

Linear Algebra HW05

Aamod Varma

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3A

Problem 1

Proof. We know for a linear map, $T(u + v) = T(u) + T(v)$ and $T(\lambda v) = \lambda T(v)$

First we look at additivity,

Consider an arbitrary $u = (x_1, y_1, z_1)$ and $v = (x_2, y_2, z_2)$. So we have,

$$\begin{aligned} T(u + v) &= T((x_1 + x_2), (y_1 + y_2), (z_1 + z_2)) \\ &= (2(x_1 + x_2) - 4(y_1 + y_2) + 3(z_1 + z_2) + b, 6(x_1 + x_2) + c(x_1 + x_2)(y_1 + y_2)(z_1 + z_2)) \end{aligned}$$

We need the above to be equal to,

$$\begin{aligned} T(u) + T(v) &= (2x_1 - 4y_1 + 3z_1 + b, 6x_1 + cx_1y_1z_1) + (2x_2 - 4y_2 + 3z_2 + b, 6x_2 + cx_2y_2z_2) \\ &= (2(x_1 + x_2) - 4(y_1 + y_2) + 3(z_1 + z_2) + 2b, 6(x_1 + x_2) + c(x_1y_1z_1 + x_2y_2z_2)) \end{aligned}$$

Comparing each of the terms we have,

$$2(x_1 + x_2) - 4(y_1 + y_2) + 3(z_1 + z_2) + 2b = 2(x_1 + x_2) - 4(y_1 + y_2) + 3(z_1 + z_2) + b$$

$$2b = b$$

$$b = 0$$

Similarly comparing the second term we have,

$$6(x_1 + x_2) + c(x_1 + x_2)(y_1 + y_2)(z_1 + z_2) = 6(x_1 + x_2) + c(x_1y_1z_1 + x_2y_2z_2)$$

$$c(x_1 + x_2)(y_1 + y_2)(z_1 + z_2) = c(x_1y_1z_1 + x_2y_2z_2)$$

$$c((x_1 + x_2)(y_1 + y_2)(z_1 + z_2) - (x_1y_1z_1 + x_2y_2z_2)) = 0$$

For this to be true for any x, y, z we need $c = 0$. Hence for additivity we need $b = c = 0$

Now we check if $T(kv) = kT(v)$. Consider $v = (x, y, z)$. Then we have

$$T(kv) = T(kx, ky, kz) = (2kx - 4ky + 3kz + b, 6kx + k^3cxyz)$$

We need this to be equal to

$$kT(v) = k(2x - 4y + 3z + b, 6x + cxyz) = (2kx - 4ky + 3kz + bk, 6kx + kcxzy)$$

Comparing the terms we have,

$$2kx - 4ky + 3kz + bk = 2kx - 4ky + 3kz + b$$

$$bk = b$$

$$b = 0$$

$$6kx + k^3xyz = 6kx + k^3xyz$$

$$c = k^2c$$

$$c = 0$$

So we have $b = c = 0$

□

Problem 2

Proof. Similar to (1) but take $p_1 = a_1 + b_1x$ and $p_2 = a_2 + b_2x$

□

Problem 3

Proof. Consider the standard basis e_1, \dots, e_n of F^n . That is

$$e_1 = (1, \dots, 0)$$

$$e_2 = (0, 1, \dots, 0)$$

$$\dots$$

$$e_n = (0, \dots, 1)$$

We have

$$\begin{aligned} T(x_1, \dots, x_n) &= T(x_1(1, \dots, 0), x_2(0, 1, \dots, 0), \dots, x_n(0, \dots, 1)) \\ &= T(x_1e_1, \dots, x_ne_n) \\ &= x_1T(e_1) + \dots + x_nT(e_n) \end{aligned}$$

Let T map e_1 to (A_{11}, \dots, A_{m1}) and e_n to (A_{1n}, \dots, A_{mn})
So we have,

$$\begin{aligned} &= x_1(A_{11}, \dots, A_{m1}) + \dots + x_n(A_{1n}, \dots, A_{mn}) \\ &= (x_1A_{11} + \dots + x_nA_{1n}, \dots, x_1A_{m1} + \dots + x_nA_{mn}) \end{aligned}$$

