

Linear Algebra HW07

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5A

Problem 1

Proof. (a). $U \subseteq \text{null } T$ we need to show $\forall u \in U \ T(u) \in \text{null } T$.

Consider any $u \in \text{null } T$ we know that $T(u) = 0$ by definition. We know that $0 \in \text{null } T$ because $T(0) = 0$ which means that $T(u) \in \text{null } T$. Hence we show that for any $u \in U \ T(u) \in U$. Which implies that U is an invariant under T .

(b). We have $\text{range } T \subseteq U$ this means that for any $u \in V$, $T(u) \in \text{range } T \Rightarrow T(u) \in U$. So consider any $u \in U$ so we know that $T(u) \in \text{range } T \Rightarrow T(u) \in U$. Which shows us that U is invariant under T . \square

Problem 2

Proof. We need to show that $V_1 + \dots + V_m$ is invariant under T if V_1, \dots, V_m are invariant under T .

So we need to show that if $v \in V_1 + \dots + V_m$ then $T(v) \in V_1 + \dots + V_m$. If $v \in V_1 + \dots + V_m$ that means that we can write v as

$$v = v_1 + \dots + v_m, \text{ where } v_1 \in V_1, \dots, v_m \in V_m$$

Now $T(v) = T(v_1 + \dots + v_m) = T(v_1) + \dots + T(v_m)$. But we know that $v_1 \in V_1, \dots, v_m \in V_m$ which means that $T(v_1) \in V_1, \dots, T(v_m) \in V_m$ as we know that V_1, \dots, V_m are invariant subspaces. So now let us write

$$T(v_1) = v'_1, \dots, T(v_m) = v'_m$$

such that $v'_1 \in V_1, \dots, v'_m \in V_m$

So we have,

$$\begin{aligned} T(v) &= T(v_1) + \dots + T(v_m) \\ &= v'_1 + \dots + v'_m \end{aligned}$$

So we have written any $T(v)$ as $v'_1 + \dots + v'_m$ such that $v'_1 \in V_1, \dots, v'_m \in V_m$ which means that $T(v) \in V_1 + \dots + V_m$. Hence by definition this makes our subspace $V_1 + \dots + V_m$ invariant under T . \square

Problem 3

Proof. Let V_1, \dots, V_m represent every collection of subspaces that are invariant under T . We need to show that $v \in V_1 \cap \dots \cap V_m \Rightarrow T(v) \in V_1 \cap \dots \cap V_m$.

Consider an arbitrary $v \in V_1 \cap \dots \cap V_m$. Now this means that

$$v \in V_1, \dots, v \in V_m$$

But because V_1, \dots, V_m are all invariant under T this means that

$$T(v) \in V_1, \dots, T(v) \in V_m$$

Now by definition this means that $T(v) \in V_1 \cap \dots \cap V_m$. Which makes the subspace $V_1 \cap \dots \cap V_m$ invariant under T . \square

Problem 4

Proof. Let us assume the contrary that $U \neq \{0\}$ and $U \neq V$. We know that $0 < \dim U < \dim V$. Let $\dim U = k$ and $\dim V = n$. So consider a basis for U as,

$$u_1, \dots, u_k$$

Now let us define an operator on U such that

$$T(u_1) = v, \dots, T(u_k) = v$$

where $v \in V - U \neq \emptyset$ (for instance let $v = v_m$)

So we constructed a map on U such that $T(U) = v \notin U$. \square

Problem 5

We have the matrix as, $\begin{bmatrix} 0 & -3 \\ 1 & 0 \end{bmatrix}$. So eigen values are $\pm i\sqrt{3}$.

However this exist outside R hence within our vector space we don't have an eigenvalue.

Problem 6

We have the matrix as, $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. So eigen values are ± 1 .

If eigen value is 1 then eigenvector is $(1, 1)$ and if its -1 then eigenvector is $(1, -1)$

Problem 7

We have the matrix as, $\begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix}$. So eigenvalues are 0 and 5. Eigenvector for 0 is $(1, 0, 0)$ and for 5 is $(0, 0, 1)$

Problem 8

We have $P^2 = P$. So consider an arbitrary v which is an eigenvector of λ . So we have,

$$Pv = \lambda v$$

But we know $P(P(v)) = P(v)$. So $P(\lambda v) = P(v)$

$$\lambda P(v) = P(v)$$

Now this is true if either $\lambda = 1$ or $P(v) = 0$. If $P(v) = 0$ that means $v \in T$ or that $\lambda = 0$ is an eigenvalue of T .

