Probability Theory: HW4

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5.18

We have $F(y) = \alpha F_1(y) + (1-\alpha)F_2(y)$ and $F(x) = \alpha F_1(x) + (1-\alpha)F_2(x)$ and if $y > x$ we get $F(y)-F(x) = \alpha F_1(y)+(1-\alpha)F_2(y)-\alpha F_1(x)+(1-\alpha)F_2(x) = \alpha(F_1(y)-F_1(x))+(1-\alpha)(F_2(y)-F_2(x))$

Now as F_1 and F_2 are distribution functions they both are monotonic non decreasing and hence $F_2(y) - F_2(x) > 0$ and $F_1(y) - F_1(x) > 0$. This gives us,

$$\begin{aligned} F(y) - F(x) &= \alpha(F_1(y) - F_1(x)) + (1-\alpha)(F_2(y) - F_2(x)) \\ &> 0 \end{aligned}$$

which means F is monotonic non decreasing

Now we need to show that as $x \to \infty$ and $x \to -\infty$ we have $F(x) = 1$ and 0 respectively. We have,

$$\lim_{x\to\infty} F_1(x) = 1, \lim_{x\to\infty} F_2(x) = 1$$

So, $\lim_{x\to\infty} \alpha F_1(x) + (1-\alpha)F_2(x) = \alpha + (1-\alpha) = 1$ and similarly as,

$$\lim_{x\to-\infty} F_1(x) = 0, \lim_{x\to-\infty} F_2(x) = 0$$

we have, $\lim_{x\to-\infty} \alpha F_1(x) + (1-\alpha)F_2(x) = \alpha 0 + (1-\alpha)0 = 0$

Now as both F_1 and F_2 are continuous from the right as F is a linear combination of those functions we have that F is continuous from the right as well.

Lastly we have for any $a < b$ that $P(X \leq b) = F(b) = \alpha F_1(b) + (1-\alpha)F_2(b)$ and similarly $P(x \leq a) = F(a) = \alpha F_1(a) + (1-\alpha)F_2(a)$. If $F_1(k) = P_1(X \leq k)$ and $F_2(k) = P_2(X \leq k)$ we get,

$$\begin{aligned} F(b) - F(a) &= \alpha(F_1(b) - F_1(a)) + (1-\alpha)(F_2(b) - F_2(a)) \\ &= \alpha(P_1(a < X \leq b)) + (1-\alpha)(P_2(a < X \leq b)) \\ &= P(a < X \leq b) \end{aligned}$$

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We know that $\mathbb{P}(X \leq x) = F(x) = \int_{-\infty}^{x} f(x)$. So to get the distribution function we have,

$$\begin{aligned} F(x) &= \int_{-\infty}^{x} f(x) \\ &= \int_{-\infty}^{x} 2x \end{aligned}$$

Now for $x < 0$ we have $F(x) = 0$ as $f(x)$ is 0. For $x > 1$ we have the following,

$$\begin{aligned} F(x) &= \int_{-\infty}^{x} f(x) = \int_{-\infty}^{0} f(x) + \int_{0}^{1} f(x) + \int_{1}^{x} f(x) \\ &= 0 + [x^2]_0^1 + 0 \\ &= 1 \end{aligned}$$

and for $0 < x < 1$ we get,

$$F(x) = [x^2]_0^x = x^2$$

So our distribution function is,

$$F(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ x^2 & \text{if } 0 < x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}$$

5.32

We need to show that F continuous. First trivially we for $-\infty < x \leq 0$ that $\frac{1}{2(1+x^2)}$ is continuous as it's a rational function with a positive denominator. Similarly we have for $0 < x < \infty$ that $\frac{1+2x^2}{2(1+x^2)}$ is continuous. So we only need to show that at the point of discontinuity that, $\lim_{x\to 0^-} F(x) = \lim_{x\to 0^+} F(x) = F(0)$. We have $F(0) = \frac{1}{2}$ by definition. And we have,

$$\lim_{x\to 0^-} F(x) = \lim_{x\to 0^-} \frac{1}{2(1+x^2)} = \frac{1}{2}$$
$$\lim_{x\to 0^+} F(x) = \lim_{x\to 0^+} \frac{1+2x^2}{2(1+x^2)} = \frac{1}{2}$$

As both the limits are equal we have that $F(x)$ is continuous through the real line.