□

Problem 4

Proof. Let us assume the contrary that v_1, \dots, v_m is linearly dependent.
This means that \exists, a_1, \dots, a_m not all zero such that,

$$a_1v_1 + \dots + a_mv_m = 0$$

Now let us apply the linear map on this vector and we get,

$$T(a_1v_1 + \dots + a_mv_m) = T(0) = 0$$

$$a_1T(v_1) + \dots + a_mT(v_m) = 0$$

Here we see that \exists scalars a_1, \dots, a_m not all zero such that the linear combination of $T(v_1), \dots, T(v_m)$ is equal to zero. This means that the list of vectors are linearly dependent. However we know that the list is linearly independent. Hence our assumption must be wrong and v_1, \dots, v_m are actually linearly independent. \square

Problem 5

Proof. We need to show additivity and homogeneity.

(1). Additivity

We need to show for any $T_1, T_2 \in L(V, W)$ that $T_1 + T_2 \in L(V, W)$. In other words we need to show that $T_1 + T_2$ is also a linear map.

Consider $v_1, v_2 \in V$ we have

$$\begin{aligned}(T_1 + T_2)(v_1 + v_2) &= T_1(v_1 + v_2) + T_2(v_1 + v_2) \\ &= T_1(v_1) + T_2(v_1) + T_1(v_2) + T_2(v_2) \\ &= (T_1 + T_2)(v_1) + (T_1 + T_2)(v_2)\end{aligned}$$

Hence we show that $T_1 + T_2$ is additive.

Now consider $v_1 \in V$ we have $(T_1 + T_2)(\lambda v_1)$ we get,

$$\begin{aligned}&= T_1(\lambda v_1) + T_2(\lambda v_1) \\ &= \lambda T_1(v_1) + \lambda T_2(v_1) \\ &= \lambda(T_1 v_1 + T_2 v_1) \\ &= \lambda(T_1 + T_2)v_1\end{aligned}$$

Which means that it is homogenous.

Hence we show that $(T_1 + T_2) \in L(V, W)$ or that $L(V, W)$ is additive.

(2). Homogenous

Consider $T \in L(V, W)$ we need to show that λT is a linear map as well.

First we show that λT is additive. Consider v_1, v_2 , we have,

$$\begin{aligned}(\lambda T)(v_1 + v_2) &= \lambda(T)(v_1 + v_2) \\ &= \lambda(Tv_1 + Tv_2) \\ &= \lambda Tv_1 + \lambda Tv_2\end{aligned}$$

Which shows that λT is additive.

Now we check homogenous, consider $v \in V$ and $k \in F$ we have,

$$\begin{aligned}(\lambda T)(kv) &= \lambda(T)(kv) \\ &= \lambda kT(v) \\ &= k(\lambda T)v\end{aligned}$$

Hence we show that λT is homogenous. This makes λT a linear map.

Therefore we show that $L(V, W)$ is a vector space. \square

Problem 6

Proof. 1. Associativity. We have $(T_1 T_2) T_3 = T_1 (T_2 T_3)$
Consider the operation on a vector v so we have, $(T_1 T_2) T_3 v$ which is,

$$((T_1 T_2)(T_3(v))) = T_1(T_2(T_3(v)))$$

Now looking at the right side we have, $T_1(T_2 T_3) = T_1(T_2(T_3(v)))$. So we showed that the LHS is equal to the RHS.

2. Identity. Consider a vector v we have,

$$T I v = T(I(v)) = T(v)$$

Now,

$$I T v = I(T(v)) = T(v) \text{ because } I v = v, \forall v$$

3. Distributive Property

To show that,

$$(S_1 + S_2)T = S_1 T + S_2 T$$

Consider an arbitrary vector v in the domain of T . We have,

$$(S_1 + S_2)T v = (S_1 + S_2)(T(v))$$

By definition of addition of linear maps we have,

$$= (S_1(T(v))) + (S_2(T(v)))$$

Similarly we have,

$$(S_1 T + S_2 T)v = S_1 T(v) + S_2 T(v) = S_1(T(v)) + S_2(T(v))$$

We see that the distributive property holds.

