

Probability Theory: HW5

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Exercise 7.60 We have X is normal with μ and σ^2 and we need to find (X^3) . We know the Moment generating function of a normal is $M_X(t) = E(e^{tX}) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$. And note that at $s = 0$ we have $(e^{sX})^{(3)} = X^3$ and hence we have $M_X(0)^{(3)} = E(X^3)$. Third derivative of $e^{\mu t + \frac{1}{2}\sigma^2 t^2}$ is $[(\mu + \sigma^2 t)^3 + 3\sigma^2(\mu + \sigma^2 t)]e^{\mu t + \frac{1}{2}\sigma^2 t^2}$ and evaluated at $s = 0$ is $\mu^3 + 3\sigma^2\mu$

Exercise 7.75

We have,

$$\frac{1}{\eta} = \frac{1}{n} \sum_{i=1}^n \frac{1}{x_i}$$

We need to show that $\sqrt[n]{x_1 \dots x_n} \geq \eta$ or that $\frac{1}{\eta} \geq \frac{1}{\sqrt[n]{x_1 \dots x_n}}$

First we know that $AM \geq GM$ which means,

$$\frac{1}{n} \sum x_i \geq \left(\prod_i x_i \right)^{\frac{1}{n}}$$

Now note if we take reciprocals then we get,

$$\frac{1}{n} \sum \frac{1}{x_i} \geq \left(\prod_i \frac{1}{x_i} \right)^{\frac{1}{n}}$$

Note the left side is $\frac{1}{\eta}$ and the right side is $\frac{1}{\sqrt[n]{x_1 \dots x_n}}$ which gives us,

$$\frac{1}{HM} \geq \frac{1}{GM}$$

or that $GM \geq HM$.

Exercise 8.10 We have N_n is the number of occurrence of 5 or 6 in n throws of a fair die. We need to show that,

$$\frac{1}{n} N_n \rightarrow \frac{1}{3} \quad \text{in mean square}$$

Let $N_n = X_1 + \dots + X_n$ where each X_i is a r.v. that takes on 1 if we get a 5 or 6 else 0. So we have $\mathbb{P}(X_i = 1) = \frac{1}{3}$ which gives us $E(X_i) = \frac{1}{3}$ and some σ^2 variance. According to theorem 8.6 we have,

$$\frac{1}{n} (X_1 + \dots + X_n) \rightarrow \mu \text{ in mean square}$$

But we have $N_n = X_1 + \dots + X_n$ and $\mu = \frac{1}{3}$ so we get,

$$\frac{1}{n} N_n \rightarrow \frac{1}{3} \quad \text{in mean square}$$

which is what we want to prove.

8.11 We assume that each X_i is uncorrelated instead of the stronger condition that it's independent. Each being uncorrelated implies that we have $Cov(X_i, X_j) = 0$. Now similarly consider we have,

$$S_n = X_1 + \dots + X_n$$

Then we get,

$$E\left(\frac{1}{n} S_n\right) = \frac{1}{n} E(X_1 + \dots + X_n) = \frac{1}{n} n\mu = \mu$$

Now for mean square we need to show that $E([Z_n - Z]^2) \rightarrow 0$ as $n \rightarrow \infty$. We have,

$$\begin{aligned} E\left(\left[\frac{1}{n}S_n - \mu\right]^2\right) &= \text{var}\left(\frac{1}{n}S_n\right) \\ &= \frac{1}{n^2}\text{var}(X_1 + \dots + X_n) \end{aligned}$$

Now note the following. As X_i, X_j are independent for any i, j then for the list X_1, \dots, X_n we have $\text{var}(X_1 + \dots + X_n) = \text{var}(X_1) + \dots + \text{var}(X_n)$. This is true because of the following proof by induction.

First note for base case we have this as trivially true as we have $\text{var}(X_1 + X_2) = \text{var}(X_1) + 2\text{Cov}(X_1, X_2) + \text{var}(X_2)$ but as X_1, X_2 are uncorrelated we have the second term is false and hence we get $\text{var}(X_1 + X_2) = \text{var}(X_1) + \text{var}(X_2)$. Now assume true for arbitrary k i.e we have $\text{var}(X_1 + \dots + X_k) = \text{var}(X_1) + \dots + \text{var}(X_k)$ now we need to show that we also have $\text{var}(X_1 + \dots + X_{k+1}) = \text{var}(X_1) + \dots + \text{var}(X_{k+1})$. First take $X_1 + \dots + X_k = Y$ then we have,

$$\begin{aligned} \text{var}(X_1 + \dots + X_{k+1}) &= \text{var}(Y + X_{k+1}) \\ &= \text{var}(Y) + \text{var}(X_{k+1}) + 2\text{Cov}(Y, X_{k+1}) \end{aligned}$$

But now note that we have $\text{Cov}(X_1 + \dots + X_k, X_{k+1}) = \text{Cov}(X_1, X_{k+1}) + \dots + \text{Cov}(X_k, X_{k+1}) = 0$ hence we have,

$$\begin{aligned} \text{var}(X_1 + \dots + X_{k+1}) &= \text{var}(Y) + \text{var}(X_{k+1}) \\ &= \text{var}(X_1) + \dots + \text{var}(X_k) + \text{var}(X_{k+1}) \end{aligned}$$

Which is the case for $k + 1$. Hence, by induction we have shown that this is true.

