Probability and Statistics

Topic 9 - Conditional Probability and the General Multiplication Rule

Aamir Alaud Din, PhD

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RECAP

- The probability of any event must be between 0 and 1, inclusive. If we let E denote any event, then $0 \leq P(E) \leq 1$.
- The sum of the probabilities of all outcomes in the sample space must equal 1. That is, if the sample space $S = \{e_1, e_2, \dots e_n\}$, then $P(e_1) + P(e_2) + \cdots + P(e_n) = 1$.
- If E and F are disjoint events, then $P(E \text{ or } F) = P(E) + P(F) - P(E \text{ and } F)$.
- If E represents any event and E^c represents the complement of E, then $P(E^c) = 1 - P(E)$.
- If E and F are independent events, then $P(E \text{ and } F) = P(E).P(F)$.

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OBJECTIVES

After learning this topic and studying, you should be able to:

1. Compute conditional probabilities

2. Compute probabilities using the general multiplication rule

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THE WHY SECTION

- Two terms that students often confuse are disjoint and independent.
- We say that two events are disjoint if they cannot occur at the same time.
- We say that two events are independent if the occurrence of one event has no effect on the probability of the other event occurring.
- The following examples illustrate the difference between these two terms in various scenarios.

Flipping a Coin

- Suppose we flip a coin once. If we define event A as the coin landing on heads and we define event B as the coin landing on tails, then event A and event B are disjoint because the coin can't possibly land on heads and tails.

- Suppose we flip a coin twice. If we define event A as the coin landing on heads on the first flip and we define event B as the coin landing on heads on the second flip, then event A and event B are independent because the outcome of one coin flip doesn't affect the outcome of the other.

Rolling a Dice

- Suppose we roll a dice once. If we let event A be the event that the dice lands on an even number and we let event B be the event that the dice lands on an odd number, then event A and event B are disjoint because the dice can't possibly land on an even number and an odd number at the same time.

- Suppose we roll a dice twice. If we define event A as the dice landing on a "5" on the first roll and we define event B as the dice landing on a "5" on the second roll, then event A and event B are independent because the outcome of one dice roll doesn't affect the outcome of the other.

Selecting a Card

- Suppose we select a card from a standard 52-card deck. If we let event A be the event that the card is a Spade and we let event B be the event that the card is a Diamond, then event A and event B are disjoint because the card can't possibly be a Spade and a Diamond at the same time.

- Suppose we select a card from a standard 52-card deck twice in a row with replacement. If we define event A as the card being a Spade on the first draw and we define event B as the card being a Spade on the second draw, then event A and event B are independent because the outcome of one draw doesn't affect the outcome of the other.

- For disjoint events we add probabilities of events and for independent events we multiply probabilities of events.

- We cannot always assume that two events are independent.
- We understant it with the help of an example.

Cards Example

- The probability of drawing a king of spade is $\frac{1}{52}$ from a deck of inverted shuffled cards. Suppose, this king is removed from the deck as its position is known. The remaining cards are now 51. Now, the probability of drawing an ace of spade is $\frac{1}{51}$ and not $\frac{1}{52}$. Note that sample space has reduced. Also note that drawing a king of ace has affected the probability of drawing an ace of spade. This is a conditional probability.

- Our objective is to learn how to find the probabilities of conditional events.

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CONDITIONAL PROBABILITY

- In the last section, we learned that when two events are independent the occurrence of one event has no effect on the probability of the second event.
- However, we cannot always assume that two events will be independent.
- Will the probability of being in a car accident change depending on driving conditions?
- We would expect that the probability of an accident will be higher for driving on icy roads than for driving on dry roads.
- According to data from the Centers for Disease Control, 33.3% of adult men in the United States are obese.
- So the probability is 0.333 that a randomly selected U.S. adult male is obese.

CONDITIONAL PROBABILITY

- However, 28% of adult men aged 20–39 are obese compared to 40% of adult men aged 40–59.
- The probability is 0.28 that an adult male is obese, given that he is aged 20–39.
- The probability is 0.40 that an adult male is obese, given that he is aged 40–59.
- The probability that an adult male is obese changes depending on his age group.
- Therefore, obesity and age are not independent.
- This is called conditional probability.

