UNIT IV FIR FILTER DESIGN

Structures of FIR – Linear phase FIR filter – Filter design using windowing techniques, Frequency sampling techniques – Finite word length effects in digital Filters

FILTERS:

Generally discrete time filter produces a discrete time output sequence y(n) for input sequence x(n). The input may be like **ramp , impulse, step** etc.

Types of Filters:

(i) Finite impulse response filter (FIR)

(ii) Infinite impulse response filter (IIR)

FIR FILTER:

In this filter, impulse response is a **finite** duration sequence. That means, it contains finite no. of non zero values.

$$h(n)=\begin{cases}1, for\ n \le 8\\ 0, otherwise\end{cases}$$

IIR FILTER:

In this filter, impulse response is a **infinite** duration sequence. That means ,it contains infinite number of non-zero values.

$$\mathbf{h(n)=a^n.\ u(n), for\ n\geq 0.}$$

Here the limit (n-value) is not given. So this is the example of IIR Filter.

Other name of Filters:

FIR filters are called as **non-recursive** filters. (all zeros).

IIR filters are called as **recursive** filters (contains poles and zeros).

Advantages of FIR Filters compared with IIR filters:

1. FIR filters have exact **linear phase.**
2. FIR filters are **stable.**

3. FIR design techniques are **linear.**

4. FIR filters can be implemented **efficiently** by using hardware.

5. When FIR filters are implemented in finite word length digital system .These filters are **free from limit** cycle oscillations.

Disadvantages of FIR Filters:

1. Implementation of narrow transition band FIR filters are **costly.**
2. When, it is implemented in software , **memory requirement** is high.
3. Execution time is high.

Equation of FIR Filters:

$$\sum_{k=0}^{N} a_k \, y(n-k) = \sum_{k=0}^{M} b_k \, x(n-k)$$

$$y_{(n)} = \sum_{k=0}^{M-1} b_k \, x(n-k)$$

b_k – is the set of filter co - efficients.

FIR Filter output depends only **present and past input** samples.

PROPERTIES OF FIR FILTER:

(i)FIR filters are inherently stable.

(ii)FIR filters have linear phase.

(iii)FIR filters need higher order for similar magnitude response compared to IIR filters.

(iv)They can be realized efficiently in hardware.

FIR filters are inherently stable:

We know that LTI system is said to be stable if bounded input produces bounded output(BIBO).

The difference equation of FIR filter of length M is given by

$$y(n) = \sum_{K=0}^{M-1} b_k \, x(n-k)$$

Applying Z transform we get

$$Y(z)=\sum_{K=0}^{M-1} b_k \; z^{-k} X(Z)$$

$$H(Z)=\frac{Y(Z)}{X(Z)}=\sum_{K=0}^{M-1} b_k \; z^{-k}$$

Applying inverse Z transform

$$h(n)=\begin{Bmatrix} b_{n}, _{0\le n\le M-1} \\ 0, otherwise \end{Bmatrix}$$

$\therefore$ $y(n)=b_0x(n)+b_1x(n-1)+\ldots\ldots\ldots\ldots+b_{M-1}x(n-M+1)$.

Y(n)=h(0) x(n)+h(1) x(n-1)+………+h(M-1) x(n-M+1) here

h(0),h(1),……….∴h(M-1) are constants that means they are bounded.For every bounded input,the output of the FIR filter is bounded.Thus FIR filters are stable.

Linear phase FIR filter:

The symmetry / anti symmetry of the unit sample response of FIR filters are related to their linearity of phase.

The unit sample response of FIR filter is symmetric if it satisfies the following condition

h(n) = h(N-1-n) ; n =0,1…….N-1 Here N- length of the filter

The unit sample response of FIR filter is antisymmetric if it satisfies the following condition. h(n) = - h(N-1-n) ; n=0,1,………N-1

The phase of FIR filter is linear if its unit sample response is symmetric or antisymmetric. This can be proved separately for even and odd length of FIR filters.

Let us consider the fourier transform of unit sample response

$$H(j\omega) = \sum_{n=0}^{N-1} h(n)\, e^{-j\omega n} \qquad (1)$$

Here N is the length of the filter

Let us assume 'N' is odd, then equation (1) can be splitted as

$$H(j\omega)=\sum_{n=0}^{\frac{N-3}{2}} h(n)\, e^{-j\omega n} + h\left(\frac{N-1}{2}\right) e^{-j\omega\left(\frac{N-1}{2}\right)} + \sum_{n=\frac{N+1}{2}}^{N-1} h(n) e^{-j\omega n} ----(2)$$

We know that h(n)=h(N-1-n) for symmetric FIR filter. Consider last summation in equation (2), we get

$$\sum_{n=\frac{N+1}{2}}^{N-1} h(n) e^{-j\omega n} = \sum_{n=\frac{N+1}{2}}^{N-1} h(N-1-n) e^{-j\omega n} \text{------------}(3)$$

Let $p = N - 1 - n$

$$n = N - 1 - p$$

Apply limits of summation will be

Lower limit n=$\frac{N+1}{2}$; $p = N - 1 - \left(\frac{N+1}{2}\right) = \frac{N-3}{2}$

Upper limit n=$N-1$; $p = N - 1 - (N-1) = 0$

Therefore eqns (3)

$$\sum_{n=\left(\frac{N+1}{2}\right)}^{N-1} h(n) e^{-j\omega n} = \sum_{p=0}^{\frac{N-3}{2}} h(p) e^{-j\omega(N-1-p)}$$

Since ‘p’ is just an index we can write ’n’ in place of ‘p’

$$\sum_{n=\left(\frac{N+1}{2}\right)}^{N-1} h(n) e^{-j\omega n} = \sum_{p=0}^{\frac{N-3}{2}} h(n) e^{-j\omega(N-1-n)}$$

Based on this equn (2) written as

$$H(j\omega) = \sum_{n=0}^{\frac{N-3}{2}} h(n) e^{-j\omega n} + h\left(\frac{N-1}{2}\right) e^{-j\omega\left(\frac{N-1}{2}\right)} + \sum_{n=0}^{\frac{N-3}{2}} h(n) e^{-j\omega(N-1-n)}$$

$$H((j\omega)=h\left(\frac{N-1}{2}\right) e^{-j\omega\left(\frac{N-1}{2}\right)} + \sum_{n=0}^{\frac{N-3}{2}} h(n)\left[e^{-j\omega n} + e^{-j\omega(N-1-n)}\right] \ldots\ldots\ldots\ldots\ldots(4)$$

We can rearrange exponential terms as

$$e^{-j\omega n} = e^{-j\omega n} e^{j\omega\left(\frac{N-1}{2}\right)} e^{-j\omega\left(\frac{N-1}{2}\right)}$$

$$e^{-j\omega n} = e^{-j\omega\left(\frac{N-1}{2}\right)} e^{-j\omega\left(n-\frac{N-1}{2}\right)} \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

Similarly

$$e^{-j\omega(N-1-n)} = e^{-j\omega(N-1)}e^{\omega jn}$$

$$=e^{-j\omega\left(\frac{N-1}{2}\right)}e^{-j\omega\left(\frac{N-1}{2}\right)}\cdot e^{j\omega n}$$

$$e^{-j\omega(N-1-n)}=e^{-j\omega\left(N-\frac{1}{2}\right)}\cdot e^{j\omega(n-.N-\frac{1}{2})}\text{.................}(6)$$

From eq (5) & (6) we can write

$$[e^{-j\omega n}+e^{-j\omega(N-1-n)}\}=e^{-j\omega(\frac{N-1}{2})}\,[e^{-j\omega(n-\frac{N-1}{2})}+e^{j\omega(n-\frac{N-1}{2})}]$$

w.k.t $\qquad e^{j\theta}+e^{-j\theta}=2\cos\theta$

Therefore,

$$[e^{-j\omega n}+e^{-j\omega(N-1-n)}]=e^{-j\omega(\frac{N-1}{2})}\,2\cos\omega(n-\frac{N-1}{2})\text{..........................}(7)$$

Putting eq(7) in (4),

$$H(j\omega)=h(\frac{N-1}{2})\,e^{-j\omega(\frac{N-1}{2})}+\sum_{n=0}^{\frac{N-3}{2}}h(n)\,e^{-j\omega(\frac{N-1}{2})}\,2\cos\omega(n-\frac{N-1}{2})$$

$$H(j\omega)=e^{-j\omega(\frac{N-1}{2})}\left\{h\left(\frac{N-1}{2}\right)+\;2\sum_{n=0}^{\frac{N-3}{2}}h(n)\cos\omega\left(n-\frac{N-1}{2}\right)\right\}$$

Polar form of H $(j\omega)$ can be expressed as

$$H(j\omega)\text{ or }H(\omega)=|H(\omega)|\,e^{j\angle H(\omega)}\text{ ----------------}(8)$$

$$\text{Magnitude }|\mathbf{H}(\boldsymbol{\omega})| = h\left(\frac{N-1}{2}\right)+\;2\sum_{n=0}^{\frac{N-3}{2}}h(n)\cos\omega\left(n-\frac{N-1}{2}\right)$$

Phase of $H(\omega)$

$$\angle\mathbf{H}(\boldsymbol{\omega}) = \begin{cases} -\omega\left(\frac{N-1}{2}\right), & for\ |H(\omega)|>0 \\ -\omega\left(\frac{N-1}{2}\right)+\pi, & for\ |H(\omega)|<0 \end{cases}$$

Hence phase <H(w) is linear function of w. Thus phase is linearly proportional to frequency. Wher |H(w)| sign changes, phase changes by π. Hence the phase is said to be linear.

Thus FIR filters are linear phase FIR filter.

