Probability and Statistics

Topic 10 - Counting Techniques

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RECAP

Conditional Probability

The notation $P(E|F)$ is read "the probability of event F given event E." It is the probability that the event F occurs, given that the event E has occurred.

Conditional Probability

If E and F are any two events, then

$$P(F|E) = \frac{P(E \text{ and } F)}{P(E)} = \frac{N(E \text{ and } F)}{N(E)}$$

he probability of event F occurring, given the occurrence of event E, is found by dividing the probability of E and F by the probability of E, or by dividing the number of outcomes in E and E by the number of outcomes in E.

RECAP

General Multiplication Rule

The probability that two events E and F both occur is

$$P(E \text{ and } F) = P(E).P(F|E)$$

In words, the probability of E and F is the probability of event E occurring times the probability of event F occurring, given the occurrence of event E.

Independent Events

Two events E and F are independent if $P(E|F) = P(E)$ or, equivalently, if $P(F|E) = P(F)$.

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OBJECTIVES

After learning this topic and studying, you should be able to:

1. Solve counting problems using the Multiplication Rule
2. Solve counting problems using permutations
3. Solve counting problems using combinations
4. Solve counting problems involving permutations with nondistinct items
5. Compute probabilities involving permutations and combinations

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THE WHY SECTION

- We study counting since childhood, even before schooling.
- Counting appears very simple and we don't pay attention to it.
- In mathematics and specially in probability, counting is very important because we can't solve problems if we don't know the required counting techniques.
- We have to count the number of events in the sample space and also for our probability experiments.
- In how many ways can horses in a ten-horse race finish first, second, and third?
- How to count the events if the order of objects in which the objects are chosen is not important?
- What if the order of objects is important?
- We are studying this topic to learn the specail techniques to count such events.

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COUNTING USING MULTIPLICATION RULE

- Counting plays a major role in many diverse areas, including probability.
- In this section, we look at special types of counting problems and develop general techniques for solving them.
- We begin with an example that demonstrates a general counting principle.

EXAMPLE 1

The fixed-price dinner at Mabenka Restaurant provides the following choices:

Appetizer:	soup or salad
Entree:	chicken, patty, liver, beef
Dessert:	ice cream or cheesecake

How many different meals can be ordered?

COUNTING USING MULTIPLICATION RULE

Multiplication Rule for Counting

If a task consists of a sequence of choices in which there are p selections for the first choice, q selections for the second choice, r selections for the third choice, and so on, then the task of making these selections can be done in

$$p.q.r.\ldots$$

different ways.

EXAMPLE 2 [Repetition Allowed]

The International Air Transport Association (IATA) assigns three-letter codes to represent airport locations. For example, the code for Fort Lauderdale International Airport is FLL. How many different airport codes are possible?

COUNTING USING MULTIPLICATION RULE

EXAMPLE 3 [Repetition Not Allowed]

Three members from a 14-member committee are to be randomly selected to serve as chair, vice-chair, and secretary. The first person selected is the chair, the second is the vice-chair, and the third is the secretary. How many different committee structures are possible?

EXAMPLE 4

You have just been hired as a book representative for Pearson Education. On your first day, you must travel to seven schools to introduce yourself. How many different routes are possible?

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COUNTING USING PERMUTATIONS

- Examples 3 and 4 illustrate a type of counting problem referred to as a permutation.

Permutation

A permutation is an ordered arrangement in which r objects are chosen from n distinct (different) objects so that $r \leq n$ and repetition is not allowed. The symbol ${}_nP_r$, represents the number of permutations of r objects selected from n objects.

- The solution in Example 3 could be represented as

$$ {}_nP_r = {}_{14}P_3 = 14.13.12 = 2184 $$

- The solution in Example 4 could be represented as

$$ {}_nP_r = {}_7P_7 = 7.6.5.4.3.2.1 = 5040 $$

COUNTING USING PERMUTATIONS

- To arrive at the formula for ${}_nP_r$, note that there are n choices for the first selection, $n-1$ choices for the second selection, $n-2$ choices for the third selection, ..., and $n-(r-1)$ choices for the r^{th} selection.
- By multiplication rule,

$$
\begin{aligned}
{}_nP_r &= 1^{st}.2^{nd}.3^{rd}\dots r^{th}\\
&= n.(n-1).(n-2)\dots(n-(r-1))\\
&= n.(n-1).(n-2)\dots(n-r+1)
\end{aligned}
$$

- This formula in factorial notation can be written as

$$
\begin{aligned}
{}_nP_r &= n.(n-1).(n-2)\dots(n-r+1)\\
&= n.(n-1).(n-2)\dots(n-r+1).\frac{(n-r).\dots 3.2.1}{(n-r).\dots 3.2.1}\\
&= \frac{n!}{(n-r)!}
\end{aligned}
$$

COUNTING USING PERMUTATIONS

Number of Permutations of n Distinct Objects Taken r at a Time

The number of arrangements of r objects chosen from n objects, in which

1. the n objects are distinct,
2. repetition of objects is not allowed, and
3. order is important,

is given by the formula

$$_nP_r = \frac{n!}{(n-r)!}$$

EXAMPLE 5

Evaluate

(a) $_7P_5$

(b) $_5P_5$

COUNTING USING PERMUTATIONS

EXAMPLE 6

In how many ways can horses in a ten-horse race finish first, second, and third?