CONDITIONAL PROBABILITY

Conditional Probability

The notation $P(E|F)$ is read "the probability of event F given event E." It is the probability that the event F occurs, given that the event E has occurred.

- For example, $P(\mathit{obese}|20 \text{ to } 39) = 0.28$ and $P(\mathit{obese}|40 \text{ to } 59) = 0.40$.

EXAMPLE 1

Suppose a single die is rolled. What is the probability that the die comes up three? Now suppose that the die is rolled a second time, but we are told the outcome will be an odd number. What is the probability that the die comes up three?

- The data in Table 8 represent the marital status of males and females 15 years old or older in the United States in 2013.

Table 8

	Males (in millions)	Females (in millions)	Totals (in millions)
Never married	41.6	36.9	**78.5**
Married	64.4	63.1	**127.5**
Widowed	3.1	11.2	**14.3**
Divorced	11.0	14.4	**25.4**
Separated	2.4	3.2	**5.6**
Totals (in millions)	**122.5**	**128.8**	**251.3**

CONDITIONAL PROBABILITY

- To find the probability that a randomly selected individual 15 years old or older is widowed, divide the number of widowed individuals by the total number of individuals who are 15 years old or older.

$$
\begin{aligned}
P(\text{widowed}) &= \frac{14.3}{251.3} \\
&= 0.057
\end{aligned}
$$

- Suppose that we know the individual is female.
- Does this change the probability that she is widowed?
- The sample space now consists only of females, so the probability that the individual is widowed, given that the individual is female, is

CONDITIONAL PROBABILITY

$$P(\text{widowed}|\text{female}) = \frac{N(\text{widowed females})}{N(females)} = \frac{11.2}{128.8} = 0.087$$

- So, knowing that the individual is female increases the likelihood that the individual is widowed.
- This leads to the following rule.

Conditional Probability

If E and F are any two events, then

$$P(F|E) = \frac{P(E \text{ and } F)}{P(E)} = \frac{N(E \text{ and } F)}{N(E)}$$

CONDITIONAL PROBABILITY

Conditional Probability

The probability of event F occurring, given the occurrence of event E, is found by dividing the probability of E and F by the probability of E, or by dividing the number of outcomes in E and E by the number of outcomes in E.

EXAMPLE 2

The data in Table 8 on the previous page represent the marital status and gender of the residents of the United States aged 15 years old or older in 2013.

(a) Compute the probability that a randomly selected individual has never married given the individual is male.

(b) Compute the probability that a randomly selected individual is male given the individual has never married.

CONDITIONAL PROBABILITY

EXAMPLE 3

Suppose that 12.7% of all births are preterm. (The gestation period of the pregnancy is less than 37 weeks.) Also 0.22% of all births resulted in a preterm baby who weighed 8 pounds, 13 ounces or more. What is the probability that a randomly selected baby weighs 8 pounds, 13 ounces or more, given that the baby is preterm? Is this unusual? Source: Vital Statistics Reports