Problem 9

We have our basis of $P(R)$ as $1 + x$. Which can be written as $(1, 0), (0, 1)x$.

So the matrix of our linear map is $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. So $\lambda = 0$ which gives us the eigenvector as $(1, 0)$

Problem 10

$P_4(R)$ is spanned by $1+x+x^2+x^3+x^4$. Which can be written as $(1, 0, 0, 0, 0), \dots, (0, 0, 0, 0, 1)$. Now to define our linear map we need to see where our standard basis will map to. We have,

$$\begin{aligned} T(1) &= 0 \\ T(x) &= x \\ T(x^2) &= 2x^2 \\ T(x^3) &= 3x^3 \\ T(x^4) &= 4x^4 \end{aligned}$$

So the matrix of our linear map will be,

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

It is clear that our eigenvalues are 0, 1, 2, 3, 4. And its eigenvectors are $(1, 0, 0, 0, 0), \dots, (0, 0, 0, 0, 1)$

Problem 12

Proof. We know that $V = U \oplus W$. So any $v \in V$ can be written as

$$v = u + w, u \in U, w \in W$$

Consider a basis for U as u_1, \dots, u_n and a basis for W as w_1, \dots, w_m .

So we have all v can be written as $a_1u_1 + \dots + a_nu_n + b_1w_1 + \dots + b_mw_m$. Because $U \cap W = \{0\}$ we know that no $w_k \in U$ or in other words our list of vectors $u_1, \dots, u_n, w_1, \dots, w_m$ is linearly independent and is a basis for V .

Now given a basis of V we can define a linear map $P : V \rightarrow V$ as follows,

$$P(u_1) = u_1, \dots, P(u_n) = u_n$$

$$P(w_1) = 0, \dots, P(w_m) = 0$$

Hence we have defined a linear map as we have assigned vectors in V for our basis of V .

Now for any $P(u+w)$ let $u = a_1u_1 + \dots + a_nu_n$ and $w = b_1w_1 + \dots + b_mw_m$

$$\begin{aligned} P(u+w) &= P(u) + P(w) \\ &= P(a_1u_1 + \dots + a_nu_n) + P(b_1w_1 + \dots + b_mw_m) \\ &= a_1P(u_1) + \dots + a_nP(u_n) + b_1P(w_1) + \dots + b_mP(w_m) \\ &= a_1u_1 + \dots + a_nu_n + 0 \\ &= u \end{aligned}$$

Hence we define P such that $P(u+w) = u$ □

Problem 13

Proof. (1). We need to show that if $Tv = \lambda v$ then $S^{-1}TSv = \lambda v$ given that S is invertible.

Consider that $S^{-1}TSv = v'$. Because S is invertible we can apply S on both sides and get,

$$T(S(v)) = S(v')$$

But we assume that T has an eigenvalue λ . Now because S is invertible in V there exists some v such that $S(v)$ is an eigenvector of T . For that v we have $T(S(v)) = \lambda S(v)$. So we get,

$$\begin{aligned} T(S(v)) &= S(v') \\ \lambda S(v) &= S(v') \\ S(v') &= S(\lambda v) \\ v' &= \lambda v \end{aligned}$$

So for some v such that $S(v)$ is an eigenvector of T we have v is an eigenvector of $S^{-1}TS$ such that λ is the eigenvalue associated with the vector.

Hence we show that both T and $S^{-1}TS$ have the same eigenvalues.

(b). If v is an eigenvector of T then $v' = S^{-1}v$ is an eigenvector of $S^{-1}TS$. □

Problem 19

Proof. Assume λ is an eigen value, this means that

$$\lambda z_1 = 0, \lambda z_2 = z_2, \dots$$

But if $\lambda z_1 = 0$ then $z_1 = 0 \Rightarrow z_2 = 0 \Rightarrow z_3 = 0 \dots$

Hence the only possibility is $(0, \dots)$ however this isn't a valid consideration for an eigenvector. So $\nexists \lambda$ such that it is an eigenvalue. Or in other words for any λ there doesn't exist an eigenvector.

□

Problem 20

Proof. (a). Consider an arbitrary λ we need,

$$\lambda z_1 = z_2, \lambda z_2 = z_3, \dots$$

For any arbitrary z_1 and λ we can define $z_n = \lambda z_{n-1}$.

