$$

We will need the derivatives of this expression so that we can find the constants from the boundary conditions:

$$\begin{array}{lllllll}
U'(x) & = & k & (& -D_1 \sin kx & +D_2 \cos kx & +D_3 \sinh kx & +D_4 \cosh kx) \\
U''(x) & = & k^2 & (& -D_1 \cos kx & -D_2 \sin kx & +D_3 \cosh kx & +D_4 \sinh kx) \\
U'''(x) & = & k^3 & (& +D_1 \sin kx & -D_2 \cos kx & +D_3 \sinh kx & +D_4 \cosh kx)
\end{array}$$

The boundary conditions at the clamped end ($x = 0$) are:

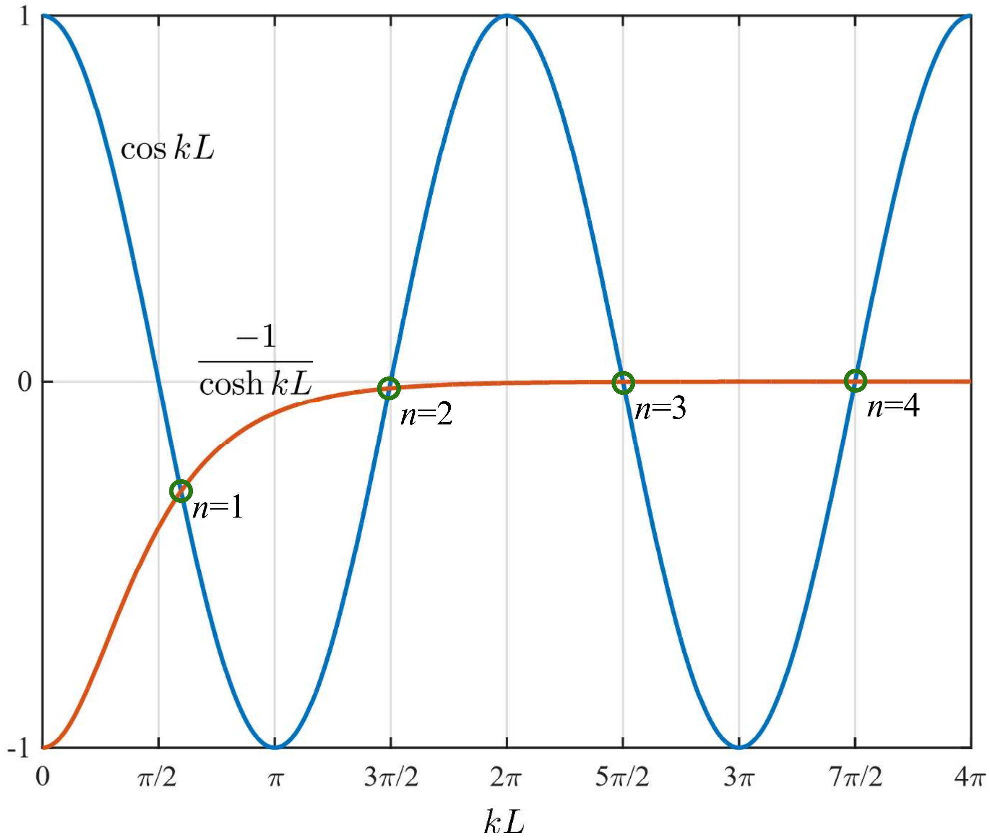
$$U(0) = 0 \Rightarrow D_1 + D_3 = 0$$
$$U'(0) = 0 \Rightarrow D_2 + D_4 = 0$$

The boundary conditions at the free end ($x = L$) are:

$$U''(0) = 0 \implies D_1 (-\cos kL - \cosh kL) + D_2 (-\sin kL - \sinh kL) = 0$$
$$U'''(0) = 0 \implies D_1 (\sin kL - \sinh kL) + D_2 (-\cos kL - \cosh kL) = 0$$

Leading to the equation whose solutions give the wavenumber k_n for each mode:

$$\cos kL \cosh kL = -1$$



We can solve this numerically by typing into Matlab:

```
>> fsolve(@(kL) cos(kL)*cosh(kL)+1,pi/2)
```

