EXAMPLE 4 Interpreting the Area under a Normal Curve

Problem The serum total cholesterol for males 20–29 years old is approximately normally distributed with mean $\mu = 180$ and $\sigma = 36.2$, based on data obtained from the National Health and Nutrition Examination Survey.

(a) Draw a normal curve with the parameters labeled.

(b) An individual with total cholesterol greater than 200 is considered to have high cholesterol. Shade the region under the normal curve to the right of $x = 200$.

(c) Suppose that the area under the normal curve to the right of $x = 200$ is 0.2903. (You will learn how to find this area in Section 7.2.) Provide two interpretations of this result.

Approach

(a) Draw the normal curve with the mean $\mu = 180$ labeled at the high point and the inflection points at $\mu - \sigma = 180 - 36.2 = 143.8$ and $\mu + \sigma = 180 + 36.2 = 216.2$.

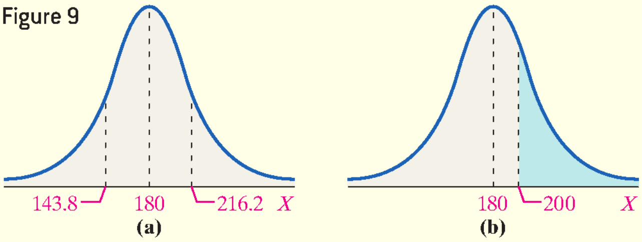
(b) Shade the region under the normal curve to the right of $x = 200$.

(c) The two interpretations of the area under a normal curve are (1) a proportion and (2) a probability.

Solution

(a) Figure 9(a) shows the graph of the normal curve.

Figure 9



(b) Figure 9(b) shows the region under the normal curve to the right of $x = 200$ shaded.

(c) The two interpretations for the area of this shaded region are (1) the proportion of 20- to 29-year-old males that have high cholesterol is 0.2903 and (2) the probability that a randomly selected 20- to 29-year-old male has high cholesterol is 0.2903. •

Historical Note

The normal probability distribution is often referred to as the Gaussian distribution in honor of Carl Gauss, the individual thought to have discovered the idea. However, it was actually Abraham de Moivre who first wrote down the equation of the normal distribution. Gauss was born in Brunswick, Germany, on April 30, 1777. Mathematical prowess was evident early in Gauss's life. At age 8 he was able to instantly add the first 100 integers. In 1799, Gauss earned his doctorate. The subject of his dissertation was the Fundamental Theorem of Algebra. In 1809, Gauss published a book on the mathematics of planetary orbits. In this book, he further developed the theory of least-squares regression by analyzing the errors. The analysis of these errors led to the discovery that errors follow a normal distribution. Gauss was considered to be "glacially cold" as a person and had troubled relationships with his family. Gauss died on February 23, 1855.

• Now Work Problems 31 and 35

7.1 Assess Your Understanding

Vocabulary and Skill Building

1. A __________ __________ __________ is an equation used to compute probabilities of continuous random variables.

2. A ______ is an equation, table, or graph used to describe reality.

3. *True or False*: The normal curve is symmetric about its mean, μ.

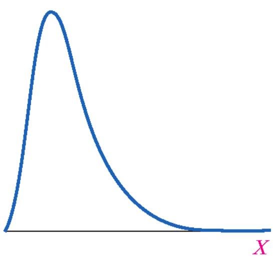
4. The area under the normal curve to the right of μ equals ______.

5. The points at $x =$ ______ and $x =$ ______ are the inflection points on the normal curve.

6. The area under a normal curve can be interpreted as a __________ or __________.

For Problems 7–12, determine whether the graph can represent a normal curve. If it cannot, explain why.

7.



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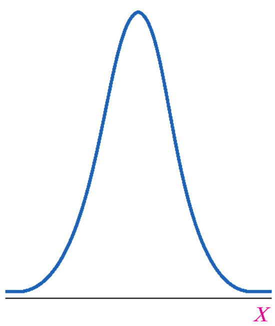

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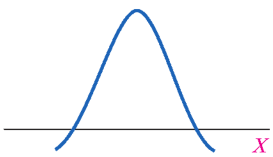

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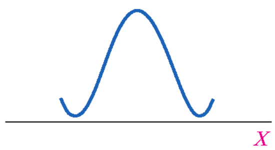

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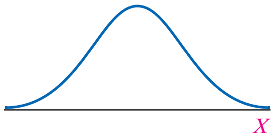

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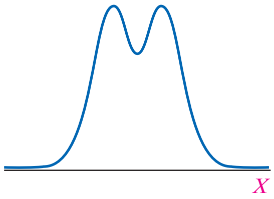


Problems 13–16 use the information presented in Examples 1 and 2.

NW **13. (a)** Find the probability that your friend is between 5 and 10 minutes late.

(b) It is 10 A.M. There is a 40% probability your friend will arrive within the next _____ minutes.

14. (a) Find the probability that your friend is between 15 and 25 minutes late.

(b) It is 10 A.M. There is a 90% probability your friend will arrive within the next _____ minutes.

15. Find the probability that your friend is at least 20 minutes late.

16. Find the probability that your friend is no more than 5 minutes late.

17. Uniform Distribution The random-number generator on calculators randomly generates a number between 0 and 1. The random variable X, the number generated, follows a uniform probability distribution.

(a) Draw the graph of the uniform density function.

(b) What is the probability of generating a number between 0 and 0.2?

(c) What is the probability of generating a number between 0.25 and 0.6?

(d) What is the probability of generating a number greater than 0.95?

(e) Use your calculator or statistical software to randomly generate 200 numbers between 0 and 1. What proportion of the numbers are between 0 and 0.2? Compare the result with part (b).

18. Uniform Distribution The reaction time X (in minutes) of a certain chemical process follows a uniform probability distribution with $5 \le X \le 10$.

(a) Draw the graph of the density curve.

