1. Let $\boldsymbol{q_1}$ $\boldsymbol{q_2}$ $\boldsymbol{q_3}$ and $\boldsymbol{v}$ represent vectors in $\mathbb{R}^5$, and let x_1 x_2 and x_3 denote scalars. Write the following vector equation as a matrix equation. Identify any symbols you choose to use.

$$x_1\boldsymbol{q_1} + x_2\boldsymbol{q_2} + x_3\boldsymbol{q_3} = \boldsymbol{v}$$

Here,

Column of Matrix A is (a1,a2...an) and vector x is represented as $\begin{bmatrix} x1 \\ . \\ . \\ xn \end{bmatrix}$

According to the definition, the weights in a linear combination of matrix A columns are represented by the entries in the vector x.

$x_1a_1 + x_2a_2 + \ldots\ldots x_na_n = Ax$

The left-hand side of the above equation is the linear combination of vectors x_1, x_2.....x_n

The vector equation in the matrix form is :

$(a_1,a_2\ldots a_n) \begin{bmatrix} x1 \\ . \\ . \\ xn \end{bmatrix} = b$

Here, for Ax to be defined the number of columns in matrix A should be equal to the number of entries in vector x

Let Q represent the column matrix where vectors q1, q2 and q3 needs to be placed

So, Q= (q1, q2, q3)

<u>Representing the vector equation in Matrix form</u>

Here, we consider x represents the vector for which the entries are:

x1, x2 and x3

Then, X= $\begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix}$. The equation can be written as:

Qx= [q1 q2 q3] $\begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix}$ = v

Therefore, Qx= v, where Q= [q1 q2 q3] , x= $\begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix}$, v= $\begin{bmatrix} v1 \\ v2 \\ v3 \\ v4 \\ v5 \end{bmatrix}$

Here, Q is a 5 X 3 Matrix , x is a 3 X 1 vector and v is a 5 X 1 vector in R^5

2. Let A be a 3×4 matrix, let $\boldsymbol{y_1}$ and $\boldsymbol{y_2}$ be vectors in $\mathbb{R}^3$, and let $\boldsymbol{w} = \boldsymbol{y_1} + \boldsymbol{y_2}$. Suppose $\boldsymbol{y_1} = A\boldsymbol{x_1}$ and $\boldsymbol{y_2} = A\boldsymbol{x_2}$ for some vectors $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$ in $\mathbb{R}^4$. What fact allows you to conclude that the system $A\boldsymbol{x} = \boldsymbol{w}$ is consistent?

The columns of Matrix A are represented as $[a_1, a_2....a_n]$ and the vector is represented as $\begin{bmatrix} x1 \\ . \\ . \\ xn \end{bmatrix}$

According to the definition, the entries in vector x are the weights in the linear combination of the columns of matrix A. The matrix equation, expressed as a vector equation, can be written as:

$$Ax= [a_1, a_2....a_n] \begin{bmatrix} x1 \\ . \\ . \\ xn \end{bmatrix} = x_1a_1 + x_2a_2.....x_na_n$$

The number of columns in Matrix A must be equal to the number of entries in vector x for Ax to be defined.

Identifying the conditions for a unique solution

For the equation Ax=b to have a unique solution, the associated system of equations must not have any free variables. This implies that each column of A is a pivot column, meaning every variable is a basic variable.

Writing the vectors using their properties

Let's consider Matrix A of order m X n. Let P and Q be vectors in R^m. Using the distributive property, we can write:

A(P+Q) = AP + AQ

Checking if the system is consistent

Substituting $P = x_1$ and $Q= x_2$ into the equation A(P+Q) = AP + AQ

$A (x_1 + x_2) = Ax_1 + Ax_2$

Now, using $y_1=Ax_1$ and $y_2=Ax_2$:

$A (x_1 + x_2) = Ax_1 + Ax_2$

$Ax= y_1 + y_2$

Ax=w

Hence, the system Ax=w is consistent.

