Complex Analysis

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MATH - 4320, Fall 2024

Contents

Chapter 1

Complex Numbers

1.12 Regions in the Complex Plane

Definition 1 (Epsilon neighborhood)**.** An epsilon neighborhood around a point z_0 is the set of all z such that,

$$|z - z_0| < \varepsilon$$

Definition 2 (Deleted neighborhood)**.** A deleted neighborhood around a point z_0 is the set of all z such that,

$$0 < |z - z_0| < \varepsilon$$

Remark. A deleted neighborhood is essentially an epsilon neighborhood but does not include the point z_0

Definition 3 (Interior point)**.** z_0 is an interior point when there exists a neighborhood of z_0 that contains only points of S

Definition 4 (Exeterior point)**.** z_0 is an exterior point when there exists a neighborhood of z_0 that contains no points of S

Definition 5 (Boundary point)**.** z_0 is a boundary point otherwise, i.e. all of the neighborhoods of z_0 contains a point in S and a point not in S

Definition 6 (Open set)**.** S is an open set if $\forall z \in S, \exists \varepsilon$ s.t. $B_\varepsilon(z) \subset S$

Remark. We can also say that an open set does not contain any of its boundary points.

Definition 7 (Closed set)**.** A set is closed if it doesn't contain its boundary points.

Definition 8 (Connected Set)**.** An open set is connected if z_1, z_2 can be joined by a polygonal line, consisting of finite number of line segments, joined end to end.

Definition 9 (domain)**.** A non empty open set that is connected is called a domain

Definition 10 (region)**.** A domain together with some, none, or all of its boundary points is referred to as a region

Definition 11 (accumulation point)**.** An accumulation point or limit point of a set S is z_0 if, each deleted neighborhood of z_0 contains at least one point of S

Remark. A closed set contains all of its accumulation points, but the opposite may not be true.

Remark. Every boundary point is not an accumulation point.

Example. Consider the set, $S = 5 \cup (0,1)$

Here, the boundary points are $5, 0$ and 1 because they ε-neighborhood defined around these points contains both inerior points and exterior points.

However 5 is not an accumulation point because the deleted-neighborhood does not contain any interior points (as it removes 5). $\diamond$

Chapter 2

Analytic functions

13. Functions and Mappings

A translation translate a complex number to another location preserving direction and magnitude.

Example. $f(z) = z_0 + z$ ◇

A rotation rotates the complex number changing magnitude or direction.

Example. $f(z) = z_0 z$ This function rotates z by multiplying it with z_0. We can see this when representing it in euler notation as follows,

$$z_0 z = r r_0 e^{i(\theta+\theta_0)}.$$

◇

Example. $f(z) = z^2$

$$z = re^{i\theta}$$
$$z^2 = r^2 e^{2i\theta}$$

So magnitude is squared and angle is doubled ◇

A reflection will reflect z along the x axis.

Example. $f(z) = \bar{z}$ reflects z along the x axis. ◇

An analytic function is a differentiable function in the complex space.

$$f(z) = w.$$

$$f(x + iy) = u + iv.$$
$$= u(x, y) + iv(x, y).$$
$$u(z) = iv(z).$$

15. Limits

If a function f is defined at all points z in some deleted neighborhood of point z_0. Then, $f(z)$ has a limit w_0 as z approaches z_0, or

$$\lim_{z \to z_0} f(z) = w_0.$$

Essentially this means that the point $w = f(z)$ can be made arbitrary close to w_0 if we choose a point z close enouhg to z_0 but distinct from it (deleted neighborhood).

Definition 12 (Limit)**.** The limit of a function $f(z)$ as z goes to z_0 is w_0 if, $\forall \varepsilon > 0, \exists \delta > 0, s.t.$

$$|f(z) - w_0| < \varepsilon \text{ whenever, } 0 < |z - z_0| < \delta.$$

Remark. Essentially this menas that for every ε-neighborhood, $|f(z) - w_0| < \varepsilon$ there is a deleted-neighborhood, $0 < |z - z_0| < \delta$ of z_0 such that every point z in it has an image w in the ε-neighborhood

Remark. All points in the deleted-neighborhood are to be considered but their images need not fill up the ε-neighborhood

Theorem 1. When a limit of a function $f(z)$ exists at a point z_0, it is unique.

