Probability and Statistics

Topic 8 - Independence and the Multiplication Rule

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TABLE OF CONTENTS

TABLE OF CONTENTS

RECAP

- For disjoint events E and F, the probability of two events is obtained by adding the individual probabilities of the two events *i.e.*,

$$P(E \text{ or } F) = P(E) + P(F)$$

- For non-disjoint events E and F, the probability is computed by the formula

$$P(E \text{ or } F) = P(E) + P(F) - P(E \text{ and } F)$$

- Contingency tables are two-way tables and are very helpful in a lot of statistical studies, specially surveys.

- The probability of the complement of an event E^c given the probability of event E is computed by the relation

$$P(E^c) = 1 - P(E)$$

TABLE OF CONTENTS

OBJECTIVES

After learning this topic and studying, you should be able to:

1. Identify independent events
2. Use the multiplication rule for independent events
3. Compute at-least probabilities

TABLE OF CONTENTS

THE WHY SECTION

- In the previous topic we learnt to compute the probabilities of events that were mutually exclusive.
- We also learnt to compute the probabilities of mutually exclusive events in which some of the outcomes were common.
- For flipping of a coin, if event E is the head or tail and event F is also a head or tail, and we used the addition rule.
- Suppose the event E is head and the event F is also a head, what will be the probability that the first two flips will both be heads?
- Now, $P(E) = \frac{1}{2}$ and $P(F) = \frac{1}{2}$.
- Based on previous section, $P(E \text{ and } F) = \frac{1}{2} + \frac{1}{2} = 1$.

THE WHY SECTION

- In other words we are saying that there is 100% probability that the first two outcomes of flipping will be heads, which is logically as well as really not possible always.
- We are interested to find the probabilities of such events and these events are independent events.
- Both outcomes are independent of each other.
- Logically the possibility of these outcomes is 25% only and mathematically this is possible if we multiply the probabilities of these outcomes i.e.,

$$P(E \text{ and } F) = P(E).P(F) = 0.5 \times 0.5 = 0.25$$

- Our objective to study this section is to compute probabilities of such (independent) events.

TABLE OF CONTENTS

INDEPENDENT EVENTS

- We use the Addition Rule for Disjoint Events to compute the probability of observing an outcome in event E or event F.
- We now describe a probability rule for computing the probability that E and F both occur.
- Before we can present this rule, we must discuss the idea of independent events.

Independt Events

Two events E and F are independent if the occurrence of event E in a probability experiment does not affect the probability of event F. Two events are dependent if the occurrence of event E in a probability experiment affects the probability of event F.

- Think about flipping a fair coin twice.

INDEPENDENT EVENTS

- Does the fact that you obtained a head on the first toss have any effect on the likelihood of obtaining a head on the second toss?
- Not unless you are a master coin flipper who can manipulate the outcome of a coin flip!
- For this reason, the outcome from the first flip is independent of the outcome from the second flip.
- Let's look at other examples.
- Our objective to solve this example is to understand the difference between mutually exclusive and independent events so that we can apply relevant formulas to compute the probabilities of events.

INDEPENDENT EVENTS

EXAMPLE 1

1. Suppose you flip a coin and roll a die. The events “obtain a head” and “roll a 5” are independent because the results of the coin flip do not affect the results of the die toss.

2. Are the events “earned a bachelor’s degree” and “earn more than \$100,000 per year” independent? No, because knowing that an individual has a bachelor’s degree affects the likelihood that the individual is earning more than \$100,000 per year.

3. Two 24-year-old male drivers who live in the United States are randomly selected. The events “male 1 gets in a car accident during the year” and “male 2 gets in a car accident during the year” are independent because the males were randomly selected. This means what happens with one of the drivers has nothing to do with what happens to the other driver.

INDEPENDENT EVENTS

- In Example 1 part 1, we are able to conclude that the events "male 1 gets in an accident" and "male 2 gets in an accident" are independent because the individuals are randomly selected.
- By randomly selecting the individuals, it is reasonable to conclude that the individuals are not related in any way (related in the sense that they do not live in the same town, attend the same school, and so on).
- If the two individuals did have a common link between them (such as they both lived on the same city block), then knowing that one male had a car accident may affect the likelihood that the other male had a car accident.
- After all, they could hit each other!

INDEPENDENT EVENTS

Disjoint Events versus Independent Events

- Disjoint events and independent events are different concepts.
- Recall that two events are disjoint if they have no outcomes in common, that is, if knowing that one of the events occurs, we know the other event did not occur.
- Independence means that one event occurring does not affect the probability of the other event occurring.
- Therefore, knowing two events are disjoint means that the events are not independent.
- Consider the experiment of rolling a single die.
- Let E represent the event “roll an even number,” and let F represent the event “roll an odd number.”