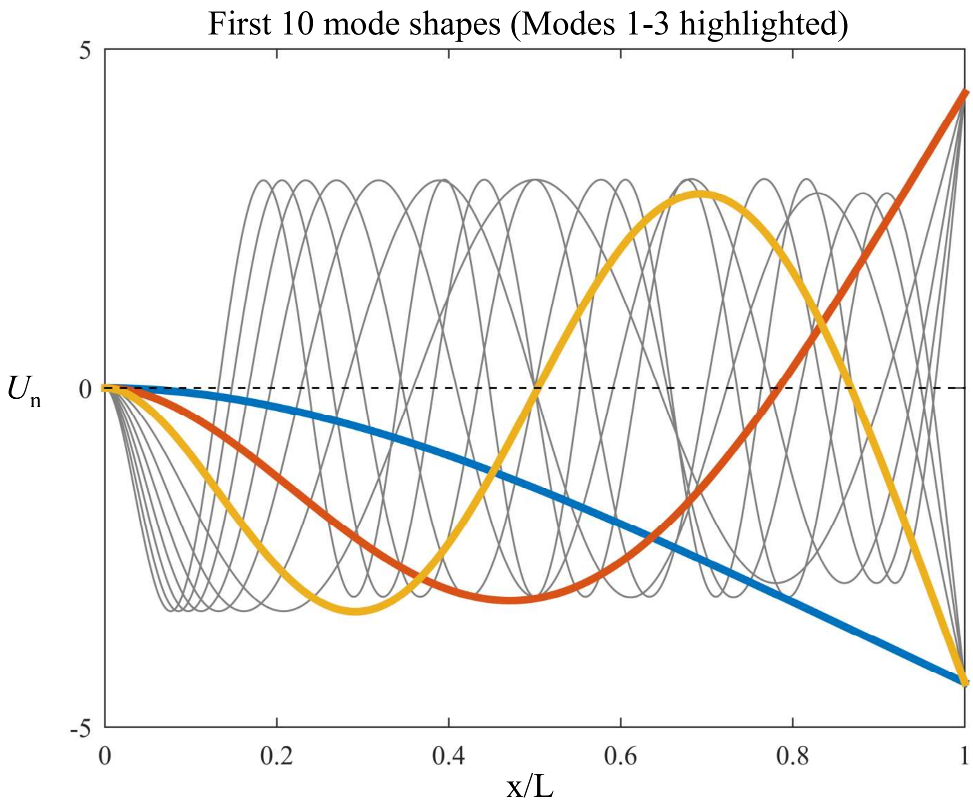
and using different starting guesses for each mode.

Or use the approximation: $kL \approx (n-\frac{1}{2})\pi$

Mode	Approximate k_nL	Numerical k_nL	Error (%)
1	1.5708	1.8751	19.3726
2	4.7124	4.6941	0.3881
3	7.8540	7.8548	0.0104
4	10.9956	10.9960	0.0039
5	14.1372	14.1370	0.0012
6	17.2788	17.2790	0.0014
7	20.4204	20.4200	0.0017
8	23.5619	23.5620	0.0002
9	26.7035	26.7040	0.0017
10	29.8451	29.8450	0.0004

Reconstruct mode shapes from analytic expression

$$U(x) = D_1 \cos kL + D_2 \sin kL - D_1 \cosh kL - D_2 \sinh kL, \quad \text{where} \quad D_2 = D_1 \frac{\sin kL - \sinh kL}{\cos kL + \cosh kL}$$



It is sometimes useful to have an analytic approximation of the mode shape. We can use Rayleigh's principle to do this:

Let
$$\begin{aligned}
u(x) &= a_3x^3 + a_2x^2 + a_1x + a_0\\
u'(x) &= 3a_3x^2 + 2a_2x + a_1\\
u''(x) &= 6a_3x + 2a_2\\
u'''(x) &= 6a_3
\end{aligned}$$

$u(0) = 0 \Rightarrow a_0 = 0$

$u'(0) = 0 \Rightarrow a_1 = 0$

$u''(L) = 0 \Rightarrow 6a_3L + 2a_2 = 0$

$u'''(L) = 0 \Rightarrow 6a_3 = 0 \Rightarrow a_2 = 0$!

