Complex Analysis

Aamod Varma

MATH - 4320, Fall 2024

Contents

Chapter 1

Complex Numbers

1.12 Regions in the Complex Plane

Definition 1.1 (Epsilon neighborhood)**.** An epsilon neighborhood around a point z_0 is the set of all z such that,

$$|z - z_0| < \varepsilon$$

Definition 1.2 (Deleted neighborhood)**.** A deleted neighborhood around a point z_0 is the set of all z such that,

$$0 < |z - z_0| < \varepsilon$$

Remark. A deleted neighborhood is essentially an epsilon neighborhood but does not include the point z_0

Definition 1.3 (Interior point)**.** z_0 is an interior point when there exists a neighborhood of z_0 that contains only points of S

Definition 1.4 (Exeterior point)**.** z_0 is an exterior point when there exists a neighborhood of z_0 that contains no points of S

Definition 1.5 (Boundary point)**.** z_0 is a boundary point otherwise, i.e. all of the neighborhoods of z_0 contains a point in S and a point not in S

Definition 1.6 (Open set)**.** S is an open set if $\forall z \in S, \exists \varepsilon$ s.t. $B_\varepsilon(z) \subset S$

Remark. We can also say that an open set does not contain any of its boundary points.

Definition 1.7 (Closed set). A set is closed if it doesn't contain its boundary points.

Definition 1.8 (Connected Set). An open set is connected if z_1, z_2 can be joined by a polygonal line, consisting of finite number of line segments, joined end to end.

Definition 1.9 (domain). A non empty open set that is connected is called a domain

Definition 1.10 (region). A domain together with some, none, or all of its boundary points is referred to as a region

Definition 1.11 (accumulation point). An accumulation point or limit point of a set S is z_0 if, each deleted neighborhood of z_0 contains at least one point of S

Remark. A closed set contains all of its accumulation points, but the opposite may not be true.

Remark. Every boundary point is not an accumulation point.

Example. Consider the set, $S = 5 \cup (0, 1)$
Here, the boundary points are $5, 0$ and 1 because they ε-neighborhood defined around these points contains both inerior points and exterior points.
However 5 is not an accumulation point because the deleted-neighborhood does not contain any interior points (as it removes 5). $\diamond$

Chapter 2

Analytic functions

2.1 13. Functions and Mappings

A translation translate a complex number to another location preserving direction and magnitude.

Example. $f(z) = z_0 + z$ ◇

A rotation rotates the complex number changing magnitude or direction.

Example. $f(z) = z_0 z$ This function rotates z by multiplying it with z_0. We can see this when representing it in euler notation as follows,

$$z_0 z = r r_0 e^{i(\theta+\theta_0)}.$$

◇

Example. $f(z) = z^2$

$$z = re^{i\theta}$$
$$z^2 = r^2 e^{2i\theta}$$

So magnitude is squared and angle is doubled ◇

A reflection will reflect z along the x axis.

Example. $f(z) = \bar{z}$ reflects z along the x axis. ◇

An analytic function is a differentiable function in the complex space.

$$f(z) = w.$$

$$f(x + iy) = u + iv.$$
$$= u(x, y) + iv(x, y).$$
$$u(z) = iv(z).$$

2.2 15. Limits

If a function f is defined at all points z in some deleted neighborhood of point z_0. Then, $f(z)$ has a limit w_0 as z approaches z_0, or

$$\lim_{z \to z_0} f(z) = w_0.$$

Essentially this means that the point $w = f(z)$ can be made arbitrary close to w_0 if we choose a point z close enouhg to z_0 but distinct from it (deleted neighborhood).

Definition 2.1 (Limit). The limit of a function $f(z)$ as z goes to z_0 is w_0 if, $\forall \varepsilon > 0, \exists \delta > 0, s.t.$

$$|f(z) - w_0| < \varepsilon \text{ whenever, } 0 < |z - z_0| < \delta.$$

Remark. Essentially this menas that for every ε-neighborhood, $|f(z) - w_0| < \varepsilon$ there is a deleted-neighborhood, $0 < |z - z_0| < \delta$ of z_0 such that every point z in it has an image w in the ε-neighborhood

Remark. All points in the deleted-neighborhood are to be considered but their images need not fill up the ε-neighborhood

Theorem 2.2. When a limit of a function $f(z)$ exists at a point z_0, it is unique.

