Real Analysis: HW5

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Exercise 2.7.9

(a) We have $\lim \left|\frac{a_{n+1}}{a_n}\right| = r < 1$ and that $r < r' < 1$. Using the limit definition we have, $\forall \varepsilon > 0$, $\exists N$ such that for $n > N$ we have,

$$\left|\left|\frac{a_{n+1}}{a_n}\right| - r\right| < \varepsilon$$

So we have,

$$\left|\frac{a_{n+1}}{a_n}\right| < r + \varepsilon$$

For any choice of ε. Now as $r' > r$ we have $r' - r > 0$ and let $r' - r = \varepsilon$. So we can find an N such that if $n > N$ then,

$$\begin{aligned}\left|\frac{a_{n+1}}{a_n}\right| &< r + \varepsilon \\ \left|\frac{a_{n+1}}{a_n}\right| &\leq r' \\ |a_{n+1}| &\leq |a_n|\, r'\end{aligned}$$

(b) We know that $|a_N|$ is constant and as $r' < 1$ we can do the following for a given N and taking $n > m > N$

$$\begin{aligned}|s_n - s_m| &= (r' + \cdots + (r')^n) - (r' + \cdots + (r')^m) \\ &= (r')^{m+1} + \cdots + (r')^n \\ &= (r')^{m+1}(1 + \cdots + (r')^{n-m-1}) \\ &= (r')^{m+1}\frac{1 - (r')^{n-m}}{1 - r'}\end{aligned}$$

Now we have $\frac{1-(r')^{n-m}}{1-(r')} \leq \frac{1}{1-(r')}$ which is a constant say M. So we have,

$$|s_n - s_m| \leq (r')^{m+1} M$$

Now we know that $(r')^m$ converges to zero as $r' < 1$, i.e if we have $N > \ln(\varepsilon)/\ln(r')$ then we get $(r')^n < \varepsilon$ for any ε. So choose ε as ε/M so we get,

$$|s_n - s_m| \leq (r')^{m+1}M \leq \varepsilon\frac{M}{M} = \varepsilon$$

Hence by cauchy convergence test we get $\sum(r')^n$ converges. Now as $|a_N|$ is a constant value multiplying that with the series also results in a convergent series (can easily show this by choosing our epsilon as $\varepsilon/(M|a_N|)$).

(c). We know from above that we have some N such that for $n \geq N$ we have $|a_{n+1}| \leq |a_n|r'$. Now this further implies that $|a_{n+1}| \leq |a_n|r' \leq |a_{n-1}|(r')^2 \leq \dots |a_N|(r')^{n-N-1}$ So for some n consider N to n.

$$\begin{aligned}
|a_N| + |a_{N+1}| + \cdots + |a_n| &\leq |a_N| + |a_N|(r') + |a_N|(r')^2 + \cdots + |a_N|r^{n-N} \\
|a_N| + |a_{N+1}| + \cdots + |a_n| &\leq |a_N|((r') + (r')^2 + \cdots + (r')^{n-N}) \\
\sum_{k=N}^{n} |a_k| &\leq |a_N| \sum_{k=1}^{n-N} (r')^k
\end{aligned}$$

Now we know that as $n \to \infty$ as N is constant we have $\sum(r')^k$ converges as $|r'| < 1$. Now as $|a_N|$ is a constant by comparison test we have the partial sums on the left side smaller than the right side so the the series $\sum_{k=N}|a_k|$ converges to some value. Now as $\sum_{k=1}^{N}|a_k|$ is a constant value we have $\sum|a_k|$ converges absolutely. Now we know the absolute convergence implies that the series is convergent without the absolute value so we have $\sum a_n$ converges.

Exercise 2.7.12

We have,

$$\begin{aligned}
\sum_{j=m}^{n} x_j y_j &= x_m y_m + \cdots + x_n y_n \\
&= (s_m - s_{m-1})y_m + (s_{m+1} - s_m)y_{m+1} + \cdots + (s_n - s_{n-1})y_n \\
&= (s_m y_m - s_{m-1}y_m) + (s_{m+1}y_{m+1} - s_m y_{m+1}) + \cdots + (s_n y_n - s_{n-1}y_n)
\end{aligned}$$

Now we see above that for ever pair of subtractions we have s_m can be taken common so we have,

$$\begin{aligned}
\sum_{j=m}^{n} x_j y_j &= (s_m y_m - s_{m-1}y_m) + (s_{m+1}y_{m+1} - s_m y_{m+1}) + \cdots + (s_n y_n - s_{n-1}y_n) \\
&= -s_{m-1}y_m + s_m(y_m - y_{m+1}) + \cdots + s_n(y_n - y_{n+1}) + s_n(y_{n+1}) \\
&= s_n y_{n+1} - s_{m-1}y_m + \sum_{j=m}^{n} s_j(y_j - y_{j+1})
\end{aligned}$$

Which is our desired result.