(ii) For even value of' N' & symmetry case

$$H(\omega)=e^{-j\omega\left(\frac{N-1}{2}\right)}\left\{2\sum_{n=0}^{\frac{N}{2}-1} h(n)\cos\omega\left(n-\left(\frac{N-1}{2}\right)\right)\right\}$$

Comparing this eq'n with eq'n 8

Magnitude $|H(\omega)|=2\sum_{n=0}^{\frac{N}{2}-1} h(n)\cos\omega\left(n-\frac{N-1}{2}\right)$

$$\text{Phase } \angle H(\omega)=\begin{cases}-\omega\left(\frac{N-1}{2}\right) & \text{for } |H(\omega)|>0\\ -\omega\left(\frac{N-1}{2}\right)+\pi & \text{for } |H(\omega)|<0\end{cases}$$

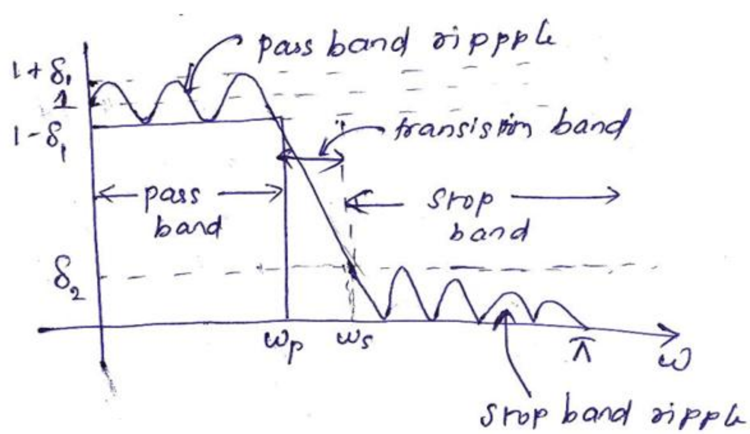
(iii) If N is odd & anti symmetry

$$H(\omega)=h\left(\frac{N-1}{2}\right)+2\sum_{n=0}^{\frac{N-3}{2}} h(n)\sin\left(\frac{N-1}{2}-n\right)\omega$$

(iv)If n is even & anti symmetry

$$H(\omega)=2\sum_{n=0}^{\frac{N}{2}-1} h(n)\sin\left(\frac{N-1}{2}-n\right)\omega$$

MAGNITUDE SPECIFICATION:



Magnitude specification of FIR filter can be expressed as:

$$\begin{cases}1-\delta_1 \le |H(\omega)| \le 1+\delta_1 \text{ for } 0 \le \omega \le \omega_p\\ 0 \le |H(\omega)| \le \delta_2 \text{ for } \omega_s \le \omega \le \pi\end{cases}$$

Formula for order 'N'

$$N=\frac{-10log_{10}(\delta1-\delta2)-15}{14\Delta f}$$

Here $\Delta f = \frac{\omega_{s}-\omega_{p}}{2\pi}$ $---$ *transistion band*

Or Δf = $f_s - f_p$ where $\omega_s = 2\pi\ f_s$ & $\omega_p = 2\pi\ f_p$

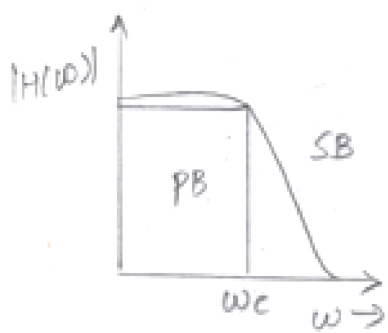
Another formula for N

$$N = k\left(\frac{2\pi}{\omega_2-\omega_1}\right)$$

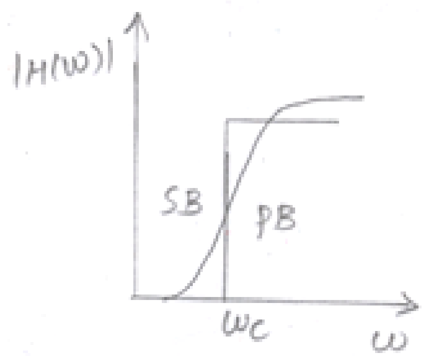
Width of main lobe = $k\left(\frac{2\pi}{N}\right)$

DIGITAL FILTER DESIGN.

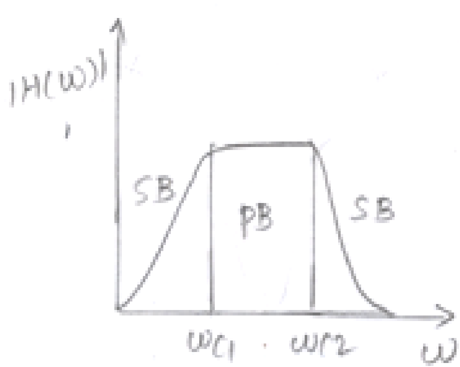
Low pass filter



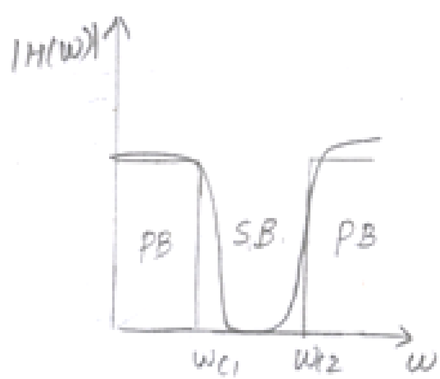
High Pass filter.



Band pass filter



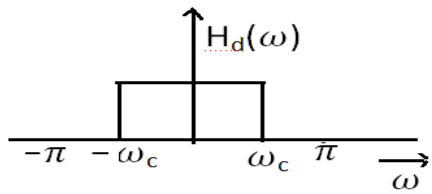
Band reject filter



Ideal frequency response

Low pass filter:-

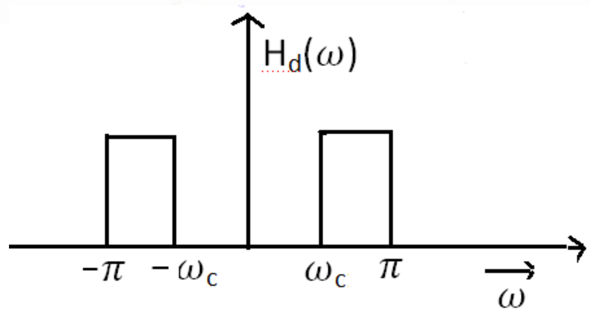
$$H_d(\omega) = \begin{cases} C e^{-j\alpha\omega} & ; -\omega_c \le \omega \le \omega_c \\ 0 & ; -\pi \le \omega < -\omega_c \ \& \ \omega_c < \omega < \pi \end{cases}$$



Here $\alpha = \frac{N-1}{2}$

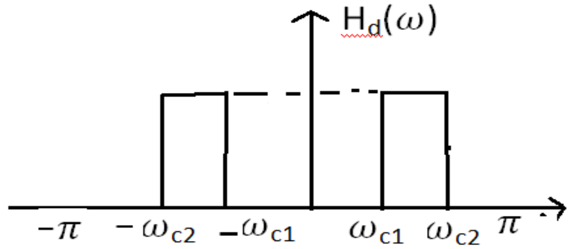
High pass filter:-

$$H_d(\omega) = \begin{cases} C e^{-j\alpha\omega} & ; -\pi \le \omega \le -\omega_c \ \& \ \omega_c < \omega < \pi \\ 0 & ; -\omega_c < \omega < \omega_c . \end{cases}$$



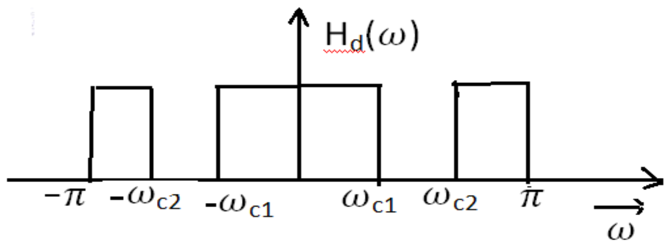
Band pass filter:-

$$H_d(\omega) = \begin{cases} C e^{-j\alpha\omega} & , -\omega_{c2} \le \omega \le -\omega_{c1} \ \& \ \omega_{c1} \le \omega \le \omega_{c2} \\ 0 & , -\pi \le \omega < -\omega_{c2} \ \& \ -\omega_{c1} \le \omega < \omega_{c1} \ \& \ \omega_{c2} < \omega \end{cases}$$



Band Stop filter:-

$$H_d(\omega) = \begin{cases} Ce^{-j\alpha\omega} ; & -\pi \le \omega \le -\omega_{c2} \ \& \ -\omega_{c1} \le \omega \le \omega_{c1} \ \& \\ & \omega_{c2} \le \omega \le \pi \\ 0 ; & -\omega_{c2} < \omega < -\omega_{c2} \ \& \ \omega_{c1} < \omega < \omega_{c2} \end{cases}$$



Design of FIR filter:

(i) Fourier series method

(ii) Windowing techniques

(iii) Frequency sampling techniques

Algorithm to Design an Ideal Filter Using Fourier Method

1. The problem gives the desired frequency response $H_d(e^{j\omega})$ and the order of the filter.

2. Calculate the desired impulse response using the formula,

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})e^{j\omega n}\,d\omega$$

3. Calculate the actual impulse response $h(n)$ from $h_d(n)$;

$$h(n) = \frac{1}{2\pi}\begin{cases} h_d(n), |n| \leq \left(\frac{N-1}{2}\right) \\ 0, \quad \text{otherwise} \end{cases}$$ When n≠0 & n=∞

obtain the filter coefficients for $n = 0, \pm 1, \pm 2, \ldots\ldots \pm\left(\frac{N-1}{2}\right)$

