ODEs: Applications of Separable Equations

Aamir Alaud Din, PhD

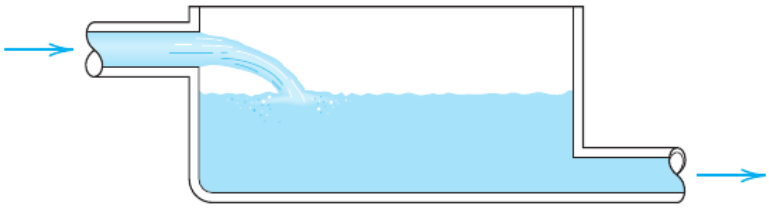
February 21, 2024

Example-4 Radiocarbon Dating[1] (p-13, EK)

In September 1991 the famous Iceman (Oetzi), a mummy from the Neolithic period of the Stone Age found in the ice of the Oetztal Alps (hence the name "Oetzi") in Southern Tyrolia near the Austrian–Italian border, caused a scientific sensation. When did Oetzi approximately live and die if the ratio of carbon ${}^{14}_{6}\mathrm{C}$ to carbon ${}^{12}_{6}\mathrm{C}$ in this mummy is 52.5% of that of a living organism?

Physical Information. In the atmosphere and in living organisms, the ratio of radioactive carbon ${}^{14}_{6}\mathrm{C}$ (made radioactive by cosmic rays) to ordinary carbon ${}^{12}_{6}\mathrm{C}$ is constant. When an organism dies, its absorption of ${}^{14}_{6}\mathrm{C}$ by breathing and eating terminates. Hence one can estimate the age of a fossil by comparing the radioactive carbon ratio in the fossil with that in the atmosphere. To do this, one needs to know the half-life of ${}^{14}_{6}\mathrm{C}$, which is 5715 years (CRC Handbook of Chemistry and Physics, 83rd ed., Boca Raton: CRC Press, 2002, page 11–52, line 9).

Example-5 Mixing Problem (p-14, EK)



Mixing problems occur quite frequently in chemical industry. We explain here how to solve the basic model involving a single tank. The tank in Fig. 11 contains 1000 gal of water in which initially 100 lb of salt is dissolved. Brine runs in at a rate of 10 gal¿ min, and each gallon contains 5 lb of dissoved salt. The mixture in the tank is kept uniform by stirring. Brine runs out at 10 gal¿min. Find the amount of salt in the tank at any time t.

Example-6 Heating an Office Building (Newton's Law of Cooling) (p-15, EK)

Suppose that in winter the daytime temperature in a certain office building is maintained at 70°F. The heating is shut off at 10 PM. and turned on again at 6 AM. On a certain day the temperature inside the building at 2 AM. was found to be 65°F. The outside temperature was 50°F at 10P M. and had dropped to 40°F by 6 AM. What was

[1]Method by WILLARD FRANK LIBBY (1908–1980), American chemist, who was awarded for this work the 1960 Nobel Prize in chemistry.

the temperature inside the building when the heat was turned on at 6 AM.?

Physical information. Experiments show that the time rate of change of the temperature T of a body B (which conducts heat well, for example, as a copper ball does) is proportional to the difference between T and the temperature of the surrounding medium (Newton's law of cooling).

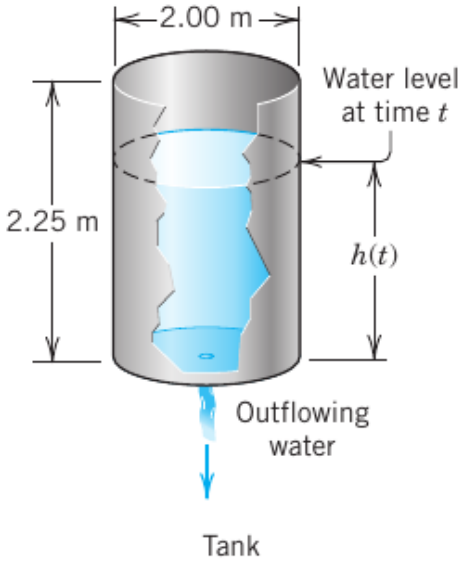
Example-7 Leaking Tank. Outflow of Water Through a Hole (Torricelli's Law) (p-16, EK)