Now To show that $S(T_1 + T_2) = S T_1 + S T_2$. Consider v we have,

$$S(T_1 + T_2)v = S(T_1(v) + T_2(v)) = S(T_1(v)) + S(T_2(v))$$

And we have,

$$(S T_1 + S T_2)v = S T_1(v) + S T_2(v) = S(T_1(v)) + S(T_2(v))$$

We see that the property holds again. □

Problem 7

Proof. As T is a linear map from V to itself and V is one dimensional say with basis $\{v'\}$. Then T is defined as

$$T(v') = \lambda v'$$

for some λ

Now we know $\forall v \in V$ we can write v as a linear combination of the basis of V , or

$$v = kv'$$

for some $k \in F$.

We have,

$$T(v') = \lambda v'$$

$$kT(v') = k\lambda v'$$

$$T(kv') = \lambda(kv')$$

$$T(v) = \lambda(v)$$

□

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Proof. Consider the function that maps any vector (x, y) to the $\max(|x|, |y|)$. We can see that this satisfies homogeneity. For instance consider $(2, 6)$. Our function maps this to 6. Now consider $(2 \times 3, 6 \times 3)$ which is mapped to 18 which is 3×6 as we saw above.

Now consider two vectors $(1, 0)$ and $(0, 4)$. Our function maps both these vectors to 1 and 4 respectively. However it maps its sum $(1, 4)$ to 4 $\neq 1 + 4$. Hence it does not follow additivity. Hence not a linear space. □

Problem 9

Proof. Consider the function that maps any complex number $x + iy$ to x . First we show this function is linear.

Consider two complex numbers $x_1 + iy_1$ and $x_2 + iy_2$. We have

$$f(z_1 + z_2) = f((x_1 + x_2) + i(y_1 + y_2))$$

$$= x_1 + x_2$$

$$= f(x_1 + iy_1) + f(x_2 + iy_2)$$

$$= f(z_1) + f(z_2)$$

Now we show it is not homogeneous.

Consider $\lambda = i$ then we have,

$$f(\lambda z_1) = f(-y_1 + ix_1)$$

$$= -y_1$$

however we know that $\lambda f(z_1) = ix_1 \neq -y_1$

Hence it is not homogeneous. □

Problem 10

Proof. We show counter example. Assume $q = 1 + x$, $p_1 = x$ and $p_2 = 2x$. We have,

$$q(p_1) = q(x) = 1 + x$$

$$q(p_2) = q(2x) = 1 + 2x$$

$$q(p_1 + p_2) = q(3x) = 1 + 3x$$

It is easy to see that $1 + 3x \neq 2 + 3x$

Hence T is not additive and not linear. \square

Problem 12

Proof. First consider the basis of U as u_1, \dots, u_n . Now let us extend this basis to V as follows, $u_1, \dots, u_n, v_{n+1}, \dots, v_m$. We need to show that T is not a linear map.

We know that $T(v) = S(v)$ for any $v \in U$. So we have

$$T(u_1) = S(u_1) \neq 0$$

Now consider v_m we have

$$T(v_m) = 0$$

by definition.

Now consider the sum of these vectors and we have $T(u_1 + v_m)$. We know that $u_1 + v_m$ cannot be in U as it cannot be represented as a linear combination of u_1, \dots, u_n as v_m is linearly independent with u_1, \dots, u_n . Hence $u_1 + v_m \in V$ but $\notin U$. Therefore by definition we have $T(u_1 + v_m) = 0$.

However we know that $T(u_1) + T(v_m) = S(u_1)$. But $S(u_1) \neq 0$. Which shows us that T is not additive. Hence T is not a linear map. \square

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Proof. First consider the basis of U as u_1, \dots, u_n . Now let us extend this basis of U to span V and we have v_{n+1}, \dots, v_m .

