Linear Alebgra 5B

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5B

Problem 1

Proof. $\Rightarrow$. We have 9 is an eigenvalue of T^2 which implies that $T^2v = 9v$ for some $v \in V$. So we have $(T^2 - 9I) = 0$ so $(T + 3I)(T - 3I) = 0$. So we have either $Tv = -3v$ or $Tv = 3v$ which means either 3 is an eigenvalue or -3 is an eigenvalue.
$\Leftarrow$. Now consider 3 or -3 is an eigenvalue so we have,

$$Tv = 3v \Rightarrow T(Tv) = T(3v) = 3T(v) = 9(v)$$

which means that 9 is an eigenvalue of T.
If -3 is an eigenvalue we have,

$$Tv = -3v \Rightarrow T(Tv)) = T(-3v) = -3T(v) = 9v$$

so 9 is an eigenvalue of T. $\square$

Problem 2

Proof. We are given that T has no eigenvalue. We need to show that every subspace of V invariant under T is either $\{0\}$ or infinite-dimentional.
Consider a finite subspace $U \subset V$ that is invariant under T. We know that the minimal polynomial of V is a polynomial multiple of that of U. First because U is a complex finite subspace of V we know that it has to have eigenvalues which are zeroes of its minimal polynomial. As it is also the zeroes of the minimal polynomial of T this means that they are the eigenvalues of T but this contradicts our assumption that T has no eigenvalues. $\square$

Problem 3

Proof. (a). We have $T(x_1, \ldots, x_n) = (x_1 + \cdots + x_{n,}, \ldots, x_1 + \cdots + x_n)$
So we have,

$$x_1 + \cdots + x_n = \lambda x_1$$

$$\cdots$$

$$x_1 + \cdots + x_n = \lambda x_n$$

If we add all we get,

$$n(x_1 + \cdots + x_n) = \lambda(x_1 + \cdots + x_n)$$

So we have either $x_1 + \cdots + x_n = 0$ where $\lambda = 0$or $x_1 = \cdots = x_n$ where $\lambda = n$.
(b).
If $n = 1$ then the minimal polynomial is $z - 1$ but if its greater than 1 then our minimal polynomial will be $z(z - n) = z^2 - zn$ $\square$

4

Proof. $\Rightarrow$. We have $\alpha = p(\lambda)$ for some eigenvalue λ of T. By definition we have,

$$Tv = \lambda v \text{ for some } v \in V$$

Applying P on both sides we have,

$$P(Tv) = P(\lambda v)$$
$$P(T)v = P(\lambda)v$$

If we take $\alpha = p(\lambda)$ we have,

$$P(T)v = \alpha v$$

for some v.
This makes α an eigenvalue of $P(T)$
$\Leftarrow$
First consider α is an eigenvalue of $p(T)$ we need to show that $\alpha = p(\lambda)$ for some eigenvalue λ of T.
So we have,

$$p(T)v = \alpha v$$

Consider q such that $q = p - \alpha$. So for some v we have $q(T)v = 0$. So $q(T)$ is the minimal polynomial (or multiple of it) of T. Which means that $\exists \lambda q(\lambda) = 0$. This means that

$$q(\lambda) = p(\lambda) - \alpha = 0$$

$$p(\lambda) = \alpha$$

□

Problem 5

Proof. Consider the operator $T(x, y) = (-y, x)$. Consider the polynomial $p = z^2$. So we have $p(T) = T^2 = -I$. So -1 is an eigenvalue of $p(T)$. However if $F = R$ then T does not have an eigenvalue hence $\nexists \lambda$ such that $\alpha = p(\lambda)$ □

Problem 6

Proof. We have $T(w, z) = (-z, w)$. Consider $e_1 = (1, 0)$ we have,

$$T(e_1) = (0, 1)$$

$$T^2(e_1) = T(0, 1) = (-1, 0) = -e_1$$

So we have $T^2 + I = 0$ so the minimal polynomial is $p(z) = z^2 + 1$ □

Problem 7

Proof. (b). We need to show that if S or T is invertible then the minimal polynomial of ST is equal to that of TS.
First we assume S is invertible. Let p be the minimal polynomial of ST and q for TS. So we have,

$$p(TS) = S^{-1}p(ST)S = 0$$

which means that p is also a polynomial multiple of q. So,

$$p = rq$$

for some r.
Now we have,

$$q(ST) = Sq(TS)S^{-1} = 0$$

Which means that q is a polynomial multiple of p. So we have,

$$q = kp$$

So we have,

$$p = rkp \Rightarrow rk = 1$$

This can only be true if both r and k are constants. Because the minimal polynomial is a monic polynomial it has to be $r = k = 1$
So we have,

$$p = q$$

□

Problem 8

Proof. We have $T \in L(R^2)$ such that it is the counterclockwise rotation by 1 degrees.
So if we consider $e_1 = (1, 0)$ we have,

$$T(e_1) = (\cos 1, \sin 1)$$
$$T^2(e_1) = (\cos 2, \sin 2)$$

So we have $1 - 2\cos(1)z + z^2 = 0$ or $1 - 2\cos(\frac{\pi}{180})z + z^2 = 0$ □

