UNIVERSITY OF CAMBRIDGE

3F1, Signals and Systems

PART III

From filters to controllers: Nyquist stability criterion

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November 5, 2018

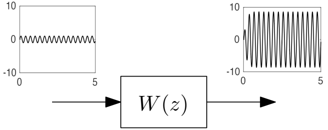
Goal of the lecture:

Filters/controllers for closed loop.

Inferring closed loop stability from
the analysis of the open loop.

Filter design for signal processing

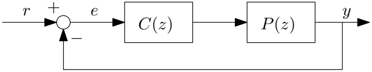
Design the filter $W(s)$ so that the filtered signal is...



Example: enhance/reduce the low frequencies of your music, band-pass filter to select a specific radio station

Filter design for control

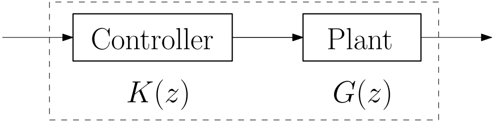
Design the controller $C(s)$ so that the closed loop $W_{r,e} = \frac{1}{1+PC} \dots$



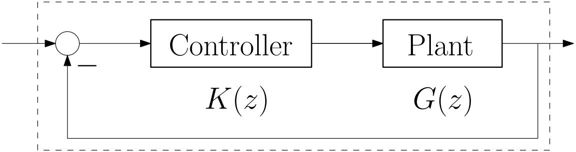
Example: shape the frequency response of the open loop PC to achieve performance/robustness of closed-loop tracking error $W_{r,e}$

Contro design is a form of filter design...

Control in open loop: $W(z) = G(z)K(z)$



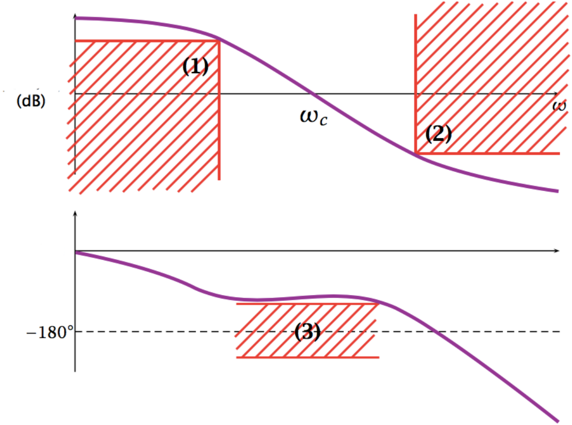
Control design in closed loop: $W(z) = \dfrac{G(z)K(z)}{1 + G(z)K(z)}$



regulation, stability, disturbance rejection...

(Recall from IB Paper 6)

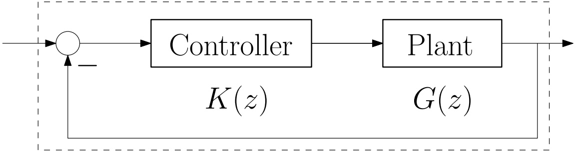
Feedback system design, loop shaping approach



Design $K(z)$ to 'shape' the open loop G(z)K(z)
so that the closed loop behavior...

The simplest question when we design a controller:

given $K(z)$ and $G(z)$, is the closed loop stable?



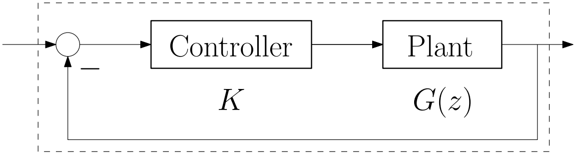
Nyquist Stability Criterion answers the stability question

Predicts the stability of the closed loop

$$W(z) = \frac{G(z)K}{1 + G(z)K}$$

from the "Nyquist diagram" of the open loop

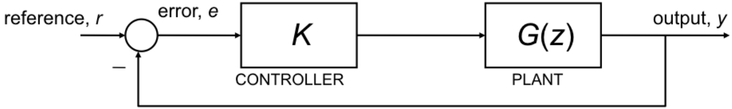
$$G(z)K$$

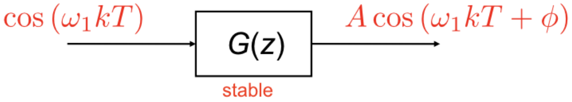


The test in discrete time is analogous to the continuous time case.

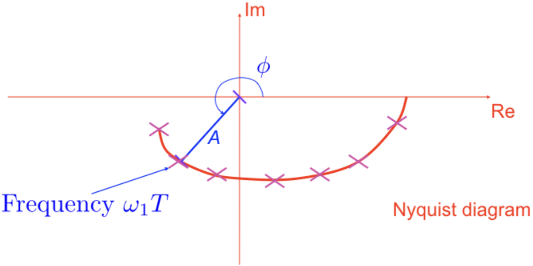
Nyquist diagram

The Nyquist diagram can be determined experimentally (or from the Bode diagram...).



