MATH3012

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February 10, 2025

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Problem 13

(a). $p(0,0) : 0 - 0 = 0 + 0^2$ which is $0 = 0$ which is **true**

(b). $p(1,1) : 1 - 1 = 1 + 1^2$ which is $0 = 2$ which is **false**

(c). $p(0,1) : 1 - 0 = 1 + 0^2$ which is $1 = 1$ which is **true**

(d). For any y we have,

$$y - 0 = y + 0^2 \text{ or } y = y$$

which means that,

$$p(0,y)$$

is **true**

(e). We need to check the existance $\exists y$ s.t. $p(1,y)$ or that,

$$y - 1 = y + 1$$

which is equivalent to,

$$2 = 0$$

which is **false** so,

$$\nexists y \text{ such that } p(1,y)$$

(f). Need to check if $\forall x, \exists y$ s.t. $p(x,y)$

Let us take any arbitrary x we have,

$$\begin{aligned} y - x &= y + x^2 \\ 0 &= x + x^2 \end{aligned}$$

Which is not true for any x, hence we can't find a y for any x such that $p(x,y)$ is true. So it is **false**

(g). Need to show $\exists y$ such that $\forall x$ we have $p(x,y)$

So let the y' be such a y where this is true so we have for any $x \in Z$

$$y' - x = y' + x^2$$
$$0 = x^2 + x$$

Now we can choose $x = 1$ and see that is not true - so we get a contradiction. Hence $\nexists y$ such that $\forall x$ we have $p(x,y)$. So **false**

(h). To show that $\forall y$ we have x such that $p(x,y)$. Take any y so we have,

$$y - x = y + x^2$$
$$0 = x + x^2$$

A solution to this can be if $x = 0$. So we see that for any $y \in Z$ we have $x = 0$ such that,

$$p(x,y)$$

So the statement is **true**

Problem 14

(a). We need to see if $\forall x, \exists y, \exists z(x = 7y + 5z)$

So essentially for any choice of x we need to be able to find y, z such that $7y + 5z = x$. First we see that $GCD(7, 5) = 1$. This means that we have y, z such that,

$$7y' + 5z' = 1$$

More specifically if $y' = 3, z' = -4$ we have $7 \cdot 3 - 5 \cdot 4 = 1$.

So now for any x we also have,

$$x(7 \cdot 3 + 5 \cdot -4) = x \cdot 1 = x$$
$$7 \cdot 3x + 5 \cdot -4x = x$$

So using any choice of x we have shown, $\exists y = 3x$ and $\exists z = -4x$ such that,

$$7y + 5z = x$$

Statement is **True**

(b). This statement is **False**. First we see that 4 and 6 share common factor of 2. Hence GCD is not 1. So $\nexists y, z$ such that,

$$4y + 6z = 1$$

This is a counterexample to the claim. In general however because both y, z are integers that means $4y$ and $6z$ are even and their sum would be even. So for any choice of x which is odd cannot be represented.

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Problem 4

Consider writing the set S as the set of sets,

$$A = \{\{3\}, \{7, 103\}, \ldots, \{55\}\}$$

We know that $|A| = 14$. Now consider the function f that maps a subset S' to A. Now we have f is not injective i.e. two elements in S' mapping to the same element in A (only possible for sets except 3,55) if $|S'| > |A|$ and the smallest would be $|A| + 1 = 14 + 1 = 15$. So $|S'|$ must at least be 15 to ensure that two elements in S' are mapped to the same in A which must imply that they add up to 110.

Problem 6

Proof. Consider the set,

$$A = \{\{1, 2\}, \{3, 4\}, \ldots, \{199, 200\}\}$$

Such that $|A| = 100$

Now if we choose 101 integers then at least two integers will be in the same set in A. So at least two integers will be consecutive elements. And we know $GCD(n, n + 1) = 1$. ☐

Problem 10

First let us divide our triangle into 9 different smaller equilateral triangles of side 1/3. Now let us consider a function that takes a set of points in our main triangle and assigns it to one of the 9 triangles. The smallest cardinality of our domain would be 10 for our function to be not injective. So this would mean that at least two points would be assigned the same triangle. As sides of the triangle are 1/3 in length that means that those points must at least be smaller than 1/3