choose $a_3 = 1$

$a_2 = -3L$

This leads to:

$u = x^3 - 3Lx^2$

$u^2 = x^6 - 6Lx^5 + 9L^2x^4$

$\tilde{T} = \frac{1}{2}\rho A\int_0^L u^2\,dx$ — DATASHEET

$$= \frac{1}{2}\rho A\int_0^L (x^6 - 6Lx^5 + 9L^2x^4)\mathrm{d}x$$

$$= \frac{1}{2}\rho A\left[\frac{x^7}{7} - Lx^6 + \frac{9L^2x^5}{5}\right]_0^L$$

$$= \frac{33}{70}\rho A L^7$$

$u'' = 6x - 6L$

$u''^2 = 36(x^2 - 2Lx + L^2)$

$V = \frac{1}{2}EI\int_0^L u''^2\,dx$ — DATASHEET

$$= \frac{1}{2}36EI\int_0^L (x^2 - 2Lx + L^2)\mathrm{d}x$$

$$= 18EI\left[\frac{x^3}{3} - Lx^2 + L^2x\right]_0^L$$

$$= 6EIL^3$$

$$\omega_1^2 \approx \frac{V}{\tilde{T}} = \frac{6EIL^3}{\frac{33}{70}\rho AL^7} = \frac{140EI}{11\rho AL^4} \simeq 12.73\,\frac{EI}{\rho AL^4}$$

actual: $kL = 1.8751$, $\omega_1^2 = k^4\frac{EI}{\rho A} = \underbrace{1.8751^4}_{12.36}\frac{EI}{\rho AL^4}$

To use the transfer function modal summation formula we need to mass normalise the mode shapes:

$$u_1(x) = \lambda\left(x^3 - 3Lx^2\right)$$

$$\int_0^L u_1^2(x)\,\rho A\,dx = 1$$

But we've already done this integral, so: $\lambda^2 \cdot \frac{33}{35}\rho A = 1$

$$\lambda = \sqrt{\frac{35}{33\rho A}}$$

This can be used to approximate the driving point transfer function *G*, valid for frequencies near the first natural frequency:

$$G(x,y,\omega) = \sum_n \frac{u_n(x)\,u_n(y)}{\omega_n^2 - \omega^2}$$

$$G(x,x,\omega) \simeq \frac{35}{33\rho A}\,\frac{\left(x^3 - 3Lx^2\right)^2}{\left(\frac{140}{11}\frac{EI}{\rho AL^4} - \omega^2\right)} = \tilde{G}$$

In practice each blade is connected to the next through friction dampers located under a platform. The platform adds mass to the each blade, lowering its natural frequencies. The effect on the first natural frequency can be approximated using Rayleigh's principle.

Consider adding a mass *M* at position $x = x_0$. The potential energy is unchanged, and the 'derivative-free kinetic energy' becomes:

$$\tilde{T} = \tilde{T}_0 + \tfrac{1}{2}Mx_0^2$$

$$\text{so } \omega_1^2 \simeq \frac{V}{\tilde{T}} = \frac{\left(6\,EIL^3\right)}{\left(\frac{33}{70}\rho AL^7 + \frac{1}{2}Mx_0^2\right)}$$

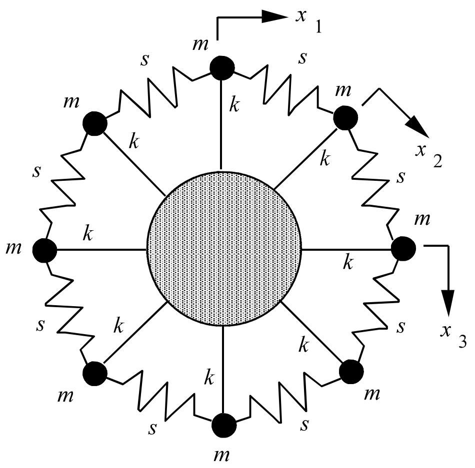
In addition each blade is spinning at some angular speed Ω, so that each blade is under tension. We know that for a stretched string then tension is what gives it stiffness. The same is true here: the tension adds stiffness to the system, but not uniformly because the tension is a function of radial distance from the axis. Without having to re-derive the equations of motion the PDE for a beam under tension is given by:

$$\rho A\ddot{y} - Py'' + EIy^{iv} = f(x,t)$$

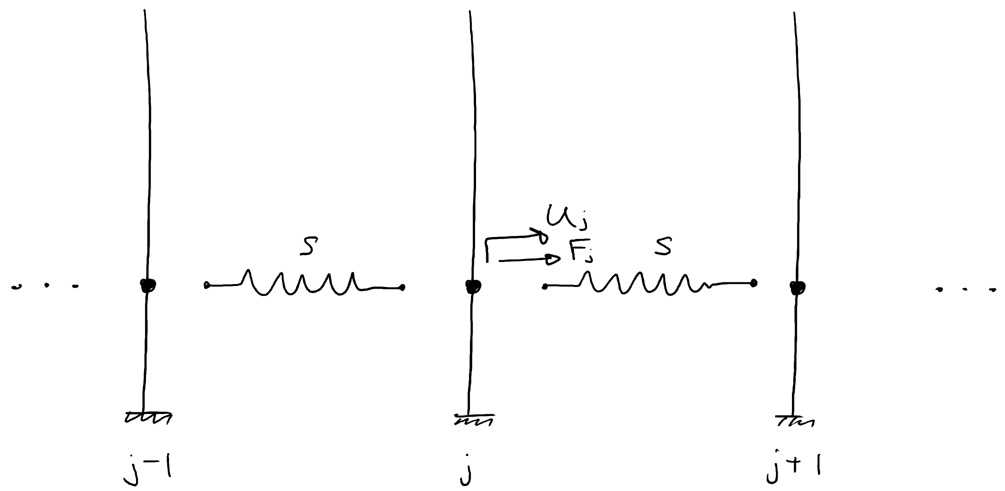
where *P* the tension and is a function of *x*. The effect of adding this stiffness is to increase the natural frequencies. By how much? We could use Rayleigh's principle again to make an estimate.

The behaviour of friction dampers is intrinsically nonlinear: for small amplitude motion there is no slip and each blade is effectively connected to its neighbour as if through a stiff spring; and for large amplitude motion the effect of friction becomes small and it is as if there was no coupling through the damper. These two extremes are linear and can be understood via the tools from this course. For methods for dealing with the fully nonlinear response, then consider taking 4C7…

In the meantime, consider the small amplitude 'no-slip' regime when each blade is coupled to its neighbours via a spring of stiffness s. What is the effect of coupling N blades in a circular array? You have already seen a rather similar question in Examples Paper 2:



Now we consider what happens if each blade is a general system with known driving point transfer function at the point of coupling.



$$u_j = GF_j$$

or $u_j - GF_j = 0$

$$F_j = s(u_{j+1} - u_j) + s(u_{j-1} - u_j)$$

$$= s(u_{j+1} - 2u_j + u_{j-1})$$

This can be written in matrix form:

$$\begin{bmatrix} \\ \underline{u} \\ \\ \end{bmatrix} + Gs \begin{bmatrix} 2 & -1 & & & -1 \\ -1 & 2 & -1 & 0 & \\ & -1 & \ddots & \ddots & \\ & 0 & \ddots & \ddots & -1 \\ -1 & & & -1 & 2 \end{bmatrix} \begin{bmatrix} \\ \underline{u} \\ \\ \end{bmatrix} = \begin{bmatrix} \\ 0 \\ \\ \end{bmatrix}$$

And because the system is circularly symmetric:

assume $\quad u_j = \cos j\lambda$

There are two unknowns, G and λ so we need two independent equations:

$$\text{(A) row 1:} \quad \cos\lambda \quad + \quad Gs\,(2\cos\lambda - \cos 2\lambda - \cos N\lambda) \quad = 0$$

$$\text{(B) row j:} \quad \cos j\lambda \quad + \quad Gs\,(2\cos j\lambda - \cos(j-1)\lambda - \cos(j+1)N\lambda) \quad = 0$$

