San Francisco Bay University

MATH203 - Linear Algebra
Homework #3

Due day: 7/2/2024

Instruction:

1. Homework answer sheet should contain the original questions and corresponding answers.
2. Recommend writing the Python or Matlab program for any calculation, and then copy & paste source code and running results on the answer sheet
3. Overdue homework submission could not be accepted.
4. Takes academic honesty and integrity seriously (Zero Tolerance of Cheating & Plagiarism)

1. Please use as many columns of A as possible to construct a matrix B with the property that the equation $B\boldsymbol{x} = 0$ has only the trivial solution. Solve $B\boldsymbol{x} = 0$ to verify your work.

   a. $A = \begin{bmatrix} 8 & -3 & 0 & -7 & 2 \\ -9 & 4 & 5 & 11 & -7 \\ 6 & -2 & 2 & -4 & 4 \\ 5 & -1 & 7 & 0 & 10 \end{bmatrix}$

   b. $A = \begin{bmatrix} 12 & 10 & -6 & -3 & 7 & 10 \\ -7 & -6 & 4 & 7 & -9 & 5 \\ 9 & 9 & -9 & -5 & 5 & -1 \\ -4 & -3 & 1 & 6 & -8 & 9 \\ 8 & 7 & -5 & -9 & 11 & -8 \end{bmatrix}$

For:

a. $\boldsymbol{A} = \begin{bmatrix} \mathbf{8} & \mathbf{-3} & \mathbf{0} & \mathbf{-7} & \mathbf{2} \\ \mathbf{-9} & \mathbf{4} & \mathbf{5} & \mathbf{11} & \mathbf{-7} \\ \mathbf{6} & \mathbf{-2} & \mathbf{2} & \mathbf{-4} & \mathbf{4} \\ \mathbf{5} & \mathbf{-1} & \mathbf{7} & \mathbf{0} & \mathbf{10} \end{bmatrix}$

Initial matrix A:

$$A = \begin{bmatrix} 8 & -3 & 0 & -7 & 2 \\ -9 & 4 & 5 & 11 & -7 \\ 6 & -2 & 2 & -4 & 4 \\ 5 & -1 & 7 & 0 & 10 \end{bmatrix}$$

Performing row operations:
1. $R1 = R1/8$:

$$A = \begin{bmatrix} 1 & -\frac{3}{8} & 0 & -\frac{7}{8} & \frac{1}{4} \\ -9 & 4 & 5 & 11 & -7 \\ 6 & -2 & 2 & -4 & 4 \\ 5 & -1 & 7 & 0 & 10 \end{bmatrix}$$

2. $R2 = R2 + 9R1$:

$$A = \begin{bmatrix} 1 & -\frac{3}{8} & 0 & -\frac{7}{8} & \frac{1}{4} \\ 0 & \frac{5}{8} & 5 & \frac{25}{8} & -\frac{19}{4} \\ 6 & -2 & 2 & -4 & 4 \\ 5 & -1 & 7 & 0 & 10 \end{bmatrix}$$

3. $R3 = R3 - 6R1$:

$$A = \begin{bmatrix} 1 & -\frac{3}{8} & 0 & -\frac{7}{8} & \frac{1}{4} \\ 0 & \frac{5}{8} & 5 & \frac{25}{8} & -\frac{19}{4} \\ 0 & \frac{1}{4} & 2 & \frac{5}{4} & \frac{5}{2} \\ 5 & -1 & 7 & 0 & 10 \end{bmatrix}$$

4. $R4 = R4 - 5R1$:

$$A = \begin{bmatrix} 1 & -\frac{3}{8} & 0 & -\frac{7}{8} & \frac{1}{4} \\ 0 & \frac{5}{8} & 5 & \frac{25}{8} & -\frac{19}{4} \\ 0 & \frac{1}{4} & 2 & \frac{5}{4} & \frac{5}{2} \\ 0 & \frac{7}{8} & 7 & \frac{35}{8} & \frac{35}{4} \end{bmatrix}$$

5. $R2 = 8R2/5$:

$$A = \begin{bmatrix} 1 & -\frac{3}{8} & 0 & -\frac{7}{8} & \frac{1}{4} \\ 0 & 1 & 8 & 5 & -\frac{38}{5} \\ 0 & \frac{1}{4} & 2 & \frac{5}{4} & \frac{5}{2} \\ 0 & \frac{7}{8} & 7 & \frac{35}{8} & \frac{35}{4} \end{bmatrix}$$

6. $R1 = R1 + 3R2/8$:

$$A = \begin{bmatrix} 1 & 0 & 3 & 1 & -\frac{13}{5} \\ 0 & 1 & 8 & 5 & -\frac{38}{5} \\ 0 & \frac{1}{4} & 2 & \frac{5}{4} & \frac{5}{2} \\ 0 & \frac{7}{8} & 7 & \frac{35}{8} & \frac{35}{4} \end{bmatrix}$$

7. $R3 = R3 - R2/4$:

$$A = \begin{bmatrix} 1 & 0 & 3 & 1 & -\frac{13}{5} \\ 0 & 1 & 8 & 5 & -\frac{38}{5} \\ 0 & 0 & 0 & 0 & \frac{22}{5} \\ 0 & \frac{7}{8} & 7 & \frac{35}{8} & \frac{35}{4} \end{bmatrix}$$