Proof. Suppose,

$$\lim_{z \to z_0} f(z) = w_0 \text{ and } \lim_{z \to z_0} f(z) = w_1.$$

This means that,

$$|f(z) - w_0| < \varepsilon \text{ when } 0 < |z - z_0| < \delta_0.$$

$$|f(z) - w_1| < \varepsilon \text{ when } 0 < |z - z_1| < \delta_1.$$

So,

$$|f(z) - w_0| + |f(z) - w_1| < 2\varepsilon.$$

We know that,

$$w_1 - w_0 = (f(z) - w_0) - (f(z) - w_1) \leq |f(z) - w_0| - |f(z) - w_1|$$

So,

$$w_1 - w_0 < 2\varepsilon, \text{where } \varepsilon \text{ can be chosen arbitrary small.}$$

Hence,

$$w_1 - w_0 = 0, \text{or, } w_1 = w_0.$$

□

Example. Show that, $f(z) = \frac{i\bar{z}}{2}$ in the open disk $|z| < 1$, then

$$\lim_{z \to 1} f(z) = \frac{i}{2}$$

$$\left| f(z) - \frac{i}{2} \right| = \left| \frac{i\bar{z}}{2} - \frac{i}{2} \right| = \frac{|z-1|}{2}.$$

Hence, for any z and ε,

$$\left| f(z) - \frac{i}{2} \right| < \varepsilon \text{ when } 0 < |z-1| < 2\varepsilon.$$

$\diamond$

Example. $f(z) = \frac{z}{\bar{z}}$ The limit,

$$\lim_{z \to 0} f(z).$$

does not exist.
Assume that it exists, that implies that by letting the point $z = (x, y)$ we can approach the point, $(0, 0)$ in any manner and we would get the same limit. Now if we approach the point from the $x-$axis where $z = (x, 0)$ we get,

$$\lim_{x \to 0} f((x, 0)) = \frac{x + 0i}{x - 0i} = 1.$$

But if we approach it from the $y-$ axis where, $z = (0, y)$ we get,

$$\lim_{y \to 0} f((0, y)) = \frac{0 + iy}{0 - iy} = -1.$$

But we know that the limit should be unique, hence this implies that the limit does not exist. $\diamond$

2.3 19. Derivatives

Theorem 2.3. If a function $f(z)$ is continuous and non-zero at a point z_0 then, there exists a neighborhood where, $f(z) \neq 0$ throughout.

Proof. We know that $f(z)$ is continuous which means that, $\varepsilon > 0, \exists \delta$ such that,

$$|f(z) - f(z_0)| < \varepsilon, \text{ when } 0 < |z - z_0| < \delta.$$

But if we take, $\varepsilon = \frac{f(z_0)}{2}$ then we have,

$$|f(z) - f(z_0)| < \frac{|f(z_0)|}{2}.$$

However, if $f(z) = 0$ for this neighborhood then we have,

$$|f(z_0)| < \frac{|f(z_0)|}{2}.$$

which is a contradiction.

□

Theorem 2.4. f is continuous on R which is closed and bounded, $\exists M > 0$, real $|f(z)| \leq M, \forall z \in R$ equality holds for at least one z.

Definition 2.5 (Derivative). f is differntiable at z_0 when $f'(z_0)$ exists where,

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

Remark. Can also solve,

$$\lim_{z_0 \to 0} \frac{f(z + z_0) - f(z)}{z_0}$$

.