Problem 10

Proof. We have V is finite and $T \in L(V)$ and $v \in V$. We need to show that,

$$span(v, Tv, \ldots, T^m v) = span(v, Tv, \ldots, T^{\dim V-1}v)$$

if $m \geq \dim V - 1$
First we show that for any subspace $U_k = \{v, Tv, \ldots, T^k v\}$ if $T^{k+1} \in U_k$

then for any $m >= k+1$, $T^m \in U_k$. We do induction to show this, we already assume the base case is true if $m = k+1$ we have $T^{k+1}v \in U_k$. Now assume it is true for an arbitrary n so we have,

$$T^n \in U_k$$

This means that,

$$T^n v = a_1 v + \cdots + a_{k+1} T^k v$$

Now apply T on both sides we get,

$$T^{n+1} v = a_1 T v + \cdots + a_{k+1} T^{k+1} v$$

Now because we know that $T^{k+1} \in U_k$ we know that T^{n+1} is a linear combination of elements in U_k which must mean that $T^{n+1}v \in U_k$. Hence by induction it is true for any $n \geq k+1$.
Now first if $m = \dim V$ then we know that the list,

$$v, Tv, \ldots, T^m v$$

is linearly dependent which means that $\exists n \in \{0, \ldots, m\}$ such that $T^{n+1} \in span(v, \ldots, T^n)$. Now based on what we proved above this must mean for any $m \geq n+1$, $T^m \in span(v, Tv, \ldots, T^k) = span(v, Tv, \ldots, T^{\dim V - 1})$

□

Problem 13

Proof. We have V is finite dimensional and we need to show there is $r \in P(F)$ such that $p(T) = r(T)$ and deg r less that minimal polynomial of T. Consider any arbitrary $p \in P(F)$. Let $q \in P(F)$ be the unique minimal polynomial of T such that $q(T) = 0$. Now we can divide p by q and uniquely write it as,

$$p = kq + r$$

Such that $deg(r) < deg(q)$.
So

$$p(T) = k(T)q(T) + r(T)$$

But we know $q(T) = 0$ so we have,

$$p(T) = r(T)$$

where $deg(r) < deg(q)$ □

Problem 14

Proof. We have the minimal polynomial of T as,

$$4 + 5z - 6z^2 - 7z^3 + 2z^4 + z^5$$

The minimal polynomial of T^{-1} will be,

$$\frac{1}{4} + \frac{1}{2}z - \frac{7}{4}z^2 - \frac{3}{2}z^3 + \frac{5}{4}z^4 + z^5$$

□

Problem 16

Proof. Consider $e_1 = (1, \dots,)$. Our matrix is $n - 1 \times n$ dimension.
So we have,

$$\begin{aligned} Te_1 &= e_2 \\ T^2 e_1 &= Te_2 = e_3 \\ &\dots \\ T^{n-1} e_1 &= Te_{n-1} = e_n \\ T^n e_1 &= Te_n = (-a_0, \dots, -a_{n-1}) \end{aligned}$$

Now we can represent T^n as,

$$T^n e_1 = -a_0 e_1 + \cdots + -a_{n-1} e_{n-1}$$

which gives us,

$$T^n e_1 = -a_0 e_1 + \cdots + -a_{n-1} T^{n-1} e_1$$

So our minimal polynomial is,

$$p(z) = a_0 + \cdots + a_{n-1} z^{n-1} + z^n$$

Which gives us $p(T) = 0$ □

Problem 17

Proof. We need to show that the minimal polynomial of $T - \lambda I$ is,

$$q(z) = p(z + \lambda)$$

given that p is the minimal polynomial of T.
We have,

$$q(T - \lambda I) = p(T - \lambda I + \lambda I) = p(T) = 0$$

So we have q is a polynomial multiple of the minimal polynomial of $T - \lambda I$. This means that $deg(s) \leq deg(q) = deg(p)$. Now we need to show that q is the minimal polynomial.
If s is the minimal polynomial of $T - \lambda I$ consider the polynomial,

$$r(z) = s(z - \lambda)$$

So we have,

$$r(T) = s(T - \lambda I) = 0$$

This means that $deg(p) \leq deg(r) = deg(s)$. SO we have $deg(q) \leq deg(s)$ and $deg(s) \leq deg(q) \Rightarrow deg(q) = deg(s)$. Or that $s = q$ and q is the minimal polynomial of $T - \lambda I$

