# Real Analysis: HW8

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## Exercise 3.4.7

(a). Consider $x, y \in \mathbb{Q}$. Now because of the density of the irrationals we know there exists some $i$ such that $x < i < y$ and $i \in I$. Now consider the sets $A = \mathbb{Q} \cap (-\infty, i)$ and $B = \mathbb{Q} \cap (i, \infty)$. First we easily see that $\mathbb{Q} = A \cup B$. Now, note that any limit point of $A$ will lie in $(-\infty, i]$ and that of $B$ will lie in $[i, \infty)$ because of the order limit theorems. However both these sets are disjoint from $(i, \infty)$ and $(-\infty, i)$ respectively which gives us $\overline{A} \cap B$ and $A \cap \overline{B}$ as empty which makes $A, B$ be separated. Hence, we have $\mathbb{Q}$ is totally disconnected.

(b). Yes the set of irrationals is totally disconnected as well by using the same reasoning as above. For any pair of irrational numbers $x, y$ we can find a rational number $q$, in between and we can construct $A = I \cap (-\infty, q), B = (q, \infty)$ such that they are separated and their union is $I$.

## Exercise 4.2.5

(a). We need $\lim_{x \to 2}(3x + 4) = 10$.

For any $\varepsilon > 0$ consider $\delta < \frac{\varepsilon}{3}$, then we have for $|x - 2| < \delta$ that,

$$\begin{aligned} |3x + 4 - 10| &= |3x - 6| \\ &= 3\,|x - 2| \\ &< 3\delta \\ &< 3\frac{\varepsilon}{3} \\ &< \varepsilon \end{aligned}$$

Hence, for any $\varepsilon$ we found a $\delta$ such that for $|x - 2| < \delta$ we have $|(3x+4) - 10| < \varepsilon$ which implies that the limit as $x$ goes to 2 of $3x + 4$ is 10.

(b). We need $\lim_{x \to 0} x^3 = 0$. For any $\varepsilon$ consider $\delta = \varepsilon^{1/3}$ such that we have for $x < \delta < \varepsilon^{1/3}$,

$$\left|x^3 - 0\right| = \left|x^3\right|$$

Now as $|x| < \varepsilon^{\frac{1}{3}}$ we have $|x^3| < |\varepsilon^3|$ which give such,

$$\left|x^3 - 0\right| < \varepsilon^{\frac{1}{3}3} = \varepsilon$$

(c). We need $\lim_{x\to 2}(x^2+x-1) = 5$. For any $\varepsilon > 0$ consider $\delta = \min(1, \frac{\varepsilon}{6})$. Now we have,

$$\begin{aligned}\left|x^2+x-1-5\right| &= \left|x^2+x-6\right| \\ &= |(x-2)(x+3)|\end{aligned}$$

Now as $|x-2| < \delta$ we have $|x-2| < \min(1, \varepsilon/6)$. As we're taking $|x-2| < 1$ we have $|x+3| \le |x-2+5| \le |x-2|+5 < 6$ which gives us,

$$|(x-2)(x+3)| < 6|x-2|$$

Now we also have $|x-2| < \frac{\varepsilon}{6}$ which means, $6|x-2| < \varepsilon$. Hence we get,

$$|(x^2+x-1)-5| = |(x-2)(x+3)| < 6|x-2| < \varepsilon$$

(d). We have $\lim_{x\to 3}\frac{1}{x} = \frac{1}{3}$. Consider $\delta = \min(1, 6\varepsilon)$. So we have for $x < \delta$ that,

$$\left|\frac{1}{x} - \frac{1}{3}\right| = \left|\frac{x-3}{3x}\right|$$

But as we have $|x-3| < 1$ we get $-1 < x-3 < 1$ or $2 < x < 4$ which means that $\frac{1}{12} < \frac{1}{3x} < \frac{1}{6}$. So we have,

$$\left|\frac{1}{x} - \frac{1}{3}\right| < \frac{1}{6}|x-3|$$

But as $|x-3| < \varepsilon$ we have,

$$\begin{aligned}\left|\frac{1}{x} - \frac{1}{3}\right| &< \frac{1}{6}|x-3| \\ &< \frac{1}{6}6\varepsilon \\ &= \varepsilon\end{aligned}$$

## Exercise 4.2.10

(a). For right hand limit we have,

Let $f : A \to R$, and let $c$ be a limit point of the domain $A$. We say $\lim_{x\to a^+} f(x) = L$ provided that, for all $\varepsilon > 0$, there is $\delta > 0$ such that whenever $0 < x - c < \delta$ (and $x \in A$) we have $|f(x) - L| < \varepsilon$.

