Probability Theory

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Contents

Chapter 1

Introduction

1.1 Sample Spaces and Sigma-Algebras

Definition 1.1 (Sample Space). A sample space Ω is any set and its elements are called outcomes.

Example. Flip a coin twice, sample space is,

$$\{HH, HT, TH, TT\}$$

$\diamond$

Events are subsets of sample space such that,

1. The whole space Ω should be an event (The event that something happened).
2. If an event $A \in \Omega$ then $A^c \in \Omega$
3. If $A, B \in \Omega$ then $A \cup B \in \Omega$

Definition 1.2 (Algebra). An algebra is a collection Σ of subsets of Ω satisfying the following,
1. $\Omega \in \Sigma$
2. If $A \in \Sigma$ then $A^c \in \Sigma$
3. If $A, B \in \Sigma$ then $A \cup B \in \Sigma$

Definition 1.3 (Sigma-algebra). A sigma-algebra is an algebra such that if whenever $A_1, A_2, \cdots \in \Sigma$ we also have $\bigcup_{n=0}^{\infty} A_n \in \Sigma$ we call Σ a sigma-algebra.

Remark. The key differnet is a sigma-algebra allows for a countably infinite union and intersection of elements while a ordinary algebra allows for a finite intersection and union.

Remark.

Some consequences are,

1. $\phi \in \Sigma$
2. If $A, B \in \Sigma$ then $A \cap B \in \Sigma$

Proof. $A \cap B = (A^c \cup B^c)^c$ □

3. If Σ is a sigma-algebra then $A_1, \dots \in \Sigma$ means that $\bigcap_{n=1}^{\infty} \in \Sigma$

Proof. $\bigcap_n A_n = (\bigcup_n A_n^c)^c$ □

Example.

1. If Ω is any set, then $\{\phi, \Omega\}$ is a sigma-algebra (the trivial sigma-algebra)

2. If Ω is any set, then the power set $P(\Omega)$ is a sigma-algebra.

3. Let $\Omega = (0, 1]$ and define Σ as finite disjoint unions of half-open intervals. ◇

Consider $\Sigma_0 = \{(a_1, b_1] \cup \dots \cup (a_n, b_n] : n \in \mathbb{N}, 0 \leq a_i \leq b_i \leq 1, \forall i, (a_i, b_i] \cap (a_j, b_j] = \phi, \forall i \neq j\}$

Proposition 1.4. Σ_0 is an algebra but not a sigma-algebra.

Proof. $\Omega = (0, 1] \in \Sigma_0$.
If $A \in \Sigma_0$, wirte it as $A = (a_1, b_1] \cup \dots \cup (a_n, b_n], a_1 \leq b_1 \leq a_2 \leq b_2 \dots \leq a_n \leq b_n$
Then $A^c = (0, a_1] \cup (b_1, a_2] \cup \dots \cup (b_n, 1] \in \Sigma_0$
Now if $A, A^c \in \Sigma_0$ we have,

$$\begin{aligned} A &= (a_1, b_1] \cup \dots \cup (a_n, b_n] \\ A^c &= (a'_1, b'_1] \cup \dots \cup (a'_m, b'_m] \end{aligned}$$

□

Definition 1.5. If A is a collection of subsets of Σ then the sigma-algebra generated by A written as $\sigma(A)$, is the intersection of all sigma-algebras that contain A.

Proof. If e is a collection of sigma-algebras of Ω then $\bigcup_{\Sigma \in e} \Sigma$ is a sigma-algebra □

Example.

1. The sigma-alg generated by $\{\phi\}$ is $\{\phi, \Omega\}$

2. The sigma-algebra generated by open subsets of $\mathbb{R}^d$ is called the Bore sigma-algebra

3. If $A \subset B$ then $\sigma(A) \subset \sigma(B)$

4. If Σ is a sigma-algebra then $\sigma(\Sigma) = \Sigma$

5. $\sigma(A)$ is the "smallest sigma-algebra containing A". If Σ is some sigma-algebra s.t. $A \subset \Sigma$ then $\sigma(A) \subset \Sigma$ ◇

1.2 Probability Measures

Definition 1.6. Let Ω be a set and Σ be a sigma-alebra on Ω. A function $\mathbb{P} : \Sigma \to \mathbb{R}$ is a probability measure if,

1. $0 \leq \mathbb{P}(A) \leq 1, \forall A \in \Sigma$,

2. $\mathbb{P}(\Omega) = 1$,

3. (Countable additivity) If $A_1, A_2, \dots$ is a sequence of pairwise disjoint elements of Σ_1 then,

$$\mathbb{P}(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mathbb{P}(A_n)$$

Some properties are,

- $A \in \Sigma, \mathbb{P}(A^c) = 1 - \mathbb{P}(A)$. So, $\mathbb{P}(\phi) = 1 - \mathbb{P}(\Omega)$

  Proof. $1 = \mathbb{P}(\Omega) = \mathbb{P}(A \cup A^c) = \mathbb{P}(A) + \mathbb{P}(A^c)$ □

- (inclusion-exclusion). If $A, B \in \Sigma$ then,

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$$

  Proof. $\mathbb{P}(A \cup B) = \mathbb{P}(A - B) + \mathbb{P}(B - A) + \mathbb{P}(A \cap B)$

$$= \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$$

  □

- (general inclusion-exclusion)

$$\mathbb{P}(\bigcup_{k=1}^{n} A_k) = \sum_{k=1}^{n} \mathbb{P}(A_u) - \sum_{i<j} \mathbb{P}(A_i \cup A_j) + \sum_{i<j<k} \mathbb{P}(A_i \cup A_j \cup A_k) + \dots$$

$$+(-1)^{n+1} \mathbb{P}(A_1 \cup \dots \cup A_n)$$

  Proof. Using induction □

- $A_1, A_2, \dots \in \Sigma$ then,

$$\mathbb{P}(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \mathbb{P}(A_n)$$