Paper 3C6: VIBRATION

Examples paper 2— Vibration modes and response

*Tripos standard questions are marked **

Discrete systems

1. Two harmonic oscillators are weakly coupled together through a spring of stiffness s, as shown in Fig. 1. Both oscillators have mass m, and their springs have respective stiffnesses k_1 and k_2, with $k_1 >> s, k_2 >> s$. The displacements of the two masses are x and y. Find the mass and stiffness matrices for this system, and show that the natural frequencies are determined by the equation

$$\begin{vmatrix} \omega_1^2 - \omega^2 & -\Omega^2 \\ -\Omega^2 & \omega_2^2 - \omega^2 \end{vmatrix} = 0$$

where $\omega_1^2 = (k_1 + s)/m, \omega_2^2 = (k_2 + s)/m, \Omega^2 = s/m$.

Obtain the natural frequencies and mode shapes for the cases (i) $\omega_1^2 = \omega_2^2$ (do this one by inspection) and (ii) $|\omega_1^2 - \omega_2^2| >> \Omega^2$.

* Sketch a graph showing (on the same axes) the variation of both natural frequencies with k_2 over the range zero to $2k_1$, and explain what happens to the mode shapes over this same range. [Hint: use "frequency squared" as the vertical axis. Draw first the two lines corresponding to approximation (ii) above, then add the points from approximation (i), then "join the dots".]

If the only damping of these modes is associated with energy dissipation in the spring s, say qualitatively how you would expect the modal damping factors of the two modes to vary over the range of the plot.

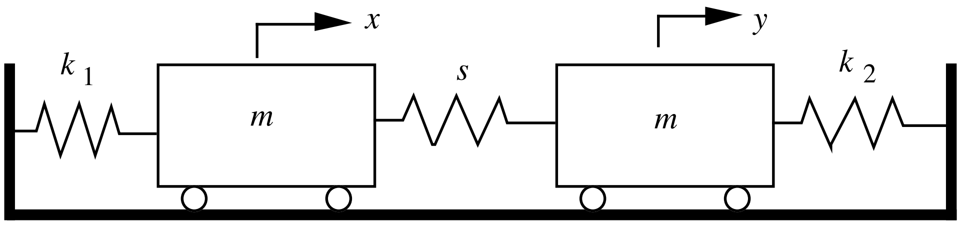

Fig. 1

*2 The vibration of a turbine fan can be modelled as sketched in Figure 2. A set of N identical masses m are attached to a fixed circular hub through leaf springs of stiffness k. The "blades" undergo small vibration with circumferential displacements of the blade tips $x_1, x_2, \cdots, x_N$. Each of these blades is connected to its two neighbours by springs of stiffness s, defined so that for example the energy stored in the spring between blades 1 and 2 is $(s/2)(x_1 - x_2)^2$. In terms of $x_1, x_2, \cdots, x_N$, obtain the mass and stiffness matrices. (In the figure $N = 8$, but do the calculation for a general number N.)

Verify that the vector with components $x_j = \cos j\lambda$ is a mode shape for certain values of the constant λ and find the corresponding natural frequency.

[Hint: you will need to use the identity for $\cos A + \cos B$ from the Maths Data Book.]

Explain carefully how N distinct mode shapes can be found in this way. Deduce that all N natural frequencies of this system lie within a finite range, even if $N \to \infty$. Sketch all the mode shapes for the case $N = 6$.