Let us define our linear map for our basis as follows,

$$T(u_1) = S(u_1), \dots, T(u_n) = S(u_n)$$

and

$$T(v_{n+1}) = 0, \dots, T(v_m) = 0$$

So we have defined a linear map such that for any $u \in U$ say $a_1u_1 + \dots + a_nu_n$ we have,

$$T(u) = T(a_1u_1 + \dots + a_nu_n)$$

$$T(u) = a_1T(u_1) + \dots + a_nT(u_n)$$

$$= a_1S(u_1) + \dots + a_nS(u_n)$$

$$= S(a_1u_1 + \dots + a_nu_n)$$

$$= S(u)$$

□

3B

2

Proof. We need to show that $(ST)^2 = 0$. Or that,

$$S(T(S(T(v)))) = 0$$

We are given that, $\text{range } S \subseteq \text{null } T$. Or that for any $v \in \text{domain } S$, $S(v) = u$ then $T(u) = 0$.

We know that $T(v) = v_0$. Then we have $S(v_0)$ is a vector in null space of T . Which means that $T(S(v_0)) = 0$. We know that if L is a linear map then $L(0) = 0$. So $S(T(S(v_0))) = S(0) = 0$

□

9

Proof. We have,

$$(v_1, \dots, v_n) \text{ is linearly independent}$$

This means that,

$$a_1 v_1 + \dots + a_n v_n = 0$$

then $a_1 = \dots = a_n = 0$

Let us apply the linear map on both sides and we get,

$$T(a_1 v_1 + \dots + a_n v_n) = T(0) = 0$$

$$= T(a_1 v_1) + \dots + T(a_n v_n) \text{ as } T \text{ is a linear map}$$

$$= a_1 T(v_1) + \dots + a_n T(v_n) = 0$$

We know from before that $a_1 = \dots = a_n = 0$. This means that

$$T(v_1), \dots, T(v_n)$$

is linearly independent as the only way to represent 0 is having all the coefficients as 0.

□

10

Proof. First we know that $\dim(\text{range } T) = \dim(V) = n$. So it is enough to show that $T(v_1), \dots, T(v_n)$ are n linearly independent vectors in range T .

If v_1, \dots, v_n span V then we know that v_1, \dots, v_n are linearly independent. So,

$$a_1v_1 + \dots + a_nv_n = 0$$

such that $a_1 = \dots = a_n = 0$

Applying the operator on both sides we get,

$$T(a_1v_1 + \dots + a_nv_n) = T(0) = 0$$

$$= T(a_1v_1) + \dots + T(a_nv_n) = 0$$

$$= a_1T(v_1) + \dots + a_nT(v_n) = 0$$

We know from above, $a_1 = \dots = a_n = 0$ which means that $T(v_1), \dots, T(v_n)$ is linearly independent set of vectors in range T such that $\dim(a_1T(v_1) + \dots + a_nT(v_n)) = \dim(\text{range}(V)) = n$ which makes it span range T . \square

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We have $\text{null } T = \{(x_1, x_2, x_3, x_4) \in F^4 : x_1 = 5x_2, x_3 = 7x_4\}$

So this means that we have two independent variables which implies that the null space has dimension of two.

So we have range of T as dimension of 2. Because \dim of range is equal to the dimension of the codomain the linear map is surjective.

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we know $\dim V = \dim (\text{null}(T)) + \dim (\text{range}(T))$

If null space and range of T are finite dimensional that means that $\dim V$ is a finite number. Or that V is a finite dimensional space.

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Given $P(P(v)) = P(v)$ we need to show that $V = \text{null } P \oplus \text{range } P$.

We have to show two things, $\text{null } P \cap \text{range } P = \{0\}$ and $\forall v, v = u + w, u \in \text{null } P, w \in \text{range } P$.

1. Assume $v \in \text{null } P \cap \text{range } P$. So that means $v \in \text{null } P$ and $v \in \text{range } P$. If $v \in \text{null } P$ then,

$$P(v) = 0$$

If $v \in \text{range}(P)$ then $\exists w \in V, v = P(w)$. We are given that $P(P(v)) = P(v)$ and we know that $P(v) = 0$ and $P(w) = v$. So we get,

$$P(v) = P(P(w)) = P(w)$$

Or in other words $P(v) = 0$ so $P(w) = v = 0$. Hence we show that their intersection only consist of the zero vector.

