Linear Alebgra HW05

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3A

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Proof. We know for a linear map, $T(u+v) = T(u) + T(v)$ and $T(\lambda v) = \lambda T(v)$
First we look at additivity,
Consider an arbitrary $u = (x_1, y_1, z_1)$ and $v = (x_2, y_2, z_2)$. So we have,

$$T(u+v) = T((x_1+x_2), (y_1+y_2), (z_1+z_2))$$

$= (2(x_1+x_2)-4(y_1+y_2)+3(z_1+z_2)+b, 6(x_1+x_2)+c(x_1+x_2)(y_1+y_2)(z_1+z_2))$

We need the above to be equal to,

$$T(u)+T(v) = (2x_1-4y_1+3z_1+b, 6x_1+cx_1y_1z_1)+(2x_2-4y_2+3z_2+b, 6x_2+cx_2y_2z_2)$$

$$= (2(x_1+x_2)-4(y_1+y_2)+3(z_1+z_2)+2b, 6(x_1+x_2)+c(x_1y_1z_1+x_2y_2z_2)$$

Comparing each of the terms we have,

$$2(x_1+x_2)-4(y_1+y_2)+3(z_1+z_2)+2b = 2(x_1+x_2)-4(y_1+y_2)+3(z_1+z_2)+b$$

$$2b = b$$

$$b = 0$$

Similarly comparing the second term we have,

$$6(x_1+x_2)+c(x_1+x_2)(y_1+y_2)(z_1+z_2) = 6(x_1+x_2)+c(x_1y_1z_1+x_2y_2z_2)$$

$$c(x_1+x_2)(y_1+y_2)(z_1+z_2) = c(x_1y_1z_1+x_2y_2z_2)$$

$$c((x_1+x_2)(y_1+y_2)(z_1+z_2) - (x_1y_1z_1+x_2y_2z_2)) = 0$$

For this to be true for any x, y, z we need $c = 0$. Hence for additvity we need $b = c = 0$
Now we check if $T(kv) = kT(v)$. Consider $v = (x, y, z)$. Then we have

$$T(kv) = T(kx, ky, kz) = (2kx - k4y + 3kz + b, 6kx + k^3cxyz)$$

We need this to be equal to

$$kT(v) = k(2x-4y+3z+b, 6x+cxyz) = (2kx-4ky+3kz+bk, 6kx+kcxyz)$$

Comparing the terms we have,

$$2kx - 4ky + 3kz + bk = 2kx - 4ky + 3kz + b$$

$$bk = b$$

$$b = 0$$

$6kx + kcxyz = 6kx + k^3cxyz$

$$c = k^2c$$

$$c = 0$$

So we have $b = c = 0$

□

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Proof. 1. Associativity. We have $(T_1T_2)T_3 = T_1(T_2T_3)$
Consider the operation on a vector v so we have, $(T_1T_2)T_3v$ which is,

$$((T_1T_2)(T_3(v)) = T_1(T_2(T_3(v)))$$

Now looking at the right side we have, $T_1(T_2T_3) = T_1(T_2(T_3(v)))$. So we showed tha tthe LHS is equal to the RHS.
2. Identity. Consider a vector v we have,

$$TIv = T(I(v)) = T(v)$$

Now,

$$ITv = I(T(v)) = T(v) \text{ because } Iv = v, \forall v$$

3. Distributive Property
To show that,

$$(S_1 + S_2)T = S_1T + S_2T$$

Consider an abitrary vector v in the domain of T. We have,

$$(S_1 + S_2)Tv = (S_1 + S_2)(T(v))$$

By definitino of addition of linera maps we have,

$$= (S_1(T(v))) + (S_2(T(v)))$$

Simliary we have,

$$(S_1T + S_2T)v = S_1T(v) + S_2T(v) = S_1(T(v)) + S_2(T(v))$$

We see that the distributive property holds.
Now To show that $S(T_1 + T_2) = ST_1 + ST_2$. Consider v we have,

$$S(T_1 + T_2)v = S(T_1(v) + T_2(v)) = S(T_1(v)) + S(T_2(v))$$

And we have,

$$(ST_1 + ST_2)v = ST_1(v) + ST_2(v) = S(T_1(v)) + S(T_2(v))$$

We see that the property holds again.

