# Complex Analysis

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# Contents

# Chapter 1

# Complex Numbers

## 1.12 Regions in the Complex Plane

**Definition 1.1** (Epsilon neighborhood). An epsilon neighborhood around a point $z_0$ is the set of all $z$ such that,

$$|z - z_0| < \varepsilon$$

**Definition 1.2** (Deleted neighborhood). A deleted neighborhood around a point $z_0$ is the set of all $z$ such that,

$$0 < |z - z_0| < \varepsilon$$

**Remark.** A deleted neighborhood is essentially an epsilon neighborhood but does not include the point $z_0$

**Definition 1.3** (Interior point). $z_0$ is an interior point when there exists a neighborhood of $z_0$ that contains only points of $S$

**Definition 1.4** (Exeterior point). $z_0$ is an exterior point when there exists a neighborhood of $z_0$ that contains no points of $S$

**Definition 1.5** (Boundary point). $z_0$ is a boundary point otherwise, i.e. all of the neighborhoods of $z_0$ contains a point in $S$ and a point not in $S$

**Definition 1.6** (Open set). $S$ is an open set if $\forall z \in S, \exists \varepsilon$ s.t. $B_\varepsilon(z) \subset S$

**Remark.** We can also say that an open set does not contain any of its boundary points.

**Definition 1.7** (Closed set). A set is closed if it doesn't contain its boundary points.

**Definition 1.8** (Connected Set). An open set is connected if $z_1, z_2$ can be joined by a polygonal line, consisting of finite number of line segments, joined end to end.

**Definition 1.9** (domain). A non empty open set that is connected is called a domain

**Definition 1.10** (region). A domain together with some, none, or all of its boundary points is referred to as a region

**Definition 1.11** (accumulation point). An accumulation point or limit point of a set $S$ is $z_0$ if, each deleted neighborhood of $z_0$ contains at least one point of $S$

**Remark.** A closed set contains all of its accumulation points, but the opposite may not be true.

**Remark.** Every boundary point is not an accumulation point.

**Example.** Consider the set, $S = 5 \cup (0, 1)$
Here, the boundary points are $5, 0$ and $1$ because they $\varepsilon$-neighborhood defined around these points contains both inerior points and exterior points.
However 5 is not an accumulation point because the deleted-neighborhood does not contain any interior points (as it removes 5 ). $\diamond$

# Chapter 2

# Analytic functions

## 13. Functions and Mappings

A translation translate a complex number to another location preserving direction and magnitude.

**Example.** $f(z) = z_0 + z$ ◇

A rotation rotates the complex number changing magnitude or direction.

**Example.** $f(z) = z_0 z$ This function rotates $z$ by multiplying it with $z_0$. We can see this when representing it in euler notation as follows,

$$z_0 z = r r_0 e^{i(\theta+\theta_0)}.$$

◇

**Example.** $f(z) = z^2$

$$z = re^{i\theta}$$
$$z^2 = r^2 e^{2i\theta}$$

So magnitude is squared and angle is doubled ◇

A reflection will reflect $z$ along the $x$ axis.

**Example.** $f(z) = \bar{z}$ reflects $z$ along the $x$ axis. ◇

An analytic function is a differentiable function in the complex space.

$$f(z) = w.$$

$$f(x+iy) = u + iv.$$
$$= u(x,y) + iv(x,y).$$
$$u(z) = iv(z).$$

## 15. Limits

If a function $f$ is defined at all points $z$ in some deleted neighborhood of point $z_0$. Then, $f(z)$ has a limit $w_0$ as $z$ approaches $z_0$, or

$$\lim_{z \to z_0} f(z) = w_0.$$

Essentially this means that the point $w = f(z)$ can be made arbitrary close to $w_0$ if we choose a point $z$ close enouhg to $z_0$ but distinct from it (deleted neighborhood).

**Definition 2.1** (Limit). The limit of a function $f(z)$ as $z$ goes to $z_0$ is $w_0$ if, $\forall \varepsilon > 0, \exists \delta > 0, s.t.$

$$|f(z) - w_0| < \varepsilon \text{ whenever, } 0 < |z - z_0| < \delta.$$

**Remark.** Essentially this menas that for every $\varepsilon$-neighborhood, $|f(z) - w_0| < \varepsilon$ there is a deleted-neighborhood, $0 < |z - z_0| < \delta$ of $z_0$ such that every point $z$ in it has an image $w$ in the $\varepsilon$-neighborhood

**Remark.** All points in the deleted-neighborhood are to be considered but their images need not fill up the $\varepsilon$-neighborhood

