Probability Theory: Hw1

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Exercise 1.10

Given $A, B \in \mathscr{F}$ and we need to show that $A \triangle B \in \mathscr{F}$. Now if $x \in A \triangle B$ then we know that $x \in (A \cup B) \setminus (A \cap B)$. By definition we have $A \cup B \in \mathscr{F}$ (closure under countable union) and we also have $A^c, B^c \in \mathscr{F}$(closure under complement) $\Rightarrow (A^c \cup B^c) \in \mathscr{F} \Rightarrow (A \cap B)^c \Rightarrow A \cap B \in \mathscr{F}$. So now let $C = A \cup B$ and $D = A \cap B$. It is enough to show that if $C, D \in \mathscr{F}$ then $C \setminus D \in \mathscr{F}$. We have $C \setminus D = C \cap D^c$. We know $D^c \in \mathscr{F}$ and $\mathscr{F}$ is closed under intersection as shown above which means that $C \cap D^c \in F \Rightarrow C \setminus D \in \mathscr{F} \Rightarrow (A \cup B) \setminus (A \cap B) \in \mathscr{F} \Rightarrow A \triangle B \in \mathscr{F}$

Exercise 1.17

First given that $\mathscr{F}$ is the power set of Ω.
1. We have $\mathbb{Q}(A) = \sum_{i:\omega_i \in A} p_i$ for $A \in \mathscr{F}$ and we know that $p_i \geq 0$ for any i so sum of non-negative numbers are also non-negative which means that $\mathbb{Q}(A) \geq 0$ for $A \in \mathscr{F}$
2. We have $\mathbb{Q}(\Omega) = \sum_{i:\omega_i \in \Omega} p_i = p_1 + \cdots + p_n = 1$. Similarly we have $\mathbb{Q}(\phi) = \sum_{i:\omega_i \in \phi} p_i = 0$.
3. We need to show that given disjoint events $A_1, A_2, \cdots \in \mathscr{F}$ we have, $\mathbb{Q}\big(\bigcup_{i=1}^{\infty} A_i\big) = \sum_{i=1}^{\infty} \mathbb{Q}(A_i)$.

$$
\begin{aligned}
\mathbb{Q}\big(\bigcup_{i=1}^{\infty} A_i\big) &= \mathbb{Q}(A_1 \cup A_2 \dots) \\
&= \sum_{i:\omega_i \in (A_1 \cup A_2 \dots)} p_i \\
&\text{Now since } A_1, \dots \text{ are pairwise disjoint we can write,} \\
&= \sum_{i:\omega_i \in (A_1)} p_i + \sum_{i:\omega_i \in (A_2)} p_i + \dots \\
&= \mathbb{Q}(A_1) + \mathbb{Q}(A_2) + \dots \\
&= \sum_{i=1}^{\infty} \mathbb{Q}(A_i)
\end{aligned}
$$

Exercise 1.21

We need to find,

$$
\begin{aligned}
&P(A \cap B \cap C^c) + P(A \cap B^c \cap C) + P(A^c \cap B \cap C) \\
&= P((A \cap B) \setminus C) + P((A \cap C) \setminus B) + P((C \cap B) \setminus A) \\
&= P(A \cap C) - P(A \cap B \cap C) + P(A \cap B) - P(A \cap B \cap C) + P(B \cap C) - P(A \cap B \cap C) \\
&= .3 - .1 + .4 - .1 + .2 - .1 = .6
\end{aligned}
$$

Exercise 1.27

First the ways to distribute 4 aces among 4 players would be 4!. Now with the remaining 48 cards, the ways to split it among 4 people random is, $\binom{48}{12}\binom{36}{12}\binom{24}{12}\binom{12}{12}$. Similarly the total ways to split 52 cards among 4 people w 13 each would be $\binom{52}{13}\binom{39}{13}\binom{26}{13}\binom{13}{13}$. So the probability would be,

$$
\frac{\binom{48}{12}\binom{36}{12}\binom{24}{12}4!}{\binom{52}{13}\binom{39}{13}\binom{26}{13}} = 0.1055
$$

Exercise 1.30

(a). Getting at least one six with 4 throws of a die.

