Chapter 1

Algorithms with numbers

One of the main themes of this chapter is the dramatic contrast between two ancient problems that at first seem very similar:

> Factoring: Given a number N, express it as a product of its prime factors.
>
> Primality: Given a number N, determine whether it is a prime.

Factoring is hard. Despite centuries of effort by some of the world's smartest mathematicians and computer scientists, the fastest methods for factoring a number N take time exponential in the number of bits of N.

On the other hand, we shall soon see that we *can* efficiently test whether N is prime! And (it gets even more interesting) this strange disparity between the two intimately related problems, one very hard and the other very easy, lies at the heart of the technology that enables secure communication in today's global information environment.

En route to these insights, we need to develop algorithms for a variety of computational tasks involving numbers. We begin with basic arithmetic, an especially appropriate starting point because, as we know, the word *algorithms* originally applied only to methods for these problems.

1.1 Basic arithmetic

1.1.1 Addition

We were so young when we learned the standard technique for addition that we would scarcely have thought to ask *why* it works. But let's go back now and take a closer look.

It is a basic property of decimal numbers that

> *The sum of any three single-digit numbers is at most two digits long.*

Quick check: the sum is at most $9+9+9=27$, two digits long. In fact, this rule holds not just in decimal but in *any* base $b \geq 2$ (Exercise 1.1). In binary, for instance, the maximum possible sum of three single-bit numbers is 3, which is a 2-bit number.

Bases and logs

Naturally, there is nothing special about the number 10—we just happen to have 10 fingers, and so 10 was an obvious place to pause and take counting to the next level. The Mayans developed a similar positional system based on the number 20 (no shoes, see?). And of course today computers represent numbers in binary.

How many digits are needed to represent the number $N \geq 0$ in base b? Let's see—with k digits in base b we can express numbers up to $b^k - 1$; for instance, in decimal, three digits get us all the way up to $999 = 10^3 - 1$. By solving for k, we find that $\lceil \log_b(N+1) \rceil$ digits (about $\log_b N$ digits, give or take 1) are needed to write N in base b.

How much does the size of a number change when we change bases? Recall the rule for converting logarithms from base a to base b: $\log_b N = (\log_a N)/(\log_a b)$. So the size of integer N in base a is the same as its size in base b, times a constant factor $\log_a b$. In big-O notation, therefore, the base is irrelevant, and we write the size simply as $O(\log N)$. When we do not specify a base, as we almost never will, we mean $\log_2 N$.

Incidentally, this function $\log N$ appears repeatedly in our subject, in many guises. Here's a sampling:

1. $\log N$ is, of course, the power to which you need to raise 2 in order to obtain N.
2. Going backward, it can also be seen as the number of times you must halve N to get down to 1. (More precisely: $\lceil \log N \rceil$.) This is useful when a number is halved at each iteration of an algorithm, as in several examples later in the chapter.
3. It is the number of bits in the binary representation of N. (More precisely: $\lceil \log(N+1) \rceil$.)
4. It is also the depth of a complete binary tree with N nodes. (More precisely: $\lfloor \log N \rfloor$.)
5. It is even the sum $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N}$, to within a constant factor (Exercise 1.5).

This simple rule gives us a way to add two numbers in any base: align their right-hand ends, and then perform a single right-to-left pass in which the sum is computed digit by digit, maintaining the overflow as a carry. Since we know each individual sum is a two-digit number, *the carry is always a single digit*, and so at any given step, three single-digit numbers are added. Here's an example showing the addition $53 + 35$ in binary.

$$
\begin{array}{rccccccccc}
\text{Carry:} & 1 & & & 1 & 1 & 1 & & \\
 & & 1 & 1 & 0 & 1 & 0 & 1 & (53) \\
 & & 1 & 0 & 0 & 0 & 1 & 1 & (35) \\
\hline
 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & (88)
\end{array}
$$

Ordinarily we would spell out the algorithm in pseudocode, but in this case it is so familiar that we do not repeat it. Instead we move straight to analyzing its efficiency.

Given two binary numbers x and y, how long does our algorithm take to add them? This is the kind of question we shall persistently be asking throughout this book. We want the answer expressed as a function of *the size of the input*: the number of bits of x and y, the number of keystrokes needed to type them in.

Suppose x and y are each n bits long; in this chapter we will consistently use the letter n for the sizes of numbers. Then the sum of x and y is $n+1$ bits at most, and each individual bit of this sum gets computed in a fixed amount of time. The total running time for the addition algorithm is therefore of the form $c_0 + c_1 n$, where c_0 and c_1 are some constants; in other words, it is *linear*. Instead of worrying about the precise values of c_0 and c_1, we will focus on the big picture and denote the running time as $O(n)$.

Now that we have a working algorithm whose running time we know, our thoughts wander inevitably to the question of whether there is something even better.

Is there a faster algorithm? (This is another persistent question.) For addition, the answer is easy: in order to add two n-bit numbers we must at least read them and write down the answer, and even that requires n operations. So the addition algorithm is optimal, up to multiplicative constants!

Some readers may be confused at this point: Why $O(n)$ operations? Isn't binary addition something that computers today perform by just one instruction? There are two answers. First, it is certainly true that in a single instruction we can add integers whose size in bits is within the *word length* of today's computers—32 perhaps. But, as will become apparent later in this chapter, it is often useful and necessary to handle numbers much larger than this, perhaps several thousand bits long. Adding and multiplying such large numbers on real computers is very much like performing the operations bit by bit. Second, when we want to understand algorithms, it makes sense to study even the basic algorithms that are encoded in the hardware of today's computers. In doing so, we shall focus on the *bit complexity* of the algorithm, the number of elementary operations on individual bits—because this accounting reflects the amount of hardware, transistors and wires, necessary for implementing the algorithm.

1.1.2 Multiplication and division

Onward to multiplication! The grade-school algorithm for multiplying two numbers x and y is to create an array of intermediate sums, each representing the product of x by a single digit of y. These values are appropriately left-shifted and then added up. Suppose for instance that we want to multiply 13×11, or in binary notation, $x = 1101$ and $y = 1011$. The multiplication would proceed thus.

$$
\begin{array}{rrrrrrrrl}
 & & & & 1 & 1 & 0 & 1 & \\
 & & & \times & 1 & 0 & 1 & 1 & \\
\hline
 & & & & 1 & 1 & 0 & 1 & \text{(1101 times 1)} \\
 & & & 1 & 1 & 0 & 1 & & \text{(1101 times 1, shifted once)} \\
 & & 0 & 0 & 0 & 0 & & & \text{(1101 times 0, shifted twice)} \\
+ & 1 & 1 & 0 & 1 & & & & \text{(1101 times 1, shifted thrice)} \\
\hline
1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & \text{(binary 143)}
\end{array}
$$

In binary this is particularly easy since each intermediate row is either zero or x itself, left-shifted an appropriate amount of times. Also notice that left-shifting is just a quick way to multiply by the base, which in this case is 2. (Likewise, the effect of a right shift is to divide by the base, rounding down if needed.)

The correctness of this multiplication procedure is the subject of Exercise 1.6; let's move on and figure out how long it takes. If x and y are both n bits, then there are n intermediate rows, with lengths of up to $2n$ bits (taking the shifting into account). The total time taken to add up these rows, doing two numbers at a time, is

$$\underbrace{O(n) + O(n) + \cdots + O(n)}_{n-1 \text{ times}},$$

which is $O(n^2)$, *quadratic* in the size of the inputs: still polynomial but much slower than addition (as we have all suspected since elementary school).

But Al Khwarizmi knew another way to multiply, a method which is used today in some European countries. To multiply two decimal numbers x and y, write them next to each other, as in the example below. Then repeat the following: divide the first number by 2, rounding down the result (that is, dropping the $.5$ if the number was odd), and double the second number. Keep going till the first number gets down to 1. Then strike out all the rows in which the first number is even, and add up whatever remains in the second column.

$$
\begin{array}{rrl}
11 & 13 & \\
5 & 26 & \\
2 & 52 & \text{(strike out)} \\
1 & 104 & \\
\hline
 & 143 & \text{(answer)}
\end{array}
$$

But if we now compare the two algorithms, binary multiplication and multiplication by repeated halvings of the multiplier, we notice that they are doing the same thing! The three numbers added in the second algorithm are precisely the multiples of 13 by powers of 2 that were added in the binary method. Only this time 11 was not given to us explicitly in binary, and so we had to extract its binary representation by looking at the parity of the numbers obtained from it by successive divisions by 2. Al Khwarizmi's second algorithm is a fascinating mixture of decimal and binary!

