Intro to Proofs: HW06

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11.1.1

11.1.4

$A = \{0, 1, 2, 3, 4, 5\}$ and

$R = \{(0, 0), (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (0, 4), (4, 0), (1, 3), (3, 1), (1, 5), (5, 1), (2, 4), (4, 2)\}$

11.1.11

Consider $|A| = n$ then the number of pairs of elements (a, b) where $a, b \in A$ is n^2. Now a relation can be defined as containing any possible combination of this set. So the set of relations would be the power set of the set of the pairs which would be

$$2^{n^2}$$

11.2.4

1. Reflexivity
For reflexivity we need $\forall a \in A, aRa$. We see that aRa, bRb, cRc, dRd therefore $A$is reflexive.
2. Symmetric
We need for any $aRb \Rightarrow bRa$. We see this is true by observation. For any pair (x, y) we see that (y, x) is in R as well. So it is symmetric.
3. Transitivity
We need $aRb, bRc \Rightarrow aRc$. We see that this is also true for any pair of relations in our form. So it is transitive.
Also notice that R contains all possible pairs of elements of A.

11.2.5

1. Reflexive
For reflexivity we need $\forall x \in \mathbb{R}, xRx$. However in our relation thish is only true for two values $0, \sqrt{2}$. Hence it is not reflexive. So for instance, $(1, 1) \notin R$ but $1 \in \mathbb{R}$.
2. Symmetric
We see that for any aRb, bRa is true. This is automatically true for $(0, 0)$ and $(\sqrt{2}, \sqrt{2})$ as they are symmetric. We see for $(0, \sqrt{2})$ that $(\sqrt{2}, 0)$ is also in R same can be said for $(\sqrt{2}, 0)$. So it is symmetric.
3. Transivity
We see that for any aRb, bRc that aRc is true. So it is transitive.

11.2.6

We have $R = \{(x, x) : x \in \mathbb{Z}\}$
1. Reflexive.
We see that $\forall x \in$, $(x, x) \in R$ by definition. So it is reflexive.
2. Symmetric
As our relation only contains pairs of (x, x) it is automatically symmetric as interchanging the numbers gives us the same pair. So it is symmetric.
3. Transitive

For transitivity we need, $aRb, bRc \Rightarrow aRc$. However in ours list $a = b$ by definition and $b = c$ by definition hence $a = c \Rightarrow aRc$. Hence it is transitive.

11.2.13

We have $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x - y \in \mathbb{Z}\}$
1. Reflexive
We need to show that $\forall x \in R$, xRx.
Take any $x \in R$ we have $x - x = 0 \in Z$. Hence x is related to itself or xRx is true. So it is reflexive.
2. Symmetric
For any xRy we need to show yRx.
Take a given xRy this implies that $x - y \in \mathbb{Z}$ so $x - y = k$ such that $k \in Z$. Now multiplying -1 on both sides we get, $y - x = -k$. If $k \in Z$ then $-k \in Z$ so $y - x \in \mathbb{Z}$ which means that yRx.
Therefore R is symmetric.
3. Transitive
We need to show $aRb, bRc \Rightarrow aRc$. aRb means $a-b \in \mathbb{Z}$ or $a-b = k_1$. Similarly, $b - c = k_2$.
Now add the two equation we get, $a - b + b - c = k_1 + k_2$ or $a - c = k_1 + k_2 \in \mathbb{Z}$. Therefore aRc is true. Hence R is transitive.

11.2.14

Proof. We are given that R is symmetric and transitive. We also know that $\exists a \in A$ such that $aRx, \forall x \in A$.
Now because it is symetric we know that $aRx \Rightarrow xRa$. So we have aRx, xRa is true.
If it is symmetric then $aRb, bRc \Rightarrow aRc$. So $xRa, aRx \Rightarrow xRx$ where x is any element of A.
Se showed that $\forall x \in A, xRx$ is true. Hence it is reflexive. □

11.2.15

Proof. We disprove by counterexample. Consider $A = \{0, 1\}$
The relation,

$$R = \{(0, 0)\}$$

is symmetriic because $(0, 0) \in R \Rightarrow (0, 0) \in R$ and transitive. Howeer it is not reflexive as $(1, 1) \notin R$. □

11.2.18

For $>$ we have, Reflexive: no, Symmetric: no, Transitive: yes. And for $\geq$ we have, Reflexive: yes, Symmetric: no, Transitive: yes.

