COMPSCI/SFWRENG 2FA3
Discrete Mathematics with Applications II
Winter 2021

Assignment 4

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Assignment 4 consists of two problems. You must write your solutions to the problems using LaTeX.

Please submit Assignment 4 as two files, `Assignment_4_`*`YourMacID`*`.tex` and `Assignment_4_`*`YourMacID`*`.pdf`, to the Assignment 4 folder on Avenue under Assessments/Assignments. *`YourMacID`* must be your personal MacID (written without capitalization). The `Assignment_4_`*`YourMacID`*`.tex` file is a copy of the LaTeX source file for this assignment (`Assignment_4.tex` found on Avenue under Contents/Assignments) with your solution entered after each problem. The `Assignment_4_`*`YourMacID`*`.pdf` is the PDF output produced by executing

```
pdflatex Assignment_4_YourMacID
```

This assignment is due **Sunday, February 28, 2021 before midnight.** You are allow to submit the assignment multiple times, but only the last submission will be marked. **Late submissions and files that are not named exactly as specified above will not be accepted!** It is suggested that you submit your preliminary `Assignment_4_`*`YourMacID`*`.tex` and `Assignment_4_`*`YourMacID`*`.pdf` files well before the deadline so that your mark is not zero if, e.g., your computer fails at 11:50 PM on February 28.

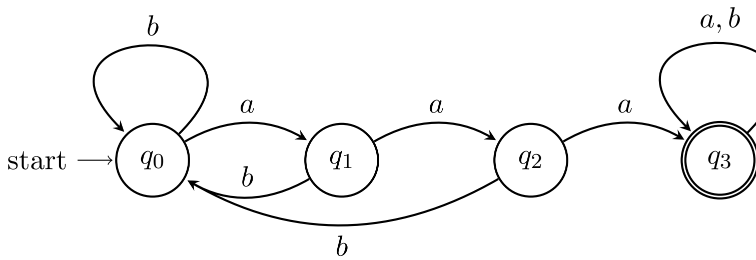
Although you are allowed to receive help from the instructional staff and other students, your submission must be your own work. Copying will be treated as academic dishonesty! If any of the ideas used in your submission were obtained from other students or sources outside of the lectures and tutorials, you must acknowledge where or from whom these ideas were obtained.

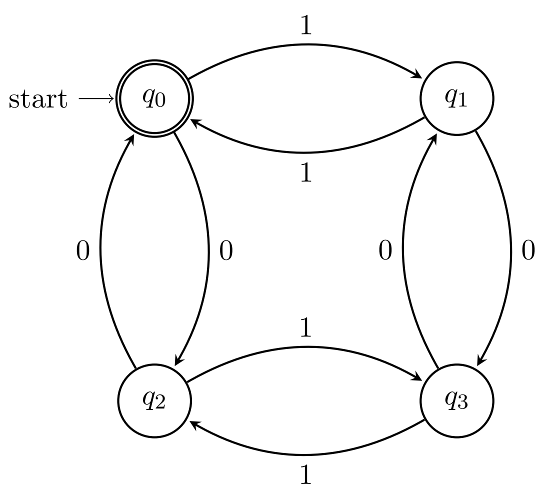
Presenting DFAs and NFAs Transition Diagrams

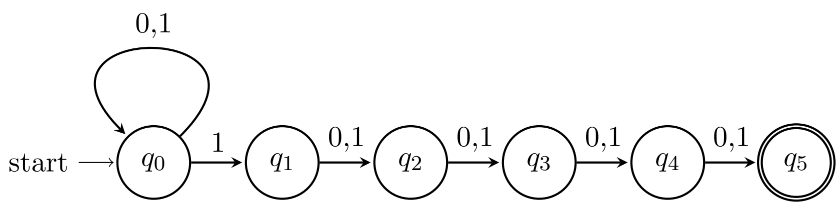
In this assignment you are asked to present DFAs as transition diagrams. You are can do this in one of two ways.

The first way is to present the diagram using the LaTeX graphics package TikZ. The TikZ code can either be written by hand or automatically generated using the finsm system available at `http:finsm.io`.

Here are some examples of how it can be used to create DFA and NFA transition diagrams that appear in the lectures slides:







The second way is to take a picture of a hand-written transition diagram and then embed it into your assignment using the following LaTeX code:

```
\begin{center}
\includegraphics[scale = 0.5]{diagram.jpg}
\end{center}
```

Please make sure your diagram is legible.

