Real Analysis

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Contents

1 The Real Numbers **2**
1.1 $\sqrt{2}$ 2

Chapter 1

The Real Numbers

1.1 $\sqrt{2}$

Theorem 1.1. $\sqrt{2}$ is irrational.

Proof. Assume that $\sqrt{2}$ is rational such that it can be written in the form $\frac{p}{q}$ where both p and q are coprime and $q \neq 0$. So we have,

$$\begin{aligned}\sqrt{2} &= \frac{p}{q}\\ 2 &= \frac{p^2}{q^2}\\ 2q^2 &= p^2\end{aligned}$$

If $2|p^2$ then it must mean that $2|p$ so we can write p like $2a$ where a is some rational. So,

$$\begin{aligned}2q^2 &= (2a)^2 = 4a^2\\ q^2 &= 2a^2\end{aligned}$$

Similarly if $2|q^2$ then $2|q$ implying that both p and q have a common factor 2. However, this contradicts our assumption as we assume p and q are co-prime numbers. Hence our assumption that $\sqrt{2}$ is rational must be false. Hence, $\sqrt{2}$ is irrational. □

Definition 1.2 (Rational Numbers). $Q = \left\{\text{all fractions } \frac{p}{q} \text{ where } p \text{ and } q \text{ are integers with } q \neq 0\right\}$

Definition 1.3 (Field). A field is any set where addition and multiplication are well-defined operations that are commutative, associative and distributive. There must be an additive identity and every element must have an additive inverse. There must be a multiplicative identity and multiplicative inverse for all nonzero elements in the field.

Example. Some examples structures that are and are not a field are,

1. $\mathbb{Q}$ is a field.
2. $\mathbb{Z}$ is not a field.
3. $\mathbb{N}$ is not a field.
4. $\{0, 1, 2, 3, 4\}$ is a field under addition and multiplication modulo 5.

$\diamond$

What $\mathbb{Q}$ is lacking

$\mathbb{Q}$ is a field is useful to carry out various operations. However, we can't take things like square roots or so, i.e. there are a lot of holes in $\mathbb{Q}$. When we fill these gaps we get $\mathbb{R}$ (will be done more rigorously later on). Every hole can be defined as a new irrational number.