Problem 14

Let x_i be the total number of resume that he sent out from the start to the i'th day such that $1 \leq i \leq 42$. We know that,

$$1 \leq x_1 < \cdots < x_{42} \leq 60$$

$$x_1 + 23 < \cdots < x_{42} + 23 \leq 83$$

Together we have 84 numbers between 1 and 83. So that must mean that for some x_i we have x_j such that,

$$x_i = x_j + 23$$

Or that,

$$x_i - x_j = 23$$

Which is means that there are cumulative days between i and j where he sent out exactly 23 resumes.

Keller and Trotter

Chapter 2

Problem 16

(a).

$$\binom{62}{4}$$

(b).

$$\binom{67}{4}$$

(c). We consider another variable x_6 to maintain the balance. As the inequality is not strict $x_1, \ldots, x_6$ can equal to 0. Hence our solution is,

$$\binom{68}{63}$$

(d). We need x_2 to be at least 10. So let us first allocate x_2 with 10 let this be x_2'

$$x_1 + x_2' + x_3 + x_4 + x_5 = 53$$

So now we need to find solutions to the above which is,

$$\binom{57}{53}$$

(e). First we have total solutions as,

$$\binom{67}{4}$$

Now we have solutions where $x_2 >= 10$ as,

$$\binom{57}{4}$$

So the difference is,

$$\binom{67}{4} - \binom{57}{4}$$

Problem 20

Consider choosing a team k in size with a singular member in the team as the captain. Now we can choose k people from n using,

$$\binom{n}{k}$$

Now as any of the k members can be a captain we have,

$$k\binom{n}{k}$$

Now another way of choosing the team is to find the number of ways to choose a $k-1$ members from a smaller set of $n-1$ people adding on a captain (the one not in the $n-1$). Ways to choose $k-1$ teams from an arbitrary $n-1$ group is,

$$\binom{n-1}{k-1}$$

Now we have n ways to have a captain sit out to make $n-1$. So for each member we have another group to give us,

$$n\binom{n-1}{k-1}$$

Problem 29

Our multinomial formula gives us,

$$\binom{100}{k_1, k_2, k_3}(2x)^{k_1}(3y^2)^{k_2}z^{k_3}$$

So we have $k_1 = 15, k_2 = 60, k_3 = 25$.

$$\binom{100}{15, 60, 25}2^{15}x^{15}3^{60}y^{120}z^{25}$$

So our coefficient must be,

$$\binom{100}{15, 60, 25}2^{15}3^{60}$$

Problem 31

(a). We have OVERNUMEROUSNESSES which has,

$$O : 2, V : 1, E : 4, R : 2, N : 2, U : 2, M : 1, S : 4$$

Total letters are 18 so number of rearrangements are,

$$\frac{18!}{2!2!2!2!4!4!}$$

(b). We have OPHTHALMOOTORHINOLARYNGOLOGY which has,

$$(P = 1), (M = 1), (I = 1), (T = 2), (A = 2), (R = 2), (N = 2), (Y = 2), (G = 2), (H = 3), (L = 3), (O = 7)$$

and total number of letters are 28 so our answer is,

$$\frac{28!}{7!3!3!2!2!2!2!2!2!}$$

(c). We have HONORIFICABILITUDINITATIBUS which has,

$$(H = 1), (R = 1), (F = 1), (C = 1), (L = 1), (D = 1), (S = 1), (O = 2)$$

$$(N = 2), (A = 2), (B = 2), (U = 2), (T = 3), (I = 7)$$

And total letters of, 27 so our answer is,

$$\frac{27!}{7!3!2!2!2!2!2!}$$

Section 3.11

Problem 1

We have for a given n the number of identifier can be constructed in three different ways, ones that begin with any upper other than D and followed by a valid identifier of $n-1$ is,

$$25 \times r(n-1)$$

in number

Ones that begin with $1C, 2K$ or $7J$ and followed by any valid identifier of $n-2$ is,

$$3 \times r(n-2)$$

Ones that begin with D and followed by string of $n-1$ is,

$$10^{n-1}$$

So together we have,

$$r(n) = 25r(n-1) + 3r(n-2) + 10^{n-1}$$

We have,

$$r(2) = 25(26) + 3(1) + 10^1 = 663$$
$$r(3) = 25(663) + 3(26) + 10^2 = 16753$$
$$r(4) = 25(16753) + 3(663) + 10^3 = 421814$$
$$r(5) = 25(421814) + 3(16753) + 10^4 = 10605609$$