(b) What is the probability that the reaction time is between 6 and 8 minutes?

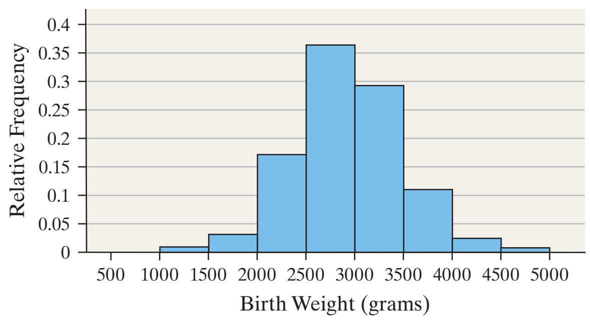
(c) What is the probability that the reaction time is between 5 and 8 minutes?

(d) What is the probability that the reaction time is less than 6 minutes?

In Problems 19–22, determine whether or not the histogram indicates that a normal distribution could be used as a model for the variable.

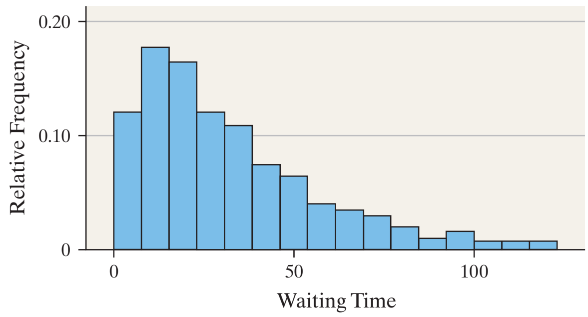
19. Birth Weights The relative frequency histogram represents the birth weights (in grams) of babies whose term was 36 weeks.

Birth Weights of Babies Whose Term Was 36 Weeks



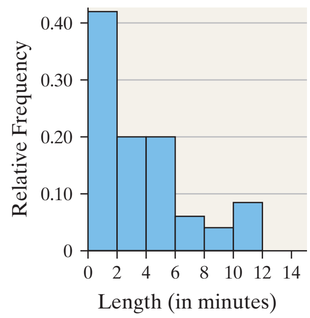
20. Waiting in Line The relative frequency histogram represents the waiting times (in minutes) to ride the American Eagle Roller Coaster for 2000 randomly selected people on a Saturday afternoon in the summer.

Waiting Time for the American Eagle Roller Coaster



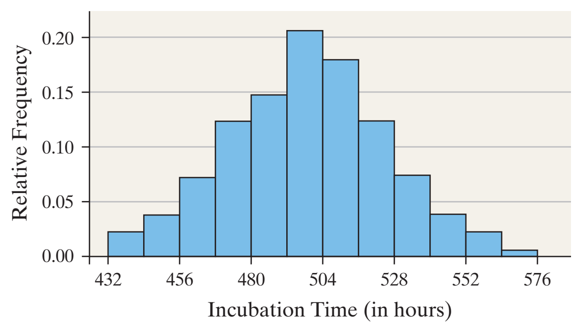
21. Length of Phone Calls The relative frequency histogram represents the length of phone calls on my wife's cell phone during the month of September.

Length of Phone Calls



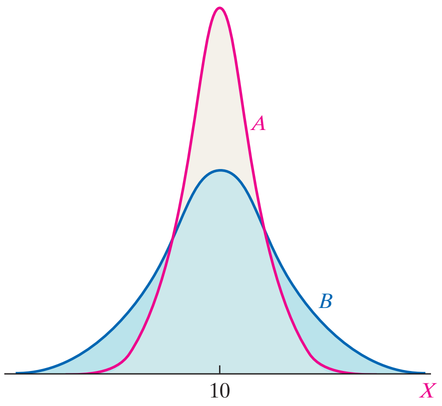
22. Incubation Times The relative frequency histogram represents the incubation times of a random sample of Rhode Island Red hens' eggs.

Rhode Island Red Hen Incubation Times

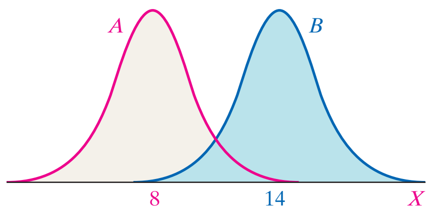


23. One graph in the figure on the following page represents a normal distribution with mean $\mu = 10$ and standard deviation $\sigma = 3$. The other graph represents a normal distribution with mean $\mu = 10$ and standard deviation $\sigma = 2$.

Determine which graph is which and explain how you know.

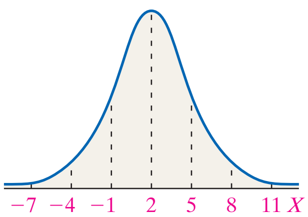


24. One graph in the figure below represents a normal distribution with mean $\mu = 8$ and standard deviation $\sigma = 2$. The other graph represents a normal distribution with mean $\mu = 14$ and standard deviation $\sigma = 2$. Determine which graph is which and explain how you know.

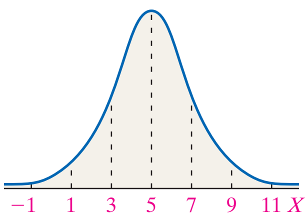


In Problems 25–28, the graph of a normal curve is given. Use the graph to identify the values of μ and σ.

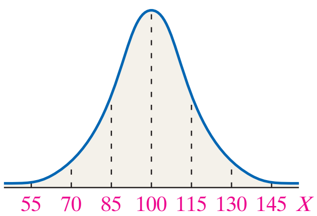
NW **25.**



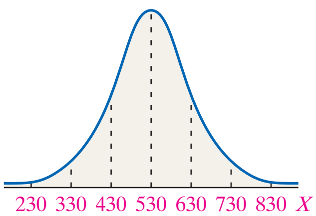
26.



