Assignment 3

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Assignment 1 consists of some background definitions, two sample problems, and two required problems. You must write your solutions to the required problems using LaTeX. Use the solutions of the sample problems as a guide.

Please submit Assignment 1 as two files, `Assignment_1_`*`YourMacID`*`.tex` and `Assignment_1_`*`YourMacID`*`.pdf`, to the Assignment 1 folder on Avenue under Assessments/Assignments. *`YourMacID`* must be your personal MacID (written without capitalization). The `Assignment_1_`*`YourMacID`*`.tex` file is a copy of the LaTeX source file for this assignment (`Assignment_1.tex` found on Avenue under Contents/Assignments) with your solution entered after each required problem. The `Assignment_1_`*`YourMacID`*`.pdf` is the PDF output produced by executing

```
pdflatex Assignment_1_YourMacID
```

This assignment is due **Sunday, February 14, 2021 before midnight.** You are allow to submit the assignment multiple times, but only the last submission will be marked. **Late submissions and files that are not named exactly as specified above will not be accepted!** It is suggested that you submit your preliminary `Assignment_1_`*`YourMacID`*`.tex` and `Assignment_1_`*`YourMacID`*`.pdf` files well before the deadline so that your mark is not zero if, e.g., your computer fails at 11:50 PM on February 14.

Although you are allowed to receive help from the instructional staff and other students, your submission must be your own work. Copying will be treated as academic dishonesty! If any of the ideas used in your submission were obtained from other students or sources outside of the lectures and tutorials, you must acknowledge where or from whom these ideas were obtained.

Background

A *word* over an alphabet Σ of symbols is a string

$$a_1 a_2 a_3 \cdots a_n$$

of symbols from Σ. For example, if $\Sigma = \{a, b, c\}$, then the following are words over Σ among many others:

- $cbaca$.
- ba.
- $acbbca$.
- a
- ϵ (which denotes the empty word).

Let Σ^* be the set of all words over Σ (which includes ϵ, the empty word). Associated with each word $w \in \Sigma^*$ is a set of positions. For example, $\{1, 2, 3\}$ is set of positions of the word abc with the symbol a occupying position 1, b occupying position 2, and c occupying position 3. If $u, v \in \Sigma^*$, uv is the word in Σ^* that results from concatenating u and v. For example, if $u = aba$ and $v = bba$, then $uv = ababba$.

A *language* L over Σ is a subset of Σ^*. A language can be specified by a first-order formula in which the quantifiers range over the set of positions in a word. In order to write such formulas we need some predicates on positions. $\mathsf{last}(x)$ asserts that position x is the last position in a word. For $a \in \Sigma$, $a(x)$ asserts that the symbol a occupies position x. For example, the formula

$$\forall x \,.\, \mathsf{last}(x) \rightarrow a(x)$$

says the symbol a occupies the last position of a word. This formula is true, e.g., for the words aaa, a, and bca.

The language over Σ specified by a formula is the set of words in Σ^* for which the formula is true. For example, if $a \in \Sigma$, then $\forall x \,.\, \mathsf{last}(x) \rightarrow a(x)$ specifies the language $\{ua \mid u \in \Sigma^*\}$.

Problems

1. **[12 points]** Let Σ be a finite alphabet and Σ^* be the set of words over Σ. Define $u \leq v$ to be mean there are $x, y \in \Sigma^*$ such that $xuy = v$. That is, $u \leq v$ holds iff u is a subword of v.

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   a. Prove that $(\Sigma^*, \leq)$ is a weak partial order.

   Proof. To prove that $(\Sigma^*, \leq)$ is a weak partial order, we must prove it is reflexive, antisymmetric, and transitive.

   Reflexive: $\forall\, u \in \Sigma^* \,.\, \exists x, y \in \Sigma^* \,.\, u \leq u$

   For all $u \in \Sigma^*$:

   $$\begin{aligned}
   &\exists x, y \in \Sigma^* \,.\, u \leq u \\
   &= \exists x, y \in \Sigma^* \,.\, u \leq xuy && \langle \text{def. of a subword} \rangle \\
   &= u \leq xuy && \langle \exists - \textit{introduction}; \text{ x,y = x,y} \rangle \\
   &= true && \langle \text{def. of a subword} \rangle
   \end{aligned}$$

   So reflexivity of $(\Sigma^*, \leq)$ holds.

   Antisymmetric: $\forall\, u, v \in \Sigma^* \,.\, \exists x, y \in \Sigma^* \,.\, u \leq v \wedge v \leq u \implies u = v$

   For all $u, v \in \Sigma^*$:

   $$\begin{aligned}
   &\exists x, y \in \Sigma^* \,.\, u \leq v \wedge v \leq u \implies u = v \\
   &= \exists x, y \in \Sigma^* \,.\, u \leq xuy \wedge xuy \leq u \implies u = xuy && \langle \text{def. of a subword; v=xuy} \rangle \\
   &= u \leq \epsilon u \epsilon \wedge \epsilon u \epsilon \leq u \implies u = \epsilon u \epsilon && \langle \exists - \textit{introduction}; \text{ x,y} = \epsilon, \epsilon \rangle \\
   &= u \leq u \wedge u \leq u \implies u = u && \langle \text{empty word} \rangle \\
   &= true \wedge true \implies true && \langle \text{def. of a subword} \rangle \\
   &= true
   \end{aligned}$$

So antisymmetry of $(\Sigma^*, \leq)$ holds.

