COMPSCI/SFWRENG 2FA3
Discrete Mathematics with Applications II
Winter 2021

Assignment 1

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Assignment 1 consists of some background definitions, two sample problems, and two required problems. You must write your solutions to the required problems using LaTeX. Use the solutions of the sample problems as a guide.

Please submit Assignment 1 as two files, `Assignment_1_`*`YourMacID`*`.tex` and `Assignment_1_`*`YourMacID`*`.pdf`, to the Assignment 1 folder on Avenue under Assessments/Assignments. *`YourMacID`* must be your personal MacID (written without capitalization). The `Assignment_1_`*`YourMacID`*`.tex` file is a copy of the LaTeX source file for this assignment (`Assignment_1.tex` found on Avenue under Contents/Assignments) with your solution entered after each required problem. The `Assignment_1_`*`YourMacID`*`.pdf` is the PDF output produced by executing

```
pdflatex Assignment_1_YourMacID
```

This assignment is due **Sunday, January 31, 2021 before midnight.** You are allow to submit the assignment multiple times, but only the last submission will be marked. **Late submissions and files that are not named exactly as specified above will not be accepted!** It is suggested that you submit your preliminary `Assignment_1_`*`YourMacID`*`.tex` and `Assignment_1_`*`YourMacID`*`.pdf` files well before the deadline so that your mark is not zero if, e.g., your computer fails at 11:50 PM on January 31.

Although you are allowed to receive help from the instructional staff and other students, your submission must be your own work. Copying will be treated as academic dishonesty! If any of the ideas used in your submission were obtained from other students or sources outside of the lectures and tutorials, you must acknowledge where or from whom these ideas were obtained.

Background

1. The notation $\sum_{i=m}^{n} f(i)$ is defined by:

$$\sum_{i=m}^{n} f(i) = \begin{cases} 0 & \text{if } m > n \\ \left(\sum_{i=m}^{n-1} f(i)\right) + f(n) & \text{if } m \leq n \end{cases}$$

2. The notation $\prod_{i=m}^{n} f(i)$ is defined by:

$$\prod_{i=m}^{n} f(i) = \begin{cases} 1 & \text{if } m > n \\ \left(\prod_{i=m}^{n-1} f(i)\right) * f(n) & \text{if } m \leq n \end{cases}$$

3. The factorial function $\mathsf{fact} : \mathbb{N} \to \mathbb{N}$ is defined by:

$$\mathsf{fact}(n) = \begin{cases} 1 & \text{if } n = 0 \\ \mathsf{fact}(n-1) * n & \text{if } n > 0 \end{cases}$$

4. The Fibonacci sequence $\mathsf{fib} : \mathbb{N} \to \mathbb{N}$ is defined by:

$$\mathsf{fib}(n) = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ \mathsf{fib}(n-1) + \mathsf{fib}(n-2) & \text{if } n \geq 2 \end{cases}$$

Sample Problems

1. Prove $\sum_{i=0}^{n-1} 2^i = 2^n - 1$ for all $n \in \mathbb{N}$.

   Proof. Let $P(n) \equiv \sum_{i=0}^{n-1} 2^i = 2^n - 1$. We will prove $P(n)$ for all $n \in \mathbb{N}$ by weak induction.

   Base case: $n = 0$. We must show $P(0)$.

$$\begin{aligned} & \sum_{i=0}^{0-1} 2^i && \langle \text{LHS of } P(0) \rangle \\ = & \sum_{i=0}^{-1} 2^i && \langle \text{arithmetic} \rangle \\ = & \ 0 && \langle \text{definition of } \textstyle\sum_{i=m}^{n} f(i) \text{ when } m > n \rangle \\ = & \ 1 - 1 && \langle \text{arithmetic} \rangle \\ = & \ 2^0 - 1 && \langle \text{arithmetic; RHS of } P(0) \rangle \end{aligned}$$

   So $P(0)$ holds.

Induction step: $n \geq 0$. Assume $P(n)$. We must show $P(n+1)$.

$$\begin{aligned}
& \sum_{i=0}^{(n+1)-1} 2^i && \langle \text{LHS of } P(n+1) \rangle \\
&= \sum_{i=0}^{n} 2^i && \langle \text{arithmetic} \rangle \\
&= \left(\textstyle\sum_{i=0}^{n-1} 2^i\right) + 2^n && \langle \text{definition of } \textstyle\sum_{i=m}^{n} f(i) \rangle \\
&= (2^n - 1) + 2^n && \langle \text{induction hypothesis: } P(n) \rangle \\
&= 2 * 2^n - 1 && \langle \text{arithmetic} \rangle \\
&= 2^{n+1} - 1 && \langle \text{arithmetic; RHS of } P(n+1) \rangle
\end{aligned}$$

So $P(n+1)$ holds.