Number of total outcomes are 6^4. The total outcomes with no six are 5^4 so the total outcomes with at least one six is $6^4 - 5^4$. The probability of this would be,

$$\frac{6^4 - 5^4}{6^4}$$

(b). Total outcomes with 24 throws of two dice. One throw of two dice has $6^2 = 36$ possibilities so 24 throws would have 36^{24} possibilities. Throws with no double six would have in each throw only 35 possibilities which would make a total of 35^{24} possibilities. So probability of no double six would be,

$$\frac{36^{24} - 35^{24}}{36^{24}}$$

Comparing the two values we see that probability of (a) is higher than (b).

Exercise 1.44

We need to show that A, B are independent if and only if A and B^c are independent.
(i). If A and B are independent then we have $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$. We can write A is the union of two disjoint events $A = (A \cap B) \cup (A \cap B^c)$. As they are disjoint we have,

$$\begin{aligned}
\mathbb{P}(A) &= \mathbb{P}((A \cap B) \cup (A \cap B^c)) \\
&= \mathbb{P}((A \cap B)) + \mathbb{P}(A \cap B^c)) \\
&= \mathbb{P}(A)\mathbb{P}(B) + \mathbb{P}(A \cap B^c)) \\
\mathbb{P}(A \cap B^c) &= \mathbb{P}(A) - \mathbb{P}(A)\mathbb{P}(B) \\
\mathbb{P}(A \cap B^c) &= \mathbb{P}(A)(1 - \mathbb{P}(B)) \\
\mathbb{P}(A \cap B^c) &= \mathbb{P}(A)\mathbb{P}(B^c)
\end{aligned}$$

Which means that A and B^c are independent as well.
(ii). We use a similar argument as above and write $A = (A \cap B) \cup (A \cap B^c)$ and we know that $\mathbb{P}(A \cap B^c) = \mathbb{P}(A)\mathbb{P}(B^c)$ so we have,

$$\begin{aligned}
\mathbb{P}(A) &= \mathbb{P}((A \cap B) \cup (A \cap B^c)) \\
&= \mathbb{P}((A \cap B)) + \mathbb{P}(A \cap B^c)) \\
&= \mathbb{P}(A)\mathbb{P}(B^c) + \mathbb{P}(A \cap B)) \\
\mathbb{P}(A \cap B) &= \mathbb{P}(A) - \mathbb{P}(A)\mathbb{P}(B^c) \\
\mathbb{P}(A \cap B) &= \mathbb{P}(A)(1 - \mathbb{P}(B^c)) \\
\mathbb{P}(A \cap B) &= \mathbb{P}(A)\mathbb{P}(B)
\end{aligned}$$

Which shows that A and B are independent as well.

Exercise 1.52

We have two urns,

1. 3 white; 4 black
2. 2 white; 6 black

(a). Let W be the event that a random ball from placed into II from I is white and B be that it's black. And let A be the event that the ball picked from Urn II is black. So we need to find $\mathbb{P}(A|W)P(W) + \mathbb{P}(A|B)P(B) = \frac{3}{7}\frac{6}{9} + \frac{4}{7}\frac{7}{9} = \frac{46}{63}$

(b). If I and II are events of picking Urn I and Urn II respectively and B is the event of picking a black ball. We need to find $\mathbb{P}(I|B)$. We have,

$$\begin{aligned}
\mathbb{P}(I|B) &= \frac{\mathbb{P}(B|I)\mathbb{P}(I)}{\mathbb{P}(B)} \\
&= \frac{\mathbb{P}(B|I)\mathbb{P}(I)}{\mathbb{P}(B|I)\mathbb{P}(I) + \mathbb{P}(B|II)\mathbb{P}(II)} \\
&= \frac{\frac{4}{7}}{\frac{4}{7} + \frac{6}{8}} \\
&= \frac{16}{37}
\end{aligned}$$

Problem 9

Two people toss a coin n times each. First we compute the total possible outcomes. Each coin has two options heads or tails, combined there are $2n$ coins. This gives us 2^{2n} possible outcomes.