8. $R4 = R4 - 7R2/8$:

$$A = \begin{bmatrix} 1 & 0 & 3 & 1 & -\frac{13}{5} \\ 0 & 1 & 8 & 5 & -\frac{38}{5} \\ 0 & 0 & 0 & 0 & \frac{22}{5} \\ 0 & 0 & 0 & 0 & \frac{77}{5} \end{bmatrix}$$

Since the element at row 3 and column 4 (pivot element) equals 0, we need to swap the rows

9. $R3 = 5R3/22$:

$$A = \begin{bmatrix} 1 & 0 & 3 & 1 & -\frac{13}{5} \\ 0 & 1 & 8 & 5 & -\frac{38}{5} \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \frac{77}{5} \end{bmatrix}$$

10. $R1 = R1 + 13R3/5$:

$$A = \begin{bmatrix} 1 & 0 & 3 & 1 & 0 \\ 0 & 1 & 8 & 5 & -\frac{38}{5} \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \frac{77}{5} \end{bmatrix}$$

11. $R2 = R2 + 38R3/5$:

$$A = \begin{bmatrix} 1 & 0 & 3 & 1 & 0 \\ 0 & 1 & 8 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \frac{77}{5} \end{bmatrix}$$

12. $R4 = R4 - \frac{77}{5}R3$:

$$A = \begin{bmatrix} 1 & 0 & 3 & 1 & 0 \\ 0 & 1 & 8 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The pivot columns are 1,2 and 5. So, the matrix B is:

$$B = \begin{bmatrix} 8 & -3 & 2 \\ -9 & 4 & -7 \\ 6 & -2 & 4 \\ 5 & -1 & 10 \end{bmatrix}$$

The System $Bx = 0$

$$B = \begin{bmatrix} 8 & -3 & 2 \\ -9 & 4 & -7 \\ 6 & -2 & 4 \\ 5 & -1 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We again need to find the row-reduced form of the matrix B

$$B = \begin{bmatrix} 8 & -3 & 2 \\ -9 & 4 & -7 \\ 6 & -2 & 4 \\ 5 & -1 & 10 \end{bmatrix}$$

Divide row 1 by 8: $R_1 = \dfrac{R_1}{8}$.

$$\begin{bmatrix} 1 & -\frac{3}{8} & \frac{1}{4} \\ -9 & 4 & -7 \\ 6 & -2 & 4 \\ 5 & -1 & 10 \end{bmatrix}$$

Add row 1 multiplied by 9 to row 2: $R_2 = R_2 + 9R_1$.

$$\begin{bmatrix} 1 & -\frac{3}{8} & \frac{1}{4} \\ 0 & \frac{5}{8} & \frac{-19}{4} \\ 6 & -2 & 4 \\ 5 & -1 & 10 \end{bmatrix}$$

Subtract row 1 multiplied by 6 from row 3: $R_3 = R_3 - 6R_1$.

$$\begin{bmatrix} 1 & -\frac{3}{8} & \frac{1}{4} \\ 0 & \frac{5}{8} & \frac{-19}{4} \\ 0 & \frac{-1}{4} & \frac{5}{2} \\ 5 & -1 & 10 \end{bmatrix}$$

Subtract row 1 multiplied by 5 from row 4: $R_4 = R_4 - 5R_1$.

$$\begin{bmatrix} 1 & -\frac{3}{8} & \frac{1}{4} \\ 0 & \frac{5}{8} & \frac{-19}{4} \\ 0 & \frac{1}{4} & \frac{5}{2} \\ 0 & \frac{7}{8} & \frac{35}{4} \end{bmatrix}$$

Multiply row 2 by $\frac{8}{5}$: $R_2 = \frac{8R_2}{5}$.

$$\begin{bmatrix} 1 & -\frac{3}{8} & \frac{1}{4} \\ 0 & 1 & -\frac{38}{5} \\ 0 & \frac{1}{4} & \frac{5}{2} \\ 0 & \frac{7}{8} & \frac{35}{4} \end{bmatrix}$$

Add row 2 multiplied by $\frac{3}{8}$ to row 1: $R_1 = R_1 + \frac{3R_2}{8}$.

$$\begin{bmatrix} 1 & 0 & -\frac{13}{5} \\ 0 & 1 & -\frac{38}{5} \\ 0 & \frac{1}{4} & \frac{5}{2} \\ 0 & \frac{7}{8} & \frac{35}{4} \end{bmatrix}$$

Subtract row 2 multiplied by $\frac{1}{4}$ from row 3: $R_3 = R_3 - \frac{R_2}{4}$.

$$\begin{bmatrix} 1 & 0 & -\frac{13}{5} \\ 0 & 1 & -\frac{38}{5} \\ 0 & 0 & \frac{22}{5} \\ 0 & \frac{7}{8} & \frac{35}{4} \end{bmatrix}$$

Subtract row 2 multiplied by $\frac{7}{8}$ from row 4: $R_4 = R_4 - \frac{7R_2}{8}$.

$$\begin{bmatrix} 1 & 0 & -\frac{13}{5} \\ 0 & 1 & -\frac{38}{5} \\ 0 & 0 & \frac{22}{5} \\ 0 & 0 & \frac{77}{5} \end{bmatrix}$$

Multiply row 3 by $\frac{5}{22}$: $R_3 = \frac{5R_3}{22}$.