Therefore, $P(n)$ holds for all $n \in \mathbb{N}$ by weak induction. □

2. Prove that, if $n \in \mathbb{N}$ with $n \geq 2$, then n is a prime number or a product of prime numbers.

Proof. Let $P(n)$ hold iff n is a product of prime numbers. We will prove $P(n)$ for all $n \in \mathbb{N}$ with $n \geq 2$ by strong induction.

Base case: $n = 2$. We must show $P(2)$. Since 2 is a prime number, $P(2)$ obviously holds.

Induction step: $n > 2$. Assume $P(2), P(3), \ldots, P(n-1)$ hold. We must show $P(n)$.

Case 1: n is a prime number. Then $P(n)$ obviously holds.

Case 2: n is not a prime number. Then $n = x * y$ where $x, y \in \mathbb{N}$ with $2 \leq x, y \leq n-1$. Thus, by the induction hypothesis ($P(x)$ and $P(y)$),

$$x = p_0 * \cdots * p_i$$

and

$$y = q_0 * \cdots * q_j$$

where $p_0, \ldots, p_i, q_0, \ldots, q_j$ are prime numbers. Then

$$n = x * y = p_0 * \cdots * p_i * q_0 * \cdots * q_j$$

and so $P(n)$ holds since n is a product of prime numbers.

Therefore, $P(n)$ holds for all $n \in \mathbb{N}$ with $n \geq 2$ by strong induction. □

Required Problems

1. **[10 points]** Prove

$$\sum_{i=0}^{n} i * \mathsf{fact}(i) = \mathsf{fact}(n+1) - 1$$

for all $n \in \mathbb{N}$.

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Proof. Let $P(n) \equiv \sum_{i=0}^{n} i * \mathsf{fact}(i) = \mathsf{fact}(n+1) - 1$. We will prove $P(n)$ for all $n \in \mathbb{N}$ by weak induction.

Base case: $n = 0$. We must show $P(0)$.

$$\begin{aligned}
&\sum_{i=0}^{0} i * \mathsf{fact}(i) && \langle \text{LHS of } P(0) \rangle \\
&= 0 * \mathsf{fact}(0) + \sum_{i=0}^{0-1} i * \mathsf{fact}(i) && \langle \text{def. of } \textstyle\sum_{i=m}^{n} f(i) \text{ when } m \leq n \rangle \\
&= 0 * \mathsf{fact}(0) + \sum_{i=0}^{-1} i * \mathsf{fact}(i) && \langle \text{arithmetic} \rangle \\
&= 0 * \mathsf{fact}(0) + 0 && \langle \text{def. of } \textstyle\sum_{i=m}^{n} f(i) \text{ when } m > n \rangle \\
&= 0 && \langle \text{arithmetic} \rangle \\
&= 1 - 1 && \langle \text{arithmetic} \rangle \\
&= \mathsf{fact}(0) - 1 && \langle \text{def. of factorial when n=0} \rangle \\
&= \mathsf{fact}(0) * 1 - 1 && \langle \text{arithmetic} \rangle \\
&= \mathsf{fact}(1-1) * 1 - 1 && \langle \text{arithmetic} \rangle \\
&= \mathsf{fact}(1) - 1 && \langle \text{def. of factorial when } n > 0 \rangle \\
&= \mathsf{fact}(0+1) - 1 && \langle \text{arithmetic; RHS of } P(0) \rangle
\end{aligned}$$

So $P(0)$ holds.

