

Complex Analysis

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Chapter 1

Complex Numbers

1.12 Regions in the Complex Plane

Definition 1.1 (Epsilon neighborhood). An epsilon neighborhood around a point z_0 is the set of all z such that,

$$|z - z_0| < \varepsilon$$

Definition 1.2 (Deleted neighborhood). A deleted neighborhood around a point z_0 is the set of all z such that,

$$0 < |z - z_0| < \varepsilon$$

Remark. A deleted neighborhood is essentially an epsilon neighborhood but does not include the point z_0

Definition 1.3 (Interior point). z_0 is an interior point when there exists a neighborhood of z_0 that contains only points of S

Definition 1.4 (Exterior point). z_0 is an exterior point when there exists a neighborhood of z_0 that contains no points of S

Definition 1.5 (Boundary point). z_0 is a boundary point otherwise, i.e. all of the neighborhoods of z_0 contains a point in S and a point not in S

Definition 1.6 (Open set). S is an open set if $\forall z \in S, \exists \varepsilon$ s.t. $B_\varepsilon(z) \subset S$

Remark. We can also say that an open set does not contain any of its boundary points.

Definition 1.7 (Closed set). A set is closed if it doesn't contain its boundary points.

Definition 1.8 (Connected Set). An open set is connected if z_1, z_2 can be joined by a polygonal line, consisting of finite number of line segments, joined end to end.

Definition 1.9 (domain). A non empty open set that is connected is called a domain

Definition 1.10 (region). A domain together with some, none, or all of its boundary points is referred to as a region

Definition 1.11 (accumulation point). An accumulation point or limit point of a set S is z_0 if, each deleted neighborhood of z_0 contains at least one point of S

Remark. A closed set contains all of its accumulation points, but the opposite may not be true.

Remark. Every boundary point is not an accumulation point.

Example. Consider the set, $S = 5 \cup (0, 1)$

Here, the boundary points are 5, 0 and 1 because they ε -neighborhood defined around these points contains both interior points and exterior points.

However 5 is not an accumulation point because the deleted-neighborhood does not contain any interior points (as it removes 5). \diamond

Chapter 2

Analytic functions

2.1 13. Functions and Mappings

A translation translate a complex number to another location preserving direction and magnitude.

Example. $f(z) = z_0 + z$ ◇

A rotation rotates the complex number changing magnitude or direction.

Example. $f(z) = z_0 z$ This function rotates z by multiplying it with z_0 . We can see this when representing it in euler notation as follows,

$$z_0 z = r r_0 e^{i(\theta + \theta_0)}.$$

Example. $f(z) = z^2$ ◇

$$z = r e^{i\theta}$$

$$z^2 = r^2 e^{2i\theta}$$

So magnitude is squared and angle is doubled ◇

A reflection will reflect z along the x axis.

Example. $f(z) = \bar{z}$ reflects z along the x axis. ◇

An analytic function is a differentiable function in the complex space.

$$f(z) = w.$$

$$f(x + iy) = u + iv.$$

$$= u(x, y) + iv(x, y).$$

$$u(z) = iv(z).$$

2.2 15. Limits

If a function f is defined at all points z in some deleted neighborhood of point z_0 . Then, $f(z)$ has a limit w_0 as z approaches z_0 , or

$$\lim_{z \rightarrow z_0} f(z) = w_0.$$

Essentially this means that the point $w = f(z)$ can be made arbitrary close to w_0 if we choose a point z close enough to z_0 but distinct from it (deleted neighborhood).

Definition 2.1 (Limit). The limit of a function $f(z)$ as z goes to z_0 is w_0 if, $\forall \varepsilon > 0, \exists \delta > 0$, s.t.

$$|f(z) - w_0| < \varepsilon \text{ whenever, } 0 < |z - z_0| < \delta.$$

Remark. Essentially this means that for every ε -neighborhood, $|f(z) - w_0| < \varepsilon$ there is a deleted-neighborhood, $0 < |z - z_0| < \delta$ of z_0 such that every point z in it has an image w in the ε -neighborhood

Remark. All points in the deleted-neighborhood are to be considered but their images need not fill up the ε -neighborhood

Theorem 2.2. When a limit of a function $f(z)$ exists at a point z_0 , it is unique.

