MATH 4320 HW14-16

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Problem 2

Let,

$$f(z) = \frac{z^{-1/2}}{z^2+1} = \frac{e^{(-1/2)\log z}}{z^2+1}$$

Now consider the indented contour and we have,

$$\int_{L_1} f(x)dx + \int_{L_2} f(x)dx + \int_{C_\rho} f(z)dz + \int_{C_R} f(z)dz = \int_C f(z)dz$$

Rearranging we have,

$$\int_{L_1} f(x)dx + \int_{L_2} f(x)dx = \int_C f(z)dz - \int_{C_\rho} f(z)dz - \int_{C_R} f(z)dz$$

First we calculate $\int_C f(z)dz$. Within our contour the only singularity is when $z = i$. So the integral is equal to,

$$\int_C f(z) = 2\pi i Res_{z=i} f(z)$$

Now

$$Res_{z=i} f(z) = \frac{e^{-1/2\log(i)}}{2i} = \frac{e^{-1/2(i\pi/2)}}{2i}$$
$$= \frac{e^{-i\pi/4}}{2i} = (\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}})\frac{1}{2i}$$

This gives us,

$$\int_C f(z) = 2\pi i \cdot (\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}})\frac{1}{2i}$$

$$= \frac{\pi}{\sqrt{2}} - \frac{\pi i}{\sqrt{2}} = \frac{\pi}{\sqrt{2}}(1-i)$$

Now let us look at,

$$\int_{L_1} f(x)dx + \int_{L_2} f(x)dx$$

We can write these integrals using the parameterization $z = re^{i0}, \rho \le r \le R$ and $z = re^{i\pi}, \rho \le r \le R$

$$= \int_\rho^R \frac{e^{-1/2\log(r)}}{r^2+1} + \int_\rho^R \frac{e^{-1/2\log(re^{\pi})}}{r^2+1}$$
$$= \int_\rho^R \frac{e^{-1/2\log(r)}}{r^2+1} + \int_\rho^R \frac{e^{-1/2\log(r)}e^{-i\pi/2}}{r^2+1}$$
$$= (1+e^{-i\pi/2})\int_\rho^R \frac{e^{-1/2\log(r)}}{r^2+1}$$
$$= (1-i)\int_\rho^R \frac{e^{-1/2\log(r)}}{r^2+1}$$

So we have,

$$= (1-i)\int_\rho^R \frac{e^{-1/2\log(r)}}{r^2+1} = \frac{\pi}{\sqrt{2}}(1-i) - \int_{C_\rho} f(z)dz - \int_{C_R} f(z)dz$$

Now we can bound $f(z)$ as follows because we can say $|\sqrt{z}| \geq |\sqrt{\rho}|$ and $|z^2+1| \geq |1-\rho^2|$ so,

$$f(z) \leq \frac{1}{\sqrt{\rho}(1-\rho^2)}$$

Or,

$$\int_{C_\rho} f(z) \leq \frac{2\pi\rho}{\sqrt{\rho}(1-\rho^2)} = \frac{2\pi\sqrt{\rho}}{1-\rho^2}$$

Now as $\rho \to 0$ we have $1-\rho^2$ goes to 1 while the numerator vanishes to zero. Hence we can say that,

$$\lim_{\rho\to 0}\int_{C_\rho} f(z) = 0$$

Similarly we have,

$$\int_{C_R} f(z) \leq \frac{2\pi\sqrt{R}}{R^2-1}$$

As the power of the denominator in terms of R is higher we have,

$$\lim_{R\to\infty} f(z)dz = 0$$

This gives us,

$$(1-i)\int_0^\infty \frac{e^{-1/2\log(r)}}{r^2+1} = \frac{\pi}{\sqrt{2}}(1-i)$$
$$\int_0^\infty \frac{e^{-1/2\log(r)}}{r^2+1} = \frac{\pi}{\sqrt{2}}$$

0.0.1 Problem 1

We have,

$$\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta}$$

Let $z = e^{i\theta}$ which gives us, $\frac{dz}{iz} = d\theta$ and $\sin\theta = \frac{z-z^{-1}}{2i}$. So we can write,

$$\begin{aligned}
&\int_C \frac{dz}{iz(5+4\frac{z-z^{-1}}{2i})} \\
&= \int_C \frac{dz}{z(5i+2(z-z^{-1}))} \\
&= \int_C \frac{dz}{5zi+2z^2-2} \\
&= \int_C \frac{dz}{2(z+\frac{i}{2})(z+2i)}
\end{aligned}$$

Taking our unit circle we can see that $z = -\frac{i}{2}$ is inside our contour hence the integral evaluates to,

$$\begin{aligned}
&2\pi i\frac{1}{2(2i-i/2)} \\
&= \pi i\frac{2}{3i} \\
&= \frac{2\pi}{3}
\end{aligned}$$

Problem 3

We have,

$$\int_0^{2\pi} \frac{\cos^2 3\theta d\theta}{5 - 4\cos 2\theta}$$