Now we find it's density function. We have for $-\infty < x \leq 0$ that $F(x) = \frac{1}{2(1+x^2)}$ so we get,

$$\begin{aligned} f(x) &= \frac{d}{dx}F(x) = \frac{d}{dx}\left(\frac{1}{2(1+x^2)}\right) \\ &= -\frac{x}{(1+x^2)^2} \end{aligned}$$

And for $0 < x < \infty$ we get,

$$\begin{aligned} f(x) &= \frac{d}{dx}F(x) = \frac{d}{dx}\frac{1+2x^2}{2(1+x^2)} \\ &= \frac{x}{(1+x^2)^2} \end{aligned}$$

So we get,

$$f(x) = \begin{cases} \frac{-x}{(1+x^2)^2} & \text{for } -\infty < x \leq 0 \\ \frac{x}{(1+x^2)^2} & \text{for } 0 < x < \infty \end{cases}$$

5.54

(a). We have $A = g(X) = 2X + 5$ so $g^{-1}(A) = \frac{A-5}{2}$ and,

$$\begin{aligned} f_A(a) &= f_X(g^{-1}(a))\frac{d}{da}[g^{-1}(a)] \\ &= \lambda e^{-\lambda(a-5)/2}\frac{1}{2} \\ &= \frac{\lambda}{2}e^{\frac{-\lambda}{2}(a-5)} \end{aligned}$$

So,

$$f_A(a) = \begin{cases} \frac{\lambda}{2}e^{\frac{-\lambda}{2}(a-5)} & a \geq 1 \\ 0 & a < 5 \end{cases}$$

(b). We have $B = g(X) = e^X$ so $g^{-1}(B) = \log B$. So we get,

$$\begin{aligned} f_B(b) &= f_X(g^{-1}(b))\frac{d}{db}[g^{-1}(b)] \\ &= \lambda e^{-\lambda \log b}\frac{1}{b} \\ &= \lambda e^{\log b^{-\lambda}}\frac{1}{b} \\ &= \lambda b^{-(\lambda+1)} \end{aligned}$$

So,

$$f_B(b) = \begin{cases} \lambda b^{-(\lambda+1)} & b \geq 1 \\ 0 & b < 1 \end{cases}$$

(c). We have $C = g(X) = (1+X)^{-1}$ so $\frac{1}{1+X} = C$ and $X = \frac{1}{C} - 1$. This give us,

$$\begin{aligned} f_C(c) &= f_X(g^{-1}(c)) \frac{d}{dc}[g^{-1}(c)] \\ &= \lambda e^{-\lambda\left(\frac{1}{c}-1\right)} \left| -\frac{1}{c^2} \right| \\ &= \frac{\lambda}{c^2} e^{-\lambda\left(\frac{1}{c}-1\right)} \end{aligned}$$

which is,

$$f_C(c) = \begin{cases} \frac{\lambda}{c^2} e^{-\lambda\left(\frac{1}{c}-1\right)} & 0 < c \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(d). $Y = g(X) = (1+X)^{-2}$ so $g^{-1}(Y) = Y^{-1/2} - 1$. And we have $\frac{d}{dy} g^{-1}(Y) = -\frac{1}{2} y^{-3/2}$

$$f_Y(y) = \lambda e^{-\lambda(y^{-1/2}-1)} \frac{1}{2} y^{-3/2}$$

So,

$$f_Y(y) = \begin{cases} \lambda e^{-\lambda(y^{-1/2}-1)} \frac{1}{2} y^{-3/2} & 0 < y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

6.14

We have $\mathbb{P}(a < X \leq b, c < Y \leq d)$. We can write the interval $(a,b] \times (c,d]$ as $(-\infty, b] \times [-\infty, d] - (-\infty, a] \times (-\infty, d] - (-\infty, b] \times (-\infty, c] + (-\infty, a] \times (-\infty, c]$. This gives us,
We have $P(-\infty < X \leq b, -\infty < Y \leq d) = P(-\infty < X \leq b, -\infty < Y \leq d) - P(-\infty < X \leq b, -\infty < Y \leq c) - P(-\infty < X \leq a, -\infty < Y \leq d) + P(-\infty < X \leq a, -\infty < Y \leq c) = F(b,d) - F(a,d) - F(b,c) + F(a,c)$,

