Homework 4, Math 4150

1. Exercise Set 3.3, #40 (with some additional parts).

   (a) Let $a \in \mathbb{Z}$ with $a > 0$. Use induction to prove that

$$\sum_{m=1}^{a} m = \frac{a(a+1)}{2},$$

and that

$$\sum_{m=1}^{a} m^3 = \left(\frac{a(a+1)}{2}\right)^2.$$

Solution. First we show, $\sum_{m=1}^{a} m = \frac{a(a+1)}{2}$. Take $a = 1$ we have $\sum_{m=1}^{1} = 1$ and $\frac{a(a+1)}{2} = \frac{1(1+1)}{2} = \frac{2}{2} = 1$. Hence base case is true. Now, assume true for some arbitrary n, so we have,

$$\sum_{m=1}^{n} m = \frac{n(n+1)}{2}$$

Now we need to show the case for $n+1$ is true. First add $n+1$ to both sides above. We get,

$$\begin{aligned}\left(\sum_{m=1}^{n} m\right) + (n+1) &= \frac{n(n+1)}{2} + (n+1) \\ \left(\sum_{m=1}^{n+1} m\right) &= (n+1)(\frac{n}{2} + 1) \\ &= \frac{(n+1)(n+2)}{2}\end{aligned}$$

which is the case for $n+1$. Hence by induction we have $\sum_{m=1}^{a} = \frac{a(a+1)}{2}$.

Now we need to show that $\sum_{m=1}^{a} m^3 = \left(\frac{a(a+1)}{2}\right)^2$. First we verify base case where $a = 1$ so we have $\sum_{m=1}^{1} m^3 = 1^3 = 1$ and we have $\left(1(1+1)\frac{1}{2}\right)^2 = 1^2 = 1$. So base case is true. Now assume true for case $a = n$. So we have,

$$\sum_{m=1}^{n} m^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Now we need to show case $a = n+1$ is true. We add $(n+1)^3$ to both sides to get,

$$\begin{aligned}
\left(\sum_{m=1}^{n} m^3\right)(n+1)^3 &= \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3 \\
\sum_{m=1}^{n+1} m^3 &= \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3 \\
&= (n+1)^2\left(\left(\frac{n}{2}\right)^2 + (n+1)\right) \\
&= (n+1)^2\left(\frac{n^2+4n+4}{2^2}\right) \\
&= (n+1)^2\left(\frac{(n+2)^2}{2^2}\right) \\
&= \left(\frac{(n+1)(n+2)}{2}\right)^2
\end{aligned}$$

Which is the case for $n+1$. Hence by induction we have $\sum_{m=1}^{a} m^3 = \left(\frac{a(a+1)}{2}\right)^2$

(b) Let $n \in \mathbb{Z}$ with $n > 0$. Use the previous part to prove that

$$\Big(\sum_{d|n,d>0} v(d)\Big)^2 = \sum_{d|n,d>0} (v(d))^3.$$

[Hint: It suffices to prove the equation in (b) for powers of prime numbers (justify this)]

Solution. As v is multiplicative it is enough to show the above for prime powers. So consider $n = p^a$ we have,

$$\begin{aligned}
\left(\sum_{d|p^a} v(d)\right)^2 &= (v(p^a) + v(p^{a-1}) + \dots p + 1)^2 \\
&= ((a+1) + a + (a-1) + \dots + 2 + 1)^2 \\
&= \left(\frac{(a+1)(a+2)}{2}\right)^2 \\
&= \sum_{m=1}^{a+1} m^3 \\
&= 1^3 + (2)^3 + \dots + (a+1)^3 \\
&= v(1)^3 + v(p)^3 + \dots + v(p^{a-1})^3 + v(p)^3 \\
&= \sum_{d|p^a,d>0} (v(d))^3
\end{aligned}$$

Now we justify why the multiplicativity of v justifies showing only for prime powers. We have $n = p_1^{a_1} \dots p_n^{a_n}$. Now if $d \mid n$ then $d = p_1^{b_1} \dots p_n^{b_n}$ where $1 \leq b_k \leq a_k$ so $v(d) = v(p_1^{b_1}) \dots v(p_n^{b_n})$. Now,

$$
\begin{aligned}
\left(\sum_{d|n, d>0} v(d) \right)^2 &= \left(\sum_{d|p_1^{a_1} \dots p_k^{a_k}, d>0} v(d) \right)^2 \\
&= \left(\sum_{d|p_1^{a_1}} v(d) \cdots \sum_{d|p_k^{a_k}} v(d) \right)^2 \\
&= \left(\sum_{d|p_1^{a_1}} v(d) \right)^2 \dots \left(\sum_{d|p_k^{a_k}} v(d) \right)^2 \\
&= \sum_{d|p_1^{a_1}} (v(d))^3 \cdots \sum_{d|p_k^{a_k}} (v(d))^3 \\
&= \sum_{d|n} (v(d))^3
\end{aligned}
$$

true as $v(d)$ is multiplicative implies that $v(d)^3$ is also multiplicative.

