Probability and Statistics

Topic 13 - The Binomial Probability Distribution

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November 14, 2024

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RECAP

- The distribution of a discrete random variable can't be continuous but lines at discrete values.
- The mean of a discrete random variable is given by the formula

$$\mu_X = \sum [x.P(x)]$$

- The standard deviation of a discrete random variable is given by the equation

$$\begin{aligned}\sigma_X &= \sqrt{\sum[(x-\mu_X)^2.P(x)]} \\ &= \sqrt{\sum [x^2.P(x)] - \mu_X^2}\end{aligned}$$

- Mean of discrete random variable is also called the expected value as it tells the expected value of the random variable that would happen in the long run.

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OBJECTIVES

After learning this topic and studying, you should be able to:

1. Determine whether a probability experiment is a binomial experiment
2. Compute probabilities of binomial experiments
3. Compute the mean and standard deviation of a binomial random variable
4. Graph a binomial probability distribution

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THE WHY SECTION

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BINOMIAL PROBABILITY EXPERIMENT

- In Topic 12, we stated that probability distributions could be presented using tables, graphs, or mathematical formulas.
- In this section, we introduce a specific type of discrete probability distribution that can be presented using a formula, the binomial probability distribution.
- The binomial probability distribution is a discrete probability distribution that describes probabilities for experiments in which there are two mutually exclusive (disjoint) outcomes.
- These two outcomes are generally referred to as success (such as making a free throw) and failure (such as missing a free throw).
- Experiments in which only two outcomes are possible are referred to as binomial experiments, provided that certain criteria are met.

BINOMIAL PROBABILITY EXPERIMENT

Criteria for a Binomial Probability Experiment

An experiment is said to be a binomial experiment if

1. The experiment is performed a fixed number of times. Each repetition of the experiment is called a trial.
2. The trials are independent. This means that the outcome of one trial will not affect the outcome of the other trials.
3. For each trial, there are two mutually exclusive (disjoint) outcomes: success or failure.
4. The probability of success is the same for each trial of the experiment.

- Let the random variable X be the number of successes in n trials of a binomial experiment.

BINOMIAL PROBABILITY EXPERIMENT

- Then X is called a binomial random variable.
- Before introducing the method for computing binomial probabilities, it is worthwhile to introduce some notation.

Notation Used in the Binomial Probability Distribution

- There are n independent trials of the experiment.
- Let p denote the probability of success for each trial so that $1-p$ is the probability of failure for each trial.
- Let X denote the number of successes in n independent trials of the experiment. So $0 ... x ... n$.

BINOMIAL PROBABILITY EXPERIMENT

EXAMPLE 1

Determine which of the following probability experiments qualify as binomial experiments. For those that are binomial experiments, identify the number of trials, probability of success, probability of failure, and possible values of the random variable X.

- Note that the word success does not necessarily imply something positive.
- Success means that an outcome has occurred that corresponds with p, the probability of success.
- For example, a probability experiment might be to randomly select ten 18-year-old male drivers.
- If X denotes the number who have been involved in an accident within the last year, a success would mean obtaining an 18-year-old male who was involved in an accident.
- This outcome is not positive, but it is a success as far as the experiment goes.

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PROBABILITIES OF BINOMIAL EXPERIMENTS

- Now we will compute probabilities for a binomial random variable X.
- We present three methods for obtaining binomial probabilities: (1) the binomial probability distribution formula, (2) a table of binomial probabilities, and (3) technology.
- We develop the binomial probability formula in Example 2.

EXAMPLE 2

According to the American Red Cross, 7% of people in the United States have blood type O-negative. A simple random sample of size 4 is obtained, and the number of people X with blood type O-negative is recorded. Construct a probability distribution for the random variable X.

PROBABILITIES OF BINOMIAL EXPERIMENTS

- Notice some interesting results in Example 2.
- Consider the probability of obtaining $x = 1$ success:

$$P(1) = 4(0.07)^1 (0.93)^3$$

4 is the number of ways we obtain 1 success in 4 trials of the experiment. Here, it is ${}_4C_1$.

0.07 is the probability of success and the exponent 1 is the number of successes.

0.93 is the probability of failure and the exponent 3 is the number of failures.

- The coefficient 4 is the number of ways of obtaining one success in four trials.
- In general, the coefficient is ${}_nC_r$, the number of ways of obtaining x successes in n trials.