At n=0 & n=∞ ; h(n) becomes indeterminent form, hence

apply l'hospital rule, $\lim_{\theta \to 0} \frac{\sin\theta}{\theta}$ =1

Similarly

$\lim_{\theta \to 0} \frac{\sin A\theta}{\theta}$ =A

4. The FIR filter transfer function can be calculated using the formula,

$$H(Z) = h(0) + \sum_{n=1}^{\frac{N-1}{2}} h(n)\left[Z^{n} + Z^{-n}\right]$$

5. Obtain the realizable (causal) filter by shifting the transfer function by $Z^{-\left(\frac{N-1}{2}\right)}$ unit

$$H'(Z) = Z^{-\left(\frac{N-1}{2}\right)} H(Z)$$

6. The impulse response is calculated as:
   (a) For symmetric & odd.

$$H(e^{j\omega}) = h\left(\frac{N-1}{2}\right) + 2\sum_{n=1}^{\frac{N-1}{2}} h\left(\frac{N-1}{2} - n\right)\cos\omega n$$
$$h(n) = h(N-1-n)$$

   (b) For symmetric & even,

$$H(e^{j\omega}) = \sum_{n=0}^{N/2} 2h\left(\frac{N}{2} - n\right)\cos\left(n - \frac{1}{2}\right)\omega$$
$$h(n) = h(N-1-n)$$

(c) **For antisymmetric & odd**

$$H(e^{j\omega}) = \sum_{n=1}^{\frac{N-1}{2}} 2h\left(\frac{N-1}{2}-n\right)\sin \omega n$$

$$h(n) = -h(N-1-n)$$

(d) **For antisymmetric & even**

$$H(e^{j\omega}) = \sum_{n=0}^{N/2} 2h\left(\frac{N}{2}-n\right)\sin\left(n-\frac{1}{2}\right)\omega$$

$$h(n) = -(h(N-1-n)$$

7. **Obtain the filter coefficients from step(5) and substitute in step(6).**

8. **The frequency response plot can be obtained by varying the value of ω from 0 to π in a regular step.**

Example:

Design an ideal low pass filter using Fourier series method of $N = 9$ whose desired frequency response is

$$H_d(e^{j\omega}) = \begin{cases} 1, & \frac{\pi}{3} \geq \omega \geq -\frac{\pi}{3} \\ 0, & \text{otherwise} \end{cases}$$

(i) Determine the impulse response $h(n)$

(ii) Determine $H(Z)$

(iii) Plot the magnitude response $|H(e^{j\omega})|$

Solution The desired impulse response can be obtained from

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})e^{j\omega n}\, d\omega$$

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi/3}^{\pi/3} e^{j\omega n}\, d\omega$$

$$h_d(n) = \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-\pi/3}^{\pi/3}$$

$$h_d(n)=\frac{\sin\frac{\pi}{3}n}{\pi n}, \qquad \infty \geq n \geq -\infty$$

The desired filter coefficients are calculated for $n = 0, \pm 1, \pm 2, \pm 3, \pm 4$

$$h_d(0)=\underset{n\to 0}{\mathrm{Lt}}\ \frac{\sin\frac{\pi}{3}n}{\pi n}=\frac{1}{3}=0.333$$

$$h_d(1)=\frac{\sin\frac{\pi}{3}}{\pi}=0.276=h_d(-1)$$

$$h_d(2)=\frac{\sin\frac{2\pi}{3}}{2\pi}=0.138=h_d(-2)$$

$$h_d(3)=\frac{\sin\pi}{3\pi}=0=h_d(-3)$$

$$h_d(4)=\frac{\sin\frac{4\pi}{3}}{4\pi}=-0.069=h_d(-4)$$

$$h(n)=h_d(n)$$

Therefore, $h(n)=\begin{cases} h_d(n) & |n|\leq 4 \\ 0, & \text{otherwise} \end{cases}$

$$h(0)=h_d(0)=0.333$$

$$h(1)=h_d(-1)=h_d(1)=0.276$$

$$h(2)=h_d(-2)=h_d(2)=0.138$$

$$h(3)=h_d(-3)=h_d(3)=0$$

$$h(4)=h_d(-4)=h_d(4)=-0.069$$

The FIR filter transfer function can be calculated from

$$H(Z)=h(0)+\sum_{n=1}^{\frac{N-1}{2}} h(n)(Z^{n}+Z^{-n})$$

$$H(Z) = h(0) + \sum_{n=1}^{4} h(n)(Z^n + Z^{-n})$$

$$H(Z) = 0.333 + h(1)(Z + Z^{-1}) + h(2)(Z^2 + Z^{-2}) + h(3)(Z^3 + Z^{-3}) + h(4)(Z^4 + Z^{-4})$$

$$H(Z) = 0.333 + 0.276(Z + Z^{-1}) + 0.138(Z^2 + Z^{-2}) - 0.069(Z^4 + Z^{-4})$$

$$H'(z) = Z^{-\left(\frac{N-1}{2}\right)} H(z)$$

$$H'(z) = z^{-4}(H(z))$$

$$H'(z) = 0.333z^{-4} + 0.276z^{-3} + 0.276z^{-5} + 0.138z^{-2} + 0.138z^{-6} - 0.069 - 0.069z^{-8}$$

Rearranging the above equation,

$$H'(z) = -0.069 + 0.138z^{-2} + 0.276z^{-3} + 0.333z^{-4} + 0.276z^{-5} + 0.138z^{-6} - 0.069z^{-8}$$

The symmetric condition for impulse response is

$$h(n) = h(N-1-n)$$

$\therefore$ filter coefficient of causal filter are

$h(0) = h(8) = -0.069$ $\quad h(1) = h(7) = 0$ $\quad h(2) = h(6) = 0.138$ $\quad h(3) = h(5) = 0.276$

$$h(4) = 0..333$$

The frequency response for symmetric impulse response with odd length is given by

$$\mathrm{H}\left(e^{j\omega}\right) = h\left(\frac{N-1}{2}\right) + 2\sum_{n=1}^{\frac{N-1}{2}} h\left(\frac{N-1}{2} - n\right)\cos\omega n$$

For N=9

$$H\left(e^{j\omega}\right) = 0.333 + 2h(3)\cos\omega + 2h(2)\cos 2\omega + 2h(1)\cos 3\omega + 2h(0)\cos 4\omega$$

$$H\left(e^{j\omega}\right) = 0.333 + 2\times(0.276)\cos\omega + 2(0.138)\cos 2\omega + 2 \times (-0.069)\cos 4\omega$$

$$H\left(e^{j\omega}\right) = 0.333 + 0.552\cos 3\omega + 0.276\cos 2\omega - 0.138\cos 4\omega$$

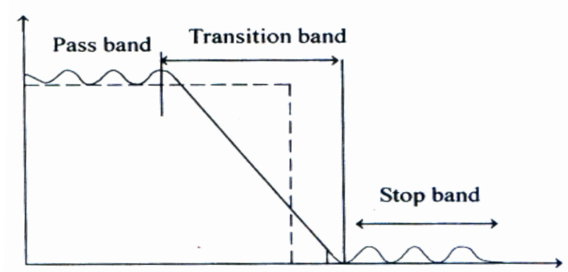
Magnitude response:

W.k.t $|H(e^{j\omega})| = 20\log(H(e^{j\omega}))$ here $H(e^{j\omega}) = H(\omega)$

ω	0	10	20	30	40	50
$20log\lvert H(\omega)\rvert$	0.1975	0.2589	0.3338	0.1553	-0.5981	-2.2750

Windowing Technique

In this tresponse ,we have some oscillations in pass band and stop band .these oscillations are due to the truncation of the Fourier series at n=$\pm\frac{N-1}{2}$, N is the length of sequence. These oscillations are called Gibb's oscillations.



Response of Low Pass Filter

These Gibb's oscillations can be reduced by multiplying the Filter response ($h_d(n)$) with windowing sequence ($w(n)$).

The types of window are given below.

1. Rectangular window ($w_R(n)$)
2. Triangular window (or) Bartlett window ($w_t(n)$)
3. raised cosine window
4. hanning window
5. hamming window
6. blackman window

Windows and their spectrums:

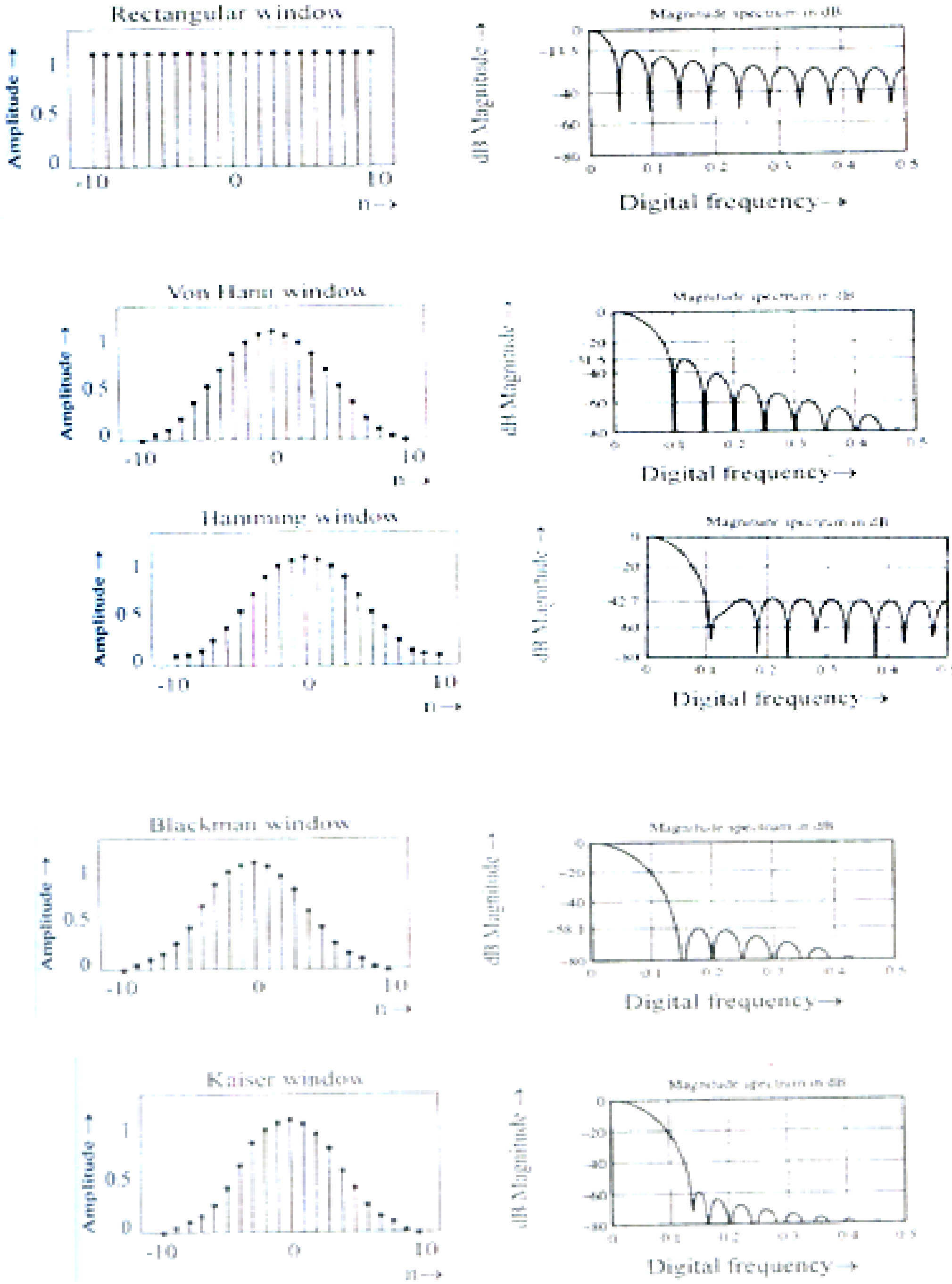


Fig. Some Windows and their Spectrum

Characteristics of Window Function

- ✓ Centre lobe of frequency response of window should contain most of the energy and it should be narrow.

- ✓ The side lobes of frequency response should be decreased in energy as ω tends to π as in following figure.

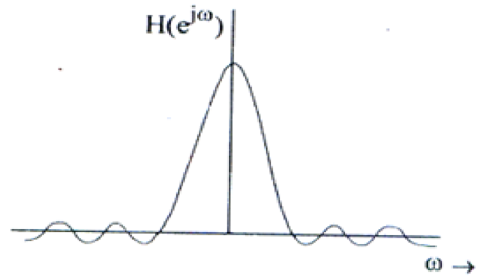


Fig.

- ✓ Highest side lobe level should be small.

Algorithm to design an ideal filter using any window method

1.The problem gives desired frequency response $H_d(e^{j\omega})$,type of window and the order of the filter.

2.Calculate desired impulse response using the formula,

$$h_d(n)=\frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})e^{j\omega n}\,d\omega$$

Obtain the desired filter coefficients for $n=0,\pm1,\pm2,\ldots\ldots\ldots\pm\left(\frac{N-1}{2}\right)$

3.Obtain the FIR filter coefficient by multiplying $h_d(n)$ with respective window function ,that is ,

$$h(n)=h_d(n)\times w(n)$$

where, for rectangular window:

$$W_R(n)=\begin{cases} 1, & |n| \le \left(\frac{N-1}{2}\right) \\ 0, & otherwise \end{cases}$$

a)For Hanning window: $\alpha = 0.5$

$$w_{hn}(n)=0.5+0.5\cos\left(\frac{2\pi n}{N-1}\right),\quad |n| \le \left(\frac{N-1}{2}\right)$$

b)For Hamming window: $\alpha = 0.54$

$$w_{hm}(n)=0.54+0.46\, cos\left(\frac{2\pi n}{N-1}\right),\ \ |n| \le \left(\frac{N-1}{2}\right)$$

4)The FIR filter transfer function can be calculated using the formula

$$H(Z)=h(0)+\sum_{n=1}^{\left(\frac{N-1}{2}\right)} h(n)[Z^{n} + Z^{-n}]$$

5)Obtain the realizable(causal) filter by shifiting the transfer function $Z^{-\left(\frac{N-1}{2}\right)}$

$$H'(Z)=Z^{-\left(\frac{N-1}{2}\right)}H(Z)$$

6)The impulse response is calculated as:

a)For Symmetric&odd,

$$H(e^{j\omega})=h\left(\frac{N-1}{2}\right) + 2\sum_{n=1}^{\left(\frac{N-1}{2}\right)} h\left(\frac{N-1}{2} - n\right) cos\omega n$$

$$H(n)=h(N-1-n)$$

b)For Symmetric and even,

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N}{2}} 2h\left(\frac{N}{2} - n\right)\cos\left(n - \frac{1}{2}\right)\omega$$

c)For anti-symmetric and odd,

$$H(e^{j\omega})=\sum_{n=1}^{\frac{N-1}{2}} 2h\left(\frac{N}{2} - n\right) sin\omega n$$

$$h(n)=-h(N-1-n)$$

d)For anti-symmetric and even ,

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N}{2}} 2h\left(\frac{N}{2} - n\right)\sin\left(n - \frac{1}{2}\right)\omega$$

7)Obtain the filter coefficients from step(5)and substitute in step(6)

8)The frequency response plot can be obtained by varying the value of ω from 0 to π in a regular step.

PROBLEM: 1

Design an ideal low pass filer using Fourier series method of N=9 whose desired frequency response is

$$H_d\left(e^{j\omega}\right) = \begin{cases} 1, & \frac{\pi}{3} \geq \omega \geq \frac{-\pi}{3} \\ 0, & otherwise \end{cases}$$

(i). Determine the impulse response h(n)

(ii) Determine H(Z)

(iii) Plot the magnitude response $|H(e^{j\omega})|$

Solution:

The desired impulse response can be obtained from

$$h_{d(n)} = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d\left(e^{j\omega}\right) e^{j\omega n}\, d\omega$$

$$h_{d(n)} = \frac{1}{2\pi}\int_{\frac{-\pi}{3}}^{\frac{\pi}{3}} e^{j\omega n}\, d\omega$$

$$h_{d(n)} = \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{\frac{-\pi}{3}}^{\frac{\pi}{3}}$$

$$h_d(n) = \frac{\sin n\frac{\pi}{3}}{\pi n} \quad , \quad \infty \geq n \geq -\infty$$

The desired filter coefficients are calculated for n= 0, ±1, ±2, ±3, ±4

$$h_d(0) = \frac{0}{0} \text{ = in determinant form = apply l'hospital rule}$$

$$\lim_{\theta\to 0}\frac{\sin A\,\theta}{\theta} = A$$

$$h_d(0) = \frac{\sin 0\frac{\pi}{3}}{\pi 0} = \frac{1}{3} = 0.333, \quad \infty \geq n \geq -\infty$$

$$h_d\ (0) = 0.333$$

$$h_d(1) = \frac{\sin\frac{\pi}{3}}{\pi n} = 0.276 = h_d(-1)$$

$$h_d(2) = \frac{\sin\frac{2\pi}{3}}{2\pi} = 0.138 = h_d(-2)$$

$$h_d(3) = \frac{\sin\pi}{3\pi} = 0 = h_d(-3)$$

$h_d(4)=\frac{\sin\frac{4\pi}{3}}{4\pi}$ = -0.069= $h_d(-4)$

$$h(n) = h_d(n)$$

Therefore $h(n) = \begin{cases} h_d(n), & |n| \le 4 \\ 0, & otherwise \end{cases}$

h(0)= h_d (0)= 0.333 h(1)= $h_d(-1)=h_d(1)$ = 0.276

h(2)= $h_d(-2)=h_d(2)$ = 0.138 h(3)= $h_d(-3)=h_d(3)$ = 0

h(4)= $h_d(-4)=h_d(4)$ =- 0.069

The fir filter transfer function can be calculated from

$$H(Z) = h(0) + \sum_{n=1}^{\frac{N-1}{2}} h(n)\,(Z^n + Z^{-n})$$

$$H(Z) = h(0) + \sum_{n=1}^{4} h(n)\,(Z^n + Z^{-n})$$

$$H(Z) = 0.333 + h(1)\,(Z + Z^{-1}) + h(2)\,(Z^2 + Z^{-2}) + h(3)\,(Z^3 + Z^{-3}) + h(4)\,(Z^4 + Z^{-4})$$

$$H(Z) = 0.333 + 0.276\,(Z + Z^{-1}) + 0.138\,(Z^2 + Z^{-2}) + 0\,(Z^3 + Z^{-3}) - 0.069\,(Z^4 + Z^{-4})$$

$$H'(Z) = Z^{-\left(\frac{N-1}{2}\right)} H(Z)$$

$$H'(Z) = Z^{-(4)} H(Z)$$

$$H'(z) = 0.333z^{-4} + 0.276z^{-3} + 0.276z^{-5} + 0.138z^{-2} + 0.138z^{-6} - 0.069 \; - 0.069z^{-8}$$

Rearranging the above equation

$$H'(z) = -0.069 + 0.138z^{-2} + 0.276z^{-3} + 0.333z^{-4} + 0.276z^{-5} + 0.138z^{-6} - 0.069z^{-8}$$

The symmetric condition for impulse response is

$$h(n) = h(N-1-n)$$

∴ filter coefficient of causal filter are

$h(0) = h(8) = -0.069$ $h(1) = h(7) = 0$

$h(2) = h(6) = 0.138$ $h(3) = h(5) = 0.276$

$h(4) = 0..333$

For N=9

The frequency response for symmetric impulse response with odd length is given by

$$H\left(e^{j\omega}\right)=h\left(\frac{N-1}{2}\right)+2\sum_{n=1}^{\frac{N-1}{2}} h\left(\frac{N-1}{2}-n\right)\cos\omega n$$

$$H\left(e^{j\omega}\right)=0.333+2h(3)\cos\omega+2h(2)\cos 2\omega+2h(1)\cos 3\omega+2h(0)\cos 4\omega$$

$$H\left(e^{j\omega}\right)=0.333+2\times(0.276)\cos\omega+2(0.138)\cos 2\omega+2\times(-0.069)\cos 4\omega$$

$$H\left(e^{j\omega}\right)=0.333+0.552\cos 3\omega+0.276\cos 2\omega-0.138\cos 4\omega$$