Example. Find derivative of, $f(z) = \frac{1}{z}$

$$\lim_{z_0 \to 0} (\frac{1}{z + z_0} - \frac{1}{z}) \frac{1}{z_0}$$
$$\lim_{z_0 \to 0} \frac{z - z - z_0}{z(z + z_0)} \frac{1}{z_0}$$
$$\lim_{z_0 \to 0} \frac{-1}{z(z + z_0)}$$
$$= \frac{-1}{z^2}$$

◇

Example. $f(z) = \bar{z}$

$$\lim_{z_0 \to 0} \frac{\overline{z + z_0} - \bar{z}}{z_0}$$

Go from x and y axis.
From x,

$$\lim_{x_0 \to 0} \frac{\bar{z} + x_0 - \bar{z}}{x_0} = 1.$$

Similarly if we go from y we get -1, so the derivative doesn't exist.

◇

If we have a function $f(z) = u(x, y) + iv(x, y)$ then,

$$z_0 = x_0 + iy_0.$$

$$\Delta z = \Delta x + i\Delta y.$$

We have to show the following exist,

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}.$$

$$= \frac{u(x_0+\Delta x, y_0+\Delta y) + iv(x_0+\Delta x, y_0+\Delta y) - (u(x_0,y_0)+iv(x_0,y_0))}{\Delta x + i\Delta y}.$$

Horizontally, $\Delta y = 0$.
So,

$$\frac{u(x_0+\Delta x, y_0) - u(x_0,y_0)}{\Delta x} + \frac{i(v(x_0+\Delta x, y_0) - v(x_0,y_0))}{\Delta x}.$$

$$= u_x + iv_x.$$

Similary, if we go vertically, $\Delta x = 0$ and we get,

$$= v_y - iu_y.$$

Theorem 2.6. If, $f(z) = u + iv, f'(z)$ exists at, $z_0 = x_0 + iy_0$. Then, u_x, u_y, v_x, v_y eixsts at (x_0, y_0) and must satisify the Cauchy-Reimann equation.

$$f'(z_0) = u_x + iv_x \text{ at } (x_0, y_0).$$

Theorem 2.7. $f(z) = u(x,y)+iv(x,y)$ defined throughout the ε-neighborhood of $z_0 = x_0 + iy_0$,
(a) u_x, u_y, v_x, v_y exists everywhere in the neighborhood
(b) u_x, u_y, v_x, v_y continuous at (x_0, y_0) and satisfy the Cauchy-Remainn equations

$$u_x = v_y, u_y = -v_x \text{ at } (x_0, y_0)$$

Then $f'(z_0)$ exists and,

$$f'(z_0) = u_x + iv_x \text{ at } (x_0, y_0).$$

Proof. We need to show,

$$\lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}.$$

$$= \lim_{\Delta z \to 0} \frac{u(x_0+\Delta x, y_0+\Delta y) + iv(x_0+\Delta x, y_0+\Delta y) - (u(x_0,y_0)+iv(x_0,y_0))}{\Delta z}.$$

Using taylor expansion we know,

$$f(x+\Delta x) = f(x) + \Delta x f'(x) + \frac{(\Delta x)^2}{2} f''(x).$$

$$u(x_0+\Delta x, y_0+\Delta y) - u(x_0,y_0) =$$

$$= u(x_0,y_0) + \Delta x u_x(x_0,y_0) + \frac{(\Delta x)^2}{2} u_{xx}(x_0,y_0) + \Delta y u_y(x_0,y_0) + \frac{(\Delta y)^2}{2} u_{yy}(x_0,y_0).$$

We can write the limit as,

$$\frac{u_x(x_0,y_0)\Delta x + u_y(x_0,y_0)\Delta y + \varepsilon_1(\Delta x^2) + \varepsilon_2(\Delta y^2)}{\Delta z} +$$

$$i\frac{v_x(x_0,y_0)\Delta x + v_y(x_0,y_0)\Delta y + \varepsilon_3(\Delta x) + \varepsilon_4(\Delta y)}{\Delta z}.$$

