Linear Alebgra HW06

Aamod Varma

October 15, 2024

3C

6

Proof. Consider a case where range T = W. In this case let $w_1, \ldots, w_m$ be a basis for W which also would be a basis for range T, we can find $v_1, \ldots, v_n$ such that,

$$Tv_1 = w_1, \ldots, Tv_m = w_m, \ldots, Tv_n = 0$$

So with respect to the basis $w_1, \ldots, w_m$ in the first column we can have all zeroes except for the first row first element.
Now if dim range T < dim W. We can find a v_1 such that $T(v_1) = 0$. Now the matrix for this will have all in the first column as zero (as none of them dependent on w_1) □

6

We know that $T(v_1) = A_{1,1}w_1 + \cdots + A_{n,1}w_m$. We need to show there exists a basis of W such that all values except for possibly $A_{1,1}$ is zero.
Consider the case when $T(v_1) = 0$ then we have $A_{1,1} = \cdots = A_{n,1} = 0$ for any arbitrary basis of W.
If $T(v_1) \neq 0$ then consider all of $A_{2,1}, \ldots, An, 1 = 0$ except for $A_{1,1}$ (the element in the first column and row). In this case consdier,

$$T(v_1) = w_1$$

where w_1 is an arbitrary vector in W. Now we can extend w_1 to a basis of W. So in both cases we have a basis of W, $w_1, \ldots, w_n$ such that only possibly the first row first column element is zero.

10

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and let $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$
We get,

$$AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

and

$$BA = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

So $AB \neq BA$

8

Proof. We need to show $(AB)_{j,.} = A_{j,.}B$

We have,

$$\begin{aligned}(AB)_{j,\cdot} &= (\sum_{k=1}^{n} A_{j,k}B_{k,1}, \dots, \sum_{k=1}^{n} A_{j,k}B_{k,p},) \\ &= (A_{j,1}, \dots, A_{j,n})B \\ &= A_{j,\cdot}B\end{aligned}$$

□

13

We know that

$$(AA)_{j,k} = \sum_{r=1}^{n} A_{j,r}A_{r,k}$$

$$(A(AA))_{j,k} = \sum_{r=1}^{n} A_{j,r}(AA)_{r,k}$$

$$(A^3)_{j,k} = \sum_{p=1}^{n} A_{j,r}(\sum_{x=1}^{n} A_{r,x}A_{x,k})$$

$$(A^3)_{j,k} = \sum_{p=1}^{n}\sum_{x=1}^{n} A_{j,r}A_{r,x}A_{x,k}$$

16

Proof. $\Rightarrow$
Take A with rank 1 then we can decompose it to matirces R and C s.t,

$$R = (m \times 1), C = (1 \times n)$$

Let $R = (c_1, \dots, c_m)^T$ and $C = (d_1, \dots, d_n)$
So we have

$$\begin{aligned}A_{jk} = (RC)_{jk} &= \sum_{n=1}^{1} R_{m,1}C_{1,n} \\ &= c_j d_k\end{aligned}$$

$\Leftarrow$
If $A_{j,k} = c_j d_k$ then we can write A in terms of two matrices R times C such that R $= (c_1, \dots, c_m)^T$ and $C = (d_1, \dots, d_m)$

□

3

Proof. We need to show, $a \Leftrightarrow b$
$a \Rightarrow b$.
If T is invertible we know that it is injective and surjective. If T is surjective

then we know that null T = 0. So we have,

$$dimrange(T) = dim(V)$$

To show that $T(v_1), \dots, T(v_n)$ is a basis of V we need to show that it is linearly independent and spans V.
To show linear independence we need to show that $c_1, \dots, c_n = 0$ if,

$$c_1(Tv_1) + \cdots + c_nT(v_n) = 0$$

This is equal to,

$$T(c_1v_1 + \cdots + c_nv_n) = 0$$

We know that the null T is $\{0\}$ so we have,

$$c_1v_1 + \cdots + c_nv_n = 0$$

We know that $v_1, \dots, v_n$ is a basis for V. So it is linearly independent which means that, $c_1, \dots, c_n = 0$. So we have $T(v_1), \dots, T(v_n)$ is a linearly independent set.
The lenght of our list is the same as the length of the basis. Which means that we have linearly independent set of vectors are also a basis for V.
Another way of showing spanning is taking any $v \in V$ we can write it as $a_1v_1, \dots, a_nv_n = v$. We can applying T on both sides and show that v can be represnted as a linear combination of $T(v_k)$
$a \Leftarrow b$
If $Tv_1, \dots, Tv_n$ is a basis for V.
We need to show that T is injective and surjective. First consider an arbitary $v \in V$ such that $T(v) = 0$. We know that $v = a_1v_1 + \cdots + a_nv_n$ where $v_1, \dots, v_n$ is a basis for V. So we get,