Magnitude response:

W.k.t $\left|H\left(e^{j\omega}\right)\right|=20\log\left(H\left(e^{j\omega}\right)\right)$ here $H\left(e^{j\omega}\right)=H(\omega)$

ω	0	10	20	30	40	50
$20log\|H(\omega)\|$	0.1975	0.2589	0.3338	0.1553	-0.5981	-2.2750

PROBLEM: 2

Design an ideal HPF whose desired frequency response

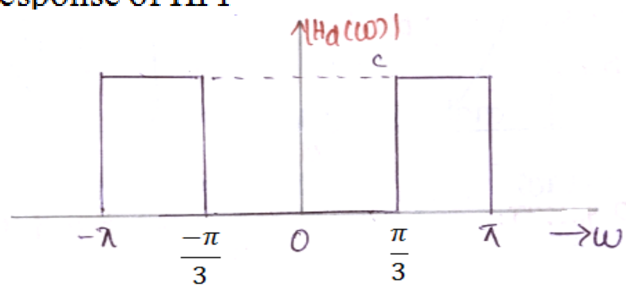
$$H_d\left(e^{j\omega}\right) = \begin{cases}1, & \pi \le |\omega| \le \frac{\pi}{3}\\ 0, & \text{else}\end{cases}$$

Using hanning window (i) determine the impulse response for N=9

(ii) Determine H(Z)

Solution:

STEP 1: frequency response of HPF



STEP 2:

$$H_d(e^{j\omega}) = \begin{cases} 1, & \pi \le |\omega| \le \frac{\pi}{3} \\ 0, & \text{else} \end{cases}$$

The desired impulse response of HPF is given by

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$h_d(n) = \frac{1}{2\pi} \left[\int_{-\pi}^{\frac{-\pi}{3}} e^{j\omega n}\, d\,\omega \quad + \int_{\frac{\pi}{3}}^{\pi} e^{j\omega n}\, d\,\omega \right]$$

$$h_d(n) = \frac{\sin \pi n - \sin \frac{\pi n}{3}}{\pi n}$$

$$h_d(0) = \lim_{n \to 0} \left[\frac{\sin \pi n - \sin \frac{\pi n}{3}}{\pi n} \right] = 1 - \frac{1}{3} = 0.067 \quad \text{apply l'hospital rule}$$

$$h_d(1) = \frac{\sin \pi - \sin \frac{\pi}{3}}{\pi} = -0.276 = h_d(-1)$$

$$h_d(2) = \frac{\sin 2\pi - \sin \frac{2\pi}{3}}{2\pi} = -0.138 = h_d(-2)$$

$$h_d(3) = \frac{\sin 3\pi - \sin \frac{3\pi}{3}}{3\pi} = 0 = h_d(-3)$$

$$h_d(4) = \frac{\sin 4\pi - \sin \frac{4\pi}{3}}{4\pi} = 0.09 = h_d(-4)$$

STEP 3:

$$h(n) = \begin{cases} h_d(n)\omega_{hn}(n), & |n| \le \frac{N-1}{2} \\ 0, & \text{else} \end{cases}$$

The hanning window function is given by

$$w_{hn}(n) = 0.5 + 0.5 \cos\left(\frac{2\pi n}{N-1}\right), \ \left(\frac{N-1}{2}\right) \ge n \ge -\left(\frac{N-1}{2}\right)$$

$$w_{hn}(n) = 0.5 + 0.5 \cos\left(\frac{\pi n}{4}\right) \quad , 4 \ge n \ge -4$$

For n=0,

$$w_{hn}(0) = = 0.5 + 0.5 = 1$$

For n=1,

$$w_{hn}(1) = w_{hn}(-1) = 0.5 + 0.5 \cos\left(\frac{\pi}{4}\right) = 0.854$$

For n=2,

$$w_{hn}(2)=w_{hn}(-2) = 0.5 + 0.5\cos\left(\frac{\pi}{2}\right) = 0.5$$

For n=3,

$$w_{hn}(3)=w_{hn}(-3) = 0.5 + 0.5\cos\left(\frac{3\pi}{4}\right) = 0.147$$

For n=4,

$$w_{hn}(4)=w_{hn}(-4) = 0.5 + 0.5\cos(\pi) = 0$$

The filter coefficients obtained due to hanning window are

$$\mathrm{h}(n) = h_d(n) \times w_{hn}(n)$$

$$\mathrm{h}(0) = h_d(0) \times w_{hn}(0) = 0.667 \times 1 = 0.667$$

$$\mathrm{h}(1) = h_d(1) \times w_{hn}(1) =-0.276 \times 0.854 = -0.236= \mathrm{h}(-1)$$

$$\mathrm{h}(2) = h_d(2) \times w_{hn}(2) =-0.138 \times 0.5 = -0.069= \mathrm{h}(-2)$$

$$\mathrm{h}(3) = h_d(3) \times w_{hn}(3)\ 0= \mathrm{h}(-3)$$

$$\mathrm{h}(4) = h_d(4) \times w_{hn}(4)\ 0= \mathrm{h}(-4)$$

Step 4:

The FIR filter transfer function can be calculated

$$\mathrm{H}(Z) = \mathrm{h}(0) + \sum_{n=1}^{\frac{N-1}{2}} h(n)\,(Z^n + Z^{-n})$$

$$\mathrm{H}(Z) = \mathrm{h}(0) +\sum_{n=1}^{4} h(n)\,(Z^n + Z^{-n})$$

$$\mathrm{H}(Z) = \mathrm{h}(0) + \mathrm{h}(1)(Z^1 + Z^{-1}) + \mathrm{h}(2)(Z^2 + Z^{-2})+ \mathrm{h}(3)(Z^3 + Z^{-3}) + \mathrm{h}(4)(Z^4 + Z^{-4})$$

$$\mathrm{H}(Z) = 0.067 -0.236(Z^1 + Z^{-1})- 0.069(Z^2 + Z^{-2})$$

Step 5:

$$\mathrm{H'}(z)= Z^{-\left(\frac{N-1}{2}\right)}\mathrm{H}(z)$$

$$H'(Z) = Z^{-4}\,\mathrm{H}(Z)$$

$$H'(Z) =0.667\,z^{-4} - 0.236z^{-3} -0.236z^{-5}- 0.069z^{-2} -0.069z^{-6}$$

$$H'(Z) =- 0.069z^{-2} - 0.236z^{-3} +0.667\,z^{-4}-0.236z^{-5}-0.069z^{-6}$$

Since it satisfies the symmetric condition

$h(n) = h(N-1-n)$

$h(0)= h(8) = 0$; $h(1)= h(7) = 0$

$h(2)= h(6) = -0.069$ $h(3)= h(5) = -0.236$ $h(4)= 0.667$

The frequency response of symmetric impulse response of odd length sequence is given by

$$H(e^{j\omega}) = h\left(\frac{N-1}{2}\right) + 2\sum_{n=1}^{\frac{N-1}{2}} h\left(\frac{N-1}{2} - n\right)\cos\omega n$$

$$H(e^{j\omega}) = h\left(\frac{N-1}{2}\right) + 2\sum_{n=1}^{4} h(4-n)\cos\omega n$$

The frequency response of HPF is given by

$$H(e^{j\omega}) = 0.667 + 2\,h(3)\cos\omega + 2\,h(2)\cos 2\omega + 2\,h(1)\cos 3\omega + 2\,h(0)\cos 4\omega$$

$$H(e^{j\omega}) = 0.667 + 2\times(-0.236)\cos\omega + 2\times(-0.069)\cos 2\omega$$

$$H(e^{j\omega}) = 0.667 - 0.472\cos\omega - 0.138\cos 2\omega$$

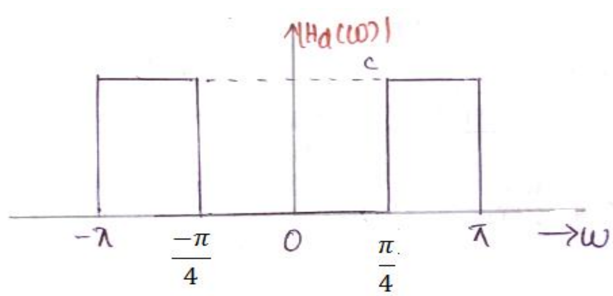
PROBLEM: 3

A high pass filter with frequency response

$H_d(e^{J\omega}) = \begin{cases} 1, & \frac{\pi}{4} \le |\omega| \le \pi \\ 0, & \text{else} \end{cases}$ Find h(z) at N=11 using hamming window technique.

