Homework 7, Math 4150

1. Exercise Set 6.1, #3 Apples and oranges at a grocery store cost 14 cents and 17 cents each respectively. A customer spends \$2.90 for apples and oranges. How many pieces of each fruit did the customer buy?

   Solution. We can represent this as the following equations taking \$2.9 as 290 cents to give us,

   $$14x + 17y = 290$$

   Now we need to find positive solutions to x, y. First note that we have $(14, 17) = 1$ and we have $17 \cdot 5 + 14 \cdot (-6) = 1$. Now we can expand this solution to get,

   $$14 \cdot (-1740) + 17 \cdot 1450 = 290$$

   However now note that we can represent solutions for this in the following manner,

   $$\begin{aligned} x &= x_0 + (b/d)n = (-1740) + (17) \cdot n \\ y &= y_0 - (b/d)n = (1450) - (14) \cdot n \end{aligned}$$

   As we need both as positive we need $1450 - 14n > 0$ and $-1750 + 17n > 0$. First condition gives us,

   $$\begin{aligned} 1450 - 14n &> 0 \\ 1450 &> 14n \\ 103.5 &> n \\ n &\leq 103 \end{aligned}$$

   and the second gives us,

   $$\begin{aligned} -1750 + 17n &> 0 \\ 17n &> 1750 \\ n &> 102.9 \\ n &\geq 103 \end{aligned}$$

   So we have $103 \leq n \leq 103$ which means $n = 103$ and pluging this back in we get the only solution,

   $$x = (-1740) + 17 \cdot 103 = 11 \text{ and } y = (1450) - 14 \cdot 103 = 8$$

   So the customers buys 11 apples and 8 oranges.

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2. Exercise Set 6.2, #11(a),(d), #12(d). Prove or disprove the following statements:

(a) The Diophantine equation $3x^2 - 7y^2 = 2$ has no integral solutions.
Solution. Consider the above modulo 7. So we have the following,

$$3x^2 - 2 = 7y^2$$
$$3x^2 \equiv 2 \pmod 7$$

But as $(3, 7) = 1$, 3 has an inverse modulo 7 and is (-2) so we have,

$$x^2 \equiv 2(-2) \pmod 7$$
$$x^2 \equiv 3 \pmod 7$$

However, now note the following. Both $3, 7$ are $\equiv 3 \pmod 4$ so using quadratic reciprocity we have,

$$\left(\frac{3}{7}\right) = -\left(\frac{7}{3}\right) = -\left(\frac{1}{3}\right) = -1$$

Hence 3 is not a quadratic residue modulo 7 and there is no x that satisfies the equation above. Hence, the Diophantine equation does not have integral solutions.

(b) The Diophantine equation $x^3 - 5 = 7y^3$ has no integral solutions. **Solution.** Take the equation modulo 7 to get,

$$x^3 \equiv 5 \pmod 7$$

Now we know that 3 is a primitive root of 7 so we can rewrite the above as follows,

$$ind_3 x^3 \equiv ind_3 5 \pmod 6$$

But we have $3^5 \equiv 5 \pmod 7$ so $ind_3 5 = 5$ which gives us,

$$3 ind_3 x \equiv 5 \pmod 6$$

Now note that this equation does not have a solution for $ind_3 x$ as $gcd(6, 3) = 3$ does not divide 5. Hence, the equation is not solvable.

(c) The Diophantine equation $x^2 + 2y^2 + 3 = 8z$ has no integral solutions.
Take the equation modulo 8 and we have,

$$x^2 + 2y^2 + 3 \equiv 0 \pmod 8$$
$$x^2 + 2y^2 \equiv 5 \pmod 8$$

Now note that modulo 8 we have the quadratic residues for 8 are $0^2, 1^2, \ldots, 7^2 = \{0, 1, 4\}$ and hence the possible values of $2y^2$ are $0, 1 \cdot 2, 4 \cdot 2 = 0$. Hence we have $x^2 \equiv 0, 1, 4 \pmod 8$ and $2y^2 \equiv 0, 2 \pmod 8$ and note that all combinations cannot equal 5. Hence no solution exists.