27.



28.



In Problems 29 and 30, draw a normal curve and label the mean and inflection points.

29. $\mu = 30$ and $\sigma = 10$

30. $\mu = 50$ and $\sigma = 5$

Applying the Concepts

NW **31. You Explain It! Cell Phone Rates** Monthly charges for cell phone plans in the United States are normally distributed with mean $\mu = \$62$ and standard deviation $\sigma = \$18$.
Source: Based on information from *Consumer Reports*

(a) Draw a normal curve with the parameters labeled.
(b) Shade the region that represents the proportion of plans that charge less than \$44.
(c) Suppose the area under the normal curve to the left of $x = \$44$ is 0.1587. Provide two interpretations of this result.

32. You Explain It! Refrigerators The lives of refrigerators are normally distributed with mean $\mu = 14$ years and standard deviation $\sigma = 2.5$ years.
Source: Based on information from *Consumer Reports*

(a) Draw a normal curve with the parameters labeled.
(b) Shade the region that represents the proportion of refrigerators that last for more than 17 years.
(c) Suppose the area under the normal curve to the right of $x = 17$ is 0.1151. Provide two interpretations of this result.

33. You Explain It! Birth Weights The birth weights of full-term babies are normally distributed with mean $\mu = 3400$ grams and $\sigma = 505$ grams.
Source: Based on data obtained from the *National Vital Statistics Report,* Vol. 48, No. 3

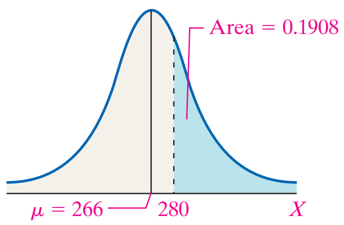
(a) Draw a normal curve with the parameters labeled.
(b) Shade the region that represents the proportion of full-term babies who weigh more than 4410 grams.
(c) Suppose the area under the normal curve to the right of $x = 4410$ is 0.0228. Provide two interpretations of this result.

34. You Explain It! Height of 10-Year-Old Males The heights of 10-year-old males are normally distributed with mean $\mu = 55.9$ inches and $\sigma = 5.7$ inches.

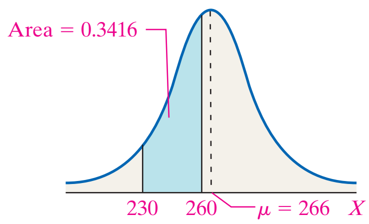
(a) Draw a normal curve with the parameters labeled.
(b) Shade the region that represents the proportion of 10-year-old males who are less than 46.5 inches tall.
(c) Suppose the area under the normal curve to the left of $x = 46.5$ is 0.0496. Provide two interpretations of this result.

NW **35. You Explain It! Gestation Period** The lengths of human pregnancies are normally distributed with $\mu = 266$ days and $\sigma = 16$ days.

(a) The figure represents the normal curve with $\mu = 266$ days and $\sigma = 16$ days. The area to the right of $x = 280$ is 0.1908. Provide two interpretations of this area.

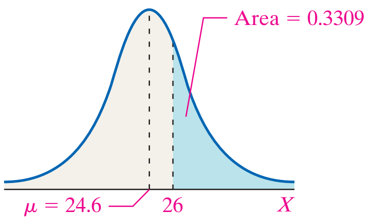


(b) The figure represents the normal curve with $\mu = 266$ days and $\sigma = 16$ days. The area between $x = 230$ and $x = 260$ is 0.3416. Provide two interpretations of this area.

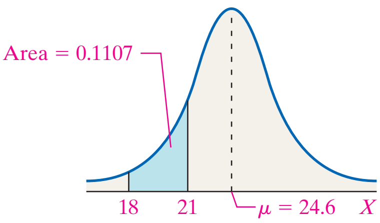


36. You Explain It! Miles per Gallon Elena conducts an experiment in which she fills up the gas tank on her Toyota Camry 40 times and records the miles per gallon for each fill-up. A histogram of the miles per gallon indicates that a normal distribution with a mean of 24.6 miles per gallon and a standard deviation of 3.2 miles per gallon could be used to model the gas mileage for her car.

(a) The figure represents the normal curve with $\mu = 24.6$ miles per gallon and $\sigma = 3.2$ miles per gallon. The area under the curve to the right of $x = 26$ is 0.3309. Provide two interpretations of this area.



(b) The figure below represents the normal curve with $\mu = 24.6$ miles per gallon and $\sigma = 3.2$ miles per gallon. The area under the curve between $x = 18$ and $x = 21$ is 0.1107. Provide two interpretations of this area.



37. Hitting with a Pitching Wedge In the game of golf, distance control is just as important as how far a player hits the ball. Michael went to the driving range with his range finder and hit 75 golf balls with his pitching wedge and measured the distance each ball traveled (in yards). He obtained the following data:

100	97	101	101	103	100	99	100	100
104	100	101	98	100	99	99	97	101
104	99	101	101	101	100	96	99	99
98	94	98	107	98	100	98	103	100
98	94	104	104	98	101	99	97	103
102	101	101	100	95	104	99	102	95
99	102	103	97	101	102	96	102	99
96	108	103	100	95	101	103	105	100
94	99	95						

(a) Use statistical software to construct a relative frequency histogram. Comment on the shape of the distribution. Draw a normal density curve on the relative frequency histogram.

(b) Do you think the normal density curve accurately describes the distance Michael hits with a pitching wedge? Why?

38. Heights of Five-Year-Old Females The following data represent the heights (in inches) of 80 randomly selected five-year-old females.