Problem 3

First we have $g(1) = 3$ and $g(2) = 9$ and $g(3) = 26$ as everything except 102 is legal. Now we partition our string of length n into three different cases,

1. If it ends with a 1. Then we have $g(n-1)$ valid strings

2. If it ends with a 0. Then we have $g(n-1)$ valid strings

3. If it ends with a 2 then we cannot have 10 preceding it. The number of strings ending with 2 would be $g(n-1)$ and the number of (n-1) length strings ending with 10 would be $g(n-3)$. So number would be $g(n-1) - g(n-3)$.

So our solution is,

$$3g(n-1) - g(n-3)$$

Problem 4

First we have $t(1) = 1, t(2) = 1 + 4 = 5$ and $t(3) = 5 + 4 \times 1 + 2$. Now our recursion we have three cases, first we have the nth column complete free and the $n - 1$ is filled completely. This gives us,

$$t(n-1) \text{ tiles}$$

Now consider n - 2 columns are filled and we can fill the remaining 2 columns by placing an L and a square in 4 ways. This gives us,

$$4t(n-2)$$

Now consider $n - 3$ columns are filled and we fill the remaining three columns just by using two L tiles. We can orient it 2 ways. This gives us,

$$2t(n-3)$$

So in total we have,

$$t(n-1) + 4t(n-2) + 2t(n-3) \text{ total tiling's}$$

Problem 6

We need to find $d = gcd(5544, 910)$ and integers a, b such that $5544a+910b = d$. Using euclidean algorithm we have,

$$\begin{aligned}
5544 &= 910 \times 6 + 84 \\
910 &= 84 \times 10 + 70 \\
84 &= 70 \times 1 + 14 \\
70 &= 14 \times 5 + 0
\end{aligned}$$

As remainder is 0 in the last our GCD is 14 so $d = 14$. Now working backwards we have,

$$\begin{aligned}
70 &= 84 - 14 \\
910 &= 84 \times 10 + 84 - 14 \\
910 &= 84 \times 11 - 14 \\
5544 &= 910 \times 6 + (910 + 14)/11 \\
5544 \times 11 &= 910 \times 67 + 14 \\
14 &= 5544 \times 11 - 910 \times 67
\end{aligned}$$

So we have $a = 11$ and $b = -67$

Problem 8

We know that if gcd of m, n is d then there exists a', b' such that $ma' + nb' = d$. Now if are given that $am + bn = 36$ we could have some $a = ka'$ and $b = kb'$ such that $36 = kd$. So we know that,

$$d \leq 36$$

Now as k should be an integer as well this means that

$$d | 36$$

. So the gad will be one of the divisors of 36 (including itself).

Problem 10

Our base case is when $n = 4$. We see that $2^4 = 16$ and $4! = 24$. We verify that $16 < 24$. Now consider an arbitrary case where $n = k$ and assume it holds true, so we have,

$$2^k < k!$$

Take the $n = k + 1$ case we have,

$$2^{k+1} < (k + 1)!$$

Now if we have,

$$2^k < k!$$

Let us multiply $(k + 1)$ on both sides to get,

$$2^k(k + 1) < k!(k + 1)$$
$$2^k(k + 1) < (k + 1)!$$

As $k > 4$ we also have,

$$2^k 2 < 2^k(k + 1)$$

So we get,

$$2^k 2 = 2^{k+1} < 2^k(k + 1) < (k + 1)!$$

or,

$$2^{k+1} < (k + 1)!$$

Which is the $k + 1$ case. Hence we show that if its true for an arbitrary $n > 4$ then it must be true for $n + 1$. Hence by inducting we show its true for all $n \geq 4$

Problem 11

We have,

$$\sum_{n=0}^{n} 2^i = 2^{n+1} - 1$$

First let us assume the base case which is $n = 1$. We have,

$$\sum_{n=0}^{1} 2^i = 2^0 + 2^1 = 1 + 2 = 3$$

and,

$$2^{1+1} - 1 = 4 - 1 = 3$$

So it is true for the $n = 1$ case.

