Applications of Separable Equations

Example-4 (p-13, EK)

Suppose $A_0(t)$ is the initial amount of ${}^{14}_{6}C$. $A(t)$ is the amount of ${}^{14}_{6}C$ at any time t. Since the rate of decay of ${}^{14}_{6}C$ i.e., $\frac{dA(t)}{dt}$ is proportional to the amount of ${}^{14}_{6}C$ present at any time t, i.e.,

$$\frac{dA(t)}{dt} \propto A(t).$$

$$\frac{dA(t)}{dt} = kA(t)$$

or $$\frac{dA}{dt} = kA$$

$$\frac{dA}{A} = k\,dt$$

$$\int \frac{dA}{A} = k\int dt + c_1$$

$$\ln|A| = kt + c_1$$

$$A = ce^{kt}$$

Initial condition:

At time $t=0$, $A=A_0$ $\quad (A(0)=A_0)$.

$$A_0 = ce^0$$
$$c = A_0.$$

So, $A = A_0 e^{kt}$

The half-life of $^{14}_{6}C$ is 5715 years. So, at $t = t_{1/2} = 5715$ years, $A = A_0/2$. i.e.,

$A(5715) = \frac{A_0}{2}$. So,

$$\frac{A_0}{2} = A_0 e^{k(5715)} = A_0 e^{5715k}.$$

$$e^{5715k} = 0.5$$

$$5715k = \ln(0.5) \Rightarrow k = \frac{\ln(0.5)}{5715}$$

$$k = -0.0001213.$$

$$\therefore \quad A = A_0 e^{-0.0001213t}$$

We have to find time t for $A = 52.5\% A_0$.

$$A = 0.525 A_0.$$

$$\therefore \quad 0.525 A_0 = A_0 e^{-0.0001213t}.$$

$$e^{-0.0001213t} = 0.525$$

$$-0.0001213t = \ln(0.525).$$

$$t = \frac{\ln(0.525)}{-0.0001213}$$

$$t = 5312 \approx 5300 \text{ years}.$$

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Example-5 (p-14, EK)

Let $y(t)$ be the amount of salt in the tank at any time t.

Since the salt is entering the tank and the salt is also already available in the tank. The amounts of entering salt and the salt in the tank (brine in the tank) are different. Moreover, the salt is also leaving the tank. So, the amount of salt in the tank is changing with time. This change is

$$\frac{dy}{dt} = \text{Salt inflow rate} - \text{Salt outflow rate.}$$

$$\text{Salt inflow rate} = \frac{10 \text{ gal brine}}{\text{min}} \times \frac{5 \text{ lb salt}}{\text{gal brine}} = \frac{50 \text{ lb salt}}{\text{min}}.$$

$$\text{Salt outflow rate} = \frac{10 \text{ gal brine}}{\text{min}} \times \frac{y \text{ lb salt}}{1000 \text{ gal brine}}$$

$$= 0.01y \text{ lb/min}.$$

$$\frac{dy}{dt} = 50 - 0.01y.$$

$$\frac{dy}{50-0.01y} = dt$$

$$\frac{-0.01\,dy}{50-0.01y} = -0.01\,dt$$

$$-0.01\int \frac{dy}{50-0.01y}. = -0.01\int dt + c_1$$

$$\ln(50-0.01y) = -0.01t + c_1$$

$$50 - 0.01y = ce^{-0.01t}$$

$$0.01y = 50 - ce^{-0.01t}$$

The initial condition is $y(0) = 100$

$$0.01 \times 100 = 50 - ce^{-0.01(0)}$$

$$1 = 50 - c$$

$$c = 49.$$

$$0.01y = 50 - 49e^{-0.01t}$$

$$y = 5000 - 4900e^{-0.01t}$$

At $t = \infty$,

$$y = 5000 - 4900\left(\frac{1}{e^{0.01t}}\right)$$

$$y = 5000 \text{ lb}$$

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Example-6 (p-15, EK)