**Theorem 2.2.** When a limit of a function $f(z)$ exists at a point $z_0$, it is unique.

**Proof.** Suppose,

$$\lim_{z \to z_0} f(z) = w_0 \text{ and } \lim_{z \to z_0} f(z) = w_1.$$

This means that,

$$|f(z) - w_0| < \varepsilon \text{ when } 0 < |z - z_0| < \delta_0.$$

$$|f(z) - w_1| < \varepsilon \text{ when } 0 < |z - z_1| < \delta_1.$$

So,

$$|f(z) - w_0| + |f(z) - w_1| < 2\varepsilon.$$

We know that,

$$w_1 - w_0 = (f(z) - w_0) - (f(z) - w_1) \leq |f(z) - w_0| - |f(z) - w_1|$$

So,

$$w_1 - w_0 < 2\varepsilon, \text{where } \varepsilon \text{ can be chosen arbitrary small.}$$

Hence,

$$w_1 - w_0 = 0, \text{or, } w_1 = w_0.$$

□

**Example.** Show that, $f(z) = \frac{i\bar{z}}{2}$ in the open disk $|z| < 1$, then

$$\lim_{z \to 1} f(z) = \frac{i}{2}$$

$$\left| f(z) - \frac{i}{2} \right| = \left| \frac{i\bar{z}}{2} - \frac{i}{2} \right| = \frac{|z-1|}{2}.$$

Hence, for any $z$ and $\varepsilon$,

$$\left| f(z) - \frac{i}{2} \right| < \varepsilon \text{ when } 0 < |z-1| < 2\varepsilon.$$

$\diamond$

**Example.** $f(z) = \frac{z}{\bar{z}}$ The limit,

$$\lim_{z \to 0} f(z).$$

does not exist.
Assume that it exists, that implies that by letting the point $z = (x, y)$ we can approach the point, $(0, 0)$ in any manner and we would get the same limit. Now if we approach the point from the $x-$axis where $z = (x, 0)$ we get,

$$\lim_{x \to 0} f((x, 0)) = \frac{x + 0i}{x - 0i} = 1.$$

But if we approach it from the $y-$ axis where, $z = (0, y)$ we get,

$$\lim_{y \to 0} f((0, y)) = \frac{0 + iy}{0 - iy} = -1.$$

But we know that the limit should be unique, hence this implies that the limit does not exist. $\diamond$

# 19. Derivatives

**Theorem 2.3.** If a function $f(z)$ is continuous and non-zero at a point $z_0$ then, there exists a neighborhood where, $f(z) \neq 0$ throughout.

**Proof.** We know that $f(z)$ is continuous which means that, $\varepsilon > 0, \exists \delta$ such that,

$$|f(z) - f(z_0)| < \varepsilon, \text{ when } 0 < |z - z_0| < \delta.$$

But if we take, $\varepsilon = \frac{f(z_0)}{2}$ then we have,

$$|f(z) - f(z_0)| < \frac{|f(z_0)|}{2}.$$

However, if $f(z) = 0$ for this neighborhood then we have,

$$|f(z_0)| < \frac{|f(z_0)|}{2}.$$

which is a contradiction.

□

**Theorem 2.4.** $f$ is continuous on $R$ which is closed and bounded, $\exists M > 0$, real $|f(z)| \leq M, \forall z \in R$ equality holds for at least one $z$.

**Definition 2.5** (Derivative). $f$ is differntiable at $z_0$ when $f'(z_0)$ exists where,

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

**Remark.** Can also solve,

$$\lim_{z_0 \to 0} \frac{f(z + z_0) - f(z)}{z_0}$$

.

**Example.** Find derivative of, $f(z) = \frac{1}{z}$

$$\lim_{z_0 \to 0} (\frac{1}{z + z_0} - \frac{1}{z}) \frac{1}{z_0}$$
$$\lim_{z_0 \to 0} \frac{z - z - z_0}{z(z + z_0)} \frac{1}{z_0}$$
$$\lim_{z_0 \to 0} \frac{-1}{z(z + z_0)}$$
$$= \frac{-1}{z^2}$$

◇

**Example.** $f(z) = \bar{z}$

$$\lim_{z_0 \to 0} \frac{z \bar{+} z_0 - \bar{z}}{z_0}$$

Go from $x$ and $y$ axis.
From $x$,

$$\lim_{x_0 \to 0} \frac{\bar{z} + x_0 - \bar{z}}{x_0} = 1.$$

Similarly if we go from $y$ we get $-1$, so the derivative doesn't exist.