11.3.3

We have $A = \{a, b, c, d, e\}$ and R has three equivalence classes. It is also given that $(a, d), (b, c) \in R$
Let the three equivalence classes be, $\{a, d\}, \{b, c\}, \{e\}$
So the equivalence relation defined using these classes would be,

$$R = \{(a, a), (b, b), (c, c), (d, d), (e, e), (a, d), (d, a), (b, c), (c, b)\}$$

We see that R is an equivalence relation because it is reflexive symmetric and transitive.

11.3.6

Given $A = \{a, b, c\}$. The differenr equivalence relation on A will currespond to the different equivalence classes in A. These are,

$$\begin{aligned}
\{a\}, \{b\}, \{c\} \text{ where R } &= \{(a, a), (b, b), (c, c)\} \\
\{c\}, \{a, b\} \text{ where R } &= \{(a, a), (b, b), (c, c), (a, b), (b, a)\} \\
\{a\}, \{b, c\} \text{ where R } &= \{(a, a), (b, b), (c, c), (b, c), (c, b)\} \\
\{b\}, \{a, c\} \text{ where R } &= \{(a, a), (b, b), (c, c), (a, c), (c, a)\} \\
\{a, b, c\} \text{ where R } &= \{(a, a), (b, b), (c, c), (a, b), (b, a), (a, c), (c, a), (b, c), (c, b)\}
\end{aligned}$$

11.3.9

We have a relation R on $\mathbb{Z}$ such that

$$xRy \text{ iff } 4|(x + 3y)$$

First we see reflexivity, we need xRx. We see that for any $x \in \mathbb{Z}$,

$$x + 3y = x + 3x = 4x \Rightarrow 4|(x + 3x) \Rightarrow xRx$$

So, R is reflexive.
Now for it to be symmetric we need $xRy \Rightarrow yRx$. Assuem xRy is true for some $x, y \in \mathbb{Z}$. We have,

$$xRy \Rightarrow x + 3y = 4k$$

Now,

$$y + 3x = y + 3(4k - 3y) = y + 12k - 9y = 12k - 8y = 4(3k - 2y) = 4k_0 \Rightarrow yRx$$

So, R is symmetric
For transitivity we need, $xRy, yRz \Rightarrow xRz$.

$$xRy \Rightarrow x + 3y = 4k_1 \text{ and } yRz \Rightarrow y + 3z = 4k_2$$

$$\begin{aligned}
x + 3y + y + 3z &= 4(k_1 + k_2) \\
x + 3z = 4(k_1 + k_2 - y) &= 4k_3 \Rightarrow xRz
\end{aligned}$$

So R is transitive.

The equivalnece classes of R would be,

$$[0] = xR0 = \{4|x + 0 : x \in \mathbb{Z}\} = \{\dots, -8, -4, 0, 4, 8, \dots\}$$

$$[1] = xR1 = \{4|x + 3 : x \in \mathbb{Z}\} = \{\dots, -7, -3, 1, 5, 9, \dots\}$$

$$[2] = xR2 = \{4|x + 6 = 4|x + 2 : x \in \mathbb{Z}\} = \{\dots - 6, -2, 2, 6, 10, \dots\}$$

$$[3] = xR3 = \{4|x + 9 = 4|x + 1 : x \in \mathbb{Z}\} = \{\dots - 1, , 3, 7, 11, 15, \dots\}$$

11.3.12

Take $A = (a, b, c)$. Let our two relations currespond to the equivalence classes as follows,

$$\{a, b\}, \{c\} \text{ so } R_1 = \{(a, a), (b, b), (c, c), (a, b), (b, a)\}$$

$$\{a\}, \{b, c\} \text{ so } R_2 = \{(a, a), (b, b), (c, c), (b, c), (c, b)\}$$

And we have,

$$R_3 = R_1 \cup R_2 = \{(a, a), (b, b), (c, c), (a, b), (b, a), (b, c), (c, b)\}$$

We see that in this union transitivity does not hold as $(a, b) \in R_3$ and $(b, c) \in R_3$ but $(a, c) \notin R_3$.
Hence we disprove by counterexample

11.4.2

All the parittions are as follows,

$$\{a\}, \{b\}, \{c\}$$

$$\{a, b\}, \{c\}$$

$$\{a, c\}, \{b\}$$

$$\{b, c\}, \{a\}$$

$$\{b, c, a\}$$

11.4.6

The currensponding equivalcne relation would be,

$R = \{(0, 0), (-1, -1), (1, 1), (-1, 1), (1, -1), \dots, (-k, -k), (k, k), (-k, k), (k, -k)\}$

For any $k \in \mathbb{Z}$