Figure 1.1 Multiplication à la Français.

```
function multiply(x, y)
Input:  Two n-bit integers x and y, where y ≥ 0
Output:  Their product

if y = 0:  return 0
z = multiply(x, ⌊y/2⌋)
if y is even:
   return 2z
else:
   return x + 2z
```

The same algorithm can thus be repackaged in different ways. For variety we adopt a third formulation, the recursive algorithm of Figure 1.1, which directly implements the rule

$$x \cdot y \;=\; \begin{cases} 2(x \cdot \lfloor y/2 \rfloor) & \text{if } y \text{ is even} \\ x + 2(x \cdot \lfloor y/2 \rfloor) & \text{if } y \text{ is odd.} \end{cases}$$

Is this algorithm correct? The preceding recursive rule is transparently correct; so checking the correctness of the algorithm is merely a matter of verifying that it mimics the rule and that it handles the base case ($y = 0$) properly.

How long does the algorithm take? It must terminate after n recursive calls, because at each call y is halved—that is, its number of bits is decreased by one. And each recursive call requires these operations: a division by 2 (right shift); a test for odd/even (looking up the last bit); a multiplication by 2 (left shift); and possibly one addition, a total of $O(n)$ bit operations. The total time taken is thus $O(n^2)$, just as before.

Can we do better? Intuitively, it seems that multiplication requires adding about n multiples of one of the inputs, and we know that each addition is linear, so it would appear that n^2 bit operations are inevitable. Astonishingly, in Chapter 2 we'll see that we *can* do significantly better!

Division is next. To divide an integer x by another integer $y \neq 0$ means to find a quotient q and a remainder r, where $x = yq + r$ and $r < y$. We show the recursive version of division in Figure 1.2; like multiplication, it takes quadratic time. The analysis of this algorithm is the subject of Exercise 1.8.

1.2 Modular arithmetic

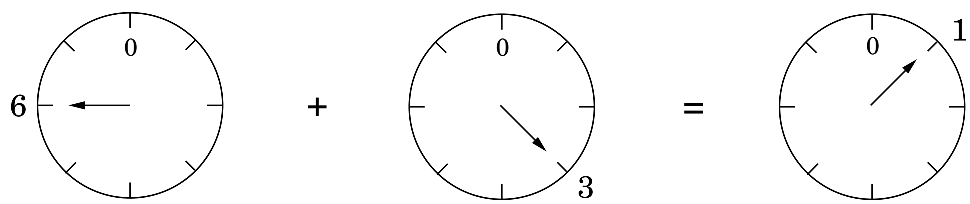
With repeated addition or multiplication, numbers can get cumbersomely large. So it is fortunate that we reset the hour to zero whenever it reaches 24, and the month to January after every stretch of 12 months. Similarly, for the built-in arithmetic operations of computer pro-

Figure 1.2 Division.

```
function divide(x, y)
Input:  Two n-bit integers x and y, where y ≥ 1
Output:  The quotient and remainder of x divided by y

if x = 0:  return (q, r) = (0, 0)
(q, r) = divide(⌊x/2⌋, y)
q = 2 · q,  r = 2 · r
if x is odd:  r = r + 1
if r ≥ y:  r = r − y,  q = q + 1
return (q, r)
```

Figure 1.3 Addition modulo 8.



cessors, numbers are restricted to some size, 32 bits say, which is considered generous enough for most purposes.

For the applications we are working toward—primality testing and cryptography—it is necessary to deal with numbers that are significantly larger than 32 bits, but whose range is nonetheless limited.

Modular arithmetic is a system for dealing with restricted ranges of integers. We define x *modulo* N to be the remainder when x is divided by N; that is, if $x = qN + r$ with $0 \leq r < N$, then x modulo N is equal to r. This gives an enhanced notion of equivalence between numbers: x and y are *congruent modulo* N if they differ by a multiple of N, or in symbols,

$$x \equiv y \pmod N \iff N \text{ divides } (x - y).$$

For instance, $253 \equiv 13 \pmod{60}$ because $253 - 13$ is a multiple of 60; more familiarly, 253 minutes is 4 hours and 13 minutes. These numbers can also be negative, as in $59 \equiv -1 \pmod{60}$: when it is 59 minutes past the hour, it is also 1 minute short of the next hour.

One way to think of modular arithmetic is that it limits numbers to a predefined range $\{0, 1, \ldots, N - 1\}$ and wraps around whenever you try to leave this range—like the hand of a clock (Figure 1.3).

Another interpretation is that modular arithmetic deals with all the integers, but divides them into N *equivalence classes*, each of the form $\{i + kN : k \in \mathbb{Z}\}$ for some i between 0 and

$N-1$. For example, there are three equivalence classes modulo 3:

$$\begin{array}{ccccccccc} \cdots & -9 & -6 & -3 & 0 & 3 & 6 & 9 & \cdots \\ \cdots & -8 & -5 & -2 & 1 & 4 & 7 & 10 & \cdots \\ \cdots & -7 & -4 & -1 & 2 & 5 & 8 & 11 & \cdots \end{array}$$

Any member of an equivalence class is substitutable for any other; when viewed modulo 3, the numbers 5 and 11 are no different. Under such substitutions, addition and multiplication remain well-defined:

Substitution rule *If* $x \equiv x' \pmod N$ *and* $y \equiv y' \pmod N$, *then:*

$$x + y \equiv x' + y' \pmod N \quad \textit{and} \quad xy \equiv x'y' \pmod N.$$

(See Exercise 1.9.) For instance, suppose you watch an entire season of your favorite television show in one sitting, starting at midnight. There are 25 episodes, each lasting 3 hours. At what time of day are you done? Answer: the hour of completion is $(25 \times 3) \bmod 24$, which (since $25 \equiv 1 \bmod 24$) is $1 \times 3 = 3 \bmod 24$, or three o'clock in the morning.

It is not hard to check that in modular arithmetic, the usual associative, commutative, and distributive properties of addition and multiplication continue to apply, for instance:

$$\begin{array}{rl} x + (y + z) \equiv (x + y) + z \pmod N & \text{Associativity} \\ xy \equiv yx \pmod N & \text{Commutativity} \\ x(y + z) \equiv xy + yz \pmod N & \text{Distributivity} \end{array}$$

Taken together with the substitution rule, this implies that while performing a sequence of arithmetic operations, it is legal to reduce intermediate results to their remainders modulo N at any stage. Such simplifications can be a dramatic help in big calculations. Witness, for instance:

$$2^{345} \equiv (2^5)^{69} \equiv 32^{69} \equiv 1^{69} \equiv 1 \pmod{31}.$$

1.2.1 Modular addition and multiplication

To *add* two numbers x and y modulo N, we start with regular addition. Since x and y are each in the range 0 to $N-1$, their sum is between 0 and $2(N-1)$. If the sum exceeds $N-1$, we merely need to subtract off N to bring it back into the required range. The overall computation therefore consists of an addition, and possibly a subtraction, of numbers that never exceed $2N$. Its running time is linear in the sizes of these numbers, in other words $O(n)$, where $n = \lceil \log N \rceil$ is the size of N; as a reminder, our convention is to use the letter n to denote input size.

To *multiply* two mod-N numbers x and y, we again just start with regular multiplication and then reduce the answer modulo N. The product can be as large as $(N-1)^2$, but this is still at most $2n$ bits long since $\log(N-1)^2 = 2\log(N-1) \leq 2n$. To reduce the answer modulo N, we

Two's complement

Modular arithmetic is nicely illustrated in *two's complement*, the most common format for storing signed integers. It uses n bits to represent numbers in the range $[-2^{n-1}, 2^{n-1}-1]$ and is usually described as follows:

- Positive integers, in the range 0 to $2^{n-1}-1$, are stored in regular binary and have a leading bit of 0.
- Negative integers $-x$, with $1 \le x \le 2^{n-1}$, are stored by first constructing x in binary, then flipping all the bits, and finally adding 1. The leading bit in this case is 1.

(And the usual description of addition and multiplication in this format is even more arcane!)

Here's a much simpler way to think about it: any number in the range -2^{n-1} to $2^{n-1}-1$ is stored modulo 2^n. Negative numbers $-x$ therefore end up as $2^n - x$. Arithmetic operations like addition and subtraction can be performed directly in this format, ignoring any overflow bits that arise.

compute the remainder upon dividing it by N, using our quadratic-time division algorithm. Multiplication thus remains a quadratic operation.

Division is not quite so easy. In ordinary arithmetic there is just one tricky case—division by zero. It turns out that in modular arithmetic there are potentially other such cases as well, which we will characterize toward the end of this section. Whenever division is legal, however, it can be managed in cubic time, $O(n^3)$.