$$\begin{bmatrix} 1 & 0 & -\frac{13}{5} \\ 0 & 1 & -\frac{38}{5} \\ 0 & 0 & 1 \\ 0 & 0 & \frac{77}{5} \end{bmatrix}$$

Add row 3 multiplied by $\frac{13}{5}$ to row 1: $R_1 = R_1 + \frac{13R_3}{5}$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{38}{5} \\ 0 & 0 & 1 \\ 0 & 0 & \frac{77}{5} \end{bmatrix}$$

Add row 3 multiplied by $\frac{38}{5}$ to row 2: $R_2 = R_2 + \frac{38R_3}{5}$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & \frac{77}{5} \end{bmatrix}$$

Subtract row 3 multiplied by $\frac{77}{5}$ from row 4: $R_4 = R_4 - \frac{77R_3}{5}$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus the solution is:

$$x_1 = 0, x_2 = 0, x_3 = 0$$

Therefore,

$$Bx = 0$$

has a trivial solution.

b. For $A = \begin{bmatrix} 12 & 10 & -6 & -3 & 7 & 10 \\ -7 & -6 & 4 & 7 & -9 & 5 \\ 9 & 9 & -9 & -5 & 5 & -1 \\ -4 & -3 & 1 & 6 & -8 & 9 \\ 8 & 7 & -5 & -9 & 11 & -8 \end{bmatrix}$

Initial matrix A:

$$A = \begin{bmatrix} 12 & 10 & -6 & -3 & 7 & 10 \\ -7 & 6 & 4 & 7 & -9 & 5 \\ 9 & 9 & -9 & -5 & 5 & -1 \\ -4 & -3 & 1 & 6 & -8 & 9 \\ 8 & 7 & -5 & -9 & 11 & -8 \end{bmatrix}$$

Performing row operations:
1. $R1 = R1/12$:

$$A = \begin{bmatrix} 1 & 5/6 & -1/2 & -1/4 & 7/12 & 5/6 \\ -7 & 6 & 4 & 7 & -9 & 5 \\ 9 & 9 & -9 & -5 & 5 & -1 \\ -4 & -3 & 1 & 6 & -8 & 9 \\ 8 & 7 & -5 & -9 & 11 & -8 \end{bmatrix}$$

2. $R2 = R2 + 7R1$:

$$A = \begin{bmatrix} 1 & 5/6 & -1/2 & -1/4 & 7/12 & 5/6 \\ 0 & -1/6 & 1/2 & 21/4 & -59/12 & 65/6 \\ 9 & 9 & -9 & -5 & 5 & -1 \\ -4 & -3 & 1 & 6 & -8 & 9 \\ 8 & 7 & -5 & -9 & 11 & -8 \end{bmatrix}$$

3. $R3 = R3 - 9R1$:

$$A = \begin{bmatrix} 1 & 5/6 & -1/2 & -1/4 & 7/12 & 5/6 \\ 0 & -1/6 & 1/2 & 21/4 & -59/12 & 65/6 \\ 0 & 3/2 & -9/2 & -11/4 & -1/4 & -17/2 \\ -4 & -3 & 1 & 6 & -8 & 9 \\ 8 & 7 & -5 & -9 & 11 & -8 \end{bmatrix}$$

4. $R4 = R4 + 4R1$:

$$A = \begin{bmatrix} 1 & 5/6 & -1/2 & -1/4 & 7/12 & 5/6 \\ 0 & -1/6 & 1/2 & 21/4 & -59/12 & 65/6 \\ 0 & 3/2 & -9/2 & -11/4 & -1/4 & -17/2 \\ 0 & 1/3 & -1 & 5 & -17/3 & 37/3 \\ 8 & 7 & -5 & -9 & 11 & -8 \end{bmatrix}$$

5. $R5 = R5 - 8R1$:

$$A = \begin{bmatrix} 1 & 5/6 & -1/2 & -1/4 & 7/12 & 5/6 \\ 0 & -1/6 & 1/2 & 21/4 & -59/12 & 65/6 \\ 0 & 3/2 & -9/2 & -11/4 & -1/4 & -17/2 \\ 0 & 1/3 & -1 & 5 & -17/3 & 37/3 \\ 0 & 1/3 & -1 & -7 & 19/3 & -44/3 \end{bmatrix}$$

6. $R2 = -6R2$:

$$A = \begin{bmatrix} 1 & 5/6 & -1/2 & -1/4 & 7/12 & 5/6 \\ 0 & 1 & -3 & -63/2 & 59/2 & -65 \\ 0 & 3/2 & -9/2 & -11/4 & -1/4 & -17/2 \\ 0 & 1/3 & -1 & 5 & -17/3 & 37/3 \\ 0 & 1/3 & -1 & -7 & 19/3 & -44/3 \end{bmatrix}$$

8. $R1 = R1 - 5R2/6$:

$$A = \begin{bmatrix} 1 & 0 & 2 & 26 & -24 & 55 \\ 0 & 1 & -3 & -63/2 & 59/2 & -65 \\ 0 & 3/2 & -9/2 & -11/4 & -1/4 & -17/2 \\ 0 & 1/3 & -1 & 5 & -17/3 & 37/3 \\ 0 & 1/3 & -1 & -7 & 19/3 & -44/3 \end{bmatrix}$$

