3A3: The equations of fluid flow & their numerical solution

Examples Paper 2

February 2016

1 (a) Show that a second-order central difference scheme in space and a first order forward difference scheme in time for the one dimensional convection equation

$$\frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} = 0$$

is equivalent to solving the equation

$$\frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} = -A^2\frac{\Delta t}{2}\frac{\partial^2 U}{\partial x^2} + O(\Delta x^2)$$

(b) Comment on the stability of the scheme.

(c) If a diffusion (damping) term, $\nu\partial^2 u/\partial x^2$, is added to the right-hand side of the convection equation and is discretised by a second order central difference scheme, show that the resulting difference equation is given by

$$u_i^{n+1} = u_i^n - \frac{c}{2}\left(u_{i+1}^n - u_{i-1}^n\right) + \alpha\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)$$

where $c = A\Delta t/\Delta x$ and $\alpha = \nu\Delta t/\Delta x^2$.

(d) Use the result derived in part (a) to show that for stability

$$\alpha > c^2/2$$

(e) Modify the `convection.m` file for the scheme derived in part (c) and run it for $c = 0.5$ with three values of α:

(i) $\alpha = 0$

(ii) $\alpha = c^2/4$

(iii) $\alpha = c^2$

Comment on the results.

2 Show that for the linear convection equation the MacCormack scheme is identical to the Lax-Wendroff scheme.

3 The upwind scheme for the inviscid Burgers equation is given by

$$u_i^{n+1} = u_i^n - u_i^n \frac{\Delta t}{\Delta x}\left(u_i^n - u_{i-1}^n\right)$$

Modify the `convection.m` file for this scheme. Set $0 \leq x \leq 2$. Use 101 grid points in the x direction. Starting from $t = 0$ compute the solution at $t = 1$. Perform numerical experiments with different number of timesteps, nt, to get to $t = 1$. Find the minimum number of nt needed for the scheme to be stable.

4 Solve the shock-tube problem using the parameters given in `Algorithm 1` of the handouts. Plot your result for density, pressure and velocity at $t = 0.2$.

5 Consider the mass conservation equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial f}{\partial x} = 0$$

and its discretisation by a finite volume scheme

$$\frac{\partial \bar{\rho}}{\partial t} + \frac{1}{\Delta x_i}\left(f_i^+ - f_i^-\right) = 0$$

where i is the index of a cell centre, $\Delta x_i = x_i^+ - x_i^-$, and $^+$ and $^-$ represent downstream and upstream edges of the cell, respectively. $\bar{\rho}$ represents the average density in the cell.

Show that the discretisation in space is identical to a second order central finite-difference scheme if the cell spacing is uniform and the edge fluxes (f) are approximated by averages from neighbouring cell centres.