Real Analysis: HW8

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1 Exercise 4.4.1

(a) We have $f(x) = x^3$. Now we need for any arbitrary point $x_0 \in \mathbb{R}$ we have for any $\varepsilon > 0$ a $\delta > 0$ such that if $|x - x_0| < \delta$ then we get $|f(x) - f(x_0)| < \varepsilon$.

First for an arbitrary x_0 let, $M = (1 + x_0)^2 + x_0^2 + |x_0(1 + x_0)|$ and take $\delta = \min\{\frac{\varepsilon}{M}, 1\}$. Then we get the following,

$$\begin{aligned} |x^3 - x_0^3| &= |(x - x_0)(x^2 + x_0^2 + xx_0)| \\ &= |x - x_0||(x^2 + x_0^2 + xx_0| \end{aligned}$$

Now as we have $|x - x_0| < \delta = \min \frac{\varepsilon}{M}, 1$ then we have $|x - x_0| < 1$. But this can be written as $x_0 - 1 \leq x \leq 1 + x_0$ and this gives us $|x^2 + x_0^2 + xx_0| \leq |(1 + x_0)^2 + x^2 + (1 + x_0)x_0| \leq M$. So we have,

$$\begin{aligned} |x^3 - x_0^3| &\leq |x - x_0||x^2 + x_0^2 + xx_0| \\ &\leq |x - x_0|M \end{aligned}$$

Now as we have $|x - x_0| \leq \frac{\varepsilon}{M}$ we have,

$$\begin{aligned} |x^3 - x_0^3| &\leq |x - x_0|M \\ &\leq \frac{\varepsilon}{M}M = \varepsilon \end{aligned}$$

So for any x_0 if we define M as above then for any ε by choosing $\delta = \min\{\frac{\varepsilon}{M}, 1\}$ we get $|f(x) - f(x_0)| < \varepsilon$ which makes f continuous on all points in R.

(b). Choose $\varepsilon = 1$ and let the sequence be $(x_n) = n$ and $(y_n) = n + \frac{1}{n}$. Then we have $|x_n - y_n| = |\frac{1}{n}| \to 0$. But now we get,

$$\begin{aligned} |f(x_n) - f(y_n)| &= \left|(n + \frac{1}{n})^3 - n^3\right| \\ &= \left|n^3 + \frac{1}{n^3} + \frac{3n}{n^2} + \frac{3n^2}{n} - n^3\right| \\ &= \left|3n + \frac{1}{n^3} + \frac{3}{n}\right| \\ &\geq |3n| \end{aligned}$$

Now as $n \to \infty$ we have $3n$ is unbounded and hence we get $|f(x_n) - f(y_n)| \geq |3n| \geq \varepsilon = 1$. So $f(x) = x^3$ is not uniformly continuous in R.

(c). Consider any bounded subset of R. So we have $|x| \leq M$ for some $M > 0$. Now note that for any point in this subset we have $x_0 \leq M$ as well. So now let $M_0 = (1+x_0)^2 + x_0^2 + |x_0(1+x_0)| \leq (1+M)^2 + M^2 + M(1+M)$ and we can choose $\delta = \min\{\frac{\varepsilon}{M_0}, 1\}$.

Note that for any point x_0 we have M_0 is independent of x_0 or x, i.e. M_0 is a constant given the subset. Hence, we have as $|x - x_0| \leq 1$ and $x \leq M$ which gives us $|x^2 + x_0^2 + xx_0| \leq M_0$ and similarly as $|x - x_0| < \frac{\varepsilon}{M_0}$ we get $|x^3 - x_0^3| \leq |x - x_0||x^2 + x_0^2 + xx_0| \leq \frac{\varepsilon}{M_0} M_0 = \varepsilon$ and hence we found a fixed δ that works with any x_0 in the subset.