Proof. Suppose,

$$\lim_{z \rightarrow z_0} f(z) = w_0 \text{ and } \lim_{z \rightarrow z_0} f(z) = w_1.$$

This means that,

$$|f(z) - w_0| < \varepsilon \text{ when } 0 < |z - z_0| < \delta_0.$$

$$|f(z) - w_1| < \varepsilon \text{ when } 0 < |z - z_1| < \delta_1.$$

So,

$$|f(z) - w_0| + |f(z) - w_1| < 2\varepsilon.$$

We know that,

$$w_1 - w_0 = (f(z) - w_0) - (f(z) - w_1) \leq |f(z) - w_0| + |f(z) - w_1|$$

So,

$$w_1 - w_0 < 2\varepsilon, \text{ where } \varepsilon \text{ can be chosen arbitrary small.}$$

Hence,

$$w_1 - w_0 = 0, \text{ or, } w_1 = w_0.$$

□

Example. Show that, $f(z) = \frac{i\bar{z}}{2}$ in the open disk $|z| < 1$, then

$$\lim_{z \rightarrow 1} f(z) = \frac{i}{2}$$

$$\left| f(z) - \frac{i}{2} \right| = \left| \frac{i\bar{z}}{2} - \frac{i}{2} \right| = \frac{|z-1|}{2}.$$

Hence, for any z and ε ,

$$\left| f(z) - \frac{i}{2} \right| < \varepsilon \text{ when } 0 < |z-1| < 2\varepsilon.$$

◇

Example. $f(z) = \frac{z}{\bar{z}}$ The limit,

$$\lim_{z \rightarrow 0} f(z).$$

does not exist.

Assume that it exists, that implies that by letting the point $z = (x, y)$ we can approach the point, $(0, 0)$ in any manner and we would get the same limit.

Now if we approach the point from the x -axis where $z = (x, 0)$ we get,

$$\lim_{x \rightarrow 0} f((x, 0)) = \frac{x + 0i}{x - 0i} = 1.$$

But if we approach it from the y -axis where, $z = (0, y)$ we get,

$$\lim_{y \rightarrow 0} f((0, y)) = \frac{0 + iy}{0 - iy} = -1.$$

But we know that the limit should be unique, hence this implies that the limit does not exist. ◇

2.3 19. Derivatives

Theorem 2.3. If a function $f(z)$ is continuous and non-zero at a point z_0 then, there exists a neighborhood where, $f(z) \neq 0$ throughout.

Proof. We know that $f(z)$ is continuous which means that, $\varepsilon > 0, \exists \delta$ such that,

$$|f(z) - f(z_0)| < \varepsilon, \text{ when } 0 < |z - z_0| < \delta.$$

But if we take, $\varepsilon = \frac{f(z_0)}{2}$ then we have,

$$|f(z) - f(z_0)| < \frac{|f(z_0)|}{2}.$$

However, if $f(z) = 0$ for this neighborhood then we have,

$$|f(z_0)| < \frac{|f(z_0)|}{2}.$$

which is a contradiction. □

Theorem 2.4. f is continuous on R which is closed and bounded, $\exists M > 0$, real $|f(z)| \leq M, \forall z \in R$ equality holds for at least one z .

Definition 2.5 (Derivative). f is differentiable at z_0 when $f'(z_0)$ exists where,

$$f'(z_0) = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

Remark. Can also solve,

$$\lim_{z_0 \rightarrow 0} \frac{f(z + z_0) - f(z)}{z_0}$$

Example. Find derivative of, $f(z) = \frac{1}{z}$

$$\begin{aligned} \lim_{z_0 \rightarrow 0} \left(\frac{1}{z + z_0} - \frac{1}{z} \right) \frac{1}{z_0} \\ \lim_{z_0 \rightarrow 0} \frac{z - z - z_0}{z(z + z_0)} \frac{1}{z_0} \\ \lim_{z_0 \rightarrow 0} \frac{-1}{z(z + z_0)} \\ = \frac{-1}{z^2} \end{aligned}$$

◇

Example. $f(z) = \bar{z}$

$$\lim_{z_0 \rightarrow 0} \frac{z + \bar{z}_0 - \bar{z}}{z_0}$$

Go from x and y axis.