□

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Proof. Let V be a one dimentional vector space. This means that the basis of V contains a single vector, let the basis be $\{v\}$. Now we are considering a linear map from V to itself.
So assume that the linear map T maps some v_0 in V to w_0. We need to show that $w_0 = \lambda v_0$ for some $\lambda \in F$. Because T maps V to itself we known that that $w_0 \in V$ for any w_0. If $w_0 \in V$ then wek now that it can be written as a linear compbination of its basis. As the basis only has one vector we can write $w_0 = \lambda_1 v$. Similarly as $v_0 \in V$ we can write $v_0 = \lambda_2 v$. So we have,

$$\frac{v_0}{\lambda_2} = v$$

$$w_0 = \lambda_1 \frac{v_0}{\lambda_2} = \lambda v_0$$

□

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Proof. Consider the function that maps any vecotor (x, y) to the $max(|x|, |y|)$. We can see that this satisfies homogeneity. For instance consider $(2, 6)$. Our function maps this to 6. Now consider $(2 \times 3, 6 \times 3)$ which is mapped to 18 which is 3×6 as we saw above.
Now consider two vector $(1, 0)$ and $(0, 4)$. Our function maps both these vectors to 1 and 4 respectively. However it maps its sum $(1, 4)$ to $4 \neq 4 + 1$. Hence it does not follow additvity. Hence not a linear space. □

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Proof. First let us define a linear map S from U to W that maps all $u \in U$ to a $w \in W$.
We need to extend this map to T from U to V such that all values from V can be mapped to a $w \in W$ such that $T(u) = S(u)$ is true for any $u \in U$.
Let us define a map T as follows,

$$T(a_1u_1+\cdots+a_ku_k+b_1v_1+\cdots+b_{n-k}v_{n-k}) = T(a_1u_1)+\cdots+T(a_ku_k)+T(b_1v_1)+\cdots+T(b_{n-k}v_{n-k})$$

such that $T(k_1v_1) = \cdots = T(k_nv_{n-k}) = 0$ and $T(u) = S(u)$ for any $u \in U$
Now we need to show that this map is a linear maps.
1. Addivitiy, we need to show that $T(a+b) = T(a)+T(b)$. Consider $a \in V$ s.t. $a = a_1u_1 + \ldots b_{n-k}v_{n-k}$ and $b = c_1u_1 + \cdots + d_{n-k}v_{n-k}$
So

$$T(a_1u_1 + \cdots + b_{n-k}v_{n-k} + c_1u_1 + \cdots + d_{n-k}v_{n-k}) =$$

$$= T(a_1u_1) + \ldots T(a_nu_n) + T(c_1u_1) + \cdots + T(c_nu_n) + 0$$

as $T(kv_k) = 0$

By definition,

$$T(a+b) = T((a_1+c_1)u_1+\cdots+(b_{n-k}+d_{n-k})v_{n-k}) = T((a_1+c_1)u_1)+\cdots+T((b_{n-k}+d_{n-k})v_{n-k})$$

$$= T((a_1+c_1)u_1)+\cdots+T((a_n+c_n)u_n)$$

$$= T(a_1u_1)+\ldots T(a_nu_n)+T(c_1u_1)+\cdots+T(c_nu_n)$$

So we have shown that it is linear.
Now we need to show its homogenous. We need to show that $T(\lambda(v)) = \lambda T(v)$
We have,

$$T(\lambda(a_1u_1+\cdots+b_{n-k}v_{n-k})) = T(\lambda a_1u_1+\cdots+\lambda b_{n-k}v_{n-k})$$

