1. Determine if the systems in (a) and (b) are consistent. Do not completely solve the systems. Notice that the detailed calculation process must be shown in your answer.

(a) $x_1 + 3x_3 = 2$
$x_2 - 3x_4 = 3$
$-2x_2 + 3x_3 + 2x_4 = 1$
$3x_1 + 7x_4 = 5$

Converting the given equation into Augmented Matrix to see if we can avoid a row of the form [0 0 0 0 | non-zero], which would indicate inconsistency, we get:

$$\left[\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & -2 & 3 & 2 & 1 \\ 3 & 0 & 0 & 7 & 5 \end{array}\right]$$

R4 → R4 -3R1

$$\left[\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & -2 & 3 & 2 & 1 \\ 0 & 0 & -9 & 7 & -1 \end{array}\right]$$

R3→ R3 +2R2

$$\left[\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & 0 & 3 & -4 & 7 \\ 0 & 0 & -9 & 7 & -1 \end{array}\right]$$

R4→ R4+3R3

$$\left[\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & 0 & 3 & -4 & 7 \\ 0 & 0 & 0 & -5 & 20 \end{array}\right]$$

Since there is no row of the form [0 0 0 0 | non-zero], the system is consistent.

(b) $x_1 - 2x_4 = -3$

$2x_2 + 2x_3 = 0$

$x_3 + 3x_4 = 1$

$-2x_1 + 3x_2 + 2x_3 + x_4 = 5$

Converting the given equation into Augmented Matrix to perform row operations to transform this matrix into row-echelon form to determine if the system is consistent.

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -2 & -3 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ -2 & 3 & 2 & 1 & 5 \end{array}\right]$$

R4→ R4 + 2R1

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -2 & -3 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 3 & 2 & -3 & -1 \end{array}\right]$$

R2 →1/2R2

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -2 & -3 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 3 & 2 & -3 & -1 \end{array}\right]$$

R4→ R4-3R2

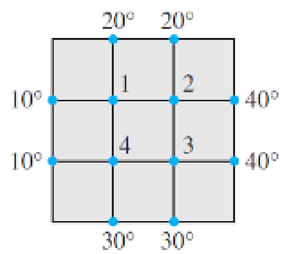
$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -2 & -3 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & -1 & -3 & -1 \end{array}\right]$$

R4→ R4+R3

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -2 & -3 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Here, we got the resulting matrix in row-echelon form. Since there are no rows that correspond to an equation of the form 0x1+0x2+0x3+0x4=b, where b is not equal to 0, the system is consistent and has infinite solutions.

2. An important concern in the study of heat transfer is to determine the steady-state temperature distribution of a thin plate when the temperature around the boundary is known. Assume the plate shown in the figure represents a cross section of a metal beam, with negligible heat flow in the direction perpendicular to the plate. Let T_1...... T_4 denote the temperatures at the four interior nodes of the mesh in the figure. The temperature at a node is approximately equal to the average of the four nearest nodes to the left, above, to the right, and below. For instance, $T_1 = \frac{(10+20+T_2+T_4)}{4}$ or $4T_1 - T_2 - T_4 = 30$



Write a system of four equations whose solution gives estimates for the temperatures T_1...... T_4. Notice that the detailed calculation process must be shown in your answer.

Given the temperature at a node is approximately equal to the average of the four nearest nodes, we can write a system of equations for the temperatures at the four interior nodes T1, T2, T3 and T4.

For T1, we have

$$T1 = \frac{10 + 20 + T2 + T4}{4}$$

4T1-T2-T4=30.....................1

Similarly considering the temperature at neighboring nodes for all other values of T

For T2, considering the temperatures of 20, 40, T1 and T3, we get.

$$T2 = \frac{20 + 40 + T1 + T3}{4}$$

4T2-T1-T3=60...................2

For T3, considering the temperatures of 30, 40, T4 and T2, we get.

$$T3 = \frac{30 + 40 + T2 + T4}{4}$$

4T3-T4-T2=70………………3

For T4, considering the temperatures of 10, 30, T1 and T3, we get.

$$T4 = \frac{10 + 30 + T1 + T3}{4}$$

4T4-T1-T3=40…………….4

Combining these equations, we get the following system of linear equations.

4T1-T2-T4=30

-T1+4T2-T3=60

-T2+4T3-T4=70

-T1-T3+4T4=40

This is the system of four equations whose solution gives the estimates for the temperatures T1 to T4 .

