Real Analysis: HW3

Aamod Varma

September 15, 2025

Exercise 2.3.4

We are given that $(a_n) \to 0$
(a) We have $\lim\left(\frac{1+2a_n}{1+3a_n-4a_n^2}\right)$

Using the Algebraic Limit Theorem we know that as $\lim a_n = 0$ then using 2.3.3 (i) we have $\lim 2a_n = 2 \cdot 0 = 0$ and if we consider the sequence that just returns the constant value 1, using 2.3.3 (ii) we have $\lim(1+2a_n) = 1+0 = 1$. Similarly we have $\lim 3a_n = 3 \cdot 0 = 3$ and using 2.3.3 (iii) we have $\lim a_n^2 = 0^2 = 0$ and $\lim 4a_n^2 = 0$. So we have $\lim(1+3a_n-4a_n^2) = 1$. Now as both the limits $\lim(1+2a_n)$ and $\lim(1+3a_n-4a_n^2)$ is defined, using 2.3.3 (iv) we have,

$$\lim\left(\frac{1+2a_n}{1+3a_n-4a_n^2}\right) = \frac{1}{1} = 1$$

(b) We have $\lim\left(\frac{(a_n+2)^2-4}{a_n}\right)$
We can expand the top as $a_n^2+4-4+4a_n = a_n(a_n+4))$. So the a_n cancels out and we have $\lim a_n + 4$ which is 4.
(c) We have $\lim\left(\frac{\frac{2}{a_n}+3}{\frac{1}{a_n}+5}\right)$

We can write this as $\lim\left(\frac{2+3a_n}{1+5a_n}\right)$. And looking at numerator we have $\lim 3a_n = 3 \cdot 0 = 0$ which means $\lim 2+3a_n = 2+0 = 2$. Similarly, $\lim 5a_n = 5 \cdot 0 = 0$ and $\lim 1+5a_n = 1+0 = 1$. Which gives us $\lim\left(\frac{\frac{2}{a_n}+3}{\frac{1}{a_n}+5}\right) = 2$

Exercise 2.3.5

We are given that (x_n) and (y_n) are convergent and (z_n) is defined as,

$$x_1, y_1, x_2, y_2, \ldots, x_n, y_n, \ldots$$

We can write this as follows, for a given n we have

$$z_n = \begin{cases} x_{\frac{n+1}{2}} & \text{if } n \text{ is odd} \\ y_{\frac{n}{2}} & \text{if } n \text{ is even} \end{cases}$$

We need to show that (z_n) converges if and only if (x_n) and (y_n) are both convergent with the same limit.

($\Rightarrow$) First assume that (z_n) converges to z. This means that $\forall \varepsilon > 0$ we can find some N such that $\forall n > N$ we have,

$$|z_n - z| < \varepsilon$$

Now for $n > N$ consider we have the following,

$$\left|x_{\frac{n+1}{2}} - z\right| < \varepsilon \qquad \text{if } n \text{ is odd}$$
$$\left|y_{\frac{n}{2}} - z\right| < \varepsilon \qquad \text{if } n \text{ is even}$$

Or in other words if we take $N' = \frac{N+1}{2}$ we have $\forall\varepsilon$ if $n > N_1$ that,

$$|x_n - z| < \varepsilon$$
$$|y_n - z| < \varepsilon$$

This means that both x_n and y_n converge to z by definition.

($\Leftarrow$) Now let us assume that (x_n) and (y_n) both converge to the same limit z. So we have $\forall\varepsilon > 0$ there is some N_1 such that for $n > N_1$

$$|x_n - z| < \varepsilon$$

Similarly we have, $\forall\varepsilon > 0$ exists N_2 such that for $n > N_2$ we have,

$$|y_n - z| < \varepsilon$$

Now consider $N = \max(N_1, N_2)$ so we have $\forall\varepsilon > 0$ if $n > N$ then,

$$|x_n - z| < \varepsilon$$
$$|y_n - z| < \varepsilon$$

As we have

$$z_n = \begin{cases} x_{\frac{n+1}{2}} & \text{if } n \text{ is odd} \\ y_{\frac{n}{2}} & \text{if } n \text{ is even} \end{cases}$$

So $z_{2n+1} = x_n$ if n is odd and $z_{2n} = y_n$ if n is even. This means that if we take $N' = 2N + 1$ then we have for $n > 2N + 1$ that z_n is the x or y that lies after the N as we defined above. Or in other words we get if n is even then $z_n = y_{\frac{n}{2}}$, but as $n > 2N + 1$ we have $\frac{n}{2} > N$ which means that,

$$|z_n - z| = \left|y_{n/2} - z\right| < \varepsilon$$

And similar if n is odd then $z_n = x_{\frac{n+1}{2}}$ and we have $n > 2N+1$ so $\frac{n+1}{2} > N+1$ which means that,

$$|z_n - z| = \left|x_{n+1/2} - z\right| < \varepsilon$$

So we have for any $n > N' = 2N + 1$ that,

$$|z_n - z| < \varepsilon$$

which means that (z_n) converges to limit z.

