# Real Analysis: HW1

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## Exercise 1.2.2

We need to show that there is no rational number $r$ satisfying $2^r = 3$. Let's assume on the contrary that there exists a rational number $r = \frac{p}{q}$ where $p, q \in \mathbb{Z}$ are coprime and $q > 0$. So we have,

$$\begin{aligned} 2^{p/q} &= 3 \\ 2^p &= 3^q \end{aligned}$$

Here the right hand side is $3^q$ and $q > 0$ so it's a positive integer. This also means that the left hand side must be positive which implies that $p \geq 0$. Now we see that $2^p$ has only 2 as a prime factor and $3^q$ has only 3 as its prime factor. So the only solution to this equation is if bothsides are equal to 1 which is when $p, q = 0$. But this contradicts our assumption that $q > 0$. Hence our assumption must be wrong and there is no rational number $r$ that satisfies the equation.

## Exercise 1.3.3

(a). We have, $A$ is nonempty and bounded below and $B = \{b \in \mathbb{R} : b \text{ is a lower bound for } A\}$. We need to show that $\sup B = \inf A$.
As $A$ is bounded below there exists a infimum say $\inf A = x$. Now as $x$ is the greatest lowerbound we have $x \geq b, \forall b \in B$. This means that $x$ is an upper bound for $B$. We need to now show that $x$ is the smallest upperbound for $B$. Consider for instance there exists an upperbound $y$ such that $b \leq y < x$. As $y < x$ this means that $y \leq a, \forall a \in A$, this means that $y \in B$ as y is a lowerbound for $A$. So we have $y \geq x$ as $x \in B$ and $x \geq y$ as $y \in B$ which means that $x = y$, a contradiction as we assumes $y < x$. Implies that there is no $y < x$ which means that $x$ is the smallest upperbound of $B$.

## Exercise 1.3.8

(a) We have $\{m/n : m, n \in \mathbb{N} \text{ with } m < n\}$. Here suprema is 1 and infima is 0.
(b) $\{(-1)^m/n : m, n \in \mathbb{N}\}$. Here suprema is 1 and infima is -1.
(c) $\{n/(3n+1) : n \in \mathbb{Z})$. Here when $n = 1$ and $-1$ we have minimum and maximum value which is $\frac{-1}{2}$ and $\frac{1}{2}$ which are the infima and suprema respectively.
(d) $\{m/(m+n) : m, n \in \mathbb{Z}\}$. Here suprema and infima don't exist as we can make it arbitrarily large and small.

## Exercise 1.4.1

(a) Given $a, b \in \mathbb{Q}$. We need to show that $ab$ and $a + b$ in $\mathbb{Q}$ as well.
If $a, b \in \mathbb{Q}$ then we have $a = \frac{p_1}{q_1}, p_1, q_1 \in \mathbb{Z}, q_1 > 0$ and $b = \frac{p_2}{q_2}, p_2, q_2 \in \mathbb{Z}, q_2 > 0$.
So we have,

$$\begin{aligned} fab &= \frac{p_1}{q_1} \cdot \frac{p_2}{q_2} \\ &= \frac{p_1 p_2}{q_1 q_2} \end{aligned}$$

Now as $p_1, p_2 \in \mathbb{Z}$ it must mean that $p_1p_2 \in \mathbb{Z}$. And as $q_1, q_2 \in \mathbb{Z}$ and $> 0$ we have $q_1q_2 > 0$. Hence we showed that $ab = \frac{p_3}{q_3}$ where $p_3 = p_1p_2$ and $q_3 = q_1q_2$ such that $p_3, q_3 \in \mathbb{Z}$ and $q_3 > 0$.
Now for $a + b$ we have,

$$\begin{aligned} a + b &= \frac{p_1}{q_1} + \frac{p_2}{q_2} \\ &= \frac{p_1q_2 + p_2q_1}{q_1q_2} \end{aligned}$$