INDEPENDENT EVENTS

- We can see that E and F are mutually exclusive (disjoint) because they have no outcomes in common.
- In addition, $P(E) = \frac{1}{2}$ and $P(F) = \frac{1}{2}$.
- However, if we are told that the roll of the die is going to be an even number, then what is the probability of event F?
- Because the outcome will be even, the probability of event F is now 0 (and the probability of event E is now 1).
- So knowledge of event E changes the likelihood of observing event F.

TABLE OF CONTENTS

MULTIPLICATION RULE FOR INDEPENDENT EVENTS

- Suppose that you flip a fair coin twice.
- What is the probability that you obtain a head on both flips, that is, a head on the first flip and you obtain a head on the second flip?
- If H represents the outcome "heads" and T represents the outcome "tails," the sample space of this experiment is

$$S = \{HH, HT, TH, TT\}$$

- There is one outcome with both heads.
- Because each outcome is equally likely, we have

MULTIPLICATION RULE FOR INDEPENDENT EVENTS

$$
\begin{aligned}
P(\text{heads on Flip 1 and heads on Flip 2}) &= \frac{N(\text{heads on Flip1 and heads on Flip 2})}{N(S)} \\
&= \frac{1}{4}
\end{aligned}
$$

- We may have intuitively figured this out by recognizing $P(head) = \frac{1}{2}$ for each flip.
- So, it seems reasonable that

$$
\begin{aligned}
P(\text{heads on Flip 1 and heads on Flip 2}) &= P(\text{heads on Flip 1}).P(\text{heads on Flip 2}) \\
&= \frac{1}{2}.\frac{1}{2} \\
&= \frac{1}{4}
\end{aligned}
$$

MULTIPLICATION RULE FOR INDEPENDENT EVENTS

- Because both approaches result in the answer, $\frac{1}{4}$, we conjecture that $P(E \text{ and } F) = P(E).P(F)$, which is true.

Multiplication Rule for Independent Events

If E and F are independent events, then

$$P(E \text{ and } F) = P(E).P(F)$$

- We can extend the Multiplication Rule for three or more independent events.

Multiplication Rule for Independent Events

If events E_1, E_2, E_3, $\ldots$, E_n are independent, then

$$P(E_1 \text{ and } E_2 \text{ and } E_3 \text{ and } \ldots E_n) = P(E_1).P(E_2).P(E_3)\ldots P(E_n)$$

MULTIPLICATION RULE FOR INDEPENDENT EVENTS

EXAMPLE 2

In the game of roulette, the wheel has slots numbered 0, 00, and 1 through 36. A metal ball rolls around a wheel until it falls into one of the numbered slots. What is the probability that the ball will land in the slot numbered 17 two times in a row?

EXAMPLE 3

The probability that a randomly selected 24-year-old male will survive the year is 0.9986 according to the National Vital Statistics Report, Vol. 56, No. 9. (a) What is the probability that three randomly selected 24-year-old males will survive the year? (b) What is the probability that 20 randomly selected 24-year-old males will survive the year?

TABLE OF CONTENTS

AT-LEAST PROBABILITIES

- Usually probabilities involving the phrase at least use the Complement Rule.
- The phrase at least means “greater than or equal to.”
- For example, a person must be at least 17 years old to see an R-rated movie.
- This means that the person’s age must be greater than or equal to 17 to watch the movie.
- The example given below will clarify the at-least probabilities.

EXAMPLE 4

Compute the probability that at least one male out of 1000 aged 24 years will die during the course of the year if the probability that a randomly selected 24-year-old male survives the year is 0.9986.

TABLE OF CONTENTS

SUMMARY

- The probability of any event must be between 0 and 1, inclusive. If we let E denote any event, then $0 \leq P(E) \leq 1$.
- The sum of the probabilities of all outcomes in the sample space must equal 1. That is, if the sample space $S = \{e_1, e_2, \dots e_n\}$, then $P(e_1) + P(e_2) + \cdots + P(e_n) = 1$.
- If E and F are disjoint events, then $P(E \text{ or } F) = P(E) + P(F) - P(E \text{ and } F)$.
- If E represents any event and E^c represents the complement of E, then $P(E^c) = 1 - P(E)$.
- If E and F are independent events, then $P(E \text{ and } F) = P(E).P(F)$.


Thank You!