Induction step: $n \geq 0$. Assume $P(n)$. We must show $P(n+1)$.

$$
\begin{aligned}
& \sum_{i=0}^{n+1} i * \mathsf{fact}(i) && \langle \text{LHS of } P(n+1) \rangle \\
&= (n+1) * \mathsf{fact}(n+1) + \sum_{i=0}^{(n+1)-1} i * \mathsf{fact}(i) && \langle \text{def. of } \textstyle\sum_{i=m}^{n} f(i) \text{ when } m \leq n \rangle \\
&= (n+1) * \mathsf{fact}(n+1) + \sum_{i=0}^{n} i * \mathsf{fact}(i) && \langle \text{arithmetic} \rangle \\
&= (n+1) * \mathsf{fact}(n+1) + \mathsf{fact}(n+1) - 1 && \langle \text{induction hypothesis: } P(n) \rangle \\
&= n * \mathsf{fact}(n+1) + \mathsf{fact}(n+1) + \mathsf{fact}(n+1) - 1 && \langle \text{arithmetic} \rangle \\
&= n * \mathsf{fact}(n+1) + 2\mathsf{fact}(n+1) - 1 && \langle \text{arithmetic} \rangle \\
&= (n+2) * \mathsf{fact}(n+1) - 1 && \langle \text{arithmetic} \rangle \\
&= (n+2) * \mathsf{fact}((n+2)-1) - 1 && \langle \text{arithmetic} \rangle \\
&= \mathsf{fact}(n+2) - 1 && \langle \text{def. of factorial when } n > 0 \rangle \\
&= \mathsf{fact}((n+1)+1) - 1 && \langle \text{arithmetic; RHS of } P(n+1) \rangle
\end{aligned}
$$

So $P(n+1)$ holds.
Therefore, $P(n)$ holds for all $n \in \mathbb{N}$ by weak induction.

2. **[10 points]** Prove that, for all $n \in \mathbb{N}$, $\mathsf{fib}(n)$ is even if $n = 3k$ for some $k \in \mathbb{N}$, is odd if $n = 3k+1$ for some $k \in \mathbb{N}$, and is odd if $n = 3k+2$ for some $k \in \mathbb{N}$.

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   Lemma 1. Proving that an even number + an even number results in an even number.
   We know that anything multiplied by 2 is even, so let $x = 2i$ and $y = 2j$.

$$
\begin{aligned}
& x + y \\
&= (2i) + (2j) \\
&= 2(i + j)
\end{aligned}
$$

   Since the result is also multiplied by 2, we know that adding 2 even numbers results in an even number.

   Lemma 2. Proving that an odd number + an odd number results in an even number.
   We know that anything multiplied by 2 is even, and adding 1 to an

even number results in an odd number, so let $x = 2i+1$ and $y = 2j+1$.

$$\begin{aligned}
&x + y \\
&= (2i + 1) + (2j + 1) \\
&= 2i + 2j + 2 \\
&= 2(i + j + 1)
\end{aligned}$$

Since the result is also multiplied by 2, we know that adding 2 odd numbers results in an even number.

Lemma 3. Proving that an odd number + an even number results in an odd number.
We know that anything multiplied by 2 is even, and adding 1 to an even number results in an odd number, so let $x = 2i$ and $y = 2j + 1$.

$$\begin{aligned}
&x + y \\
&= (2i) + (2j + 1) \\
&= (2i + 2j) + 1 \\
&= 2(i + j) + 1
\end{aligned}$$

Since the result is a number multiplied by 2, or an even number, and 1 is added to that even number, we know that an odd number + an even number results in an odd number.

Proof for Q2. Let $P(n) \equiv \mathsf{fib}(n)$. Let $P(n)$ hold iff $\mathsf{fib}(n)$ is even when $n = 3k$ for some $k \in \mathbb{N}$, is odd when $n = 3k + 1$ for some $k \in \mathbb{N}$, and is odd when $n = 3k + 2$ for some $k \in \mathbb{N}$. We will prove $P(n)$ for all $n \in \mathbb{N}$ using strong induction.

Base case (1): We must show $P(n)$, where $n = 3k$, is even when $k = 0$.

$$\begin{aligned}
&\mathsf{fib}(3k) && \langle\text{LHS of } P(3k)\rangle \\
&= \mathsf{fib}(3 * 0) && \langle\text{sub k=0}\rangle \\
&= \mathsf{fib}(0) && \langle\text{arithmetic}\rangle \\
&= 0 && \langle\text{def. of fib. when } n = 0\rangle \\
&= \mathit{even} && \langle\text{zero is even; RHS of } P(3k)\rangle
\end{aligned}$$

So $P(3k)$ holds.