Problems

1. **[10 points]** Construct a deterministic finite automaton (DFA) A for the alphabet $\Sigma = \{0, 1\}$ such that $L(A)$ is the set of all strings x in Σ^* for which $\#0(x)$ is even and $\#1(x)$ is divisible by 3. Present A as a transition diagram.

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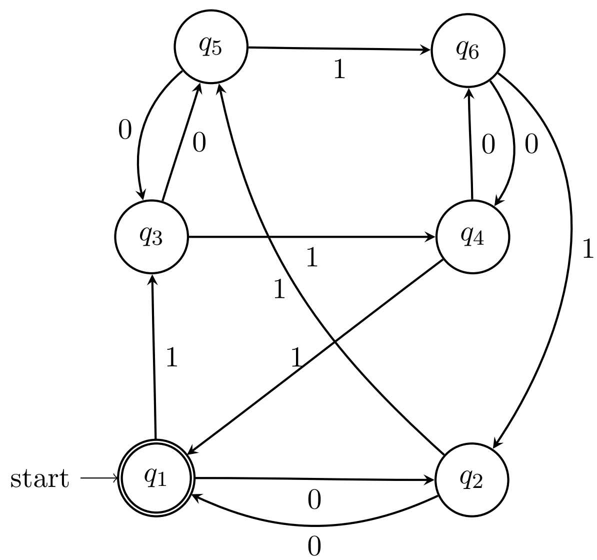
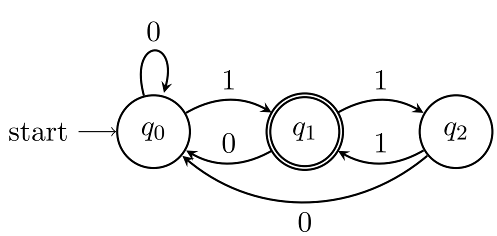


$letM = (Q, \Sigma, \delta, s, F)$
$where$
$Q = \{q_1, q_2, q_3, q_4, q_5, q_6\}$
$\Sigma = \{0, 1\}$
$\delta = (above)$
$s = q_1$
$F = \{q_1\}$

2. **[10 points]** Let B be the DFA given by the following transition diagram:



Prove that $L(B)$ is the language of all binary strings that end with an odd number of 1s. Hint: Use weak induction on the length of the input string, and let $P(n)$ be the statement that, for all input strings w with $|w| = n$, the following conditions hold:

a. If $\delta^*(q_0, w) = q_0$, then w is ϵ or ends with 0.

b. If $\delta^*(q_0, w) = q_1$, then w ends with an odd number of 1s.

c. If $\delta^*(q_0, w) = q_2$, then w ends with an even number of 1s.

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Proof. Let $L(B) = \{w \in \{0,1\}^* | w \text{ ends with an odd number of } 1s\}$. Then I need to show that $w \in \Sigma^*$ is $w \in L(B)$ if it ends in an odd number of 1s, or $w \notin L(B)$ if it does not end in an odd number of 1s.

$P_1 \equiv$ if $\delta^*(q_0, w) = q_0 \implies w = \epsilon \lor w$ ends in a 0 $(w \notin L(B))$
$P_2 \equiv$ if $\delta^*(q_0, w) = q_1 \implies w$ ends in an odd number of 1s $(w \in L(B))$
$P_3 \equiv$ if $\delta^*(q_0, w) = q_2 \implies w$ ends in an even number of 1s $(w \notin L(B))$

Let $|w| = n$ be the length of our input string w. We will prove P_1, P_2, and P_3 hold by weak induction on the length of our input string w. $P(n)$ holds if P_1, p_2, and P_3 hold.

Base Case (P_1) $|w| = n = 0$, $w = \epsilon$

$$\begin{aligned} & \delta(q_0, w) && \langle \text{LHS of } P_1 \rangle \\ = {} & \delta(q_0, \epsilon) && \\ = {} & q_0 && \langle \text{def. of } \delta^* \rangle \end{aligned}$$

This is trivially obvious by the definition of δ^* since $w = \epsilon$.
So, P_1 holds for $w = \epsilon$.