Exercise 2.4.2

(a) We have the sequence defined by $y_{n+1} = 3 - y_n$ where $y_1 = 1$. We can write $y_{n+2} = 3 - y_{n+1} = 3 - (3 - y_n) = y_n$. So if $y_1 = 1, y_2 = 2$ then we have $(y_{2n+1}) = 1$ and $(y_{2n}) = 2$ for any $n \in \mathbb{Z}$. But this means that we found two subsequence that converge to different values which means that our original sequence doesn't converge.

In other words the argument is incorrect in assuming that (y_n) and (y_{n+1}) have the same limit which we showed is not true above.

(b) Now we have $y_1 = 1$ and $y_{n+1} = 3 - \frac{1}{y_n}$. In this case the method above can be used as the values y_n, y_{n+1} don't oscillate like above.

Exercise 2.5.6

We need to show that $b^{\frac{1}{n}}$ exists for all $b \geq 0$ and find the value of the limit. If $b > 1$ then

$$b^1 > \sqrt[2]{b} > \sqrt[3]{b} > \cdots > \sqrt[n]{b} > \ldots$$

else we have,

$$b^1 < \sqrt[2]{b} < \sqrt[3]{b} < \cdots < \sqrt[n]{b} < \ldots$$

So in both cases we see that it is monotone. Now we see that if $b > 1$ then $b^{\frac{1}{n}} > 1$ and hence 1 is a lower bound for it. Similarly we see that if $0 < b < 1$ then $b^{\frac{1}{n}} < 1$ which means 1 is an upperbound for it. In both cases we see that the sequence is bounded. Using the monotone convergence theorem this means that it has a limit.

Now consider the subsequence defined as below,

$$b^{\frac{1}{2}}, b^{\frac{1}{4}}, b^{\frac{1}{6}}, \ldots, b^{\frac{1}{2i}}, \ldots$$

We see that this is equivalent to $(\sqrt{b^{1/n}})$ as we have the i'th element of our subsequence defined as $\sqrt{b^{1/i}} = b^{1/2i}$. Now, as we know that the sequence is convergent we know that the subsequence converges to the same limit. So we have the following,

$$(b^{1/n}) \to L \quad \text{and} \quad (\sqrt{b^{1/n}}) \to L$$

But we know that if $(x_n) \to x$ then $(\sqrt{x_n}) \to \sqrt{x}$ which means that $\sqrt{L} = L$ or that $L = L^2$ and $L(L-1) = 0$. So we have either $L = 0$ or $L = 1$. If $b > 1$ we know that $b^{\frac{1}{n}} > 1$ as if that weren't the case we would have $b^{(1/n)n} = b < 1$ which is false. So we have $L = 1$. Now for $b < 1$ we know that $b^{1/n} > 0$ which means the only options is for $L = 1$.

If $b = 1$ then it trivially converges to 1. And if $b = 0$ then the sequence has value 0 everywhere and hence converges to 0.

Exercise 2.5.8

(a) Zero peak terms: $x_i = i$

$$0, 1, 2, 3, \ldots$$

Here as it's increase there is no x_i such that the terms after it are smaller than equal to it.

One peak term:

$$1, 5, 2, 2, 2, \ldots$$

Here, 5 is the only peak term.

Two peak terms:

$$1, 5, 2, 3, 2, 2, 2, \ldots$$

Here 5 and 3 are peak terms.

Infinitely many peak terms: $x_i = (-1)^i$

$$-1, 1, -1, 1, -1, 1, \ldots$$

Here every 1 is a peak terms as for any value after it (either -1, 1) we have $1 \geq 1$ and $1 \geq -1$ and we see here that this sequence is not monotone as it's oscillating.

(b) Given a sequence we have two cases, either it has infinitely many peak terms or it doesn't.

Case 1: If it has infinitely many peak terms say specifically $p_1, p_2, \ldots, p_n, \ldots$. Then $(p_1, p_2 \ldots)$ would be a monotonically decreasing subsequence of our original sequence as for any p_i we have $p_i \geq p_j$ if $j > i$. So this means we have a monotone subequence that is bounded which implies that it is convergent.

Case 2: If it does not have infinitely many peak terms then it means it's finitely many peak terms. So there is some N for which if $n > N$ there are no peak terms. If there are no peak terms then there is no p_i such that $p_i \geq p_j$ for all $j > i$. So we can find a j such that $p_j > p_i$ and $j > i$. So consider this subsequence such that every later term is strictly greater than the previous terms. So we have a sequence that is monotonically increasing and bounded which means it is convergent.