3. Mark each statement True or False. Justify each answer.
   a. If $\boldsymbol{x}$ is a nontrivial solution of $A\boldsymbol{x} = \mathbf{0}$, then every entry in $\boldsymbol{x}$ is nonzero.
   b. The equation $\boldsymbol{x} = x_2\boldsymbol{u} + x_3\boldsymbol{v}$, with x_2 and x_3 free (and neither $\boldsymbol{u}$ nor $\boldsymbol{v}$ a multiple of the other), describes a plane through the origin.
   c. The equation $A\boldsymbol{x} = b$ is homogeneous if the zero vector is a solution.
   d. The effect of adding $\mathbf{p}$ to a vector is to move the vector in a direction parallel to $\mathbf{p}$.
   e. The solution set of $A\boldsymbol{x} = b$ is obtained by translating the solution set of $A\boldsymbol{x} = 0$.

a. If $\boldsymbol{x}$ is a nontrivial solution of $A\boldsymbol{x} = \mathbf{0}$, then every entry in $\boldsymbol{x}$ is nonzero.

False.

Just because x is a nontrivial solution to Ax=0 does not mean that every entry in x must be nonzero. A nontrivial solution simply means that not all entries in x are zero; some entries could still be zero.

b. The equation $\boldsymbol{x} = x_2\boldsymbol{u} + x_3\boldsymbol{v}$, with x_2 and x_3 free (and neither $\boldsymbol{u}$ nor $\boldsymbol{v}$ a multiple of the other), describes a plane through the origin.

True

The equation x= x_2u+ x_3v, where x_2 and x_3 are free variables and neither u nor v is a multiple of the other (implying that u and v are linearly independent), does indeed describe a plane through the origin in three-dimensional space. This is because any linear combination of two non-collinear vectors will form a plane.

c. The equation $A\boldsymbol{x} = b$ is homogeneous if the zero vector is a solution.

True.

An equation Ax=b is homogeneous if b is the zero vector. If the zero vector is a solution, then A .0 = b implies that b must be the zero vector, which means the equation is homogeneous.

d. The effect of adding **p** to a vector is to move the vector in a direction parallel to **p**.

True.

Adding a vector **p** to another vector moves the second vector in the direction parallel to **p** by the definition of vector addition.

e. The solution set of $A\boldsymbol{x} = b$ is obtained by translating the solution set of $A\boldsymbol{x} = 0$.

True.

The solution set of Ax=b can be obtained by translating the solution set of the homogeneous equation Ax=0 .Assuming that the equation Ax=b is consistent, if Ax=b has at least one solution, then the solution set can be described as the set of all vectors of the form x_p + v, where x_p is a particular solution to Ax=b and v is any solution to Ax=0.

4. Given $A = \begin{bmatrix} -2 & -6 \\ 7 & 21 \\ -3 & -9 \end{bmatrix}$ find solutions of $A\boldsymbol{x} = 0$

Given the matrix A:

A= $\begin{bmatrix} -2 & -6 \\ 7 & 21 \\ -3 & -9 \end{bmatrix}$

Let, x= $\begin{bmatrix} x1 \\ x2 \end{bmatrix}$

Solving for Ax=0, we get system of linear equation as

$\begin{bmatrix} -2 & -6 \\ 7 & 21 \\ -3 & -9 \end{bmatrix}$ $\begin{bmatrix} x1 \\ x2 \end{bmatrix}$ = $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Representing this in homogenous form, we get:

$-2x_1 - 6x_2 = 0$..........1

$7x_1 + 21x_2 = 0$...........2

$-3x_1 - 9x_2 = 0$...........3

Here, we see that each row of A is a multiple of the first row. Therefore, all three equations are linearly dependent, and we can use just the first equation to find the solution set. This also means the system has a single unique equation

Dividing 1st equation by -2, we get:

$x_1 + 3x_2 = 0$

$x_1 = -3x_2$

Thus, the solution set is all the scalar multiples of the vector x= $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$

It can be written as:

x= $x_2 \begin{bmatrix} -3 \\ 1 \end{bmatrix}$, for all $x_2 \in R$

Hence, the solution set to Ax=0 is the span of vector $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$, which forms a one-dimensional subspace of R^2

5. Suppose an economy has four sectors, Agriculture (A), Energy (E), Manufacturing (M), and Transportation (T). Sector A sells 10% of its output to E and 25% to M and retains the rest. Sector E sells 30% of its output to A, 35% to M, and 25% to T and retains the rest. Sector M sells 30% of its output to A, 15% to E, and 40% to T and retains the rest. Sector T sells 20% of its output to A, 10% to E, and 30% to M and retains the rest.

a. Construct the exchange table for this economy.
b. Find a set of equilibrium prices for the economy.

a. The exchange table for this economy, also known as the transaction matrix, is constructed as follows:

Output	A	E	M	T	Purchased by
A	0.65	0.30	0.30	0.20	A
E	0.10	0.10	0.15	0.10	E
M	0.25	0.35	0.15	0.30	M
T	0.00	0.25	0.40	0.40	T

Here, we use the table to develop a linear system by assuming variables for the sector outputs, such as pA, pE, pM, and pT for Agriculture, Energy, Manufacturing, and Transportation respectively. The total input to Agriculture is

0.65pA+0.30pE+0.30pM+0.20pT.