9. $R3 = R3 - 3R2/2$:

$$A = \begin{bmatrix} 1 & 0 & 2 & 26 & -24 & 55 \\ 0 & 1 & -3 & -63/2 & 59/2 & -65 \\ 0 & 0 & 0 & 89/2 & -89/2 & 89 \\ 0 & 1/3 & -1 & 5 & -17/3 & 37/3 \\ 0 & 1/3 & -1 & -7 & 19/3 & -44/3 \end{bmatrix}$$

10. $R4 = R4 - R2/3$:

$$A = \begin{bmatrix} 1 & 0 & 2 & 26 & -24 & 55 \\ 0 & 1 & -3 & -63/2 & 59/2 & -65 \\ 0 & 0 & 0 & 89/2 & -89/2 & 89 \\ 0 & 0 & 0 & 31/2 & -31/2 & 34 \\ 0 & 1/3 & -1 & -7 & 19/3 & -44/3 \end{bmatrix}$$

11. $R5 = R5 - R2/3$:

$$A = \begin{bmatrix} 1 & 0 & 2 & 26 & -24 & 55 \\ 0 & 1 & -3 & -63/2 & 59/2 & -65 \\ 0 & 0 & 0 & 89/2 & -89/2 & 89 \\ 0 & 0 & 0 & 31/2 & -31/2 & 34 \\ 0 & 0 & 0 & 7/2 & -7/2 & 7 \end{bmatrix}$$

Since the elements at row 3 and column 3 equals 0, we need to swap the rows

12. $R3 = 2R3/89$:

$$A = \begin{bmatrix} 1 & 0 & 2 & 26 & -24 & 55 \\ 0 & 1 & -3 & -63/2 & 59/2 & -65 \\ 0 & 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 31/2 & -31/2 & 34 \\ 0 & 0 & 0 & 7/2 & -7/2 & 7 \end{bmatrix}$$

13. $R1 = R1 - 26R3$:

$$A = \begin{bmatrix} 1 & 0 & 2 & 0 & 2 & 3 \\ 0 & 1 & -3 & -63/2 & 59/2 & -65 \\ 0 & 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 31/2 & -31/2 & 34 \\ 0 & 0 & 0 & 7/2 & -7/2 & 7 \end{bmatrix}$$

14. $R2 = R2 + 63R3/2$:

$$A = \begin{bmatrix} 1 & 0 & 2 & 0 & 2 & 3 \\ 0 & 1 & -3 & 0 & -2 & -2 \\ 0 & 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 31/2 & -31/2 & 34 \\ 0 & 0 & 0 & 7/2 & -7/2 & 7 \end{bmatrix}$$

15. $R4 = R4 - 31R3/2$:

$$A = \begin{bmatrix} 1 & 0 & 2 & 0 & 2 & 3 \\ 0 & 1 & -3 & 0 & -2 & -2 \\ 0 & 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 7/2 & -7/2 & 7 \end{bmatrix}$$

16. $R5 = R5 - 7R3/2$:

$$A = \begin{bmatrix} 1 & 0 & 2 & 0 & 2 & 3 \\ 0 & 1 & -3 & 0 & -2 & -2 \\ 0 & 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Since, the element at row 4 and column 5 equals 0, we need to swap the rows

17. $R4 = R4/3$:

$$A = \begin{bmatrix} 1 & 0 & 2 & 0 & 2 & 3 \\ 0 & 1 & -3 & 0 & -2 & -2 \\ 0 & 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$R1 = R1 - 3R4$:

$$A = \begin{bmatrix} 1 & 0 & 2 & 0 & 2 & 0 \\ 0 & 1 & -3 & 0 & -2 & -2 \\ 0 & 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$R2 = R2 + 2R4$:

$$A = \begin{bmatrix} 1 & 0 & 2 & 0 & 2 & 0 \\ 0 & 1 & -3 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$R3 = R3 - 2R4$:

$$A = \begin{bmatrix} 1 & 0 & 2 & 0 & 2 & 0 \\ 0 & 1 & -3 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The pivot columns are 1,2,4 and 5. So, Matrix B is

$$B = \begin{bmatrix} 12 & 10 & -3 & 10 \\ -7 & -6 & 7 & 5 \\ 9 & 9 & -5 & -1 \\ -4 & -3 & 6 & 9 \\ 8 & 7 & -9 & -8 \end{bmatrix}$$

1. $R1 = R1/12$:

$$\begin{bmatrix} 1 & 5/6 & -1/4 & 5/6 \\ -7 & -6 & 7 & 5 \\ 9 & 9 & -5 & -1 \\ -4 & -3 & 6 & 9 \\ 8 & 7 & -9 & -8 \end{bmatrix}$$

2. $R2 = R2 + 7R1$:

$$: \begin{bmatrix} 1 & 5/6 & -1/4 & 5/6 \\ 0 & -1/6 & 21/4 & 65/6 \\ 9 & 9 & -5 & -1 \\ -4 & -3 & 6 & 9 \\ 8 & 7 & -9 & -8 \end{bmatrix}$$

3. $R3 = R3 - 9R1$:

$$\begin{bmatrix} 1 & 5/6 & -1/4 & 5/6 \\ 0 & -1/6 & 21/4 & 65/6 \\ 0 & 3/2 & -11/4 & -17/2 \\ -4 & -3 & 6 & 9 \\ 8 & 7 & -9 & -8 \end{bmatrix}$$