We know $u_x(x_0, y_0) = v_y(x_0, y_0)$ and $u_y(x_0, y_0) = -v_x(x_0, y_0)$, so,

$$\frac{u_x(x_0,y_0)\Delta x + -v_x(x_0,y_0)\Delta y + \varepsilon_1(\Delta x^2) + \varepsilon_2(\Delta y^2)}{\Delta z} +$$

$$i\frac{v_x(x_0,y_0)\Delta x + u_x(x_0,y_0)\Delta y + \varepsilon_3(\Delta x) + \varepsilon_4(\Delta y)}{\Delta z}.$$

$$= \frac{u_x(x_0,y_0)(\Delta x + i\Delta y) + u_y(x_0,y_0)(\Delta y - i\Delta x) + \varepsilon_4}{\Delta z}.$$

and $\Delta z = \Delta x + i\Delta y$

$$= \frac{u_x(x_0,y_0)(\Delta x + i\Delta y) + u_y(x_0,y_0)(\Delta y - i\Delta x) + \varepsilon_4}{\Delta x + i\Delta y}.$$

□

Definition 2.8 (Analytic function)**.** A function f is analytic in an open set S, if f has derivative everywhere in S. It is analytic at a point z_0 if it is analytic in some neighborhood of z_0

Remark. Analytic functino has to be on an open set.

Remark. For it to be analytic at z_0 derivative should exist in the neighborhood of z_0 (not just the point z_0)

Example. $f(z) = (|z|)^2 = \sqrt{x^2+y^2}^2$

$$u = x^2 + y^2, v = 0$$

$$u_x = 2x, u_y = 2y.$$

$$v_x = 0, v_y = 0.$$

So the Cauchy-Reimann equation is only satisfied at $(0, 0)$
$f'(0) = 0$ and it exists. ◇

Remark. $f(z) = |z|^2$ is not analytic anywhere. So even if the derivative exists at $z = 0$. The function is not analytic at $z = 0$ (or at any point)
Because, (1). $f'(z)$ exists at $z = 0$
(2). u_x, u_y, v_x, v_y exists $\not\Rightarrow f'(z)$
(3). $f(z)$ is continuous $\not\Rightarrow f'(z)$
Essentially it only exists for $z = 0$ and not in the neighborhood around it.

Definition 2.9 (Entire function)**.** A function f is analytic at each point in the entire plane.

Definition 2.10 (Singular point)**.** z_0 is a singular point if f fails to be analytic at z_0 but is analytic at some point in every neighborhood at z_0

Example. $f(z) = 2 + 3z^2 + z^3$
Is analytic everywhere so it is an entire function ◇

Example. $f(z) = \frac{1}{z}$
Is analytic at all non-zero, but $z = 0$ is a sigular point ◇

Example. $f(z) = |z|^2 = x^2 + y^2$
Is not analytic, no singular points either. ◇

2.4 Harmonic Function

Definition 2.11 (Harmonic function). A real valued functino of $H(x, y)$ is said to be harmonic if in a given domain of the x, y plane, it has a continuous partial derivative of the first and second order $(H_x, H_y, H_{xx}, H_{yy}, H_{xy})$ and satisfies,

$$H_{xx}(x, y) + H_{yy}(x, y) = 0 \text{ Laplace equation.}$$

Theorem 2.12. If $f = u(x, y) + i(x, y)$ is analytic in a domain D, then u, v are harmonic in D

Theorem 2.13. If $f'(z) = 0$ everywhere in D then $f(z)$ is a constant in D.

Proof. Consider $f(z) = u(x, y) + iv(x, y)$ given that

$$f'(z) = u_x + iv_x = 0$$

Using Cauchy-Reimann equation we have, $u_y, v_y = 0$. So all of the first order derivative s are equal to 0 in D.
$U(x, y)$ is constant along any line L, extending from p to p'. Let the vector from p to p' be u. So we have,

$$\frac{du}{ds} = (\text{grad } u)u$$

$$\text{grad } u = u_x i + u_y j = 0$$

So u is a constant (a) on L. Similarly for $v = b$

$$f(z) = a + bi$$

□

Lemma 2.14. Suppose,
(a). $f(z)$ is analytic throughout D
(b). $f(z) = 0$ at each point at the domain or line segment containing D
Then $f(z) \equiv 0$ in D