From x ,

$$\lim_{x_0 \rightarrow 0} \frac{\bar{z} + x_0 - \bar{z}}{x_0} = 1.$$

Similarly if we go from y we get -1 , so the derivative doesn't exist.

◇

If we have a function $f(z) = u(x, y) + iv(x, y)$ then,

$$z_0 = x_0 + iy_0.$$

$$\Delta z = \Delta x + i\Delta y.$$

We have to show the following exist,

$$\lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}.$$

$$= \frac{u(x_0 + \Delta x, y_0 + \Delta y) + iv(x_0 + \Delta x, y_0 + \Delta y) - (u(x_0, y_0) + iv(x_0, y_0))}{\Delta x + i\Delta y}.$$

Horizontally, $\Delta y = 0$.

So,

$$\begin{aligned} \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} + \frac{i(v(x_0 + \Delta x, y_0) - v(x_0, y_0))}{\Delta x} \\ = u_x + iv_x. \end{aligned}$$

Similary, if we go vertically, $\Delta x = 0$ and we get,

$$= v_y - iu_y.$$

Theorem 2.6. If, $f(z) = u + iv$, $f'(z)$ exists at, $z_0 = x_0 + iy_0$. Then, u_x, u_y, v_x, v_y exists at (x_0, y_0) and must satisfy the Cauchy-Reimann equation.

$$f'(z_0) = u_x + iv_x \text{ at } (x_0, y_0).$$

Theorem 2.7. $f(z) = u(x, y) + iv(x, y)$ defined throughout the ε -neighborhood of $z_0 = x_0 + iy_0$,

- (a) u_x, u_y, v_x, v_y exists everywhere in the neighborhood
- (b) u_x, u_y, v_x, v_y continuous at (x_0, y_0) and satisfy the Cauchy-Reimann equations

$$u_x = v_y, u_y = -v_x \text{ at } (x_0, y_0)$$

Then $f'(z_0)$ exists and,

$$f'(z_0) = u_x + iv_x \text{ at } (x_0, y_0).$$

Proof. We need to show,

$$\lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}.$$

$$= \lim_{\Delta z \rightarrow 0} \frac{u(x_0 + \Delta x, y_0 + \Delta y) + iv(x_0 + \Delta x, y_0 + \Delta y) - (u(x_0, y_0) + iv(x_0, y_0))}{\Delta z}.$$

Using taylor expansion we know,

$$f(x + \Delta x) = f(x) + \Delta x f'(x) + \frac{(\Delta x)^2}{2} f''(x).$$

$$\begin{aligned} u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0) = \\ = u(x_0, y_0) + \Delta x u_x(x_0, y_0) + \frac{(\Delta x)^2}{2} u_{xx}(x_0, y_0) + \Delta y u_y(x_0, y_0) + \frac{(\Delta y)^2}{2} u_{yy}(x_0, y_0). \end{aligned}$$

We can write the limit as,

$$\frac{u_x(x_0, y_0)\Delta x + u_y(x_0, y_0)\Delta y + \varepsilon_1(\Delta x^2) + \varepsilon_2(\Delta y^2)}{\Delta z} +$$

$$i \frac{v_x(x_0, y_0)\Delta x + v_y(x_0, y_0)\Delta y + \varepsilon_3(\Delta x) + \varepsilon_4(\Delta y)}{\Delta z}.$$

We know $u_x(x_0, y_0) = v_y(x_0, y_0)$ and $u_y(x_0, y_0) = -v_x(x_0, y_0)$, so,

$$\frac{u_x(x_0, y_0)\Delta x + -v_x(x_0, y_0)\Delta y + \varepsilon_1(\Delta x^2) + \varepsilon_2(\Delta y^2)}{\Delta z} +$$

$$i \frac{v_x(x_0, y_0)\Delta x + u_x(x_0, y_0)\Delta y + \varepsilon_3(\Delta x) + \varepsilon_4(\Delta y)}{\Delta z}.$$

$$= \frac{u_x(x_0, y_0)(\Delta x + i\Delta y) + u_y(x_0, y_0)(\Delta y - i\Delta x) + \varepsilon_4}{\Delta z}.$$

and $\Delta z = \Delta x + i\Delta y$

$$= \frac{u_x(x_0, y_0)(\Delta x + i\Delta y) + u_y(x_0, y_0)(\Delta y - i\Delta x) + \varepsilon_4}{\Delta x + i\Delta y}.$$

□

Definition 2.8 (Analytic function). A function f is analytic in an open set S , if f has derivative everywhere in S . It is analytic at a point z_0 if it is analytic in some neighborhood of z_0

Remark. Analytic function has to be on an open set.

