Intro to Proofs

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Contents

Real Numbers

Definition 1 (Properties of real numbers)**.** Properties of $\mathbb{R}$ are

(d). $\exists$ an order on $\mathbb{R}$ which means $\forall x, y \in \mathbb{R}, x < y$ or $, x > y,$ or $x = y$
Ordering follows the following properties,

(1). $x < y, y < z \Rightarrow x < z$ (transitivity)

(2). $x < y \Rightarrow x + z < y + z, \forall z \in \mathbb{R}$

(3). $x < y, z > 0 \Rightarrow xz < yz$

Theorem 1. $xy = 0 \Leftrightarrow x = 0$ or $y = 0$

Proof. $\Leftarrow$ Without loss of generality take, $x = 0$ Then we get,

$$0y.$$

We can write this as,

$$(0 + 0)y = 0y + 0y.$$

So,

$$0y = 0y + 0y.$$

Or, m

$\Rightarrow$

Assume the contrary that, $x \neq 0$ and $y \neq 0$ We have, $xy = 0$. Without loss of generality we take the multiplicative inverse of x so,

$$\frac{xy}{x} = \frac{0}{x}.$$

We showed that $0(k) = 0$ so $y = 0$

Which contradicts our assumption, hence our asusmptoin must be wrong and $x = 0$ or $y = 0$

$\square$

Theorem 2. $(-)x = -x$

Proof. We start with $(-1)x$ and add x to both sides so,

$$(-1)x + x = x(1 - 1) = 0x = 0.$$

So we showed that $(-1)x$ is the additive identity of x.

We know that the additive identity is unique for any x

Therefore, $(-1)x = -x$ □

Theorem 3. $\forall x < y, z < 0$

$$xz > yz.$$

Theorem 4. $\forall x \in \mathbb{R}, x^2 \geq 0$ and if $x \neq 0$ then $x^2 > 0$

Theorem 5. $x^2 = -(-x^2)$

Case 1, $x > 0$:

$$x > 0$$

$$x \times x > x$$

$$x \times x > 0x$$

$$x^2 > 0$$

Case 2, $x < 0$:

Example. $\forall a, b > 0$

$$\frac{a+b}{2} \geq \sqrt{ab}$$

Proof.

$$0 \leq (\sqrt{a} - \sqrt{b})^2 = a - 2\sqrt{ab} + b.$$

□

◇

Example. $x^2 - x + 1$ ◇

Theorem 6. $\forall x, y \in \mathbb{R}$ we have,

$$|x| \geq x \text{ and } |x + y| \leq |x| + |y|.$$

Proof. □

Proof related to Sets

Theorem 7.

$$A \cup B \backslash (A \cap B) = (A \backslash B) \cup (B \backslash A).$$

We need to show that,

$$A \cup B \backslash (A \cap B) \subseteq (A \backslash B) \cup (B \backslash A).$$

and,

$$(A \backslash B) \cup (B \backslash A) \subseteq A \cup B \backslash (A \cap B).$$

□

Theorem 8. $A \subseteq B \Leftrightarrow A \cup B = B$

Proof. $\Rightarrow$ Take $\forall x \in A \cup B$, so either

Case 1, $x \in A$:

We know that by deifnition if, $A \subseteq B$ then for $x \in A, x \in B$ so $x \in B$

Case 2, $x \in B$: If $x \in B$ then we don't need to go further.

So we get $\forall x \in A \cup B$, $x \in B$

$\Leftarrow$

$\forall x \in A \Rightarrow x \in A \cup B = B$

So, $x \in B$ which means that, $A \subseteq B$

□