Intro to Proofs: HW09

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Section 12.5

Problem 2

We have $f : \mathbb{R} - \{2\} \to \mathbb{R} - \{5\}$ defined by,

$$f(x) = \frac{5x+1}{x-2}$$

We know that it is bijective hence the inverse exists, we have,

$$\begin{aligned} y &= \frac{5x+1}{x-2} \\ yx - 2y &= 5x + 1 \\ -1 - 2y &= 5x - yx \\ -(1+2y) &= x(5-y) \\ x &= -\frac{1+2y}{5-y} \end{aligned}$$

when $y \in \mathbb{R} - \{5\}$
So we have,

$$f^{-1}(y) = -\frac{1+2y}{5-y}$$

Problem 4

The function $f : \mathbb{R} \to (0, \infty)$ is defined as $f(x) = e^{x^3+1}$ is bijective.
So we have,

$$\begin{aligned} y &= e^{x^3+1} \\ \ln(y) &= ln(e^{x^3+1} \\ \ln(y) &= x^3 + 1 \\ \ln(y) - 1 &= x^3 \\ x &= (\ln(y) - 1)^{\frac{1}{3}} \end{aligned}$$

where $y \in (0, \infty)$
So we have,

$$f^{-1}(y) = (\ln y - 1)^{\frac{1}{3}}$$

Problem 8

Our function takes any $X \in P(\mathbb{Z})$ and maps it to $\overline{X} \in P(\mathbb{Z})$.
The function is bijective as it is injective and surjective. It is injective because if $\theta(X_1) = \theta(X_2)$. This means that $\overline{X_1} = \overline{X_2} \Rightarrow X_1 = X_2$ which means its injective.
Now it is surjective because for any Y in the co domain we can find $X = \overline{Y}$ in the domain such that $\overline{X} = \overline{\overline{Y}} = Y$. This shows surjectivity.
Hence it is bijective and inverse exists. So,

$$Y = \overline{X}$$
$$\overline{Y} = \overline{\overline{X}}$$
$$X = \overline{Y}$$

So we have a function $\theta^{-1}(Y) = \overline{Y}$ which is the inverse of our function.

12.6

Problem 5

Proof. We have a function $f : A \to B$ and a subset $X \subseteq A$. We need to show that $X \subseteq f^{-1}(f(X))$
Essentially we show that $x \in X \Rightarrow x \in f^{-1}(f(X))$
Now if $x \in X$ then $x \in A$ so for any $x \in A$ we have $f(x) \in f(X) \subseteq B$. Now by definition of inverse we have $f^{-1}(f(X)) = \{x \in A : f(x) \in f(X)\}$. So as $f(x) \in f(X)$ we have $x \in f^{-1}f(X)$ which gives us $X \subseteq f^{-1}(f(X))$ $\square$

Problem 6

Proof. Consider the function defined from $A = \{1\}$ to $B = \{a, b\}$. Let $f(1) = b$.
Now let $Y = B$. So we have $f^{-1}(Y) = \{x \in A : f(x) \in Y\} = \{1\}$. However $f(f^{-1}(Y)) = \{a\} \neq Y$ $\square$

Problem 8

Proof. Consider the function f defined from $A = \{1, 2, 3\}$ to $B = \{a, b\}$. Such that,

$$f(1) = b, f(2) = a, f(3) = b$$

Now let $W = \{1, 2\}$ and $X = \{2, 3\}$. First we have $W \cap X = \{2\}$ and $f(W \cap X) = \{a\}$
However consider $f(W) = \{a, b\}$ and $f(X) == \{a, b\}$ which means that $f(W) \cap f(X) = \{a, b\}$
We see that this is not equal to the set above. $\square$

Problem 9

Proof. We need to show $f(W \cup X) = f(W) \cup f(X)$. First let $y \in f(W \cup X)$. This means that $y \in B$ such that $\exists x \in W \cup X$ and $f(x) = y$. So we have $x \in W$ or $x \in X$. If $x \in X$ then by definition we know that $f(x) \in f(X)$ and similarly if $x \in W$ then $f(x) \in f(W)$. So we have either $y \in f(X)$ or