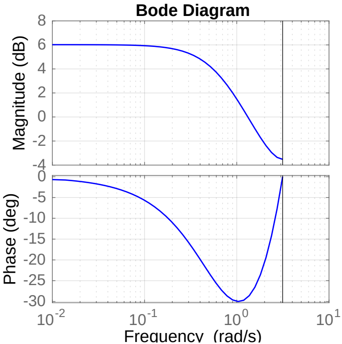


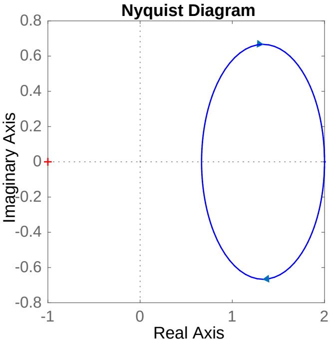
Experiment (if open loop stable):



Example

$$G(z) = \frac{1}{1 - 0.5z^{-1}}$$





The encirclement property

The Encirclement Property

For any rational function $F(z)$ then the number of encirclements of the origin by $F(e^{j\theta})$ as θ increases from 0 to 2π tells us something about the number of poles and zeros of $F(z)$ inside the unit circle, as follows.

Let

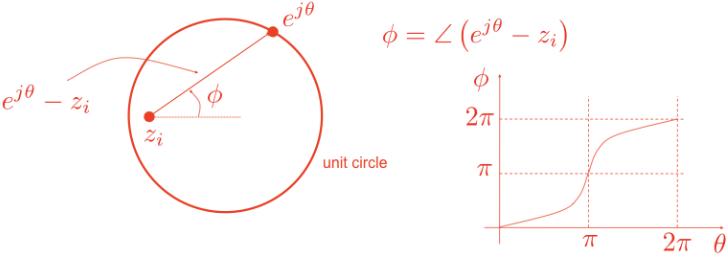
$$F(z) = \frac{A(z - z_1)(z - z_2)\dots(z - z_m)}{(z - p_1)(z - p_2)\dots(z - p_n)}$$

then

$$\text{(i)}\quad \angle F(e^{j\theta}) = \angle A + \sum_{i=1}^{m} \angle(e^{j\theta} - z_i) - \sum_{i=1}^{n} \angle(e^{j\theta} - p_i)$$

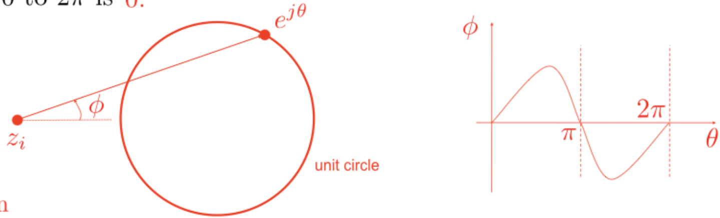
$$F(z) = \frac{A(z-z_1)(z-z_2)\dots(z-z_m)}{(z-p_1)(z-p_2)\dots(z-p_n)}$$

(ii) If $|z_i| < 1$ then the increase in $\angle(e^{j\theta} - z_i)$ as θ increases from 0 to 2π is 2π.



(iii) If $|z_i| > 1$ then the increase in $\angle(e^{j\theta} - z_i)$ as θ increases from 0 to 2π is 0.

No net change in ϕ during one whole revolution



$$F(z) = \frac{A(z-z_1)(z-z_2)\dots(z-z_m)}{(z-p_1)(z-p_2)\dots(z-p_n)}$$

(iv) Hence the increase in $\angle F(e^{j\theta})$ as θ increases from 0 to 2π is

$$\begin{aligned} = 2\pi(&\text{ number of zeros of } F(z) \text{ inside the unit circle} \\ &- \text{number of poles of } F(z) \text{ inside the unit circle }) \end{aligned}$$

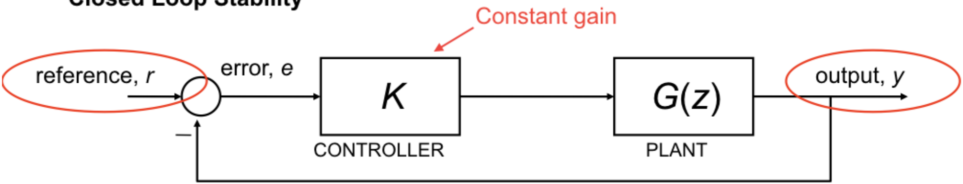
(v) Hence

Number of counterclockwise encirclements of the origin by $F(e^{j\theta})$ as θ increases from 0 to 2π is $=$ number of zeros of $F(z)$ inside the unit circle $-$ number of poles of $F(z)$ inside the unit circle

