Topology

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Chapter 1

Preliminaries

1.1 Axiom of Choice

Axiom of choice. Given a collection $\mathcal{A}$ of disjoint nonempty sets, there exists a set C consisting exactly one element from each element of $\mathcal{A}$.

Note. This means that the set C is such that C is contained in the union of the elements of $\mathcal{A}$ and for each $A \in \mathcal{A}$ the set $C \cap A$ has a single element.

Lemma 1.1 (Choice function)**.** Given a collection $\mathcal{B}$ of nonempty sets (not necessarily disjoint), there exists a function,

$$c : \mathcal{B} \to \bigcup_{B \in \mathcal{B}} B$$

such that $c(B)$ is an element of B, for each $B \in \mathcal{B}$

Note. the function c is called a *choice function* for the collection $\mathcal{B}$

Remark. The difference from the A.O.C is that in this lemma the sets need not be disjoint. So we have have $\mathcal{B}$ as the powerset of any set (excluding ϕ that is)

Proof. For a given B in $\mathcal{B}$ consider,

$$B' = \{(B, x) : x \in B\}$$

Now note that B' is nonempty as B is nonempty. Also note that for distinct B_1, B_2 we have B'_1, B'_2 are distinct as well. Now consider,

$$\mathcal{C} = \{B' : B \in \mathcal{B}\}$$

Note that sets in $\mathcal{B}$ are disjoint. So we can use axiom of choice there is a set c consisting of exactly one element from each elements of $\mathcal{C}$. Now note we have an assignment for each element of $\mathcal{C}$ to an ordered pair (B, x) and hence c is a rule for a function from the collection $\mathcal{B}$ to the set $\bigcup_{B \in \mathcal{B}} B$ □

1.2 Well Ordered sets

Definition 1.2. A set A with an order relation $<$ is said to be well-ordered if every nonempty subset of A has a smallest element.

Example. $\{1, 2\} \times \mathbb{Z}_+$ in the dictionary ordering, so we have,

$$(1, 1), (1, 2), (1, 3), \ldots, (2, 1), (2, 2), (2, 3)$$

where,

$$(a, b) < (c, d) \text{ if } a < c \text{ or } a = c \text{ and } b < d$$

$\diamond$

Example. The set of integers is not well-ordered in the usual sense. For instance the subset containing the negative integers has no smallest element. Neither is the real numbers in $[0, 1]$ either as any open subset does not have a smallest element. $\diamond$

Theorem 1.3. Every nonempty finite ordered set has the order type of a section $\{1, \ldots, n\}$ of $\mathbb{Z}_+$, so it is well ordered.

Proof. First we claim that every finite ordered set has a largest element (by induction). Secondly we can construct an order preserving bisection of A with $\{1, \ldots, n\}$ for some n by induction as well. Trivial for $n = 1$, assume for $n - 1$, now for case n we isolate the largest element, use $n - 1$ case and define $f : A \to \{1, \ldots, n\}$ by setting $f(x) = f'(x)$ and $f(b) = n$ where $f' : A - \{b\} \to \{1, \ldots, n - 1\}$. $\square$

Theorem 1.4. If A is a set, there exists an order relation on A that is a well-ordering

Chapter 2

Topological Spaces and Continuous Functions

2.1 Topological Spaces

Definition 2.1. A ***topology*** on a set X is a collection $\mathcal{T}$ of aubsets of X having the following properties,

(1) $\emptyset$ and X are in $\mathcal{T}$

(2) The union of the elements of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$

(3) The intersection of the elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$

A set X for which a topology $\mathcal{T}$ has been specified is called a topological space.

Note. A topological space is an ordered pair $(X, \mathcal{T})$

Remark. Note that properties (2) and (3) are closely related to the properties of open sets in analysis. In fact this is how we define open sets more generally.

Definition 2.2. If X is a topological space with topology $\mathcal{T}$ then a subset U of X is an **open set** of X if U belongs to the collection $\mathcal{T}$.

Remark. This is a definition for open sets in the context of a topology.