Now we need to count how many outcomes where there are an equal number of heads. We see that given a person, there are $\binom{n}{k}$ ways that person can get k heads. So for each person there are $\binom{n}{k}$ ways to get k heads. So between them given a fixed k there are $\binom{n}{k}^2$ ways they both have k heads. Now because k is not fixed and can go from 1 to n we have,

$$\binom{n}{1}^2 + \dots + \binom{n}{n}^2 = \sum_{k=1}^{n} \binom{n}{k} = \binom{2n}{n}$$

So we have our answer is,

$$\binom{2n}{n}\frac{1}{2^{2n}}$$

Problem 14

(a). We'll show using induction that,

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_i A_i - \sum_{i<j} \mathbb{P}(A_i \cap A_j) + \dots (-1)^{n+1}\mathbb{P}\left(\bigcap_i A_i\right)$$

Proof. First we verify for our base case. Consider two sets A_1 and A_2. We can write $A_1 = A_1 \setminus A_2 \cup (A_1 \cap A_2)$ and $A_2 = A_2 \setminus A_1 \cup (A_1 \cap A_2)$ and $A_1 \cup A_2 = A_1 \setminus A_2 \cup A_2 \setminus A_1 \cup A_1 \cap A_2$. So we have,

$$\begin{aligned}
\mathbb{P}(A_1) &= \mathbb{P}(A_1 \setminus A_2) + \mathbb{P}(A_1 \cap A_2) \qquad \text{As they are disjoint} \\
\mathbb{P}(A_2) &= \mathbb{P}(A_2 \setminus A_1) + \mathbb{P}(A_1 \cap A_2) \\
\mathbb{P}(A_2 \cap A_1) &= \mathbb{P}(A_2 \setminus A_1) + \mathbb{P}(A_2 \setminus A_1) + \mathbb{P}(A_1 \cap A_2)
\end{aligned}$$

We can rewrite the first two equations to get,

$$\begin{aligned}\mathbb{P}(A_1 \setminus A_2) &= \mathbb{P}(A_1) - \mathbb{P}(A_1 \cap A_2)\\ \mathbb{P}(A_2 \setminus A_1) &= \mathbb{P}(A_2) - \mathbb{P}(A_1 \cap A_2)\end{aligned}$$

Plugging this back in to the third equation we get,

$$\begin{aligned}\mathbb{P}(A_2 \cap A_1) &= \mathbb{P}(A_1) - \mathbb{P}(A_1 \cap A_2) + \mathbb{P}(A_2 \setminus A_1) + \mathbb{P}(A_2) - \mathbb{P}(A_1 \cap A_2) + \mathbb{P}(A_1 \cap A_2)\\ &= \mathbb{P}(A_1) + \mathbb{P}(A_2) - \mathbb{P}(A_1 \cap A_2)\end{aligned}$$

Hence we show the base case is true.
Now let us assume it's true for some arbitrary n so we have,

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_i A_i - \sum_{i<j} \mathbb{P}(A_i \cap A_j) + \dots (-1)^{n+1}\mathbb{P}\left(\bigcap_i A_i\right)$$

We will now show that it will also hold true for $n+1$, we have,

$$\begin{aligned}\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) &= \mathbb{P}\left(\bigcup_{i=1}^{n} A_i \cup A_{n+1}\right)\\ &= \mathbb{P}\left(A_0 \cup A_{n+1}\right) \text{ taking } A_0 = \bigcup_{i=1}^{n} A_i\end{aligned}$$

