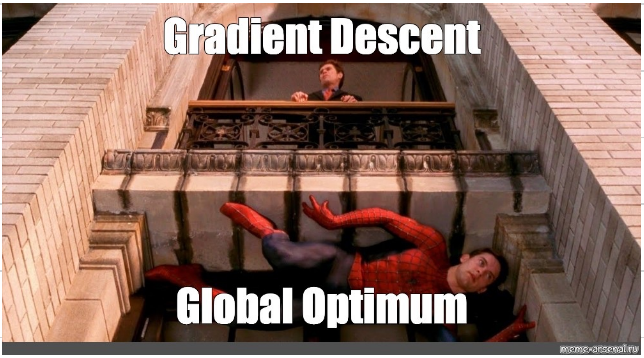
Session 9

OPTIMIZATION-4 CONSTRAINED OPTIMIZATION

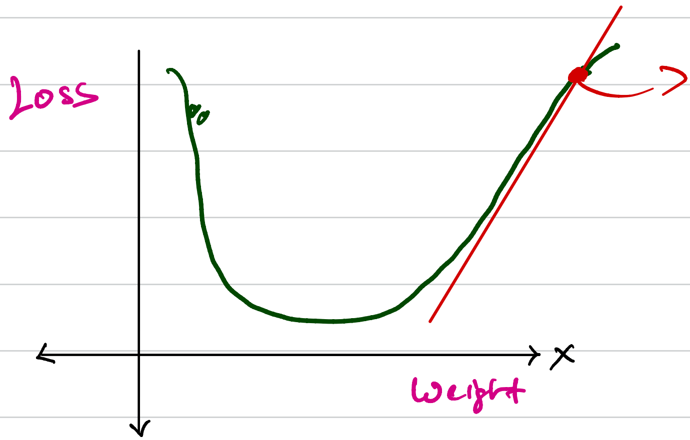
Feb 15, 2024



AGENDA

1. Optimization Problem
2. How To Solve Constraint Optimization Problem

Gradient Descent



Rate at which loss is INCREASING w.r.t. the parameter at that point

$$\boxed{w_{new} = w_{old} - \eta \left(\frac{\partial L}{\partial w_{old}}\right)}$$

Gradient descent

multiple diff ways of how to calculate gradient

Ex. Q. $f(x_1, x_2, x_3, x_4) = a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4$

$\vec{a}^T \cdot \vec{x}$

$$\frac{\partial f}{\partial \vec{x}} = \vec{a}$$

$$\frac{\partial f}{\partial x_1} = a_1 \quad / \quad \frac{\partial f}{x_2} = a_2 \ldots$$

$$\frac{\partial f}{\vec{x}} = \underline{\underline{\vec{a}}}$$

Ex-2 Q $\rightarrow$ $f(x_1, x_2 \ldots x_d) = \vec{x}^T \cdot \vec{x}$

$$\frac{\partial f}{\partial x} = \underline{2\vec{x}}$$

$\downarrow$

∇_x

Optimization Problem

$$\min_{\vec{w}, w_0} - \sum y_i \left(\frac{w^T \cdot x_i + w_0}{\|w\|} \right)$$

$$f'(w_0, w_1, \ldots w_n)$$

$$= \frac{f(w_1 + \Delta w_1, w_2 + \Delta w_2 \ldots w_n + \Delta w_n) - f(w_1, w_2 \ldots w_n)}{\Delta}$$

$\Delta \sim 0$

derivative of this

$$\frac{\partial f}{\partial w_1} = \left(\frac{-\sum y_i (w_1 x_1 + w_2 x_2 + \ldots w_0)}{\sqrt{w_1^2 + w_2^2 + \ldots w_n^2}} \right)$$

$$\left(\frac{f(w_i)}{g(w_i)} \right) \Rightarrow$$ Computationally O. expensive

$$w_1 x_1 + w_2 x_2 + w_0 = 0$$

$$\begin{array}{ccc} w_1 & w_2 & w_0 \\ \downarrow & \downarrow & \downarrow \\ 3 & 4 & 4 \end{array}$$

$$\Rightarrow \quad \frac{3x + 4y + 4 = 0}{\|w\|}$$

$w_1 = 3/5$

$w_2 = 4/5$

$w_0 = 5/5$

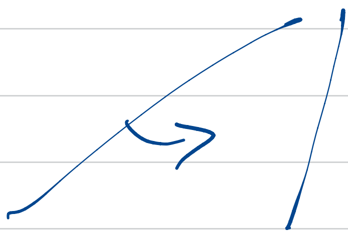
$$\|w\| = \sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2}$$

$$= 1$$

$$\frac{3}{5}x + \frac{4}{5}y + \frac{4}{5} = 0$$

$$f(\vec{w}, w_0)$$

$$\min_{\vec{w}, w_0} \left\{ - \sum y_i \left(w^T x + w_0 \right) \right\} \left\{ \text{as } \|w\| = 1 \right\}$$



$$\frac{\partial f}{\partial \vec{w}}$$

$$\frac{\partial f}{\partial \vec{w_j}} \Rightarrow -\sum_{i=1}^{n} y_i \left(\cancel{w_1} \overset{1}{x_1} + \cancel{w_2} x_2 - \cdots \cancel{w_n} \overset{0}{x_n} \right)$$

$$\Rightarrow -\sum_{i=1}^{n} y_i \, x_1$$

$$w_{j\,new} = w_{j\,old} - \eta \left(-\sum_{i=1}^{n} y_i \cdot x_j \right)$$

$j=1$

$$\overset{3/5 \rightarrow 0.7}{w_1} = w_1 - \eta \times \left(-\sum_{i=1}^{n} y_i \times x_1 \right)$$