PROBABILITIES OF BINOMIAL EXPERIMENTS

- The second factor in the formula, $(0.07)^1$, is the probability of successes, p, raised to the number of successes, n.
- The third factor in the formula, $(0.93)^3$, is the probability of successes, $(1-p)$, raised to the number of failures, $(n-x)$.
- The following binomial probability distribution function (pdf) formula holds for all binomial experiments.

Binomial Probability Distribution Function

The probability of obtaining x successes in n independent trials of a binomial experiment is given by

$$P(x) = {}_nC_r p^x (1-p)^{n-x} \qquad x = 0, 1, 2, \ldots, n$$

where p is the probability of success.

PROBABILITIES OF BINOMIAL EXPERIMENTS

- When reading probability problems, pay special attention to key phrases that translate into mathematical symbols.
- Table 9 lists various phrases and their corresponding mathematical equivalent.

Table 9

Phrase	Math Symbol
"at least" or "no less than" or "greater than or equal to"	$\geq$
"more than" or "greater than"	$>$
"fewer than" or "less than"	$<$
"no more than" or "at most" or "less than or equal to"	$\leq$
"exactly" or "equals" or "is"	$=$

EXAMPLE 3

According to CTIA, 41% of all U.S. households are wireless-only households (no landline). In a random sample of 20 households, what is the probability that

(a) exactly 5 are wireless-only?

(b) fewer than 3 are wireless-only?

(c) at least 3 are wireless-only?

(d) the number of households that are wireless-only is between 5 and 7, inclusive?

PROBABILITIES OF BINOMIAL EXPERIMENTS

Obtaining Binomial Probabilities from Tables

- Another method for obtaining probabilities is the binomial probability table.
- Table III in Appendix A gives probabilities for a binomial random variable X taking on a specific value, such as $P(10)$, for select values of n and p.
- Table IV in Appendix A gives cumulative probabilities of a binomial random variable X.
- This means that Table IV gives "less than or equal to" binomial probabilities, such as $P(X \leq 6)1$.
- We illustrate how to use Tables III and IV in Example 4.
- Statistical software and graphing calculators also have the ability to compute binomial probabilities.

PROBABILITIES OF BINOMIAL EXPERIMENTS

EXAMPLE 4

According to the Gallup Organization, 65% of adult Americans are in favor of the death penalty for individuals convicted of murder. In a random sample of 15 adult Americans, what is the probability that

(a) exactly 10 favor the death penalty?

(b) no more than 6 favor the death penalty?

EXAMPLE 5

Solve the problem using any programming language/software of your choice. Python and R are highly recommended.

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MEAN AND STANDARD DEVIATION OF A BINOMIAL RANDOM VARIABLE

- We discussed finding the mean (or expected value) and standard deviation of a discrete random variable in Topic 12.
- These formulas can be used to find the mean and standard deviation of a binomial random variable, but a simpler method exists.

Mean (or Expected Value) and Standard Deviation of a Binomial Random Variable

A binomial experiment with n independent trials and probability of success p has a mean and standard deviation given by the formulas

$$\mu_X = np \qquad \text{and} \qquad \sigma_X = \sqrt{np(1-p)}$$

MEAN AND STANDARD DEVIATION OF A BINOMIAL RANDOM VARIABLE

EXAMPLE 6

According to CTIA, 41% of all U.S. households are wireless-only households. In a simple random sample of 300 households, determine the mean and standard deviation number of wireless-only households.

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GRAPHING A BINOMIAL PROBABILITY DISTRIBUTION

- To graph a binomial probability distribution, first find the probabilities for each possible value of the random variable.
- Then follow the same approach that was used to graph discrete probability distributions.

EXAMPLE 7

(a) Graph the binomial probability distribution with $n = 10$ and $p = 0.2$ Comment on the shape of the distribution.