$$= T(\lambda a_1u_1)+\cdots+T(\lambda b_{n-k}v_{n-k})$$

We know $T(\lambda v_k) = 0$ so this is equal to,

$$= T(\lambda a_1u_1)+\ldots T(\lambda a_nu_n)$$

$$= S(\lambda a_1u_1)+\ldots T(\lambda a_nu_n)$$

$$= \lambda S(a_1u_1)+\ldots \lambda S(a_nu_n)$$

$$= \lambda T(a_1u_1)+\cdots+\lambda T(a_nu_n)$$

$$= \lambda(T(a_1u_1)+\ldots T(a_nu_n)+T(b_1v_1)+\cdots+T(b_{n-k}v_{n-k}))$$

$$= \lambda(T(a_1u_1+\cdots+a_nu_n+b_1v_1+\cdots+b_{n-k}v_{n-k}))$$

$$= \lambda(T(a+b))$$

Hence it is homogenous.
So we have construted a linear map from V to W that has $T(u) = S(u)$ for all $u \in U$

□

3B

2

Proof. We need to show that $(ST)^2 = 0$. Or that,

$$S(T(S(T(v)))) = 0$$

We are given that, range S $\subseteq$ null T. Or that for any $v \in$ domain S. $S(v) = u$ then $T(u) = 0$.
We know that $T(v) = v_0$. Then we have $S(v_0)$ is a vector in null space of T. Which means that $T(S(v_0)) = 0$. We know that if L is a linear map then $L(0) = 0$. So $S(T(S(v_0))) = S(0) = 0$

□

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Proof. We have,

$$(v_1, \ldots, v_n) \text{ is linearly independent}$$

This means that,

$$a_1 v_1 + \cdots + a_n v_n = 0$$

then $a_1 = \cdots = a_n = 0$
Let us apply the linear map on both sides and we get,

$$T(a_1 v_1 + \cdots + a_n v_n) = T(0) = 0$$

$$= T(a_1 v_1) + \cdots + T(a_n v_n) \text{ as T is a linear map}$$

$$= a_1 T(v_1) + \cdots + a_n T(v_n) = 0$$

We know from before that $a_1 = \cdots = a_n = 0$. This means that

$$T(v_1), \ldots, T(v_n)$$

is linearly independent as the only way to represent 0 is having all the coefficients as 0.

□

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Proof. First we know that dim(range T) = dim(V) = n. So it is enough to show that $T(v_1), \ldots, T(v_n)$ are n linearly indepenmdent vectors in range T.
If $v_1, \ldots, v_n$ span V then we know that $v_1, \ldots, v_n$ are linearly independent. So,

$$a_1 v_1 + \cdots + a_n v_n = 0$$

such that $a_1 = \cdots = a_n = 0$
Applying the operator on both sides we get,

$$T(a_1 v_1 + \cdots + a_n v_n) = T(0) = 0$$

$$= T(a_1 v_1) + \cdots + T(a_n v_n) = 0$$

$$= a_1 T(v_1) + \cdots + a_n T(v_n) = 0$$

We know from above, $a_1 = \cdots = a_n = 0$ which means that $T(v_1), \ldots, T(v_n)$ is linearly independent set of vectors in range T such that $dim(a_1 T(v_1) + \cdots + a_n T(v_n)) = dim(range(V)) = n$ which makes it span range T.

□

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We have null T = $\{(x_1, x_2, x_3, x_4) \in F^4 : x_1 = 5x_2, x_3 = 7x_4\}$
So this means that we have two independent variables which implies that the null space has dimension of two.

So we have range of T as dimension of 2. Because dim of range is equal to the dimension of the codomain the linear map is surrjective.

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we know dim V = dim (null(T)) + dim (range(T))
If null space and range of T are finite dimensional that means that dim V is a finite number. Or that V is a finite dimentional space.

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Given $P(P(v)) = P(v)$ we need to shwo that $V = nullP \oplus rangeP$.
We have to show two things, $nullP \cap rangeP = \{0\}$ and $\forall v, v = u + v, u \in nullP, v \in rangeP$.
1. Assue $v \in nullP \cap rangeP$. So that means $v \in nullP$ and $v \in rangeP$. If $v \in nullP$ then,

$$P(v) = 0$$

If $v \in range(P)$ then $\exists w \in V, v = P(w)$. We are given that $P(P(v)) = P(v)$ and we know that $P(v) = 0$ and $P(w) = v$. So we get,