Chapter 3

Elementary Functions

3.1 Exponential Function

The exponential function is e^z. But we can write this as

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(cos(y) + isin(y))$$

We can also write,

$$e^z = \rho e^{i\phi} \text{ where } \rho = |e^x| \text{ and } \phi = y$$

For a function, $e^{z_1}e^{z_2}$ we can write,

$$e^{z_1}e^{z_2} = e^{x_1+iy_1}e^{x_2+iy_2}$$

$$= e^{x_1+x_2}e^{i(y_1+y_2)}.$$

$$= e^{z_1+z_2}.$$

The derivative if e^z is an entire function

$$\frac{d}{dx}e^z = e^z \text{which is an entire function.}$$

$$e^{z+2} = e^z + e^2 = e^z$$

3.2 Log Function

The log function is $f(z) = log(z) = w = u + iv$. We know

$$e^w = z = e^{u+iv} = e^u e^{iv}.$$

We see that $r = e^u$ and $\theta = v + 2n\pi$

$$r = e^u \Rightarrow ln(r) = u$$

Similarly,

$$\theta = v + 2n\pi.$$

So we have,

$$f(z) = \log(z) = \ln|z| + i\arg(z).$$

and the principal direction is,

$$f(z) = \log(z) = \ln|z| + i\theta, \quad -\pi < \theta < \pi.$$

Some properties are,
(1).$e^{\log z} = z, (z \neq 0)$
(2).$|e^z| = e^x$
(3).$\log(e^z) = \ln|e^z| + i\arg(e^z)$

$$= \ln|e^x| + i(y + 2n\pi), n = 0, \pm 1, \pm 2.$$

$$= \ln e^x + iy + i2n\pi.$$

$$= z + 2n\pi.$$

Branches

The principal branch is

$$\log z = \ln r + i\theta \text{ where } r > 0, -\pi < \theta < \pi.$$

A branch cut is a portion of a line or curve that is introduced in order to deifne a branch F of a multiple-valued function f.
Points on the branch cut for F are singular points of F and any point that is common to all branches of f are called branch points.

Example.

$$\frac{d}{dz}\log z = \frac{1}{z}, \text{ where } |z| > 0$$

The branches can be $\alpha < \arg z < \alpha + 2\pi$ ◇

Property. $\log z_1 z_2 = \log z_1 + \log z_2$

Proof.

$$\log z_1 z_2 = \ln(r_1 r_2) + i(\theta_1 + \theta_2 + 2n\pi)$$

$$= \log z_1 z_2 = \ln(r_1) + \ln(r_2) + i(\theta_1 + \theta_2 + 2n\pi)$$

$$= \log z_1 z_2 = \ln(r_1) + i(\theta_1 + 2n\pi) + \ln(r_2) + i(\theta_2 + 2n\pi)$$

$$= \log z_1 z_2 = \ln(r_1) + i(\theta_1 + 2n\pi) + \ln(r_2) + i(\theta_2 + 2n\pi)$$

$$= \log z_1 z_2 = \log z_1 + \log z_2$$

□

Property. $\log|z_1 z_2| = \log|z_1| + \log|z_2|$

Property. $\log(\frac{z_1}{z_2)} = \log(z_1) - \log(z_2)$

Property. $z^n = e^{n\log(z)}$

3.3 Power Function

We have a complex number c and we have $f(z) = z^c$. By definition we have $z^c = e^{c \log z}$
The derivative is $\frac{d}{dz}f(z) = \frac{d}{dz}(z^c)$

$$\frac{d}{dz}e^{c\log z} = e^{c\log z}\frac{d}{dz}c\log z = e^{c\log z}\frac{c}{z}$$

But we can write $\frac{e^{c\log z}c}{e^{\log z}} = ce^{(c-1)\log z} = cz^{c-1}$. The principal value of $z^c = e^{cLogz}$
If the function is $f(z) = c^z$ then we have

$$\frac{d}{dz}c^z = \frac{d}{z}e^{z\log c} = e^{z\log c}\frac{d}{dz}z\log c = e^{z\log c}\log c = c^z\log c$$