Similar to above we have $p_3 = p_1q_2 + p_2q_1$ and we know that a linear combination of integers is also an integer so $p_3 \in \mathbb{Z}$. WE also have $q_3 = q_1q_2$ and as both $q_1, q_2 > 0$ we have $q_3 = q_1q_2 > 0$ . So we are able to write $a + b = \frac{p_3}{q_3}$ where $p_3, q_3 \in \mathbb{Z}$ and $q_3 > 0$.
(b) We have $a \in Q$ and $t \in I$ we need to show that $a + t \in I$ and $at \in I$ given $a \neq 0$.
Consider to the contrary that $a + t \notin I$. This means that $a + t$ is of form $\frac{p}{q}$ where $p, q \in \mathbb{Z}$ and $q > 0$. So we have,

$$\begin{aligned} a + t &= \frac{p}{q} \\ t &= \frac{p}{q} + (-a) \end{aligned}$$

As per (a) we know that the sum of two rationals is also rational. This implies that $\frac{p}{q} + (-a)$ is rational which implies that $t$ is rational. But this is a contradiction as we know that $t \in I$. Hence our assumption must be wrong and $a + t \in I$.
Now consider that $at \notin I \Rightarrow at \in \mathbb{Q}$. So we have $at = \frac{p}{q}$ for $p, q \in \mathbb{Z}$ and $q > 0$. So,

$$\begin{aligned} at &= \frac{p}{q} \\ t &= \frac{p}{q} \cdot \frac{1}{a} \text{ which is defined as } a \neq 0 \end{aligned}$$

We know from above that product of two rationals is also rational which means that $\frac{p}{q} \cdot \frac{1}{a}$ is rational or that $t$ is rational. A contradiction as we know that $t \in I$ so our assumption must be wrong and $at \in I$.
(c) No, $I$ is not closed under addition and multiplication. For instance, consider the following example where $a = \sqrt{2} + 1$ and $b = 1 - \sqrt{2}$. We have $a + b = 2$. Here $a, b \in I$ but $a + b = 2 \in \mathbb{Q}$ which shows that it is not closed under addition. Now consider $ab = (1 + \sqrt{2})(1 - \sqrt{2}) = 1^2 - \sqrt{2}^2 = -1 \in \mathbb{Q}$. So here we have $a, b \in I$ but $ab \in \mathbb{Q}$ which shows that irrationals are not closed under multiplication either.

## Exercise 1.4.4

We have $a < b$ where $a, b \in \mathbb{R}$ and $T = Q \cap [a, b] = \{x : x \in \mathbb{Q} \text{ and } x \in [a, b]\}$. We need to show that $\sup T = b$. We have to show two things that $b$ is an upper bound and $b$ is the smallest upper bound. Now we know that $\forall x \in [a, b]$ that

$x \leq b$ by definition of the closed interval. And as all $x \in T$ we have $x \in [a, b]$ this means that $\forall x \in T$ we have $x \leq b$. This makes $b$ an upper bound for $T$.

We now have two cases, either $b \in \mathbb{Q}$ or $b \notin \mathbb{Q}$. If $b \in \mathbb{Q}$ then we have $b \geq x, \forall x \in T$ and $b \in T$ which makes $b$ the supremum as if any other strictly smaller upper bound than $b$ exists then it's not a lower bound anymore as $b \in T$ would be greater than it.
Now consider the case where $b \notin \mathbb{Q}$. Let us assume to the contrary that $b$ is not the smallest upperbound and there exists some $q < b$ such that $q \geq x, \forall x \in T$. However, as $q, b \in \mathbb{R}$ we know that there must exist some $a \in \mathbb{Q}$ such that $q < a < b$ because of the density of the rationals in reals. Now as $a < b, \in [a, b]$ and $a \in \mathbb{Q}$ so we have $a \in T$. So we showed that there is some $a \in T$ such that $q < a$ thus making $q$ not an upperbound anymore. So our assumption that $q < b$ where $q \geq x, \forall x \in T$ exists is wrong which must mean that $b$ is the smallest upper bound. Hence, $b$ is the suprema of $T$.
$\frac{x}{y}$