Remark. For it to be analytic at z_0 derivative should exist in the neighborhood of z_0 (not just the point z_0)

Example. $f(z) = (|z|)^2 = \sqrt{x^2 + y^2}^2$

$$u = x^2 + y^2, v = 0$$

$$u_x = 2x, u_y = 2y.$$

$$v_x = 0, v_y = 0.$$

So the Cauchy-Reimann equation is only satisfied at $(0, 0)$

$f'(0) = 0$ and it exists. ◇

Remark. $f(z) = |z|^2$ is not analytic anywhere. So even if the derivative exists at $z = 0$. The function is not analytic at $z = 0$ (or at any point)

Because, (1). $f'(z)$ exists at $z = 0$

(2). u_x, u_y, v_x, v_y exists $\nRightarrow f'(z)$

(3). $f(z)$ is continuous $\nRightarrow f'(z)$

Essentially it only exists for $z = 0$ and not in the neighborhood around it.

Definition 2.9 (Entire function). A function f is analytic at each point in the entire plane.

Definition 2.10 (Singular point). z_0 is a singular point if f fails to be analytic at z_0 but is analytic at some point in every neighborhood at z_0

Example. $f(z) = 2 + 3z^2 + z^3$

Is analytic everywhere so it is an entire function

◇

Example. $f(z) = \frac{1}{z}$

Is analytic at all non-zero, but $z = 0$ is a singular point

◇

Example. $f(z) = |z|^2 = x^2 + y^2$

Is not analytic, no singular points either.

◇

2.4 Harmonic Function

Definition 2.11 (Harmonic function). A real valued function of $H(x, y)$ is said to be harmonic if in a given domain of the x, y plane, it has a continuous partial derivative of the first and second order ($H_x, H_y, H_{xx}, H_{yy}, H_{xy}$) and satisfies,

$$H_{xx}(x, y) + H_{yy}(x, y) = 0 \text{ Laplace equation.}$$

Theorem 2.12. If $f = u(x, y) + i(v(x, y))$ is analytic in a domain D , then u, v are harmonic in D

Chapter 3

Elementary Functions

3.1 Exponential Function

The exponential function is e^z . But we can write this as

$$e^z = e^{x+iy} = e^x e^{iy} = e^x (\cos(y) + i \sin(y))$$

We can also write,

$$e^z = \rho e^{i\phi} \text{ where } \rho = |e^x| \text{ and } \phi = y$$

For a function, $e^{z_1} e^{z_2}$ we can write,

$$\begin{aligned} e^{z_1} e^{z_2} &= e^{x_1+iy_1} e^{x_2+iy_2} \\ &= e^{x_1+x_2} e^{i(y_1+y_2)} \\ &= e^{z_1+z_2}. \end{aligned}$$

The derivative of e^z is an entire function

$$\frac{d}{dz} e^z = e^z \text{ which is an entire function.}$$

$$e^{z+2} = e^z + e^2 = e^z$$

3.2 Log Function

The log function is $f(z) = \log(z) = w = u + iv$. We know

$$e^w = z = e^{u+iv} = e^u e^{iv}.$$

We see that $r = e^u$ and $\theta = v + 2n\pi$

$$r = e^u \Rightarrow \ln(r) = u$$

Similarly,

$$\theta = v + 2n\pi.$$

So we have,

$$f(z) = \log(z) = \ln |z| + i \arg(z).$$

and the principal direction is,

$$f(z) = \log(z) = \ln |z| + i\theta, \quad -\pi < \theta < \pi.$$

Some properties are,

$$(1). e^{\log z} = z, (z \neq 0)$$

$$(2). |e^z| = e^x$$

$$(3). \log(e^z) = \ln |e^z| + i \arg(e^z)$$

$$= \ln |e^x| + i(y + 2n\pi), n = 0, \pm 1, \pm 2.$$

$$= \ln e^x + iy + i2n\pi.$$

$$= z + 2n\pi.$$

Branches

The principal branch is

$$\log z = \ln r + i\theta \text{ where } r > 0, -\pi < \theta < \pi.$$

A branch cut is a portion of a line or curve that is introduced in order to define a branch F of a multiple-valued function f .