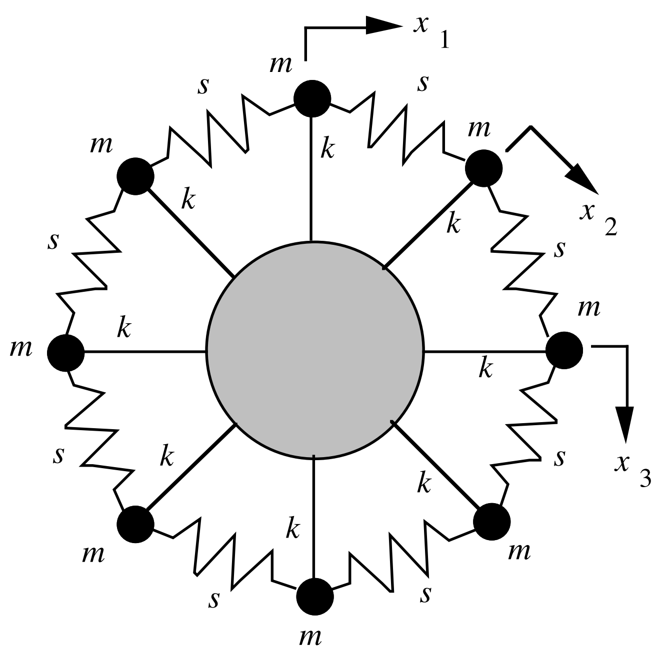

Fig. 2

*3 A discrete model for the lowest two modes of a stretched string may be constructed as follows. Suppose a string of length L has tension P and mass per unit length m, and is fixed at the points $x = 0, L$. The (small) transverse displacement $y(x,t)$ can be represented approximately by the polynomial

$$y(x,t) = a_1(t)\frac{x}{L} + a_2(t)\frac{x^2}{L^2} + a_3(t)\frac{x^3}{L^3} \ .$$

Use the boundary condition at $x = L$ to provide a relation between the three coefficients a_1, a_2 and a_3, and thus reduce the expression to involve only a_1 and a_2.

Given that the potential and kinetic energies of the string are given by the expressions

$$V = \frac{P}{2}\int_0^L \left(\frac{\partial y}{\partial x}\right)^2 dx, T = \frac{m}{2}\int_0^L \left(\frac{\partial y}{\partial t}\right)^2 dx \ ,$$

obtain the mass and stiffness matrices for the two-DOF problem in terms of a_1 and a_2.

Find the two natural frequencies, either by hand or by typing the matrices into MATLAB, and compare them with the exact results from the lecture notes. (Type "help eig" in MATLAB to see how to calculate eigenvalues and eigenvectors.)

[This problem illustrates that generalised coordinates do not have to be just displacements or rotations. A generalisation of this approach is the basis of the *Finite-element method*, the commonest computational approach to the vibration analysis of complicated engineering structures.]

4 (a) Verify that the transfer function term

$$H_n = \frac{a_n}{\omega_n^2 + 2i\zeta_n\omega_n\omega - \omega^2}$$

can be written in partial fraction form as $H_n = Y_n + X_n$ where

$$Y_n = \frac{b_n}{\omega - \omega_n(1 + i\zeta_n)} \quad \text{and} \quad X_n = -\frac{b_n}{\omega - \omega_n(-1 + i\zeta_n)},$$

provided the damping is small ($\zeta_n << 1$). Find the approximate value of b_n. Explain the significance of these two terms.

(b) The peak value of $|Y_n|$ occurs when $\omega = \omega_n$. Find the frequencies on either side of this peak value at which $|Y_n|$ has fallen to $1/\sqrt{2}$ of the peak value (i.e. the half-power points).

(c) Find the phase of Y_n at the peak frequency, and find the frequencies on either side of this at which the phase has changed by $\pi/4$.

(d) Show that the Nyquist plot of $Y_n(\omega)$ in the complex plane is a circle, centred on $\dfrac{ib_n}{2\omega_n\zeta_n}$, and show that the two half-power points from (a) lie on a diameter of the circle.

[*Hint*: Think where the circle would have to be positioned, then work backwards from an equation for the supposed circle in terms of the real and imaginary parts of $Y_n(\omega)$.]

Continuous systems

5 Bending vibration of an undamped Euler beam of length L satisfies the standard differential equation

$$\rho A \frac{\partial^2 w}{\partial t^2} + EI \frac{\partial^4 w}{\partial x^4} = 0 .$$

Find equations satisfied by the natural frequencies of the beam for the following end conditions:

(i) clamped at both ends;
(ii) clamped at one end, pinned at the other;
(iii) pinned at one end, free at the other.

Hence check that the natural frequencies in case (i) are the same as for the free-free case treated in lectures, except for two additional zero frequencies in the free-free case. Why does this happen? Similarly, check that cases (ii) and (iii) have the same natural frequencies except for an additional zero frequency in case (iii).

For each case, obtain an approximate expression for the *n*th natural frequency.

Sketch (without calculation) the first three mode shapes for each case.

6. It is a useful fact that the tweezers supplied with Swiss Army knives vibrate like a tuning fork at a frequency of about 230 Hz†. The tweezers comprise two flat stainless-steel cantilevers which are 30mm long.

(i) Take the formula for the frequency of the lowest mode of a cantilever beam from the lecture notes, and show that for a beam of solid rectangular section this can be written as $f_1 \approx 0.162\, d\, c / L^2$ (Hz) where d is its thickness, L its length and $c = \sqrt{E/\rho}$ is the speed of sound in the material.