$yf(W) \Rightarrow y \in f(X) \cup f(W)$. This shows us that $f(W \cup X) \subseteq f(W) \cup f(X)$. Now consider $y \in f(W) \cup f(X)$ this means that either $y \in f(W)$ or $y \in f(X)$. If $y \in f(W)$ then $\exists x \in W \subseteq A$ such that $f(x) = y$. If $x \in W$ then $x \in W \cup X$. So we have $y \in f(W)$ implies $\exists x \in W \cup X$ such that $f(x) = y$ this means that $y \in f(W \cup X)$. Similarly if $y \in f(X)$ we have $\exists x \in X$ such that $f(x) = y$, but $x \in X \Rightarrow x \in X \cup W$ so we have $y \in f(X \cup W)$. So we have shown that $y \in f(W) \cup f(X) \Rightarrow y \in f(W \cup X) \Rightarrow f(W) \cup f(X) \subseteq f(W \cup X)$ □

Problem 10

Proof. We need to show $f^{-1}(Y \cap Z) = f^{-1}(Y) \cap f^{-1}(Z)$
Consider $x \in f^{-1}(Y \cap Z)$. This means that $\exists y \in Y \cap Z$ such that $f(x) = y$. Now if $y \in Y \cap Z$ means that $y \in Y$ and $y \in Z$. So we have,
(1). x such that $f(x) = y$ where $y \in f^{-1}(Y)$ which implies that $x \in f^{-1}(Y)$.
(2). x such that $f(x) = y$ where $y \in f^{-1}(Z)$ which implies that $x \in f^{-1}(Z)$.
Both these imply that $x \in f^{-1}(Y) \cap f^{-1}(Z)$ which imply,

$$f^{-1}(Y \cap Z) \subseteq f^{-1}(Y) \cap f^{-1}(Z)$$

Now assume $x \in f^{-1}(Y) \cap f^{-1}(Z)$. This means that $x \in f^{-1}(Y)$ and $y \in f^{-1}(Z)$
(1). $x \in f^{-1}(Y)$ then this means $\exists y \in Y$ such that $f(x) = y$
(2) $x \in f^{-1}(Z)$ means that $\exists y \in Z$ such that $f(x) = y$
Now as both these are true we have $y \in Y$ and $y \in Z$ which imply that $y \in Y \cap Z$. So we have $f(x)$ such that $f(x) = y \in Y \cap Z$ which is equivalent to saying $x \in f^{-1}(Y \cap Z)$ which implies,

$$f^{-1}(Y) \cap f^{-1}(Z) \subseteq f^{-1}(Y \cap Z)$$

So we show that,

$$f^{-1}(Y) \cap f^{-1}(Z) = f^{-1}(Y \cap Z)$$

□

Problem 11

Proof. We need to show that $f^{-1}(Y \cup Z) = f^{-1}(Y) \cup f^{-1}(Z)$.
First take $x \in f^{-1}(Y \cup Z)$. This for some $y \in Y \cup Z$ we have $f(x) = y$. S o we have two cases $y \in Y$ or $y \in Z$.
(1). If $y \in Y$ then this means that we have x such that $f(x) = y \in Y$. Or $x \in f^{-1}(Y)>$
(2). If $y \in Z$ then this means that we have x such that $f(x) = y \in Z$. Or $x \in f^{-1}(Z)>$
So either way we can say that $x \in f^{-1}(Y) \cup f^{-1}(Z)$ which implies that,

$$f^{-1}(Y \cup Z) \subseteq f^{-1}(Y) \cup f^{-1}(Z)$$

Now consider $x \in f^{-1}(Y) \cup f^{-1}(Z)$. So either, $x \in f^{-1}(Y)$ or $x \in f^{-1}(Z)$.
(1). If $x \in f^{-1}(Y)$ then $f(x) = y \in Y$.
(2). If $x \in f^{-1}(Z)$ then $f(x) = y \in Z$
So we have either $f(x) \in Y$ or $f(x) \in Z$ which means that $f(x) \in Y \cup Z$. Now by definition we have $x \in f^{-1}(Y \cup Z)$ this implies that,