Base case (2): We must show $P(n)$, where $n = 3k + 1$, is odd when $k = 0$.

$$\begin{aligned}
&\mathsf{fib}(3k+1) && \langle \text{LHS of } P(3k+1) \rangle \\
&= \mathsf{fib}(3*0+1) && \langle \text{sub k=0} \rangle \\
&= \mathsf{fib}(1) && \langle \text{arithmetic} \rangle \\
&= 1 && \langle \text{def. of fib. when } n = 1 \rangle \\
&= \mathit{odd} && \langle \text{one is odd; RHS of } P(3k+1) \rangle
\end{aligned}$$

So $P(3k + 1)$ holds.

Base case (3): We must show $P(n)$, where $n = 3k + 2$, is odd when $k = 0$.

$$\begin{aligned}
&\mathsf{fib}(3k+2) && \langle \text{LHS of } P(3k+2) \rangle \\
&= \mathsf{fib}(3*0+2) && \langle \text{sub k=0} \rangle \\
&= \mathsf{fib}(2) && \langle \text{arithmetic} \rangle \\
&= \mathsf{fib}(2-1) + \mathsf{fib}(2-2) && \langle \text{def. of fib. when } n \geq 2 \rangle \\
&= \mathsf{fib}(1) + \mathsf{fib}(0) && \langle \text{arithmetic} \rangle \\
&= 1 + 0 && \langle \text{def. of fib. when } n = 0, n = 1 \rangle \\
&= 1 && \langle \text{arithmetic} \rangle \\
&= \mathit{odd} && \langle \text{one is odd; RHS of } P(3k+2) \rangle
\end{aligned}$$

So $P(3k + 2)$ holds.

Induction step (1): $n \geq 0$. Assume $P(n)$ holds for all $m \leq n$. We must show $P(n)$ holds for $n = 3k$. Assume $3k - 1$ and $3k - 2$ ($3k - 1 < 3k$ and $3k - 2 < 3k$).

$$\begin{aligned}
&\mathsf{fib}(3k) && \langle \text{LHS of } P(3k) \rangle \\
&= \mathsf{fib}(3k-1) + \mathsf{fib}(3k-2) && \langle \text{def. of fib. when } n \geq 2 \rangle \\
&= \mathit{odd} + \mathit{odd} && \langle \text{assumption } P(3k-1) \text{ and } P(3k-2) \rangle \\
&= \mathit{even} && \langle \text{lemma 2; RHS of } P(3k) \rangle
\end{aligned}$$

So $P(3k)$ holds.

Induction step (2): $n \geq 0$. Assume $P(n)$ holds for all $m \leq n$. We must show $P(n)$ holds for $n = 3k + 1$. Assume $3k$ and $3k - 1$ ($3k < 3k + 1$ and $3k - 1 < 3k + 1$).

$$
\begin{aligned}
&\mathsf{fib}(3k+1) && \langle \text{LHS of } P(3k+1)\rangle \\
&= \mathsf{fib}((3k+1)-1) + \mathsf{fib}((3k+1)-2) && \langle \text{def. of fib. when } n \geq 2\rangle \\
&= \mathsf{fib}(3k) + \mathsf{fib}(3k-1) && \langle \text{arithmetic}\rangle \\
&= \mathit{even} + \mathit{odd} && \langle \text{assumption } P(3k) \text{ and } P(3k-1)\rangle \\
&= \mathit{odd} && \langle \text{lemma 3; RHS of } P(3k+1)\rangle
\end{aligned}
$$

So $P(3k+1)$ holds.

Induction step (3): $n \geq 0$. Assume $P(n)$ holds for all $m \leq n$. We must show $P(n)$ holds for $n = 3k+2$. Assume $3k$ and $3k+1$ ($3k < 3k+2$ and $3k+1 < 3k+2$).

$$
\begin{aligned}
&\mathsf{fib}(3k+2) && \langle \text{LHS of } P(3k+2)\rangle \\
&= \mathsf{fib}((3k+2)-1) + \mathsf{fib}((3k+2)-2) && \langle \text{def. of fib. when } n \geq 2\rangle \\
&= \mathsf{fib}(3k+1) + \mathsf{fib}(3k) && \langle \text{arithmetic}\rangle \\
&= \mathit{odd} + \mathit{even} && \langle \text{assumption } P(3k) \text{ and } P(3k+1)\rangle \\
&= \mathit{odd} && \langle \text{lemma 3; RHS of } P(3k+2)\rangle
\end{aligned}
$$

So $P(3k+2)$ holds.

Therefore, $P(n)$ holds for all $n \in \mathbb{N}$ by strong induction.