44.5	42.4	42.2	46.2	45.7	44.8	43.3	39.5
45.4	43.0	43.4	44.7	38.6	41.6	50.2	46.9
39.6	44.7	36.5	42.7	40.6	47.5	48.4	37.5
45.5	43.3	41.2	40.5	44.4	42.6	42.0	40.3
42.0	42.2	38.5	43.6	40.6	45.0	40.7	36.3
44.5	37.6	42.2	40.3	48.5	41.6	41.7	38.9
39.5	43.6	41.3	38.8	41.9	40.3	42.1	41.9
42.3	44.6	40.5	37.4	44.5	40.7	38.2	42.6
44.0	35.9	43.7	48.1	38.7	46.0	43.4	44.6
37.7	34.6	42.4	42.7	47.0	42.8	39.9	42.3

(a) Use statistical software to construct a relative frequency histogram. Comment on the shape of the distribution. Draw a normal density curve on the relative frequency histogram.

(b) Do you think the normal density curve accurately describes the heights of five-year-old females? Why?

Retain Your Knowledge

39. Cardiac Arrest Researchers conducted a prospective cohort study in which male patients who had an out-of-hospital cardiac arrest were submitted to therapeutic hypothermia (intravenous infusion of cold saline followed by surface cooling with the goal of maintaining body temperature of 33 degrees Celsius for 24 hours. Note that normal body temperature is 37 degrees Celsius). The survival status, length of stay in the intensive care unit (ICU), and time spent on a ventilator were measured. Each of these variables was compared to a historical cohort of patients who were treated prior to the availability of therapeutic hypothermia. Of the 52 hypothermia patients, 37 survived; of the 74 patients in the control group, 43 survived. The median length of stay among survivors for the hypothermia patients was 14 days versus 21 days for the control group. The time on the ventilator among survivors for the hypothermia group was 219 hours versus 328 hours for the control group.

Source: Storem, Christian, et al. Mild Therapeutic Hypothermia Shortens Intensive Care Unit Stay of Survivors After Out-of-Hospital Cardiac Arrest Compared to Historical Controls. *Critical Care* 2008, 12:R78 BioMed Central

(a) What does it mean to say this is a prospective cohort study?

(b) What is the explanatory variable in the study? Is it qualitative or quantitative?

(c) What are the three response variables in the study? For each, state whether the variable is qualitative or quantitative.

(d) Is time on the ventilator a statistic or parameter? Explain.

(e) To what population does this study apply?

(f) Based on the results of this study, what is the probability a randomly selected male who has an out-of-hospital cardiac arrest and submits to therapeutic hypothermia will survive? What about those who do not submit to therapeutic hypothermia?

Preparing for This Section Before getting started, review the following:

- z-scores (Section 3.4, pp. 183–184)
- Percentiles (Section 3.4, p. 184)
- Complement Rule (Section 5.2, p. 295)

Objectives

1. Find and interpret the area under a normal curve
2. Find the value of a normal random variable

If X is a normally distributed random variable, the area under the normal curve represents the proportion of a population with a certain characteristic, or the probability that a randomly selected individual from the population has the characteristic.

The question then is, "How do I find the area under the normal curve?" We have two options—by-hand calculations with the aid of a table or technology.

1 Find and Interpret the Area under a Normal Curve

We use z-scores to help find the area under a normal curve by hand. Recall, the z-score allows us to transform a random variable X with mean μ and standard deviation σ into a random variable Z with mean 0 and standard deviation 1.

Standardizing a Normal Random Variable

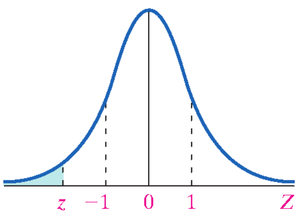
Suppose that the random variable X is normally distributed with mean μ and standard deviation σ. Then the random variable

$$Z = \frac{X - \mu}{\sigma}$$

is normally distributed with mean $\mu = 0$ and standard deviation $\sigma = 1$. The random variable Z is said to have the **standard normal distribution**.

This result is powerful! If a normal random variable X has mean different from 0 or a standard deviation different from 1, we can transform X into a **standard normal random variable Z** whose mean is 0 and standard deviation is 1. Then we can use Table V (found on the inside back cover of the text and in Appendix A) to find the area to the left of a specified z-score, z, as shown in Figure 10, which is also the area to the left of the value of x in the distribution of X. The graph in Figure 10 is called the **standard normal curve**.

Figure 10



For example, IQ scores can be modeled by a normal distribution with $\mu = 100$ and $\sigma = 15$. An individual whose IQ is 120, is $z = \frac{x - \mu}{\sigma} = \frac{120 - 100}{15} = 1.33$ standard deviations above the mean (recall, we round z-scores to two decimal places). We look in Table V and find the area under the standard normal curve to the left of $z = 1.33$ is 0.9082. See Figure 11. Therefore, the area under the normal curve to the left of $x = 120$ is 0.9082 as shown in Figure 12.

To find the area to the right of the value of a random variable, use the Complement Rule and determine one minus the area to the left. For example, to find the area under the normal curve with mean $\mu = 100$ and standard deviation $\sigma = 15$ to the right of $x = 120$, compute

$$\begin{aligned} \text{Area} &= 1 - 0.9082 \\ &= 0.0918 \end{aligned}$$

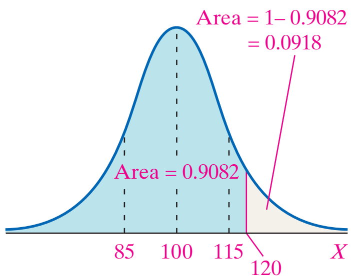
as shown in Figure 12.

Figure 11

Standard Normal Distribution

z	.00	.01	.02	.03	.04	.05	.06
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.927[illegible]
1.5	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

Figure 12



EXAMPLE 1 Finding Area under a Normal Curve

Problem A pediatrician obtains the heights of her three-year-old female patients. The heights are approximately normally distributed, with mean 38.72 inches and standard deviation 3.17 inches. Use the normal model to determine the proportion of the three-year-old females that have a height less than 35 inches.

