Linear Alebgra HW04

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2B

Problem 4

(a). We are given $U = \{(z_1, z_2, z_3, z_4, z_5) \in \mathbb{C}^5 : 6z_1 = z_2, z_3 + 2z_4 + 3z_5 = 0\}$
The contraints are as follows, $6z_1 = z_2$ and $z_3 + 2z_4 + 3z_5 = 0$
So we can rewrite each z as

$$z_1 = \frac{z_2}{6}, z_2 = z_2, z_3 = -2z_4 - 3z_5, z_4 = z_4, z_5 = z_5.$$

We see we have two dependent variables and three independent variables which means our basis will be of length 3 dependent on z_2, z_4, z_5 as follows,

$$(\frac{1}{6}, 1, 0, 0, 0), (0, 0, -2, 1, 0), (0, 0, -3, 0, 1).$$

(b). We need to extend this basis onto $\mathbb{C}^5$. We know from (a) that our dependent variables are z_1 and z_3. So to extend our basis we need to be able to make these vectors our indepdent. For this we can add the following two vectors,

$$(1, 0, 0, 0, 0), (0, 0, 1, 0, 0).$$

These additions are linearly independent because we can't represent these vectors as a linearly combination of our previous list (in our first list it was necessarily true that $z_1 = \frac{z_2}{6}$, so if $z_1 = 1, z_2 \neq 0$, similary reasoning for z_3). We also know this new list spans $\mathbb{C}^5$ because our new additions give us control over the dependent variables from our previous list (we could also argue that because it is a linearly independent set of vectors and we have $\dim(\mathbb{C}_5)$ of them.
(c). We need to find a subspace W such that $U \oplus W = \mathbb{C}^5$. Take W from above as,

$$W = (1, 0, 0, 0, 0), (0, 0, 1, 0, 0).$$

First we need to show that $W + U = \mathbb{C}^5$. That every vector in $\mathbb{C}^5$ can be represented as $v = u + w, u \in U, w \in W$
Now, if $u \in U, u = a_1u_1 + a_2u_2 + a_3u_3$ and if $w \in W, w = b_1w_1 + b_2w_2$.
So

$$v = a_1u_1 + a_2u_2 + a_3u_3 + b_1w_1 + b_2w_2$$

But we know from above that u_1, u_2, u_3, w_1, w_2 is a basis for $\mathbb{C}^5$. Which means that the linear combination of these vectors can reprsent every vector in $\mathbb{C}^5$. So we show that all of $v \in \mathbb{C}^5$ can be written as a vecotr $u \in U$ plus a vector $w \in W$.

Problem 5

If $V = W + U$ we can say that $\forall v \in V$,

$$v = u + w \text{ for } u \in U, w \in W.$$

Now, u can be written as a linear combinatino of vectors in U and similar can be done for w.
So let $u = a_1u_1 + \cdots + a_nu_n$ and $w = b_1w_1 + \cdots + b_mw_m$. So we have a linear combination of $n+m$ vectors. We know that $\dim(V) \leq n+m$ because $\dim(V) \leq$ length of any spanning set in V.

If $n+m > \dim V$. Then we can reduce it to a linearly independent set of vector such that it still spans V. So now we have a basis of V that consists of vectors that are either in U or W. Or in other words our basis are vectors in $U \cup W$.
If $n + m = \dim V$ then we already have a linearly independent set of vectors that span V which consists of vectors either in U or V. Which meanst hat the basis are vectors in $U \cup W$.
So we have shown that there exists a basis of V in $U \cup W$ if $U + W = V$.

11

We know that $v_1, \dots, v_n$ is a basis for V. We need to show that it is also a basis for V_C. Now $V_{\mathbb{C}}$ is defined by $V \times V$ such that $(x, y) = x + iy \in V_{\mathbb{C}}$.
So we need to show that any vector of the form $u + iw \in V_C$ can be represented by a linear combinatino of $v_1, \dots, v_n$.
First we know that $u \in V, w \in V$. So we can write $u = a_1v_1 + \cdots + a_nv_n$, similarly $w = b_1v_1 + \cdots + b_nv_n$.
Now because we also define scalar multiplication with complex numbers we can write,

$$a_1v_1 + \cdots + a_nv_n + i(b_1v_1 + \cdots + b_nv_n) = u + iw.$$

Or,

$$\forall(u, w) \in V_C, u + iw = (a_1 + ib_1)v_1 + \cdots + (a_n + ib_n)v_n.$$

So we showed that we can represent all elements of V_C as a linear combination of our vectors $v_1, \dots, v_n$