The equilibrium prices must satisfy the equations:

0.65pA+0.30pE+0.30pM+0.20pT=pA

0.35pA−0.30pE−0.30pM−0.20pT=0

Similarly, we can find the total input for Energy, Manufacturing, and Transportation, and find the equations that the equilibrium prices must satisfy:

0.10pA+0.10pE+0.15pM+0.10pT= pE 0.25pA+0.35pE+0.15pM+0.30pT=pM−0.10pA+0.9pE-0.15pM-0.10pT=0-0.25pA−0.35pE+0.85pM−0.30pT=0

0.25pE+0.40pM+0.40pT=pT

−0.25pE−0.40pM+0.60pT=0

Now, We can form an augmented matrix from the system of equations using the coefficients of the variables.

$$\begin{bmatrix} 0.35 & -0.30 & -0.30 & -0.20 & 0 \\ -0.10 & 0.90 & -0.15 & -0.10 & 0 \\ -0.25 & -0.35 & 0.85 & -0.30 & 0 \\ 0 & -0.25 & -0.40 & 0.60 & 0 \end{bmatrix}$$

We can simplify the matrix by applying row operations. We multiply row 1 by 1/35, add 10 times row 1 to row 2, and then multiply row 1 by 25 and add it to row 3. After applying these row operations, the matrix becomes:

$$\begin{bmatrix} 1 & -6/7 & -6/7 & -4/7 & 0 \\ 0 & 1 & -11/38 & -11/57 & 0 \\ 0 & -395/7 & 445/7 & -310/7 & 0 \\ 0 & -25 & -40 & 60 & 0 \end{bmatrix}$$

We can perform additional row operations, such as multiplying row 2 by 7/570 and adding it to row 3 and adding 25 times row 2 to row 4. After these operations, the matrix becomes:

$$\begin{bmatrix} 1 & -6/7 & -6/7 & -4/7 & 0 \\ 0 & 1 & 0 & -572/1077 & 0 \\ 0 & 0 & 1 & -1258/1077 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We perform more row operations, such as multiplying row 3 by 38/795 and adding it to row 2, and then multiplying row 3 by 6/7 and adding it to row 1. After these operations, the matrix becomes:

$$\begin{bmatrix} 1 & 0 & 0 & -728/359 & 0 \\ 0 & 1 & 0 & -572/1077 & 0 \\ 0 & 0 & 1 & -1258/1077 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The augmented matrix is now in row echelon form. This gives us the solution:

pA= 728/359 pT

pE= 572/1077 pT

pM= 1258/1077 pT

Let pT = 100. Then,

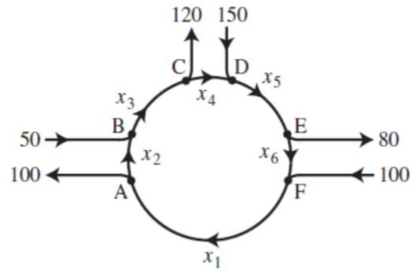
pA= 728/359 * 100 = 203

pE = 572/1077 * 100 = 53

pM = 1258/1077 * 100 = 117

Hence, the outputs of Agriculture, Energy, Manufacturing, and Transport are 203, 53, 117, and 100, respectively.

6. Intersections in England are often constructed as one-way "roundabouts," such as the following figure. Assume that traffic must travel in the directions shown. Find the general solution of the network flow. Find the smallest possible value for x_6.



Here, we mark the intersection and unknown flows in the branches as shown in the table. At the each intersection set the flow in is equal to the flow out.