To complete the suite of modular arithmetic primitives we need for cryptography, we next turn to *modular exponentiation*, and then to the *greatest common divisor*, which is the key to division. For both tasks, the most obvious procedures take exponentially long, but with some ingenuity polynomial-time solutions can be found. A careful choice of algorithm makes all the difference.

1.2.2 Modular exponentiation

In the cryptosystem we are working toward, it is necessary to compute $x^y \bmod N$ for values of x, y, and N that are several hundred bits long. Can this be done quickly?

The result is some number modulo N and is therefore itself a few hundred bits long. However, the raw value of x^y could be much, much longer than this. Even when x and y are just 20-bit numbers, x^y is at least $(2^{19})^{(2^{19})} = 2^{(19)(524288)}$, about 10 million bits long! Imagine what happens if y is a 500-bit number!

To make sure the numbers we are dealing with never grow too large, we need to perform all intermediate computations modulo N. So here's an idea: calculate $x^y \bmod N$ by repeatedly multiplying by x modulo N. The resulting sequence of intermediate products,

$$x \bmod N \to x^2 \bmod N \to x^3 \bmod N \to \cdots \to x^y \bmod N,$$

Figure 1.4 Modular exponentiation.

```
function modexp(x, y, N)
Input:  Two n-bit integers x and N, an integer exponent y
Output:  x^y mod N

if y = 0:  return 1
z = modexp(x, ⌊y/2⌋, N)
if y is even:
   return z^2 mod N
else:
   return x · z^2 mod N
```

consists of numbers that are smaller than N, and so the individual multiplications do not take too long. But there's a problem: if y is 500 bits long, we need to perform $y - 1 \approx 2^{500}$ multiplications! This algorithm is clearly exponential in the size of y.

Luckily, we *can* do better: starting with x and *squaring repeatedly* modulo N, we get

$$x \bmod N \to x^2 \bmod N \to x^4 \bmod N \to x^8 \bmod N \to \cdots \to x^{2^{\lfloor \log y \rfloor}} \bmod N.$$

Each takes just $O(\log^2 N)$ time to compute, and in this case there are only $\log y$ multiplications. To determine $x^y \bmod N$, we simply multiply together an appropriate subset of these powers, those corresponding to 1's in the binary representation of y. For instance,

$$x^{25} = x^{11001_2} = x^{10000_2} \cdot x^{1000_2} \cdot x^{1_2} = x^{16} \cdot x^8 \cdot x^1.$$

A polynomial-time algorithm is finally within reach!

We can package this idea in a particularly simple form: the recursive algorithm of Figure 1.4, which works by executing, modulo N, the self-evident rule

$$x^y = \begin{cases} (x^{\lfloor y/2 \rfloor})^2 & \text{if } y \text{ is even} \\ x \cdot (x^{\lfloor y/2 \rfloor})^2 & \text{if } y \text{ is odd.} \end{cases}$$

In doing so, it closely parallels our recursive multiplication algorithm (Figure 1.1). For instance, that algorithm would compute the product $x \cdot 25$ by an analogous decomposition to the one we just saw: $x \cdot 25 = x \cdot 16 + x \cdot 8 + x \cdot 1$. And whereas for multiplication the terms $x \cdot 2^i$ come from repeated *doubling*, for exponentiation the corresponding terms x^{2^i} are generated by repeated squaring.

Let n be the size in bits of x, y, and N (whichever is largest of the three). As with multiplication, the algorithm will halt after at most n recursive calls, and during each call it multiplies n-bit numbers (doing computation modulo N saves us here), for a total running time of $O(n^3)$.

1.2.3 Euclid's algorithm for greatest common divisor

Our next algorithm was discovered well over 2000 years ago by the mathematician Euclid, in ancient Greece. Given two integers a and b, it finds the largest integer that divides both of them, known as their *greatest common divisor* (gcd).

Figure 1.5 Euclid's algorithm for finding the greatest common divisor of two numbers.

```
function Euclid(a, b)
Input:  Two integers a and b with a ≥ b ≥ 0
Output:  gcd(a, b)

if b = 0:  return a
return Euclid(b, a mod b)
```

The most obvious approach is to first factor a and b, and then multiply together their common factors. For instance, $1035 = 3^2 \cdot 5 \cdot 23$ and $759 = 3 \cdot 11 \cdot 23$, so their gcd is $3 \cdot 23 = 69$. However, we have no efficient algorithm for factoring. Is there some other way to compute greatest common divisors?

Euclid's algorithm uses the following simple formula.

Euclid's rule *If x and y are positive integers with $x \geq y$, then* $\gcd(x, y) = \gcd(x \bmod y, y)$.

Proof. It is enough to show the slightly simpler rule $\gcd(x, y) = \gcd(x - y, y)$ from which the one stated can be derived by repeatedly subtracting y from x.

Here it goes. Any integer that divides both x and y must also divide $x - y$, so $\gcd(x, y) \leq \gcd(x - y, y)$. Likewise, any integer that divides both $x - y$ and y must also divide both x and y, so $\gcd(x, y) \geq \gcd(x - y, y)$. ∎

Euclid's rule allows us to write down an elegant recursive algorithm (Figure 1.5), and its correctness follows immediately from the rule. In order to figure out its running time, we need to understand how quickly the arguments (a, b) decrease with each successive recursive call. In a single round, arguments (a, b) become $(b, a \bmod b)$: their order is swapped, and the larger of them, a, gets reduced to $a \bmod b$. This is a substantial reduction.

Lemma *If $a \geq b$, then* $a \bmod b < a/2$.

Proof. Witness that either $b \leq a/2$ or $b > a/2$. These two cases are shown in the following figure. If $b \leq a/2$, then we have $a \bmod b < b \leq a/2$; and if $b > a/2$, then $a \bmod b = a - b < a/2$.

∎

This means that after any two consecutive rounds, both arguments, a and b, are at the very least halved in value—the length of each decreases by at least one bit. If they are initially n-bit integers, then the base case will be reached within $2n$ recursive calls. And since each call involves a quadratic-time division, the total time is $O(n^3)$.

Figure 1.6 A simple extension of Euclid's algorithm.

```
function extended-Euclid(a, b)
Input:  Two positive integers a and b with a ≥ b ≥ 0
Output:  Integers x, y, d such that d = gcd(a, b) and ax + by = d

if b = 0:  return (1, 0, a)
(x', y', d) = Extended-Euclid(b, a mod b)
return (y', x' − ⌊a/b⌋y', d)
```

1.2.4 An extension of Euclid's algorithm

A small extension to Euclid's algorithm is the key to dividing in the modular world.

To motivate it, suppose someone claims that d is the greatest common divisor of a and b: how can we check this? It is not enough to verify that d divides both a and b, because this only shows d to be a common factor, not necessarily the largest one. Here's a test that can be used if d is of a particular form.

Lemma *If d divides both a and b, and $d = ax + by$ for some integers x and y, then necessarily $d = \gcd(a, b)$.*

Proof. By the first two conditions, d is a common divisor of a and b and so it cannot exceed the greatest common divisor; that is, $d \leq \gcd(a, b)$. On the other hand, since $\gcd(a, b)$ is a common divisor of a and b, it must also divide $ax + by = d$, which implies $\gcd(a, b) \leq d$. Putting these together, $d = \gcd(a, b)$. ∎

So, if we can supply two numbers x and y such that $d = ax + by$, then we can be sure $d = \gcd(a, b)$. For instance, we know $\gcd(13, 4) = 1$ because $13 \cdot 1 + 4 \cdot (-3) = 1$. But when can we find these numbers: under what circumstances can $\gcd(a, b)$ be expressed in this checkable form? It turns out that it *always* can. What is even better, the coefficients x and y can be found by a small extension to Euclid's algorithm; see Figure 1.6.

Lemma *For any positive integers a and b, the extended Euclid algorithm returns integers x, y, and d such that $\gcd(a, b) = d = ax + by$.*

Proof. The first thing to confirm is that if you ignore the x's and y's, the extended algorithm is exactly the same as the original. So, at least we compute $d = \gcd(a, b)$.

For the rest, the recursive nature of the algorithm suggests a proof by induction. The recursion ends when $b = 0$, so it is convenient to do induction on the value of b.