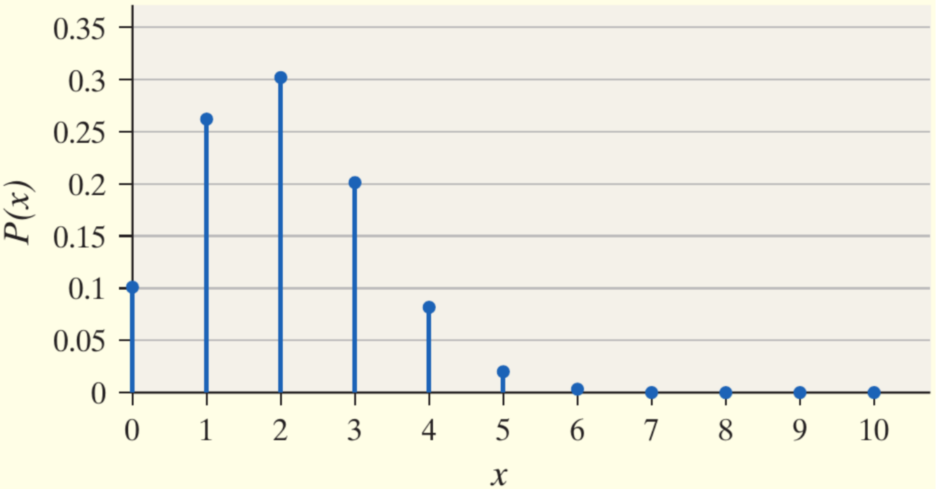
(b) Graph the binomial probability distribution with $n = 10$ and $p = 0.5$ Comment on the shape of the distribution.

(c) Graph the binomial probability distribution with $n = 10$ and $p = 0.8$ Comment on the shape of the distribution.

GRAPHING A BINOMIAL PROBABILITY DISTRIBUTION

Table 10

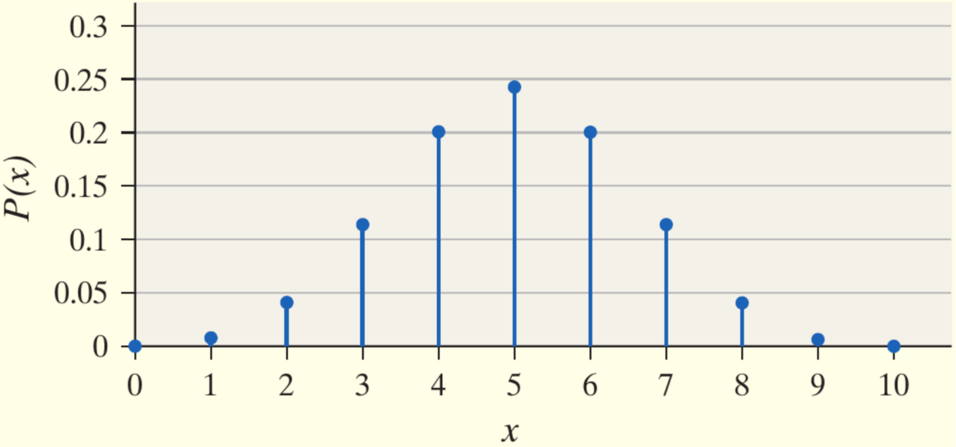
x	$P(x)$
0	0.1074
1	0.2684
2	0.3020
3	0.2013
4	0.0881
5	0.0264
6	0.0055
7	0.0008
8	0.0001
9	0.0000
10	0.0000



GRAPHING A BINOMIAL PROBABILITY DISTRIBUTION

Table 11

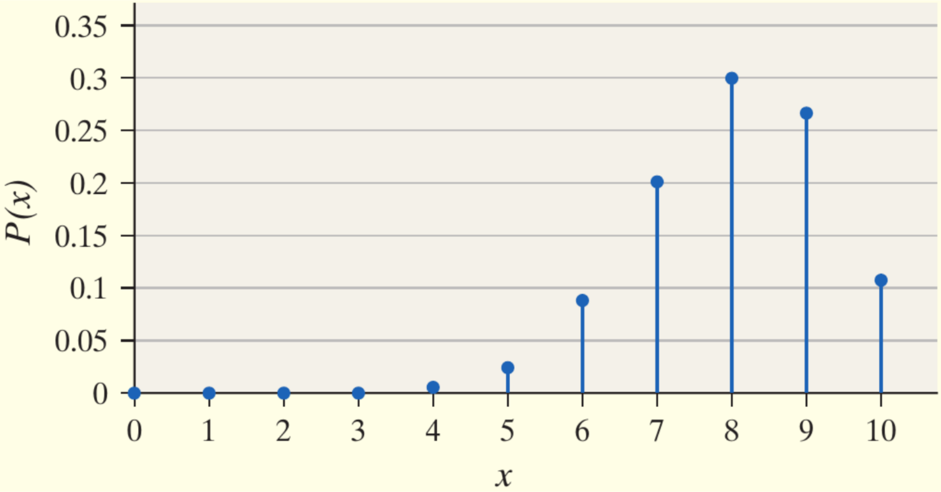
x	$P(x)$
0	0.0010
1	0.0098
2	0.0439
3	0.1172
4	0.2051
5	0.2461
6	0.2051
7	0.1172
8	0.0439
9	0.0098
10	0.0010