Now we need to show that every vector $v \in V$ can be written as $u + w$ such that $u \in \text{null } P$ and $w \in \text{range } P$.

Consider any $v \in V$ such that $P(v) = v_1$. This means that $v_1 \in \text{range } P$. Also $P(v_1) = P(P(v)) = P(v)$ so $P(v_1) = P(v)$.

Now consider $v_2 = v - v_1$. Applying the operator on both sides we get,

$$P(v_2) = P(v - v_1) = P(v) - P(v_1) = 0$$

which implies that $v_2 \in \text{null}P$.

So now we have a $v_1 + v_2 = v - v_1 + v_1 = v$ such that $v_1 \in \text{range}P$ and $v_2 \in \text{null}P$

3C

1

Proof. Consider the contrary that there exists a matrix of T such that it has less than $\dim \text{range } T$ non zero entries.

This means that there always exists a column of the matrix such that the column is filled with zeroes. Consider r to be the columns that have non-zero entries this means that $r < \dim \text{range } T$.

We know by definition of matrices that,

$$T(v_k) = A_{1k}w_1 + \dots + A_{nk}w_n$$

where w_1, \dots, w_n is a basis for $\text{range } T$.

Now this means that there are $n - r$ choices of k s.t $T(v_k) = 0$

We know by definition of a linear map that for a given basis of V ,

$$T(v_1), \dots, T(v_m) = w_m$$

such that w_1, \dots, w_m span W .

However in our case we have only r non-zero $w \in W$. Which means that the $\dim W$ is r . However this contradicts the fact that $r < \dim W$. Which means that our assumption must be wrong. □

2

Proof. \Rightarrow Consider an arbitrary basis of V v_1, \dots, v_n . We are told that $\dim (\text{range } T) = 1$. Which means that $T(v_1), \dots, T(v_n)$ spans a 1 D space or in other words,

$$T(v_1) = c_1 w$$

$$\dots$$

$$T(v_n) = c_n w$$

for some $w \in W$

Now we can adjust our basis for V by multiplying $\frac{1}{c_k}$ to each side to get a new basis, $\frac{v_1}{c_1}, \dots, \frac{v_n}{c_n}$ such that,

$$T(v'_1) = w$$

...

$$T(v'_n) = w$$

We also know that for a matrix $M(T)$ we have,

$$T(v_k) = A_{1,k}w_1 + \dots + A_{n,k}w_n$$

So from above we have,

$$T(v_k) = A_{1,k}w_1 + \dots + A_{n,k}w_n = w$$

for any $w \in W$. So we need a basis of W such that $w_1 + \dots + w_n = w$. Consider an arbitrary basis for W , w_1, \dots, w_m . Now consider the following list of vectors, we know, $\exists a_1, \dots, a_n$ s.t $a_k \neq 0$

$$a_1w_1 + \dots + a_nw_n = w$$

Now let's adjust this basis such that $w'_1 = a_1w_1, \dots, w'_n = a_nw_n$. It is easy to show that this new list of vectors is a basis.

So now we have constructed a basis for W such that $w_1 + \dots + w_n = w$.

Which means we have defined a basis for V and W such that,

$$T(v_k) = w_1 + \dots + w_n$$

\Leftarrow

Consider there exists a basis of V and W such that,

$$T(v_k) = 1w_1 + \dots + 1w_n$$

where $k \in 1, \dots, n$ and w_1, \dots, w_n is a basis for W .

This means that $T(v_k) = w$ where $w = w_1 + \dots + w_n$.