It can further be represented in Matrix form as:

$$\left[\begin{array}{cccc|c} 4 & -1 & 0 & -1 & 30 \\ -1 & 4 & -1 & 0 & 60 \\ 0 & -1 & 4 & -1 & 70 \\ -1 & 0 & -1 & -4 & 40 \end{array}\right]$$

3. Suppose a 3 x 5 coefficient matrix for a system has three pivot columns. Is the system consistent? Why or why not? Notice that the detailed explanation with the examples must be shown in your answer.

A 3 X 5 coefficient matrix represents a system with 3 equations and 5 unknowns. When we say that this matrix has three pivot columns, it means that there are three columns that can be used as leading columns in the row reduction process (i.e., columns that contain a leading 1 in each row after the matrix is put into reduced row-echelon form or some form of echelon form).

Having three pivot columns in a 3 X 5 matrix means that there are three independent equations that can define the relationships among the 5 variables. However, since there are more variables (5) than equations (3), the system is underdetermined. This implies that there will be free variables, which are variables that are not leading variables in any row and can take on any value.

Whether the system is consistent or not depends on the presence of contradictions in the augmented matrix (which includes the coefficient matrix and the constants on the right-hand side of the equations). A system is consistent if it does not lead to an equation of the form 0=b, where b is a nonzero number, during the row reduction process.

Given that the matrix has three pivot columns and assuming there are no contradictions in the augmented matrix (such as a row that reads 0 0 0 0 0 | b with b≠0) the system is consistent. This is because having pivot positions means we can solve for at least some of the variables directly or in terms of the free variables, and there are no impossible equations (like 0=b for b≠0).

For example:

Considering a system represented by a coefficient matrix that, after row reduction, looks like this:

10 a0 b | d1

01 c0 d | d2

00 01 e | d3

Here, columns 1, 2, and 4 are pivot columns, which means the corresponding variables are leading variables, and the system has no contradictory equations of the form 0=b (assuming d1, d2 and d3 do not create such a situation). This indicates the system is consistent. The variables corresponding to columns 3 and 5 are free variables and can take any value, leading to an infinite number of solutions.

In summary, a 3 X 5 coefficient matrix with three pivot columns indicates the system is consistent, provided there are no contradictory equations formed during the row reduction process. The presence of pivot columns ensures that we can express some variables in terms of others, and the absence of contradictions means there is at least one solution to the system. Since there are more variables than equations, the system will have an infinite number of solutions if it is consistent.

4. Suppose a system of linear equations has a 3 x 5 augmented matrix whose fifth column is a pivot column. Is the system consistent? Why (or why not)? Notice that the detailed explanation with the examples must be shown in your answer.

To determine if a system of linear equations is consistent, we need to examine the structure of its augmented matrix. Given that the system has a 3 x 5 augmented matrix, we can represent it as follows:

$$\left[\begin{array}{cccc|c} a11 & a12 & a13 & a14 & b1 \\ a21 & a22 & a23 & a24 & b2 \\ a31 & a32 & a33 & a34 & b3 \end{array}\right]$$

Here, the first four columns represent the coefficients of the variables x1,x2,x3,x4 and the fifth column represents the constants from the right-hand side of the equations.

A pivot column is a column in a matrix that contains a leading 1 (pivot) in row echelon form (REF) or reduced row echelon form (RREF). The presence of a pivot in the fifth column (the column of constants) indicates that one of the rows has been reduced to a form where all the coefficients of the variables are zero, but the constant term is non-zero. This would look like:

[0 0 0 0 | 1]

This row corresponds to the equation 0=1, which is a contradiction.

For a system to be consistent, there must be no row in the augmented matrix that has all zeros in the coefficient columns but a non-zero entry in the constant column. Such a row represents a contradiction, implying that the system has no solutions.

Consistent System Example.

2x+3y-z=7

4x+2y+5z=10

6x-y+3z=4

The augmented matrix form is:

$$\left[\begin{array}{ccc|c} 2 & 3 & -1 & 7 \\ 4 & 2 & 5 & 10 \\ 6 & -1 & 3 & 4 \end{array}\right]$$

If this matrix is row-reduced to REF or RREF, and there are no pivots in the fifth column, it could look like:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{array}\right]$$

Hence, This indicates a consistent system with a unique solution, as there is no contradiction.

Inconsistent System Example.

Now, let's consider a system where the augmented matrix has a pivot in the fifth column:

$$\left[\begin{array}{ccc|c} 2 & 3 & -1 & 7 \\ 4 & 2 & 5 & 10 \\ 6 & -1 & 3 & 4 \end{array}\right]$$

If this matrix is row-reduced to REF or RREF and the fifth column becomes a pivot column, it might look like:

[0 0 0 | 1]

This row represents the equation 0=1 which is a clear contradiction. Hence, we can say that the system is inconsistent.

