

Linear Algebra 3D

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Problem 1

Proof. We are given that T^{-1} is the inverse of T which means that $TT^{-1} = I$ and $T^{-1}T = I$. Now we know that if $AB = BA = I$ then B is the inverse of T . So similarly we see that T is the inverse of T^{-1} . Hence $(T^{-1})^{-1} = T$ \square

Problem 2

Proof. We need to show that $(ST)^{-1} = T^{-1}S^{-1}$. If ST is invertible we can find M such that $STM = I$ and $MST = I$. Let $M = T^{-1}S^{-1}$. So we have,

$$\begin{aligned} STM &= ST(T^{-1}S^{-1}) \\ &= STT^{-1}S^{-1} \\ &= SS^{-1} \\ &= I \end{aligned}$$

And,

$$\begin{aligned} MST &= T^{-1}S^{-1}ST \\ &= T^{-1}T \\ &= I \end{aligned}$$

Hence ST is invertible and the inverse is $T^{-1}S^{-1}$ \square

Problem 3

Proof. (a) \Rightarrow (b) We know that T is invertible hence it is both injective and surjective. We can say that Tv_1, \dots, Tv_n spans range of T . But we know that range $T = V$ as it is injective. We also know that $\dim V = n = \dim \text{range } T$.

Hence we have n vectors Tv_1, \dots, Tv_n that span V and because the number of vectors are equal to basis length the vectors themselves form a basis of V .

(b) \Rightarrow (c) (c) is just a specific case of (b)

(c) \Rightarrow (a) For some basis v_1, \dots, v_n we know that Tv_1, \dots, Tv_n is a basis of V . So that means that for any $v \in V$, v is in the span of Tv_1, \dots, Tv_n or that it is in the range of T . This means that V is surjective. As it is a map onto itself and T is surjective this means T is also injective. Hence T is invertible. \square

Problem 5

Proof. \Leftarrow

We have T is an invertible linear map such that $Tu = Su, \forall u \in U$ and we

need to show that S is injective.

For any $u_1, u_2 \in V$ we need to show that $Su_1 = Su_2 \Rightarrow u_1 = u_2$

We know that $Su_1 = Tu_1$ and $Su_2 = Tu_2$. So we have $Tu_1 = Tu_2$. But we know that T is invertible hence it is also injective. So we have $u_1 = u_2$. Hence S is injective.

\Rightarrow We know that S is injective from U to V , we need to construct a map T such that $Tu = Su$ but T is invertible (both injective and surjective).

First consider the basis of U as u_1, \dots, u_n . We know S is injective hence $\text{null } S = \{0\}$. So we know that S can be written as,

$$Su_1 = v_1, \dots, Su_n = v_n$$

Such that v_1, \dots, v_n span $\text{range } S$ and as $\dim \text{range } S = \dim U = n$. v_1, \dots, v_n are linearly independent.

Now let us first extend our basis of U to a basis of V as follows, u_1, \dots, u_m and extend our basis of $\text{range } S$ to a basis of V as v_1, \dots, v_m .

Let us define a linear map as follows,

$$Tu_1 = Su_1 = v_1$$

$$\dots$$

$$Tu_n = Su_n = v_n$$

$$Tu_k = v_k \text{ for } k > n$$

Now based on our definition we have $\text{range } T$ is spanned by v_1, \dots, v_n which spans V . Hence $\text{range } T = V$. Which both means that it is surjective and because null space is $\{0\}$ we also have injectivity. Hence T is invertible.

Based on our definition $T(u) = S(u)$ is also true, as for any $u = a_1u_1 + \dots + a_nu_n$ we have, $S(u) = a_1Su_1 + \dots + a_nSu_n = a_1Tu_1 + \dots + a_nTu_n = T(a_1u_1 + \dots + a_nu_n) = T(u)$ \square

Problem 9

Proof. We know T is surjective we need to show there is a subspace U of V such that $T|_U$ is an isomorphism of U onto W .

First because T is surjection we have $\text{range } T = W$. Consider v_1, \dots, v_n is a basis on V then we can say T is defined as,

$$Tv_1 = w_1, \dots, Tv_n = w_n$$

Now because w_1, \dots, w_n span W let us reduce it to a basis of W , take this as w_1, \dots, w_k without loss of generality. Now consider the subspace spanned by v_1, \dots, v_k . Now we need to show it is an isomorphism or that it is injective and surjective.

Firstly we know it is surjective as it maps to w_1, \dots, w_k which we know span W . We need to show that it is injective.

Because U is spanned by n vectors and W is spanned by n vectors we know by rank nullity theorem that $\dim \text{null } T = 0$ hence our map is injective. \square

Problem 11

Proof. \Rightarrow We know that ST is invertible. Assume to the contrary that S or T is not invertible.

(1). We have S is not invertible and $ST = S(T(v))$ is invertible this means that $\forall v \in V$ we can find $v' \in V$ such that $S(T(v')) = v$. So if $T(v') = v''$ then $S(v'') = v$ for any v . But this means that S is surjective. However based on our assumption we know S is not surjective. So this contradicts our assumption.

(2). Consider T is not invertible $\Rightarrow T$ is not injective. If T is not injective then $\exists v \in V$ such that $Tv = 0$. So we have $STv = S(0) = 0$ which means that $STv = 0$ for some non-zero v which makes ST not injective and not invertible. Hence a contradiction.

\Leftarrow We can show $\exists M = T^{-1}S^{-1}$ such that $STM = I, MST = I$ \square

Problem 12

We have $STU = I$ this means that ST and U are both invertible which means that S and T are also invertible. So $\exists S^{-1}$ and U^{-1} .

$$\begin{aligned} STU &= I \\ TU &= S^{-1} \\ U &= T^{-1}S^{-1} \\ US &= T^{-1}S^{-1}S \\ T^{-1} &= US \end{aligned}$$

Problem 13

Proof. Consider the backward shift operator $T((x_1, \dots)) = (x_2, \dots)$. It is not injective and hence not invertible. Let S be I and U be the forward shift operator. We have $STU = I$ but T is not invertible. \square

Problem 14

Proof. RST is a map from V to V hence RST is invertible. This means that RST is injective. First let's show that T is invertible. Assume T is not invertible means that T is not injective or $\exists v \in V$ such that $Tv = 0$ but this means that $RSTv = 0$ but this is a contradiction as RST is invertible. Now we know that T is invertible so $\text{range } T$ is V .

Now assume S is not injective $\exists v \in \text{range } T = V$ such that $S(v) = 0$ but this gives us a contradiction as well as RST is invertible. Hence S is injective. \square

Problem 15

Proof. We know that Tv_1, \dots, Tv_m span V . So,

$$a_1Tv_1 + \dots + a_nTv_m = 0$$

only if $a_1, \dots, a_n = 0$

Now let's assume the contrary that v_1, \dots, v_n are not invertible so there exists not all zero b_1, \dots, b_n such that

$$b_1v_1 + \dots + b_nv_m = 0$$

Let us take $T(v)$ so,

$$T(b_1v_1 + \dots + b_nv_n) = T(0) = 0$$

$$b_1T(v_1) + \dots + b_nT(v_n) = 0$$

If we have b_1, \dots, b_n all not zero then that means that $T(v_1), \dots, T(v_n)$ is not lin independent hence a contradiction. \square

Problem 16

Proof. We know that Tx in $F^{m,1}$ so $Tx = M(Tx)$,

$$= M(T)M(x)$$

$$Ax$$

\square

Problem 20

Problem 21