For left hand limit we have,
Let $f : A \to R$, and let $c$ be a limit point of the domain $A$. We say $\lim_{x\to a^-} f(x) = L$ provided that, for all $\varepsilon > 0$, there is $\delta > 0$ such that whenever $0 < c - x < \delta$ (and $x \in A$) we have $|f(x) - L| < \varepsilon$.

(b). Assume that we have $\lim_{x\to c} f(x) = L$. By definition this means that for any $\varepsilon$ we have some  such that if $0 < |x - c| < \delta$ then we get $|f(x) - L| < \varepsilon$. Now if we have $0 < |x-c| < \delta$ then this implies that we have both $0 < x-c < \delta$ if $x > c$ and $0 < c - x < \delta$ if $x < c$. Now by definition defined in (a) we have $\lim_{x\to c^+} f(x) = L$ and $\lim_{x\to c^-} f(x) = L$.

Now assume we have $\lim_{x\to c^+} f(x) = L$ and $\lim_{x\to c^-} f(x) = L$. So we get $0 < x - c < \delta_1$ and $0 < c - x < \delta_2$. Now we can just choose the smaller of the two deltas which will give us $0 < |x - c| < \delta$ for which we get $|f(x) - L| < \varepsilon$ for any $\varepsilon$. Which is just the definition for $\lim_{x\to c} f(x) = L$.

## Exercise 4.2.11

We have $\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = L$. So for any $\varepsilon$ we have, $0 < |x - c| < \delta_1$ we have $|f(x) - L| < \varepsilon/3$ and for $0 < |x - c| < \delta_2$ we have $|h(x) - L| < \varepsilon/3$. So take $\delta = \min(\delta_1, \delta_2)$. This gives us for $0 < |x - c| < \delta$ that, $|h(x) - L| < \varepsilon/3$ and $|f(x) - L| < \varepsilon/3$.

Now also note that $f(x) \le g(x) \le h(x)$ which means $g(x) - f(x) \le h(x) - f(x)$ this gives us that $|g(x) - f(x)| \le |h(x) - L + L - f(x)| \le |h(x) - L| + |f(x) - L|$. Now if $0 < |x - c| < \delta$ we get $|g(x) - f(x)| < 2\varepsilon/3$. Now consider the following when $0 < |x - c| < \delta$,

$$\begin{aligned}|g(x) - L| &= |g(x) - f(x) + f(x) - L| \\ &< |g(x) - f(x)| + |f(x) - L| \\ &< 2\varepsilon/3 + \varepsilon/3 = \varepsilon\end{aligned}$$

Hence we got for any $\varepsilon$ a $\delta$ such that whenever $0 < |x - c| < \delta$ we have, $|g(x) - L| < \varepsilon$ which means that $\lim_{x\to c} g(x) = L$

## Exercise 4.3.11

(a). We show that $f$ is continuous by showing that for any $k$ we have for any $\varepsilon$ a $\delta$ such that for $|x - k| < \delta$ we get $|f(x) - f(k)| < \varepsilon$.

Take $\delta = \varepsilon$, so we have $|x - k| < \varepsilon$. Now by definition of $f$ we have some $c \in (0, 1)$ such that,

$$|f(x) - f(k)| \le c|x - k|$$

now as $c < 1$ this means that $c|x - k| < |x - k|$ so,

$$\begin{aligned}|f(x) - f(k)| &\le c|x - k| \\ &\le |x - k| \\ &< \varepsilon\end{aligned}$$

So for any $\varepsilon$ we got a $\delta$ such that when $|x - k| < \delta$ we have $|f(x) - f(k)| < \varepsilon$ which by definition means that $f$ is continuous.

(b). We have the sequence $(y_1, f(y_1), f(f(y_1)), \dots)$. We need to show that $(y_n)$ is a Cauchy sequence. So we need for any $\varepsilon$ some $N$ that if $m, n > N$ then we get $|y_m - y_n| < \varepsilon$. First notice that we have for some arbitrary $n$ that,

$$\begin{aligned} |y_{n+2} - y_{n+1}| &= |f(f(y_n)) - f(y_n)| \\ &< c|f(y_n) - y_n| \\ &= c|y_{n+1} - y_n| \end{aligned}$$

Similarly note that we can do a similar bounding to get $|y_{n+1} - y_n| < c|y_n - y_{n-1}|$. Recursively doing this we get,

$$|y_{n+2} - y_{n+1}| < c^n|y_2 - y_1|$$

Take $|y_2 - y_1| = M$, now as $c < 1$ we can choose $n$ to be arbitrary large to get $c^n M < \varepsilon$ we see this as follows. $c^n < \varepsilon/M$ so $n\log(c) < \log(\varepsilon/M)$. As $c < 1$ we have $\log(c) < 0$ so $n|\log c| > \log(\varepsilon/M)$ and we have $n > \log(\varepsilon/M)/|\log c|$. So for any $\varepsilon$ take $N = \log(\varepsilon/M)/|\log c| + 2$ which gives us for any $n > N$ that, $|y_{n+1} - y_n| < \varepsilon$.