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3. Exercise Set 6.3, #16(b). Find all solutions in positive integers to the Diophantine equation $x^2 + 3y^2 = z^2$.
[HINT: Parallel the proof of Theorem 6.3 that concerns primitive Pythagorean triples.]
Solution. First let us consider the primitive solution such that we have $(x, y, z) = 1$ which means we need either x or y is even. Now rewrite the equation as follows,

$$3y^2 = z^2 - x^2$$
$$y^2 = \frac{1}{3}(z+x)(z-x)$$

Now note that $2 \mid y$ and as z, x are odd we have $2 \mid z + x$ and $2 \mid z - x$ this gives us,

$$\left(\frac{y}{2}\right)^2 = \frac{1}{3}\left(\frac{z+x}{2}\right)\left(\frac{z-x}{2}\right)$$

Now note that we have $\left(\frac{z+x}{2}, \frac{z-x}{2}\right) = 1$ as if they had a divisor d then it would divide the sum and difference of the two or we would have $d \mid z$ and $d \mid x$ which means x, y, z is not the primitive solution. Hence we have them as coprime. So this means we have two cases, either $3 \mid \frac{z+x}{2}$ or $3 \mid \frac{z-x}{2}$.

Case 1: $3 \mid \frac{z+x}{2}$. Now note that we have $\left(\frac{1}{3}\frac{z+x}{2}, \frac{z-x}{2}\right) = 1$. And as the product of the two is a square and they are coprime each of them themselves must be perfect squares. So there is some m, n such that we have $m^2 = \frac{1}{3}\frac{z+x}{2}$ and $n^2 = \frac{z-x}{2}$. Now we have that we have $\frac{z+x}{2} = 3m^2$ and $\frac{z-x}{2} = n^2$. So their sum is $3m^2 + n^2 = z$ and differnece is $3m^2 - n^2 = x$ and note we also have $y^2 = 4m^2n^2$ or $y = 2mn$. Hence we have the solution,

$$x = 3m^2 - n^2, y = 2mn, z = 3m^2 + n^2$$

Case 2: $3 \mid \frac{z-x}{2}$. Now note that we have $\left(\frac{z+x}{2}, \frac{1}{3}\frac{z-x}{2}\right) = 1$. And as the product of the two is a square and they are coprime each of them themselves must be perfect squares. So there is some m, n such that we have $m^2 = \frac{z+x}{2}$ and $n^2 = \frac{1}{3}\frac{z-x}{2}$. Now we have that we have $\frac{z-x}{2} = 3n^2$ and $\frac{z+x}{2} = m^2$. So their sum is $3n^2 + m^2 = z$ and differnece is $3n^2 - m^2 = x$ and note we also have $y^2 = 4m^2n^2$ or $y = 2mn$. Hence we have the solution,

$$x = 3n^2 - m^2, y = 2mn, z = 3n^2 + m^2$$

Now for both cases we have solutions of the form,

$$x = 3m^2 - n^2, y = 2mn, z = 3m^2 + n^2$$

Now note that we have a case where x is even and y, z are odd. But we see that in this case consider the equation modulo 4 we get $x^2 + 3y^2 \equiv z^2 \pmod 4$. Now if x is even then x^2 has to be $0 \pmod 4$ and if y^2, z^2 are odd they have to be $\equiv 1 \pmod 4$. But this means we have $x^2 \equiv 0 \pmod 4$ and $3y^2 \equiv 3 \pmod 4$ and $z^2 \equiv 1 \pmod 4$ and putting the three together we get $x^2 + 3y^2 - z^2 \equiv 0 + 3 - 1 \equiv 2 \pmod 4$ which gives us $0 \equiv 2 \pmod 4$ which is an obvious contradiction. Hence, the only possibility is the only above where x, z is odd and y is even.

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4. Exercise Set 6.4, #21 Prove that the Diophantine equation $x^4 - y^4 = z^2$ has no solutions in non-zero integers x, y, z.
[HINT: The proof of Theorem 6.4 i.e. "Fermat Descent" might give you some ideas on how to get started].

Solution. First we rewrite the equation as follows,

$$x^4 = z^2 + y^4$$

Now if we consider $(x^2)^2 = z^2 + (y^2)^2$ and consider the solution where $x^2 = x_0, z = z_0, y^2 = y_0$ don't share a common GCD with either y^2 or z being even. Now as all the solutions x_0, y_0, z_0 are integer solutions, by the well ordering principle we can choose the minimal solution such that x_0 has the minimal possible value. But we found another solution x_1, y_1, z_1 such that we have $x_1 < x_0$ which contradicts our assumption that x_0 is the minimal. Hence our assumption must be wrong.