4. $R4 = R4 + 4R1$:

$$B = \begin{bmatrix} 1 & 5/6 & -1/4 & 5/6 \\ 0 & -1/6 & 21/4 & 65/6 \\ 0 & 3/2 & -11/4 & -17/2 \\ 0 & 1/3 & 5 & 37/3 \\ 8 & 7 & -9 & -8 \end{bmatrix}$$

5. $R5 = R5 - 8R1$:

$$B = \begin{bmatrix} 1 & 5/6 & -1/4 & 5/6 \\ 0 & -1/6 & 21/4 & 65/6 \\ 0 & 3/2 & -11/4 & -17/2 \\ 0 & 1/3 & 5 & 37/3 \\ 0 & 1/3 & -7 & -44/3 \end{bmatrix}$$

6. $R2 = -6R2$:

$$B = \begin{bmatrix} 1 & 5/6 & -1/4 & 5/6 \\ 0 & 1 & -63/2 & -65 \\ 0 & 3/2 & -11/4 & -17/2 \\ 0 & 1/3 & 5 & 37/3 \\ 0 & 1/3 & -7 & -44/3 \end{bmatrix}$$

7. $R1 = R1 - 5R2/6$:

$$B = \begin{bmatrix} 1 & 0 & 26 & 55 \\ 0 & 1 & -63/2 & -65 \\ 0 & 3/2 & -11/4 & -17/2 \\ 0 & 1/3 & 5 & 37/3 \\ 0 & 1/3 & -7 & -44/3 \end{bmatrix}$$

8. $R3 = R3 - 3R2/2$:

$$B = \begin{bmatrix} 1 & 0 & 26 & 55 \\ 0 & 1 & -63/2 & -65 \\ 0 & 0 & 89/2 & 89 \\ 0 & 1/3 & 5 & 37/3 \\ 0 & 1/3 & -7 & -44/3 \end{bmatrix}$$

9. $R4 = R4 - R2/3$:

$$B = \begin{bmatrix} 1 & 0 & 26 & 55 \\ 0 & 1 & -63/2 & -65 \\ 0 & 0 & 89/2 & 89 \\ 0 & 0 & 31/2 & 34 \\ 0 & 1/3 & -7 & -44/3 \end{bmatrix}$$

10. $R5 = R5 - R2/3$:

$$B = \begin{bmatrix} 1 & 0 & 26 & 55 \\ 0 & 1 & -63/2 & -65 \\ 0 & 0 & 89/2 & 89 \\ 0 & 0 & 31/2 & 34 \\ 0 & 0 & 7/2 & 7 \end{bmatrix}$$

11. $R3 = 2R3/89$:

$$B = \begin{bmatrix} 1 & 0 & 26 & 55 \\ 0 & 1 & -63/2 & -65 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 31/2 & 34 \\ 0 & 0 & 7/2 & 7 \end{bmatrix}$$

11. $R1 = R1 - 26R3$:

$$B = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & -63/2 & -65 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 31/2 & 34 \\ 0 & 0 & 7/2 & 7 \end{bmatrix}$$

12. $R2 = R2 + 63R3/2$:

$$B = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 31/2 & 34 \\ 0 & 0 & 7/2 & 7 \end{bmatrix}$$

13. $R4 = R4 - 31R3/2$:

$$B = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 7/2 & 7 \end{bmatrix}$$

14. $R5 = R5 - 7R3/2$:

$$B = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

15. $R4 = R4/3$:

$$B = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

16. $R1 = R1 - 3R4$:

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

17. $R2 = R2 + 2R4$:

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

18. $R3 = R3 - 2R4$:

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

In the system $B\mathbf{x} = 0$:

$$\begin{bmatrix} 12 & 10 & -3 & 10 \\ -7 & -6 & 7 & 5 \\ 9 & 9 & -5 & -1 \\ -4 & -3 & 6 & 9 \\ 8 & 7 & -9 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus, the solution is $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, $x_4 = 0$.

Hence, $B\mathbf{x} = 0$ has only the trivial solution.

2. **Let T: $\mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and let $\{v_1, v_2, v_3\}$ be a linearly dependent set in $\mathbb{R}^n$. Explain why the set $\{T(v_1), T(v_2), T(v_3)\}$ is linearly dependent.**

   Note: column vectors are written as rows, such as $x = (x_1, x_2)$ and $T(x)$ is written as $T(x_1, x_2)$

Since $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly dependent set in R^n, there exist scalars c_1, c_2, c_3, not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}.$$

Applying the linear transformation T to both sides of the equation and using the properties of linearity ($T(a+b) = T(a) + T(b)$ and $T(ca) = cT(a)$ for any vectors $\mathbf{a}, \mathbf{b}$ and scalar c), we get

$$T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3) = T(\mathbf{0}).$$

This simplifies to

$$c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + c_3T(\mathbf{v}_3) = \mathbf{0}.$$

Since $T(\mathbf{0}) = \mathbf{0}$ for any linear transformation T, we have

$$c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + c_3T(\mathbf{v}_3) = \mathbf{0}.$$

This shows that the set $\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$ is linearly dependent in R^n, as there exists a nontrivial combination of these vectors that results in the zero vector.