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We have,

$$f(x,y) = \begin{cases} e^{-x-y} & x, y > 0 \\ 0 & \text{otherwise} \end{cases}$$

So for $\mathbb{P}(X+Y \leq 1)$ we have $0 < y \leq 1$ then we need $x + y \leq 1$ or $0 < x \leq 1 - y$ which gives us,

$$\begin{aligned}
\mathbb{P}(X+Y \leq 1) &= \int_0^1 \int_0^{1-y} e^{-x-y} dxdy \\
&= \int_0^1 \frac{1}{e^y} \int_0^{1-y} e^{-x} dxdy \\
&= \int_0^1 \frac{1}{e^y} \int_0^{y-1} -e^t dtdy \\
&= \int_0^1 \frac{1}{e^y} \cdot -[e^t]_0^{y-1} dy \\
&= \int_0^1 \frac{1}{e^y} \cdot (1 - e^{y-1}) dy \\
&= \int_0^{-1} -(e^t - e^{-1}) dt \\
&= [-e^t + e^{-1} t]_0^{-1} \\
&= [-\frac{1}{e} - e^{-1}] - [-1] \\
&= 1 - \frac{2}{e}
\end{aligned}$$

For $\mathbb{P}(X > Y)$ we have for any $0 < x$ that y goes from $0 < y < x$ so we get,

$$\begin{aligned}
\mathbb{P}(X > Y) &= \int_0^\infty \int_0^x e^{-x-y} dydx \\
&= \int_0^\infty \frac{1}{e^x} \int_0^x e^{-y} dydx \\
&= \int_0^\infty \frac{1}{e^x} \int_0^{-x} -e^t dtdx \\
&= \int_0^\infty \frac{1}{e^x} [-e^t]_0^{-x} dx \\
&= \int_0^\infty \frac{1}{e^x} (1 - e^{-x}) dx \\
&= \int_0^\infty \frac{1}{e^x} (1 - e^{-x}) dx \\
&= \int_0^\infty e^{-x} dx - \int_0^\infty e^{-2x} dx \\
&= 1 - \frac{1}{2} = \frac{1}{2}
\end{aligned}$$

Problem 7

We have,

$$\begin{aligned}\int_0^\infty [1 - F_X(x)]dx &= \int_0^\infty [1 - P(X \leq x)]dx \\ &= \int_0^\infty P(X > x)dx \\ &= \int_0^\infty \int_x^\infty f_X(t)dtdx \\ &= \int_0^\infty \int_0^t f_X(t)dxdt \\ &= \int_0^\infty [xf_X(t)]_0^t dt \\ &= \int_0^\infty tf_X(t)dt \\ &= \int_0^\infty xf_X(x)dx \\ &= E(X)\end{aligned}$$

Problem 11

Let X be the r.v that determines that angle. We have x goes from $[0, 2\pi]$ and the field of view is π. As he chooses it uniformly we have,

$$f_X(x) = \begin{cases} \frac{1}{\pi} & x \in [0, \pi] \\ 0 & \text{otherwise} \end{cases}$$

Now we know that given $X = x$ we have the horizontal distance from the center is $\left|\frac{1}{\tan x}\right|$. So if Y is the r.v for the horizontal distance then we get $Y = \left|\frac{1}{\tan X}\right|$ so $|\tan X| = \frac{1}{Y}$ and $X = tan^{-1}[\frac{1}{Y}]$ for $x \in (0, \pi/2)$ and $\pi - tan^{-1}[\frac{1}{Y]}$ for $x \in (\pi/2, \pi)$. Now consider

$$\begin{aligned}f_Y(y) &= f_X(g^{-1}(y)\frac{d}{dy}g^{-1}(y) \\ &= \frac{1}{\pi}\frac{1}{1 + (\frac{1}{y})^2}\frac{1}{y^2} \\ &= \frac{1}{\pi}\frac{1}{1 + y^2}\end{aligned}$$

Which gives us $F_Y(y) = \int_0^y f_y(y) = \frac{1}{\pi}\int_0^y \frac{1}{1+y^2}dy = \frac{1}{\pi}[tan^{-1}(y)]_0^y = \frac{1}{\pi}tan^{-1}(y)$