Proof. Suppose,

$$\lim_{z \to z_0} f(z) = w_0 \text{ and } \lim_{z \to z_0} f(z) = w_1.$$

This means that,

$$|f(z) - w_0| < \varepsilon \text{ when } 0 < |z - z_0| < \delta_0.$$

$$|f(z) - w_1| < \varepsilon \text{ when } 0 < |z - z_1| < \delta_1.$$

So,

$$|f(z) - w_0| + |f(z) - w_1| < 2\varepsilon.$$

We know that,

$$w_1 - w_0 = (f(z) - w_0) - (f(z) - w_1) \leq |f(z) - w_0| - |f(z) - w_1|$$

So,

$$w_1 - w_0 < 2\varepsilon, \text{where } \varepsilon \text{ can be chosen arbitrary small.}$$

Hence,

$$w_1 - w_0 = 0, \text{or, } w_1 = w_0.$$

□

Example. Show that, $f(z) = \frac{i\bar{z}}{2}$ in the open disk $|z| < 1$, then

$$\lim_{z \to 1} f(z) = \frac{i}{2}$$

$$\left| f(z) - \frac{i}{2} \right| = \left| \frac{i\bar{z}}{2} - \frac{i}{2} \right| = \frac{|z-1|}{2}.$$

Hence, for any z and ε,

$$\left| f(z) - \frac{i}{2} \right| < \varepsilon \text{ when } 0 < |z-1| < 2\varepsilon.$$

◇

Example. $f(z) = \frac{z}{\bar{z}}$ The limit,

$$\lim_{z \to 0} f(z).$$

does not exist.

Assume that it exists, that implies that by letting the point $z = (x, y)$ we can approach the point, $(0, 0)$ in any manner and we would get the same limit.

Now if we approach the point from the $x-$axis where $z = (x, 0)$ we get,

$$\lim_{x \to 0} f((x, 0)) = \frac{x + 0i}{x - 0i} = 1.$$

But if we approach it from the $y-$ axis where, $z = (0, y)$ we get,

$$\lim_{y \to 0} f((0, y)) = \frac{0 + iy}{0 - iy} = -1.$$

But we know that the limit should be unique, hence this implies that the limit does not exist. ◇

19. Derivatives

Theorem 2. If a function $f(z)$ is continuous and non-zero at a point z_0 then, there exists a neighborhood where, $f(z) \neq 0$ throughout.

Proof. We know that $f(z)$ is continuous which means that, $\varepsilon > 0, \exists \delta$ such that,

$$|f(z) - f(z_0)| < \varepsilon, \text{ when } 0 < |z - z_0| < \delta.$$

But if we take, $\varepsilon = \frac{f(z_0)}{2}$ then we have,

$$|f(z) - f(z_0)| < \frac{|f(z_0)|}{2}.$$

However, if $f(z) = 0$ for this neighborhood then we have,

$$|f(z_0)| < \frac{|f(z_0)|}{2}.$$

which is a contradiction.

□

Theorem 3. f is continuous on R which is closed and bounded, $\exists M > 0$, real $|f(z)| \leq M, \forall z \in R$ equality holds for at least one z.

Definition 13 (Derivative)**.** f is differntiable at z_0 when $f'(z_0)$ exists where,

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

Remark. Can also solve,

$$\lim_{z_0 \to 0} \frac{f(z + z_0) - f(z)}{z_0}$$

.

Example. Find derivative of, $f(z) = \frac{1}{z}$

$$\lim_{z_0 \to 0} (\frac{1}{z + z_0} - \frac{1}{z})\frac{1}{z_0}$$

$$\lim_{z_0 \to 0} \frac{z - z - z_0}{z(z + z_0)} \frac{1}{z_0}$$

$$\lim_{z_0 \to 0} \frac{-1}{z(z + z_0)}$$

$$= \frac{-1}{z^2}$$

◇

Example. $f(z) = \bar{z}$

$$\lim_{z_0 \to 0} \frac{z \bar{+} z_0 - \bar{z}}{z_0}$$

Go from x and y axis.

From x,

$$\lim_{x_0 \to 0} \frac{\bar{z} + x_0 - \bar{z}}{x_0} = 1.$$

Similarly if we go from y we get -1, so the derivative doesn't exist.

◇

If we have a function $f(z) = u(x, y) + iv(x, y)$ then,

$$z_0 = x_0 + iy_0.$$

$$\Delta z = \Delta x + i\Delta y.$$

We have to show the following exist,

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}.$$

$$= \frac{u(x_0 + \Delta x, y_0 + \Delta y) + iv(x_0 + \Delta x, y_0 + \Delta y) - (u(x_0, y_0) + iv(x_0, y_0))}{\Delta x + i\Delta y}.$$

Horizontally, $\Delta y = 0$.