2 Exercise 4.4.2

(a). $\frac{1}{x}$ is not uniformly continuous on $(0,1)$. Consider the sequence $(x_n) = \frac{1}{n+1}$ and $(y_n) = \frac{1}{n+2}$. We have $|x_n - y_n| = \frac{1}{n+1} - \frac{1}{n+2} = \frac{1}{(n+2)(n+1)} \to 0$. But we see that $|f(x_n) - f(y_n)| = |n+2-n-1| = 1$. So if we choose $\varepsilon_0 = .5$ then we have $|f(x_n) - f(y_n)| \geq \varepsilon_0$ and hence $\frac{1}{x}$ is not uniformly continuous.
(b). Is uniformly continuous on $(0,1)$. We can see this as follows,

$$\begin{aligned}
|f(x) - f(x_0)| &= \left|\sqrt{x^2+1} - \sqrt{x_0^2+1}\right| \\
&= \left|\frac{x^2 - x_0^2}{\sqrt{x^2+1} + \sqrt{x_0^2+1}}\right| \\
&= \left|\frac{(x+x_0)(x-x_0)}{\sqrt{x^2+1} + \sqrt{x_0^2+1}}\right|
\end{aligned}$$

Now note that in $(0,1)$ we have $(x + x_0)$ is bounded above by 2. And we also have $|\sqrt{x^2+1} + \sqrt{x_0^2+1}| \geq |\sqrt{1} + \sqrt{1}| \geq 2$ i.e it's bounded below by 2 and hence we can write,

$$\left|\frac{(x+x_0)(x-x_0)}{\sqrt{x^2+1} + \sqrt{x_0^2+1}}\right| \leq |x - x_0|$$

Hence, we just need to take $\delta = \varepsilon$

(c) We have $x \sin(\frac{1}{x})$. Define $h(x) = 0$ if $x = 0$ then we get that h is continuous on $[0,1]$ but we have $[0,1]$ is a compact set and hence it is uniformly continuous on $[0,1]$ which means that it is uniformly continuous on the interval $(0,1)$ as well.

3 Exercise 4.4.11

We have $B \subset R$ and $g^{-1}(B) = \{x \in \mathbb{R} : g(x) \in B\}$.

($\Rightarrow$) We have g is continuous which means that for any $x_0 \in \mathbb{R}$ for all $\varepsilon > 0$ we have a $\delta > 0$ such that $|x - x_0| < \delta \Rightarrow |g(x) - g(x_0)| < \varepsilon$. Now consider an arbitrary open set $O \subset \mathbb{R}$ so we have,

$$g^{-1}(O) = \{x \in \mathbb{R}, g(x) \in O\}$$

we need to show that this is open. First consider some $o \in g^{-1}(O)$ which maps to $g(o) \in O$. As O is open there is some $\varepsilon-$neighborhood of $g(o)$ which is contained within O. So for z such that $|g(o) - z| < \varepsilon$ is a subset of O. Now as g is continuous choose ε and we delta such that for $|o - x| < \delta$ we get $|g(o) - g(x)| < \varepsilon$. So now in O for the value such that $g(x) = z$ we have for the $x \in g^{-1}(O)$ is in the δ neighborhood of o and hence is a subset of $g^{-1}(O)$. So for any value $o \in g^{-1}(O)$ we found a delta neighborhood that is also in the set which means that it's open.

($\Leftarrow$) Consider the contrary that for any open subset of R the preimage is also open. So consider an arbitrary point in $g^{-1}(O)$ say x_0. Now we have $g(x_0) \in O$ and has an ε neighborhood in O as its open. Now consider this open subset of O, i.e. the set $(g(x_0) - \varepsilon, g(x_0) + \varepsilon)$, note that the preimage of this is a subset of $g^{-1}O$ call S and is open as well. Now since S is open there is a delta neighborhood for x_0 that is contained within S. So now for that δ we have for all points $|x - x_0| < \delta$ in S that $|g(x) - g(x_0)| < \varepsilon$ in O and hence for an arbitrary x_0 for any $\varepsilon > 0$ we found a δ such that for any x in the delta neighborhood of x_0 we have $f(x_0)$ in the ε neighborhood of $f(x)$.