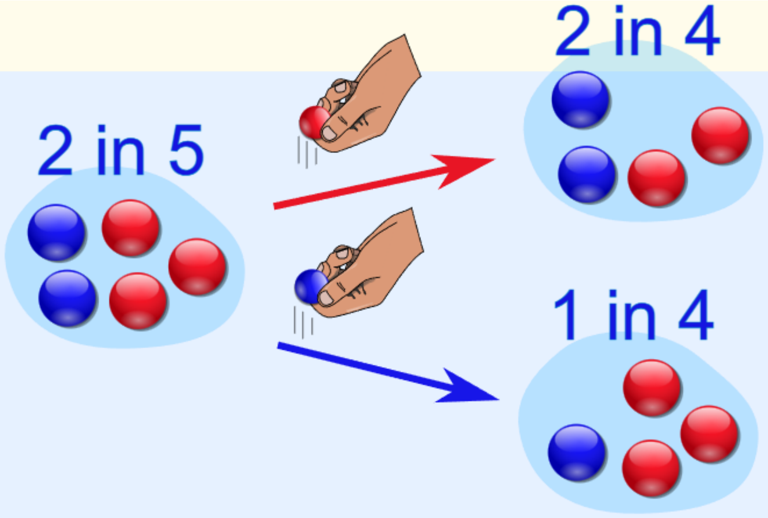
Dependent (Conditional) Events

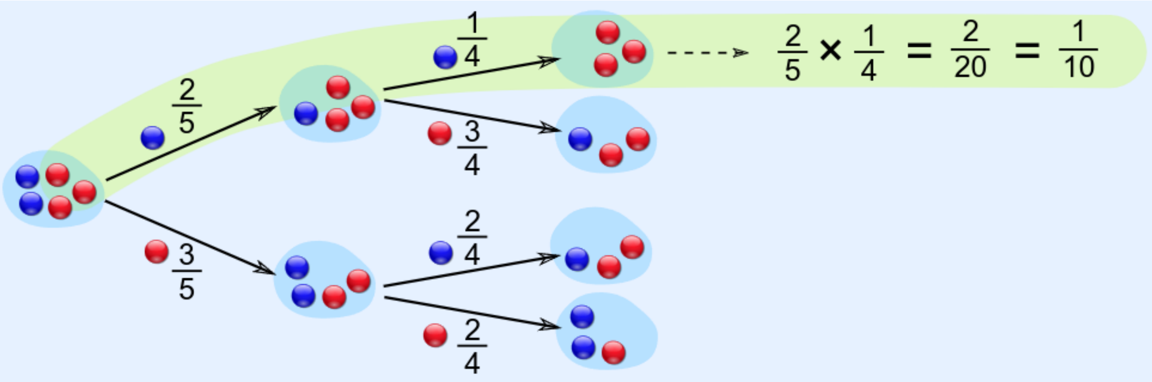
Dependent events means, the events can be affected by previous events.

CONDITIONAL PROBABILITY

- Consider 2 blue and 3 red marbles in a bag.
- Probability of drawing a blue marble is $\frac{2}{5}$.
- Once the marble is drawn, the probability of drawing a red marble is $\frac{3}{4}$.
- Now, the probability of drawing a blue marble is $\frac{1}{3}$ while that of red marble is $\frac{2}{3}$.
- This is because we are removing the marbles from the bag.
- So, the next event depends on what happened in the previous event, and is called dependent (conditional).
- In the conditional or dependent events, the sample spaces reduces in size.

CONDITIONAL PROBABILITY





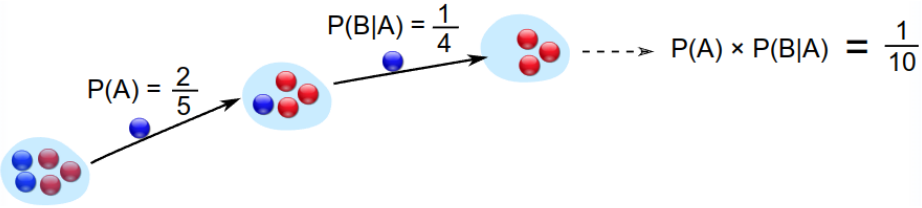
- The probability of drawing two blue marbles is 1 out of 10.
- Build the complete picture now.

CONDITIONAL PROBABILITY

- In our marbles example Event A is "get a Blue Marble first" with a probability of 2/5 *i.e.*, $P(A) = \frac{2}{5}$.
- And Event B is "get a Blue Marble second", but for that we have 2 choices.
  1. If we got a Blue Marble first the chance is now 1/4.
  2. If we got a Red Marble first the chance is now 2/4.
- So we have to say which one we want, and use the symbol "$|$" to mean "given".
- $P(B|A)$ means "Event B given Event A".
- In other words, event A has already happened, now what is the chance of event B?

CONDITIONAL PROBABILITY

- $P(B|A)$ is also called the "Conditional Probability" of B given A.
- In our case $P(B|A) = 1/4$.
- So, the probability of getting 2 blue marbles is shown below.



- Mathematically, we can write the above scenario as shown below.

"Probability Of" *"Given"*

$$P(A \text{ and } B) = P(A) \times P(B \mid A)$$

Event A *Event B*

*"Probability of **event A and event B** equals the probability of **event A** times the probability of **event B given event A**"*

CONDITIONAL PROBABILITY

EXAMPLE

Suppose you have a plan to play football today and you want to be a goalkeeper. Tha academy has two coaches Adnan and Maaz and Adnan is mostly the coach about 6 out of 10 games. If Adnan is the coach, the probability of being the goalkeeper is 50% while in case of Maaz, the probability of being the goalkeeper is only 30%. Compute the following probabilities.