SOLUTION:

STEP 1: frequency response of HPF



STEP 2:

$$H_d(e^{j\omega}) = \begin{cases} 1, & \frac{\pi}{4} \le |\omega| \le \pi \\ 0, & \text{else} \end{cases}$$

$$h_d(n)=\frac{1}{2\pi}\int_{-\pi}^{\pi} H_d\,(e^{j\omega})e^{j\omega n}d\omega$$

$$=\frac{1}{2\pi}\left\{\int_{-\pi}^{-\pi/4} e^{j\omega n}\ d\omega\ +\ \int_{\pi/4}^{\pi} e^{j\omega n}\,d\omega\right\}$$

$$=\frac{1}{2\pi}\left\{\left[\frac{e^{j\omega n}}{jn}\right]_{-\pi}^{\frac{-\pi}{4}}+\left[\frac{e^{j\omega n}}{jn}\right]_{\frac{\pi}{4}}^{\pi}\right\}$$

$$=\frac{1}{2\pi jn}\left\{e^{-\frac{j\pi n}{4}}-e^{-j\pi n}+e^{jn\pi}-e^{jn\pi/4}\right\}$$

$$\sin\theta=\frac{e^{j\theta}-e^{-j\theta}}{2j}$$

$$=\frac{1}{2j\pi n}\left\{e^{-\frac{j\pi n}{4}}-e^{-j\pi n}+e^{jn\pi}-e^{jn\pi/4}\right\}$$

$$h_d(n)=\ \frac{1}{\pi n}\left\{\sin n\pi-\sin\frac{n\pi}{4}\right\}$$

STEP 3:

$$h(n)=\begin{cases}h_d(n)\omega_H(n), & |n|\le\frac{N-1}{2}\\ 0, & \text{else}\end{cases}$$

Where $\omega_H(n)=\begin{cases}0.54+0.46\cos\frac{2\pi n}{N-1}, & |n|\le\frac{N-1}{2}\\ 0, & \text{else}\end{cases}$

At n=0, $h(n)=\frac{0}{0}=$ inderterminant form

Using L' Hospital rule, $\underset{\theta\to0}{\text{Lt}}\frac{\sin A\theta}{\theta}=A$

Apply l'hospital rule, $\lim_{n\to0}\left[\frac{\sin n\pi}{n\pi}-\frac{\sin\frac{n\pi}{4}}{n\pi}\right]=1-\frac{1}{4}$

$$h_d(0)=1-\frac{1}{4}$$

$$h_d(0)=0.75$$

For n=1, $h_d(1)=\frac{1}{\pi}\{\sin\pi-\sin\frac{\pi}{4}\}=-0.2251$

For n=2, $h_d(2)=\frac{1}{2\pi}\{\sin 2\pi-\sin\frac{\pi}{2}\}=-0.1592$

For n=3, $h_d(3)=\frac{1}{3\pi}\{\sin 3\pi-\sin\frac{3\pi}{4}\}=-0.0750$

For n=4, $h_d(4)=\frac{1}{4\pi}\{\sin 4\pi-\sin\pi\}=0$

For n=5, $h_d(5)=\frac{1}{5\pi}\{\sin 6\pi-\sin\frac{5\pi}{4}\}=0.0450$

$$W_H(n)=\begin{cases}0.54+0.46\cos\frac{2\pi n}{N-1}, & |n|\le\frac{N-1}{2}\\ 0, & \text{else}\end{cases}$$

For n=0, $W_h(0)=1$

For n=1, $W_h(1)=0.912$

For n=2, $W_h(2)=0.682$

For n=3 $W_h(3)=0.3979$

For n=4, $W_h(4)=0.1679$

For n=5, $W_h(5)=0.08$

Find $h(n)=h_d(n)\times W_h(n)$

$h(0)=h_d(0)\times w_h(0)=0.75$

$h(1)=h_d(1)\times w_h(1)=-0.2058$ hence symmetric $h(1)=h(-1)$

$h(2)=h_d(2)\times w_h(2)=-0.1086$ $h(2)=h(-2)$

$h(3)=h_d(3)\times w_h(3)=-0.0298$ $h(3)=h(=3)$

$h(4)=h_d(4)\times w_h(4)=0$ $h(4)=h(-4)$

$h(5)=h_d(5)\times w_h(5)=0.0036$ $h(5)=(-5)$

Step 4:

$$H(z)=h(0)+\sum_{n=1}^{\frac{N-1}{2}} h(n)[z^{-n}+z^{n}]$$

$$=0.75+h(1)[z^{-1}+z^{1}]+h(2)[z^{-2}+z^{2}]+ +h(3)[z^{-3}+z^{3}]\ +h(1)[z^{-4}+z^{4}]+ +h(5)[z^{-5}+z^{5}]$$

$$H(z)=0.75-0.2052[z^{-1}+z^{1}]-0.1086[z^{-2}+z^{2}]-0.0298[z^{-3}+z^{3}]+0.036[z^{-5}+z^{5}]$$

Step 5:

$$H'(z)=Z^{-\left(\frac{N-1}{2}\right)}H(z)$$

$$=Z^{-5}H(z)$$

$$=Z^{-5}[0.75-0.2052Z^{-1}-0.2052Z^{1}-0.1086Z^{-2}-0.1086Z^{2}\ {}^{-}0.0298Z^{-3}-0.0298Z3+0.036Z^{-5}+0.036Z^{5}]$$

$H'(z) = 0.75z^{-5} - 0.2052z^{-6} - 0.2052z^{-4} - 0.1086z^{-7} - 0.1086z^{-3} - 0.0298z^{-8} - 0.0298z^{-2} + 0.036z^{-10} + 0.036$

$H'(z) = 0.036z^{-10} - 0.0298z^{-8} - 0.1086z^{-7} - 0.2052z^{-6} - 0.75z^{-5} - 0.2052z^{-4} - 0.1086z^{-3} - 0.0298z^{-2} + 0.036$

Step 6:

Find Filter Co-efficients

$h(n) = h(N-1-n)$; $n = 0,1\ldots\ldots N-1$

$h(0)=h(10)=0.036$

$h(1)=h(9)=0$

$h(2)=h(8)=-0.0298$

$h(3)=h(7)=-0.1086$

$h(4)=h(6)=-0.2052$

$h(5)=0.75$

step:7

Frequency Response:

$$H(e^{j\omega}) = h\left(\frac{N-1}{2}\right) + 2\sum_{n=1}^{\frac{N-1}{2}} h\left(\frac{N-1}{2} - n\right)\cos \omega n$$

$$H(e^{j\omega}) = h\left(\frac{N-1}{2}\right) + 2\sum_{n=1}^{5} h(5-n)\cos \omega n$$

$$H(e^{j\omega}) = h(5) + 2h(4)\cos\omega + 2\,h(3)\cos 2\omega + 2\,h(2)\cos 3\omega + 2\,h(1)\cos 4\omega + 2\,h(0)\cos 5\omega$$

$$H(\omega) = 0.75 - 0.4104\cos\omega - 0.2168\cos 2\omega - 0.06\cos 3\omega + 0.0072\cos 5\omega$$

Step:8

Magnitude response:

We know that, $|H(\omega)| = 20\log H(\omega)$

ω	0	10	20	30	40	50	60
$20\log H(\omega)$	-23.1	-20.47	-15.54	-11.06	-7.51	-4.83	-2.89

Design of FIR Filters by Frequency Sampling Technique

In this method the ideal (desired) frequency response is sampled at sufficient number of points (i.e., N-points). These samples are the DFT coefficients of the impulse response of the filter. Hence the impulse response of the filter is determined by taking inverse DFT.

Let, $H_d(e^{j\omega})$ = Ideal desired frequency response

$H(k)$ = DFT sequence obtained by sampling $H_d(e^{j\omega})$

$h(n)$ = Impulse response of FIR filter.

The impulse response h(n) is obtained by taking inverse DFT of H(k). For practical realizability the samples of impulse response should be real. This can happen if all the complex terms appear in complex conjugate pairs. It can be observed that the complex DFT coefficients exists only as conjugate pairs. This suggest that the terms of H(k) can be matched by comparing the exponentials. The term $H(k)\, e^{+j2\pi nk/N}$ should be matched by the term that has the exponential $e^{-j2\pi nk/N}$ as a factor.

Procedure for Type-1 Design

1. Choose the ideal (desired) frequency response $H_d(e^{j\omega})$.

2. Sample $H_d(e^{j\omega})$ at N-points by taking $\omega=\omega_k = 2\pi k/N$ where $k = 0, 1, 2, 3, \ldots.(N-1)$, to generate the sequence H(k).

$$\therefore\ H(k) = H_d(e^{j\omega})\Big|_{\omega = \frac{2\pi k}{N}} \quad ; \quad \text{for } k = 0, 1, \ldots.. (N-1)$$

3. Compute the N samples of impulse response h(n) using the following equation.

When N is odd,

$$\text{Impulse response, } h(n) = \frac{1}{N}\left[H(0) + 2\sum_{k=1}^{\frac{N-1}{2}} \mathrm{Re}\left[H(k)\, e^{j2\pi nk/N}\right]\right]$$

When N is even,

$$\text{Impulse response, } h(n) = \frac{1}{N}\left[H(0) + 2\sum_{k=1}^{\frac{N}{2}-1} \mathrm{Re}\left[H(k)\, e^{j2\pi nk/N}\right]\right]$$

where, "Re" stands for "real part of".

4. Take Z-transform of the impulse response h(n) to get the filter transfer function, H(z).

$$\therefore\ H(z) = \mathcal{Z}\{h(n)\} = \sum_{n=0}^{N-1} h(n)\, z^{-n}$$

Procedure for Type-2 Design

1. Choose the ideal (desired) frequency response $H_d(e^{j\omega})$.

2. Sample $H_d(e^{j\omega})$ at N-points by taking $\omega = \omega_k = 2\pi(2k + 1)/2N$, where $k = 0, 1, 2, 3, \ldots..$ $(N-1)$, to generate the sequence H(k).

$$\therefore\ H(k) = H_d(e^{j\omega})\Big|_{\omega = \frac{2\pi(2k+1)}{2N}} \ ; \ \text{for } k = 0, 1, \ldots\ldots (N-1)$$

3. Compute the N samples of impulse response h(n) using the following equation, when N is odd,

Impulse response h(n) = $\frac{2}{N}\sum_{k=0}^{\frac{N-3}{2}} Re\left[H(k)e^{\frac{jn\pi(2k+1)}{N}}\right]$

When N is even,

Impulse response h(n) = $\frac{2}{N} \times 2\sum_{k=0}^{\frac{N}{2}-1} Re\left[H(k)e^{\frac{jn\pi(2k+1)}{N}}\right]$

Where “Re” stands for “real part of”.