Now using basecase we have,

$$\begin{aligned}\mathbb{P}\left(A_0 \cup A_{n+1}\right) &= \mathbb{P}(A_0) + \mathbb{P}(A_{n+1}) - \mathbb{P}(A_0 \cap A_{n+1})\\ &= \mathbb{P}(A_0) + \mathbb{P}(A_{n+1}) - \mathbb{P}(A_0 \cap A_{n+1})\\ &= \sum_i^{n} A_i - \sum_{i<j\leq n} \mathbb{P}(A_i \cap A_j) + \dots (-1)^{n+1}\mathbb{P}\left(\bigcap_{i\leq n} A_i\right) + \mathbb{P}(A_{n+1}) - \mathbb{P}(A_0 \cap A_{n+1})\\ &= \sum_i^{n+1} A_i - \sum_{i<j\leq n} \mathbb{P}(A_i \cap A_j) + \dots (-1)^{n+1}\mathbb{P}\left(\bigcap_{i\leq n} A_i\right) - \mathbb{P}(A_0 \cap A_{n+1})\end{aligned}$$

Now expanding the last term we have,

$$\begin{aligned}\mathbb{P}(A_0 \cap A_{n+1}) &= \mathbb{P}((A_1 \cup \dots \cup A_n) \cap A_{n+1})\\ &= \mathbb{P}((A_1 \cap A_{n+1}) \cup \dots \cup (A_n \cap A_{n+1}))\end{aligned}$$

If we take $A_m \cap A_{n+1}$ as B_m for $m \leq n$ then we have, $\mathbb{P}(B_1 \cup \dots \cup B_n)$ which using our induction assumption is equivalent to,

$$\sum_i^{n} B_i - \sum_{i<j\leq n} \mathbb{P}(B_i \cap B_j) + \dots (-1)^{n+1}\mathbb{P}\left(\bigcap_{i\leq n} B_i\right)$$

We expand this further to get,

$$\sum_{i}^{n}\mathbb{P}(A_i\cap A_{n+1})-\sum_{i<j\leq n}\mathbb{P}((A_i\cap A_{n+1})\cap(A_j\cap A_{n+1}))+\dots(-1)^{n+1}\mathbb{P}\left(\bigcap_{i\leq n}(A_i\cap A_{n+1})\right)$$

For some arbitrary set indexes $a,\dots,b$ we have $(A_a\cap A_{n+1})\cap\dots\cap(A_b\cap A_{n+1})=A_a\cap\dots\cap A_b\cap A_{n+1}$ so we have,

$$\sum_{i}^{n}\mathbb{P}(A_i\cap A_{n+1})-\sum_{i<j\leq n}\mathbb{P}(A_i\cap A_j\cap A_{n+1})+\sum_{i<j<k\leq n}\mathbb{P}(A_i\cap A_j\cap A_k\cap A_{n+1})+\dots(-1)^{n+1}\mathbb{P}\left(\bigcap_{i\leq n}(A_i\cap A_{n+1})\right)$$

Now putting back $\mathbb{P}(A_0\cap A_{n+1})$ which expands to the above in our original equation we get,

$$\begin{aligned}
&\sum_{i}^{n+1}A_i-\sum_{i<j\leq n}\mathbb{P}(A_i\cap A_j)+\dots(-1)^{n+1}\mathbb{P}\left(\bigcap_{i\leq n}A_i\right)\\
&\quad-\sum_{i}^{n}\mathbb{P}(A_i\cap A_{n+1})+\sum_{i<j\leq n}\mathbb{P}(A_i\cap A_j\cap A_{n+1})-\sum_{i<j<k\leq n}\mathbb{P}(A_i\cap A_j\cap A_k\cap A_{n+1})-\\
&\quad\dots(-1)^{n+1}\mathbb{P}\left(\bigcap_{i\leq n}(A_i\cap A_{n+1})\right)
\end{aligned}$$

This gives us,

$$\begin{aligned}
\mathbb{P}\left(A_0\cup A_{n+1}\right)&=\sum_{i}^{n+1}A_i-\sum_{i<j\leq n+1}\mathbb{P}(A_i\cap A_j)+\dots(-1)^{n+2}\mathbb{P}\left(\bigcap_{i\leq n+1}A_i\right)\\
\mathbb{P}\left(\bigcup_{i=1}^{n+1}A_i\right)&=\sum_{i}^{n+1}A_i-\sum_{i<j\leq n+1}\mathbb{P}(A_i\cap A_j)+\dots(-1)^{n+2}\mathbb{P}\left(\bigcap_{i\leq n+1}A_i\right)
\end{aligned}$$

Which is the case for $n+1$. Hence, we complete our induction step and show that it must be true for some arbitrary n.

