Real Analysis

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Chapter 1

Introduction

1.1 Logic and proofs

Types of proofs,

1. Direct proof
2. Argument by contradiction
3. Induction
4. Contrapositive (we show$\neg B \Rightarrow \neg A$)

Theorem 1.1. $a = b \Leftrightarrow \forall \varepsilon > 0, |a - b| < \varepsilon$

Proof. 1. To show, $a = b \Rightarrow \forall \varepsilon > 0, |a - b| < \varepsilon$.
Suppose $a = b$ so $|a - b| = 0$. We have $\forall \varepsilon > 0$ so $|a - b| = 0 < \varepsilon$
2. To show, $\forall \varepsilon > 0, |a - b| < \varepsilon \Rightarrow a = b$
Now assume this is not true, or that $a \neq b$ so $a - b \neq 0$ this means that there is a non-zero number k such that $|a - b| = \varepsilon_0$. Now take $\varepsilon = \frac{\varepsilon_0}{2}$. This gives us, $|a - b| = \varepsilon_0 > \varepsilon$ which contradicts the statement. Hence our assumption is false and we prove the results. □

Example (Induction). $x_1 = 1$ and $x_{n+1} = \frac{1}{2}x_n + 1, \forall n \in \mathbb{Z}$. Show $x_{n+1} \geq x_n \forall n \in \mathbb{N}$
Define $S = \{n \in \mathbb{N}, s.t. x_{n+1} \geq x_n\}$ clearly, $S \subseteq N$.
$x_1 = 1$ and $x_2 = \frac{x_1}{2} + 1 = 1.5$. This gives us $x_2 > x_1$ so $1 \in S$
Suppose $n \in S$ and $x_{n+1} \geq x_n$. Note that,

$$x_{n+2} = \frac{1}{2}x_{n+1} + 1$$
$$x_{n+1} = \frac{1}{2}x_n + 1$$

Then $x_{n+2} = \frac{1}{2}x_{n+1} + 1 \geq \frac{1}{2}x_n + 1 = x_{n+1}$ or $x_{n+2} \geq x_{n+1}$ which means $n + 1 \in S$. So by induction we have $S = N$ and $x_{n+1} \geq x_n, \forall n \in \mathbb{N}$ ⋄

1.2 Real Numbers

Number systems,

1. Natural numbers $\mathbb{N}$

   $1, 2, 3, \ldots$

   Can't do subtraction

2. Integers $\mathbb{Z}$

   $\ldots, -3, -2, -1, 0, 1, 2, 3 \ldots$

   Can't do division

3. Rationals $\mathbb{R}$

   $\{\frac{p}{q}$ where $p, q \in \mathbb{Z}$ but $q \neq 0\}$

   Now we have $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{R}$

   But other numbers are still not captured,

   Example. $\sqrt{2}$ is not defined in $\mathbb{R}$. However if we define $x_1 = 2$, $x_{n+1} = \frac{1}{2}(x_n + \frac{2}{x_n})$. We know $x_{n+1} \in \mathbb{R}, \forall n \in \mathbb{N}$ (we can then show that $x_n \to \sqrt{2}$). ◇

   Theorem 1.2. $\sqrt{2}$ is not rational

   Proof. Argue by contradiction □

4. Real numbers $\mathbb{R}$

   We will define $\mathbb{R}$ as $\mathbb{Q}$ with the gaps filled in.

Definition (Axiom of completeness)**.** Every non-empty subset of $\mathbb{R}$ that is bounded above has a least upper bound called the supremum.

Let $S \subseteq \mathbb{R}$ and S is bounded above. If there is $u \in \mathbb{R}$ such that $s \leq u, \forall s \in S$ then S is bounded above by u (*Similar for bounded below*)

Definition (Least upper bound or supremum)**.** We say $u \in \mathbb{R}$ is the least upper bound for S if,

1. If u is an upper bound for S
2. $u \leq v$ for any other upperbound v of S.

Similar for greatest lower bound or infimum

Example. $S = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ where $S \subseteq R$ and $S \neq \phi$. Here S is bounded above by $1.1, 1.2, 2, 3, 4, \ldots$ By AoC, $\sup S$ exists (in this case is 1). Similarly S is bounded below as well and can also be shown that $\inf S = 0$ ◇

Note. Here, $1 \in S$ but $0 \notin S$. So sup or inf may or may not be in the set.