Let T and T_A be the building's inside and ambient temperatures. The rate of change of temperature inside the building i.e., $\frac{dT}{dt}$ depends upon the difference of temperature in building and outside. So,

$$\frac{dT}{dt} \propto (T - T_A)$$

$$\frac{dT}{dt} = k(T - T_A).$$

$$\frac{dT}{T - T_A} = k\,dt$$

$$\int \frac{dT}{T - T_A} = k\int dt + c_1$$

$$\ln(T - T_A) = kt + c_1$$

$$T - T_A = ce^{kt}$$

$$T = T_A + ce^{kt}$$

We take $T_A = 45°F$, the average ambient temperature ($T_A = 50°F$ at 10PM and $40°F$ at 6AM).

$$T = 45 + ce^{kt}$$

We are interested in the building's inside temperature.

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We take 10PM as $t=0$ $(T = 70^\circ F)$.

so, $T(0) = 70$.

$$70 = 45 + ce^{0}$$

$$c = 25$$

$$T = 45 + 25e^{kt}$$

At 2AM, $T = 65^\circ F$, i.e., $T(4) = 65$

$$65 = 45 + 25e^{4k}$$

$$25e^{4k} = 20$$

$$e^{4k} = \frac{4}{5} = 0.8.$$

$$e^{4k} = 0.8.$$

$$4k = \ln(0.8).$$

$$k = \frac{\ln(0.8)}{4} = -0.056.$$

$$T = 45 + 25e^{-0.056t}$$

At 6AM, $t = 8$, so,

$$T = 45 + 25e^{-0.056(8)}$$

$$T = 61^\circ F$$

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Example-7 (p-16, EK)

Let $h(t)$ be the height of water in the tank, A the area of hole, and B the cross-sectional area of the tank. As the water flows out of the tank, the volume of water in the tank will decrease.

If the water flows out with velocity v through the hole with area A, the volume of water flowing out of the tank in time Δt is given by

$$\Delta V = A v \Delta t \quad \text{--- (1)}$$

The water flowing out of the tank is the decrease in volume of water in the tank. If B is the cross-sectional area of the tank and Δh is the change in the level of water in the tank, then the decrease in volume of water in the tank is given by

$$\Delta V = -B \Delta h \quad \text{--- (2)}$$

Negative sign shows decrease in volume of water in the tank.

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From (1) and (2)

$$-B\Delta h = Av\Delta t.$$

$$\frac{\Delta h}{\Delta t} = -\frac{A}{B}v$$

According to Torricelli's law,

$$v = 0.6\sqrt{2gh}$$

$$\therefore \frac{\Delta h}{\Delta t} = -\frac{A}{B}\cdot 0.6\sqrt{2gh}$$

$$\frac{\Delta h}{\Delta t} = -0.000664\sqrt{h}$$

In the limiting case when $\Delta t \to 0$,

$$\frac{dh}{dt} = -0.000664\sqrt{h}.$$

$$\frac{dh}{\sqrt{h}} = -0.000664\,dt.$$

$$\int h^{-\frac{1}{2}}dh = -0.000664\int dt + c_1.$$

$$2\sqrt{h} = -0.000664t + c_1$$

$$\sqrt{h} = -0.000332t + c$$

At $t = 0$ sec, $h = 2.25m = 225$ cm.

i.e., $h(0) = 225$.

$$\sqrt{225} = -0.000332(0) + c$$

$$c = 15$$

$$\sqrt{h} = -0.000332t + 15$$

$$h = (15 - 0.000332t)^2$$

When tank will be empty, h will be zero. We are required

$$h(t=?) = 0.$$

$$(15 - 0.000332t)^2 = 0.$$

$$0.000332t = 15$$

$$t = \frac{15}{0.000332}$$

$$t = 45181 \text{ sec}$$

$$t = 12.6 \text{ hours}.$$

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