5. Suppose experimental data are represented by a set of points in the plane. An interpolating polynomial for the data is a polynomial whose graph passes through every point. In scientific work, such a polynomial can be used, for example, to estimate values between the known data points. Another use is to create curves for graphical images on a computer screen. One method for finding an interpolating polynomial is to solve a system of linear equations.

   Find the interpolating polynomial $p(t) = a_0 + a_1t + a_2t^2$ for the data (1, 12), (2, 15), (3, 16). That is, find a_0, a_1, and a_2.

Converting the Polynomial equation into the Matrix form:

Given, $p(t)= a_0 + a_1t + a_2t^2$, data (1,12) , (2,15), (3,16). We can write it as

$a_0 + a_1(1) + a_2(1)^2 = 12$

$a_0 + a_1(2) + a_2(2)^2 = 15$

$a_0 + a_1(3) + a_2(3)^2 = 16$

This can be written as

$a_0 + a_1 + a_2 = 12$

$a_0 + 2a_1 + 4a_2 = 15$

$a_0 + 3a_1 + 9a_2 = 16$

Its Matrix form is:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 12 \\ 1 & 2 & 4 & 15 \\ 1 & 3 & 9 & 16 \end{array}\right]$$

R2 → -R1+R2

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 12 \\ 0 & 1 & 3 & 3 \\ 1 & 3 & 9 & 16 \end{array}\right]$$

R3 → -R1+R3

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 12 \\ 0 & 1 & 3 & 3 \\ 0 & 2 & 8 & 4 \end{array}\right]$$

R3 → ½ R3

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 12 \\ 0 & 1 & 3 & 3 \\ 0 & 1 & 4 & 2 \end{array}\right]$$

R3 → -R2+R3

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 12 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

R1 → -R3+R1

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 13 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

R2 → -3R3+R2

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 13 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

R2 → -R1+R2

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

Converting the Matrix into equation, we get

$a_0 + 0(a_1) + 0(a_2) = 7$

$0(a_0) + a_1 + 0(a_2) = 6$

$0(a_0) + 0(a_1) + a_3 = -1$

Equating the value of a_0, a_1 and a_2, we get

a_0= 7

a_1= 6

a_2= -1

Thus the equation is:

P(t)= 7 +6t – t2

6. A mining company has two mines. One day's operation at mine #1 produces ore that contains 20 metric tons of copper and 550 kilograms of silver, while one day's operation at mine #2 produces ore that contains 30 metric tons of copper and 500 kilograms of silver. Let $\boldsymbol{V_1} = \begin{bmatrix} 20 \\ 550 \end{bmatrix}$ and $\boldsymbol{V_2} = \begin{bmatrix} 30 \\ 500 \end{bmatrix}$. Then $\boldsymbol{V_1}$ and $\boldsymbol{V_2}$ represent the "output per day" of mine #1 and mine #2, respectively.

   a. What physical interpretation can be given to the vector $\boldsymbol{5V_1}$?
   b. Suppose the company operates mine #1 for x_1 days and mine #2 for x_2 days. Write a vector equation whose solution gives the number of days each mine should operate in order to produce 150 tons of copper and 2825 kilograms of silver.

a. What physical interpretation can be given to the vector $\boldsymbol{5V1}$?

Vector 5V1 represents the output of mine #1 after operating for 5 days. It gives the total amount of copper and silver produced by mine #1 over that period. Multiplying the vector by 5 scales the quantities of copper and silver by 5 times the daily output, we get

5V1= 5 $\begin{bmatrix} 20 \\ 550 \end{bmatrix}$ = $\begin{bmatrix} 5 * 20 \\ 5 * 550 \end{bmatrix}$ = 5 $\begin{bmatrix} 100 \\ 2750 \end{bmatrix}$

Thus, this means that after 5 days of operation, mine #1 has produced 100 metric tons of copper and 2750 kilograms of silver.

b. Suppose the company operates mine #1 for $x1$ days and mine #2 for $x2$ days. Write a vector equation whose solution gives the number of days each mine should operate in order to produce 150 tons of copper and 2825 kilograms of silver.