(b). For an arbitrary λ every eigenvector is of form,

$$(z_1, \lambda z_1, \lambda^2 z_1, \lambda^3 z_1, \dots)$$

□

Problem 21

Proof. (a). \Leftarrow We have T is invertible and λ is an eigenvalue of T . So $\exists v$ such that,

$$T(v) = \lambda v$$

Now because T is invertible we can apply T^{-1} and we get,

$$\begin{aligned} T^{-1}Tv &= T^{-1}(\lambda v) \\ v &= T^{-1}(\lambda v) \\ v &= \lambda T^{-1}(v) \\ \frac{v}{\lambda} &= T^{-1}(v) \end{aligned}$$

So we've shown that $\frac{1}{\lambda}$ is an eigenvalue of T^{-1}

\Rightarrow The argument is exactly the same as (a) because $T^{-1} = T'$ is also an invertible linear map and just consider $\lambda' = \frac{1}{\lambda}$ as the eigenvalue of this map.

(b). Let $v \in V$ such that $T(v) = \lambda v$. Then we have

$$\begin{aligned} T^{-1}Tv &= T^{-1}\lambda v \\ v &= \lambda T^{-1}v \\ \frac{v}{\lambda} &= T^{-1}v \end{aligned}$$

So we've shown that for any arbitrary eigenvector v with eigenvalue λ v is also an eigenvector of T^{-1} with eigenvalue of $\frac{1}{\lambda}$

□

Problem 21

Proof. Consider two cases either $w = -u$ or $w \neq -u$.

If $w = -u$ then we have $T(u) = -3u$ and $T(w) = -3w$ which makes -3 an eigenvalue with eigenvector u .

Now if $w \neq -u$ then we have,

$$T(u) = 3w, T(w) = 3u$$

$$T(u) + T(w) = 3(u + w)$$

$$T(u + w) = 3(u + w)$$

And because $u \neq w$ we know that $u + w \neq 0$ which is required for an eigenvector. So here we show that $u + w$ is an eigenvector and the associated eigenvalue for this is 3 . □

Problem 25

Proof. Let u be an eigenvector such that $T(u) = \lambda_1 u$ and $T(w) = \lambda_2 w$

We are told that $T(u + w) = \lambda_3(u + w)$. So we have,

$$T(u) + T(w) = \lambda_3(u + w)$$

$$\lambda_1 u + \lambda_2 w = \lambda_3(u + w)$$

$$\lambda_1 u + \lambda_2 w = \lambda_3 u + \lambda_3 w$$

Now if u, w are linearly dependent (one is in the span of the other) then it is trivial to show that $\lambda_1 = \lambda_2$ and that $\lambda_3 = \lambda_1 = \lambda_2$.

Now if u, w are linearly independent this means that neither are in the span of each other, $\nexists k$ s.t. $u = kw$.

Hence the only solution to the equation $(\lambda_1 - \lambda_3)u + (\lambda_2 - \lambda_3)w = 0$ is if the coefficients are equal to 0 . Or

$$\lambda_1 = \lambda_3$$

$$\lambda_2 = \lambda_3$$

But this then means that $\lambda_1 = \lambda_2$. So we show that in both cases $\lambda_1 = \lambda_2$ □

Problem 28

Proof. First within the range of T we can construct T such that we have at most $\dim \text{range } T$ eigenvectors corresponding to each subspace spanned by the basis of $\text{range } T$. That is, we can construct $\dim \text{range } T$ invariant subspaces from V to itself such that $T(v) \in \text{range } T$.

Now consider the case when $\text{range } T$ doesn't span V then $\exists v$ such that $T(v) = 0$. Hence we have another eigenvalue 0 such that $T(v) = 0v$ and $v \in T$.

So we have shown that we have at most $\dim \text{range } T + 1$ eigenvalues. \square

Problem 30

Proof. For $(T-2I)(T-3I)(T-4I) = 0$ to be true we have either $(T-2I) = 0$ or $(T-3I) = 0$ or $(T-4I) = 0$.

Let us consider each case, 1. $(T-2I) = 0$ or

$$(T-2I)v = 0v = 0$$

$$Tv - 2v = 0$$

$$Tv = 2v$$

Which means that 2 is an eigenvalue or that $\lambda = 2$. We can use similar reasoning for (2) and (3) to conclude that either $\lambda = 2$ or 3 or 4. \square