Probability Theory: HW1

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Exercise 1.10

Given $A, B \in \mathscr{F}$ and we need to show that $A \triangle B \in \mathscr{F}$. Now if $x \in A \triangle B$ then we know that $x \in (A \cup B) \setminus (A \cap B)$. By definition we have $A \cup B \in \mathscr{F}$ (closure under countable union) and we also have $A^c, B^c \in \mathscr{F}$(closure under complement) $\Rightarrow$ $(A^c \cup B^c) \in \mathscr{F} \Rightarrow (A \cap B)^c \Rightarrow A \cap B \in \mathscr{F}$. So now let $C = A \cup B$ and $D = A \cap B$. It is enough to show that if $C, D \in \mathscr{F}$ then $C \setminus D \in \mathscr{F}$. We have $C \setminus D = C \cap D^c$. We know $D^c \in \mathscr{F}$ and $\mathscr{F}$ is closed under intersection as shown above which means that $C \cap D^c \in F \Rightarrow C \setminus D \in \mathscr{F} \Rightarrow (A \cup B) \setminus (A \cap B) \in \mathscr{F} \Rightarrow A \triangle B \in \mathscr{F}$

Exercise 1.17

First given that $\mathscr{F}$ is the powerset of Ω.
1. We have $\mathbb{Q}(A) = \sum_{i:\omega_i \in A} p_i$ for $A \in \mathscr{F}$ and we know that $p_i \geq 0$ for any i so sum of non-negative numbers are also non-negative which means that $\mathbb{Q}(A) \geq 0$ for $A \in \mathscr{F}$
2. We have $\mathbb{Q}(\Omega) = \sum_{i:\omega_i \in \Omega} p_i = p_1 + \cdots + p_n = 1$. Similarly we have $\mathbb{Q}(\phi) = \sum_{i:\omega_i \in \phi} p_i = 0$.
3. We need to show that given disjoint events $A_1, A_2, \cdots \in \mathscr{F}$ we have, $\mathbb{Q}\big(\bigcup_{i=1}^{\infty} A_i\big) = \sum_{i=1}^{\infty} \mathbb{Q}(A_i)$.

$$
\begin{aligned}
\mathbb{Q}\big(\bigcup_{i=1}^{\infty} A_i\big) &= \mathbb{Q}(A_1 \cup A_2 \dots) \\
&= \sum_{i:\omega_i \in (A_1 \cup A_2 \dots)} p_i \\
&\text{Now since } A_1, \dots \text{ are pairwise disjoint we can write,} \\
&= \sum_{i:\omega_i \in (A_1)} p_i + \sum_{i:\omega_i \in (A_2)} p_i + \dots \\
&= \mathbb{Q}(A_1) + \mathbb{Q}(A_2) + \dots \\
&= \sum_{i=1}^{\infty} \mathbb{Q}(A_i)
\end{aligned}
$$

Exercise 1.21

We need to find,

$$
\begin{aligned}
&P(A \cap B \cap C^c) + P(A \cap B^c \cap C) + P(A^c \cap B \cap C) \\
&= P((A \cap B) \setminus C) + P((A \cap C) \setminus B) + P((C \cap B) \setminus A) \\
&= P(A \cap C) - P(A \cap B \cap C) + P(A \cap B) - P(A \cap B \cap C) + P(B \cap C) - P(A \cap B \cap C) \\
&= .3 - .1 + .4 - .1 + .2 - .1 = .6
\end{aligned}
$$

Exercise 1.27

First the ways to distribute 4 aces among 4 players would be 4!. Now with the remaining 48 cards, the ways to split it among 4 people random is, $\binom{48}{12}\binom{36}{12}\binom{24}{12}\binom{12}{12}$.

Similalry the total ways to split 52 cards among 4 people w 13 each would be $\binom{52}{13}\binom{39}{13}\binom{26}{13}\binom{13}{13}$. So the probability would be,

$$\frac{\binom{48}{12}\binom{36}{12}\binom{24}{12}4!}{\binom{52}{13}\binom{39}{13}\binom{26}{13}} = 0.1055$$

Exercise 1.30

Exercise 1.44

Exercise 1.52

Problem 9

Problem 14

Problem 17