3. Show that the transformation T defined by $T(x_1, x_2) = (4x_1 - 2x_2, 3|x_2|)$ is not linear.

To show that the transformation T defined by $T(x_1, x_2) = (4x_1 - 2x_2, 3|x_2|)$ is not linear, we need to check whether T satisfies the properties of linearity: additivity and homogeneity (or scalar multiplication).

Additivity

A transformation T is additive if for all vectors $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$, the following holds:

$$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$$

Let's check this property for T:

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2)$$

$$T(\mathbf{u} + \mathbf{v}) = T(u_1 + v_1, u_2 + v_2) = (4(u_1 + v_1) - 2(u_2 + v_2), 3|u_2 + v_2|)$$

$$T(\mathbf{u}) = T(u_1, u_2) = (4u_1 - 2u_2, 3|u_2|)$$

$$T(\mathbf{v}) = T(v_1, v_2) = (4v_1 - 2v_2, 3|v_2|)$$

$$T(\mathbf{u}) + T(\mathbf{v}) = (4u_1 - 2u_2, 3|u_2|) + (4v_1 - 2v_2, 3|v_2|) = (4u_1 - 2u_2 + 4v_1 - 2v_2, 3|u_2| + 3|v_2|)$$

For T to be additive, we need:

$$(4(u_1 + v_1) - 2(u_2 + v_2), 3|u_2 + v_2|) = (4u_1 - 2u_2 + 4v_1 - 2v_2, 3|u_2| + 3|v_2|)$$

To see if this holds, consider $\mathbf{u} = (1, 1)$ and $\mathbf{v} = (-1, -1)$:

$$T(\mathbf{u} + \mathbf{v}) = T(0, 0) = (4 \cdot 0 - 2 \cdot 0, 3|0|) = (0, 0)$$

$$T(\mathbf{u}) = T(1, 1) = (4 \cdot 1 - 2 \cdot 1, 3|1|) = (2, 3)$$

$$T(\mathbf{v}) = T(-1, -1) = (4 \cdot (-1) - 2 \cdot (-1), 3|-1|) = (-4 + 2, 3) = (-2, 3)$$

$$T(\mathbf{u}) + T(\mathbf{v}) = (2, 3) + (-2, 3) = (0, 6)$$

Since $(0, 0) \neq (0, 6)$, T does not satisfy additivity.

Homogeneity

A transformation T is homogeneous if for all vectors $\mathbf{u} = (u_1, u_2)$ and scalar c, the following holds:

$$T(c\mathbf{u}) = cT(\mathbf{u})$$

Let's check this property for T:

$$c\mathbf{u} = (cu_1, cu_2)$$

$$T(c\mathbf{u}) = T(cu_1, cu_2) = (4cu_1 - 2cu_2, 3|cu_2|)$$

$$cT(\mathbf{u}) = cT(u_1, u_2) = c(4u_1 - 2u_2, 3|u_2|) = (c(4u_1 - 2u_2), c \cdot 3|u_2|)$$

For T to be homogeneous, we need:

$$(4cu_1 - 2cu_2, 3|cu_2|) = (c(4u_1 - 2u_2), c \cdot 3|u_2|)$$

To see if this holds, consider $c = -1$ and $\mathbf{u} = (1, 1)$:

$$T(c\mathbf{u}) = T(-1, -1) = (4 \cdot (-1) - 2 \cdot (-1), 3| - 1|) = (-4 + 2, 3) = (-2, 3)$$

$$cT(\mathbf{u}) = -1 \cdot T(1, 1) = -1 \cdot (4 \cdot 1 - 2 \cdot 1, 3|1|) = -1 \cdot (2, 3) = (-2, -3)$$

Since $(-2, 3) \neq (-2, -3)$, T does not satisfy homogeneity.

Since T does not satisfy both additivity and homogeneity, it is not a linear transformation.

4. The given matrix determines a linear transformation T. Find all $\mathbf{x}$ such that $T(x) = \mathbf{0}$.

$$A = \begin{bmatrix} 4 & -2 & 5 & -5 \\ -9 & 7 & -8 & 0 \\ -6 & 4 & 5 & 3 \\ 5 & -3 & 8 & -4 \end{bmatrix}$$

The given matrix determines a linear transformation T. We need to find all x such that $T(x) = 0$.

Given matrix:

$$A = \begin{pmatrix} 4 & -2 & 5 & -5 \\ -9 & 7 & -8 & 0 \\ -6 & 4 & 5 & 3 \\ 5 & -3 & 8 & -4 \end{pmatrix}$$

We solve the homogeneous system $A\vec{x} = 0$ by performing row reduction:

1. Divide row 1 by 4:

$$\begin{pmatrix} 1 & -\frac{1}{2} & \frac{5}{4} & -\frac{5}{4} \\ -9 & 7 & -8 & 0 \\ -6 & 4 & 5 & 3 \\ 5 & -3 & 8 & -4 \end{pmatrix}$$

2. Add 9 times row 1 to row 2:

$$\begin{pmatrix} 1 & -\frac{1}{2} & \frac{5}{4} & -\frac{5}{4} \\ 0 & \frac{5}{2} & \frac{13}{4} & -\frac{45}{4} \\ -6 & 4 & 5 & 3 \\ 5 & -3 & 8 & -4 \end{pmatrix}$$