1. The probability of being goalkeeper given Adnan is the coach.
2. The probability of being goalkeeper given Maaz is the coach.
3. The probability of not being the goalkeeper if Adnan is the coach.
4. The probability of not being the goalkeeper if Maaz is the coach.

Also prove that the model developed is the probability model.

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GENERAL MULTIPLICATION RULE

- If we solve the Conditional Probability Rule for $P(E \text{ and } F)$, we obtain the General Multiplication Rule.

General Multiplication Rule

The probability that two events E and F both occur is

$$P(E \text{ and } F) = P(E).P(F|E)$$

In words, the probability of E and F is the probability of event E occurring times the probability of event F occurring, given the occurrence of event E.

GENERAL MULTIPLICATION RULE

EXAMPLE 4

The probability that a driver who is speeding gets pulled over is 0.8. The probability that a driver gets a ticket, given that he or she is pulled over, is 0.9. What is the probability that a randomly selected driver who is speeding gets pulled over and gets a ticket?

EXAMPLE 5

Suppose that of 100 circuits sent to a manufacturing plant, 5 are defective. The plant manager receiving the circuits randomly selects 2 and tests them. If both circuits work, she will accept the shipment. Otherwise, the shipment is rejected. What is the probability that the plant manager discovers at least 1 defective circuit and rejects the shipment?

GENERAL MULTIPLICATION RULE

EXAMPLE 6

In a study to determine whether preferences for self are more or less prevalent than preferences for others, researchers first asked individuals to identify the person who is most valuable and likeable to you, or your favorite other. Of the 1519 individuals surveyed, 42 had chosen themselves as their favorite other. Source: Gebauer JE, et al. Self-Love or Other-Love? Explicit Other-Preference but Implicit Self-Preference. PLoS ONE 7(7): e41789. doi:10.1371/journal.prone.0041789

(a) Suppose we randomly select 1 of the 1519 individuals surveyed. What is the probability that he or she had chosen himself or herself as their favorite other?

(b) If two individuals from this group are randomly selected, what is the probability that both chose themselves as their favorite other?

(c) Compute the probability of randomly selecting two individuals from this group who selected themselves as their favorite other assuming independence.

GENERAL MULTIPLICATION RULE

- If small random samples are taken from large populations without replacement, it is reasonable to assume independence of the events.
- As a rule of thumb, if the sample size is less than 5% of the population size, we treat the events as independent.
- For example, in Example 6, we can compute the probability of randomly selecting two individuals who consider themselves their favorite other assuming independence because the sample size, 2, is only $\frac{2}{1519}$, or 0.13% of the population size, 1519.
- We can now express independence using conditional probabilities.

Independent Events

Two events E and F are independent if $P(E|F) = P(E)$ or, equivalently, if $P(F|E) = P(F)$.

GENERAL MULTIPLICATION RULE

- If either condition in our definition is true, the other is as well.
- In addition, for independent events,

$$P(E \text{ and } F) = P(E).P(F)$$

- So the Multiplication Rule for Independent Events is a special case of the General Multiplication Rule.
- Look back at Table 8 on page 308. Because $P(\textit{widowed}|\textit{female}) = 0.057$ does not equal $P(\textit{widowed}) = 0.087$, the events "widowed" and "female" are not independent.
- In fact, knowing an individual is female increases the likelihood that the individual is also widowed.

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SUMMARY

Conditional Probability

The notation $P(E|F)$ is read "the probability of event F given event E." It is the probability that the event F occurs, given that the event E has occurred.

Conditional Probability

If E and F are any two events, then

$$P(F|E) = \frac{P(E \text{ and } F)}{P(E)} = \frac{N(E \text{ and } F)}{N(E)}$$

he probability of event F occurring, given the occurrence of event E, is found by dividing the probability of E and F by the probability of E, or by dividing the number of outcomes in E and E by the number of outcomes in E.

SUMMARY

General Multiplication Rule

The probability that two events E and F both occur is

$$P(E \text{ and } F) = P(E).P(F|E)$$

In words, the probability of E and F is the probability of event E occurring times the probability of event F occurring, given the occurrence of event E.

Independent Events

Two events E and F are independent if $P(E|F) = P(E)$ or, equivalently, if $P(F|E) = P(F)$.


Thank You!