The base case $b = 0$ is easy enough to check directly. Now pick any larger value of b. The algorithm finds $\gcd(a, b)$ by calling $\gcd(b, a \bmod b)$. Since $a \bmod b < b$, we can apply the inductive hypothesis to this recursive call and conclude that the x' and y' it returns are correct:

$$\gcd(b, a \bmod b) = bx' + (a \bmod b)y'.$$

Writing $(a \bmod b)$ as $(a - \lfloor a/b \rfloor b)$, we find

$$d = \gcd(a,b) \;=\; \gcd(b, a \bmod b) = \; bx' + (a \bmod b)y' \;=\; bx' + (a - \lfloor a/b \rfloor b)y' \;=\; ay' + b(x' - \lfloor a/b \rfloor y').$$

Therefore $d = ax + by$ with $x = y'$ and $y = x' - \lfloor a/b \rfloor y'$, thus validating the algorithm's behavior on input (a, b). ∎

Example. To compute $\gcd(25, 11)$, Euclid's algorithm would proceed as follows:

$$\begin{aligned} \underline{25} &= 2 \cdot \underline{11} + 3 \\ \underline{11} &= 3 \cdot \underline{3} + 2 \\ \underline{3} &= 1 \cdot \underline{2} + 1 \\ \underline{2} &= 2 \cdot \underline{1} + 0 \end{aligned}$$

(at each stage, the gcd computation has been reduced to the underlined numbers). Thus $\gcd(25, 11) = \gcd(11, 3) = \gcd(3, 2) = \gcd(2, 1) = \gcd(1, 0) = 1$.

To find x and y such that $25x + 11y = 1$, we start by expressing 1 in terms of the last pair $(1, 0)$. Then we work backwards and express it in terms of $(2, 1)$, $(3, 2)$, $(11, 3)$, and finally $(25, 11)$. The first step is:

$$1 = \underline{1} - \underline{0}.$$

To rewrite this in terms of $(2, 1)$, we use the substitution $0 = 2 - 2 \cdot 1$ from the last line of the gcd calculation to get:

$$1 = \underline{1} - (\underline{2} - 2 \cdot \underline{1}) = -1 \cdot \underline{2} + 3 \cdot \underline{1}.$$

The second-last line of the gcd calculation tells us that $1 = 3 - 1 \cdot 2$. Substituting:

$$1 = -1 \cdot \underline{2} + 3(\underline{3} - 1 \cdot \underline{2}) = 3 \cdot \underline{3} - 4 \cdot \underline{2}.$$

Continuing in this same way with substitutions $2 = 11 - 3 \cdot 3$ and $3 = 25 - 2 \cdot 11$ gives:

$$1 = 3 \cdot \underline{3} - 4(\underline{11} - 3 \cdot \underline{3}) = -4 \cdot \underline{11} + 15 \cdot \underline{3} = -4 \cdot \underline{11} + 15(\underline{25} - 2 \cdot \underline{11}) = 15 \cdot \underline{25} - 34 \cdot \underline{11}.$$

We're done: $15 \cdot 25 - 34 \cdot 11 = 1$, so $x = 15$ and $y = -34$.

1.2.5 Modular division

In real arithmetic, every number $a \neq 0$ has an inverse, $1/a$, and dividing by a is the same as multiplying by this inverse. In modular arithmetic, we can make a similar definition.

> We say x is the *multiplicative inverse* of a modulo N if $ax \equiv 1 \pmod{N}$.

There can be at most one such x modulo N (Exercise 1.23), and we shall denote it by a^{-1}. However, this inverse does not always exist! For instance, 2 is not invertible modulo 6: that is, $2x \not\equiv 1 \bmod 6$ for every possible choice of x. In this case, a and N are both even and thus then $a \bmod N$ is always even, since $a \bmod N = a - kN$ for some k. More generally, we can be certain that $\gcd(a, N)$ divides $ax \bmod N$, because this latter quantity can be written in the form $ax + kN$. So if $\gcd(a, N) > 1$, then $ax \not\equiv 1 \bmod N$, no matter what x might be, and therefore a cannot have a multiplicative inverse modulo N.

In fact, this is the only circumstance in which a is not invertible. When $\gcd(a, N) = 1$ (we say a and N are *relatively prime*), the extended Euclid algorithm gives us integers x and y such that $ax + Ny = 1$, which means that $ax \equiv 1 \pmod N$. Thus x is a's sought inverse.

Example. Continuing with our previous example, suppose we wish to compute $11^{-1} \bmod 25$. Using the extended Euclid algorithm, we find that $15 \cdot 25 - 34 \cdot 11 = 1$. Reducing both sides modulo 25, we have $-34 \cdot 11 \equiv 1 \bmod 25$. So $-34 \equiv 16 \bmod 25$ is the inverse of $11 \bmod 25$.

Modular division theorem *For any $a \bmod N$, a has a multiplicative inverse modulo N if and only if it is relatively prime to N. When this inverse exists, it can be found in time $O(n^3)$ (where as usual n denotes the number of bits of N) by running the extended Euclid algorithm.*

This resolves the issue of modular division: when working modulo N, we can divide by numbers relatively prime to N—and only by these. And to actually carry out the division, we multiply by the inverse.

Is your social security number a prime?

The numbers $7, 17, 19, 71$, and 79 are primes, but how about 717-19-7179? Telling whether a reasonably large number is a prime seems tedious because there are far too many candidate factors to try. However, there are some clever tricks to speed up the process. For instance, you can omit even-valued candidates after you have eliminated the number 2. You can actually omit all candidates except those that are themselves primes.

In fact, a little further thought will convince you that you can proclaim N a prime as soon as you have rejected all candidates *up to* $\sqrt{N}$, for if N can indeed be factored as $N = K \cdot L$, then it is impossible for both factors to exceed $\sqrt{N}$.

We seem to be making progress! Perhaps by omitting more and more candidate factors, a truly efficient primality test can be discovered.

Unfortunately, there is no fast primality test down this road. The reason is that we have been trying to tell if a number is a prime *by factoring it.* And factoring is a hard problem!

Modern cryptography, as well as the balance of this chapter, is about the following important idea: ***factoring is hard and primality is easy.*** We cannot factor large numbers, but we can easily test huge numbers for primality! (Presumably, if a number is composite, such a test will detect this *without finding a factor.*)

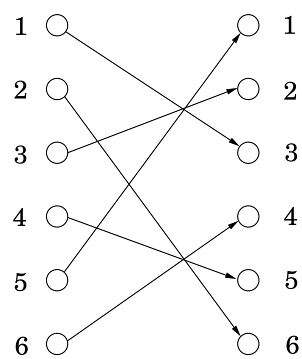
1.3 Primality testing

Is there some litmus test that will tell us whether a number is prime without actually trying to factor the number? We place our hopes in a theorem from the year 1640.

Fermat's little theorem *If p is prime, then for every $1 \leq a < p$,*

$$a^{p-1} \equiv 1 \pmod{p}.$$

Proof. Let S be the nonzero integers modulo p; that is, $S = \{1, 2, \ldots, p-1\}$. Here's the crucial observation: the effect of multiplying these numbers by a (modulo p) is simply to permute them. For instance, here's a picture of the case $a = 3, p = 7$:



Let's carry this example a bit further. From the picture, we can conclude

$$\{1, 2, \ldots, 6\} \; = \; \{3 \cdot 1 \bmod 7,\ 3 \cdot 2 \bmod 7, \ldots,\ 3 \cdot 6 \bmod 7\}.$$

Multiplying all the numbers in each representation then gives $6! \equiv 3^6 \cdot 6! \pmod 7$, and dividing by $6!$ we get $3^6 \equiv 1 \pmod 7$, exactly the result we wanted in the case $a = 3, p = 7$.

Now let's generalize this argument to other values of a and p, with $S = \{1, 2, \ldots, p-1\}$. We'll prove that when the elements of S are multiplied by a modulo p, the resulting numbers are all distinct and nonzero. And since they lie in the range $[1, p-1]$, they must simply be a permutation of S.

The numbers $a \cdot i \bmod p$ are distinct because if $a \cdot i \equiv a \cdot j \pmod p$, then dividing both sides by a gives $i \equiv j \pmod p$. They are nonzero because $a \cdot i \equiv 0$ similarly implies $i \equiv 0$. (And we *can* divide by a, because by assumption it is nonzero and therefore relatively prime to p.)