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2. Exercise Set 3.4, #48. Let $n \in \mathbb{Z}$ with $n > 0$. Prove that

$$\frac{\sigma(n!)}{n!} \geq \sum_{i=1}^{n} \frac{1}{i}.$$

Solution.

We know that σ is the sum of divisors. So $\sigma(n!) = \sigma(1 \cdot 2 \cdot \cdots \cdot (n-1) \cdot (n))$. Now note that out of all the divisors of $n!$ we have $n!, n!/2, \ldots n!/n$ are all unique divisors of $n!$. So so $\sigma(n!)$ is the sum of all divisors, it is at least as big as $n! + n!/2 \cdots + n!/n$. Hence weget,

$$\begin{aligned}
\sigma(n!) &\geq n! + \frac{n!}{2} + \cdots + \frac{n!}{n} \\
\frac{\sigma(n!)}{n!} &\geq 1 + \frac{1}{2} + \cdots + \frac{1}{n} \\
&= \sum_{i=1}^{n} \frac{1}{i}
\end{aligned}$$

3. Exercise Set 3.5, # 58.
Definiton: Let $k, n \in \mathbb{Z}$ with $k, n > 1$. Then n is said to be k-perfect if $\sigma(n) = kn$ (Note: A 2-perfect number is perfect in the usual sense).

(a) Prove that 120 is 3-perfect.
Solution. We have $120 = 2^3 3^1 5^1$ so we have $\sigma(120) = \frac{2^4-1}{1}\frac{3^2-1}{2}\frac{5^2-1}{4} = 15 \cdot 4 \cdot 6 = 360 = 3 \cdot 120$ and hence we show it is 3-perfect.

(b) Find all 3-perfect numbers of the form $2^k 3p$ where p is an odd prime number.
Solution. For $2^k 3p$ to be a 3-perfect number we have,

$$\sigma(2^k 3p) = \left(\frac{2^{k+1}-1}{1}\right)\left(\frac{3^2-1}{2}\right)\left(\frac{p^2-1}{p-1}\right) = 2^k 3^2 p$$

And we have,

$$\begin{aligned}
\left(\frac{2^{k+1}-1}{1}\right)\left(\frac{3^2-1}{2}\right)\left(\frac{p^2-1}{p-1}\right) &= 2^k 3^2 p \\
(2^{k+1}-1)(4)(p+1) &= 2^k 9p \\
(2 - \frac{1}{2^k})(4p+4) &= 9p \\
8(p+1) - \frac{4(p+1)}{2^k} &= 9p \\
(8-p)2^k &= 4(p+1)
\end{aligned}$$

We see that $4(p+1)$ is positive so $8 - p > 0$ so $p < 8$. The only odd primes smaller than 8 are $3, 5, 7$. For each of those numbers we have $52^k = 4(4)$ clearly does not have a solution for k. For 5 we have $3 \cdot 2^k = 4(6)$. Here $k = 3$ and for 7 we have $2^k = 4(8)$ so $k = 5$. So we have two numbers, $2^3 3 \cdot 5 = 120$ and $2^5 3 \cdot 7 = 672$

(c) Let $n \in \mathbb{Z}$ with $n > 1$ and let p be a prime number not dividing n. Prove that if n is p-perfect, then pn is $(p+1)$-perfect.
Solution.
We have n is p perfect. Let $n = p_1^{a_1} \dots p_n^{a_n}$ so we have $\sigma(n) = \sigma(p_1^{a_1}) \dots \sigma(p_n^{a_n}) = \frac{p_1^{a_1+1}-1}{p_1-1} \dots \frac{p_n^{a_n+1}-1}{p_n-1}$. As n is p perfect we have, $\sigma(n) = pn$. Now consider pn. As $p \nmid n$ we have $pn = p_1^{a_1} \dots p_n^{a_n} p$. So we can write,

$$\begin{aligned}
\sigma(pn) &= \sigma(p_1^{a_1}) \dots \sigma(p_n^{a_n})\sigma(p) \\
&= \frac{p_1^{a_1+1}-1}{p_1-1} \dots \frac{p_n^{a_n+1}-1}{p_n-1}\frac{p^2-1}{p-1} \\
&= \frac{p_1^{a_1+1}-1}{p_1-1} \dots \frac{p_n^{a_n+1}-1}{p_n-1}(p+1) \\
&= \sigma(n)(p+1) \\
&= pn(p+1)
\end{aligned}$$

as $\sigma(n) = pn$. But this means that we have $\sigma(pn) = (p+1)(pn)$ which by definition means that pn is $p+1$-perfect .

