Linear Algebra

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Theorem 0.1. If V_1, V_2 are subspace of V, then $\dim(V_1 + V_2) = \dim V_1 + \dim V_2 + \dim(V_1 \cap V_2)$

Proof. There exists a subspace of W_1 of V_1 s.t.

$$(V_1 \cap V_2) \oplus W_1 = V_1.$$

Similarly we can find W_2 s.t.

$$(V_1 \cap V_2) \oplus W_2 = V_2.$$

We can say $\dim V_1 = \dim(V_1 \cap V_2) + \dim W_1$ and $\dim V_2 = \dim(V_1 \cap V_2) + \dim W_2$
$\dim V_1 + V_2 = \dim W_1 + \dim W_2 + \dim V_1 \cap V_2$
Since, $\dim W_2 + \dim(V_1 \cap V_2) = \dim V_2$, we need to show that

$$\dim(V_1 + V_2) = \dim W_1 + \dim V_2.$$

It is enough to show that $V_1 + V_2 = W_1 \oplus V_2$
(1) To show $V_1 + V_2 = W_1 + V_2$

1. W_1, V_2 are subspace of $V_1 + V_2 \Rightarrow W_1 + V_2 \subseteq V_1 + V_2$
2. For any $v = v_1 + v_2$ we can write, $v = w_1 + v_{12}$

$$v_{12} \in V_2, v_2 \in V_2 \Rightarrow v_{12} + v_2 \in V_2.$$

So, $V_1 + V_2 \subseteq W_1 + V_2$
(2) To show $W_1 \cap V_2 = \phi$
Let $w \in W_1 \cap V_2 \Rightarrow w \in W_1 \subseteq V_1$

□

Theorem 0.2. A list $v_1, \ldots, v_n$ in V is a basis of $V \Leftrightarrow \forall v \in V, v$ can be written uniquely in the form,

$$v = a_1 v_1 + \cdots + a_n v_n.$$

Proof. □

Polynomials

Definition 0.3 (Polynomial)**.** A polynomial p is a are functions from $F \to F$ s.t. p can be written as

$$p = a_n z^n + a_{n-1} z^{n-1} \cdots + a_0.$$

Remark. View, $a_n z^n + \cdots + a_0$ as $(a_0, a_1, \ldots, a_n)$

Remark. $P(F) =$ the set of polynomials with coeff. in

Example.

$$\deg(z^2 + z + 1) = 2$$
$$\deg(1) = 0$$
$$\deg(0) = -\infty$$

◇

Dimention of $P_n(F) = \{p \in P(F) : \deg p \leq n\}$ is $n + 1$
$1, z, z_2 \ldots z^n$ is a standard basis of $P_n(F)$

Example. $U = \{p \in P_3(F) : p'(5) = 0\}$
U is a subspace
A basis is, $1, (z-5)^2, (z-5)^3$

Proof. First we show its linearly indpendent, $a_0 + a_2(z-5)^2 + a_3(z-5)^3 = 0$

$$a_0 = a_2 = a_3 = 0$$

We show its spanning,

□

◇

Theorems

We start with,

Theorem 0.4. If a list, $v_1, \ldots, v_m$ spans V then we can shrink it (and keep it spanning), if its not linearly independent.

Theorem 0.5. If a list $v_1, \ldots, v_m$ is linearly independent, then we can extend it (and it is still linearly independent) if the list is not spanning.

Theorem 0.6. Length of linearly independent set $\leq$ length of spanning set

Definition 0.7. A vector space is finite-dimentional if some list of vectors span the space

Definition 0.8. A basis is a list of vectors that are linearly independent and spanning

We prove the following,

Proposition 0.9. Every spanning list can be reduced to a basis.

Proof. We know from theorem 1 that we can shrink every spanning list and keep it spanning, if its not linearly independent.
So given a spanning list, we can remove a vector and have it spanning.
We then have two cases, (1) the resultant list is linearly independent or (2) the list is linearly dependent.
If the list is linearly independent, then we have a spanning list that is lienarly independent which makes it a basis.
If the list is linearly dependent, then we can shrink it once again and keep it spanning and apply the same cases as above.
We know that this won't go on until the list is empty because from theorem 3 we know that the length of the spanning set is lowerbounded by the length of linearly independent sets. So there will be a point where our spanning set will be linearly independent. $\square$

Proposition 0.10. Every linearly independent list extends to a basis.