3. Add 6 times row 1 to row 3:

$$\begin{pmatrix} 1 & -\frac{1}{2} & \frac{5}{4} & -\frac{5}{4} \\ 0 & \frac{5}{2} & \frac{13}{4} & -\frac{45}{4} \\ 0 & 1 & \frac{25}{4} & -\frac{9}{2} \\ 5 & -3 & 8 & -4 \end{pmatrix}$$

4. Subtract 5 times row 1 from row 4:

$$\begin{pmatrix} 1 & -\frac{1}{2} & \frac{5}{4} & -\frac{5}{4} \\ 0 & \frac{5}{2} & \frac{13}{4} & -\frac{45}{4} \\ 0 & 1 & \frac{25}{4} & -\frac{9}{2} \\ 0 & -\frac{1}{2} & \frac{7}{4} & \frac{9}{4} \end{pmatrix}$$

5. Multiply row 2 by $\frac{2}{5}$:

$$\begin{pmatrix} 1 & -\frac{1}{2} & \frac{5}{4} & -\frac{5}{4} \\ 0 & 1 & \frac{13}{10} & -\frac{9}{2} \\ 0 & 1 & \frac{25}{4} & -\frac{9}{2} \\ 0 & -\frac{1}{2} & \frac{7}{4} & \frac{9}{4} \end{pmatrix}$$

6. Add $\frac{1}{2}$ times row 2 to row 1:

$$\begin{pmatrix} 1 & 0 & \frac{19}{10} & -\frac{7}{2} \\ 0 & 1 & \frac{13}{10} & -\frac{9}{2} \\ 0 & 1 & \frac{25}{4} & -\frac{9}{2} \\ 0 & -\frac{1}{2} & \frac{7}{4} & \frac{9}{4} \end{pmatrix}$$

7. Subtract row 2 from row 3:

$$\begin{pmatrix} 1 & 0 & \frac{19}{10} & -\frac{7}{2} \\ 0 & 1 & \frac{13}{10} & -\frac{9}{2} \\ 0 & 0 & \frac{56}{10} & 0 \\ 0 & -\frac{1}{2} & \frac{7}{4} & \frac{9}{4} \end{pmatrix}$$

8. Subtract $\frac{1}{2}$ times row 2 from row 4:

$$\begin{pmatrix} 1 & 0 & \frac{19}{10} & -\frac{7}{2} \\ 0 & 1 & \frac{13}{10} & -\frac{9}{2} \\ 0 & 0 & \frac{56}{10} & 0 \\ 0 & 0 & \frac{12}{5} & 0 \end{pmatrix}$$

9. Multiply row 3 by $\frac{5}{56}$

$$\begin{pmatrix} 1 & 0 & \frac{19}{10} & -\frac{7}{2} \\ 0 & 1 & \frac{13}{10} & -\frac{9}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{12}{5} & 0 \end{pmatrix}$$

10. Subtract $\frac{19}{10}$R3 from R1

$$\begin{pmatrix} 1 & 0 & 0 & -\frac{7}{2} \\ 0 & 1 & \frac{13}{10} & -\frac{9}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{12}{5} & 0 \end{pmatrix}$$

11. Subtract $\frac{13}{10}$R3 from R2

$$\begin{pmatrix} 1 & 0 & 0 & -\frac{7}{2} \\ 0 & 1 & 0 & -\frac{9}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{12}{5} & 0 \end{pmatrix}$$

12. Subtract $\frac{12}{5}$R3 from R4

$$\begin{pmatrix} 1 & 0 & 0 & -\frac{7}{2} \\ 0 & 1 & 0 & -\frac{9}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

From the row echelon form, we have the following system of equations:

$$\begin{cases} x_1 - \frac{7}{2}x_4 = 0 \\ x_2 - \frac{9}{2}x_4 = 0 \\ x_3 = 0 \end{cases}$$

Solving these equations, we get:

$$x_3 = 0$$

$$x_2 = \frac{9}{2}x_4$$

$$x_1 = \frac{7}{2}x_4$$

Let $t = x_4$. Then,

$$x_1 = \frac{7}{2}t$$

$$x_2 = \frac{9}{2}t$$

$$x_3 = 0$$

$$x_4 = t$$

Therefore, the solution set is:

$$x = \begin{pmatrix} \frac{7}{2}t \\ \frac{9}{2}t \\ 0 \\ t \end{pmatrix}$$

5. T: $\mathbb{R}^2 \to \mathbb{R}^4$, $T(e_1) = (3,1,3,1)$ and $T(e_2) = (-5,2,0,0)$, where $\boldsymbol{e_1} = (1,0)$ and $\boldsymbol{e_2} = (0,1)$, assume that T is a linear transformation. Find the standard matrix of T.

Given the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^4$ with the following images of basis vectors:

$$T(e_1) = (3,1,3,1)$$

$$T(e_2) = (-5,2,0,0)$$

To find the standard matrix A of T:

1. Expressing $T(e_1)$ and $T(e_2)$ in terms of the standard basis vectors of $\mathbb{R}^4$:

$$T(e_1) = 3\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + 1\begin{pmatrix}0\\1\\0\\0\end{pmatrix} + 3\begin{pmatrix}0\\0\\1\\0\end{pmatrix} + 1\begin{pmatrix}0\\0\\0\\1\end{pmatrix} = \begin{pmatrix}3\\1\\3\\1\end{pmatrix}$$

$$T(e_2) = -5\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + 2\begin{pmatrix}0\\1\\0\\0\end{pmatrix} + 0\begin{pmatrix}0\\0\\1\\0\end{pmatrix} + 0\begin{pmatrix}0\\0\\0\\1\end{pmatrix} = \begin{pmatrix}-5\\2\\0\\0\end{pmatrix}$$