By-Hand Approach

Step 1 Draw a normal curve and shade the desired area.

Step 2 Convert the value of x to a z-score using

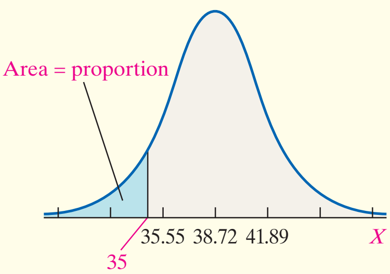
$$z = \frac{x - \mu}{\sigma}$$

Step 3 Use Table V to find the area to the left of the z-score found in Step 2.

By-Hand Solution

Step 1 Figure 13 shows the normal curve with the area to the left of 35 shaded.

Figure 13



Step 2 Convert $x = 35$ to a z-score.

$$z = \frac{x - \mu}{\sigma} = \frac{35 - 38.72}{3.17} = -1.17$$

Technology Approach

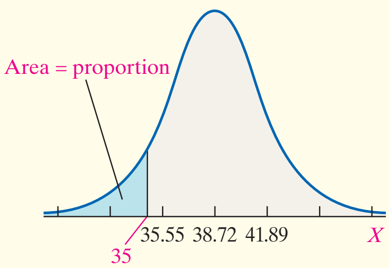
Step 1 Draw a normal curve and shade the desired area.

Step 2 Use a statistical spreadsheet or calculator with advanced statistical features to find the area. The steps for determining the area under any normal curve using the TI-83/84 Plus graphing calculator, Minitab, Excel, and StatCrunch are found in the Technology Step-by-Step on pages 401–402.

Technology Solution

Step 1 Figure 14 shows the normal curve with the area to the left of 35 shaded.

Figure 14



(continued)

Step 3 Look up $z = -1.17$ in Table V and find the entry. The area to the left of $z = -1.17$ is 0.1210. See Figure 15. Therefore, the area to the left of $x = 35$ is 0.1210.

Figure 15

z	.00	.01	.02	.03	.04	.05	.06	.07	.08
−3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003
−3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004
−3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005
−1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694
−1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838
−1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003
−1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190
−1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401

The normal model indicates that the proportion of the pediatrician's three-year-old females who are less than 35 inches tall is 0.1210.

Step 2 Figure 16 shows the results from Minitab. The area under the normal curve to the left of 35 is 0.1203.

Figure 16

Cumulative Distribution Function

```
Normal with mean = 38.72 and standard deviation = 3.17

 x  P( X <= x )
35     0.120297
```

The normal model indicates that the proportion of the pediatrician's three-year-old females who are less than 35 inches tall is 0.1203.

CAUTION!
Notice the by-hand solution and technology solution in Example 1 differ. The difference exists because we rounded the z-score in Step 2 of the by-hand solution, which leads to rounding error.

Table 2

Height (inches)	Relative Frequency
29.0–29.9	0.005
30.0–30.9	0.005
31.0–31.9	0.005
32.0–32.9	0.025
33.0–33.9	0.02
34.0–34.9	0.055
35.0–35.9	0.075
36.0–36.9	0.09
37.0–37.9	0.115
38.0–38.9	0.15
39.0–39.9	0.12
40.0–40.9	0.11
⋮	⋮
47.0–47.9	0.005

According to the results of Example 1, the proportion of three-year-old females who are shorter than 35 inches is approximately 0.12. If the normal curve is a good model for determining proportions (or probabilities), then about 12% of the three-year-olds in Table 1 (from Section 7.1) should be shorter than 35 inches. For convenience, part of Table 1 is repeated in Table 2.

The relative frequency distribution in Table 2 shows that $0.005 + 0.005 + 0.005 + 0.025 + 0.02 + 0.055 = 0.115 = 11.5\%$ of the three-year-old females are less than 35 inches tall. The results based on the normal curve are close to the actual results. The normal curve accurately models the heights.

If we wanted to know the proportion of three-year-old females whose height is greater than 35 inches, use the Complement Rule and find the proportion is $1 - 0.1210 = 0.879$ (using the "by-hand" computation).

Because the area under the normal curve represents a proportion, we can also use the area to find percentile ranks of scores. Recall that the kth percentile divides the lower $k\%$ of a data set from the upper $(100 - k)\%$. In Example 1, 12% of the females have a height less than 35 inches, and 88% of the females have a height greater than 35 inches, so a child whose height is 35 inches is at the 12th percentile.

EXAMPLE 2 Finding the Probability of a Normal Random Variable

Problem For the pediatrician presented in Example 1, find the probability that a randomly selected three-year-old girl is between 35 and 40 inches tall, inclusive. That is, find $P(35 \le X \le 40)$.

By-Hand Approach

Step 1 Draw a normal curve and shade the desired area.

Step 2 Convert the values of x to z-scores using

$$z = \frac{x - \mu}{\sigma}$$

Step 3 Use Table V to find the area to the left of each z-score found in Step 2. Use this result to find the area between the z-scores.

Technology Approach

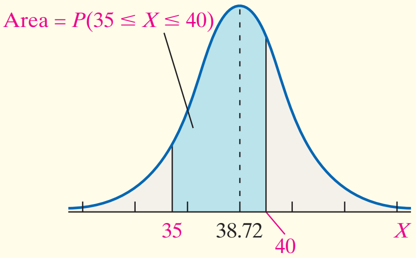
Step 1 Draw a normal curve and shade the desired area.

Step 2 Use a statistical spreadsheet or calculator with advanced statistical features to find the area. The steps for determining the area under any normal curve using the TI-83/84 Plus graphing calculator, Minitab, Excel, and StatCrunch are found in the Technology Step-by-Step on pages 401–402.

By-Hand Solution

Step 1 Figure 17 shows the normal curve with the area between 35 and 40 shaded.