$$f^{-1}(Y) \cup f^{-1}(Z) \subseteq f^{-1}(Y \cup Z)$$

This implies that,

$$f^{-1}(Y) \cup f^{-1}(Z) = f^{-1}(Y \cup Z)$$

□

Problem 12

Proof. (1). We need to show that,

$$f \text{ is injective} \Leftrightarrow X = f^{-1}(f(X))$$

(a). $\Rightarrow$
First consider $x \in X$. We have $f(X) = Y = \{f(x) : x \in X\}$. Now $f^{-1}(Y)$ is defined as $\{x : f(x) \in Y\}$. But for any $x \in X$ we have $f(x) \in Y$ by definition, hence $x \in X$ implies that $x \in f^{-1}(f(X))$.
Now consider $a \in f^{-1}f(X)$ so a is in the set $\{x : f(x) \in f(X)\}$. However $f(X)$ is defined as $\{f(x) : x \in X\}$. So if $f(a) \in f(X)$ then it means that we have $x \in X$ such that $f(x) = f(a)$. But because f is injective we have $x = a$ which means that $a \in X$. Hence we can say that, $X = f^{-1}(f(X))$
(b). $\Leftarrow$
We have $X = f^{-1}(f(X))$ and we need to show that f is injective. Assume that f is not injective. That means $\exists x_1 \neq x_2$ such that $f(x_1) = f(x_2)$. Now consider a subset X such that $x_1 \in X$ and $x_2 \notin X$. Now because $x \in X$ we have $f(x_1) \in f(X)$ but this also means that $f(x_2) \in f(X)$. Now by definition of f^{-1} we have $\{x : f(x) \in f(X)\}$ and as $f(x_1) = f(x_2) \in f(X)$ we have $x_1, x_2 \in f^{-1}(f(X))$ which means that $x_2 \in X$ but this contradicts the fact that $x_2 \notin X$ assumption hence our assumption must be wrong and f is injective.
(2). Now we show that,

$$f \text{ is surjective} \Leftrightarrow f(f^{-1}(Y)) = Y$$

(a). $\Rightarrow$
Assume f is surjective. Now consider $y \in f(f^{-1}(Y))$. By definition that means $\exists x$ such that $f(x) = y$ and that $x \in f^{-1}(Y)$. Now by definition of the inverse this set is such that $\{x : f(x) \in Y\}$ so $x \in f^{-1}(Y) \Rightarrow f(x) = y \in Y$. So we have $f(f^{-1}(Y)) \subseteq Y$
Now consider $y \in Y$. Because f is surjective we have $\exists x$ such that $f(x) = y$ which means that $x \in f^{-1}(Y)$. Now by definition of f we have the set $\{f(x) : x \in f^{-1}(Y)\}$ so $f(x)$ such that $x \in f^{-1}(Y)$. But we have $f(x) = y$ which means that $y \in f(f^{-1}(Y))$. Which gives us, $Y \subseteq f(f^{-1}(Y))$

(b) $\Leftarrow$
Now assume f is not surjective which means that, $\exists y \in B$ such that there is not $x \in A$ such that $f(x) = y$. Now now consider $Y = \{y\}$. But because there is no x we have $f^{-1}(Y) = \{\}$ which means that $f(f^{-1}(Y)) = \{\} \neq Y$. Hence f has to be surjective. $\square$