◇

If we have a function $f(z) = u(x, y) + iv(x, y)$ then,

$$z_0 = x_0 + iy_0.$$

$$\Delta z = \Delta x + i\Delta y.$$

We have to show the following exist,

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}.$$

$$= \frac{u(x_0+\Delta x, y_0+\Delta y) + iv(x_0+\Delta x, y_0+\Delta y) - (u(x_0,y_0)+iv(x_0,y_0))}{\Delta x + i\Delta y}.$$

Horizontally, $\Delta y = 0$.
So,

$$\frac{u(x_0+\Delta x, y_0) - u(x_0,y_0)}{\Delta x} + \frac{i(v(x_0+\Delta x, y_0) - v(x_0,y_0))}{\Delta x}.$$

$$= u_x + iv_x.$$

Similary, if we go vertically, $\Delta x = 0$ and we get,

$$= v_y - iu_y.$$

**Theorem 2.6.** If, $f(z) = u + iv, f'(z)$ exists at, $z_0 = x_0 + iy_0$. Then, $u_x, u_y, v_x, v_y$ eixsts at $(x_0, y_0)$ and must satisify the Cauchy-Reimann equation.

$$f'(z_0) = u_x + iv_x \text{ at } (x_0, y_0).$$

**Theorem 2.7.** $f(z) = u(x,y)+iv(x,y)$ defined throughout the $\varepsilon$-neighborhood of $z_0 = x_0 + iy_0$,
(a) $u_x, u_y, v_x, v_y$ exists everywhere in the neighborhood
(b) $u_x, u_y, v_x, v_y$ continuous at $(x_0, y_0)$ and satisfy the Cauchy-Remainn equations

$$u_x = v_y, u_y = -v_x \text{ at } (x_0, y_0)$$

Then $f'(z_0)$ exists and,

$$f'(z_0) = u_x + iv_x \text{ at } (x_0, y_0).$$

**Proof.** We need to show,

$$\lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}.$$

$$= \lim_{\Delta z \to 0} \frac{u(x_0+\Delta x, y_0+\Delta y) + iv(x_0+\Delta x, y_0+\Delta y) - (u(x_0,y_0)+iv(x_0,y_0))}{\Delta z}.$$

Using taylor expansion we know,

$$f(x+\Delta x) = f(x) + \Delta x f'(x) + \frac{(\Delta x)^2}{2} f''(x).$$

$$u(x_0+\Delta x, y_0+\Delta y) - u(x_0,y_0) =$$

$$= u(x_0,y_0) + \Delta x u_x(x_0,y_0) + \frac{(\Delta x)^2}{2} u_{xx}(x_0,y_0) + \Delta y u_y(x_0,y_0) + \frac{(\Delta y)^2}{2} u_{yy}(x_0,y_0).$$

We can write the limit as,

$$\frac{u_x(x_0,y_0)\Delta x + u_y(x_0,y_0)\Delta y + \varepsilon_1(\Delta x^2) + \varepsilon_2(\Delta y^2)}{\Delta z} +$$

$$i\frac{v_x(x_0,y_0)\Delta x + v_y(x_0,y_0)\Delta y + \varepsilon_3(\Delta x) + \varepsilon_4(\Delta y)}{\Delta z}.$$

We know $u_x(x_0,y_0) = v_y(x_0,y_0)$ and $u_y(x_0,y_0) = -v_x(x_0,y_0)$, so,

$$\frac{u_x(x_0,y_0)\Delta x + -v_x(x_0,y_0)\Delta y + \varepsilon_1(\Delta x^2) + \varepsilon_2(\Delta y^2)}{\Delta z} +$$

$$i\frac{v_x(x_0,y_0)\Delta x + u_x(x_0,y_0)\Delta y + \varepsilon_3(\Delta x) + \varepsilon_4(\Delta y)}{\Delta z}.$$

$$= \frac{u_x(x_0,y_0)(\Delta x + i\Delta y) + u_y(x_0,y_0)(\Delta y - i\Delta x) + \varepsilon_4}{\Delta z}.$$

and $\Delta z = \Delta x + i\Delta y$

$$= \frac{u_x(x_0,y_0)(\Delta x + i\Delta y) + u_y(x_0,y_0)(\Delta y - i\Delta x) + \varepsilon_4}{\Delta x + i\Delta y}.$$

□

**Definition 2.8** (Analytic function)**.** A function $f$ is analytic in an open set $S$, if $f$ has derivative everywhere in $S$. It is analytic at a point $z_0$ if it is analytic in some neighborhood of $z_0$