Proof. We know from theorem 2 that we can extend a linearly independent set keeping it linearly independent if the list is not spanning.
We given a linearly independent list, we can add a vector such that the new list is linearly indendent. We have two cases now (1) It is spanning, (2) It is not spanning.
If it is spanning, then we have a linearly independent set of vectors that are spanning which forms a basis.
If it is not spanning, then we can repeat this and extend our list once more keeping it linearly independent.
Using theorem 3 we know that the length of the set of linearly independent set of vectors is upperbounded by the length of a spanning set. Hence there will come a point where our list of vectors are linearly indpendent and spanning.
Hence they will form a basis. □

Theorem 0.11. Every vector space has a basis

Proof. We know by definition a finite-dimenstional vector space can be spanned by a list of vectors.
Now from proposition 1 we know that every spanning list can be reduced to a basis.
Hence the list spanning any finite-dimentional vector space can be reduced to a basis for the vector space. □

Theorem 0.12. Any two basis of a vector space V have the same length

Proof. Lets assume that we can have two basis of different lengths.
Let a and b be the lengths of the basis.
We know that both the sets are linearly indpendnet and spanning.
Without loss of generality assume, $a < b$
Using theorem 3 we also know that the length of a linearly independent list is always smaller than or equal to the length of the spanning list.
Now, a is a spanning list because it is a basis and b is a linearly independent list as it is a basis.
So this means that the length of a spanning list is smaller than the length of a linearly independent list.
But this contradicts theorem 3, hence our assumption must be wrong and the length of the basis must be equal. □

Definition 0.13. The length of any basis V is the dimension of V denoyed by $\dim V$

Proposition 0.14. Any linearly independent list of length $\dim V$ is a basis.

Proof. Assume there exists a linearly independent list of length $\dim V thatisnotabasis.$ That means that we are able to extend the list according to theorem 2 such that it is still linearly independent and not spanning.
So now we have a linearly indpendent list of vectors of length $\dim V + 1$. where $\dim V$ is the lenght of the basis (which is a spanning set)
However, this contradicts theorem 3 that states that the length of a linearly independent list is always smaller than or equal to the length of a spanning list. Hence our assumptino must be wrong and any liinearly independent lsit of length $\dim V$ is a basis. $\square$

Proposition 0.15. Any spanning list of length $\dim V$ is a basis.

Proof. Assume the contrary.
Using theorem 1 we can shrink the list and keep it spanning.
But this contradicts theorem 3 as now we have a spanning set that is smaller than an linearly independnet list of vectors (the basis).
So our assumption must be wrong and any spanning list of length $\dim V$ is a basis. $\square$

Lemma 0.16. Every subspace of a finite-dimentional vector space V is a finite dimentional.

Proof. Consider any vector in our subspace, u. As this set is linearly in dependent we can keep extending this set and keep it linearly indpendent. However we know that any linearly independent set is always smaller than or equal to the spanning set.
This set exist is linearly independent in the bigger subspace as well. So we know there exist an upperbound for our linearly indpendent set.
So we can keep adding a new vector, u_k to our exisisting linearly indpendent list. However we know this process will terminate as there exists an upperbound which is the length of the spanning list in V. Therefore U has to be finite-dimentional. $\square$

Lemma 0.17. If U is a subspace of V then, $\dim U \leq \dim V$

Proof. Assume the contrary that $\dim U > \dim V$
Then we have a linearly independent set of vectors that span U. But we know that they are linearly indpendent in V as well as $U \subseteq V$. So this means that there exists a linearly independent set of vectors in V which is larger than the length of the spanning set ($\dim V$). But this contradicts theorem 3.

Hence our assumption must be wrong and $\dim U \leq \dim V$ □

Lemma 0.18. If U is a subspace of V and $\dim U = \dim V$, then $U = V$

Proof. We can take a basis for U, $u_1, u_2, \ldots, u_n$ that spans U. Now these are also a linearly independent set of vectors in V. So this means that these vectors span V as they are equal to $\dim V$. So this means that every vector in V is a linearly combination of this span. Which implies that $U = V$

□

Chapter 3

Linear Maps

3A - Vector space of Linear Maps

Definition 3.1 (Linear Maps). A linear map is a function from V to W with the following properties,

1. $T(u + v) = T(u) + T(v)$
2. $T(kv) = kT(v)$

Lemma 3.2. $T(0) = 0$

Proof.

$$T(0v) = 0T(v) = 0$$

□

$L(V, W)$ is all the linear maps from V to W. $L(V)$ is the notatino for $L(V, V)$

Theorem 3.3. $L(V, W)$ is a linear space

Proof. For $(S + T)(v) = S(v) + T(v)$ □

Some properties of linear maps are,

Property.

$$(T_1T_2)T_3 = T_1(T_2T_3).$$

Property.