We now have two ways to write set S:

$$S \; = \; \{1, 2, \ldots, p-1\} \; = \; \{a \cdot 1 \bmod p,\ a \cdot 2 \bmod p, \ldots,\ a \cdot (p-1) \bmod p\}.$$

We can multiply together its elements in each of these representations to get

$$(p-1)! \equiv a^{p-1} \cdot (p-1)! \pmod{p}.$$

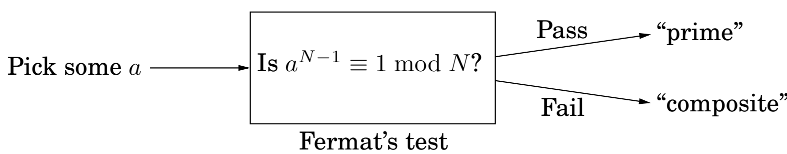
Dividing by $(p-1)!$ (which we can do because it is relatively prime to p, since p is assumed prime) then gives the theorem. ∎

This theorem suggests a "factorless" test for determining whether a number N is prime:

Figure 1.7 An algorithm for testing primality.

```
function primality(N)
Input:  Positive integer N
Output:  yes/no

Pick a positive integer a < N at random
if a^(N-1) ≡ 1 (mod N):
   return yes
else:
   return no
```



The problem is that Fermat’s theorem is not an if-and-only-if condition; it doesn’t say what happens when N is *not* prime, so in these cases the preceding diagram is questionable. In fact, it *is* possible for a composite number N to pass Fermat’s test (that is, $a^{N-1} \equiv 1 \bmod N$) for certain choices of a. For instance, $341 = 11 \cdot 31$ is not prime, and yet $2^{340} \equiv 1 \bmod 341$. Nonetheless, we might hope that for composite N, *most* values of a will fail the test. This is indeed true, in a sense we will shortly make precise, and motivates the algorithm of Figure 1.7: rather than fixing an arbitrary value of a in advance, we should choose it *randomly* from $\{1, \ldots, N-1\}$.

In analyzing the behavior of this algorithm, we first need to get a minor bad case out of the way. It turns out that certain extremely rare composite numbers N, called *Carmichael numbers*, pass Fermat’s test for *all* a relatively prime to N. On such numbers our algorithm will fail; but they are pathologically rare, and we will later see how to deal with them (page 38), so let’s ignore these numbers for the time being.

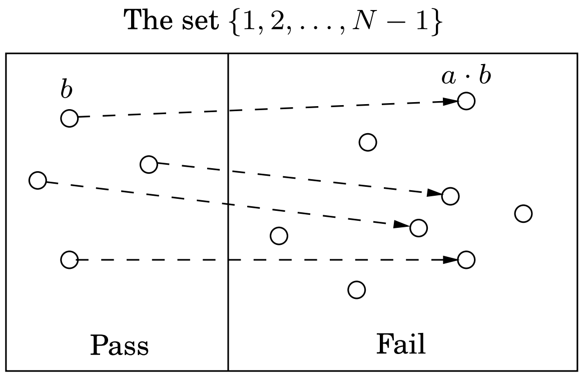
In a Carmichael-free universe, our algorithm works well. Any prime number N will of course pass Fermat’s test and produce the right answer. On the other hand, any non-Carmichael composite number N must fail Fermat’s test for some value of a; and as we will now show, this implies immediately that N fails Fermat’s test for *at least half the possible values of a!*

Lemma *If $a^{N-1} \not\equiv 1 \bmod N$ for some a relatively prime to N, then it must hold for at least half the choices of $a < N$.*

Proof. Fix some value of a for which $a^{N-1} \not\equiv 1 \bmod N$. The key is to notice that every element $b < N$ that passes Fermat’s test with respect to N (that is, $b^{N-1} \equiv 1 \bmod N$) has a twin, $a \cdot b$, that fails the test:

$$(a \cdot b)^{N-1} \equiv a^{N-1} \cdot b^{N-1} \equiv a^{N-1} \not\equiv 1 \bmod N.$$

Moreover, all these elements $a \cdot b$, for fixed a but different choices of b, are distinct, for the same reason $a \cdot i \not\equiv a \cdot j$ in the proof of Fermat's test: just divide by a.



The one-to-one function $b \mapsto a \cdot b$ shows that at least as many elements fail the test as pass it. ∎

Hey, that was group theory!

For any integer N, the set of all numbers mod N that are relatively prime to N constitute what mathematicians call a *group*:

- There is a multiplication operation defined on this set.
- The set contains a neutral element (namely 1: any number multiplied by this remains unchanged).
- All elements have a well-defined inverse.

This particular group is called the *multiplicative group of* N, usually denoted $\mathbb{Z}_N^*$.

Group theory is a very well developed branch of mathematics. One of its key concepts is that a group can contain a *subgroup*—a subset that is a group in and of itself. And an important fact about a subgroup is that its size must divide the size of the whole group.

Consider now the set $B = \{b : b^{N-1} \equiv 1 \bmod N\}$. It is not hard to see that it is a subgroup of $\mathbb{Z}_N^*$ (just check that B is closed under multiplication and inverses). Thus the size of B must divide that of $\mathbb{Z}_N^*$. Which means that if B doesn't contain all of $\mathbb{Z}_N^*$, the next largest size it can have is $|\mathbb{Z}_N^*|/2$.

We are ignoring Carmichael numbers, so we can now assert

If N is prime, then $a^{N-1} \equiv 1 \bmod N$ for all $a < N$.
If N is not prime, then $a^{N-1} \equiv 1 \bmod N$ for at most half the values of $a < N$.

The algorithm of Figure 1.7 therefore has the following probabilistic behavior.

$$\begin{aligned}
\Pr(\text{Algorithm 1.7 returns } \texttt{yes} \text{ when } N \text{ is prime}) &= 1 \\
\Pr(\text{Algorithm 1.7 returns } \texttt{yes} \text{ when } N \text{ is not prime}) &\leq \frac{1}{2}
\end{aligned}$$

Figure 1.8 An algorithm for testing primality, with low error probability.

```
function primality2(N)
Input:  Positive integer N
Output:  yes/no

Pick positive integers a_1, a_2, ..., a_k < N at random
if a_i^{N-1} ≡ 1 (mod N) for all i = 1, 2, ..., k:
   return yes
else:
   return no
```

We can reduce this *one-sided error* by repeating the procedure many times, by randomly picking several values of a and testing them all (Figure 1.8).

$$\Pr(\text{Algorithm 1.8 returns } \texttt{yes} \text{ when } N \text{ is not prime}) \quad \leq \quad \frac{1}{2^k}$$

This probability of error drops exponentially fast, and can be driven arbitrarily low by choosing k large enough. Testing $k = 100$ values of a makes the probability of failure at most 2^{-100}, which is miniscule: far less, for instance, than the probability that a random cosmic ray will sabotage the computer during the computation!

1.3.1 Generating random primes

We are now close to having all the tools we need for cryptographic applications. The final piece of the puzzle is a fast algorithm for choosing random primes that are a few hundred bits long. What makes this task quite easy is that primes are abundant—a random n-bit number has roughly a one-in-n chance of being prime (actually about $1/(\ln 2^n) \approx 1.44/n$). For instance, *about 1 in 20 social security numbers is prime!*

Lagrange's prime number theorem *Let $\pi(x)$ be the number of primes $\leq x$. Then $\pi(x) \approx x/(\ln x)$, or more precisely,*

$$\lim_{x\to\infty} \frac{\pi(x)}{(x/\ln x)} = 1.$$

Such abundance makes it simple to generate a random n-bit prime:

- Pick a random n-bit number N.
- Run a primality test on N.
- If it passes the test, output N; else repeat the process.

Carmichael numbers

The smallest Carmichael number is 561. It is not a prime: $561 = 3 \cdot 11 \cdot 17$; yet it fools the Fermat test, because $a^{560} \equiv 1 \pmod{561}$ for all values of a relatively prime to 561. For a long time it was thought that there might be only finitely many numbers of this type; now we know they are infinite, but exceedingly rare.

There *is* a way around Carmichael numbers, using a slightly more refined primality test due to Rabin and Miller. Write $N - 1$ in the form $2^t u$. As before we'll choose a random base a and check the value of $a^{N-1} \bmod N$. Perform this computation by first determining $a^u \bmod N$ and then repeatedly squaring, to get the sequence:

$$a^u \bmod N,\ a^{2u} \bmod N,\ \ldots,\ a^{2^t u} = a^{N-1} \bmod N.$$

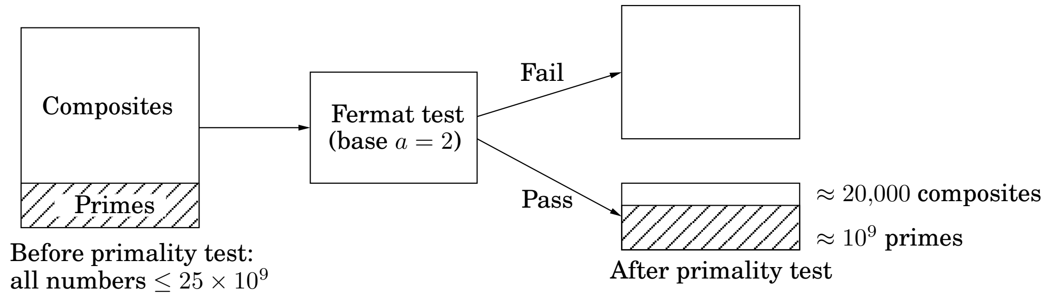
If $a^{N-1} \not\equiv 1 \bmod N$, then N is composite by Fermat's little theorem, and we're done. But if $a^{N-1} \equiv 1 \bmod N$, we conduct a little follow-up test: somewhere in the preceding sequence, we ran into a 1 for the first time. If this happened after the first position (that is, if $a^u \bmod N \neq 1$), and if the preceding value in the list is not $-1 \bmod N$, then we declare N composite.