**Remark.** Analytic functino has to be on an open set.

**Remark.** For it to be analytic at $z_0$ derivative should exist in the neighborhood of $z_0$ (not just the point $z_0$)

**Example.** $f(z) = (|z|)^2 = \sqrt{x^2+y^2}^2$

$$u = x^2 + y^2, v = 0$$

$$u_x = 2x, u_y = 2y.$$

$$v_x = 0, v_y = 0.$$

So the Cauchy-Reimann equation is only satisfied at $(0,0)$
$f'(0) = 0$ and it exists. ◇

**Remark.** $f(z) = |z|^2$ is not analytic anywhere. So even if the derivative exists at $z = 0$. The function is not analytic at $z = 0$ (or at any point)
Because, (1). $f'(z)$ exists at $z = 0$
(2). $u_x, u_y, v_x, v_y$ exists $\not\Rightarrow f'(z)$
(3). $f(z)$ is continuous $\not\Rightarrow f'(z)$
Essentially it only exists for $z = 0$ and not in the neighborhood around it.

**Definition 2.9** (Entire function)**.** A function $f$ is analytic at each point in the entire plane.

**Definition 2.10** (Singular point)**.** $z_0$ is a singular point if $f$ fails to be analytic at $z_0$ but is analytic at some point in every neighborhood at $z_0$

**Example.** $f(z) = 2 + 3z^2 + z^3$
Is analytic everywhere so it is an entire function $\diamond$

**Example.** $f(z) = \frac{1}{z}$
Is analytic at all non-zero, but $z = 0$ is a sigular point $\diamond$

**Example.** $f(z) = |z|^2 = x^2 + y^2$
Is not analytic, no singular points either. $\diamond$

## Harmonic Function

**Definition 2.11** (Harmonic function)**.** A real valued functino of $H(x, y)$ is said to be harmonic if in a given domain of the $x, y$ plane, it has a continuous partial derivative of the first and second order $(H_x, H_y, H_{xx}, H_{yy}, H_{xy})$ and satisfies,

$$H_{xx}(x, y) + H_{yy}(x, y) = 0 \text{ Laplace equation.}$$

**Theorem 2.12.** If $f = u(x, y) + i(x, y)$ is analytic in a domain $D$, then $u, v$ are harmonic in D

# Elementary Functions

## Exponential Function

The exponential function is $e^z$. But we can write this as

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(cos(y) + isin(y))$$

We can also write,

$$e^z = \rho e^{i\phi} \text{ where } \rho = |e^x| \text{ and } \phi = y$$

For a function, $e^{z_1}e^{z_2}$ we can write,

$$e^{z_1}e^{z_2} = e^{x_1+iy_1}e^{x_2+iy_2}$$

$$= e^{x_1+x_2}e^{i(y_1+y_2)}.$$

$$= e^{z_1+z_2}.$$

The derivative if $e^z$ is an entire function

$$\frac{d}{dx}e^z = e^z \text{which is an entire function.}$$

$$e^{z+2} = e^z + e^2 = e^z$$

## Log Function

The log function is $f(z) = log(z) = w = u + iv$. We know

$$e^w = z = e^{u+iv} = e^u e^{iv}.$$

We see that $r = e^u$ and $\theta = v + 2n\pi$

$$r = e^u \Rightarrow ln(r) = u$$

Similarly,

$$\theta = v + 2n\pi.$$

So we have,

$$f(z) = \log(z) = \ln|z| + i\arg(z).$$

and the principal direction is,

$$f(z) = \log(z) = \ln|z| + i\theta, \quad -\pi < \theta < \pi.$$

Some properties are,
(1).$e^{\log z} = z, (z \neq 0)$
(2).$|e^z| = e^x$
(3).$\log(e^z) = \ln|e^z| + i\arg(e^z)$

$$= \ln|e^x| + i(y + 2n\pi), n = 0, \pm 1, \pm 2.$$

$$= \ln e^x + iy + i2n\pi.$$

$$= z + 2n\pi.$$

## Branches

The principal branch is

$$\log z = \ln r + i\theta \text{ where } r > 0, -\pi < \theta < \pi.$$

A branch cut is a portion of a line or curve that is introduced in order to deifne a branch $F$ of a multiple-valued function $f$.
Points on the branch cut for $F$ are singular points of $F$ and any point that is common to all branches of $f$ are called branch points.

**Example.**

$$\frac{d}{dz}\log z = \frac{1}{z}, \text{ where } |z| > 0$$

The branches can be $\alpha < \arg z < \alpha + 2\pi$ ◇