2C

Problem 1

We know that $\dim(\mathbb{R}^2) = 2$ which means that for a given subspace V we have three cases,

$$\dim(V) = 0, \dim(V) = 1, \dim(V) = 2.$$

If $\dim(V) = 0$ then our vector space if $V = \{0\}$ by definition.
If $\dim(V) = 1$ then that means our vector space contains one vector so V is spanned by $\{v\}$. First we knwo that $0 \in V$ as V is a subspace (we can take the coefficient to be 0). Now for any vector $v \in V, kv \in V$. We know that this defines any line in $\mathbb{R}^2$ that goes through the origin.
If $\dim(V) = 2$ we also know that $U \subseteq V$. If $U \subseteq V$ and $\dim(U) = \dim(V)$ then we know that $U = V$. So, U determines $\mathbb{R}^2$

Problem 4

(a). A basis of U would be one where $p''(6) = 0$. First we know that a basis of $P_4(R)$ is $1, x, x^2, x^3$ which can also be written as $1, (x-6), (x-6)^2, (x-6)^3$ where $x \in R$
So any p is written as

$$p(x) = 1a_1 + a_2(x-6) + a_3(x-6)^2 + a_4(x-6)^3$$

$$p''(6) = 2a_3$$

So we see that for it to be equal to 0, $a_3 = 0$. Which means our basis is,

$$1, (x-6), (x-6)^3.$$

(b). As we discussed above, adding $(x-6)^2$ to the list will give us a basis for $P_4(R)$
So our basis is,

$$1, (x-6), (x-6)^2, (x-6)^3$$

(c). Our subspace W would be spanned by $(x-6)^2$. We first show that $W+U = P_4(R)$. To do this we need to show any $p \in P_4(R)$ can be represented as,

$$p = u + w, u \in U, w \in W.$$

We know for $u \in U, u = a_1 + a_2(x-6) + a_3(x-6)^3$ and for $w \in W, w = b_1(x-6)^2$. So,

$$p = a_1 + a_2(x-6) + a_3(x-6)^3 + b_1(x-6)^2.$$

Which is a linear combination of the basis of $P_4(R)$ which means that $u + w$ can represent any vector $p \in P_4(R)$ and hence we can say $U + W = P_4(F)$
Now we need to show that $U \oplus W = P_4(F)$. To show this we can show that there is only one way of representing 0 as $u + w$.
Now if $u + w = 0$ as we did abov ewe can write,

$$0 = a_1 + a_2(x-6) + a_3(x-6)^3 + b_1(x-6)^2$$

First we know that $a_3 = 0$ as we can't represent x^3 using any of the other terms. Similary we can show that $b_1 = 0, a_2 = 0, a_1 = 0$. Hence the only way of repenseting 0 is to have all coefficients as 0.
Which means that $U \oplus W = P_4(R)$

Problem 8

Given $v_1, \dots, v_m$ is linearly independent in V and $w \in V$. Let U be the subspace spanned by $v_1, \dots, v_m$. We know $U \subseteq V$ so we have two cases, either,

$$w \in U \text{ or } w \notin U$$

Case 1: $w \in U$
First we know that $a_1 v_1 + \cdots + a_m v_m = u, \forall u \in U$

$$span(v_1 + w, \dots, v_m + w) = b_1(v_1 + w) + \cdots + b_m(v_m + w).$$

Now if $-w \in [v_1, \dots, v_m]$ Then we have for some n, $b_n(v_n - v_n) = 0$
So we have $m-1$ linearly independent vectors. And we know that the dimension of the span of the subspace defined by $m-1$ linearly independent vecots is $m-1$.
Else if $-w \notin [v_1, \dots, v_m]$ then we are able to remove kw from our list and still have it spanning and linearly indpendnet with m vectors. Hence the dimension of our spanning set is still m.
Case 2: $w \notin U$. If $w \notin U$ then we have,

$$b_1 v_1 + \cdots + b_m v_m + w(b_1 + \cdots + b_m).$$

As w is not in U, the new list is linearly indpendent with $m+1$ lineralt indpendent vectors. So the diension is $m+1$
So we showed that the dimensoin of the span is greater than or equal to $m-1$

Problem 10

To show that it is a basis all we need to show is that it is a linearly indpendent set of polynoamials. Because linearly indpendent set of polynoamisl of $\dim(P_m(F)$ will span the space.
To show linear independnece we need to show that there is only one unique way of preresenting 0. So consider,

$$a_0p_0 + a_1pp_1 + \cdots + a_mp_m.$$

Now in our list, $p_0, \ldots, p_m$ there is only one polynomials that represents degree 0 (constant term). That is, when $k = 0$ we get,

$$p_0 = (1 - x)^m.$$

and one of the terms is 1^m. Notice that every other polynomials has a degree $n > 0$ attached to the left (x^n).
So in our polynomial expantion the coefficient of x^0 is only a_0. So to have it equal to 0 we need $a_0 = 0$.
Now similary we see that the term with degree 1 can only be represented by

$$p_1 = x(1 - x)^{m-1}.$$

This means that the only coefficient of x is a_1 and as we need the sum of all terms to be equal to 0 we have $a_1 = 0$.
We can continue this till $n = m$ and we get that for all a we have $a_n = 0$ for us to have the linear combination of our polynomials equal to 0.
So we have shown that our set is linealry indpendent and because it has the same lenght as the dimension of the basis of $P_m(F)$ we can say that it is a basis itself.