(we have assumed there are no poles or zeros on the unit circle)

Closed loop stability

Closed Loop Stability



$$Y(z) = \frac{KG(z)}{1 + KG(z)} R(z)$$

The closed-loop poles are the roots of

$$1 + KG(z) = 0$$

If

$$G(z) = \frac{A(z - z_1)(z - z_2)\dots(z - z_m)}{(z - p_1)(z - p_2)\dots(z - p_n)} = \frac{b(z)}{a(z)}$$

assume causal ($m \le n$)

then

$$1 + KG(z) = \frac{a(z) + Kb(z)}{a(z)}$$

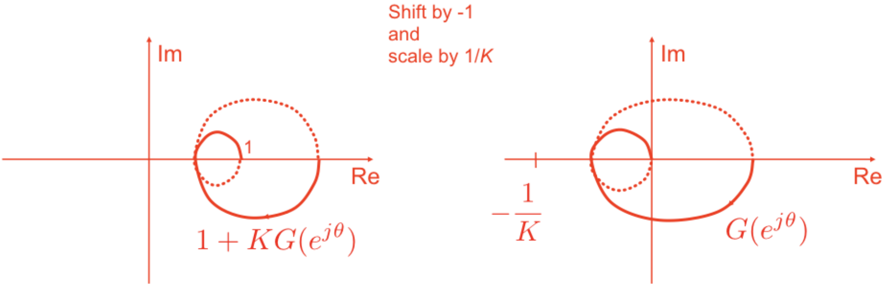
roots are closed loop poles

roots are open loop poles

Hence (by the encirclement property):

The number of counter clockwise encirclements of the origin by $1 + KG(e^{j\theta})$ is

$$= \{\text{ number of closed-loop poles inside the unit circle}\}$$
$$- \ \{\text{ number of open-loop poles inside the unit circle}\}$$



Hence the number of counter clockwise encirclements of the $-1/K$ point by $G(e^{j\theta})$ is

= { number of closed-loop poles inside the unit circle }

- { number of open-loop poles inside the unit circle }

= { number of open-loop poles outside the unit circle }

- { number of closed-loop poles outside the unit circle }

$$\text{since } CLP_{in} - OLP_{in} = (n - CLP_{out}) - (n - OLP_{out})$$

Hence we get the **discrete-time Nyquist Stability Criterion**

The above closed-loop system will be stable if (and only if) the number of counter clockwise encirclements of the $-1/K$ point by $G(e^{j\theta})$ as θ increases from 0 to 2π $=$ the number of open-loop unstable poles.

Nyquist stability criterion

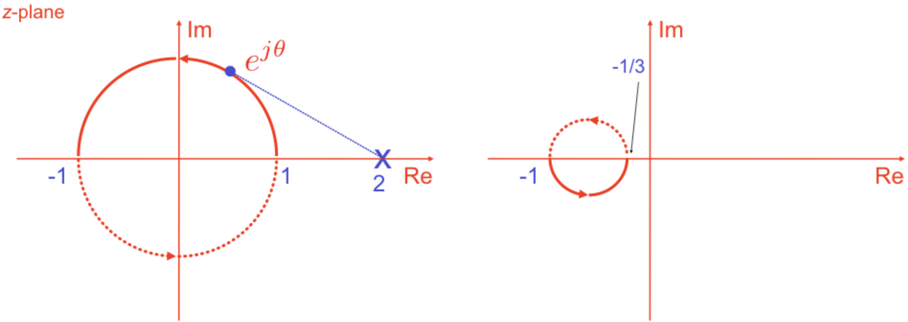
The above closed-loop system will be stable if (and only if)

the number of counter clockwise encirclements of the $-1/K$ point by $G(e^{j\theta})$ as θ increases from 0 to 2π
$=$ the number of open-loop unstable poles.

Example.

$$G(z) = \frac{1}{z-2} \Rightarrow G(e^{j\theta}) = \frac{1}{e^{j\theta}-2}$$

Unstable. 1 pole outside the unit circle.



Closed-loop stability: $-1 < -\frac{1}{K} < -\frac{1}{3} \quad \Leftrightarrow \quad 1 < K < 3$

$$\text{check:}\, 1 + K\frac{1}{z-2} = \frac{z-2+K}{z-2} \quad \Rightarrow \quad 1 < K < 3$$

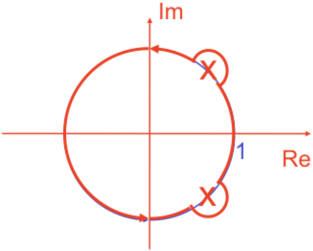
Interpretation

Note the following points:

- If the open-loop system is stable, then just plot $G(e^{j\theta})$ for $0 \leq \theta \leq \pi$, and use the same rule as for continuous-time: 'Leave the $-1/K$ point on your left as you move along the Nyquist diagram'.
- Although plotting the discrete-time Nyquist diagram is different from plotting the continuous-time one ($G(e^{j\theta})$ instead of $G(j\omega)$), its interpretation is exactly the same: gain and phase margins can be read off it in the same way, and have the same meanings. Gain and phase margins can also be found from the Bode diagram in exactly the same way as for continuous time.
- For physical systems $\overline{G(e^{j\theta})} = G(e^{-j\theta})$. Thus to find a complete Nyquist diagram you only need to plot $G(e^{j\theta})$ for $0 \leq \theta \leq \pi$ (which corresponds to frequencies used on Bode plot) and then draw in complex conjugate locus.