Definition. $S \subseteq \mathbb{R}$ then we say a real numbers $m \in S$ is a maximum if $\forall s \in S$ we have

$$s \leq m$$

Similar for minimum

Note. Following are true,

1. $m \in S$

2. m might not exist, consider,

$$S = [0, 1)$$

This does not have a maximum element but $\sup S = 1$

It does have a minimum element which is also equal to the infinium, $\inf S = 0$

Note. Following are true of AoC,

1. AoC doesn't hold for $\mathbb{Q}$
2. AoC will be basic to take limits.

Example. Consider $\phi \neq A \subseteq R$, and is bounded above. Let $c \in \mathbb{R}$. Define

$$A + c = \{a + c, a \in A\}$$

We show that $\sup(A) + c = \sup(A + c)$ ◇

Proof. Denote $s = \sup A$, so we have $s \geq a, \forall a \in A$.
1. To show $s + c$ is an upperbound. Above definition gives us, $s + c \geq a + c, \forall a \in A$. By definition we have $s + c$ is an upperbound of $A + c$.
2. To show $s + c$ is the smallest upperbound of $A + c$. Let b be an arbitrary upper bound of $A + c$. So $a + c \leq b, \forall a \in A$. Therefore $a \leq b - c, \forall a \in A$ where $b - c$ is an upperbound of A. But s is the least upper bound which means that $s \leq b - c$ or that $s + c \leq b$. So we showed that b must be greater than or equal to $s + c$. Hence, $s + c$ is the least upper bound. So $s + c = \sup(A + c)$ □

Lemma 1.3. Assume $s \in \mathbb{R}$ is an upperbound for a set $A \subseteq R$ and $A \neq \phi$. Then $s = \sup(A)$ if and only if $\forall \varepsilon, \exists a \in A, s.t\ a > s - \varepsilon$

Proof. (1) $\Rightarrow$ (2)
Assume $s = \sup(A)$, given $\varepsilon > 0$ we have $s - \varepsilon < a$. So $s - \varepsilon$ cannot be an upper bound of A. This means that $\exists a \in A$ such that $a > s - \varepsilon$.
(2) $\Rightarrow$ (1)
We have s such that $s - \varepsilon < a$ for some $a \in A$ and $\forall \varepsilon$. We need to show that s is the least upperbound. Let b be an arbitrary upperbound. Suppose $b < s$ so we have $\varepsilon = s - b > 0$ and $b = s - \varepsilon$ however we have some $a \in A$ such that $a > s - \varepsilon = b$ so $a > b$ which makes b not an upperbound and hence breaks our assumption. So $s \leq b$ □

1.3 Consequences of Completeness

Theorem 1.4 (Nested Interval property). For any $n \in \mathbb{N}$, assume that we are given interval $I_n = [a_n, b_n] = \{x \in \mathbb{R} | a_n \leq x \leq b_n\}$ where $a_n \leq b_n$
Assume that $I_n \supseteq I_{n+1}, \forall n \in \mathbb{N}$ such that,

$$\ldots I_3 \subseteq I_2 \subseteq I_1$$

Then,

$$\bigcap_{n=1}^{\infty} I_n \neq \phi$$

Note. This means that for any $I_n, I_{n'}$ we have either $I_n \subseteq I_{n'}$ or $I_{n'} \subseteq I_n$