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4. Exercise Set 3.6, #65.

Definiton: Let $n \in \mathbb{Z}$ with $n > 0$. Von Mangoldt's function, denoted $\Lambda(n)$, is given by

$$\Lambda(n) := \begin{cases} \ln p & \text{if} \quad n = p^a \text{ for some } p \text{ prime and } a \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}.$$

Prove that

$$\Lambda(n) = -\sum_{d|n, d>0} \mu(d) \ln d.$$

Solution.

First trivially if $n = 1$ we have $\Lambda(n) = \ln(1) = 0$ and as the only divisor of 1 is itself we have $-\mu(1)\ln(1) = 0$ and hence we have $\Lambda(n) = -\sum_{d|n,d>0} \mu(d) \ln d$.

Now consider if $n = p_1 \dots p_k$ if $p_1, \dots, p_k$ are distinct. Then we have $\Lambda(n) = 0$ if $k > 1$ else $\Lambda(n) = \ln(p_1)$. We have the divisors of n are $p_1 \dots p_i$ for $1 \leq i \leq k$. For any arbitrary divisors we have $\mu(d) = \mu(p_1 \dots p_i) = (-1)^i$ and $\ln(d) = \ln(p_1 \dots p_i) = \ln(p_1) + \dots + \ln(p_i)$. First, if $k = 1$ we get the only divisor is p_1 so $-\sum_{d|n,d>0} \mu(d)\ln(d) = -\mu(p_1)\ln(p_1) = -(-1)^1 \ln(p_1) = \ln(p_1) = \Lambda(n)$. For $k > 1$ we get,

$$\begin{aligned} -\sum_{d|n,d>0} \mu(d) \ln d &= -\left(\sum_i \ln(p_i) - \sum_{1 \leq i < j \leq k} \ln(p_i p_j) + \dots (-1)^{k+1} \ln(p_1 \dots p_k) \right) \\ &= -\left(\sum_i \ln(p_i) - \sum_{1 \leq i < j \leq k} (\ln(p_i) + \ln(p_j)) + \dots (-1)^k (\ln(p_1) + \dots + \ln(p_k)) \right) \\ &= 0 \end{aligned}$$

Lastly if $p^2 \mid n$ for some p then we have either $n = p^a$ for $a \geq 2$ in which case we have $-\sum \mu(d)\ln(d) = -(\mu(1)\ln(1) + \mu(p)\ln(p) + \dots \mu(p^a)\ln(p^a))$. But we know that $\mu(p^a)$ for $a > 2$ is zero so we get $= -(-\ln(p) + 0 \dots 0) = \ln(p)$ which is our desired solution. Now if $n = p_1^{a_1} \dots p_n^{a_n}$ for all $a_1 \dots a_n \geq 2$ which is the case where we have some $p_k^2 \mid n$. Here note that across all the divisors of n the ones with power greater than 1 for any of the primes will go to zero in the sum as we have $\mu(d)$ for that divisor as zero. So the only divisors that remain are $p_1 \dots p_i$ for $1 \leq i \leq k$. Now note that in this case the above expansion will apply and we get the sum as going to zero.

5. Exercise Set 3.6, #69. This exercise presents an interpretation of the Möbius Inversion Formula from the point of view of convolutions.
Definition: Let $n \in \mathbb{Z}$ with $n > 0$, and let f and g be arithmetic functions. The convolution of f and g, denoted $f \star g$, is given by

$$(f \star g)(n) := \sum_{d|n, d>0} f(d) g\left(\frac{n}{d}\right).$$

(a) **Commutativity**: Let f and g be arithmetic functions. Prove that $f \star g = g \star f$.
Solution. We have,

$$\begin{aligned}(f \star g)(n) &= \sum_{d|n, d>0} f(d) g\left(\frac{n}{d}\right) \\ &= \sum_{d|n, d>0} f\left(\frac{n}{d}\right) g(d) \\ &= \sum_{d|n, d>0} g(d) f\left(\frac{n}{d}\right) \\ &= (g \star f)(n)\end{aligned}$$

Hence, it is commutative.