$$P(v) = P(P(w)) = P(w)$$

Or in other words $P(v) = 0$ so $P(w) = v = 0$. Hence we show that their intersectino only consist of the zero vector.
Now we need to show that every vecotr $v \in V$ can be written as $u + w$ such that $u \in$ null P and $v \in$ range P.
Consider any $v \in V$ such that $P(v) = v_1$. This means that $v_1 \in rangeP$. Also $P(v_1) = P(P(v)) = P(v)$ so $P(v_1) = P(v)$.
Now consider $v_2 = v - v_1$. Appling the operator on both sides we get,

$$P(v_2) = P(v - v_1) = P(v) - P(v_1) = 0$$

which implies that $v_2 \in nullP$.
So now we have a $v_1 + v_2 = v - v_1 + v_1 = v$ such that $v_1 \in rangeP$ and $v_2 \in nullP$

3C

1

> **Proof.** Consider the contrary that there exists a matrix of T such that it has less then dim range T non zero entries.
> This means that there always exists a column of the matrix such that the column is filled with zeroes. Consider r to be the columns that have non-zero entries this means that $r <$ dim range T.
> We know by definition of matrices that,
>
> $$T(v_k) = A_{1k}w_1 + \cdots + A_{nk}w_n$$
>
> where $w_1, \ldots, w_n$ is a basis for range T.

Now this menas that are n - r choices of k s.t $T(v_k) = 0$
We know by definitnio of a linear map that for a given basis of V,

$$T(v_1) = w_1, \dots, T(v_m) = w_m$$

such that $w_1, \dots, w_m$ span W.
However in our case we have only r non-zero $w \in W$. Which means that the dim W is r. However this contradicts the fact that r < dim W. Which means that our assumption must be wrong.

$\square$

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Proof. $\Rightarrow$ Consider an arbitrary basis of V $v_1, \dots, v_n$. We are told that dim (range T) = 1. Which means that $T(v_1), \dots, T(v_n)$ spans a 1 D space or in other words,

$$T(v_1) = c_1 w$$
$$\dots$$
$$T(v_n) = c_n w$$

for some $w \in W$
Now we can adjust our basis for V by multiplying $\frac{1}{c_k}$ to each side to get a new basis, $\frac{v_1}{c_1}, \dots, \frac{v_n}{c_n}$ such that,

$$T(v_1') = w$$

$$\dots$$

$$T(v_n') = w$$

We also know that for a matrix $M(T)$ we have,

$$T(v_k) = A_{1,k} w_1 + \cdots + A_{n,k}$$

So from above we have,

$$T(v_k) = A_{1,k} w_1 + \cdots + A_{m,k} = w$$

for any $w \in W$. So we need a basis of W such that $w_1 + \cdots + w_m = w$
Consider an arbitrary basis for W, $w_1, \dots, w_m$. Now consider the following list of vectors, we know , $\exists a_1, \dots, a_n$ s.t $a_k \neq 0$

$$a_1 w_1 + \cdots + a_n w_n = w$$

.
Now lets adjust this basis such that $w_1' = a_1 w_1, \dots, w_n' = a_n w_n$. It is easy to show that this new list of vectors is a basis.

So now we have constructed a basis for W such that $w_1 + \cdots + w_n = w$. Which means we have defined a basis for V and W such that,

$$T(v_k) = w_1 + \cdots + w_n$$

$\Leftarrow$
Conisder there exists a basis of V and W such that,

$$T(v_k) = 1w_1 + \cdots + 1w_n$$

where $k \in 1, \ldots, n$ and $w_1, \ldots, w_n$ is a basis for W.
This means that $T(V_k) = w$ where $w = w_1 + \cdots + w_n$.
We know that $T(v_1), \ldots, T(v_n)$ spans the range of T which means that the range of T is spanned by $w \Rightarrow$ dim range T = 1 $\square$

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(a). We define $M(S)$ and $M(T)$ as follows,

$$Sv_k = M_{s(1,k)}w_1 + \cdots + M_{s(m,k)}w_m$$
$$Tv_k = M_{t(1,k)}w_1 + \cdots + M_{t(m,k)}w_m$$