(ii) For most engineering metals $c \approx 3000$ to 5000m/s . Use data from the Materials Data Book to find the appropriate value for the case for stainless steel. Thus estimate the thickness of each 'fork' of the tweezers††.

(iii) The note ‘A’ has a frequency of 220Hz. Given that there are exactly 12 equal semitones in an octave (frequency doubles in an octave) how many semitones above ‘A’ is the tweezer's note?

†Clamp the base of the tweezers between your teeth to hear the note clearly. The sound travels to your ears by 'bone conduction' and the tweezers are perfectly audible even in a noisy room.
††It can be difficult to measure material thickness accurately without a micrometer. Even more commonly, you may know the geometric dimensions of a specimen but be unsure of the Young’s modulus. Vibration methods on the lines of this question can often be used to determine any one unknown quantity which appears in a frequency equation.

7 It is sometimes useful to make a "quick and dirty" estimate of the lowest resonance frequency of a system, to establish whether you need to worry about resonance or not. This can sometimes be done by estimating a static stiffness of the system, then using "$\sqrt{k/m}$" with an estimate of the effective moving mass. You expect m to be some fraction of the total mass of the system. This question explores what fraction that should be, by analysing two problems to which the exact answer is already known.

(a) An Euler beam of length L, mass per unit length m and flexural rigidity EI is pinned at both ends. The fundamental natural frequency is to be estimated by modelling the beam as a massless beam with a concentrated mass M at its centre. Using information from the structures data book for the stiffness of the beam find the natural frequency of this system, and hence deduce what value of M is needed in order to obtain the correct frequency.

(b) Perform the corresponding calculation for the fundamental frequency of a cantilever beam of length L, modelled as a massless beam with a mass M at the free end.

8 In the lectures, the formula for orthogonality of mode shapes of continuous systems was deduced by analogy with discrete systems. This question shows how that result can be confirmed directly, for the case of bending vibration.

(a) Prove that the mode shapes of an undamped, simply-supported, uniform Euler beam of length L satisfy the orthogonality condition

$$\int_0^L \rho A u_j(x) u_k(x) dx = 0 \quad \text{for} \quad j \neq k$$

where ρ is the density and A the cross-sectional area, and $u_j(x), u_k(x)$ denote two mode shapes. Note that you already know the mode shapes for this system!

(b) This result also holds for any other combination of free, pinned and clamped end conditions for the beam. Prove this by the following steps. Write down the differential equations satisfied by the modes $u_j(x), u_k(x)$, with corresponding natural frequencies ω_j, ω_k (which are assumed to be different). Multiply the j equation by $u_k(x)$, multiply the k equation by $u_j(x)$, and subtract one from the other. Now relate $\int_0^L \frac{d^4 u_j}{dx^4} u_k dx$ to $\int_0^L \frac{d^2 u_j}{dx^2} \frac{d^2 u_k}{dx^2} dx$ using integration by parts, and hence deduce the required result.

Answers

1 (i) $\omega^2 = \omega_1^2 \pm \Omega^2$, modes $[1 \quad \mp 1]^t$; (ii) $\omega = \omega_1, \omega_2$, modes $[1 \quad 0]^t, [0 \quad 1]^t$.

2 $\lambda = \dfrac{2n\pi}{N}, \quad n = 0, 1, 2, \cdots$ Most values of λ correspond to two distinct modes.

$\omega^2 = \dfrac{k}{m} + \dfrac{2s}{m}(1 - \cos\lambda)$.

3 $K = \dfrac{P}{30L}\begin{bmatrix} 24 & 9 \\ 9 & 4 \end{bmatrix}, M = \dfrac{mL}{420}\begin{bmatrix} 32 & 11 \\ 11 & 4 \end{bmatrix}, \omega^2 = 10P/mL^2, 42P/mL^2$.

4 (a) $b_n = -a_n / 2\omega_n$; (b) and (c) $\omega_n(1 \pm \zeta_n)$.

5 (i) $\cosh aL \cos aL = +1$, where $a^4 = \dfrac{m\omega_j^2}{EI}$.

(ii) & (iii) $\tanh aL = \tan aL$.

6 (ii) 0.25mm

(iii) just less than one semitone (almost a B flat)

7 (a) 0.493 of the beam's total mass;

(b) 0.243 of the beam's total mass.