The important key points for problems involving permutations are:

- Logical understanding.
- Conditions for permutations.
- Definitely practice, practice, and practice.

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COUNTING USING COMBINATIONS

- In a permutation, order is important.
- For example, the arrangements ABC and BAC are considered different arrangements of the letters A, B, and C.
- If order is unimportant, we do not distinguish ABC from BAC.
- In poker, the order in which the cards are received does not matter.
- The combination of the cards is what matters.

Combination

A combination is a collection, without regard to order, in which r objects are chosen from n distinct objects with $r \leq n$ and without repetition. The symbol ${}_nC_r$ represents the number of combinations of n distinct objects taken r at a time.

COUNTING USING COMBINATIONS

EXAMPLE 7

Roger, Ken, Tom, and Jay are going to play golf. They will randomly select teams of two players each. List all possible team combinations. That is, list all the combinations of the four people Roger, Ken, Tom, and Jay taken two at a time. What is ${}_4C_2$?

Number of Combinations of n Distinct Objects Taken r at a Time

The number of different arrangements of r objects chosen from n objects, in which

1. the n objects are distinct,
2. repetition of objects is not allowed, and
3. order is not important,

is given by the formula

$$ {}_nC_r = \frac{n!}{r!(n-r)!} $$

COUNTING USING COMBINATIONS

- The logical derivation of the formula for ${}_nC_r$ is given on page 322 of the book and the students are advised to understand it because it only involves logic and nothing else.
- Solve example 7 by using the formula for combinations.

EXAMPLE 8

Evaluate

(a) ${}_4C_1$

(b) ${}_6C_4$

(b) ${}_6C_2$

EXAMPLE 9

How many different simple random samples of size 4 can be obtained from a population whose size is 20?

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COUNTING USING PERMUTATIONS WITH NONDISTINCT ITEMS

- Sometimes we want to arrange objects in order, but some of the objects are not distinguishable.

EXAMPLE 10

A DNA sequence consists of a series of letters representing a DNA strand that spells out the genetic code. There are four possible letters (A, C, G, and T), each representing a specific nucleotide base in the DNA strand (adenine, cytosine, guanine, and thymine, respectively). How many distinguishable sequences can be formed using two As, two Cs, three Gs, and one T?

- Example 10 suggests a general result.
- Had the letters in the sequence each been different, $8P8 = 8!$ possible sequences would have been formed.
- This is the numerator of the answer.

COUNTING USING PERMUTATIONS WITH NONDISTINCT ITEMS

- The presence of two As, two Cs, and three Gs reduces the number of different sequences, as the entries in the denominator illustrate.

Permutations with Nondistinct Items

The number of permutations of n objects of which n_1 are of one kind, n_2 are of a second kind, $\ldots$, and n_k are of a k^{th} kind is given by

$$\frac{n!}{n_1!.n_2!.\ldots.n_k!}$$

where $n = n_1 + n_2 + \cdots + n_k$.

EXAMPLE 11

How many different vertical arrangements are there of 10 flags if 5 are white, 3 are blue, and 2 are red?

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PROBABILITIES INVOLVING PERMUTATIONS AND COMBINATIONS

- The counting techniques presented in this section can be used along with the classical method to compute certain probabilities.
- Recall that this method stated the probability of an event E is the number of ways event E can occur divided by the number of different possible outcomes of the experiment provided each outcome is equally likely.

EXAMPLE 12

In the Illinois Lottery, an urn contains balls numbered 1–52. From this urn, six balls are randomly chosen without replacement. For a \$1 bet, a player chooses two sets of six numbers. To win, all six numbers must match those chosen from the urn. The order in which the balls are picked does not matter. What is the probability of winning the lottery?

PROBABILITIES INVOLVING PERMUTATIONS AND COMBINATIONS

EXAMPLE 13

A shipment of 120 fasteners that contains 4 defective fasteners was sent to a manufacturing plant. The plant's quality-control manager randomly selects and inspects 5 fasteners. What is the probability that exactly 1 of the inspected fasteners is defective?

- This section is very important in your practical/professional life if you have to work in the field where you have to use the counting techniques.
- The mastering of these techniques depends upon
  1. The logical understanding of the ideas.
  2. Understanding the differences in counting techniques.
  3. Of course, practice, practice, and practice.
  4. If you want to understand the importance of practice, look at the number of problems at the end of each section.

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SUMMARY

Table 10

	Description	Formula
Combination	The selection of r objects from a set of n different objects when the order in which the objects are selected does not matter (so AB is the same as BA) and an object cannot be selected more than once (repetition is not allowed)	${}_nC_r = \frac{n!}{r!(n-r)!}$
Permutation of Distinct Items with Replacement	The selection of r objects from a set of n different objects when the order in which the objects are selected matters (so AB is different from BA) and an object may be selected more than once (repetition is allowed)	n^r
Permutation of Distinct Items without Replacement	The selection of r objects from a set of n different objects when the order in which the objects are selected matters (so AB is different from BA) and an object cannot be selected more than once (repetition is not allowed)	${}_nP_r = \frac{n!}{(n-r)!}$
Permutation of Nondistinct Items without Replacement	The number of ways n objects can be arranged (order matters) in which there are n_1 of one kind, n_2 of a second kind, . . . , and n_k of a kth kind, where $n = n_1 + n_2 + \cdots + n_k$	$\frac{n!}{n_1!n_2!\cdots n_k!}$


Thank You!