Now using this statement we have,

$$\begin{aligned} E\left(\left[\frac{1}{n}S_n - \mu\right]^2\right) &= \frac{1}{n^2}\text{var}(X_1 + \dots + X_n) \\ &= \frac{1}{n^2}(\text{var}X_1 + \dots + \text{var}X_n) \\ &= \frac{1}{n^2}n\sigma^2 \\ &= \frac{1}{n}\sigma^2 \end{aligned}$$

Now note that trivially we have $\frac{1}{n}\sigma^2 \rightarrow 0$ as $n \rightarrow \infty$. Hence, we show just being uncorrelated is enough for convergence in mean square.

Exercise 8.32 Let $S = X_1 + \dots + X_{12000}$ where X_i is a Bernoulli with $p = \frac{1}{6}$ and takes on 1 if we hit a 6. Now note that we have $E(X_i) = \frac{1}{6}$ and $\text{Var}(X_i) = \frac{5}{36}$. Now we know that using CLT for large enough n we have the distribution of the standardized version of S goes to a normal standard.

$$S' = \frac{S - n\mu}{\sqrt{N}\sigma} = \frac{S - 2000}{\sqrt{12000 \cdot \frac{5}{6}}} = \frac{6S - 12000}{\sqrt{60000}}$$

So for n large enough we have S' is equivalent to a normal standard. Now note,

$$\begin{aligned} \mathbb{P}(1900 < S < 2200) &= \mathbb{P}(6 \cdot 1900 < 6 \cdot S < 6 \cdot 2200) \\ &= \mathbb{P}(-600 < 6 \cdot S - 12000 < 1200) \\ &= \mathbb{P}\left(\frac{-600}{\sqrt{60000}} < \frac{6 \cdot S - 12000}{\sqrt{60000}} < \frac{1200}{\sqrt{60000}}\right) \\ &= \mathbb{P}(-2.449 < S' < 4.899) \end{aligned}$$

And as S' is standard normal this is equal to $\int_{-2.449}^{4.889} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$ so we have $a = -2.449$ and $b = 4.889$.

Problem 8 We have $S = X_1 + \dots + X_N$ where both N and X_i are r.v. Now the moment generating function of S would be,

$$\begin{aligned}
M_S(t) &= E(e^{tS}) \\
&= E(e^{t(X_1 + \dots + X_n)}) \\
&= \sum_{n=1}^{\infty} E(e^{t(X_1 + \dots + X_n)}) \mathbb{P}(N = n) \\
&= \sum_{n=1}^{\infty} E(e^{tX_1} \dots e^{tX_n}) \mathbb{P}(N = n) \\
&= \sum_{n=1}^{\infty} E(e^{tX_1}) \dots E(e^{tX_n}) \mathbb{P}(N = n) \quad \text{cause of independence of } X_i \\
&= \sum_{n=1}^{\infty} M_X(t)^n \mathbb{P}(N = n)
\end{aligned}$$

Now note that the MGF of N is,

$$\begin{aligned}
M_N(s) &= E(e^{sN}) \\
&= \sum_{n=1}^{\infty} e^{sn} \mathbb{P}(N = n)
\end{aligned}$$

Now note that we need s in this equation such that $e^{sn} = M_X(t)^n$ so we have $e^s = M_X(t)$ or $s = \log M_X(t)$. Which will give us $M_N(s) = \sum_{n=1}^{\infty} M_X(t)^n \mathbb{P}(N = n) = M_S(t)$. So we have,

$$M_S(t) = M_N(\log M_{X_1}(t))$$

Or in terms of the GF of N we have,

$$M_S(t) = G_N(M_{X_1}(t))$$

Problem 11 We have,

$$M(s, t) = E(e^{sX + tY})$$

First we know that we can write $f(x, y) = f_{Y|X}(y | x)f_X(x)$ where Y given $X = x$ is a normal with mean ρx and variance $1 - \rho^2$ and X is just standard normal. So we have,

$$\begin{aligned}
M(s, t) &= E(e^{sX + tY}) \\
&= E(E[e^{sX + tY} | X]) \\
&= E(e^{sX} E(e^{tY} | X))
\end{aligned}$$

But note the $e^{tY} | X$ has MGF of,

$$e^{\rho xt + \frac{1}{2}(1-\rho^2)t^2}$$

So we have,

$$\begin{aligned}
M(s, t) &= E(e^{sx} e^{\rho xt + \frac{1}{2}(1-\rho^2)t^2}) \\
&= E(e^{\rho xt + \frac{1}{2}(1-\rho^2)t^2 + sx}) \\
&= E(e^{\rho xt + sx} e^{\frac{1}{2}(1-\rho^2)t^2}) \\
&= e^{\frac{1}{2}(1-\rho^2)t^2} E(e^{\rho xt + sx}) \\
&= e^{\frac{1}{2}(1-\rho^2)t^2} e^{\frac{1}{2}(\rho t + s)^2} \\
&= e^{\frac{1}{2}(1-\rho^2)t^2 + \frac{1}{2}(\rho t + s)^2} \\
&= e^{\frac{1}{2}(s^2 + 2\rho st + t^2)}
\end{aligned}$$

Note that for standard bivariate in X, Y we can write $U = X\sigma_1 + \mu_1$ and $V = Y\sigma_2 + \mu_2$ where U, V are our required bivariate with the relevant parameters. So now we have,

$$\begin{aligned}
M_{U,V}(s, t) &= E(e^{sU+tV}) \\
&= E(e^{s(X\sigma_1+\mu_1)+t(Y\sigma_2+\mu_2)}) \\
&= E(e^{sX\sigma_1+tY\sigma_2}e^{s\mu_1+t\mu_2}) \\
&= e^{(s\mu_1+t\mu_2)}E(e^{sX\sigma_1+tY\sigma_2}) \\
&= e^{(s\mu_1+t\mu_2)}M_{X,Y}(s\sigma_1, t\sigma_2)
\end{aligned}$$