$\square$

Problem 19

Proof. Consider the mapping $\phi \in L(P(F), L(V))$ defined as $\phi(q) = q(T)$. Now we see that the range of ϕ is E. We know that $null\phi = \{pq : q \in P(F)\}$ because $p(T)q(T) = 0q(T) = 0$ as p is the minimal polynomial of T. Now for any $x \in P$ such that degree x is greater than p we can write it as,

$$x = x'p + r$$

where degree of r is smaller than p.
So we have $x(T) = r(T)$ so we can consider the subspace $P(F) - null\phi$ which has dimension p as for any r, deg$(r) \leq deg(p)$. And we have an isomorphism from $P(F) - null(\phi)$ to E. which gives us our result.

$\square$

Problem 20

Proof. We have $T \in L(F^4)$ such that $3, 5, 8$ are its eigenvalues. First because its F^4 we know the highest degree of the minimal polynomial is 4. We also know that the eigenvalues are zeroes of our minimal polynomial. So the minimal polynomial is,

$$p(z) = s(z-3)(z-5)(z-8)$$

Where $s \in \{1, z-3, z-5, z-8\}$. In either case we have $k(z) = (z-3)^2(z-5)^2(z-8)^2$ is a polynomial multiple of the minimal polynomial which makes $k(T) = 0$ $\square$

Problem 21

Proof. We need to show the minimal polynomial of T has degree at most $1 + \dim rangeT$.
Let p be minimal polynomial of V and q be of T_{rangeT}. We have,

$$q(T)Tv = q(T_{rangeT})Tv = 0$$

So $q(T)T = 0$ and as p is the minimal polynomial we have,

$$deg(p) \leq deg(xq(x)) = 1 + deg(q) \leq 1 + dim(rangeT)$$

$\square$

Problem 22

Proof. We need to show T is invertible only of $I \in span(T, T^2, \ldots, T^{dimV})$.
If T is invertible that means that p has a non-zero constant term. Now the minimal polynomial of T can be written as,

$$c + c_1 z + \cdots + z^n$$

where $n = \dim V$. So we have,

$$p(T) = cI + c_1 T + \cdots + T^n = 0$$

$$cI = -c_1 T + \cdots + -T^n \Rightarrow I \in span(T, \ldots, T^n)$$

Now assume $I \in span(T, T^2, \ldots, T^n)$. So we can write,

$$I = c_1 T + \cdots + c_n T^n$$

or

$$r(T) = b_0 I + b_1 T + \cdots + T^n = 0$$

So $r = b_0 + b_1 z + \cdots + z^n$
This must be a polynomial multiple of the minimal polynomial. So,

$$r(z) = k(z)p(z)$$

We know that $r(0) \neq 0$, so,

$$r(0) = b_0 = k(0)p(0) \Rightarrow p(0) \neq 0 \text{ and } k(0) \neq 0$$

So p has a non-zero constant term which means that T is invertible. □

Problem 23

Proof. We need to show that $span(v, Tv, \ldots, T^{n-1}v)$ is invariant under T.
We have n vector $v, \ldots, T^{n-1}v$. Consider they are linearly independent, this means their span is V which makes them invariant under T.
If the list of vectors are not linearly independent that means $\exists k$ such that $T^{k+1}v \in span(v, Tv, \ldots, T^k v)$. If that is the case then we can show by induction that for any $m \geq k+1, T^m v \in span(v, \ldots, T^k)$. Base case is true as $T^{k+1} \in span(v, Tv, \ldots, T^k v)$. Now consider an arbitrary $n > k+1$ such that,

$$T^n = a_1 v + \cdots + a_n T^k v$$

now we have,

$$T^{n+1} = a_1 T(v) + \cdots + a_n T^{k+1} v \in span(v, Tv, \ldots, T^k)$$

as T^{k+1} is in the span.
Now using this we can conclude that $span(v, Tv, \ldots, T^{n-1}) = span(v, Tv, \ldots, T^k)$.
For any $v \in span(v, Tv, \ldots, T^k)$ we have shown that Tv is also in the span

hence making it invariant.

$\square$