MATH 4320 HW14-16

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Problem 2

Let,

$$f(z) = \frac{z^{-1/2}}{z^2+1} = \frac{e^{(-1/2)\log z}}{z^2+1}$$

Now consider the indented contour and we have,

$$\int_{L_1} f(x)dx + \int_{L_2} f(x)dx + \int_{C_\rho} f(z)dz + \int_{C_R} f(z)dz = \int_C f(z)dz$$

Rearranging we have,

$$\int_{L_1} f(x)dx + \int_{L_2} f(x)dx = \int_C f(z)dz - \int_{C_\rho} f(z)dz - \int_{C_R} f(z)dz$$

First we calculate $\int_C f(z)dz$. Within our contour the only singularity is when $z = i$. So the integral is equal to,

$$\int_C f(z) = 2\pi i Res_{z=i} f(z)$$

Now

$$Res_{z=i} f(z) = \frac{e^{-1/2\log(i)}}{2i} = \frac{e^{-1/2(i\pi/2)}}{2i}$$
$$= \frac{e^{-i\pi/4}}{2i} = (\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}})\frac{1}{2i}$$

This gives us,

$$\int_C f(z) = 2\pi i \cdot (\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}})\frac{1}{2i}$$

$$= \frac{\pi}{\sqrt{2}} - \frac{\pi i}{\sqrt{2}} = \frac{\pi}{\sqrt{2}}(1-i)$$

Now let us look at,

$$\int_{L_1} f(x)dx + \int_{L_2} f(x)dx$$

We can write these integrals using the parameterization $z = re^{i0}, \rho \le r \le R$ and $z = re^{i\pi}, \rho \le r \le R$

$$= \int_\rho^R \frac{e^{-1/2\log(r)}}{r^2+1} + \int_\rho^R \frac{e^{-1/2\log(re^{\pi})}}{r^2+1}$$
$$= \int_\rho^R \frac{e^{-1/2\log(r)}}{r^2+1} + \int_\rho^R \frac{e^{-1/2\log(r)}e^{-i\pi/2}}{r^2+1}$$
$$= (1+e^{-i\pi/2})\int_\rho^R \frac{e^{-1/2\log(r)}}{r^2+1}$$
$$= (1-i)\int_\rho^R \frac{e^{-1/2\log(r)}}{r^2+1}$$

So we have,

$$= (1-i)\int_\rho^R \frac{e^{-1/2\log(r)}}{r^2+1} = \frac{\pi}{\sqrt{2}}(1-i) - \int_{C_\rho} f(z)dz - \int_{C_R} f(z)dz$$

Now we can bound $f(z)$ as follows because we can say $|\sqrt{z}| \geq |\sqrt{\rho}|$ and $|z^2+1| \geq |1-\rho^2|$ so,

$$f(z) \leq \frac{1}{\sqrt{\rho}(1-\rho^2)}$$

Or,

$$\int_{C_\rho} f(z) \leq \frac{2\pi\rho}{\sqrt{\rho}(1-\rho^2)} = \frac{2\pi\sqrt{\rho}}{1-\rho^2}$$

Now as $\rho \to 0$ we have $1-\rho^2$ goes to 1 while the numerator vanishes to zero. Hence we can say that,

$$\lim_{\rho\to 0} \int_{C_\rho} f(z) = 0$$

Similarly we have,

$$\int_{C_R} f(z) \leq \frac{2\pi\sqrt{R}}{R^2-1}$$

As the power of the denominator in terms of R is higher we have,

$$\lim_{R\to\infty} f(z)dz = 0$$

This gives us,

$$(1-i)\int_0^\infty \frac{e^{-1/2\log(r)}}{r^2+1} = \frac{\pi}{\sqrt{2}}(1-i)$$
$$\int_0^\infty \frac{e^{-1/2\log(r)}}{r^2+1} = \frac{\pi}{\sqrt{2}}$$

0.0.1 Problem 1

We have,

$$\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta}$$

Let $z = e^{i\theta}$ which gives us, $\frac{dz}{iz} = d\theta$ and $\sin\theta = \frac{z-z^{-1}}{2i}$. So we can write,

$$\int_C \frac{dz}{iz(5+4\frac{z-z^{-1}}{2i})}$$
$$= \int_C \frac{dz}{z(5i+2(z-z^{-1}))}$$
$$= \int_C \frac{dz}{5zi+2z^2-2}$$
$$= \int_C \frac{dz}{2(z+\frac{i}{2})(z+2i)}$$