$$TI = IT = T.$$

Property.

$$(S_1 + S_2)T = S_1T + S_2T$$

Lemma 3.4. Suppose $v_1, \dots, v_n$ is a basis of v, then for any $w_1, \dots, w_n \in W$. There exists a unique linear map $T : V \to W$ s.t.

$$Tv_k = w_k \text{ for } k = 1, \dots, n.$$

Proof. □

3B - Null Spaces and Ranges

Definition 3.5 (Null space)**.** The null space of T is defined as the vectors in V such that $Tv = 0$

Lemma 3.6. null T is a subspace of domain of T

Definition 3.7 (Range)**.** $\text{range}(T) = \{Tv : v \in V\}$

Lemma 3.8. range (T) is a subspace of W

Proof. 1. $0 \in$ range(T) because $T(0) = 0$
2. For $a, b \in$ range (T) we can say,

$$a + b = T(v_1) + T(v_2) = T(v_1 + v_2).$$

So, $a + b \in \text{range}(T)$
3. For $a \in$ range (T) we can say,

$$a = T(v_1).$$

$ka = kT(v_1) = T(kv_1)$ so $ka \in \text{range}(T)$ □

Definition 3.9 (Injective)**.** A functino $T : V \to W$ is called injective if

$$Tu = Tv \Rightarrow u = v.$$

or,

$$u \neq v \Rightarrow Tu \neq Tv.$$

Definition 3.10 (Surjective)**.** A function $T : V \to W$ is called surjective if,

$$\forall w \in W, \exists \text{ unique } v \in V \text{ such that } Tv = w.$$

Or, $\text{range}(T) = W$

Lemma 3.11. T is injective $\Leftrightarrow$ null space of T is $\{0\}$

Proof. (1). $null(T) = \{0\} \Rightarrow T$ is injective.
Then for any $w \in range(T)$,

$$T^{-1}(w) = \{ \text{ a single element in } V \ \}$$

(2). T is injective $\Rightarrow null(T) = \{0\}$
We know $T(0) = \{0\}$ then $v = 0$ because by definition an injective functino only has one vector curresponding to a vector in its range. □

Lemma 3.12. If $T(v) = w$ then, $T^{-1}(w) = v + null(T)$

Example. $T : \mathbb{R}^3 \to \mathbb{R}, T(x, y, z) = z$ ◇

Proof. (1) $v + null(T) \subseteq T^{-1}(w)$
Start with any $v + v_0 \in v + null(T)$ ($v_0 \in null(T)$)
We know $T(w) = \{v \in V : T(v) = w\}$. So it is enough to show that $T(v + v_0) = w$. So we can write,

$$T(v + v_0) = T(v) + T(v_0) = w + 0 = w$$

(2) $T^{-1}(w) \subseteq v + null(v)$
For any $x \in T^{-1}(w)$ we know that $T(x) = w = T(v)$, So we have,

$$T(x - v) = 0$$

Taking, $x - v = v_0$ for some $v_0 \in null(T)$. So, $x \in v + null(T)$ □

Theorem 3.13 (Fundamental theorem of linear maps). V is a finite dimensional vector space. $T \in L(v, w)$. Then

$$\dim(V) = \dim null(T) + \dim range(T)$$

Proof. Let $v_1, v_2, \ldots, v_k$ be a basis of $null(T)$.
Extend the basis to a basis of V. So

$$v_1, \ldots, v_k, w_1, \ldots, w_m$$

We want to show,

$$T(w_1), \ldots, T(w_m) \text{ is a basis of } range(T)$$

(1). To show the list is spanning

Let $x \in range(T)$ then

$$x = T(a_1v_1, \cdots + a_kv_k + b_1w_1 + \cdots + b_mw_m)$$

$$= a_1T(v_1) + \cdots + a_kT(v_k) + b_1T(w_1) + \cdots + b_mT(w_m)$$

$$= b_1T(w_1) + \cdots + b_mT(w_m)$$

So x is a linear combinatino of $Tw_1, \ldots, Tw_m$
(2). To show the list is linearly independent.
Let,

$$b_1T(w_1) + \cdots + b_mT(w_m) = 0$$

$$T(b_1w_1) + \cdots + (b_mw_m) = 0$$

So, $b_1w_1 + \cdots + b_mw_m \in nullT$. We can say, $b_1w_1 + \cdots + b_mw_m = a_1v_1 + \cdots + a_nv_n$ for some $a_1, \ldots, a_n \in F$. Since, $v_1, \ldots, v_m, w_1, \ldots, w_m$ is a basis. $b_1 = b_2 \cdots = b_m = 0$

□