Intro to Proofs

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Contents

Real Numbers

Definition 0.1 (Properties of real numbers). Properties of $\mathbb{R}$ are
(d). $\exists$ an order on $\mathbb{R}$ which means $\forall x, y \in \mathbb{R}, x < y$ or $, x > y,$ or $x = y$
Ordering follows the following properties,
(1). $x < y, y < z \Rightarrow x < z$ (transitivity)
(2). $x < y \Rightarrow x + z < y + z, \forall z \in \mathbb{R}$
(3). $x < y, z > 0 \Rightarrow xz < yz$

Theorem 0.2. $xy = 0 \Leftrightarrow x = 0$ or $y = 0$

Proof. $\Leftarrow$ Without loss of generality take, $x = 0$ Then we get,

$$0y.$$

We can write this as,

$$(0 + 0)y = 0y + 0y.$$

So,

$$0y = 0y + 0y.$$

Or, m
$\Rightarrow$
Assume the contrary that, $x \neq 0$ and $y \neq 0$ We have, $xy = 0$. Without loss of generality we take the multiplicative inverse of x so,

$$\frac{xy}{x} = \frac{0}{x}.$$

We showed that $0(k) = 0$ so $y = 0$
Which contradicts our assumption, hence our asusmptoin must be wrong and $x = 0$ or $y = 0$

$\square$

Theorem 0.3. $(-)x = -x$

Proof. We start with $(-1)x$ and add x to both sides so,

$$(-1)x + x = x(1-1) = 0x = 0.$$

So we showed that $(-1)x$ is the additive identity of x.
We know that the additive identity is unique for any x
Therefore, $(-1)x = -x$ □

Theorem 0.4. $\forall x < y, z < 0$

$$xz > yz.$$

Theorem 0.5. $\forall x \in \mathbb{R}, x^2 \geq 0$ and if $x \neq 0$ then $x^2 > 0$

Theorem 0.6. $x^2 = -(-x^2)$
Case 1, $x > 0$:

$$x > 0$$

$$x \times x > x$$

$$x \times x > 0x$$

$$x^2 > 0$$

Case 2, $x < 0$:

Example. $\forall a, b > 0$

$$\frac{a+b}{2} \geq \sqrt{ab}$$

◇

Proof.

$$0 \leq (\sqrt{a} - \sqrt{b})^2 = a - 2\sqrt{ab} + b.$$

□

Example. $x^2 - x + 1$ ◇

Theorem 0.7. $\forall x, y \in \mathbb{R}$ we have,

$$|x| \geq x \text{ and } |x+y| \leq |x| + |y|.$$

Proof. □

Proof related to Sets

Theorem 0.8.

$$A \cup B \backslash (A \cap B) = (A \backslash B) \cup (B \backslash A).$$

Proof. We need to show that,

$$A \cup B \backslash (A \cap B) \subseteq (A \backslash B) \cup (B \backslash A).$$

and,

$$(A \backslash B) \cup (B \backslash A) \subseteq A \cup B \backslash (A \cap B).$$

$\square$

Theorem 0.9. $A \subseteq B \Leftrightarrow A \cup B = B$

Proof. $\Rightarrow$ Take $\forall x \in A \cup B$, so either
Case 1, $x \in A$:
We know that by deifnition if, $A \subseteq B$ then for $x \in A, x \in B$ so $x \in B$
Case 2, $x \in B$: If $x \in B$ then we don't need to go further.
So we get $\forall x \in A \cup B$, $x \in B$
$\Leftarrow$
$\forall x \in A \Rightarrow x \in A \cup B = B$
So, $x \in B$ which means that, $A \subseteq B$

$\square$

Disproofs

If we need to show existance, $\exists x.P(x)$. We can show using,
1. Direct constructions
2. Indirectly (contradiction). For instance we can show that, $\forall x$, $\not P(x)$ is false

Example. $\exists a, b, c \in R - Q$ s.t. $a^{bc} \in Q$ $\diamond$

Example. Pigeonhole principle
Suppose there are m balls in n boxes, $m > n \geq 1$ then, $\exists$ a box where there are at least, $\frac{m}{n} + 1$ balls $\diamond$

Proof. Assume pigeonhole is false.
Then, there are at most $\frac{m}{n}$ balls in each box.
In case 1 where $\frac{m}{n} \notin N \Rightarrow$ total balls $\leq n[\frac{m}{n}] = \frac{nm}{n} = m$ which is a contradiction.
In case 2 where $\frac{m}{n} \in N$ there are at most $\frac{m}{n} - 1$ balls in each box. So total number of balls are $\frac{nm}{n} - n = m - n$ which is contradictory. $\square$

To disprove $\forall x P(x)$ we can show that, $\exists \not P(x)$

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Example. 100 can't be written as the sum of two even integers and an odd integer. $\diamond$

Proof. Suppose it's false $\Rightarrow \exists a, b, c \in Z$ s.t. $2|a, 2|b, 2 \not| c$ and $100 = a+b+c$
But, $2|a, 2|b \Rightarrow 2|a+b$ but $2 \not| c \Rightarrow 2 \not| (a+b)+c = 100$
So we get, $2 \not| 100$ which is a contradiction.
Which means that the original statmeent is true. $\square$

Example. $\nexists$ the smallest positive real number
The smallest positive real number is defined as $x \in R$ s.t. $x > 0$ and $\forall y > 0, x \leq y$ $\diamond$

Proof. Let's assume it is true which mean that $\exists x \in R$ s.t. $x > 0$ and $\forall y > 0$, $x \leq y$
We know that $x > 0 \Rightarrow \frac{x}{2} > 0$
So if we set $y = \frac{x}{2}$ then we get

$$x \leq \frac{x}{2}.$$

Which is a contradiction.
Hence it cannot be the case that there exists the smalest positive number.
$\square$

Example. $\nexists f(x)$: a polynomial with integer coefficients s.t. $\forall n, f(n)$ is prime $\diamond$

Proof. $\square$

Example. Let $f(x) = x^3 + 2x - 5$ then $\exists$ unique $x_0 \in [1,2]$ s.t. $f(x_0) = 0$ $\diamond$

Proof. Using intermediate value theorem.

$$f(1) = -2$$
$$f(2) = 7$$

To show unique we need to show its strictly increasing. $\square$