Now let us assume it is true for an arbitrary $n = k$ case which gives us,

$$\sum_{i=0}^{k} 2^i = 2^{k+1} - 1$$

We need to show its true for $k + 1$ case which is,

$$\sum_{i=0}^{k+1} 2^i = 2^{k+2} - 1$$

Our $n = k$ case gives us,

$$\sum_{i=0}^{k} 2^i = 2^0 + \cdots + 2^k = 2^{k+1} - 1$$

Let us add 2^{k+1} on both sides to get,

$$2^0 + \cdots + 2^k + 2^{k+1} = 2^{k+1} - 1 + 2^{k+1}$$

$$2^0 + \cdots + 2^k + 2^{k+1} = 2 \times 2^{k+1} - 1$$

$$2^0 + \cdots + 2^k + 2^{k+1} = 2^{k+2} - 1$$

We can rewrite the left hand side to get,

$$2^0 + \cdots + 2^{k+1} = \sum_{i=0}^{k+1} 2^i = 2^{k+2} - 1$$

Which is the case for $n = k + 1$. Hence we show that if true for $n = k$ then it must be true for $n = k + 1$. So by induction we see that it must be true for all $n \geq 1$.

Problem 12

Method 1. First we see that,

$$7 \equiv 1 \mod 3$$

Similarly we have,

$$4 \equiv 1 \mod 3$$

So we known that $7^n \equiv 1^n \mod 3$ or $7^n \equiv 1 \mod 3$. Similarly we have $4^n \equiv 1 \mod 3$. This gives us,

$$7^n - 4^n \equiv 0 \mod 3$$

or that,

$$3|7^n - 4^n$$

for any positive n.

Method 2. We need to show that for all positive integers n we have,

$$3|7^n - 4^n$$

First for base case we have $n = 1$ to get,

$$3|7 - 4 \Rightarrow 3|3 \text{ which is true}$$

Now assume true for arbitrary $n = k$ to get,

$$7^k - 4^k = 3m \text{ for some } m \in Z$$

We need to show that for some m' that,

$$7^{k+1} - 4^{k+1} = 3m'$$

First we have,

$$7^k - 4^k = 3m$$

We multiply both sides by 7 to get,

$$\begin{aligned}
7 \times 7^k - 7 \times 4^k &= 3(7m) \\
7^{k+1} - (4+3) \times 4^k &= 3(7m) \\
7^{k+1} - 4 \times 4^k - 3 \times 4^k &= 3(7m) \\
7^{k+1} - 4^{k+1} - 3 \times 4^k &= 3(7m) \\
7^{k+1} - 4^{k+1} &= 3(7m) + 3 \times 4^k \\
7^{k+1} - 4^{k+1} &= 3(7m + 4^k)
\end{aligned}$$

Or in other words $3|7^{k+1} - 4^{k+1}$ which is the $n = k + 1$ case.

Hence we concluded by inducting it is true for any positive n.

Problem 15

We need to show the following is true for all positive integers,

$$9|n^3 + (n+1)^3 + (n+2)^3$$

Check base case first for $n = 1$ we have,

$$1 + 2^3 + 3^3 = 36$$

and $9|36$ so base case is true.

Assume true for $n = k$ case we have,

$$9|k^3 + (k+1)^3 + (k+2)^3$$

We need to show true for $n = k + 1$ case or,

$$9|(k+1)^3 + (k+2)^3 + (k+3)^3$$

From the $n = k$ case we know that $\exists m \in Z$ such that,

$$k^3 + (k+1)^3 + (k+2)^3 = 9m$$

Now let us add $(k+3)^3$ to both sides to get,

$$\begin{aligned} k^3 + (k+1)^3 + (k+2)^3 + (k+3)^3 &= 9m + (k+3)^3 \\ (k+1)^3 + (k+2)^3 + (k+3)^3 &= 9m + (k+3)^3 - k^3 \end{aligned}$$