And we define $M(S+T)$,

$$(S+T)v_k = M_{st(1,k)}w_1 + \cdots + M_{st(m,k)}w_m$$

Further expanding we get,

$$\begin{aligned}(S+T)v_k &= Sv_k + Tv_k \\ &= M_{s(1,k)}w_1 + \cdots + M_{s(m,k)}w_m + M_{t(1,k)}w_1 + \cdots + M_{t(m,k)}w_m \\ &= (M_{s(1,k)} + M_{t(1,k)})w_1 + \ldots (M_{s(m,k)} + M_{t(m,k)})w_m\end{aligned}$$

Comparing the two results we get,

$$M_{st(1,k)}w_1 = (M_{s(1,k)} + M_{t(1,k)})w_1$$
$$\ldots$$
$$M_{st(m,k)}w_m = (M_{s(m,k)} + M_{t(m,k)})w_m$$

Which means that for any

$$M_{st(r,c)} = M_{s(r,c)} + M_{t(r,c)} \Rightarrow M(S+T) = M(T) + M(S)$$

(b). Similary we have,

$$Tv_k = M_{t(1,k)}w_1 + M_{t(m,k)}w_m$$

and

$$\lambda Tv_k = M'_{t(1,k)}w_1 + M'_{t(m,k)}w_m)$$

Now multiplying λ we get,

$$\lambda Tv_k = \lambda(M_{t(1,k)}w_1 + M_{t(m,k)}w_m)$$
$$\lambda Tv_k = (\lambda M_{t(1,k)})w_1 + (\lambda M_{t(m,k)})w_m$$

So we see that for all entires of our matrix $M'_{r,c} = \lambda M_{r,c}$ which means that,

$$M(\lambda T) = \lambda M(T)$$

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We have matrix M such that,

$$\begin{aligned}
Tv_1 &= M_{1,1}w_1 + 0M_{2,1}w_2 + 0M_{3,1}w_3 \\
Tv_2 &= M_{1,2}w_1 + 0M_{2,2}w_2 + 0M_{3,2}w_3 \\
Tv_3 &= M_{1,3}w_1 + 0M_{2,3}w_2 + 0M_{3,3}w_3 \\
Tv_4 &= M_{1,4}w_1 + 0M_{2,4}w_2 + 0M_{3,4}w_3
\end{aligned}$$

Consider the basis of W as $(1, x, x^2)$. So we have $T(v_1) = 1, T(v_2) = x, T(v_3) = x^2, T(v_4) = 0$. Our basis in V is $(x, \frac{1}{2}x^2, \frac{1}{3}x^3, 1)$

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Proof. First let us take dim V = n, dim W = m, dim range T = k. Let $w_1, \ldots, w_k$ be a basis for range T. Now according to the lienar map lemma we can find $v_1, \ldots, v_k$ such that,

$$T(v_r) = w_r$$

We know dim range T + dim null T = dim V, so dim null T = n - r. and we know that none of our current $v_r \in nullT$ as $w_r \neq 0$. Which means that we can exted our linearly independe list, $v_1, \ldots, v_k$ to $v_1, \ldots, v_k, v_{k+1}, \ldots, v_n$ such that

$$Tv_{k+1} = 0, \ldots, Tv_n = 0$$

Similarly let us extend our basis for W from $w_1, \ldots, w_r$ to $w_1, \ldots, w_r, w_{r+1}, \ldots, w_m$
Now let us define our matrix such that,

$$\begin{aligned}
Tv_1 &= 1w_1 + \cdots + 0w_r + \cdots + 0w_n \\
&\cdots \\
Tv_r &= 0w_1 + \cdots + 1w_r + \cdots + 0w_n \\
Tv_{r+1} &= 0w_1 + \cdots + 0w_r + \cdots + 0w_n \\
&\cdots \\
Tv_m &= 0w_1 + \cdots + 0w_r + \cdots + 0w_n
\end{aligned}$$

Now we have defined a matrix such that the $A_{k,k}$element is 1 for $k \leq$ dim (range T) and all the other elements are 0.

□