Note. Using this terminology we can say that a toplogical space is a set X with a collection of subsets of X called open sets such that $\emptyset$ and X are both open and such that arbitrary unions and finite intersections of open sets are open.

Example. If X is a set, the collection of all subsets of X is a topology on X called the ***discrete topology***. The collection with just $\emptyset$ and X is called the ***trivial topology*** $\diamond$

Example. If X is a set and $\mathcal{T}_f$ is the collection of all subsets U of X such that $X - U$ is either finite or all of X, then $\mathcal{T}_f$ is a topology on X called the ***finite complement topology***. First we have $X, \emptyset$ is in $\mathcal{T}_f$, since $X - X$ and $X - \emptyset$ are finite. Now given an indexed family of elements of $\mathcal{T}_f$ i.e. $\{U_\alpha\}$ that satisfy this property we need to show (2) and (3) hold.We see,

$$X - \bigcup U_\alpha = \bigcap (X - U_\alpha)$$

This is finite as each $X - U_\alpha$ is finite by assumption of U. Similarly given $U_1, \ldots, U_n$ we see,

$$X - \bigcap^n U_t = \bigcup^n (X - U_i)$$

again the latter is finite as union of finite sets is finite. $\diamond$

Definition 2.3. Given $\mathcal{T}$ and $\mathcal{T}'$ are two topologies on X. If $\mathcal{T} \subset \mathcal{T}'$. Then $\mathcal{T}'$ is ***finer*** than $\mathcal{T}$. If $\mathcal{T}'$ properly contains $\mathcal{T}$ then it's ***strictly finer***. We also use the terminology ***coarser*** and ***striclty coarser*** the other way around. $\mathcal{T}$ is ***comparable*** with $\mathcal{T}'$ if either $\mathcal{T} \subset \mathcal{T}'$ or $\mathcal{T}' \subset \mathcal{T}$

Note. This means that the "finest" topology is the discrete topology and the "coarsest" is the trivial.

2.2 Basis for a Topology

Definition 2.4. If X is a set, a ***basis*** for a topology on X is a collection $\mathcal{B}$ of subsets of X (called the basis elements) such that,

1. For each $x \in X$, there is at least one basis element B containing x.
2. If x belongs to $B_1 \cap B_2$, then there exists a basis element B_3 containing x such that $B_3 \subset B_1 \cap B_2$

Remark. If $\mathcal{B}$ satisfies the above two condition, then the we define the ***topology T generated by B*** as follows: A subset U of X is said to be open in X (in other words in $\mathcal{T}$) if for each $x \in U$, there is $B \in \mathcal{B}$ such that $x \in B$ and $B \subset U$. Each basis element is itself an element in $\mathcal{T}$.

Example. If $\mathcal{B}$ is the collection of all circular regions in the plane (interiors of circles). Then $\mathcal{B}$ satisfies both conditions of a basis. In the topology generated, a subset U of the plane is open if every x in U lies in some circular region contained in U (note this implies that the border points cannot be included). $\diamond$

Example. If X is any set, the collection of all one-point subsets of X is a basis for the discrete topology on X. $\diamond$

We need to check that the collection $\mathcal{T}$ generated by $\mathcal{B}$ is a topology. First if $U = \emptyset$, then clearly it satisfies our condition and hence is in $\mathcal{T}$ if $U = X$ then clearly for each $x \in U$ there is a basis element B (by definition of a basis) containing X and trivially $B \subset X$ and hence $X \in \mathcal{T}$.

Now take two elements U_1, U_2 in $\mathcal{T}$. We need to show that $U_1 \cap U_2 \in \mathcal{T}$. First choose B_1, B_2 containing x such that $B_1 \subset U_1$ and $B_2 \subset U_2$. The second condition gives us B_3 containing x such that $B_3 \subset B_1 \cap B_2$. Now $x \in B_3$ and $B_3 \subset U_1 \cap U_2$ so we have $U_1 \cap U_2 \in \mathcal{T}$.

2.3 The Order Topology

2.4 The Product Topology

2.5 The Subspace Topology