Figure 17



Step 2 Convert $x_1 = 35$ and $x_2 = 40$ to z-scores.

$$z_1 = \frac{x_1 - \mu}{\sigma} = \frac{35 - 38.72}{3.17} = -1.17$$

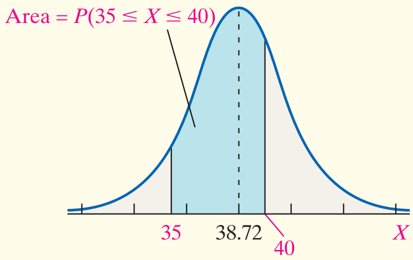
$$z_2 = \frac{x_2 - \mu}{\sigma} = \frac{40 - 38.72}{3.17} = 0.40$$

Step 3 Table V shows that the area to the left of $z_2 = 0.4$ (or $x_2 = 40$) is 0.6554 and the area to the left of $z_1 = -1.17$ (or $x_1 = 35$) is 0.1210, so the area between $z_1 = -1.17$ and $z_2 = 0.40$ is $0.6554 - 0.1210 = 0.5344$. The probability a randomly selected three-year-old female is between 35 and 40 inches tall is 0.5344. That is, $P(35 \le X \le 40) = P(-1.17 \le Z \le 0.40) = 0.5344$.

Technology Solution

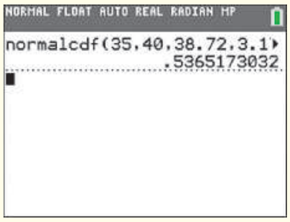
Step 1 Figure 18 shows the normal curve with the area between 35 and 40 shaded.

Figure 18



Step 2 Figure 19 shows the results from a TI-84 Plus C graphing calculator.

Figure 19



The area between $x = 35$ and $x = 40$ is 0.5365. The probability a randomly selected three-year-old female is between 35 and 40 inches tall is 0.5365. That is, $P(35 \le X \le 40) = 0.5365$.

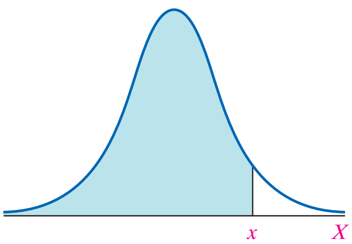
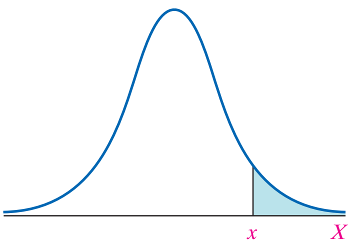
Interpretation If we randomly selected 100 three-year-old females, we would expect about 53 or 54 of them to be between 35 and 40 inches tall. •

• **Now Work Problem 39**

According to the relative frequency distribution in Table 2, the proportion of three-year-old females with heights between 35 and 40 inches is $0.075 + 0.09 + 0.115 + 0.15 + 0.12 = 0.55$. This is very close to the probability found in Example 2.

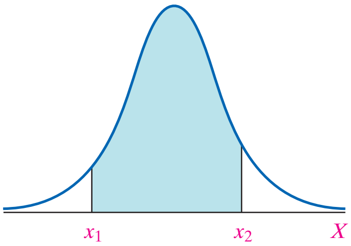
We summarize the methods for obtaining the area under a normal curve in Table 3.

Table 3

Problem	Approach	Solution
Find the area to the left of x.	Shade the area to the left of x. 	• Convert the value of x to a z-score. Use Table V to find the row and column that correspond to z. The area to the left of x is the value where the row and column intersect. or • Use technology to find the area.
Find the area to the right of x.	Shade the area to the right of x. 	• Convert the value of x to a z-score. Use Table V to find the area to the left of z (which is also the area to the left of x). The area to the right of z (also x) is 1 minus the area to the left of z. or • Use technology to find the area.

(continued)

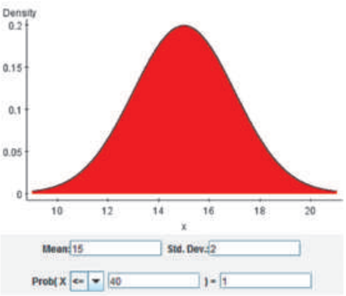
Table 3 (Continued)

Problem	Approach	Solution
Find the area between x_1 and x_2.	Shade the area between x_1 and x_2. 	• Convert the values of x to z-scores. Use Table V to find the area to the left of z_1 and the area to the left of z_2. The area between z_1 and z_2 is (area to the left of z_2) − (area to the left of z_1). or • Use technology to find the area.

Some Cautionary Thoughts

The normal curve extends indefinitely in both directions. For this reason, there is no range of values of a normal random variable for which the area under the curve is 1. For example, if asked to find the area under a normal curve to the left of $x = 40$ with $\mu = 15$ and $\sigma = 2$, StatCrunch (as well as other software and calculators) will state the area is 1, because it can only compute a limited number of decimal places. See Figure 20. However, the area under the curve to the left of $x = 40$ is not 1; it is some value slightly less than 1. So we will follow the practice of reporting such areas as >0.9999. Similarly, if software reports an area of 0, we will report the area as <0.0001.

Figure 20



When finding area under the normal curve by hand using Table V, we will report any area to the left of $z = -3.49$ (the smallest value of z in the table) or to the right of $z = 3.49$ (the largest value of z in the table) as <0.0001. Any area under the normal curve to the left of $z = 3.49$ or to the right of $z = -3.49$ is stated as >0.9999.

2 Find the Value of a Normal Random Variable

Often, we do not want to find the proportion, probability, or percentile given a value of a normal random variable. Rather, we want to find the value of a normal random variable that corresponds to a certain proportion, probability, or percentile. For example, we might want to know the height of a three-year-old girl who is at the 20th percentile. Or we might want to know the scores on a standardized exam that separate the middle 90% of scores from the bottom and top 5%.