3.4 Trignometric Function

We know that $e^{i\theta} = \cos\theta + i\sin\theta$ and $e^{-i\theta} = \cos\theta - i\sin\theta$. So we can write,

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

and,

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

We have $\frac{d}{dz}\sin z = \cos z$ and $\frac{d}{dz}\cos z = -\sin z$

Property. $\sin(-z) = -\sin(z)$ and $\cos(-z) = \cos(z)$

Property. $\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$

Property. $\sin(2z) = 2\sin(z)\cos(z)$

Property. $\sin(z + \frac{\pi}{2}) = \cos(z)$

Consider the hyperbolic sin and cos functions,

$$\sinh z = \frac{e^z - e^{-z}}{2}, \cosh z = \frac{e^z + e^{-z}}{2}$$

We can write $\sin z = \sin(x + iy)$. Now expanding this we get,

$$\sin(x)\cos(iy) + \cos(x)\sin(iy) = \sin(x)\cosh(y) + i\cos x\sinh(y)$$

And we have,

$$|\sin z|^2 = \sin^2 x + \sinh^2 y$$
$$|\cos z|^2 = \cos^2 x + \cosh^2 y$$

3.5 Inverse Trignometric Functions

The function is $w = f(z) = \sin^{-1} z$. So we have

$$\sin(w) = z = \frac{e^{iw} - e^{-iw}}{2}$$

We know $2iz = (e^{iw} - e^{-iw}) \times e^{iw}$,

$$2ize^{iw} = e^{2iw} - e^{0}$$

$$e^{iw2} - 2ize^{iw} - 1 = 0$$

Solving this we get,

$$e^{iw} = iz \pm (1 - z^2)^{\frac{1}{2}}$$

Chapter 4

Integrals

Consider $f(z) = f(x+iy) = u(x,y) + iv(x,y)$. We can write this as,

$$w(t) = u(t) + iv(t)$$

$$w'(t) = u'(t) + iv'(y)$$

Example. $\frac{d}{dt}(w(t))^2 = \frac{d}{dt}(u+iv)^2$

$$\begin{aligned}
&= \frac{d}{dt}(u^2 - v^2 + 2uvi) \\
&= 2uu' - 2vv' + i(2u'v + 2uv') \\
&= 2(u+iv)(u'+iv') \\
&= 2w(t)w'(t)
\end{aligned}$$

◇

4.1 Definite Integrals

The integeral of $w(t)$ with respect to t is,

$$\int_a^b w(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt$$

Exercise. Find c such that,

$$\int_a^b w(t)dt = w(c)(b-a) \text{ where } w(t) = e^{it}, a = 0, b = 2\pi$$

Solution. We have,

$$\int_0^{2\pi} e^{it} = \int_0^{2\pi} (\cos(t) + i\sin(t) = [\sin(t) - i\cos(t)]_0^{2\pi} = 0$$

Generally for arbitrary a and b we can show that, ...

Remark. In this case t is moving from 0 to 2π. But because we are in the complex plane it represents a loop.

4.2 Contour

Definition 4.1. We have $z(t) = x(t) = iy(t)$ is a contour if,
(1) C is simple arc or Jordan arc, it does not cross itself.

$$z(t_1) \neq z(t_2), t_{12}$$

(2) $z(a) = z(b)$; C simple closed curve.
It is positively oriented if the direction is anticlockwise

Example. $x = \begin{cases} x + ix, 0 \leq x \leq 1 \\ x + i, 1 \leq x \leq 2 \end{cases}$ ◇

Example. $z = re^{i\theta}, 0 \leq \theta \leq 2\pi$ ◇

Example. $z = re^{i3\theta}, 0 \leq \theta \leq 2\pi$
Not a simple arc ◇

Example. $\int_C w(z)dz = \int_{C_1} f[z(x)]z'(x)dx + \int_{C_2} f[z(x)]z'(x)dx$
Here C is the contour from example (1). ◇