Now we for any $\varepsilon$ we can find a $N$ such that $|y_{n+1} - y_n$ $m > n > N$ consider $|y_m - y_n|$ note that we can write this as,

$$\begin{aligned} |y_m - y_n| &= |y_n - y_{n+1} + y_{n+1} - y_{n+2} + \cdots - y_{m-1} + y_m| \\ &\leq |y_n - y_{n+1}| + \cdots + |y_m - y_{m-1}| \\ &\leq \varepsilon + c\varepsilon + c^2\varepsilon + \cdots + c^{m-n}\varepsilon \\ &\leq \varepsilon(1 + c + c^2 + \cdots + c^{m-n}) \\ &\leq \varepsilon\left(\frac{1 - c^{m-n+1}}{1 - c}\right) \\ &\leq \varepsilon\left(\frac{1}{1 - c}\right) \end{aligned}$$

But now as $c$ is a constant we have $\frac{1}{1-c}$ is a constant say $M'$. So we have,

$$|y_m - y_n| \leq \varepsilon M'$$

As we already established for any $\varepsilon > 0$ if $m, n > N$ we have $|y_m - y_n| < \varepsilon$ we can choose it to be $\varepsilon M'$ to get $|y_m - y_n| < \frac{\varepsilon}{M'}M' = \varepsilon$ hence completing the proof.

(c). Now we show that $y$ is a fixed point. From above we have the sequence is a convergent sequence whose limit is say $y$. Now consider the following,

$$\begin{aligned} |f(y) - y| &= |f(y) - y_n + y_n - y| \\ &\leq |f(y) - y_n| + |y_n - y| \\ &= |f(y) - f(y_{n-1})| + |y - y_n| \end{aligned}$$

Now note that we have $|f(y) - f(y_{n-1})| \leq c|y - y_{n-1}| < |y - y_{n-1}|$. Now for any $\varepsilon$ we can have for $n > N + 1$, $|y_n - y| < \frac{\varepsilon}{2}$. This gives us both, $|y - y_{n-1}| < \varepsilon/2$ and $|y - y_n| < \varepsilon/2$ which means we have,

$$
\begin{aligned}
|f(y) - y| &= |f(y) - f(y_{n-1})| + |y - y_n| \\
&< |y - y_{n-1}| + |y - y_n| \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}
$$

So we have $|f(y) - y| < \varepsilon$ for any $\varepsilon$ which is equivalent to saying $f(y) = y$.

Now assume that it is not unique, i.e. there exists another $y_k$ such that we have $f(y_k) = y_k$. However, this gives us $y_{k+1} = y_k$ which means that $f(y_{k+1}) = f(y_k) = y_k$. Or in other words for any $k' > k$ we have $f(y_k') = y_k$ i.e. we have a constant sequence of $y_k$ for $k' > k$. But this means that $y_k$ is the limit of the sequence as for any $k' > k$ we also have $|y_{k'} - y_k| < \varepsilon$ trivially as they are equal. So we have both $y$ and $y_k$ is the limit of the sequence. However, we know that a sequence with a limit has a unique limit. This means that we have $y = y_k$ and hence the there is only a unique point $y$ for which we have $f(y) = y$.

(d). Now for any arbitrary $x$ we have,

$$
\begin{aligned}
|x_n - f(y)| &\leq c|x_{n-1} - y| = c|x_{n-1} - f(y)| \\
&\leq c^2|x_{n-2} - f(y)| \\
&\leq \dots \\
&\leq c^n|x - f(y)| = c^n|x - y|
\end{aligned}
$$

But $x, y$ are constant so we have $|x_n - f(y)| = |x_n - y| \leq c^n M$. And we can choose $N$ to be arbitrarily large such that we have for some $\varepsilon$ if $n > N$ then $c^n M < \varepsilon$ hence we have $|x_n - y| < \varepsilon$ for $n > N$ as well which means the limit of the sequence $(x, f(x), f(f(x)), \dots)$ is $y$ defined in $(b)$.