Intersection	Flow in Flow out
A	x1=100+x2
B	x2=x3-50
C	x3=120+x4
D	x4=x5-150
E	x5=80+x6
F	x6=x1-100

Representing this into augmented matrix, we get

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 100 \\ 0 & 1 & -1 & 0 & 0 & 0 & -50 \\ 0 & 0 & 1 & -1 & 0 & 0 & 120 \\ 0 & 0 & 0 & 1 & -1 & 0 & -150 \\ 0 & 0 & 0 & 0 & 1 & -1 & 80 \\ -1 & 0 & 0 & 0 & 0 & 1 & -100 \end{bmatrix}$$

Now, Using R6 = R6 – R1

The matrix becomes:

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 100 \\ 0 & 1 & -1 & 0 & 0 & 0 & -50 \\ 0 & 0 & 1 & -1 & 0 & 0 & 120 \\ 0 & 0 & 0 & 1 & -1 & 0 & -150 \\ 0 & 0 & 0 & 0 & 1 & -1 & 80 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

Using R6 = R6 – R2

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 100 \\ 0 & 1 & -1 & 0 & 0 & 0 & -50 \\ 0 & 0 & 1 & -1 & 0 & 0 & 120 \\ 0 & 0 & 0 & 1 & -1 & 0 & -150 \\ 0 & 0 & 0 & 0 & 1 & -1 & 80 \\ 0 & 0 & 1 & 0 & 0 & -1 & 50 \end{bmatrix}$$

Using R6 = R6 – R3

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 100 \\ 0 & 1 & -1 & 0 & 0 & 0 & -50 \\ 0 & 0 & 1 & -1 & 0 & 0 & 120 \\ 0 & 0 & 0 & 1 & -1 & 0 & -150 \\ 0 & 0 & 0 & 0 & 1 & -1 & 80 \\ 0 & 0 & 0 & 1 & 0 & -1 & 70 \end{bmatrix}$$

Using R6 = R6 – R4

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 100 \\ 0 & 1 & -1 & 0 & 0 & 0 & -50 \\ 0 & 0 & 1 & -1 & 0 & 0 & 120 \\ 0 & 0 & 0 & 1 & -1 & 0 & -150 \\ 0 & 0 & 0 & 0 & 1 & -1 & 80 \\ 0 & 0 & 0 & 0 & 1 & -1 & 80 \end{bmatrix}$$

Using R6 = R6 – R5

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 100 \\ 0 & 1 & -1 & 0 & 0 & 0 & -50 \\ 0 & 0 & 1 & -1 & 0 & 0 & 120 \\ 0 & 0 & 0 & 1 & -1 & 0 & -150 \\ 0 & 0 & 0 & 0 & 1 & -1 & 80 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Using R4 = R4 + R5

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 100 \\ 0 & 1 & -1 & 0 & 0 & 0 & -50 \\ 0 & 0 & 1 & -1 & 0 & 0 & 120 \\ 0 & 0 & 0 & 1 & 0 & -1 & -70 \\ 0 & 0 & 0 & 0 & 1 & -1 & 80 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Using R3 = R3 + R4

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 100 \\ 0 & 1 & -1 & 0 & 0 & 0 & -50 \\ 0 & 0 & 1 & 0 & 0 & -1 & 50 \\ 0 & 0 & 0 & 1 & 0 & -1 & -70 \\ 0 & 0 & 0 & 0 & 1 & -1 & 80 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Using R2 = R2 + R3

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 100 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 50 \\ 0 & 0 & 0 & 1 & 0 & -1 & -70 \\ 0 & 0 & 0 & 0 & 1 & -1 & 80 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Using R1 = R1 + R2

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 100 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 50 \\ 0 & 0 & 0 & 1 & 0 & -1 & -70 \\ 0 & 0 & 0 & 0 & 1 & -1 & 80 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The general solution of the network flow in terms of x6 we get from above is:

x1 - x6 = 100

x2 - x6 = 0

x3 – x6 = 50

x4 - x6 = -70

x5 – x6 = 80

x6 = x6

This reflects the fact that x6 can be any value, that it is free variable and the other variables will adjust accordingly.

To find the smallest possible value for x_6, we consider the constraint that all traffic flows must be non-negative (assuming that negative traffic flow is not possible). Since, $x_4 = x_6 - 70$, the smallest x_6 can be without making x_4 negative is when

$x_6 - 70 = 0$. Thus:

$$x_6 = 70$$

This is the smallest possible value for x_6 to keep all traffic flows non-negative.