4. Take Z transform of the impulse response h(n) to get the filter transfer function H(z)

H(z) =Z$\{h(n)\}$ = $\sum_{n=0}^{N-1} h(n)\ Z^{-n}$

8.Explain frequency sampling technique for design of FIR fiters.Determine the coefficient of a linear-phase FIR filter of length N=15 which has a symmetric unit sample response and a frequency response that satisfies the condition

$$H_r(2\pi k/15) = \begin{cases} 1, & k = 0,1,2,3 \\ 0.4, & k = 4 \\ 0, & k = 5,6,7 \end{cases}$$

. ***(May/June 2011)***

problem

Determine the coefficients of a linear phase FIR filter of length N=15 which has a symmetric unit Sample response and frequency response satisfy the condition

$$H\left(\frac{2\pi k}{15}\right) = \begin{cases} 1; for\ k = 0,1,2,3 \\ 0.4\ ; \text{for k} = 4 \\ 0\ ; for\ k = 5,6,7 \end{cases}$$

Solution:

For linear phase FIR filter phase function , $\theta(\omega)$ = -αω where $\alpha = \frac{N-1}{2}$

here, N=15, $\therefore\ \alpha = \frac{15-1}{2}$ =7

also, here $\omega = \omega_k = \frac{2\pi k}{N} = \frac{2\pi k}{15}$. hence we can go for type- design.

In this problem the samples of the magnitude response of the ideal filter are directly given for Various values of k

$$H(k) = H_d(\omega)|_{\omega=\omega_k} = 1\ e^{-j\alpha\omega_k} = e^{-j7\times\frac{2\pi k}{15}}\ ;\ k=0,1,2,3$$

$$= 0.4\ e^{-j\alpha\omega_k} = 0.4\ e^{-j7\times\frac{2\pi k}{15}}\ ;\ k=4$$

$$=0 \quad ;\ k=5,6,7$$

The samples of impulse response h(n) are given by,

$$h(n) = \frac{1}{N}\left[[H(0)] + 2\sum_{k=1}^{\frac{N-1}{2}} Re\left[H(k)e^{\frac{j2\pi nk}{N}}\right]\right]$$

$$= \frac{1}{15}\left[[H(0)] + 2\sum_{k=1}^{7} Re\left[H(k)e^{\frac{j2\pi nk}{15}}\right]\right]$$

$$= \frac{1}{15}\left[[H(0)] + 2\sum_{k=1}^{3} Re\left[H(k)e^{\frac{j2\pi nk}{15}}\right] + 2\ Re\left[H(4)e^{\frac{j2\pi n\times 4}{15}}\right]\right]$$

$$\therefore h(n) = \frac{1}{15}\left[1 + 2\sum_{k=1}^{3} Re\left[e^{-j7\times\frac{2\pi k}{15}} \times e^{\frac{j2\pi nk}{15}}\right] + 2\ Re\left[0.4e^{-j7\times\frac{2\pi\times 4}{15}} \times e^{\frac{j8\pi n}{15}}\right]\right]$$

$\boxed{H(0)=1}$

$$= \frac{1}{15}\left[1 + 2\sum_{k=1}^{3} Re\left[e^{\frac{j2\pi k\ (n-7)}{15}}\right] + 2\ Re\left[0.4e^{\frac{j8\pi(n-7)}{15}}\right]\right]$$

$e^{j\theta} = \cos\theta + j\sin\theta$

$\therefore$ Re$\left[e^{j\theta}\right] = \cos\theta$

$$= \frac{1}{15}\left[1 + 2\sum_{k=1}^{3} \cos\frac{2\pi k}{15}(n-7) + 0.8\ \cos\frac{8\pi}{15}(n-7)\right]$$

$$= \frac{1}{15}\left[1 + 2\cos\frac{2\pi(n-7)}{15} + 2\cos\frac{4\pi(n-7)}{15} + 2\cos\frac{6\pi(n-7)}{15} + 0.8\cos\frac{8\pi(n-7)}{15}\right]$$

Here N=5, $\therefore$ N- 1 = 14, $\frac{N-1}{2} = 7$ Hence , calculate h(n) for n= 0 to 14

Since h(n) satisfies the symmetry condition $h(N-1-n) = h(n)$ with centre of symmetry at $(N-1)/2$, calculate h(n) for n = 0 to 7.

When $n = 0$; $h(0) = \frac{1}{15}\left[1 + 2\cos\frac{2\pi(0-7)}{15} + 2\cos\frac{4\pi(0-7)}{15} + 2\cos\frac{6\pi(0-7)}{15} + 0.8\cos\frac{8\pi(0-7)}{15}\right]$

$= -0.0141$

When $n = 1$; $h(1) = \frac{1}{15}\left[1 + 2\cos\frac{2\pi(1-7)}{15} + 2\cos\frac{4\pi(1-7)}{15} + 2\cos\frac{6\pi(1-7)}{15} + 0.8\cos\frac{8\pi(1-7)}{15}\right]$

$= -0.0019$

When $n = 2$; $h(2) = \frac{1}{15}\left[1 + 2\cos\frac{2\pi(2-7)}{15} + 2\cos\frac{4\pi(2-7)}{15} + 2\cos\frac{6\pi(2-7)}{15} + 0.8\cos\frac{8\pi(2-7)}{15}\right]$

$= 0.04$

When $n = 3$; $h(3) = \frac{1}{15}\left[1 + 2\cos\frac{2\pi(3-7)}{15} + 2\cos\frac{4\pi(3-7)}{15} + 2\cos\frac{6\pi(3-7)}{15} + 0.8\cos\frac{8\pi(3-7)}{15}\right]$

$= 0.0122$

When $n = 4$; $h(4) = \frac{1}{15}\left[1 + 2\cos\frac{2\pi(4-7)}{15} + 2\cos\frac{4\pi(4-7)}{15} + 2\cos\frac{6\pi(4-7)}{15} + 0.8\cos\frac{8\pi(4-7)}{15}\right]$

$= -0.0914$

When $n = 5$; $h(5) = \frac{1}{15}\left[1 + 2\cos\frac{2\pi(5-7)}{15} + 2\cos\frac{4\pi(5-7)}{15} + 2\cos\frac{6\pi(5-7)}{15} + 0.8\cos\frac{8\pi(5-7)}{15}\right]$

$= -0.0181$

When $n = 6$; $h(6) = \frac{1}{15}\left[1 + 2\cos\frac{2\pi(6-7)}{15} + 2\cos\frac{4\pi(6-7)}{15} + 2\cos\frac{6\pi(6-7)}{15} + 0.8\cos\frac{8\pi(6-7)}{15}\right]$

$= 0.3130$

When $n = 7$; $h(7) = \frac{1}{15}\left[1 + 2\cos\frac{2\pi(7-7)}{15} + 2\cos\frac{4\pi(7-7)}{15} + 2\cos\frac{6\pi(7-7)}{15} + 0.8\cos\frac{8\pi(7-7)}{15}\right]$

$= 0.52$

Using symmetry condition $h(N-1-n) = h(n)$

When $n = 8$, $h(8) = h(15-1-8) = h(6) = 0.3130$

When $n = 9$, $h(9) = h(15-1-9) = h(5) = -0.0181$

When $n = 10$, $h(10) = h(15-1-10) = h(4) = -0.0914$

When $n = 11$, $h(11) = h(15-1-11) = h(3) = 0.0122$

When $n = 12$, $h(12) = h(15-1-12) = h(2) = 0.04$

When $n = 13$, $h(13) = h(15-1-13) = h(1) = -0.0019$

When $n = 14$, $h(14) = h(15-1-14) = h(0) = -0.0141$

The transfer function H(z) of the filter is given by Z-transform of h(n)

$$\therefore\ H(z) = \mathcal{Z}\{h(n)\} = \sum_{n=0}^{N-1} h(n)\, z^{-n} = \sum_{n=0}^{14} h(n)\, z^{-n}$$

$$= h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(4)z^{-4} + h(5)z^{-5} + h(6)z^{-6} + h(7)z^{-7} + h(8)z^{-8} + h(9)z^{-9} + h(10)z^{-10} + h(11)z^{-11} + h(12)z^{-12} + h(13)z^{-13} + h(14)z^{-14}$$

$$= h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(4)z^{-4} + h(5)z^{-5} + h(6)z^{-6} + h(7)z^{-7} + h(6)z^{-8} + h(5)z^{-9} + h(4)z^{-10} + h(3)z^{-11} + h(2)z^{-12} + h(1)z^{-13} + h(0)z^{-14}$$

$$= h(0)\left[1 + z^{-14}\right] + h(1)\left[z^{-1} + z^{-13}\right] + h(2)\left[z^{-2} + z^{-12}\right] + +h(3)\left[z^{-3} + z^{-11}\right] + h(4)\left[z^{-4} + z^{-10}\right] + h(5)\left[z^{-5} + z^{-9}\right] + h(6)\left[z^{-6} + z^{-8}\right] + h(7)z^{-7}$$

Using symmetry condition $h(N-1-n) = h(n)$

$$= -0.0141\left[1 + z^{-14}\right] - 0.0019\left[z^{-1} + z^{-13}\right] + 0.04\left[z^{-2} + z^{-12}\right] + 0.0122\left[z^{-3} + z^{-11}\right] - 0.0914\left[z^{-4} + z^{-10}\right] - 0.0181\left[z^{-5} + z^{-9}\right] + 0.3130\left[z^{-6} + z^{-8}\right] + 0.52z^{-7}$$

Frequency Response

where, $$|H(e^{j\omega})| = h\left(\frac{N-1}{2}\right) + \sum_{n=1}^{\frac{N-1}{2}} 2h\left(\frac{N-1}{2} - n\right)\cos\omega n$$

$$= h(7) + \sum_{n=1}^{7} 2h(7-n)\cos\omega n$$

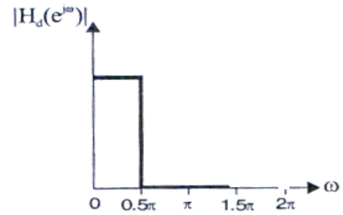
$$= h(7) + 2h(6)\cos\omega + 2h(5)\cos 2\omega + 2h(4)\cos 3\omega + 2h(3)\cos 4\omega + 2h(2)\cos 5\omega + 2h(1)\cos 6\omega + 2h(0)\cos 7\omega$$

$$= 0.52 + 2 \times 0.3130\cos\omega + 2 \times -0.0181\cos 2\omega + 2 \times -0.0914\cos 3\omega + 2 \times 0.0122\cos 4\omega + 2 \times 0.04\cos 5\omega + 2 \times -0.0019\cos 6\omega + 2 \times -0.0141\cos 7\omega$$

$$|H(e^{j\omega})| = 0.52 + 0.626\cos\omega - 0.0362\cos 2\omega - 0.1828\cos 3\omega + 0.0244\cos 4\omega + 0.08\cos 5\omega - 0.0038\cos 6\omega - 0.0282\cos 7\omega$$

Example

Design a linear phase FIR lowpass filter with a cutoff frequency of 0.5π rad/sample by taking 11 samples of ideal frequency response.