14.1

Problem 2

Proof. Consider the bijection g defined from $\mathbb{R}$ such that

$$g(x) = 2^x + \sqrt{2}$$

.
We can show that it is a bijection because its injective and surjective. First because if we have $2^{x_1} + \sqrt{2} = 2^{x_2} + \sqrt{2}$ then,

$$2^{x_1} = 2^{x_2}$$

$$log_2(2^{x_1}) = log_2(2^{x_2})$$

$$x_1 = x_2$$

Hence it is injective.
Now for any $y \in (\sqrt{2}, \infty)$ we have,

$$y = 2^x + \sqrt{2}$$

$$2^x = y - \sqrt{2} > 0$$

$$x = \log_2(y - \sqrt{2})$$

which we know is defined because the inside term is always positive. $\square$

Problem 13

First consider the bijective function $f : \mathbb{N} \to \mathbb{Z}$ that maps every number in N to an element in $\mathbb{Z}$ as follows,

$$f(x) = \frac{(-1)^n(2n-1)+1}{4}$$

We can verify this is a bijective function.
Now consider a function $g : P(N) \to P(Z)$ that maps a set from $P(N)$ to $P(Z)$ defined as,

$$g(X) = \{f(x) : x \in X\}$$

We know that $f(x) \in Z$ so any set $g(X) \in P(Z)$
Now we show this is a bijective function.
First consider two sets X_1, X_2, we need to show that $g(X_1) = g(X_2) \Rightarrow X_1 = X_2$.

We have,

$$\{f(x_1) : x_1 \in X_1\} = \{f(x_2) : x_2 \in X_2\}$$

First consider $x_1 \in X_1$ this means that $f(x_1) \in g(X_1)$. Now because the sets are equal means $\exists x_2 \in X_2$ such that $f(x_2) = f(x_1)$. However because f is injective we have $x_1 = x_2$ or $x_1 \in X_2$. This means that $X_1 \subseteq X_2$
Now we can similarly show that $X_2 \subseteq X_1$ which implies that $X_1 = X_2$
Now we need to show that g is surjective.
Consider an arbitrary Y in $P(Z)$. We need to show there is an $X \in P(N)$ such that $g(X) = Y$.
We know that because f is surjective, for any $y \in Y, \exists x \in N$ such that $f(x) = y$. Hence we define,

$$X = \{x : f(x) \in Y\}$$

Now because of how we define X we have,

$$g(X) = \{f(x) : x \in X\}$$

but $x \in X$ such that $f(x) \in Y$. Hence if $y \in g(X)$ then $\exists x \in X$ such that $f(x) = y$. But this means that $x \in X$ which implies that $f(x) \in Y$ or $y \in Y$ which shows that $f(x) \subseteq Y$.
Similarly, if $y \in Y$ we have $x \in X$ such that $f(x) = y$. But based on how $g(X)$ is defined we have $f(x)$ if $x \in X$ but $f(x) = y$ so $y \in g(X)$ hence $Y \subseteq g(X)$ or $g(X) = Y$
This shows surjection. So we have defined a bijective function fro $P(N)$ to $P(Z)$ showing their cardinality is the same.

Problem 15

Consider the function $f : \mathbb{N} \to \mathbb{Z}$ defined as,

$$f(n) = \frac{(-1)^n(2n-1)+1}{4}$$

First we show that it is a bijection.
If we have,

$$\frac{(-1)^{n_1}(2n_1-1)+1}{4} = \frac{(-1)^{n_2}(2n_2-1)+1}{4}$$

We can write this as,

$$(-1)^{n_1}(2n_1-1) = (-1)^{n_2}(2n_2-1)$$

We have $2n_1 - 1$ is always positive as $n \geq 1$ which means that for the signs to be the same we need $(-1)^{n_1} = (-1)^{n_2}$ which is only true if we have $n_2 = n_1 + 2k$ for some $k \in \mathbb{Z}$. So now consider,

$$2n_1 - 1 = 2n_2 - 1$$

$$2n_1 - 1 = 2n_1 + 4k - 1$$

$$4k = 0, k = 0$$

Hence $n_1 = n_2$

Now to show it surjective for any $z \in Z$. Consider by cases first z is positive, then we have,

$$n = 2z$$

and if z is negative or zero consider,

$$n = 1 - 2z$$

So for any $z \in Z$ we have an $n \in N$ such that $f(n) = z$.