Linear Algebra and Ordinary Differential Equations
(MATH-121)

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Outline

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Objectives

After taking this lecture and studying, you should be able to

1. Describe and identify a homogeneous linear ODE with constant coefficients.
2. Formulate auxiliary equation from the given homogeneous linear ODE with constant coefficients.
3. Solve auxiliary equation with real distinct and real repeated roots to find solution to the given homogeneous linear ODE with constant coefficients.

Outline

Homogeneous Linear ODEs with Constant Coefficients

- An n-th order homogeneous linear ODE is given by
$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = 0 \tag{1}$$

- In equation (1), the coefficients $a_n,\ a_{n-1},\ \ldots,\ a_1,\ a_0$ are functions of x.

- If the coefficients in equation (1) are constants $i.e.,\ \ c_n, c_{n-1}, \ldots, c_1, c_0\ \ etc.$, then the ODE (1) becomes
$$c_n y^{(n)} + c_{n-1} y^{(n-1)} + \ldots + c_1 y' + c_0 y = 0 \tag{2}$$

- Using equation (2), a second order homogeneous linear ODE with constant coefficients can be written as
$$ay'' + by' + cy = 0 \tag{3}$$

Outline

Auxiliary Equation

- A homogeneous linear ODE with constant coefficients has the solution of the form $y = e^{mx}$.
- As $y = e^{mx}$ is a solution, so, $y' = me^{mx}$, and $y'' = m^2e^{mx}$.
- Therefore, equation (3) becomes

$$am^2e^{mx} + bme^{mx} + ce^{mx} = 0 \Rightarrow e^{mx}\left(am^2 + bm + c\right) = 0$$

- Since e^{mx} is never zero for real values of x, therefore,

$$am^2 + bm + c = 0 \tag{4}$$

- Equation (4) is called the auxiliary equation of the ODE (3).

Auxiliary Equation

- The auxiliary equation (4) has two roots.

1. $m_1 = \frac{-b+\sqrt{b^2-4ac}}{2a}$
2. $m_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}$

- Based on discriminant $b^2 - 4ac$, following three cases exist.

1. m_1 and m_2 are real and distinct $\Rightarrow (b^2 - 4ac > 0)$.
2. m_1 and m_2 are real and equal $\Rightarrow (b^2 - 4ac = 0)$.
3. m_1 and m_2 are conjugate complex numbers $\Rightarrow (b^2 - 4ac < 0)$.

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Auxiliary Equation with Real Distinct and Real Repeated Roots

Case I: Real Distinct Roots

- If the auxiliary equation has two real and distinct roots, the two solutions of the ODE are $y_1 = e^{m_1 x}$ and $y_2 = e^{m_2 x}$.
- These are linearly independent solution in the interval $(\infty, -\infty)$ and therefore the general solution becomes

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} \tag{5}$$

Case II: Real and Repeated Roots

- If the auxiliary equation (4) has two real and repeated roots $i.e., \ \ m_1 = m_2$, then one solution of the ODE is

$$y_1 = e^{m_1 x}$$

Auxiliary Equation with Real Distinct and Real Repeated Roots

- We already know that if one solution of a second order homogeneous linear ODE is known, the second solution can be found using the method of reduction of order using

$$y_2 = e^{m_1 x} \int \frac{e^{2m_1 x}}{e^{2m_1 x}} dx = e^{m_1 x} \int dx = x e^{m_1 x} \tag{6}$$

- Therefore, the general solution of a homogeneous linear ODE with constant coefficients and with auxiliary equation having real and repeated roots is

$$y = c_1 e^{m_1 x} + c_2 x e^{m_2 x} \tag{7}$$

- Let's solve some examples.

Practice Problems

Solve the following ODEs

1. $y'' - y' - 6y = 0$ $(Ans.\ \ y = c_1 e^{3x} + c_2 e^{-2x})$
2. $4y'' - 4y' - 3y = 0$ $(Ans.\ \ y = c_1 e^{x/2} + c_2 e^{-3x/2})$
3. $y'' - 16y = 0$ $(Ans.\ \ y = c_1 e^{4x} + c_2 e^{-4x})$
4. $4y'' - 28y' - 49y = 0$ $(Ans.\ \ y = e^{7x/2}(c_1 + c_2 x))$
5. $y'' + 6y' - 9y = 0$ $(Ans.\ \ y = e^{-3x}(c_1 + c_2 x))$
6. $y'' + y' = -0.25y$ $(Ans.\ \ y = e^{-0.5x}(c_1 + c_2 x))$

THANK YOU