Taking our unit circle we can see that $z = -\frac{i}{2}$ is inside our contour hence the integral evaluates to,

$$2\pi i \frac{1}{2(2i-i/2)}$$
$$= \pi i \frac{2}{3i}$$
$$= \frac{2\pi}{3}$$

Problem 3

We have,

$$\int_0^{2\pi} \frac{\cos^2 3\theta d\theta}{5 - 4\cos 2\theta}$$

We can write $\cos 3\theta = \frac{e^{3i\theta}+e^{-3i\theta}}{2}$. If we take $e^{i\theta} = z$ then we get,

$$\begin{aligned}
\cos(3\theta) &= \frac{z^3 + z^{-3}}{2} \\
\cos^2(3\theta) &= \frac{\left(z^3 + z^{-3}\right)^2}{2} \\
&= \frac{z^6 + z^{-6} + 1}{2} \\
&= \frac{1 + \cos(6\theta)}{2}
\end{aligned}$$

We know that $\cos(6\theta) = Re(e^{6i\theta})$. So we can rewrite our integral as,

$$\frac{1}{2}\int_0^{2\pi} \frac{1}{5 - 4\cos 2\theta} d\theta + \frac{1}{2} Re \int_0^{2\pi} \frac{e^{6i\theta}}{5 - 4\cos 2\theta} d\theta$$

Taking $e^{i\theta} = z$ we can write the first integral as,

$$\int_C \frac{i}{2} \frac{z}{(z^2 - 2)(2z^2 - 1)} dz$$

Using residue the value of the integral is,

$$2\pi i Res_{z=\frac{1}{\sqrt{2}},\frac{-1}{\sqrt{2}}} \frac{iz}{2(z^2 - 2)(2z^2 - 1)}$$

which is,

$$= \frac{\pi}{3}$$

Similarly we have,

$$Re(2\pi i Res_{z=\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}} \frac{i}{2} \frac{z^7}{(z^2 - 2)(2z^2 - 1)})$$

Which will equal to,

$$\frac{\pi}{24}$$

So our integral is,

$$\begin{aligned}
&= \frac{\pi}{3} + \frac{\pi}{24} \\
&= \frac{9\pi}{24} \\
&= \frac{3\pi}{8}
\end{aligned}$$

Problem 1

(a). We have $f(z) = z^2$ which has two zeroes inside the unit circle and no poles, hence we have,

$$2\pi(Z - P) = 2\pi(2 - 0) = 4\pi$$

(b). We have $f(z) = \frac{1}{z^2}$. Inside the unit circle we have no zeroes but two poles because z^2 has two zeroes and its in the denominator. Hence we have,

$$2\pi(Z - P) = 2\pi(0 - 2) = -4\pi$$

(c). We have $f(z) = (2z-1)^7/z^3$. The numerator is a polynomial of degree 7 hence it has 7 zeroes. And the denominator is of degree 3 hence it has 3 zeroes corresponding to 3 poles of the function. Hence we have,

$$2\pi(Z - P) = 2\pi(7 - 3) = 2\pi(4) = 8\pi$$

Problem 8

We have,

$$2z^5 - 6z^2 + z + 1 = 0$$

First within the unit circle if we take $f(z) = -6z^2 + z + 1$ and $g(z) = 2z^5$. Then we have,

$$|g(z)| \leq |f(z)|$$

as

$$|6z^2| + |z| + |1| \geq |2z^5| \text{ for } z < 1$$

Now considering $z < 2$, we take $f(z) = 2z^5 + 1$ and $g(z) = -6z^2 + z$. So we have on $|z| = 2$

$$\begin{aligned} -6z^2 + z + 1 &\leq |64| + |2| + |1| \\ &= 24 + 2 + 1 \\ &\leq 32 \\ &= 2^5 \\ &= z^5 \\ &\leq z^5 \end{aligned}$$

So we z^5 denominating over the circle, hence the sum will have 5 zeroes in the circle $z < 2$. So in the annulus we have $5 - 2$ zeroes which is 3.