In the latter case, we have found a *nontrivial square root* of 1 modulo N: a number that is not $\pm 1 \bmod N$ but that when squared is equal to $1 \bmod N$. Such a number can only exist if N is composite (Exercise 1.40). It turns out that if we combine this square-root check with our earlier Fermat test, then at least three-fourths of the possible values of a between 1 and $N - 1$ will reveal a composite N, even if it is a Carmichael number.

How fast is this algorithm? If the randomly chosen N is truly prime, which happens with probability at least $1/n$, then it will certainly pass the test. So on each iteration, this procedure has at least a $1/n$ chance of halting. Therefore on average it will halt within $O(n)$ rounds (Exercise 1.34).

Next, exactly which primality test should be used? In this application, since the numbers we are testing for primality are chosen at random rather than by an adversary, it is sufficient to perform the Fermat test with base $a = 2$ (or to be really safe, $a = 2, 3, 5$), because for random numbers the Fermat test has a much smaller failure probability than the worst-case $1/2$ bound that we proved earlier. Numbers that pass this test have been jokingly referred to as "industrial grade primes." The resulting algorithm is quite fast, generating primes that are hundreds of bits long in a fraction of a second on a PC.

The important question that remains is: what is the probability that the output of the algorithm is really prime? To answer this we must first understand how discerning the Fermat test is. As a concrete example, suppose we perform the test with base $a = 2$ for all numbers $N \leq 25 \times 10^9$. In this range, there are about 10^9 primes, and about $20{,}000$ composites that pass the test (see the following figure). Thus the chance of erroneously outputting a composite is approximately $20{,}000/10^9 = 2 \times 10^{-5}$. This chance of error decreases rapidly as the length of the numbers involved is increased (to the few hundred digits we expect in our applications).



Randomized algorithms: a virtual chapter

Surprisingly—almost paradoxically—some of the fastest and most clever algorithms we have rely on *chance*: at specified steps they proceed according to the outcomes of random coin tosses. These *randomized algorithms* are often very simple and elegant, and their output is correct *with high probability*. This success probability does not depend on the randomness of the input; it only depends on the random choices made by the algorithm itself.

Instead of devoting a special chapter to this topic, in this book we intersperse randomized algorithms at the chapters and sections where they arise most naturally. Furthermore, no specialized knowledge of probability is necessary to follow what is happening. You just need to be familiar with the concept of probability, expected value, the expected number of times we must flip a coin before getting heads, and the property known as "linearity of expectation."

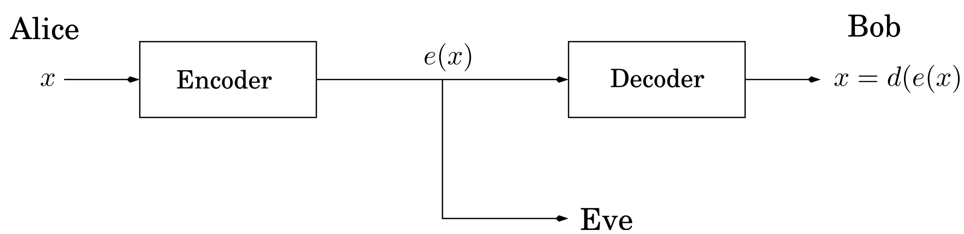
Here are pointers to the major randomized algorithms in this book: One of the earliest and most dramatic examples of a randomized algorithm is the randomized primality test of Figure 1.8. Hashing is a general randomized data structure that supports inserts, deletes, and lookups and is described later in this chapter, in Section 1.5. Randomized algorithms for sorting and median finding are described in Chapter 2. A randomized algorithm for the min cut problem is described in the box on page 150. Randomization plays an important role in heuristics as well; these are described in Section 9.3. And finally the quantum algorithm for factoring (Section 10.7) works very much like a randomized algorithm, its output being correct with high probability—except that it draws its randomness not from coin tosses, but from the superposition principle in quantum mechanics.

Virtual exercises: 1.29, 1.34, 2.24, 9.8, 10.8.

1.4 Cryptography

Our next topic, the Rivest-Shamir-Adelman (RSA) cryptosystem, uses all the ideas we have introduced in this chapter! It derives very strong guarantees of security by ingeniously exploiting the wide gulf between the polynomial-time computability of certain number-theoretic tasks (modular exponentiation, greatest common divisor, primality testing) and the intractability of others (factoring).

The typical setting for cryptography can be described via a cast of three characters: Alice and Bob, who wish to communicate in private, and Eve, an eavesdropper who will go to great lengths to find out what they are saying. For concreteness, let's say Alice wants to send a specific message x, written in binary (why not), to her friend Bob. She encodes it as $e(x)$, sends it over, and then Bob applies his decryption function $d(\cdot)$ to decode it: $d(e(x)) = x$. Here $e(\cdot)$ and $d(\cdot)$ are appropriate transformations of the messages.



Alice and Bob are worried that the eavesdropper, Eve, will intercept $e(x)$: for instance, she might be a sniffer on the network. But ideally the encryption function $e(\cdot)$ is so chosen that without knowing $d(\cdot)$, Eve cannot do anything with the information she has picked up. In other words, knowing $e(x)$ tells her little or nothing about what x might be.

For centuries, cryptography was based on what we now call *private-key protocols*. In such a scheme, Alice and Bob meet beforehand and together choose a secret codebook, with which they encrypt all future correspondence between them. Eve's only hope, then, is to collect some encoded messages and use them to at least partially figure out the codebook.

Public-key schemes such as RSA are significantly more subtle and tricky: they allow Alice to send Bob a message without ever having met him before. This almost sounds impossible, because in this scenario there is a symmetry between Bob and Eve: why should Bob have any advantage over Eve in terms of being able to understand Alice's message? The central idea behind the RSA cryptosystem is that using the dramatic contrast between factoring and primality, Bob is able to implement a *digital lock*, to which only he has the key. Now by making this digital lock public, he gives Alice a way to send him a secure message, which only he can open. Moreover, this is exactly the scenario that comes up in Internet commerce, for example, when you wish to send your credit card number to some company over the Internet.

In the RSA protocol, Bob need only perform the simplest of calculations, such as multiplication, to implement his digital lock. Similarly Alice and Bob need only perform simple calculations to lock and unlock the message respectively—operations that any pocket computing device could handle. By contrast, to unlock the message without the key, Eve must perform operations like factoring large numbers, which requires more computational power than would be afforded by the world's most powerful computers combined. This compelling guarantee of security explains why the RSA cryptosystem is such a revolutionary development in cryptography.

An application of number theory?

The renowned mathematician G. H. Hardy once declared of his work: “I have never done anything useful.” Hardy was an expert in the theory of numbers, which has long been regarded as one of the purest areas of mathematics, untarnished by material motivation and consequence. Yet the work of thousands of number theorists over the centuries, Hardy’s included, is now crucial to the operation of Web browsers and cell phones and to the security of financial transactions worldwide.

1.4.1 Private-key schemes: one-time pad and AES

If Alice wants to transmit an important private message to Bob, it would be wise of her to scramble it with an encryption function,

$$e : \langle\text{messages}\rangle \rightarrow \langle\text{encoded messages}\rangle.$$

Of course, this function must be invertible—for decoding to be possible—and is therefore a bijection. Its inverse is the decryption function $d(\cdot)$.