$$T(a_1v_1 + \cdots + a_nv_n) = 0$$

$$a_1T(v_1) + \cdots + a_nT(v_n) = 0$$

We know that $Tv_1, \dots, Tv_n$ is a basis which means that its linearly independent. So w have, $a_1, \dots, a_n = 0$. But if $a_1, \dots, a_n = 0$ then we have $v = a_1v_1 + \cdots + a_nv_n = 0$. Which means that for any $T(v) = 0$ means that $v = 0$. This means that it is injective.
We already know that if we have $V \to V$ such that both the dimensiosn are same then injective means that its surjective which means that it is invertible.
We also can show that any $w \in V$ can be written as $T(v) = w$ which m eans that it is surjective.
Take a $w \in V$ so $Tw = T(a_1v_1 + \cdots + a_nv_n) = a_1T(v_1) + \cdots + a_nT(v_n)$. We know that $T(v_1), \dots, T(v_n)$ is a basis which means that any vector $v \in V$ can be repreesnted as alinear combination of $T(v_1), \dots, T(v_n)$. So we showed that any vector $v \in V$ can be represented by a $w \in V$ such that $T(w) = v$ which means that its surjective.

$\square$

5

Proof. $\Leftarrow$ We know that T is invertible so its injective and surjective. We know for every $u \in U$, $T(u) = S(u)$.
We need to show S is injective. First consider u_1, u_2 so we need to showm $S(u_1) = S(u_2) \Rightarrow u_1 = u_2$.
If $Su_1 = Su_2$ then we can say $T(u_1) = T(u_2)$. However we nkow that T is injective so this means that $u_1 = u_2$. Hence we show that S has to be injective.
$\Rightarrow$
We have S is injective and maps a subspace of V, U onto V. We need to show that there exists a linear map T from V to itself such that it is an invertible linear map.
First consider the basis of U as $u_1, \dots, u_k$. We can extend the basis from this to,

$$u_1, \dots, u_k, v_{k+1}, \dots, v_n$$

Let us define our lienar map T such that $T(u) = S(u)$ if $u \in U$ or in other words if $u = a_1u_1 + \dots + a_ku_k$. And define $T(v) = v$ if $v \in V - U$
We need to show $T(v_1') = T(v_2') \Rightarrow v_1' = v_2'$. Let $v_1' = a_1u_1 + \dots a_nv_n$ and $v_2' = b_1u_1 + \dots + b_nv_n$. So we have,

$$\begin{aligned}
T(v_1') &= T(v_2') \\
T(a_1u_1 + \dots a_nv_n) &= T(b_1u_1 + \dots b_nv_n) \\
T(a_1u_1) + \dots + a_nT(v_n) &= T(b_1v_1) + \dots + b_nT(v_n) \\
(a_1 - b_1)T(u_1) + \dots + (a_n - b_n)T(v_n) &= 0
\end{aligned}$$

TO DO LATER

9

If T is surjective then there exists a map $S : W \to V$, TS is the identity map. Or that $T(S(v)) = v$ for $v \in V$.
Now let $U = range(S)$ we need to show that $T_{|U}$ is injective and surjective.
1. Injective.
Consider a $u \in$ null $T|_U$. So

$$T_U(u) = 0$$

$$T_U(S(w)) = w = 0$$

But

$$S(w) = u$$

and $w = 0$
which means that $u = 0$.
2. Surjective
For any $w \in W$ we have $v = S(w) \in U$ such that $T(S(w)) = w \in W$.

So we show that $v = S(w)$ for any $w \in W$
Hence we show that it is isomorphic

$\square$

11

$\Rightarrow$
We have ST is invertible. Lets assume the contrary that either S is not invertible or T is not invertible.
1. S is not invertible. Means S is not surjective. We know that ST is invertibel which means that $\forall v \in V$ $\exists v'$ s.t. $STv' = v$. Now let $Tv' = v''$. This means that $\forall v \in V, \exists v'' \in V$ s.t. $S(v'') = v$. But this makes S surjective which contradicts our assumption.
2. T is not invertible means that T is not injective or surjective. We know that ST is injective and surjective. If T is not injective then $\exists v \in V$ s.t. $T(v) = 0$. Now this means $S(T(v)) = S(0) = 0$ or that $STv = 0$ for some $v \neq 0$. But this makes ST not injective and hence not invertible which contradicts our assumption.
So by proof by contradiction our assumption ust be wrong and both S and T are invertiblie.