EXAMPLE 3 Finding the Value of a Normal Random Variable

Problem The heights of a pediatrician's three-year-old females are approximately normally distributed, with mean 38.72 inches and standard deviation 3.17 inches. Find the height of a three-year-old female at the 20th percentile.

By-Hand Approach

Step 1 Draw a normal curve and shade the desired area.

Step 2 Use Table V to find the z-score that corresponds to the shaded area.

Step 3 Obtain the normal value from the formula $x = \mu + z\sigma$.*

Technology Approach

Step 1 Draw a normal curve and shade the desired area.

Step 2 Use a statistical spreadsheet or calculator with advanced statistical features to find the score. The steps for determining the value of a normal random variable, given an area, using the TI-83/84 Plus graphing

*The formula provided in Step 3 of the by-hand approach is the formula for computing a z-score, solved for x.

$z = \dfrac{x - \mu}{\sigma}$ Formula for standardizing a value, x, for a random variable X

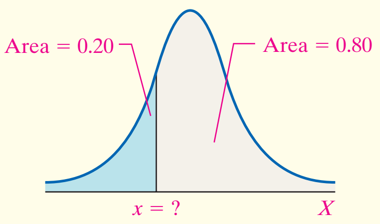
$z\sigma = x - \mu$ Multiply both sides by σ.

$x = \mu + z\sigma$ Add μ to both sides.

By-Hand Solution

Step 1 Figure 21 shows the normal curve with the unknown value of x at the 20th percentile, which separates the bottom 20% of the distribution from the top 80%.

Figure 21



Step 2 We want to find the z-score such that the area to the left of the z-score is 0.20. Refer to Table V and look in the body of the table for the area closest to 0.20. The area closest to 0.20 is 0.2005, which corresponds to a z-score of −0.84. See Figure 22.

Figure 22

Standard Normal Distribution

z	.00	.01	.02	.03	.04	.05
−3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003
−3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004
−0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711
−0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977
−0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266

Step 3 The height of a three-year-old female at the 20th percentile is

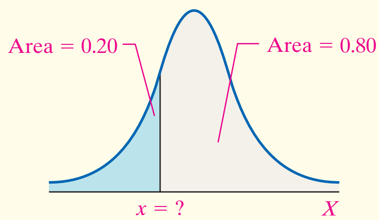
$$\begin{aligned} x &= \mu + z\sigma \\ &= 38.72 + (-0.84)(3.17) \\ &= 36.1 \text{ inches} \end{aligned}$$

calculator, Minitab, Excel, and StatCrunch are found in the Technology Step-by-Step on pages 401–402.

Technology Solution

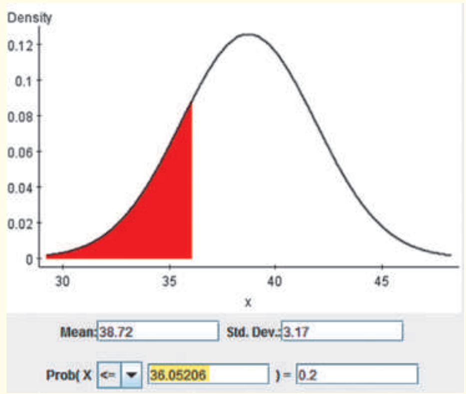
Step 1 Figure 23 shows the normal curve with the unknown value of x at the 20th percentile, which separates the bottom 20% of the distribution from the top 80%.

Figure 23



Step 2 Figure 24 shows the results obtained from StatCrunch. The height of a three-year-old female at the 20th percentile is 36.1 inches.

Figure 24



• **Now Work Problem 47(a)**

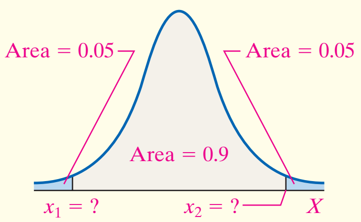
EXAMPLE 4 Finding the Value of a Normal Random Variable

Problem The scores earned on the mathematics portion of the SAT, a college entrance exam, are approximately normally distributed with mean 516 and standard deviation 116. What scores separate the middle 90% of test takers from the bottom and top 5%? In other words, find the 5th and 95th percentiles. *Source:* The College Board

By-Hand Solution

Step 1 Figure 25 shows the normal curve with the unknown values of x separating the bottom and top 5% of the distribution from the middle 90%.

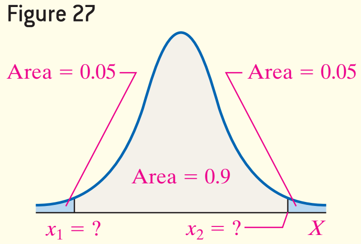
Figure 25



Technology Solution

Step 1 Figure 27 shows the normal curve with the unknown values of x separating the bottom and top 5% of the distribution from the middle 90%.

Figure 27



(continued)

Step 2 First, find the z-score that corresponds to an area of 0.05 to the left. In Table V, look in the body of the table and find that 0.0495 and 0.0505 are equally close to 0.05. See Figure 26. We agree to take the mean of the two z-scores corresponding to the areas. The z-score corresponding to an area of 0.0495 is -1.65, and the z-score corresponding to an area of 0.0505 is -1.64. The approximate z-score corresponding to an area of 0.05 to the left is $z_1 = -1.645$.

Figure 26

Standard Normal Distribution

z	.00	.01	.02	.03	.04	.05
−3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003
−3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004
−1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322
−1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401
−1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495
−1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606

Now find the z-score corresponding to an area of 0.05 to the right, which means the area to the left is 0.95. From Table V, we find an area of 0.9495 and 0.9505, which correspond to 1.64 and 1.65. The approximate z-score, such that the area to the right is 0.05, is $Z_2 = 1.645$.

