Real Analysis: HW11

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Exercise 6.4.2

(a). True. If $\sum_{n=1}^{\infty} g_n$ converges uniformly, then using the Cauchy Criterion for Uniform convergence of series we have $\forall \varepsilon > 0$ there is some $N \in \mathbb{N}$ such that for $n > m \geq N$ we have,

$$|g_{m+1} + \cdots + g_n| < \varepsilon$$

Now choose $m = n - 1$ then we have,

$$|g_n| < \varepsilon$$

for $n > N$ which is equivalent to saying $(g_n) \to 0$.

(b). True. By Weierstrass M-Test we have if $|f_n(n)| \leq M_n$ and $\sum M_n$ converges we also have $\sum f_n$ converges uniformly on A. In this cas ewe have $0 \leq f_n(x) \leq g_n(x)$ now as they're both positive we satisfy the inequality $|f_n| \leq g_n$ and we have $\sum g_n$ converges so $\sum f_n$ converges uniformly.

(c). False.

Exercise 6.4.5

(a). We have,

$$h(x) = \sum_{n=1}^{\infty} h_n = \sum \frac{x^n}{n^2} = x + \frac{x^2}{4} + \dots$$

Now we will have $h(x)$ is continuous if we show that $\sum h_n$ converges to h uniformly and each h_n is continuous. First clears we have h_n is continuous on $[-1, 1]$ as we have $h_n = \frac{x^n}{n^2}$ and x^n is obviously continuous when $n \geq 1$. Now we show uniform convergence. Note first that because our domain is $[-1, 1]$ we have for each n that $|h_n(x)| = \left|\frac{x^n}{n^2}\right| \leq \frac{1}{n^2}$. Now let $\frac{1}{n^2} = M_n$. So we have $|h_n(x)| \leq M_n$ for all n. And we also have $\sum \frac{1}{n^2}$ converges. Hence, we have uniform convergence of $\sum h_n$.

Now uniform convergence to h and continuity of each h_n ensures that h is continuous as well.

(b). Let $x_0 \in (-1, 1)$. We need to show continuity of f at x_0. First note we can find a point $y \in (-1, 1)$ such that $-y < |x_0| < y$. Now note that this gives us $x_0^n \leq a^n$ and thus $\frac{x_0^n}{n} \leq \frac{a^n}{n}$. So if we choose $M_n = \frac{a^n}{n}$ we have $|f_n| \leq M_n$. Now further note that as we have $|y| < 1$. This means we have $\sum y_n$ converges as x^n exponentially decrease faster than the denominator. Hence in that subdomain

from $[-y, y]$ we have uniform convergence of $\sum f_n$ meaning we have continuity for f.
Exercise 6.5.11 (a). We are given a series that converges to a limit L say g_n. So we have,

$$\sum^{\infty} g_n$$

converges to L. Which is the same as saying $\sum g_n x^n$ converges to L at $x = 1$. This means that using Abel's Theorem we have $g(x) = \sum_{n=0}^{\infty} g_n x^n$ converges uniformly in the in interval $[0, 1)$. But this also means that we have $g(x)$ to be continuous in this domain and hence we have the series is Able-summable to L.

(b). We have $\sum_{n=0}^{\infty}(-1)^n$. We need to consider $\sum_{n=0}^{\infty}(-1)^n x^n$ for all $x \in [0, 1)$. Now note that as $0 \leq x < 1$ we also have $(x^n) \to 0$ and by alternating series test we have that the series converges to $\frac{1}{1+x}$ as we get,

$$\begin{aligned} S &= 1 - x + x^2 - x^3 + \dots \\ Sx &= x - x^2 + x^3 + \dots \\ S(1+x) &= 1 \\ S &= \frac{1}{1+x} \end{aligned}$$

and we have this function converges to $\frac{1}{2}$ for $x \to 1^-$.