We know that $T(v_1), \dots, T(v_n)$ spans the range of T which means that the range of T is spanned by $w \Rightarrow \dim \text{range } T = 1$ \square

3

(a). We define $M(S)$ and $M(T)$ as follows,

$$Sv_k = M_{s(1,k)}w_1 + \dots + M_{s(m,k)}w_m$$

$$Tv_k = M_{t(1,k)}w_1 + \dots + M_{t(m,k)}w_m$$

And we define $M(S+T)$,

$$(S+T)v_k = M_{st(1,k)}w_1 + \dots + M_{st(m,k)}w_m$$

Further expanding we get,

$$\begin{aligned} (S+T)v_k &= Sv_k + Tv_k \\ &= M_{s(1,k)}w_1 + \dots + M_{s(m,k)}w_m + M_{t(1,k)}w_1 + \dots + M_{t(m,k)}w_m \\ &= (M_{s(1,k)} + M_{t(1,k)})w_1 + \dots + (M_{s(m,k)} + M_{t(m,k)})w_m \end{aligned}$$

Comparing the two results we get,

$$M_{st(1,k)}w_1 = (M_{s(1,k)} + M_{t(1,k)})w_1$$

...

$$M_{st(m,k)}w_m = (M_{s(m,k)} + M_{t(m,k)})w_m$$

Which means that for any

$$M_{st(r,c)} = M_{s(r,c)} + M_{t(r,c)} \Rightarrow M(S+T) = M(T) + M(S)$$

(b). Similary we have,

$$Tv_k = M_{t(1,k)}w_1 + M_{t(m,k)}w_m$$

and

$$\lambda Tv_k = M'_{t(1,k)}w_1 + M'_{t(m,k)}w_m$$

Now multiplying λ we get,

$$\lambda Tv_k = \lambda(M_{t(1,k)}w_1 + M_{t(m,k)}w_m)$$

$$\lambda Tv_k = (\lambda M_{t(1,k)})w_1 + (\lambda M_{t(m,k)})w_m$$

So we see that for all entires of our matrix $M'_{r,c} = \lambda M_{r,c}$ which means that,

$$M(\lambda T) = \lambda M(T)$$

4

We have matrix M such that,

$$Tv_1 = M_{1,1}w_1 + 0M_{2,1}w_2 + 0M_{3,1}w_3$$

$$Tv_2 = M_{1,2}w_1 + 0M_{2,2}w_2 + 0M_{3,2}w_3$$

$$Tv_3 = M_{1,3}w_1 + 0M_{2,3}w_2 + 0M_{3,3}w_3$$

$$Tv_4 = M_{1,4}w_1 + 0M_{2,4}w_2 + 0M_{3,4}w_3$$

Consider the basis of W as $(1, x, x^2)$. So we have $T(v_1) = 1, T(v_2) = x, T(v_3) = x^2, T(v_4) = 0$. Our basis in V is $(x, \frac{1}{2}x^2, \frac{1}{3}x^3, 1)$

5

Proof. First let us take $\dim V = n, \dim W = m, \dim \text{range } T = k$. Let w_1, \dots, w_k be a basis for $\text{range } T$. Now according to the lienar map lemma we can find v_1, \dots, v_k such that,

$$T(v_r) = w_r$$

We know $\dim \text{range } T + \dim \text{null } T = \dim V$, so $\dim \text{null } T = n - k$. and we know that none of our current $v_r \in \text{null } T$ as $w_r \neq 0$. Which means that we can exted our linearly independe list, v_1, \dots, v_k to $v_1, \dots, v_k, v_{k+1}, \dots, v_n$ such that

$$Tv_{k+1} = 0, \dots, Tv_n = 0$$

Similarly let us extend our basis for W from w_1, \dots, w_r to $w_1, \dots, w_r, w_{r+1}, \dots, w_m$

Now let us define our matrix such that,

$$\begin{aligned}
Tv_1 &= 1w_1 + \cdots + 0w_r + \cdots + 0w_n \\
&\quad \dots \\
Tv_r &= 0w_1 + \cdots + 1w_r + \cdots + 0w_n \\
Tv_{r+1} &= 0w_1 + \cdots + 0w_r + \cdots + 0w_n \\
&\quad \dots \\
Tv_m &= 0w_1 + \cdots + 0w_r + \cdots + 0w_n
\end{aligned}$$

Now we have defined a matrix such that the $A_{k,k}$ element is 1 for $k \leq \dim(\text{range } T)$ and all the other elements are 0. \square