Linear Alebgra HW04

Aamod Varma

September 15, 2024

2B

Problem 4

(a). We are given $U = \{(z_1, z_2, z_3, z_4, z_5) \in \mathbb{C}^5 : 6z_1 = z_2, z_3 + 2z_4 + 3z_5 = 0\}$
The contraints are as follows, $6z_1 = z_2$ and $z_3 + 2z_4 + 3z_5 = 0$
So we can rewrite each z as

$$z_1 = \frac{z_2}{6}, z_2 = z_2, z_3 = -2z_4 - 3z_5, z_4 = z_4, z_5 = z_5.$$

We see we have two dependent variables and three independent variables which means our basis will be of length 3 dependent on z_2, z_4, z_5 as follows,

$$(\frac{1}{6}, 1, 0, 0, 0), (0, 0, -2, 1, 0), (0, 0, -3, 0, 1).$$

(b). We need to extend this basis onto $\mathbb{C}^5$. We know from (a) that our dependent variables are z_1 and z_3. So to extend our basis we need to be able to make these vectors our indepdent. For this we can add the following two vectors,

$$(1, 0, 0, 0, 0), (0, 0, 1, 0, 0).$$

These additions are linearly independent because we can't represent these vectors as a linearly combination of our previous list (in our first list it was necessarily true that $z_1 = \frac{z_2}{6}$, so if $z_1 = 1, z_2 \neq 0$, similary reasoning for z_3). We also know this new list spans $\mathbb{C}^5$ because our new additions give us control over the dependent variables from our previous list (we could also argue that because it is a linearly independent set of vectors and we have $\dim(\mathbb{C}_5)$ of them.
(c). We need to find a subspace W such that $U \oplus W = \mathbb{C}^5$. Take W from above as,

$$W = (1, 0, 0, 0, 0), (0, 0, 1, 0, 0).$$

First we need to show that $W + U = \mathbb{C}^5$. That every vector in $\mathbb{C}^5$ can be represented as $v = u + w, u \in U, w \in W$
Now, if $u \in U, u = a_1u_1 + a_2u_2 + a_3u_3$ and if $w \in W, w = b_1w_1 + b_2w_2$.
So

$$v = a_1u_1 + a_2u_2 + a_3u_3 + b_1w_1 + b_2w_2$$

But we know from above that u_1, u_2, u_3, w_1, w_2 is a basis for $\mathbb{C}^5$. Which means that the linear combination of these vectors can reprsent every vector in $\mathbb{C}^5$. So we show that all of $v \in \mathbb{C}^5$ can be written as a vecotr $u \in U$ plus a vector $w \in W$.

Problem 5

If $V = W + U$ we can say that $\forall v \in V$,

$$v = u + w \text{ for } u \in U, w \in W.$$

Now, u can be written as a linear combinatino of vectors in U and similar can be done for w.
So let $u = a_1u_1 + \cdots + a_nu_n$ and $w = b_1w_1 + \cdots + b_mw_m$. So we have a linear combination of $n+m$ vectors. We know that $\dim(V) \leq n+m$ because $\dim(V) \leq$ length of any spanning set in V.

If $n+m > \dim V$. Then we can reduce it to a linearly independent set of vector such that it still spans V. So now we have a basis of V that consists of vectors that are either in U or W. Or in other words our basis are vectors in $U \cup W$.
If $n+m = \dim V$ then we already have a linearly independent set of vectors that span V which consists of vectors either in U or V. Which meanst hat the basis are vectors in $U \cup W$.
So we have shown that there exists a basis of V in $U \cup W$ if $U + W = V$.

11

We know that $v_1, \dots, v_n$ is a basis for V. We need to show that it is also a basis for V_C. Now $V_{\mathbb{C}}$ is defined by $V \times V$ such that $(x, y) = x + iy \in V_{\mathbb{C}}$.
So we need to show that any vector of the form $u + iw \in V_C$ can be represented by a linear combinatino of $v_1, \dots, v_n$.
First we know that $u \in V, w \in V$. So we can write $u = a_1v_1 + \cdots + a_nv_n$, similarly $w = b_1v_1 + \cdots + b_nv_n$.
Now because we also define scalar multiplication with complex numbers we can write,

$$a_1v_1 + \cdots + a_nv_n + i(b_1v_1 + \cdots + b_nv_n) = u + iw.$$

Or,

$$\forall (u, w) \in V_C, u + iw = (a_1 + ib_1)v_1 + \cdots + (a_n + ib_n)v_n.$$

So we showed that we can represent all elements of V_C as a linear combination of our vectors $v_1, \dots, v_n$

2C

Problem 1

We know that $\dim(\mathbb{R}^2) = 2$ which means that for a given subspace V we have three cases,

$$\dim(V) = 0, \dim(V) = 1, \dim(V) = 2.$$

If $\dim(V) = 0$ then our vector space if $V = \{0\}$ by definition.
If $\dim(V) = 1$ then that means our vector space contains one vector so V is spanned by $\{v\}$. First we knwo that $0 \in V$ as V is a subspace (we can take the coefficient to be 0). Now for any vector $v \in V, kv \in V$. We know that this defines any line in $\mathbb{R}^2$ that goes through the origin.
If $\dim(V) = 2$ we also know that $U \subseteq V$. If $U \subseteq V$ and $\dim(U) = \dim(V)$ then we know that $U = V$. So, U determines $\mathbb{R}^2$

Problem 4

(a). A basis of U would be one where $p''(6) = 0$. First we know that a basis of $P_4(R)$ is $1, x, x^2, x^3$ which can also be written as $1, (x-6), (x-6)^2, (x-6)^3$ where $x \in R$
So any p is written as

$$p(x) = 1a_1 + a_2(x-6) + a_3(x-6)^2 + a_4(x-6)^3$$

$$p''(6) = 2a_3$$

So we see that for it to be equal to 0, $a_3 = 0$. Which means our basis is,

$$1, (x-6), (x-6)^3.$$

(b). As we discussed above, adding $(x-6)^2$ to the list will give us a basis for $P_4(R)$
So our basis is,

$$1, (x-6), (x-6)^2, (x-6)^3$$

(c). Our subspace W would be spanned by $(x-6)^2$. We first show that $W+U = P_4(R)$. To do this we need to show any $p \in P_4(R)$ can be represented as,

$$p = u + w, u \in U, w \in W.$$

We know for $u \in U, u = a_1 + a_2(x-6) + a_3(x-6)^3$ and for $w \in W, w = b_1(x-6)^2$. So,

$$p = a_1 + a_2(x-6) + a_3(x-6)^3 + b_1(x-6)^2.$$

Which is a linear combination of the basis of $P_4(R)$ which means that $u + w$ can represent any vector $p \in P_4(R)$ and hence we can say $U + W = P_4(F)$
Now we need to show that $U \oplus W = P_4(F)$. To show this we can show that there is only one way of representing 0 as $u + w$.
Now if $u + w = 0$ as we did abov ewe can write,

$$0 = a_1 + a_2(x-6) + a_3(x-6)^3 + b_1(x-6)^2$$

First we know that $a_3 = 0$ as we can't represent x^3 using any of the other terms. Similary we can show that $b_1 = 0, a_2 = 0, a_1 = 0$. Hence the only way of reprenseting 0 is to have all coefficients as 0.
Which means that $U \oplus W = P_4(R)$

Problem 8

Given $v_1, \ldots, v_m$ is linearly independent in V and $w \in V$. Take $U := span($