Ideal magnitude response of FIR lowpass filter.

Solution

desired frequency response $H_d(e^{j\omega})$ of linear phase FIR lowpass filter with cutoff frequency of 0.5π rad/sample is given by,

$$H_d(e^{j\omega}) = e^{-j\alpha\omega} \quad ; \quad 0 \le \omega \le 0.5\pi$$

$$= 0 \qquad ; 0.5\pi \le \omega$$

where, $\alpha = \frac{N-1}{2} = \frac{11-1}{2} = 5$

$$\omega_k = \frac{2\pi k}{N} = \frac{2\pi k}{11} \quad ; \quad \text{for } k = 0 \text{ to } 10$$

When $k = 0$; $\omega_k = \frac{2\pi \times 0}{11} = 0$

When $k = 1$; $\omega_k = \frac{2\pi \times 1}{11} = 0.18\pi$

When $k = 2$; $\omega_k = \frac{2\pi \times 2}{11} = 0.36\pi$

When $k = 3$; $\omega_k = \frac{2\pi \times 3}{11} = 0.55\pi$

When $k = 4$; $\omega_k = \frac{2\pi \times 4}{11} = 0.73\pi$

When $k = 5$; $\omega_k = \frac{2\pi \times 5}{11} = 0.91\pi$

When $k = 6$; $\omega_k = \frac{2\pi \times 6}{11} = 1.09\pi$

When $k = 7$; $\omega_k = \frac{2\pi \times 7}{11} = 1.27\pi$

When $k = 8$; $\omega_k = \frac{2\pi \times 8}{11} = 1.45\pi$

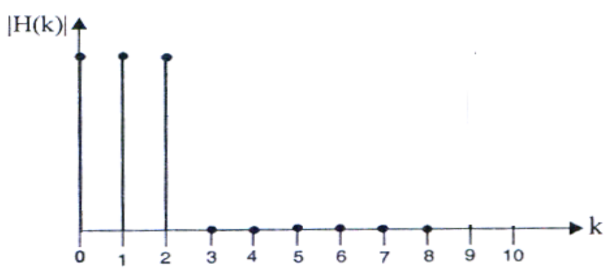
When $k = 9$; $\omega_k = \frac{2\pi \times 9}{11} = 1.64\pi$

When $k = 10$; $\omega_k = \frac{2\pi \times 10}{11} = 1.82\pi$

From the above calculations the following observations can be made.

For $k = 0$ to 2, the samples lie in the range $0 \le \omega \le 0.5\pi$

For $k = 3$ to 10, the samples lie in the range $0.5\pi < \omega$



Magnitude spectrum of H(k).

Based on the above discussions, the equation for DFT coefficients H(k) can be written as shown below.

$$H(k) = H_d(e^{j\omega})\Big|_{\omega = \omega_k} = e^{-j\alpha\omega_k} = e^{-j5\times\frac{2\pi k}{11}} \quad ; \text{ for } k = 0,\ 1,\ 2$$

$$= 0 \quad ; \text{ for } k = 3 \text{ to } 10$$

The samples of impulse response, h(n) are given by,

$$h(n) = \frac{1}{N}\left[H(0) + 2\sum_{k=1}^{\frac{N-1}{2}} \mathrm{Re}\left[H(k)\, e^{\frac{j2\pi nk}{N}}\right]\right]$$

$$= \frac{1}{11}\left[H(0) + 2\sum_{k=1}^{5} \mathrm{Re}\left[H(k)\, e^{\frac{j2\pi nk}{11}}\right]\right]$$

$$= \frac{1}{11}\left[1 + 2\sum_{k=1}^{2} \mathrm{Re}\left[e^{-j5\times\frac{2\pi k}{11}}\, e^{\frac{j2\pi nk}{11}}\right]\right]$$

$$= \frac{1}{11}\left[1 + 2\sum_{k=1}^{2} \mathrm{Re}\left[e^{\frac{j2\pi k}{11}(n-5)}\right]\right]$$

$$= \frac{1}{11}\left[1 + 2\,\mathrm{Re}\left[e^{\frac{j2\pi}{11}(n-5)}\right] + 2\,\mathrm{Re}\left[e^{\frac{j4\pi}{11}(n-5)}\right]\right]$$

$$= \frac{1}{11}\left[1 + 2\cos\frac{2\pi(n-5)}{11} + 2\cos\frac{4\pi(n-5)}{11}\right]$$

Here, $N = 11$, $\therefore N - 1 = 10$, $\frac{N-1}{2} = 5$.

Hence calculate h(n) for n = 0 to 10.

Since, h(n) satisfies the symmetry condition $h(N-1-n) = h(n)$ with centre of symmetry at $(N-1)/2$, calculate h(n) for n = 0 to to 5.

When $n = 0$; $h(0) = \frac{1}{11}\left[1 + 2\cos\frac{2\pi(0-5)}{11} + 2\cos\frac{4\pi(0-5)}{11}\right] = 0.0694$

When $n = 1$; $h(1) = \frac{1}{11}\left[1 + 2\cos\frac{2\pi(1-5)}{11} + 2\cos\frac{4\pi(1-5)}{11}\right] = -0.054$

When $n = 2$; $h(2) = \frac{1}{11}\left[1 + 2\cos\frac{2\pi(2-5)}{11} + 2\cos\frac{4\pi(2-5)}{11}\right] = -0.1094$

When $n = 3$; $h(3) = \frac{1}{11}\left[1 + 2\cos\frac{2\pi(3-5)}{11} + 2\cos\frac{4\pi(3-5)}{11}\right] = 0.0474$

When $n = 4$; $h(4) = \frac{1}{11}\left[1 + 2\cos\frac{2\pi(4-5)}{11} + 2\cos\frac{4\pi(4-5)}{11}\right] = 0.3194$

When $n = 5$; $h(5) = \frac{1}{11}\left[1 + 2\cos\frac{2\pi(5-5)}{11} + 2\cos\frac{4\pi(5-5)}{11}\right] = 0.4545$

When $n = 6$; $h(6) = h(11 - 1 - 6) = h(4) = 0.3194$

When $n = 7$; $h(7) = h(11 - 1 - 7) = h(3) = 0.0474$

When $n = 8$; $h(8) = h(11 - 1 - 8) = h(2) = -0.1094$

When $n = 9$; $h(9) = h(11 - 1 - 9) = h(1) = -0.054$

When $n = 10$; $h(10) = h(11 - 1 - 10) = h(0) = 0.0694$

The transfer function H(z) of the filter is given by $\mathcal{Z}$-transform of h(n).

$$\therefore\ H(z) = \mathcal{Z}\{h(n)\} = \sum_{n=0}^{N-1} h(n)\, z^{-n} = \sum_{n=0}^{10} h(n)\, z^{-n}$$

$$= h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(4)z^{-4} + h(5)z^{-5} + h(6)z^{-6} + h(7)z^{-7} + h(8)z^{-8} + h(9)z^{-9} + h(10)z^{-10}$$

$$= h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(4)z^{-4} + h(5)z^{-5} + h(4)z^{-6} + h(3)z^{-7} + h(2)z^{-8} + h(1)z^{-9} + h(0)z^{-10}$$

$$= h(0)\left[1 + z^{-10}\right] + h(1)\left[z^{-1} + z^{-9}\right] + h(2)\left[z^{-2} + z^{-8}\right] + h(3)\left[z^{-3} + z^{-7}\right] + h(4)\left[z^{-4} + z^{-6}\right] + h(5)z^{-5}$$

$$= 0.0694\left[1 + z^{-10}\right] - 0.054\left[z^{-1} + z^{-9}\right] - 0.1094\left[z^{-2} + z^{-8}\right] + 0.0474\left[z^{-3} + z^{-7}\right] + 0.3194\left[z^{-4} + z^{-6}\right] + 0.4545\,z^{-5}$$

where,

$$|H(e^{j\omega})| = h\left(\tfrac{N-1}{2}\right) + \sum_{n=1}^{\frac{N-1}{2}} 2h\left(\tfrac{N-1}{2} - n\right)\cos\omega n$$

$$= h(5) + \sum_{n=1}^{5} 2h(5-n)\cos\omega n$$

$$= h(5) + 2h(4)\cos\omega + 2h(3)\cos 2\omega + 2h(2)\cos 3\omega + 2h(1)\cos 4\omega + 2h(0)\cos 5\omega$$

$$= 0.4545 + 2 \times 0.3194\cos\omega + 2 \times 0.0474\cos 2\omega + 2 \times 0.1094\cos 3\omega + 2 \times -0.054\cos 4\omega + 2 \times 0.0694\cos 5\omega$$

$$= 0.4545 + 0.6388\cos\omega + 0.0948\cos 2\omega + 0.2188\cos 3\omega - 0.108\cos 4\omega + 0.1388\cos 5\omega$$