(b) **Associativity**: Let f, g, and h be arithmetic functions. Prove that $(f \star g) \star h = f \star (g \star h)$.
Solution. We have,

$$\begin{aligned}((f \star g) \star h)(n) &= \sum_{d|n, d>0} (f \star g)(d) h\left(\frac{n}{d}\right) \\ &= \sum_{d|n, d>0} \left(\sum_{e|d, e>0} f(e) g\left(\frac{d}{e}\right) \right) h\left(\frac{n}{d}\right) \\ &= \sum_{d|n, d>0} h\left(\frac{n}{d}\right) \left(\sum_{e|d, e>0} f(e) g\left(\frac{d}{e}\right) \right) \\ &= \sum_{d|n, d>0} h\left(\frac{n}{d}\right) \left(\sum_{e|d, e>0} f\left(\frac{d}{e}\right) g(e) \right) \\ &= \sum_{d|n, d>0} f\left(\frac{n}{d}\right) \left(\sum_{e|d, e>0} h\left(\frac{d}{e}\right) g(e) \right) \\ &= \sum_{d|n, d>0} f(d) (h \star g) \left(\frac{n}{d}\right) \\ &= f \star (g \star h)\end{aligned}$$

(c) **Identity**: For $n \in \mathbb{Z}$ with $n > 0$, let

$$\delta(n) = \begin{cases} 1 & \text{if} \quad n = 1 \\ 0 & \text{otherwise} \end{cases}.$$

Let f be an arithmetic function. Prove that $f \star \delta = \delta \star f = f$.
Solution. We have,

$$\begin{aligned} f \star \delta &= \sum_{d|n, d>0} f(d)\delta\left(\frac{n}{d}\right) \\ &= \sum_{d|n, d>0} f\left(\frac{n}{d}\right)\delta(d) \end{aligned}$$

Note that for all divisors except 1 we have $\delta(d) = 0$, so in our summation all the terms except for when $d = 1$ will go to zero so we get,

$$\begin{aligned} (f \star \delta)(n) &= f(1)\delta(n) + \cdots + f(n)\delta(1) \\ &= 0 + \cdots + f(n)\delta(1) \\ &= f(n) \cdot 1 \\ &= f(n) \end{aligned}$$

So we have $f \star \delta = f$. Now as we showed commutativity in (a) we have $f \star \delta = \delta \star f$and hence we have $f \star \delta = \delta \star f = f$

(d) **Inverses**: For $n \in \mathbb{Z}$ with $n > 0$, let $\mathbf{1}(n) = 1$. In other words, $\mathbf{1}$ is the arithmetic function mapping every positive integer to 1. Prove that $\mu \star \mathbf{1} = \mathbf{1} \star \mu = \delta$.
Solution.
We have,

$$\begin{aligned} \mu \star 1 &= \sum_{d|n, d>0} \mu(d) 1\left(\frac{n}{d}\right) \\ &= \sum_{d|n, d>0} \mu(d) \end{aligned}$$

Now the right term, by Proposition 3.14 is,

$$\sum_{d|n, d>0} \mu(d) = \begin{cases} 1, & \text{if } n = 1 \\ 0, & \text{otherwise} \end{cases} = \delta(n)$$

Hence, we have $\mu \star 1 = \delta$ and because of commutativity we also have $\mu \star 1 = 1 \star \mu = \delta$

(e) **Möbius Inversion Formula**: Prove the Möbius Inversion Formula using convolutions.
[Hint: Theorem 3.5 can be restated as $f = g \star \mathbf{1}$ if and only if $g = \mu \star f = f \star \mu$.]

Solution.

($\Rightarrow$) We have $f(n) = \sum_{d|n,d>0} g(d)$ which is equivalent to $f = g \star 1$. Now using the above properties we have,

$$\begin{aligned}
f &= g \star 1 \\
f \star \mu &= (g \star 1) \star \mu \\
f \star \mu &= g \star (1 \star \mu) && \text{using associativity} \\
f \star \mu &= g \star \delta && \text{using (d)} \\
f \star \mu &= g && \text{using (c)}
\end{aligned}$$

As we have commutativity, we now have $g = f \star \mu = \mu \star f$

($\Leftarrow$)

Now assume we have $g = f \star \mu = \mu \star f$ then we can write,

$$\begin{aligned}
g &= f \star \mu \\
g \star 1 &= (f \star \mu) \star 1 \\
g \star 1 &= f \star (\mu \star 1) \\
g \star 1 &= f \star (\delta) \\
g \star 1 &= f
\end{aligned}$$

So we get $f = g \star 1$ which is our solution.

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