Open loop poles on the unit circle

Open-loop poles on the unit circle



Closing the locus

In order to obtain a closed curve for the Nyquist locus and hence correctly count the encirclements, it is customary to indent the path of z around the poles on the unit circle with a small semi-circular excursion outside the unit circle. Then the open loop poles on the unit circle are counted as being stable in the stability criterion.

The "right turn" in the z-plane then gives a "right turn" in the G-plane and a circular arc of large radius is produced which continues for $m\pi$ radians where m is the multiplicity of the pole on the unit circle.

Asymptotes

If there is an open-loop pole of multiplicity one on the unit circle then the Nyquist diagram will be asymptotic to a straight line as it tends to infinity. It is possible to find the asymptote along which it tends as follows.

Suppose that $G(z)$ has a pole at $z = 1$, i.e.

$$G(z) = \frac{1}{(z-1)}\ F(z)$$

where $F(z)$ has no poles or zeros at $z = 1$.

Then for $z \approx 1$, expand $F(z)$ in a Taylor series to give

$$\begin{aligned} G(z) \quad &= \quad \frac{1}{(z-1)}\left\{F(1) + F'(1)(z-1) + \right. \\ &\qquad\qquad \left. + \frac{1}{2}F''(1)(z-1)^2 + \dots \right\} \end{aligned}$$

$$\approx \frac{F(1)}{(z-1)} + F'(1)$$

But

$$\begin{aligned}
\frac{1}{e^{j\theta}-1} &= \left\{e^{j\theta/2}(e^{j\theta/2} - e^{-j\theta/2})\right\}^{-1} \\
&= \frac{e^{-j\theta/2}}{2j\sin(\theta/2)} \qquad e^{-j\theta/2} = \cos\frac{\theta}{2} - j\sin\frac{\theta}{2} \\
&= -\frac{1}{2} - \frac{j}{2\tan(\theta/2)}
\end{aligned}$$

Hence

$$G(e^{j\theta}) \approx -\frac{1}{2}F(1) + F'(1) - j\overbrace{\frac{F(1)}{2\tan(\theta/2)}}^{\text{large as } \theta \to 0}$$

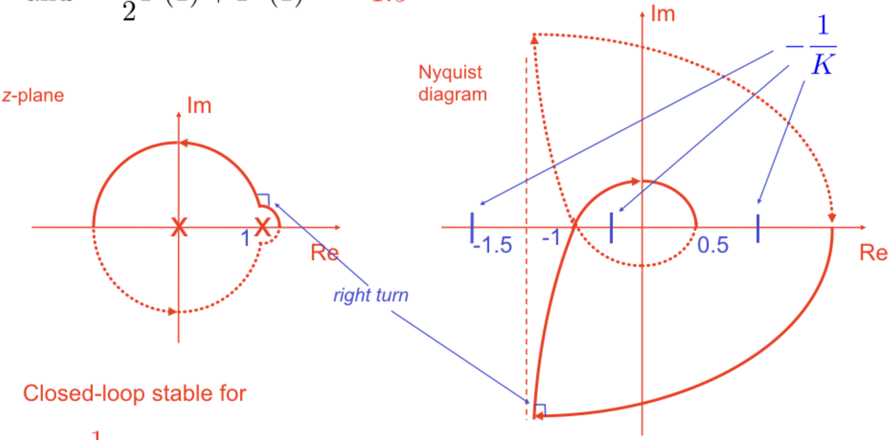
Hence the asymptote as $\theta \to 0$ will be a straight line with a constant real part of $-\frac{1}{2}F(1) + F'(1)$.

For multiple poles on the unit circle the asymptotic behaviour is more complex and requires more terms in the Taylor series expansion.

Example.

$$G(z) = \frac{1}{z(z-1)}. \quad \text{Then, } F(z) = \frac{1}{z} \quad , \quad F'(z) = -\frac{1}{z^2}$$

$$\text{and } -\frac{1}{2}F(1) + F'(1) = -1.5$$



Closed-loop stable for

$$-\frac{1}{K} < -1 \quad \Leftrightarrow 0 < K < 1$$