Step 3 The SAT mathematics score that separates the bottom 5% from the top 95% of scores is

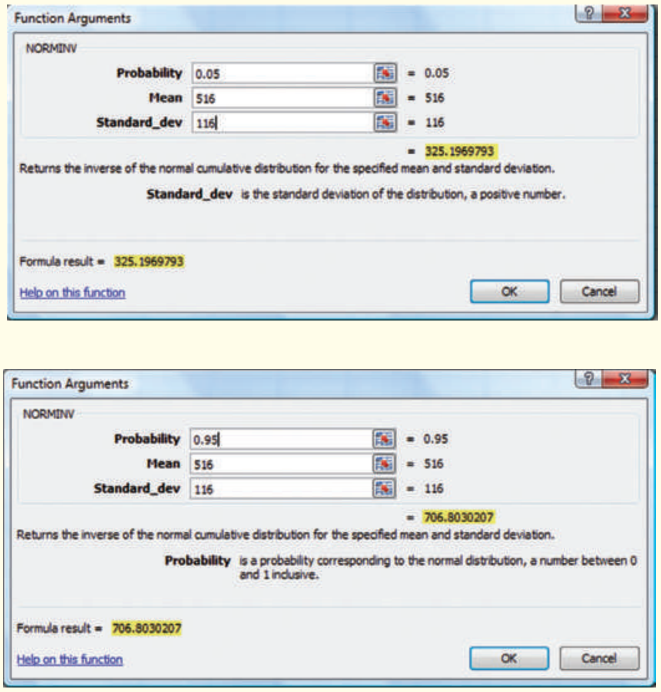
$$\begin{aligned} x_1 &= \mu + z_1\sigma \\ &= 516 + (-1.645)(116) \\ &= 325 \end{aligned}$$

The SAT mathematics score that separates the bottom 95% from the top 5% of scores is

$$\begin{aligned} x_2 &= \mu + z_2\sigma \\ &= 516 + (1.645)(116) \\ &= 707 \end{aligned}$$

Step 2 Figure 28 shows the results obtained from Excel with the values of x_1 and x_2 highlighted.

Figure 28



Interpretation SAT mathematics scores that separate the middle 90% of the scores from the bottom and top 5% are 325 and 707. Put another way, a student who scores 325 on the SAT math exam is at the 5th percentile. A student who scores 707 on the SAT math exam is at the 95th percentile. We might use these results to identify those scores that are unusual. •

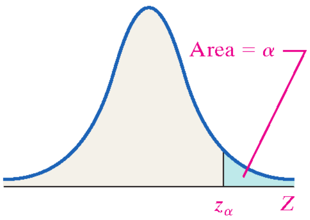
• **Now Work Problem 47(b)**

We could also obtain the by-hand solution to Example 4 using symmetry. Because the normal curve is symmetric about its mean, the z-score that corresponds to an area of 0.05 to the left will be the additive inverse (i.e., the opposite) of the z-score that corresponds to an area of 0.05 to the right. Since the area to the left of $z = -1.645$ is 0.05, the area to the right of $z = 1.645$ is 0.05.

Important Notation for the Future

In upcoming chapters, we will need to find the z-score that has a specified area to the right. We have special notation to represent this situation.

Figure 29



The notation z_α (pronounced "z sub alpha") is the z-score such that the area under the standard normal curve to the right of z_α is α. Figure 29 illustrates the notation.

EXAMPLE 5 Finding the Value of z_α

Problem Find the value of $z_{0.10}$.

Approach We wish to find the z-value such that the area under the standard normal curve to the right of the z-value is 0.10.

By-Hand Solution The area to the right of the unknown z-value is 0.10, so the area to the left of the z-value is $1 - 0.10 = 0.90$. We look in Table V for the area closest to 0.90. The closest area is 0.8997, which corresponds to a z-value of 1.28. Therefore, $z_{0.10} = 1.28$.

Technology Solution The area to the right of the unknown z-value is 0.10, so the area to the left is $1 - 0.10 = 0.90$. A TI-84 Plus C is used to find the z-value such that the area to the left is 0.90 is 1.28. See Figure 30. Therefore, $z_{0.10} = 1.28$.

Figure 30

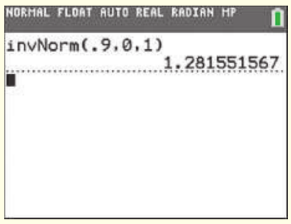
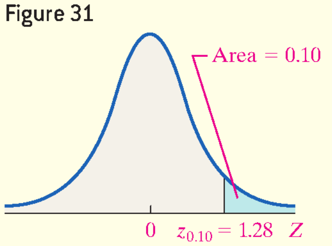


Figure 31 shows the z-value on the normal curve.

Figure 31



• Now Work Problem 19

For any continuous random variable, the probability of observing a specific value of the random variable is 0. For example, for a normal random variable, $P(a) = 0$ for any value of a, because there is no area under the normal curve associated with a single value. Therefore, the following probabilities are equivalent:

$$P(a < X < b) = P(a \le X < b) = P(a < X \le b) = P(a \le X \le b)$$

Technology Step-by-Step The Normal Distribution

TI-83/84 Plus

Finding Area under the Normal Curve

1. From the HOME screen, press 2nd VARS to access the DISTRibution menu.
2. Select `2:normalcdf(`.
3. Enter the lowerbound, upperbound, μ, and σ. Highlight Paste and hit ENTER. Hit ENTER again with the formula on the HOME screen.

Finding Normal Values Corresponding to an Area

1. From the HOME screen, press 2nd VARS to access the DISTRibution menu.
2. Select `3:invNorm(`.
3. Enter the *area left*, μ, and σ. Highlight Paste and hit ENTER. Hit ENTER again with the formula on the HOME screen.

Note: When there is no lowerbound, enter −1E99. When there is no upperbound, enter 1E99. The E shown is scientific notation; it is selected by pressing 2nd then ,.