2. Form the standard matrix A of T using the column vectors obtained above:

$$A = \begin{pmatrix}3 & -5\\1 & 2\\3 & 0\\1 & 0\end{pmatrix}$$

Therefore, the standard matrix A of the linear transformation T is:

$$A = \begin{pmatrix}3 & -5\\1 & 2\\3 & 0\\1 & 0\end{pmatrix}$$

6. $T: \mathbb{R}^2 \to \mathbb{R}^2$ first reflects points through the horizontal x_1- axis and then reflects points through the line $x_2 = x_1$. Show that the transformation is merely a rotation about the origin. What is the angle of the rotation?

Considering the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by the following operations:

1. Reflect points through the horizontal x_1-axis: $(x, y) \mapsto (x, -y)$.
2. Reflect points through the line $x_2 = x_1$: $(x, -y) \mapsto (-y, x)$.

Transformation Matrix

To find the matrix representation T of the transformation:

1. The reflection through the x_1-axis is represented by the matrix $R_{x_1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.
2. The reflection through the line $x_2 = x_1$ is represented by the matrix $R_{x_2=x_1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

The transformation matrix T is the composition of these two reflections:

$$T = R_{x_2=x_1} \cdot R_{x_1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

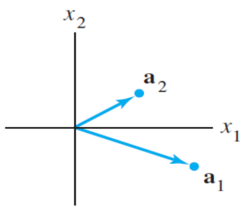
Angle of Rotation

The matrix T corresponds to a rotation by $\frac{\pi}{2}$ radians (or $90°$) counterclockwise about the origin in $\mathbb{R}^2$, given by

$$\cos\theta = 0, \quad \sin\theta = 1.$$

Therefore, the transformation T, which initially appears to involve reflections, is actually a rotation about the origin. The angle of rotation is $\frac{\pi}{2}$ radians (or $90°$).

7. Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a linear transformation with standard matrix $A = [\boldsymbol{a_1}, \boldsymbol{a_2}]$ where $\boldsymbol{a_1}$ and $\boldsymbol{a_2}$ are shown in the figure. Using the figure, draw the image of $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$ under the transformation T.



Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation with standard matrix $A = [a_1, a_2]$, where a_1 and a_2 are shown in the figure. We need to find the image of the vector $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$ under the transformation T.

First, let's find the image of the standard basis vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ under T. Recall that $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The image of $\mathbf{e}_1$ is given by $T(\mathbf{e}_1) = A\mathbf{e}_1 = a_1$. The image of $\mathbf{e}_2$ is given by $T(\mathbf{e}_2) = A\mathbf{e}_2 = a_2$.

Next, we'll find the image of the given vector $\mathbf{v} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$:

$$T(\mathbf{v}) = A\mathbf{v} = \begin{bmatrix} a_1 & a_2 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

To compute this, we multiply the matrix A by the vector $\mathbf{v}$:

$$T(\mathbf{v}) = -a_1 + 3a_2$$

The figure below shows the vectors a_1, a_2, $\mathbf{v}$, and $T(\mathbf{v})$.

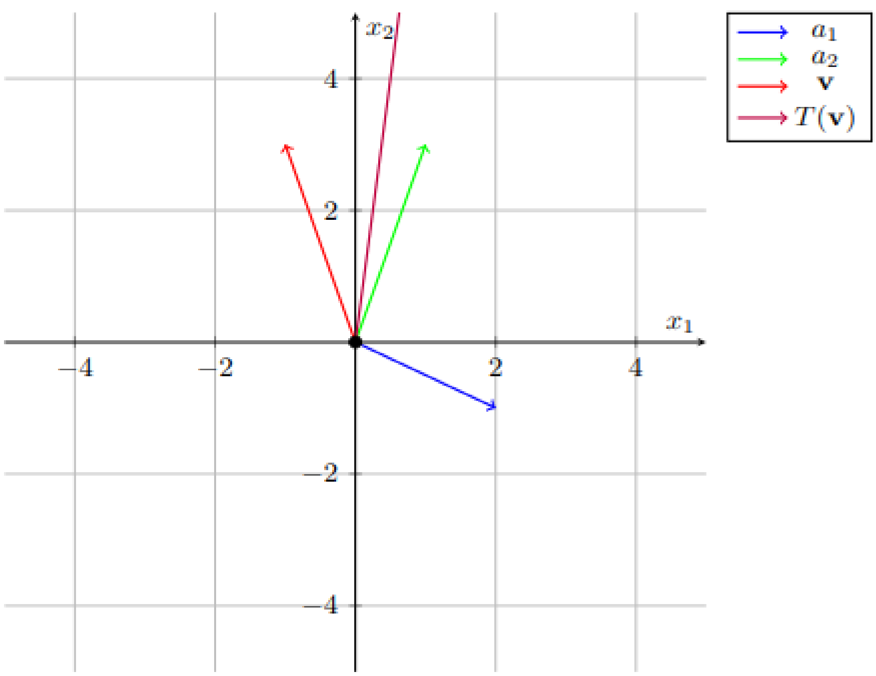


Figure 1: Transformation of $\mathbf{v}$ under T