$\square$

(b). We need to find the probability that at least one key was hung on its own hook. Let A_k be the event that the k'th key is hung on its own hook. Then $A_1\cup\dots\cup A_n$ is the event that at least one key is hung on its own hook. So we need $\mathbb{P}(\bigcup_i^n A_i)$ which is equal to $1-\mathbb{P}((\bigcup_i^n A_i)^c)=1-\mathbb{P}(\bigcap_i^n A_i^c)$. We also know the probability that a key is hung on its own hook is $\frac{1}{n}$ which gives us $\mathbb{P}(A_i^c)=\frac{n-1}{n}$. So now we have,

$$\begin{aligned}
\mathbb{P}(\bigcup_{i}^{n}A_i)&=1-\mathbb{P}((\bigcup_{i}^{n}A_i)^c)\\
&=1-\mathbb{P}(\bigcap_{i}^{n}A_i^c)\\
&=1-\mathbb{P}(A_1^c)\dots\mathbb{P}(A_n^c)\text{ as even the complement are independent}\\
&=1-\frac{(n-1)^n}{n^n}=1-\left(1-\frac{1}{n}\right)^n
\end{aligned}$$

We are given that $\lim_{N\to\infty}(1+\frac{x}{N})^N=e^x$. We see that this is same as our second term but with

$x = -1$ so we have,

$$\lim_{n\to\infty} \mathbb{P}(\bigcup_i^n A_i) = 1 - \lim_{n\to\infty}\left(1 - \frac{1}{n}\right)^n$$
$$= 1 - e^{-1} = 0.632120559$$

Now we know that only one key can be hung on each hook. We need to find probability that no key was hung on its own hook. Let A_i be the event that the i'th key was not hung on its own hook.
And we need to find $\mathbb{P}(\bigcap_i^n A_i)$. If there are n keys then the total permutations is $n!$. Now the total permutations where the ith key does not go in the $i'th$ hook would be,

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}(n-k)! = \sum_{k=0}^{n}(-1)^k\frac{n!}{k!} = n!\sum_{k=0}^{n}(-1)^k\frac{1}{k!}$$

So the probability of none going in their own hooks is,

$$\frac{1}{n!}n!\sum_{k=0}^{n}(-1)^k\frac{1}{k!} = \sum_{k=0}^{n}(-1)^k\frac{1}{k!}$$

Now we find the limit as n goes to ∞. We have,

$$\lim_{n\to\infty}\sum_{k=0}^{n}\frac{-1^k}{k!}$$

We are given that $e^x = \sum_{r=0}^{\infty}\frac{x^r}{r!}$ which is similar to our term above but with $x = -1$

$$\lim_{n\to\infty}\sum_{k=0}^{n}\frac{-1^k}{k!} = e^{-1} = \frac{1}{e} = 0.367879441$$

Problem 17

A coin is tossed repeatedly and probability of heads is p and tail is $1-p$. E is the event that r successive heads occurs before the first s successive tails. We need to show that,

$$\mathbb{P}(E|A = H) = p^{r-1} + (1 - p^{r-1})\mathbb{P}(E|A = T)$$

To get this we'll first condition the probability of E given $A = H$ on the next $r-1$ tosses being heads. So let B be the event that the next $r-1$ tosses are heads so we have,

$$\mathbb{P}(E|A = H) = \mathbb{P}(E|A = H, B)\mathbb{P}(B|A = H) + \mathbb{P}(E|A = H, B^c)\mathbb{P}(B^c|A = H)$$