Rearranging:

$$\text{(B)} \Rightarrow \quad \cancel{\cos j\lambda} + 2Gs\,\cancel{\cos j\lambda} - 2Gs\,\cancel{\cos j\lambda}\,\cos\lambda = 0$$

$$1 + 2Gs(1 - \cos\lambda) = 0$$

$$2Gs = \frac{1}{\cos\lambda - 1}$$

$$\text{(A)} \Rightarrow \quad \cos\lambda + \frac{\cos\lambda}{\cos\lambda - 1} - \frac{1}{2(\cos\lambda - 1)}\left(\cos 2\lambda + \cos N\lambda\right) = 0$$

$$\cos\lambda\left(\cos\lambda - 1\right) + \cos\lambda - \tfrac{1}{2}\left(\cos 2\lambda + \cos N\lambda\right) = 0$$

$$\cos^2\lambda = \tfrac{1}{2}\left(\cos 2\lambda + \cos N\lambda\right)$$

$$\tfrac{1}{2}\left(1 + \cos 2\lambda\right) = \tfrac{1}{2}\left(\cos 2\lambda + \cos N\lambda\right)$$

$$\Rightarrow \cos N\lambda = 1$$

$$\lambda = \frac{2n\pi}{N}, \qquad G = \frac{1}{2s(\cos\lambda - 1)}$$

So the circumferential mode shapes evaluated at the coupling points are:

n^{th} mode
j^{th} blade

$$U_n(j) = \cos j\lambda_n = \cos \frac{2nj\pi}{N}$$

Half of these repeat, so the orthogonal set of modes must be the other polarisations:

$$U_n(j) = \sin j\lambda_n = \sin \frac{2nj\pi}{N}$$

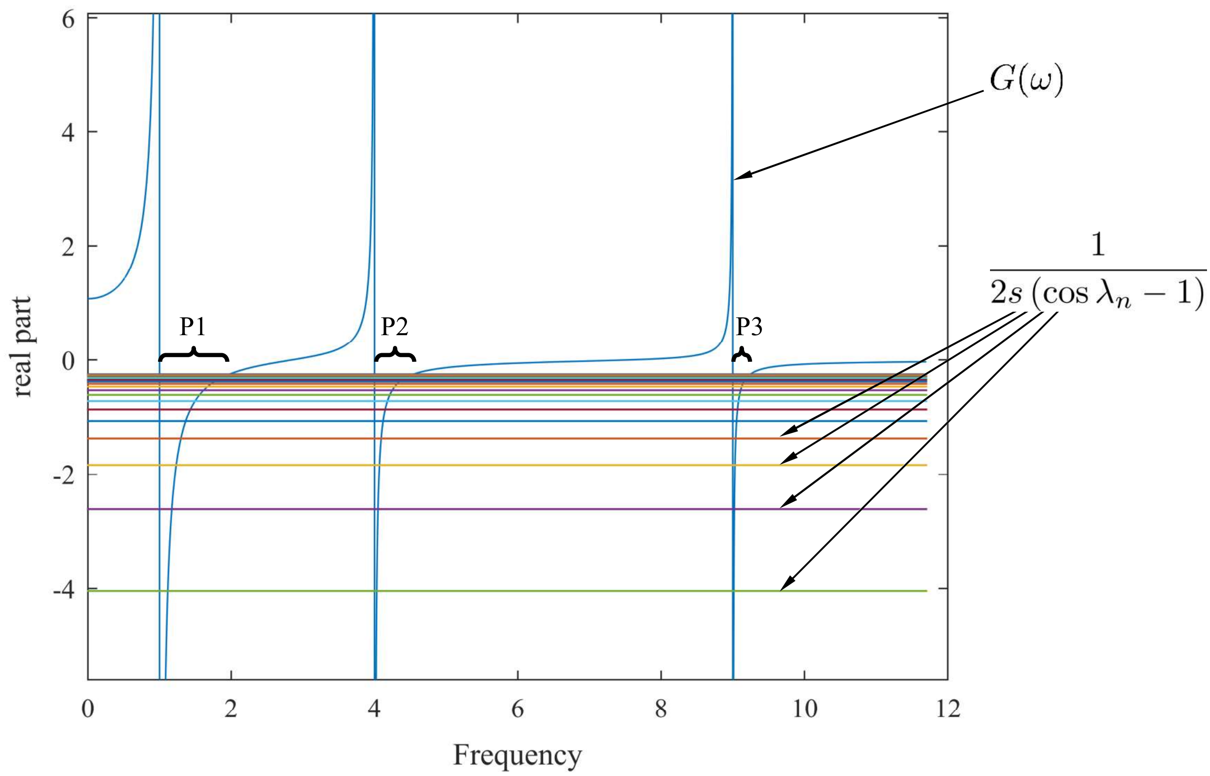
To obtain the frequencies we need to find the solutions of:

$$G(\omega) = \frac{1}{2s\left(\cos\lambda_n - 1\right)}$$

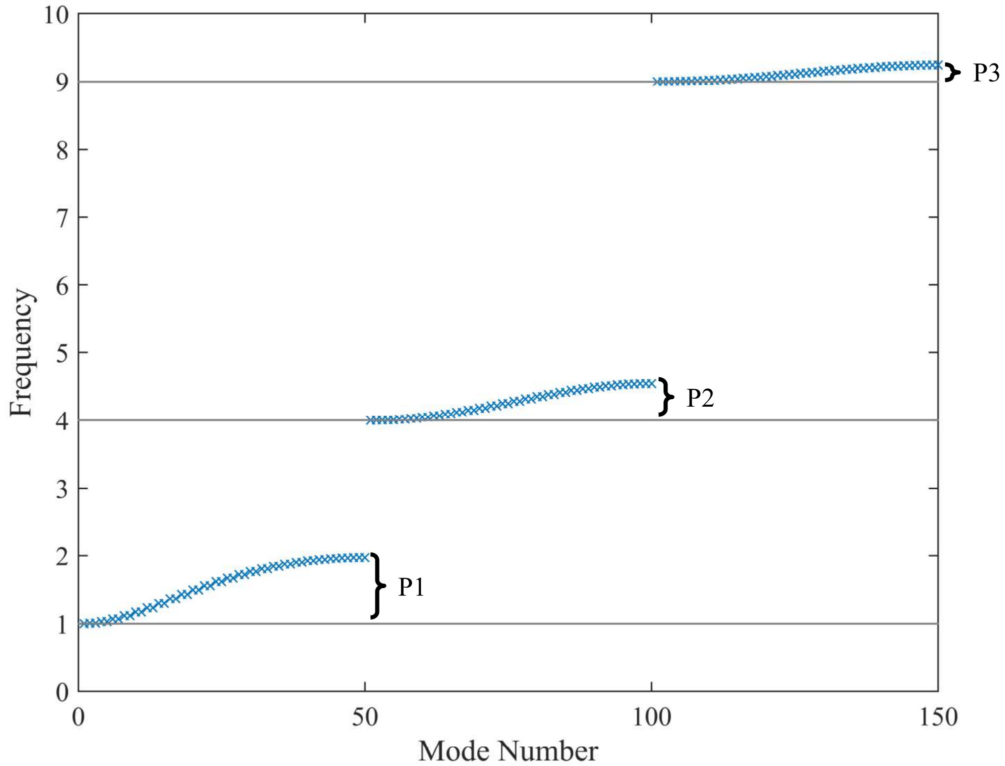
which can be solved graphically or numerically: see `H8_periodic_beams.m`.

The result is a pattern of passbands: groups of natural frequencies associated with the original uncoupled blade modes, labelled P1, P2 and P3 in the following figures. Notice that when the coupling stiffness *s* is small, then *G* must be large and the natural frequencies are all very close to the original blade modes.

By way of example, consider a three mode system with $\omega_n = [1 \quad 4 \quad 9]$:



The solutions give the natural frequencies of each mode, which fall into well-defined groups or *passbands* (P1, P2 and P3).