Exercise 2.7.13

(a). From above we have,

$$\sum_{j=m}^{n} x_j y_j = s_n y_{n+1} - s_{m-1} y_m + \sum_{j=m}^{n} s_j (y_j - y_{j+1})$$

So,

$$\sum_{j=1}^{n} x_j y_j = s_n y_{n+1} + \sum_{j=1}^{n} s_j (y_j - y_{j+1})$$

(b). We know that $y_1 \geq y_2 \geq \cdots \geq 0$ so y_n is a monotonically non-increasing sequence. So we have $0 < y_k - y_{k+1} < y_k$. Now as $\sum x_k$ converges that also means that $y_1 \sum x_k$ converges that means the sequence of partials sums of x_n that is s_n is bounded above say by M. So we have,

$$\begin{aligned}
\sum |s_m(y_m - y_{m+1})| &\leq \sum |s_m||y_m - y_{m+1}| \\
&\leq \sum M(y_m - y_{m+1}) \quad \text{as } y_m - y_{m+1} > 0 \\
&= M \sum y_m - y_{m+1}
\end{aligned}$$

Now $y_m - y_{m+1}$ is a telescoping series which means that $\sum y_m - y_{m+1}$ is convergent. Hence we have $\sum s_m(y_m - y_{m+1})$ is absolutely convergent which means that the series, $\sum_{k=1}^{\infty} s_k(y_k - y_{k+1})$ itself is convergent.

Now we know that $\sum_{k=1}^{\infty} x_k y_k = \lim_{n\to\infty}(s_n y_{n+1} + \sum_{k=1}^{n} s_k(y_k - y_{k+1}))$. And we showed that the second term converges. It is enough to show that the left term is bounded. i.e. $\lim_{n\to\infty} s_n y_{n+1}$ is bounded. First we know that y_{n+1} goes to some constant as $n \to \infty$ and it's monotonically non-increaseing and bounded below and that $\lim_{n\to\infty} s_n$ is $\sum_{k=1}^{\infty} x_k$ which means that $\lim_{n\to\infty} s_n y_{n+1}$ converges to some constant which is the product of both their limits (using algebraic limit theorem). So we have $\sum_{k=1}^{\infty} x_k y_k$ converges.

Exercise 2.7.14

(a) Abels test assumes that the series $\sum x_k$ converges and that (y_k) is non-increaseing and bounded below by 0. On the other hand Dirichle'ts Test doesn't assume that the series $\sum x_k$ converges but assumes that it is bounded. On the other hand it assumes that the sequence (y_n) is non-increasing and converges to zero.

We see that despite the change in hypothesis we can still rewrite the partial sum of $\sum_{j=1}^{n} x_j y_j$ in the same manner. And as in this case we have y_n converges to zero the first term goes to zero. And for the second term (the series) we can still use the fact that s_j is bounded above and $y_j - y_{j+1}$ is a telescoping series. So we use the same strategy to show that the series converges.
(b). The alternating series test tells that if the absolute value of the sequence is non-increasing and bounded below by zero and that it's alternating then the

series converges. Given a series $\sum(-1)^n y_n$ we can look at this as the sequence $\sum(-1)^n$ which is bounded but not convergent and the positive values (y_n) as the sequence that is non-increasing and bounded below by zero. We have by Dirichlet's Test that $\sum x_n y_n$ is convergent but $x_n y_n = (-1)^n y_n$ so by the test we have $\sum(-1)^n y_n$ is convergent based on the Dirichlet's test and hence show that the alternating series test is a special case of the Dirichlet's Test.

Exercise 3.2.2

We have,

$$A = \left\{(-1)^n + \frac{2}{n} : n = 1, 2, 3 \dots\right\}$$

$$B = \{x \in \mathbb{Q} : 0 < x < 1\}$$

(a). For set B the limit points are $[0, 1]$ as for any value in this we can find a deleted neighborhood still in B (this is because the rationals are dense in R so for any ε if can find some $q \in \mathbb{Q}$ such that $b - \varepsilon < q < b + \varepsilon$) and hence the deleted neighborhood is in Q). For A we see that if we consider the alternating elements i.e. the positive ones together and negative ones together, the limit of that sequence is 1 and -1 respectively. For instance consider the positive sub sequence, we have $1 + \frac{2}{2n}$ which is $1 + \frac{1}{n}$ and $\frac{1}{n}$ goes to zero as $n \to \infty$ as we can take $N > \frac{1}{\varepsilon}$ and the sequence converges to 1. The proof for -1 is similar. So we have 1 and -1 are the limit points of A.

(b). B is not an open set because of the density of irrationals, if we consider any point in B and any ε we can find an irrational number i such that $b-\varepsilon < i < b+\varepsilon$ i.e. inside the ε neighborhood and as B consists of only rational numbers we have the ε neighborhood is not a subset of B an hence b is not open. Similarly

we have B is not closed as it does not contain all it's limit points. For instance we can consider a sub sequence in B whose limit point is an irrational number and hence is not in B.

Now for A we see similar to above that for any ε neighborhood of any value we can find an irrational number in it which is not in A and hence A is not open. We see that althouhg $1 \in A$ we do not have $-1 \in A$ and hence this means that A does not contain all its limit points and is not closed.

(c). B does not contain any isolated points as all the points in B are limit points or in other words every ε neighborhood of all points in B intersect with B is some place other than the point itself (because of the density of the rationals in R).

We see that A contains isolated points as points other than the limit points are isolated points.
(d). Closure of A would be $A \cup \{-1\}$ and for B it would be $B \cup \{0, 1\} = [0, 1]$