We have $\mathbb{P}(E|A = H, B) = 1$ as if B happens after $A = H$ then we have r successive tosses of H. And we also have $\mathbb{P}(B|A = H) = p^{r-1}$ as B is the event that we have $r-1$ successive H and is independent of the first toss. Similarly we have $\mathbb{P}(E|A = H, B^c) = \mathbb{P}(E|A = T)$ because b^c means that we don't have $r-1$ successive heads which means we got a tails in the middle and as our events are independent this is equivalent to assuming that the first toss is tails and moving forward. Similarly we have $\mathbb{P}(B^c|A = H) = 1 - p^{r-1}$. Putting this all together we get,

$$\mathbb{P}(E|A = H) = p^{r-1} + (1 - p^{r-1})\mathbb{P}(E|A = T)$$

We following a similar argument as above and condition based on the probability of getting $s-1$ successive tails (event C) and have,

$$\mathbb{P}(E|A=T) = \mathbb{P}(E|A=T,C)\mathbb{P}(C|A=T) + \mathbb{P}(E|A=T,C^c)\mathbb{P}(C^c|A=T)$$

Same as above we get $\mathbb{P}(E|A=T,C)=0$ as that means we got s successive tails which means E cannot happen. And $\mathbb{P}(E|A=T,C^c)$ means that s successive tails did not happen meaning we got a heads which is equivalent to $\mathbb{P}(E|A=H)$. We also have $\mathbb{P}(C^c|A=T) = 1-(1-p)^{s-1}$. So we get,

$$\mathbb{P}(E|A=T) = \mathbb{P}(E|A=H)(1-(1-p)^{s-1})$$

Using the above two equations we have,

$$\begin{aligned}
\mathbb{P}(E|A=T) &= \mathbb{P}(E|A=H)(1-(1-p)^{s-1}) \\
\mathbb{P}(E|A=H) &= p^{r-1} + (1-p^{r-1})\mathbb{P}(E|A=T)
\end{aligned}$$

Putting the first in the second we get,

$$\begin{aligned}
P(E|A=H) &= p^{r-1} + (1-p^{r-1})\mathbb{P}(E|A=H)(1-(1-p)^{s-1}) \\
&= p^{r-1} + (1-(1-p)^{s-1} - p^{r-1} + p^{r-1}(1-p)^{s-1})\mathbb{P}(E|A=H)
\end{aligned}$$

So,

$$\begin{aligned}
P(E|A=H)((1-p)^{s-1} + p^{r-1} - p^{r-1}(1-p)^{s-1}) = p^{r-1} \\
\mathbb{P}(E|A=H) &= \frac{p^{r-1}}{(1-p)^{s-1} + p^{r-1} - p^{r-1}(1-p)^{s-1}} \\
\mathbb{P}(E|A=H) &= \frac{p^{r-1}}{(1-p)^{s-1} + p^{r-1}(1-(1-p)^{s-1})}
\end{aligned}$$

Similarly we have,

$$\begin{aligned}
\mathbb{P}(E|A=T) &= \mathbb{P}(E|A=H)(1-(1-p)^{s-1}) \\
&= \frac{p^{r-1}}{(1-p)^{s-1} + p^{r-1}(1-(1-p)^{s-1})}(1-(1-p)^{s-1})
\end{aligned}$$

Now using the law of total probability we have,

$$\begin{aligned}
\mathbb{P}(E) &= \mathbb{P}(E|A=H)\mathbb{P}(A=H) + \mathbb{P}(E|A=T)\mathbb{P}(A=T) \\
&= \mathbb{P}(E|A=H)p + \mathbb{P}(E|A=T)(1-p) \\
&= \frac{p^r}{(1-p)^{s-1} + p^{r-1}(1-(1-p)^{s-1})} + \frac{((1-p)-(1-p)^s)(p^{r-1})}{(1-p)^{s-1} + p^{r-1}(1-(1-p)^{s-1})} \\
&= \frac{p^r}{(1-p)^{s-1} + p^{r-1}(1-(1-p)^{s-1})} + \frac{p^{r-1} - p^r - p^{r-1}(1-p)^s}{(1-p)^{s-1} + p^{r-1}(1-(1-p)^{s-1})} \\
&= \frac{p^{r-1}(1-(1-p)^s)}{(1-p)^{s-1} + p^{r-1}(1-(1-p)^{s-1})}
\end{aligned}$$