Now consider the last two terms in the right side which is,

$$(k+3)^3 - k^3 = k^3 + 3^3 + 3k^2 \times 3 + 3 \times 3^2 k - k^3 = 3^3 + 9k^2 + 3^3 k$$

We can write this as,

$$9(3 + k^2 + 3k) = 9m' \text{ where } m' = 3 + 3k + k^2$$

So this means that,

$$(k+1)^3 + (k+2)^3 + (k+3)^3 = 9m + 9m' = 9(m + m')$$

or that,

$$9|(k+1)^3 + (k+2)^3 + (k+3)^3$$

which is the case for $n = k + 1$

So we show that if $n = k$ is true then $n = k+1$ must be true. Hence by induction we show that it must be true for all $n \geq 1$

Problem 17

We have,

$$f(n) = 2f(n-1) - f(n-2) + 6$$

for $n \geq 2$ and $f(0) = 2, f(1) = 4$. We need to show using mathematical induction that,

$$f(n) = 3n^2 - n + 2$$

First we verify the base case for $0, 1, 2$. To get,

$$f(0) = 2, f(1) = 3 - 1 + 2 = 4$$

Using recursion we have, $f(2) = 8 - 2 + 6 = 12$ and using formula we have,

$$12 - 2 + 2 = 12$$

Hence it is true for $n = 2$. Now let us assume it is true for that case $n - 1$ and n. So we have,

$$f(n) = 3n^2 - n + 2$$
$$f(n-1) = 3(n-1)^2 - n + 3$$

We need to show that it is true for the $n + 1$ case or that,

$$f(n+1) = 2f(n) - f(n-1) + 6 = 3(n+1)^2 - n + 1$$

Plugging in the solutions of the $f(n), f(n-1)$ into the recursive algorithm for $n + 1$ we have,

$$\begin{aligned}
f(n+1) = 2f(n) - f(n-1) &= 2(3n^2 - n + 2) - (3(n-1)^2 - n + 3) + 6 \\
&= 6n^2 - 2n + 4 - (3(n^2 + 1 - 2n) - n + 3) + 6 \\
&= 6n^2 - 2n + 4 - (3n^2 + 6 - 7n) + 6 \\
&= 6n^2 - 2n + 4 - 3n^2 - 6 + 7n + 6 \\
&= 6n^2 - 2n + 4 - 3n^2 + 7n \\
&= 3n^2 + 5n + 4 \\
&= 3n^2 + 6n - n + 3 + 1 \\
&= 3(n^2 + 2n + 1) - n + 1 \\
&= 3(n^2 + 2n + 1) - n + 1 \\
&= 3(n+1)^2 - n + 1
\end{aligned}$$

This is the case for $n + 1$. Hence we show that assuming true for $n - 1$ and n we show that $n+1$ is true. Hence by strong inducting we show it is true for any $n \geq 0$.

Problem 19

We have $x \in \mathbb{R}$ and $x > -1$. To show for all $n \geq 0$ that,

$$(1+x)^n \geq 1 + nx$$

We check base case first with $n = 0$ to get,

$$(x)^0 \geq 1 + 0$$

$$1 \geq 1$$

which is true.

Now we assume true for case $n = k$ to get,

$$(1+x)^k \geq 1 + kx \text{ for } x > -1$$

We need to show true for case $n = k + 1$ which is,

$$(1+x)^{k+1} \geq 1 + (k+1)x \text{ for } x > -1$$

We have,

$$(1+x)^k \geq 1 + kx$$

We know that $x > -1$ which means that $x + 1 > 0$. So multiplying $1 + x$ on both sides we get,

$$(1+x)^{k+1} \geq (1+x)(1+kx) = 1 + kx + x + (k+1)x$$

Looking at the right side we see that,

$$1 + kx + x + kx^2 = 1 + (k+1)x + kx^2 \geq 1 + (k+1)x$$

Putting the above two together we get,

$$(1+k)^{k+1} \geq 1 + (k+1)x$$

which is the case for $n = k + 1$. Hence we show if true for $n = k$ then it must be true for $n = k + 1$. Hence by induction we show that it must be true for all $n \geq 0$.