Transitive: $\forall\, u, v, w \in \Sigma^* \,.\, \exists a, b, c, d \in \Sigma^* \,.\, u \leq v \wedge v \leq w \implies u = w$

For all $u, v, w \in \Sigma^*$:

$$\begin{aligned}
&\exists a, b, c, d \in \Sigma^* \,.\, u \leq v \wedge v \leq w \implies u \leq w \\
&= \exists a, b, c, d \in \Sigma^* \,.\, u \leq v \wedge v \leq avb \implies u \leq avb && \langle \text{def. of a subword; w=avb} \rangle \\
&= \exists a, b, c, d \in \Sigma^* \,.\, u \leq cud \wedge cud \leq acudb \implies u \leq acudb && \langle \text{def. of a subword; v=cud} \rangle \\
&= u \leq cud \wedge cud \leq acudb \implies u \leq acudb && \langle \exists - introduction; \text{a,b,c,d = a,b,c,d} \rangle \\
&= true \wedge true \implies true && \langle \text{def. of a subword} \rangle \\
&= true
\end{aligned}$$

So transitivity of $(\Sigma^*, \leq)$ holds.

Therefore, $(\Sigma^*, \leq)$ is a weak partial order.

b. Prove that $(\Sigma^*, \leq)$ is not a weak total order.
Proof. To prove that $(\Sigma^*, \leq)$ is not a weak total order, we must prove it is not total.

Total: $\forall\, u \in \Sigma^* \,.\, \exists v \in \Sigma^* \,.\, u \leq v \vee v \leq u$

For all $u \in \Sigma^*$:

$$\begin{aligned}
&\exists v \in \Sigma^* \,.\, u \leq v \vee v \leq u \\
&= u \leq abc \vee abc \leq u && \langle \exists - introduction; \text{v=abc} \rangle \\
&= false \vee false && \langle \text{def. of a subword} \rangle \\
&= false
\end{aligned}$$

So totality of $(\Sigma^*, \leq)$ does not hold by proof by contradiction. Therefore, $(\Sigma^*, \leq)$ is not a weak total order.

c. Does $(\Sigma^*, \leq)$ have a minimum element? Justify your answer.
A minimum element means that element is a subset of every other element in the set. ϵ is the minimum element, assuming that since ϵ denotes an empty word, ab, $ab\epsilon$, ϵab, and $a\epsilon b$ are all equal words. Therefore, every word in Σ^* contains an empty would, or ϵ, which is why it is the minimum element.

d. Does $(\Sigma^*, \leq)$ have a maximum element? Justify your answer.
No, it does not have a maximum element. This is because the set of words Σ^* is infinite. Since there is no limit on the length of the words, there are infinite combinations of words, meaning an infinite number of words in Σ^*. A maximum element means that all the other elements in the set are a subset of that element. Since the set is infinite, there cannot be a maximum element that contains all other elements. There cannot be an infinite element that contains infinite elements.

2. **[8 points]** Let $\Sigma = \{a, b, c\}$ be a finite alphabet. Construct formulas that specify the following languages over Σ.

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   Additional predicates:

   - $\mathsf{between}(x, y) = z$ where $x, y, z \in \Sigma^*$ states that there is a word z between the words x and y. The word z begins immediately after the end of the word x, and ends right before the beginning of the word y. For example, $\mathsf{between}(ab, cd) = xyz$ would be describing $abxyzcd$.
   - $\mathsf{length}(x) = n$ where $x \in \Sigma^*$ and $n \in \mathbb{N}$ states the length of a word. For example, $\mathsf{length}(abc) = 3$ would be true.
   - The following is assuming the first position of the word is 1.

   a. $\{awa \mid w \in \Sigma^*\}$.
   $\forall x \,.\, a(1) \wedge \mathsf{last}(x) \to a(x)$

   b. $\{dwd \mid d \in \Sigma \text{ and } w \in \Sigma^*\}$.
   $\forall x \,.\, (a(1) \wedge \mathsf{last}(x) \to a(x)) \vee (b(1) \wedge \mathsf{last}(x) \to b(x)) \vee (c(1) \wedge \mathsf{last}(x) \to c(x))$

   c. $\{uaav \mid u, v \in \Sigma^*\}$.
   $\forall u, v \in \Sigma^* \,.\, \mathsf{between}(u, v) = aa$

   d. $\{uavbw \mid u, v, w \in \Sigma^*\}$.
   $\forall u, v, w \in \Sigma^* \,.\, \mathsf{between}(u, v) = a \wedge \mathsf{between}(v, w) = b$

   e. Σ^*.
   $\forall w \in \Sigma^* \,.\, w$

   f. $\Sigma^* \setminus \{\epsilon\}$.
   $\forall w \in \Sigma^* \,.\, \neg(\mathsf{length}(w) = 1 \wedge \epsilon(1))$

   g. $\{\epsilon\}$.
   $\forall w \in \Sigma^* \,.\, \mathsf{length}(w) = 1 \wedge \epsilon(1)$

   h. $\emptyset$.
   $\forall w \in \Sigma^* \,.\, \neg w$