In the *one-time pad*, Alice and Bob meet beforehand and secretly choose a binary string r of the same length—say, n bits—as the important message x that Alice will later send. Alice’s encryption function is then a *bitwise exclusive-or*, $e_r(x) = x \oplus r$: each position in the encoded message is the exclusive-or of the corresponding positions in x and r. For instance, if $r = 01110010$, then the message 11110000 is scrambled thus:

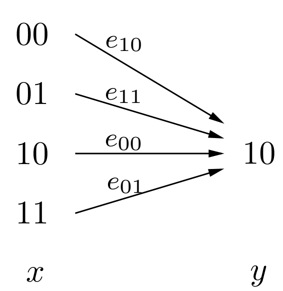
$$e_r(11110000) \;=\; 11110000 \oplus 01110010 \;=\; 10000010.$$

This function e_r is a bijection from n-bit strings to n-bit strings, as evidenced by the fact that it is its own inverse!

$$e_r(e_r(x)) \;=\; (x \oplus r) \oplus r \;=\; x \oplus (r \oplus r) \;=\; x \oplus \overline{0} \;=\; x,$$

where $\overline{0}$ is the string of all zeros. Thus Bob can decode Alice’s transmission by applying the same encryption function a second time: $d_r(y) = y \oplus r$.

How should Alice and Bob choose r for this scheme to be secure? Simple: they should pick r *at random*, flipping a coin for each bit, so that the resulting string is equally likely to be any element of $\{0,1\}^n$. This will ensure that if Eve intercepts the encoded message $y = e_r(x)$, she gets no information about x. Suppose, for example, that Eve finds out $y = 10$; what can she deduce? She doesn’t know r, and the possible values it can take all correspond to different original messages x:



So given what Eve knows, all possibilities for x are equally likely!

The downside of the one-time pad is that it has to be discarded after use, hence the name. A second message encoded with the same pad would not be secure, because if Eve knew $x \oplus r$ and $z \oplus r$ for two messages x and z, then she could take the exclusive-or to get $x \oplus z$, which might be important information—for example, (1) it reveals whether the two messages begin or end the same, and (2) if one message contains a long sequence of zeros (as could easily be the case if the message is an image), then the corresponding part of the other message will be exposed. Therefore the random string that Alice and Bob share has to be the combined length of all the messages they will need to exchange.

The one-time pad is a toy cryptographic scheme whose behavior and theoretical properties are completely clear. At the other end of the spectrum lies the *advanced encryption standard* (AES), a very widely used cryptographic protocol that was approved by the U.S. National Institute of Standards and Technologies in 2001. AES is once again private-key: Alice and Bob have to agree on a shared random string r. But this time the string is of a small fixed size, 128 to be precise (variants with 192 or 256 bits also exist), and specifies a bijection e_r from 128-bit strings to 128-bit strings. The crucial difference is that this function can be used repeatedly, so for instance a long message can be encoded by splitting it into segments of 128 bits and applying e_r to each segment.

The security of AES has not been rigorously established, but certainly at present the general public does not know how to break the code—to recover x from $e_r(x)$—except using techniques that are not very much better than the brute-force approach of trying all possibilities for the shared string r.

1.4.2 RSA

Unlike the previous two protocols, the RSA scheme is an example of *public-key cryptography*: anybody can send a message to anybody else using publicly available information, rather like addresses or phone numbers. Each person has a public key known to the whole world and a secret key known only to him- or herself. When Alice wants to send message x to Bob, she encodes it using his public key. He decrypts it using his secret key, to retrieve x. Eve is welcome to see as many encrypted messages for Bob as she likes, but she will not be able to decode them, under certain simple assumptions.

The RSA scheme is based heavily upon number theory. Think of messages from Alice to Bob as numbers modulo N; messages larger than N can be broken into smaller pieces. The encryption function will then be a bijection on $\{0, 1, \ldots, N-1\}$, and the decryption function will be its inverse. What values of N are appropriate, and what bijection should be used?

Property *Pick any two primes p and q and let $N = pq$. For any e relatively prime to $(p-1)(q-1)$:*

1. *The mapping $x \mapsto x^e \bmod N$ is a bijection on $\{0, 1, \ldots, N-1\}$.*

2. *Moreover, the inverse mapping is easily realized: let d be the inverse of e modulo $(p-$*

$1)(q-1)$. *Then for all* $x \in \{0, \ldots, N-1\}$,

$$(x^e)^d \equiv x \bmod N.$$

The first property tells us that the mapping $x \mapsto x^e \bmod N$ is a reasonable way to encode messages x; no information is lost. So, if Bob publishes (N, e) as his *public key*, everyone else can use it to send him encrypted messages. The second property then tells us how decryption can be achieved. Bob should retain the value d as his *secret key*, with which he can decode all messages that come to him by simply raising them to the dth power modulo N.

Example. Let $N = 55 = 5 \cdot 11$. Choose encryption exponent $e = 3$, which satisfies the condition $\gcd(e, (p-1)(q-1)) = \gcd(3, 40) = 1$. The decryption exponent is then $d = 3^{-1} \bmod 40 = 27$. Now for any message $x \bmod 55$, the encryption of x is $y = x^3 \bmod 55$, and the decryption of y is $x = y^{27} \bmod 55$. So, for example, if $x = 13$, then $y = 13^3 = 52 \bmod 55$. and $13 = 52^{27} \bmod 55$.

Let's prove the assertion above and then examine the security of the scheme.

Proof. If the mapping $x \mapsto x^e \bmod N$ is invertible, it must be a bijection; hence statement 2 implies statement 1. To prove statement 2, we start by observing that e is invertible modulo $(p-1)(q-1)$ because it is relatively prime to this number. To see that $(x^e)^d \equiv x \bmod N$, we examine the exponent: since $ed \equiv 1 \bmod (p-1)(q-1)$, we can write ed in the form $1 + k(p-1)(q-1)$ for some k. Now we need to show that the difference

$$x^{ed} - x \;=\; x^{1+k(p-1)(q-1)} - x$$

is always 0 modulo N. The second form of the expression is convenient because it can be simplified using Fermat's little theorem. It is divisible by p (since $x^{p-1} \equiv 1 \bmod p$) and likewise by q. Since p and q are primes, this expression must also be divisible by their product N. Hence $x^{ed} - x = x^{1+k(p-1)(q-1)} - x \equiv 0 \pmod{N}$, exactly as we need. ∎

The RSA protocol is summarized in Figure 1.9. It is certainly convenient: the computations it requires of Alice and Bob are elementary. But how secure is it against Eve?

The security of RSA hinges upon a simple assumption:

> Given N, e, and $y = x^e \bmod N$, it is computationally intractable to determine x.

This assumption is quite plausible. How might Eve try to guess x? She could experiment with all possible values of x, each time checking whether $x^e \equiv y \bmod N$, but this would take exponential time. Or she could try to factor N to retrieve p and q, and then figure out d by inverting e modulo $(p-1)(q-1)$, but we believe factoring to be hard. Intractability is normally a source of dismay; the insight of RSA lies in using it to advantage.

1.5 Universal hashing

We end this chapter with an application of number theory to the design of *hash functions.* Hashing is a very useful method of storing data items in a table so as to support insertions, deletions, and lookups.

Figure 1.9 RSA.

Bob chooses his public and secret keys.

- He starts by picking two large (n-bit) random primes p and q.
- His public key is (N, e) where $N = pq$ and e is a $2n$-bit number relatively prime to $(p-1)(q-1)$. A common choice is $e = 3$ because it permits fast encoding.
- His secret key is d, the inverse of e modulo $(p-1)(q-1)$, computed using the extended Euclid algorithm.

Alice wishes to send message x to Bob.

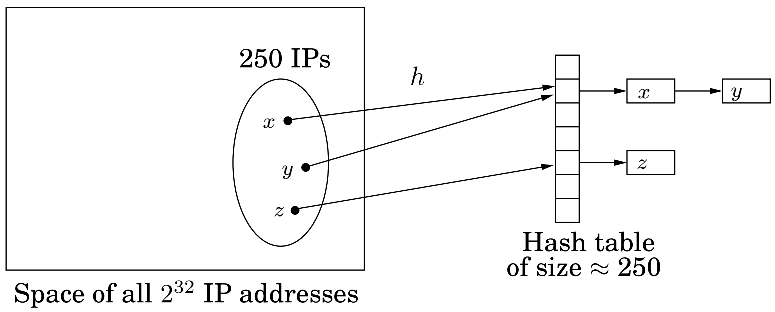
- She looks up his public key (N, e) and sends him $y = (x^e \bmod N)$, computed using an efficient modular exponentiation algorithm.
- He decodes the message by computing $y^d \bmod N$.

Suppose, for instance, that we need to maintain an ever-changing list of about 250 IP (Internet protocol) addresses, perhaps the addresses of the currently active customers of a Web service. (Recall that an IP address consists of 32 bits encoding the location of a computer on the Internet, usually shown broken down into four 8-bit fields, for example, 128.32.168.80.) We could obtain fast lookup times if we maintained the records in an array indexed by IP address. But this would be very wasteful of memory: the array would have $2^{32} \approx 4 \times 10^9$ entries, the vast majority of them blank. Or alternatively, we could use a linked list of just the 250 records. But then accessing records would be very slow, taking time proportional to 250, the total number of customers. Is there a way to get the best of both worlds, to use an amount of memory that is proportional to the number of customers and yet also achieve fast lookup times? This is exactly where hashing comes in.