Problem 14

Proof. We need to show that there exists at least one combinatino of subspaces of V_1, V_2, V_3 of dimension 7 in a 10 dimenstional vector space such that $V_1 \cap V_2 \cap V_3 \neq \{0\}$
First consider V_1 and V_2. We know that,

$$\dim(V_1 + V_2) = \dim(V_1) + \dim(V_2) - \dim(V_1 \cap V_2).$$

Now, $W = V_1 + V_2$ can be written as $w \in W$ such that

$$w = a_1v_{11} + \cdots + a_7v_{17} + b_1v_{21} + \cdots + b_7v_{27}.$$

As $v_{11}, \ldots, v_{17}$ is linearly indpendent and $v_{21}, \ldots, v_{27}$ is linealry indpendent. There could eixst at most 3 vectors from $v_{21}, \ldots, v_{27}$ such that our list is stil llinearly indepndnent which would make our W have a span of 10. If all the vectors are in V_1 then our span would be 7.
Hence we have,

$$7 \leq \dim(V_1 + V_2) \leq 10$$

Now we know,

$$\dim(V_{12}) = \dim(V_1) + \dim(V_2) - \dim(V_1 + V_2)$$

$$7 \leq \dim(V_1 + V_2) \leq 10$$

$$-7 + \dim(V_1) + \dim(V_2) \geq \dim(V_1 + V_2) \geq -10 + \dim(V_1) + \dim(V_2)$$

$$7 + \dim(V_1) + \dim(V_2) \geq \dim(V_1 + V_2) \geq 4$$

Let $V_1 \cap V_2 = W_0$. Using similar reasoning as above we know,

$$\dim(W_0 + V_3) = \dim(W_0) + \dim(V_3) - \dim(W_0 \cap V_3).$$

If we take the case of $W_0 + V_3$ we know W_0 is of dimension 4 and V_3 is of dimension 7. Similar to above $7 \leq \dim(W_0 + V_3) \leq 10$.
We know that,

$$\dim(W_0 \cap V_3) = dim(V_1 \cap V_2 \cap V_3) = \dim(w_0) + \dim(V_3) - \dim(W_0 + V_3).$$

From (1) we know that $4 \leq \dim(W_0) \leq 7$ and from above we showed that $7 \leq \dim(W_0 + V_3) \leq 10$
So we have

$$11 \leq \dim(W_0) + \dim(V_3) \leq 14.$$

and,

$$-10 \leq -\dim(W_0 + V_3) \leq -7.$$

So,

$$1 \leq \dim(W_0) + \dim(V_3) - \dim(W_0 + V_3) \leq 7.$$

Or,

$$1 \leq \dim(V_1 \cap V_2 \cap V_3) \leq 7.$$

□

Problem 18

Proof. We know that for any subspace V_1 of V we can find anoter subspace W in V such that $V_1 \oplus W = V$.
Let V_1 be the subspace spanned by any vector in V
Now consider the subspace W. We know that $W \cap V_1 = \{0\}$. Now consider a subspace in W called V_2 that is spanned by any vecotr in W. We are able to find another subspace W_1 within W such that $W_1 \oplus V_2 = W$.
So we can write $V = V_1 \oplus V_2 \oplus W_1$
Now we could continue this for W_1 until $\dim(W_{n-2}) = 1$ which will give us,

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_n.$$

Where $V_n = W_{n-2}$ □

Problem 9

We might guess this formula because if we just add the dimensions of each subspace then if there eixsts vectors that are common to any two or three sets then we are double or tripple counting them. So after adding each dimension we are checking whether there are subspaces common in each of them and removing the ones that we are conting multiples times.

Proof. The given formula doesn't work. For instance, consider,

$$V_1 := span((1,0,0)).$$

$$V_2 := span((0,1,0)).$$

$$V_3 := span((1,1,0)).$$

Now consider $W = V_1 + V_2 + V_3$. We have $w \in W$ such that

$$w = a(1,0,0) + b(0,1,0) + c(1,1,0).$$

We see that this is a linarly dependent set of vectors and we can remove any one of them such that $\dim(W) = 2$
It is easy to see, $\dim(V_1) = 1, \dim(V_2) = 1, \dim(V_3) = 1$
Now looking at $\dim(V_1 \cap V_2)$ we see that there are no common vectors as they are spaned by linealry indepndent vectors. Similarly we can say that

$$\dim(V_1 \cap V_2) = 0, \dim(V_1 \cap V_3) = 0, \dim(V_3 \cap V_2) = 0$$

Laslty it is trival to see why $\dim(V_1 \cap V_2 \cap V_3) = 0$.
So using the formula we have, $\dim(V_1+V_2+V_3) = 1+1+1-0-0-0+0 = 3$
However we know that the actual number is 2. Hence the formula is not correct.

$\square$