Proof. Take $A = \{a_n, n \in N\}$ we have $A \neq \phi$ and $A \subseteq \mathbb{R}$. A is bounded above as we have $a_1 \leq a_2 \dots a_n \leq \dots$ and $b_1 \geq b_2 \dots\ b_n \geq \dots$
So for every n, $a_n \leq b_n \leq b_1$. So b_1 is an upperbound for A. By AoC we have $\sup(A) = x \in \mathbb{R}$ exist.
Now we show that $x \in I_n, \forall n$.
Note that $\forall n, b_n$ is an upper bound for A.
$\forall m \in \mathbb{N}, a_m \in A$ and if,
$m \geq n$ then $a_m \leq b_m \leq b_n$
$m < n$ then $a_m \leq a_n < b_n$
As $\sup A = x$ then we have $x \leq b_n$ and as x is an upperbound of A we have, $a_n \leq x$ for all $n \in N$. So $x \in I_n$ hence proving the above statement. □

1.4 Density of $\mathbb{Q}$ in $\mathbb{R}$

Theorem 1.5 (Archimedean properties). The following are true,

1. Given any $x \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $n > x$
2. Given any $y \in \mathbb{R}, y > 0$, $\exists n \in \mathbb{N}$ s..t $y > \frac{1}{n}$

Proof. (2) folows by (1) by setting $x = \frac{1}{y}$.
For (1) lets assume that there is no n for some $x \in \mathbb{R}$ that satisfies the condition. So $\exists x_0 \in \mathbb{R}$ such that $\forall n \in \mathbb{N}$, $n \leq x_0$. So N is bounded above by x_0. So by AoC let $\alpha = \sup N$. Now, $\alpha - 1$ is not an upperbound for $\mathbb{N}$. So $\exists n_0 \in \mathbb{N}$ such that $n > \alpha - 1$. So $\alpha < n_0 + 1 \in \mathbb{N}$. This is a contradiction, so (1) holds. □

Theorem 1.6. $\forall a, b \in R$ (a < b), $\exists r \in \mathbb{Q}$ such that $a < r < b$.

Proof. It suffices to find $m, n \in Z$ such that,

$$a < \frac{m}{n} < b$$

Step 1: find n.
Note that $b - a > 0$ and $b - a \in R$. By (2) we have n s.t. $b - a > \frac{1}{n}$. Now fix such an n.
Step 2. find m for the fixed n.
Without loss of generality take $na > 0$ and by (1), $m_0 \in \mathbb{N}$ s.t. $M_0 > na$. Then consider a finite set $\{0, 1, \dots, M_0\}$. Now take k in this set and compare with na. Take m to be the smallest one such that $m > na$.
So we have $m > na \geq m - 1$
Step 3: Check if m, n work,
We have,

$$m > na \geq m - 1$$
$$\frac{m}{n} > a \text{ and } \frac{m}{n} \leq a + \frac{1}{n}$$

But we have $b - a > \frac{1}{n}$ so $b > a + \frac{1}{n}$ which gives us,

$$a < \frac{m}{n} \leq a + \frac{1}{n} < b$$

$\square$

Theorem 1.7. $\exists s \in R$ such that $s^2 = 2$

Proof. Let $A = \{x > 0, x \in \mathbb{R}, s.t. x^2 < 2\}$. Clearly $A \subseteq \mathbb{R}$ and is nonempty. We have A is bounded above as 2 is an upper bound.
By AoC $\sup A \in \mathbb{R}$ exists and set $s = \sup A$. Claim $s^2 = 2$.
Now we will prove this by contradiction by showing it cannot be the case that $s^2 < 2$ or $s^2 > 2$.
Now assume that $s^2 < 2$ and let $0 < \delta = 2 - s^2$. We will show that there is some $\varepsilon > 0$ such that $(s + \varepsilon)^2 < 2$ (i.e. s cannot be a supremum)

Scratchwork
We want to find ε to satisfy the $(s + \varepsilon)^2 < 2$, for this we work backwards.

$$\begin{aligned} (s+\varepsilon)^2 &< 2 \\ s^2 + \varepsilon^2 + 2s\varepsilon &< 2 \end{aligned}$$

We have $s < s^2 < 2$ (as s is definitely greater than 1). So $2s < 4$ to get,

$$s^2 + \varepsilon^2 + 2s\varepsilon < s^2 + \varepsilon^2 + 4\varepsilon < 2$$

Now let's assume that $\varepsilon < 1$ as if ≥ 1 works then trivially $\varepsilon < 1$ works as well. So we have $\varepsilon^2 < \varepsilon$ so,

$$\begin{aligned} s^2 + 5\varepsilon &< 2 \\ 5\varepsilon &< 2 - s^2 \\ \varepsilon &< \frac{\delta}{5} \end{aligned}$$

Now we can take $\varepsilon = \min\{\frac{\delta}{10}, 1\}$.