Finally the blades are excited by a rotating pattern of forces created due to the previous stator stage. This has the form:

$$F = F_0 e^{iD(\theta - \Omega t)}$$

where *D* is the number of stator vanes from the previous stage, and *Ω* is the speed of rotation of the shaft. It appears to the rotating blades as a rotating distributed sinusoidal force (sinusoidal in time and angle). Because the circumferential mode shapes of the assembly of blades are also sinusoidal, then only the particular mode from within each passband will be excited, as all other mode shapes are orthogonal to the forcing pattern.

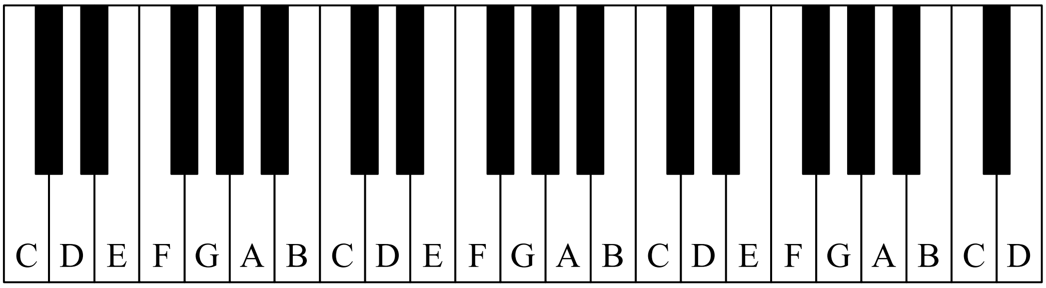
Previously we have seen that a particular mode will not be observed if it is excited or measured at a nodal point. This is a more general result: a particular mode will not be observed if it is excited by a distributed force that is orthogonal to the mode shape.

A musical interlude

Approximate measurements of vibration can often be made by ear, especially if there is a tuned instrument to hand to provide a set of reference pitches (or frequencies). Without that most of us are not very good at judging absolute pitch, but we are still usually fairly good at judging the relative pitch of two sounds. We hear the same musical interval when two tones have the same frequency ratio. An octave is a factor of 2, so going up one octave is a doubling of frequency.

Below is part of a piano keyboard. Every adjacent pair of notes, regardless of whether the keys are black or white, has frequencies in the same ratio (called a semitone). There are 12 semitones in an octave, so the ratio of frequencies for one semitone has to be $2^{1/12} \approx 1.06$. In other words a semitone corresponds to a 6% increase in frequency. The most acute musical ears can discriminate $1/50^{\text{th}}$ of a semitone, i.e. a frequency change of about 0.1%.

"C sharp" : C# = D♭ : "D flat"   one octave



The standard frequencies of the notes on a piano can be easily tabulated:

Note	Frequency (Hz)						
C	32.70	65.41	130.8	261.6	523.3	1047	2093
C#	34.65	69.30	138.6	277.2	554.4	1109	2217
D	36.71	73.42	146.8	293.7	587.3	1175	2349
D#	38.89	77.78	155.6	311.1	622.3	1245	2489
E	41.20	82.41	164.8	329.6	659.3	1319	2637
F	43.65	87.31	174.6	349.2	698.5	1397	2794
F#	46.25	92.50	185.0	370.0	740.0	1480	2960
G	49.00	98.00	196.0	392.0	784.0	1568	3136
G#	51.91	103.8	207.7	415.3	830.6	1661	3322
A	55.00	110.0	220.0	440.0	880.0	1760	3520
A#	58.27	116.5	233.1	466.2	932.3	1865	3729
B	61.74	123.5	246.9	493.9	987.8	1976	3951
Octave:	**1**	**2**	**3**	**4**	**5**	**6**	**7**

This is the system of 'equal temperament', with every semitone representing the same frequency ratio and it is the way that pianos are tuned. But to an acute musical ear some of the intervals sound out of tune. The worst offender is the interval of four semitones (the *major third*). The equal temperament formula gives a ratio $2^{4/12} = 1.2599$, but our ears prefer the ratio 5/4=1.25. To enquire more deeply opens a can of worms…

see `H8_equal_temperament.m`

And just for fun…

https://www.youtube.com/watch?v=muCPjK4nGY4