Gradient at 1st iteration

$\downarrow$

my weight chang

and norm $\neq$ 1

We can't do G.D. (Gradient descent)

unconstrained loss function is

$$\min_{\vec{w}, w_0} - \sum y_i \left(\frac{w^T \cdot x_i + w_0}{||w||} \right)$$

Constrained loss function

$$\underbrace{\min_{\vec{w}, w_0} - \sum y_i \left(w^T \cdot x_i + w_0 \right)}_{f(x)}$$

$$\text{S.T} \quad \underbrace{||w|| = 1}_{g(x)}$$

such that

$$||w|| - 1 = 0$$

LAGRANGES MULTIPLIERS

→ Converts constrained eqn into unconstrained or

$$\rightarrow \quad \underbrace{f(x) + \lambda \times g(x)}_{\text{unconstrained eqn}}$$

λ → symbol for lagrange's multiplier

Constrain $\min_{\theta} f(\theta)$; s.t. $g_1(\theta)=0$

$g_2(\theta)=0$

$g_3(\theta)=0$

L. multiplier

L. multiplier

new loss function

$$\min_{\theta} f(\theta) + \lambda_1 * g_1(\theta) + \lambda_2 * g_2(\theta) + \lambda_3 * g_3(\theta)$$

Ex.

$$\min_{x,y} f(x,y) = x^2 + y^2$$

s.t. $x + 2y - 1 = 0$

Perform partial derivative w.r.t. x, y, λ

new eqn $= x^2 + y^2 + \lambda(x + 2y - 1)$

$$\frac{\partial f(x,y,\lambda)}{\partial x} = 2x+\lambda = 0$$

$$\Rightarrow \lambda = -2x$$

$$\frac{\partial f(x,y,\lambda)}{\partial y} = 2y + 2\lambda = 0$$

$$\Rightarrow \lambda = -y$$

$$\frac{\partial f(x,y,\lambda)}{\partial \lambda} \Rightarrow x+2y-1 = 0$$

$$x = \frac{1}{5}, \quad y = \frac{2}{5}, \quad \lambda = \frac{-2}{5}$$

$\lambda = -2x$

$\lambda = -y$

$x+2y-1 = 0$

$\cancel{-}y = \cancel{-}2x$

$y = 2x$

$$x+2y-1 = 0$$

$$x+2(2x)-1 = 0$$

$$x+4x-1 = 0$$

$$x = \frac{1}{5}$$

INTUITION

→ In general you get Constraint → Eqn
→ EqN
← Think yourself

→ Let's say youre tryin → lowest point of hill-scape

Can't explore everywhere

Some constraint (stay on specific path)

Forces you to find lowest pt meanwhile also staying on path

$$w_i = w_i - \frac{\partial L}{\partial w_i} \times \eta$$

$$\lambda = \lambda - \frac{\partial L}{\partial \lambda} \times \eta$$

My orig Eqn become.

$$\boxed{L = \min_{\vec{w}, w_0, \lambda} - \sum_{i=1}^{n} (\vec{w}^T \cdot x + w_0) y_i + \lambda(||w|| - 1)}$$

Q. Find minima S.T.

$$f(x) = x^2 - 3x - 3$$

s.t. Constraint $= -x^2 + 2x + 3 = 0$

new eqn

$$h(x) = x^2 - 3x - 3 + \lambda(-x^2 + 2x + 3) = 0$$

$$\frac{\partial h}{\partial x} \Rightarrow 2x - 3 - 2x\lambda + 2\lambda = 0$$

$$\frac{\partial h}{\partial \lambda} \Rightarrow -x^2 + 2x + 3 = 0$$

$2x - 3 - 2x\lambda + 2\lambda$

$\Rightarrow 2x(1-\lambda) + 2\lambda - 3 = 0$

$-x^2 + 2x + 3$

$\Rightarrow -x^2 + 3x - x + 3$

$\Rightarrow x(-x+3) + 1(-x+3)$

$= (x+1)(-x+3)$

$\Rightarrow x = -1, 3$

for $x = 3$

$2 \times 3(1-\lambda) + 2\lambda - 3 = 0$

$\lambda = \frac{3}{4}$

$6 - 6\lambda + 2\lambda - 3 = 0$

$3 - 4\lambda = 0$

$\frac{3}{4} = \lambda$

for $x = -1$

$\Rightarrow 2x(1-\lambda) + 2\lambda - 3 = 0$

$\lambda = \frac{5}{4}$

$2(-1)(1-\lambda) + 2\lambda - 3 = 0$

$-2 + 2\lambda + 2\lambda - 3 = 0$

$-5 + 4\lambda = 0$

$\cancel{\frac{5}{4}} = \lambda$

$\lambda = \frac{5}{4}$

$x = 3, \ \lambda = \frac{3}{4}$

$x = -1, \ \lambda = \frac{5}{4}$

$x^2 - 3x - 2 + \lambda(-x^2 + 2x + 3) = 0$

① $x = 3, \quad \lambda = \frac{2}{4}; \quad \underline{\underline{-3}}$

② $x = -1, \quad \lambda = \frac{5}{4}; \quad 1$

Eqn

$$\boxed{L = \min_{\vec{w}, w_0, \lambda} - \sum_{i=1}^{n} (\vec{w}^T \cdot x + w_0) y_i + \lambda(||w|| - 1)}$$

$\frac{\partial L}{\partial \vec{w}}$

$\frac{\partial L}{\partial w_0}$ $\quad$ $ax + by + c = f(x)$

$\frac{\partial L}{\lambda}$