So,

$$\frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} + \frac{i(v(x_0 + \Delta x, y_0) - v(x_0, y_0))}{\Delta x}.$$

$$= u_x + iv_x.$$

Similary, if we go vertically, $\Delta x = 0$ and we get,

$$= v_y - iu_y.$$

Theorem 4. If, $f(z) = u + iv, f'(z)$ exists at, $z_0 = x_0 + iy_0$. Then, u_x, u_y, v_x, v_y eixsts at (x_0, y_0) and must satisify the Cauchy-Reimann equation.

$$f'(z_0) = u_x + iv_x \text{ at } (x_0, y_0).$$

Theorem 5. $f(z) = u(x, y)+iv(x, y)$ defined throughout the ε-neighborhood of $z_0 = x_0 + iy_0$,

(a) u_x, u_y, v_x, v_y exists everywhere in the neighborhood

(b) u_x, u_y, v_x, v_y continuous at (x_0, y_0) and satisfy the Cauchy-Remainn equations

$$u_x = v_y, u_y = -v_x \text{ at } (x_0, y_0)$$

Then $f'(z_0)$ exists and,

$$f'(z_0) = u_x + iv_x \text{ at } (x_0, y_0).$$

Proof. We need to show,

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}.$$

$$= \lim_{\Delta z \to 0} \frac{u(x_0 + \Delta x, y_0 + \Delta y) + iv(x_0 + \Delta x, y_0 + \Delta y) - (u(x_0, y_0) + iv(x_0, y_0))}{\Delta z}.$$

Using taylor expansion we know,

$$f(x + \Delta x) = f(x) + \Delta x f'(x) + \frac{(\Delta x)^2}{2} f''(x).$$

$$u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0) =$$

$$= u(x_0, y_0) + \Delta x u_x(x_0, y_0) + \frac{(\Delta x)^2}{2} u_{xx}(x_0, y_0) + \Delta y u_y(x_0, y_0) + \frac{(\Delta y)^2}{2} u_{yy}(x_0, y_0).$$

We can write the limit as,

$$\frac{u_x(x_0, y_0)\Delta x + u_y(x_0, y_0)\Delta y + \varepsilon_1(\Delta x^2) + \varepsilon_2(\Delta y^2)}{\Delta z} +$$

$$i\frac{v_x(x_0, y_0)\Delta x + v_y(x_0, y_0)\Delta y + \varepsilon_3(\Delta x) + \varepsilon_4(\Delta y)}{\Delta z}.$$

We know $u_x(x_0, y_0) = v_y(x_0, y_0)$ and $u_y(x_0, y_0) = -v_x(x_0, y_0)$, so,

$$\frac{u_x(x_0, y_0)\Delta x + -v_x(x_0, y_0)\Delta y + \varepsilon_1(\Delta x^2) + \varepsilon_2(\Delta y^2)}{\Delta z} +$$

$$i\frac{v_x(x_0, y_0)\Delta x + u_x(x_0, y_0)\Delta y + \varepsilon_3(\Delta x) + \varepsilon_4(\Delta y)}{\Delta z}.$$

$$= \frac{u_x(x_0, y_0)(\Delta x + i\Delta y) + u_y(x_0, y_0)(\Delta y - i\Delta x) + \varepsilon_4}{\Delta z}.$$

and $\Delta z = \Delta x + i\Delta y$

$$= \frac{u_x(x_0, y_0)(\Delta x + i\Delta y) + u_y(x_0, y_0)(\Delta y - i\Delta x) + \varepsilon_4}{\Delta x + i\Delta y}.$$

□

Definition 14 (Analytic function)**.** A function f is analytic in an open set S, if f has derivative everywhere in S. It is analytic at a point z_0 if it is analytic in some neighborhood of z_0

Remark. Analytic functino has to be on an open set.

Remark. For it to be analytic at z_0 derivative should exist in the neighborhood of z_0 (not just the point z_0)

Example. $f(z) = (|z|)^2 = \sqrt{x^2 + y^2}^2$

$$u = x^2 + y^2, v = 0$$

$$u_x = 2x, u_y = 2y.$$
$$v_x = 0, v_y = 0.$$

So the Cauchy-Reimann equation is only satisfied at $(0, 0)$

$f'(0) = 0$ and it exists. ◇

Remark. $f(z) = |z|^2$ is not analytic anywhere. So even if the derivative exists at $z = 0$. The function is not analytic at $z = 0$ (or at any point)

Because, (1). $f'(z)$ exists at $z = 0$

(2). u_x, u_y, v_x, v_y exists $\not\Rightarrow f'(z)$

(3). $f(z)$ is continuous $\not\Rightarrow f'(z)$

Essentially it only exists for $z = 0$ and not in the neighborhood around it.

Definition 15 (Entire function)**.** A function f is analytic at each point in the entire plane.

Definition 16 (Singular point)**.** z_0 is a singular point if f fails to be analytic at z_0 but is analytic at some point in every neighborhood at z_0

Example. $f(z) = 2 + 3z^2 + z^3$

Is analytic everywhere so it is an entire function ◇

Example. $f(z) = \frac{1}{z}$

Is analytic at all non-zero, but $z = 0$ is a sigular point ◇

Example. $f(z) = |z|^2 = x^2 + y^2$

Is not analytic, no singular points either. ◇

Polar Coordinates