Problem 3

Our transformation should rotate by $\frac{\pi}{2}$ anti-clockwise and shift it to the right by a unit of 1.

Firstly rotating by $\frac{\pi}{2}$ is equivalent to multiplying by $e^{i\frac{\pi}{2}}$. So we have $ze^{i\frac{\pi}{2}} = zi$

And shifting by a unit of one in the positive x axis is adding 1 so we have $f(z) = zi + 1$

Problem 5

We have the domain $x > 1$ and $y > 0$. First we know that any line $x = c$ is transformed into the circle,

$$(u - \frac{1}{2c})^2 + v^2 = (\frac{1}{2c})^2$$

so for any line $x > 1$ we have,

$$(u - \frac{1}{2})^2 + v^2 < \frac{1}{4}$$

And because we have $y > 0$ we have $C > 0$ which implies $-C < 0$ or that $v < 0$

Problem 11

We can look at it as $w = \frac{1}{z} = z^{-1} = e^{-i\theta}$. So as θ increases we have $arg(w)$ decreasing hence the orientation is opposite or negative.

Problem 5

First we show the boundary of the strip is mapped in a one to one manner onto the real axis in the w plane.

We have,

$$u = \sin(x)\cosh(y) \qquad v = \cos(x)\sinh(y)$$

So we have our first boundary as $x = \frac{-\pi}{2}$ which gives us,

$$u = -\cosh(y), v = 0$$

restricting y to be non-negative we have a point $(\frac{-\pi}{2}, y)$ mapped to $(-\cosh(y), 0)$. This means that as y increase along the boundary we have the image moving towards the left from D' towards E'. Now points $(x, 0)$ on the horizontal segment will have an image,

$$u = \sin(x), v = 0$$

But as $-\frac{\pi}{2} \leq x \leq 0$ we have $\sin(x)$ goes from -1 to 0. Which would be D' to C'.

Now each point on the interior of our domain will lie on the vertical lines $x = c_1, y > 0$. So we have,

$$u = \sin(c_1)\cosh(y), v = \cos(c_1)\sinh(y)$$

Now because y is always positive but x is always negative we have u is negative and v is positive. In other words we have,

$$\frac{u^2}{\cosh^2(y)} + \frac{v^2}{\sinh^2(y)} = 0$$

Such that it as y increases it moves to the left of the hyperbola.

Problem 4

We know that the function $f(z) = \sin(z)$ maps the semi infinite strip onto the first quadrant. This can be seen as we can write,

$$u = \sin(x)\cosh(y), v = \cos(x)\sinh(y)$$

And because $0 \leq x \leq \frac{\pi}{2}$ and $y \geq 0$. Both u and v is always greater than zero.

Now this means that any $Z = \sin(z)$ for any z in our domain is such that $arg(Z) \leq \frac{\pi}{2}$.

Now considering the function $F_0(z) = z^{1/2}$ we know that this will map any z to a point where $arg(z) = 2arg(f(z))$. Hence we know that $arg(F_0(Z)) \leq \frac{\pi}{4}$ which is the first octane in the image.

Problem 3

We know that under the mapping $w = 1/z$ a domain in the x, y plane in the form,

$$A(x^2 + y^2) + Bx + Cy + D = 0$$

will be mapped to,

$$D(u^2 + v^2) + Bu - Cv + A = 0$$

So our line $y = x - 1$ is such that $A = 0, B = 1, C = -1, D = -1$ which is mapped to,

$$-1(u^2 + v^2) + u + v = 0$$

or,

$$u^2 - u + v^2 - v = 0$$

Similarly we have $y = 0$ which is when $A = 0, B = 0, C = 1, D = 0$ which gets mapped to $v = 0$

Now at $z_0 = 1$ which is at the point $(1, 0)$, the angle is $\frac{\pi}{4}$.

Now in the image we have our first line mapping to $(u, v) = (1, 0)$ in the circle and the second line mapping to $(0, 0)$ in the line $v = 0$. The angle between these two points are $\frac{\pi}{4}$ as well.

Hence the angle is preserved and we verified the conformality of the mapping at $z = 1$.