Case 1: y^2 is odd. Using the Pythagorean triplet formulation we can find m, n co prime with one being even such that,

$$\begin{aligned} x^2 &= x_0 = m^2 + n^2 \\ y^2 &= y_0 = m^2 - n^2 \\ z &= z_0 = 2mn \end{aligned}$$

Now multiply the first two to get,

$$(xy)^2 = (m^2 + n^2)(m^2 - n^2) = m^4 - n^4$$

Or we get $m^4 = (xy)^2 + n^4$. Now note that this is a solution to our original equation and if we take $x_1 = m, z_1 = xy, y_1 = n$ then note we have $x_1 = m \leq m^2 < m^2 + n^2 = x^2 = x_0$. So we get $x_1 < x_0$. However, note that we choose x_0 such that x_0 is the minimal solution.

Case 2: Now take y^2 as even. So find m, n such that we have,

$$\begin{aligned} x^2 &= x_0 = m^2 + n^2 \\ y^2 &= y_0 = 2mn \\ z &= z_0 = m^2 - n^2 \end{aligned}$$

Note that we have two cases here either m is even or n is even. Now further consider $x^2 = m^2 + n^2$ and consider the primitive solution for this as well (so x, m, n are coprime and either m, n are even). Now note we can use the pythagorean triplet formula once again to find a, b for two cases one for m is even and n is even. For m is even we have

$$x = a^2 + b^2, m = 2ab, n = a^2 - b^2$$

and for n is even we have

$$x = a^2 + b^2, m = a^2 - b^2, n = 2ab$$

However now note from above that we have $y^2 = 2mn$. If m is even then pair 2 with m and we get $2m$ and n are perfect squares. So we have $2m = (2c)^2$ or $m = 2c^2$ for some $c \in Z^+$. Now plugging this to the case where m is even we have, $m = 2ab = 2c^2$ or $c^2 = ab$ which means both a, b are squares as well. And if n is even then we have $y^2 = m(2n)$ and $2n$ is a square so we can write $2n = (2c)^2$ or $n = 2c^2$ which gives us $n = 2ab = 2c^2$ which means $ab = c^2$ or a, b are perfect squares.

Note that in both cases we have a, b are perfect squares and either $2m$ and n or $2n$ and m are perfect squares. In the first case note we have $n = a^2 - b^2$ or $n + b^2 = a^2$. But as n, a, b are squares take $n = z_1^2, a = x_1^2, b = y_1^2$ and we have,

$$x_1^4 = z_1^2 + y_1^4$$

However, now note that we have $x_1 < x_1^2 = a < a^2 + b^2 = x$. Hence, we found a solution set x_1, y_1, z_1 such that $x_1 < x$ which is not possible as x is the smallest value by assumption. Hence, we have contradiction.

Now for the second case we have $2n$ or m are perfect. And note we have $m = a^2 - b^2$ or $a^2 = m + b^2$. As the three are squares we can take $a = x_1^2, m = z_1^2, b = y_1^2$ and we get $x_1^4 = z_1^2 + y_1^4$ and note we have $x_1 < x_1^2 = a < a^2 + b^2 = x$ and we get the same contradiction as above.

Hence, we see in both cases y being odd or even we get a contradiction. Which means our assumption that there exists a solution must be wrong and there does not exist a solution.

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5. Exercise Set 6.4, #27 Prove that the Diophantine equation $x^2 + y^2 = z^3$ has infinitely many integral solutions. [HINT: Consider $x = n^3 - 3n$ and $y = 3n^2 - 1$ for $n \in \mathbb{Z}$].

   Solution. Consider we have $x = n^3 - 3n$ and $y = 3n^2 - 1$. Now we have,

$$\begin{aligned} x^2 + y^2 &= (n^3 - 3n)^2 + (3n^2 - 1)^2 \\ &= (n^6 + 9n^2 - 6n^4) + (9n^4 + 1 - 6n^2) \\ &= n^6 + 3n^2 + 3n^4 + 1 \\ &= (n^2)^3 + 1^3 + 3(1^2 \cdot n^2) + 3(1 \cdot (n^2)^2) \\ &= (n^2 + 1)^3 \end{aligned}$$

   So take $z = n^2 + 1$ and we have $x^2 + y^2 = z^2$ and note that for each $n \in Z$ we if we take $x = n^3 - 3n, y = 3n^2 - 1, z = n^2 + 1$ we have infinitely many integral solutions to the equation $x^2 + y^2 + z^2$.