Real Analysis

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Contents

Chapter 1

Introduction

1.1 Logic and proofs

Types of proofs,

1. Direct proof
2. Argument by contradiction
3. Induction
4. Contrapositive (we show $\neg B \Rightarrow \neg A$)

Theorem 1.1. $a = b \Leftrightarrow \forall \varepsilon > 0, |a - b| < \varepsilon$

Proof. 1. To show, $a = b \Rightarrow \forall \varepsilon > 0, |a - b| < \varepsilon$.
Suppose $a = b$ so $|a - b| = 0$. We have $\forall \varepsilon > 0$ so $|a - b| = 0 < \varepsilon$
2. To show, $\forall \varepsilon > 0, |a - b| < \varepsilon \Rightarrow a = b$
Now assume this is not true, or that $a \neq b$ so $a - b \neq 0$ this means that there is a non-zero number k such that $|a - b| = \varepsilon_0$. Now take $\varepsilon = \frac{\varepsilon_0}{2}$. This gives us, $|a - b| = \varepsilon_0 > \varepsilon$ which contradicts the statement. Hence our assumption is false and we prove the results. $\square$

Example (Induction). $x_1 = 1$ and $x_{n+1} = \frac{1}{2}x_n + 1, \forall n \in \mathbb{Z}$. Show $x_{n+1} \geq x_n \forall n \in \mathbb{N}$
Define $S = \{n \in \mathbb{N}, s.t. x_{n+1} \geq x_n\}$ clearly, $S \subseteq N$.
$x_1 = 1$ and $x_2 = \frac{x_1}{2} + 1 = 1.5$. This gives us $x_2 > x_1$ so $1 \in S$
Suppose $n \in S$ and $x_{n+1} \geq x_n$. Note that,

$$x_{n+2} = \frac{1}{2}x_{n+1} + 1$$
$$x_{n+1} = \frac{1}{2}x_n + 1$$

Then $x_{n+2} = \frac{1}{2}x_{n+1} + 1 \geq \frac{1}{2}x_n + 1 = x_{n+1}$ or $x_{n+2} \geq x_{n+1}$ which means $n + 1 \in S$. So by induction we have $S = N$ and $x_{n+1} \geq x_n, \forall n \in \mathbb{N}$ $\diamond$

1.2 Real Numbers

Number systems,

1. Natural numbers $\mathbb{N}$

   $1, 2, 3, \ldots$

   Can't do subtraction

2. Integers $\mathbb{Z}$

   $\ldots, -3, -2, -1, 0, 1, 2, 3 \ldots$

   Can't do division

3. Rationals $\mathbb{R}$

   $\{\frac{p}{q}$ where $p, q \in \mathbb{Z}$ but $q \neq 0\}$

   Now we have $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{R}$

   But other numbers are still not captured,

   Example. $\sqrt{2}$ is not defined in $\mathbb{R}$. However if we define $x_1 = 2$, $x_{n+1} = \frac{1}{2}(x_n + \frac{2}{x_n})$. We know $x_{n+1} \in \mathbb{R}, \forall n \in \mathbb{N}$ (we can then show that $x_n \to \sqrt{2}$). ⋄

   Theorem 1.2. $\sqrt{2}$ is not rational

   Proof. Argue by contradiction □

4. Real numbers $\mathbb{R}$

   We will define $\mathbb{R}$ as $\mathbb{Q}$ with the gaps filled in.

Definition 1.3 (Axiom of completeness)**.** Every non-empty subset of $\mathbb{R}$ that is bounded above has a least upper bound called the supremum.

Let $S \subseteq \mathbb{R}$ and S is bounded above. If there is $u \in \mathbb{R}$ such that $s \leq u, \forall s \in S$ then S is bounded above by u (*Similar for bounded below*)

Definition 1.4 (Least upper bound or supremum)**.** We say $u \in \mathbb{R}$ is the least upper bound for S if,

1. If u is an upper bound for S
2. $u \leq v$ for any other upperbound v of S.

Similar for greatest lower bound or infimum