This is another prototype engineering problem that leads to an ODE. It concerns the outflow of water from a cylindrical tank with a hole at the bottom (Fig. 13). You are asked to find the height of the water in the tank at any time if the tank has diameter 2 m, the hole has diameter 1 cm, and the initial height of the water when the hole is opened is 2.25 m. When will the tank be empty?

Physical information. Under the influence of gravity the outflowing water has velocity

$$v(t) = 0.600\sqrt{2gh(t)}$$

where $h(t)$ is the height of the water above the hole at time t, and $g = 980cm/sec^2 = 32.17ft/sec^2$ is the acceleration of gravity at the surface of the earth.



Example-1 Bacterial Growth (p-74, DGZ)

A culture initially has P_0 number of bacteria. At $t = 1h$ the number of bacteria is measured to be $32P_0$. If the rate of growth is proportional to the number of bacteria $P(t)$ present at time t, determine the time necessary for the number of bacteria to triple.

Example-2 Half-Life of Plutonium (p-75, DGZ)

A breeder reactor converts relatively stable uranium-238 into the isotope plutonium-239. After $15years$ it is determined that 0.043% of the initial amount A_0 of the plutonium has disintegrated. Find the half-life of this isotope

if the rate of disintegration is proportional to the amount remaining.

Example-3 Age of a Fossil (p-75, DGZ)

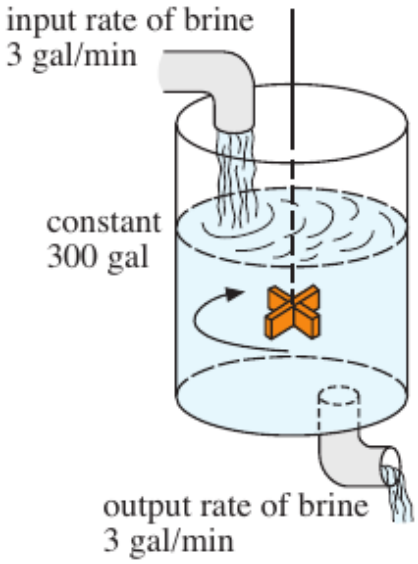
A fossilized bone is found to contain 0.1% of its original amount of C-14. Determine the age of the fossil.

Example-4 Cooling of a Cake (p-76, DGZ)

When a cake is removed from an oven, its temperature is measured at $300^{\circ}F$. Three minutes later its temperature is $200^{\circ}F$. How long will it take for the cake to cool off to a room temperature of $70^{\circ}F$?

Example-5 Mixture of Two Salt Solutions (p-77, DGZ)

The tank shown in figure holds 300 gallons of a brine solution. Salt was entering and leaving the tank; a brine



solution was being pumped into the tank at the rate of 3 gal/min, mixed with the solution there, and then the mixture was pumped out at the rate of 3 gal/min. The concentration of the salt in the inflow, or solution entering, was 2 lb/gal, and so salt was entering the tank at the rate R_{in} = (lb/gal).(3 gal/min) = 6 lb/min and leaving the tank at the rate R_{out} = (x/300 lb/gal).(3 gal/min) = x/100 lb/min. Let us pose the question: If there were 50 lb of salt dissolved initially in the 300 gallons, how much salt is in the tank after a long time?

NOTE:
Examples 1 through 5 from the book of Dennis G. Zill are left as an exercise for the students. Students are required to refer to the book and complete the modeling and simulation part of these problems. Use Microsoft Excel or any other software/program/tool to simulate the system.