1.5.1 Hash tables

Here's a high-level view of hashing. We will give a short "nickname" to each of the 2^{32} possible IP addresses. You can think of this short name as just a number between 1 and 250 (we will later adjust this range very slightly). Thus many IP addresses will inevitably have the same nickname; however, we hope that most of the 250 IP addresses of our particular customers are assigned distinct names, and we will store their records in an array of size 250 indexed by these names. What if there is more than one record associated with the same name? Easy: each entry of the array points to a linked list containing all records with that name. So the total amount of storage is proportional to 250, the number of customers, and is independent of the total number of possible IP addresses. Moreover, if not too many customer IP addresses are assigned the same name, lookup is fast, because the average size of the linked list we have to scan through is small.

But how do we assign a short name to each IP address? This is the role of a *hash function*: in our example, a function h that maps IP addresses to positions in a table of length about 250 (the expected number of data items). The name assigned to an IP address x is thus $h(x)$, and the record for x is stored in position $h(x)$ of the table. As described before, each position of the table is in fact a *bucket*, a linked list that contains all current IP addresses that map to it. Hopefully, there will be very few buckets that contain more than a handful of IP addresses.



1.5.2 Families of hash functions

Designing hash functions is tricky. A hash function must in some sense be "random" (so that it scatters data items around), but it should also be a function and therefore "consistent" (so that we get the same result every time we apply it). And the statistics of the data items may work against us. In our example, one possible hash function would map an IP address to the 8-bit number that is its last segment: $h(128.32.168.80) = 80$. A table of $n = 256$ buckets would then be required. But is this a good hash function? Not if, for example, the last segment of an IP address tends to be a small (single- or double-digit) number; then low-numbered buckets would be crowded. Taking the first segment of the IP address also invites disaster—for example, if most of our customers come from Asia.

There is nothing inherently wrong with these two functions. If our 250 IP addresses were uniformly drawn from among all $N = 2^{32}$ possibilities, then these functions would behave well. The problem is we have no guarantee that the distribution of IP addresses is uniform.

Conversely, there is no single hash function, no matter how sophisticated, that behaves well on all sets of data. Since a hash function maps 2^{32} IP addresses to just 250 names, there must be a collection of at least $2^{32}/250 \approx 2^{24} \approx 16{,}000{,}000$ IP addresses that are assigned the same name (or, in hashing terminology, "collide"). If many of these show up in our customer set, we're in trouble.

Obviously, we need some kind of randomization. Here's an idea: let us pick a hash function *at random* from some class of functions. We will then show that, no matter what set of 250 IP addresses we actually care about, most choices of the hash function will give very few collisions among these addresses.

To this end, we need to define a class of hash functions from which we can pick at random; and this is where we turn to number theory. Let us take the number of buckets to be not 250 but $n = 257$—*a prime number*! And we consider every IP address x as a quadruple $x = (x_1, \ldots, x_4)$ of integers modulo n—recall that it is in fact a quadruple of integers between 0 and 255, so there is no harm in this. We can define a function h from IP addresses to a number mod n as follows: fix any four numbers mod $n = 257$, say 87, 23, 125, and 4. Now map the IP address $(x_1, ..., x_4)$ to $h(x_1, ..., x_4) = (87x_1 + 23x_2 + 125x_3 + 4x_4) \bmod 257$. Indeed, any four numbers mod n define a hash function.

For any four coefficients $a_1, \ldots, a_4 \in \{0, 1, \ldots, n-1\}$, write $a = (a_1, a_2, a_3, a_4)$ and define h_a to be the following hash function:

$$h_a(x_1, \ldots, x_4) \;=\; \sum_{i=1}^{4} a_i \cdot x_i \bmod n.$$

We will show that if we pick these coefficients a at random, then h_a is very likely to be good in the following sense.

Property *Consider any pair of distinct IP addresses $x = (x_1, \ldots, x_4)$ and $y = (y_1, \ldots, y_4)$. If the coefficients $a = (a_1, a_2, a_3, a_4)$ are chosen uniformly at random from $\{0, 1, \ldots, n-1\}$, then*

$$\Pr\{h_a(x_1, \ldots, x_4) = h_a(y_1, \ldots, y_4)\} \;=\; \frac{1}{n}.$$

In other words, the chance that x and y collide under h_a is the same as it would be if each were assigned nicknames randomly and independently. This condition guarantees that the expected lookup time for any item is small. Here's why. If we wish to look up x in our hash table, the time required is dominated by the size of its bucket, that is, the number of items that are assigned the same name as x. But there are only 250 items in the hash table, and the probability that any one item gets the same name as x is $1/n = 1/257$. Therefore the expected number of items that are assigned the same name as x by a randomly chosen hash function h_a is $250/257 \approx 1$, which means the expected size of x's bucket is less than 2.[1]

Let us now prove the preceding property.

Proof. Since $x = (x_1, \ldots, x_4)$ and $y = (y_1, \ldots, y_4)$ are distinct, these quadruples must differ in some component; without loss of generality let us assume that $x_4 \neq y_4$. We wish to compute the probability $\Pr[h_a(x_1, \ldots, x_4) = h_a(y_1, \ldots, y_4)]$, that is, the probability that $\sum_{i=1}^{4} a_i \cdot x_i \equiv \sum_{i=1}^{4} a_i \cdot y_i \bmod n$. This last equation can be rewritten as

$$\sum_{i=1}^{3} a_i \cdot (x_i - y_i) \;\equiv\; a_4 \cdot (y_4 - x_4) \bmod n \tag{1}$$

Suppose that we draw a random hash function h_a by picking $a = (a_1, a_2, a_3, a_4)$ at random. We start by drawing a_1, a_2, and a_3, and then we pause and think: What is the probability that the last drawn number a_4 is such that equation (1) holds? So far the left-hand side of equation (1) evaluates to some number, call it c. And since n is prime and $x_4 \neq y_4$, $(y_4 - x_4)$ has a

[1]When a hash function h_a is chosen at random, let the random variable Y_i (for $i = 1, \ldots, 250$) be 1 if item i gets the same name as x and 0 otherwise. So the expected value of Y_i is $1/n$. Now, $Y = Y_1 + Y_2 + \cdots + Y_{250}$ is the number of items which get the same name as x, and by linearity of expectation, the expected value of Y is simply the sum of the expected values of Y_1 through Y_{250}. It is thus $250/n = 250/257$.

unique inverse modulo n. Thus for equation (1) to hold, the last number a_4 must be precisely $c \cdot (y_4 - x_4)^{-1} \bmod n$, out of its n possible values. The probability of this happening is $1/n$, and the proof is complete. ∎

Let us step back and see what we just achieved. Since we have no control over the set of data items, we decided instead to select a hash function h uniformly at random from among a family $\mathcal{H}$ of hash functions. In our example,

$$\mathcal{H} = \{h_a : a \in \{0, \ldots, n-1\}^4\}.$$

To draw a hash function uniformly at random from this family, we just draw four numbers $a_1, \ldots, a_4$ modulo n. (Incidentally, notice that the two simple hash functions we considered earlier, namely, taking the last or the first 8-bit segment, belong to this class. They are $h_{(0,0,0,1)}$ and $h_{(1,0,0,0)}$, respectively.) And we insisted that the family have the following property:

> For any two distinct data items x and y, exactly $|\mathcal{H}|/n$ of all the hash functions in $\mathcal{H}$ map x and y to the same bucket, where n is the number of buckets.

A family of hash functions with this property is called *universal*. In other words, for any two data items, the probability these items collide is $1/n$ if the hash function is randomly drawn from a universal family. This is also the collision probability if we map x and y to buckets uniformly at random—in some sense the gold standard of hashing. We then showed that this property implies that hash table operations have good performance *in expectation*.

This idea, motivated as it was by the hypothetical IP address application, can of course be applied more generally. Start by choosing the table size n to be some prime number that is a little larger than the number of items expected in the table (there is usually a prime number close to any number we start with; actually, to ensure that hash table operations have good performance, it is better to have the size of the hash table be about twice as large as the number of items). Next assume that the size of the domain of all data items is $N = n^k$, a power of n (if we need to overestimate the true number of data items, so be it). Then each data item can be considered as a k-tuple of integers modulo n, and $\mathcal{H} = \{h_a : a \in \{0, \ldots, n-1\}^k\}$ is a universal family of hash functions.