We can define the differential arc to be $z'(t) = x'(t) + y'(t)i$ which is continuous on $a \leq t \leq b$ then, $C : z(t)$ is a differential arc and

$$\int_a^b |z'(t)|dt = \int_a^b \sqrt{|x'(t)|^2 + |y'(t)|^2} \text{ length.}$$

$$L = \int_a^b |z'(t)|dt$$

$$t = \phi(\tau), dt = \phi'(\tau)d\tau$$

$$L = \int_a^b |z'(t)|dt = \int_\alpha^\beta |z'(\phi(\tau))|\phi'(\tau)d\tau$$

$$T = \frac{z'(t)}{|z'(t)|} \text{ tangent vector}$$

Contour: piecewise smooth arc.

4.3 Contour Integral

Consider the integer,

$$\int_C f(z)dz \text{ or } \int_{z_1}^{z_2} f(z)dz$$

We can parametrize in terms of t as,

$$\int_C f(z)dz = \int_a^b f(z(t))z'(t)dt$$

$\int_{-C} f(z)dz$ represents going backwards from the curve.
An integral along a given curve C can be written as a sum of integrals of curves within it,

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz$$

Example. $\int_{C_1} \frac{dz}{z}$ where C_1 is the upper semicircle and $\int_{C_2} \frac{dz}{z}$ is the lower semicircle.

◇

Solution. For C_1
$$z = re^{i\theta}, r = 1, 0 \leq \theta \leq \pi$$
And for C_2 we have,
$$z = re^{i\theta}, r = 1, \pi \leq \theta \leq 2\pi$$

$$dz = ire^{i\theta} d\theta$$
For C_1 we have,
$$\int_{C_1} \frac{dz}{z} = \int_0^{\pi} \frac{1}{e^{i\theta}} ie^{i\theta} d\theta$$
$$= [i\theta]_0^{\pi} = i\pi$$
Similarly for C_2,
$$\int_{C_2} \frac{dz}{z} = -\int_{\pi}^{2\pi} \frac{1}{e^{i\theta}} ie^{i\theta} d\theta$$
$$= -[i\theta]_{\pi}^{2\pi} = [i\pi - i2\pi] = -i\pi$$
We see that it is not path independent.

Theorem 4.2. Suppose a function $f(z)$ is cont. in D the following statements are equivalent,
1. $f(z)$ has an antiderivative $F(z)$ throughout D.
2. Any contours entirely in D all have the same value,
$$\int_{z_1}^{z_2} f(z)dz = F(z)]_{z_1}^{z_2} = F(z_2) - F(z_1)$$
3. $\int_C f(z) = 0$, C closed contours entirely in D

4.4 Branch Cuts

Example. $z = 3e^{i\theta}, (0 \leq \theta \leq \pi)$ ◇

Lemma 4.3. If $w(t)$ is piecewise cont. then,
$$\left| \int_a^b w(t)dt \right| \leq \int_a^b |w(t)|dt$$

Proof. Let,
$$\int_a^b w(t)dt = re^{i\theta}$$

$r = \int_a^b e^{-i\theta} w(t)dt$. Both sides of this equations are real.

$$r = \int_a^b Re[e^{i\theta} w(t)]dt$$

But,

$$Re[e^{i\theta} w(t)] \leq |e^{-i\theta} w(t)| = |e^{-i\theta}||w(t)| \leq |w(t)|$$

So,

$$r \leq \int_a^b |w(t)|dt$$

Or,

$$\left| \int_a^b w(t)dt \right| \leq \int_a^b |w(t)|dt$$

□

Theorem 4.4. C has a length L and $f(z)$ is piecewise cont. on C and let $|f(z)| \leq M$ then,

$$\left| \int_C f(z)dz \right| \leq ML$$

Proof. $\int_C f(z)dz = \int_a^b f(z(t))z'(t)dt$. So we have,

$$\left| \int_C f(z)dz \right| \leq \int_a^b |f(z(t))z'(t)|dt \leq \int_a^b |Mz'(t)|dt = ML$$