Base Case (P_2) $|w| = n = 0$, $w = \epsilon$

$$\begin{aligned} & \delta(q_0, w) && \langle \text{LHS of } P_2 \rangle \\ = {} & \delta(q_0, \epsilon) && \\ = {} & q_0 && \langle \text{def. of } \delta^* \rangle \\ \equiv {} & False && \langle q_0 \neq q_1 \rangle \end{aligned}$$

Since the LHS of the implication is false (i.e., ϵ does not end with an odd number of 1s; $False \implies x \equiv True$), and $q_0 \neq q_1$, we can conclude P_2 is true and therefore holds for $w = \epsilon$.

Base Case (P_3) $|w| = n = 0$, $w = \epsilon$

$$
\begin{aligned}
& \delta(q_0, w) && \langle \text{LHS of } P_3 \rangle \\
&= \delta(q_0, \epsilon) && \\
&= q_0 && \langle \text{def. of } \delta^* \rangle \\
&\equiv False && \langle q_0 \neq q_2 \rangle
\end{aligned}
$$

Since the LHS of the implication is false (i.e., ϵ does not end with an even number of 1s; $False \implies x \equiv True$), and $q_0 \neq q_2$, we can conclude P_3 is true and therefore holds for $w = \epsilon$.

Induction Step: Let $n = |w| > 0$. Let $w = xi$, where $x \in \Sigma^*$ and $i \in \Sigma$. Let $n = |x|$. Assume $P(n)$ holds. $|w| = n+1$, so prove $P(n+1)$ holds.

Induction Step (P_1):

$$
\begin{aligned}
& \delta^*(q_0, w) && \langle \text{LHS of } P_1 \rangle \\
&= \delta^*(q_0, xi) && \langle \text{let } w = xi \rangle \\
&= \delta(\delta^*(q_0, x), i) && \langle \text{def. of } \delta^* \rangle \\
&= q_0 && \langle \text{def. of } \delta^*\text{; RHS} \rangle
\end{aligned}
$$

For all $\delta(q, 0)$ for $q \in Q$ in B. (According to DFA B, every state transitions to state q_0 using the input 0, and no state transitions to q_0 using the input 1.) Therefore, i must be equal to 0 ($i = 0$). This means that our string $w = xi = x0$ ends in a zero. Therefore, P_1 holds.

Induction Step (P_2):

$$
\begin{aligned}
& \delta^*(q_0, w) && \langle \text{LHS of } P_2 \rangle \\
&= \delta^*(q_0, xi) && \langle \text{let } w = xi \rangle \\
&= \delta(\delta^*(q_0, x), i) && \langle \text{def. of } \delta^* \rangle \\
&= q_1 && \langle \text{def. of } \delta^*\text{; RHS} \rangle
\end{aligned}
$$

We know from DFA B that we have $\delta(q_0, 1) = q_1$ and $\delta(q_2, 1) = q_1$. There are two transitions that take us to q_1 using a 1. Therefore, $i = 1$ in both cases.

Case 1 $\delta(q_0, 1) = q_1$: $i = 1$ and $\delta^*(q_0, x) = q_0$. We can use our induction hypothesis, where $P(n+1)$ says that P_1 holds. Then $x = y0$ where $y \in \Sigma^*$. $w = xi = y01$, which ends in a single 1, which is an odd number of 1s.

Case 2 $\delta(q_2, 1) = q_1$: $i = 1$ and $\delta^*(q_0, x) = q_2$. We can use our induction hypothesis, where $P(n+1)$ says that P_3 holds. Then $x = y11$ where $y \in \Sigma^*$. $w = xi = y111$, which ends in three 1s, which is an odd number of 1s.

Therefore, P_2 holds.

Induction Step (P_3):

$$
\begin{aligned}
&\delta^*(q_0, w) && \langle \text{LHS of } P_3 \rangle \\
&= \delta^*(q_0, xi) && \langle \text{let } w = xi \rangle \\
&= \delta(\delta^*(q_0, x), i) && \langle \text{def. of } \delta^* \rangle \\
&= q_2 && \langle \text{def. of } \delta^*\text{; RHS} \rangle
\end{aligned}
$$

We know from DFA B that we have $\delta(q_1, 1) = q_2$, which means that $i = 1$. $\delta^*(q_0, x) = q_1$. We can use our induction hypothesis, where $P(n+1)$ says that P_2 holds. Then $x = y1$ where $y \in \Sigma^*$. $w = xi = y11$, which ends in a two 1s, which is an even number of 1s.

Therefore, P_3 holds.

P_1, P_2, and P_3 all hold. Therefore, $P(n)$ holds for all n by weak induction.