In order to determine the number of days each mine should operate to produce a certain amount of copper and silver; we can set up a vector equation using the output vectors V1 and V2 and the unknowns values x1 and x2. We need to produce 150 tons of copper and 2825 kilograms of silver, so we can write.

x1V1 + x2V2 = $\begin{bmatrix} 150 \\ 2825 \end{bmatrix}$

Substituting value of V1 and V2, we get

x1$\begin{bmatrix} 20 \\ 550 \end{bmatrix}$ + x2 $\begin{bmatrix} 30 \\ 500 \end{bmatrix}$ = $\begin{bmatrix} 150 \\ 2825 \end{bmatrix}$

Then, our linear equation is:

20x1 + 30x2= 150.........1

550x1 + 500x2= 2825.........2

Multiplying 1st equation by 500 and 2nd equation by 30. We get,

10000x1 + 15000x2= 75000...........3

16500x1 + 15000x2= 84750...........4

Subtracting the above equation 3 from 4, we get

16500x1-10000x1= 84750-75000

6500x1=9750

x1= 1.5

Substituting value of x1 into equation 1st, we get

20x1+30x2=150

20*1.5 + 30x2=150

30+30x2=150

x2= 120/30

x2= 4

Hence, the mining company should operate mine #1 for 1.5 days and mine #2 for 4 days to produce 150 tons of copper and 2825 kilograms of silver.

7. A steam plant burns two types of coal: anthracite (A) and bituminous (B). For each ton of A burned, the plant produces 27.6 million Btu of heat, 3100 grams (g) of sulfur dioxide, and 250g of particulate matter (solid-particle pollutants). For each ton of B burned, the plant produces 30.2 million Btu, 6400g of sulfur dioxide, and 360g of particulate matter.

   a. How much heat does the steam plant produce when it burns x_1 tons of A and x_2 tons of B?
   b. Suppose the output of the steam plant is described by a vector that lists the amounts of heat, sulfur dioxide, and particulate matter. Express this output as a linear combination of two vectors, assuming that the plant burns x_1 tons of A and x_2 tons of B.
   c. Over a certain time period, the steam plant produced 162 million Btu of heat, 23,610g of sulfur dioxide, and 1623g of particulate matter. Determine how many tons of each type of coal the steam plant must have burned. Include a vector equation as part of your solution.

From part a

We can calculate the heat produced by burning tons of A and tons of B by multiplying the amount of heat produced per ton by the number of tons burned and summing the results for A and B. The heat produced is:

27.6x1 + 30.2x2

From part b

The output of the steam plant can be expressed as a linear combination of two vectors, where each vector represents the output per ton of A and B, respectively. Let x1 be the tons of A burned and x2 be the tons of B burned. The output vector is:

$$x1 \begin{bmatrix} 27.6 \\ 3100 \\ 250 \end{bmatrix} + x2 \begin{bmatrix} 30.2 \\ 6400 \\ 360 \end{bmatrix}$$

From part c

We can get output vector as $\begin{bmatrix} 162 \\ 23610 \\ 1623 \end{bmatrix}$

By Using the matrix method, we represent the equation in the form of matrix AX=B where A is the coefficient matrix, X is the vector of unknowns, and B is the output vector.

$$A= \begin{bmatrix} 27.6 & 30.2 \\ 3100 & 6400 \\ 250 & 360 \end{bmatrix} \quad X= \begin{bmatrix} x1 \\ x2 \end{bmatrix} \quad B= \begin{bmatrix} 162 \\ 23610 \\ 1623 \end{bmatrix}$$

Solving for AX=B, To find X

$$\begin{bmatrix} 27.6 & 30.2 \\ 3100 & 6400 \\ 250 & 360 \end{bmatrix} \begin{bmatrix} x1 \\ x2 \end{bmatrix} = \begin{bmatrix} 162 \\ 23610 \\ 1623 \end{bmatrix}$$

Since we are only looking for x1 and x2 value, we only take two independent equations for our matrix. So, A and B becomes:

A= $\begin{bmatrix} 27.6 & 30.2 \\ 3100 & 6400 \end{bmatrix}$ B= $\begin{bmatrix} 162 \\ 23610 \end{bmatrix}$

X= A^{-1} B, So to find A^{-1}, we do:

A^{-1}= $\frac{1}{detA}$(adjA)

Det A= 27.6 * 6400 – 30.2* 3100

=83020

adjA= = $\begin{bmatrix} 6400 & -30.2 \\ -3100 & 27.6 \end{bmatrix}$

Then,

X= A^{-1} * B

$$= \frac{1}{83020} \begin{bmatrix} 6400 & -30.2 \\ -3100 & 27.6 \end{bmatrix} \begin{bmatrix} 162 \\ 23610 \end{bmatrix}$$

$$= \frac{1}{83020} \begin{bmatrix} 6400 * 162 - 30.2 * 23610 \\ -3100 * 162 + 27.6 * 23610 \end{bmatrix}$$

$$= \frac{1}{83020} \begin{bmatrix} 328338 \\ 83020 \end{bmatrix}$$

$$= \begin{bmatrix} 3.95 \\ 1.80 \end{bmatrix}$$

Hence, x1= 3.95, x2= 1.80

Therefore, the steam plant must have burned approximately 3.95 tons of A and 1.80 tons of B.