If we take $\varepsilon = \min\{1, \frac{\delta}{10}\}$ then we have,

$$\begin{aligned} (s+\varepsilon)^2 &= s^2 + \varepsilon^2 + 2s\varepsilon \\ (s+\varepsilon)^2 &\leq s^2 + \varepsilon^2 + 2s\varepsilon \leq s^2 + \delta < 2 \end{aligned}$$

$\square$

Exercise. Show $s^2 > 2$ is impossible. We similarly show that we can find an ε such that $(s - \varepsilon)^2 > 2$

1.5 Cardinality

Definition. We say that two sets A and B have the same cardinality if there is a bijective function $f : A \to B$. We write $A \sim B$

Remark. Types,

1. We say A is finite if $A \sim \{1, 2, \ldots, n\}$ for some $n \in N$
2. We say A is countable (countably infinite) then $A \sim N$

3. An infinite set that is not countable is called unountable.

Example. $E = \{2, 4, \dots\}$, we show $E \sim N$.
Take $f : N \to E$ defined as $f(n) = 2n$. ◇

Example. $N \sim Z$
Take $f : N \to Z$ s.t. $f(n) = \frac{n-1}{2}$ if n is odd else $-\frac{n}{2}$. ◇

Example. $(-1, 1) \sim \mathbb{R}$
Take $f(x) = \frac{x}{x^2-1}$ ◇

Theorem 1.8. Following are true,

1. $\mathbb{Q}$ is countable
2. $\mathbb{R}$ is uncountable

Proof. For $\mathbb{Q}$ define $A_1 = \{0\}$ and

$$A_n = \{\pm\frac{p}{q} : p + q = n, p, q \in \mathbb{N} \text{ and } p, q \text{ coprime}\}$$

Note that A_n is finite and $\forall x \in \mathbb{Q}$ we can find a unique $n \in \mathbb{N}$ s.t. $x \in A_n$
Now map elements of $A_0, A_1, \dots$ iterative with $1, 2, 3, \dots$. So by construction, any element from A_n will be listed. So there is a bijection between $\mathbb{Q}$ and $\mathbb{N}$ since $\mathbb{Q} = \bigcup_{n-1}^{\infty} A_n$ and A_n's are disjoint, so $\mathbb{Q} \sim \mathbb{N}$
Now we show the reals are not countable. Assume that $\mathbb{R}$ is countable and suppose there is a bijective funciton $f : \mathbb{N} \to \mathbb{R}$. Let $x_1 = f(1), x_2 = f(2), \dots$
Then $\mathbb{R} = \{x_1, x_2, \dots, \}$. Let I be a closed interval $I_1 \subseteq \mathbb{R}$ s.t. $x_1 \notin I_1$ and similarly find $I_2 \subseteq I_1$ such that $x_2 \notin I_2$. Similarly define for all n such that $I_{n+1} \subseteq I_n$ closed interval such that $x_n \notin I_{n+1}$.
Since $\forall n_0 \in N$ we have $x_{n_0} \notin I_{n_0}$ so $xn_0 \notin \bigcap_{n=1}^{\infty} I_n$. But $R = \{x_1, x_2, \dots\}$ so we have $\bigcap_{n=1}^{\infty} I_n = \phi$. However, this is a contradiction with the nested interval property. □

Theorem 1.9. Following are true,

1. $A \subseteq B$ if B is countable them A is either countable or finite.
2. If A_n is countable $\forall n \in \mathbb{N}$ then $\bigcup_{n=1}^{\infty} A_n$ is countable.