Points on the branch cut for F are singular points of F and any point that is common to all branches of f are called branch points.

Example.

$$\frac{d}{dz} \log z = \frac{1}{z}, \text{ where } |z| > 0$$

The branches can be $\alpha < \arg z < \alpha + 2\pi$

◇

Property. $\log z_1 z_2 = \log z_1 + \log z_2$

Proof.

$$\begin{aligned} \log z_1 z_2 &= \ln(r_1 r_2) + i(\theta_1 + \theta_2 + 2n\pi) \\ &= \log z_1 z_2 = \ln(r_1) + \ln(r_2) + i(\theta_1 + \theta_2 + 2n\pi) \\ &= \log z_1 z_2 = \ln(r_1) + i(\theta_1 + 2n\pi) + \ln(r_2) + i(\theta_2 + 2n\pi) \\ &= \log z_1 z_2 = \ln(r_1) + i(\theta_1 + 2n\pi) + \ln(r_2) + i(\theta_2 + 2n\pi) \\ &= \log z_1 z_2 = \log z_1 + \log z_2 \end{aligned}$$

□

Property. $\log |z_1 z_2| = \log |z_1| + \log |z_2|$

Property. $\log\left(\frac{z_1}{z_2}\right) = \log(z_1) - \log(z_2)$

Property. $z^n = e^{n \log(z)}$

3.3 Power Function

We have a complex number c and we have $f(z) = z^c$. By definition we have $z^c = e^{c \log z}$

The derivative is $\frac{d}{dz}f(z) = \frac{d}{dz}(z^c)$

$$\frac{d}{dz}e^{c \log z} = e^{c \log z} \frac{d}{dz}c \log z = e^{c \log z} \frac{c}{z}$$

But we can write $\frac{e^{c \log z} c}{e^{\log z}} = ce^{(c-1) \log z} = cz^{c-1}$. The principal value of $z^c = e^{c \operatorname{Log} z}$

If the function is $f(z) = c^z$ then we have

$$\frac{d}{dz}c^z = \frac{d}{dz}e^{z \log c} = e^{z \log c} \frac{d}{dz}z \log c = e^{z \log c} \log c = c^z \log c$$

3.4 Trigonometric Function

We know that $e^{i\theta} = \cos \theta + i \sin \theta$ and $e^{-i\theta} = \cos \theta - i \sin \theta$. So we can write,

$$\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

and,

$$\sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

We have $\frac{d}{dz} \sin z = \cos z$ and $\frac{d}{dz} \cos z = -\sin z$

Property. $\sin(-z) = -\sin(z)$ and $\cos(-z) = \cos(z)$

Property. $\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$

Property. $\sin(2z) = 2 \sin(z) \cos(z)$

Property. $\sin(z + \frac{\pi}{2}) = \cos(z)$

Consider the hyperbolic sin and cos functions,

$$\sinh z = \frac{e^z - e^{-z}}{2}, \cosh z = \frac{e^z + e^{-z}}{2}$$

We can write $\sin z = \sin(x + iy)$. Now expanding this we get,

$$\sin(x) \cos(iy) + \cos(x) \sin(iy) = \sin(x) \cosh(y) + i \cos x \sinh(y)$$

And we have,

$$\begin{aligned} |\sin z|^2 &= \sin^2 x + \sinh^2 y \\ |\cos z|^2 &= \cos^2 x + \cosh^2 y \end{aligned}$$

3.5 Inverse Trigonometric Functions

The function is $w = f(z) = \sin^{-1} z$. So we have

$$\sin(w) = z = \frac{e^{iw} - e^{-iw}}{2}$$

We know $2iz = (e^{iw} - e^{-iw}) \times e^{iw}$,

$$2ize^{iw} = e^{2iw} - e^0$$

$$e^{iw^2} - 2ize^{iw} - 1 = 0$$

Solving this we get,

$$e^{iw} = iz \pm (1 - z^2)^{\frac{1}{2}}$$