GRAPHING A BINOMIAL PROBABILITY DISTRIBUTION

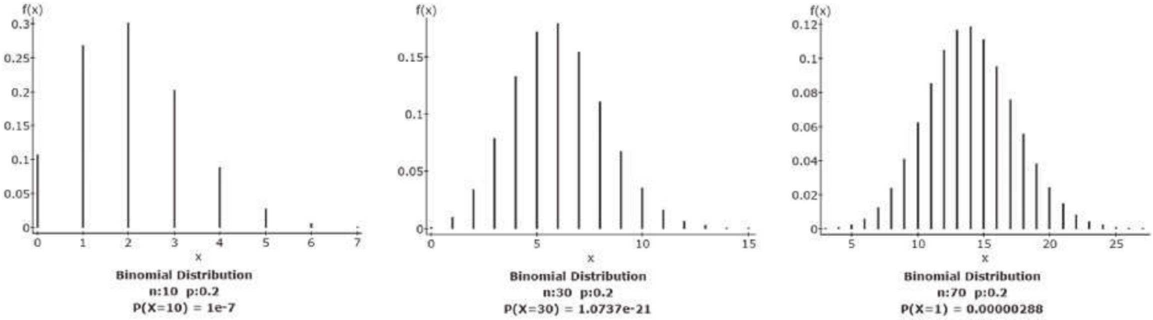
Table 12

x	$P(x)$
0	0.0000
1	0.0000
2	0.0001
3	0.0008
4	0.0055
5	0.0264
6	0.0881
7	0.2013
8	0.3020
9	0.2684
10	0.1074



GRAPHING A BINOMIAL PROBABILITY DISTRIBUTION

- Based on the results of Example 7, we conclude that the binomial probability distribution is skewed right if $p < 0.5$, symmetric and approximately bell shaped if $p = 0.5$, and skewed left if $p > 0.5$.
- Notice that first ($p = 0.2$) and third figures ($p = 0.8$) in Example 7 are mirror images of each other.
- We have seen the roles that p plays in the shape of of a binomial distribution, but what role does n in its shape?
- Following figure shows the graph of three binomial probability distributions drawn in StatCrunch.
- In Figure (a), $n = 10$ and $p = 0.2$; in Figure (b) $n = 30$ and $p = 0.2$; in Figure (b) $n = 70$ and $p = 0.2$.



(a) **(b)** **(c)**

- Figure (a) is skewed right, Figure (b) is slightly skewed right, and Figure (c) appears bell shaped.

GRAPHING A BINOMIAL PROBABILITY DISTRIBUTION

Conclusion

For a fixed p, as the number of trials n in a binomial experiment increases, the probability distribution of the random variable X becomes bell shaped. As a rule of thumb, if $np(1-p) \geq 10$, the probability distribution will be approximately bell shaped.

- This result allows us to use the Empirical Rule to identify unusual observations in a binomial experiment.
- Recall the Empirical Rule states that in a bell-shaped distribution about 95% of all observations lie within two standard deviations of the mean.
- That is, about 95% of the observations lie between $\mu - 2\sigma$ and $\mu + 2\sigma$.

GRAPHING A BINOMIAL PROBABILITY DISTRIBUTION

- Any observation that lies outside this interval may be considered unusual because the observation occurs less than 5% of the time.

EXAMPLE 8

According to CTIA, 41% of all U.S. households are wireless-only. In a simple random sample of 300 households, 143 were wireless-only. Is this result unusual?

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SUMMARY

- For a binomial probability distribution, there are two mutually exclusive outcomes which are referred to as success and failure.
- Although the success and failure are mutually exclusive, but their probabilities are different.
- The probability of obtaining x successes in n independent trials of a binomial experiment is given by

$$P(x) = {}_nC_r p^x (1-p)^{n-x} \qquad x = 0, 1, 2, \ldots, n$$

  where p is the probability of success.
- A binomial experiment with n independent trials and probability of success p has a mean and standard deviation given by the formulas

$$\mu_X = np \qquad \text{and} \qquad \sigma_X = \sqrt{np(1-p)}$$


Thank You!