□

Theorem 4.5. $f(z)$ is cont. over D then,
(a). $f(z)$ has antiderivative $F(z)$ throughout D
(b). $\int_{z_1}^{z_2} f(z)dz = F(z_2) - F(z_1)$
The antiderivative is independent to the path.
(c). $\int_C f(z)$ where c is a closed contour entirely in D

Proof. 1. (a) $\Rightarrow$ (b). We have, $c : z = z(t), z_1 = z(a), z_2 = z(b)$

$$\frac{d}{dt}[F[z(t)]] = F'[z(t)]z'(t) = f(z)z'(t)$$

So taking,

$$\int_C f(z)dz = \int_a^b f[z(t)]z'(t)dt = F[z(t)]$$

$$= F[z(t)]_a^b = F(z_2) - F(z_1)$$

2. (b) $\Rightarrow$ (c)

$$\int_{C_1} f(z)dz = F(z_2) - F(z_1)$$
$$\int_{C_2} f(z)dz = F(z_2) - F(z_1)$$

So,

$$\int_{C_1} f(z)dz = F(z_2) - F(z_1) = \int_{C_2} f(z)dz = F(z_2) - F(z_1)$$
$$\int_{C_1} f(z)dz = F(z_2) - F(z_1) - \int_{C_2} f(z)dz = F(z_2) - F(z_1) = 0$$

$C = C_1 - C_2$: a closed contour in D

$$\int_C f(z)dz = 0$$

3. (c) $\Rightarrow$ (a)
We define

$$F(z) = \int_{z_1}^{z_2} f(s)ds$$

$$\lim_{\Delta z \to 0} \frac{F(z+\Delta z) - F(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{1}{\Delta z}\left[\int_{z_0}^{z+\Delta z} f(s)ds - \int_{z_0}^{z} f(z)ds\right]$$
$$= \lim_{\Delta z \to 0} \frac{1}{\Delta z} \int_{z}^{z+\Delta z} f(s)ds$$

Remark.

$$\int_{z}^{z+\Delta z} ds = s]_z^{z+\Delta z} = \Delta z$$

Remark.

$$f(z) = \frac{1}{\Delta z} \int_{z}^{z+\Delta z} f(z)ds$$
$$\frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z) = \frac{1}{\Delta z} \int_{z}^{z+\Delta z} [f(s) - f(z)]ds$$

By cont. of $f(z), \forall \varepsilon, \exists \delta$,

$$|f(s) - f(z)| < \varepsilon \text{ whenever } |s - z| < \delta$$

□

4.5 Cauchy-Goursat Theorem

We have $\int_c f(z)dz = \int_a^b f(z(t))z'(t)dt$. We want to show that $\int_C f(z) = 0$ if C is a closed contour. Here,

$$f(z) = u(x,y) + iv(x,y)$$

$$\begin{aligned} z(t) &= x(t) + iy(t) \\ z'(t) &= (x'(t) + iy'(t))dt \end{aligned}$$

$$\begin{aligned} \int_C f(z)dz &= \int_a^b (u+iv)(x'+iy')dt \\ &= \int_a^b (ux'-vy')dt + i\int_a^b (vx'+uy')dt \\ &= \int_C udx - vdy + i\int_C vdx + udy \end{aligned}$$

$$\text{Greens theorem says that,} \int_C Pdx + Qdy = \int\int_R (Qx - Py)dA$$

$$= \int\int_R (-vx - uy)dA + \int\int_R (ux - vy)dA$$

We know that if f is analytic then it satisfies the cauchy reimann equatoin such that,

$$u_x = v_y, u_y = -v_x$$

$$\begin{aligned} &= \int\int_R (0)dA + \int\int_R (0)dA \\ &= 0 \end{aligned}$$

So for any closed curve f such that f is analytic and f' is cont. the integral over the countour is 0.

Theorem 4.6. If a function f is analytic at all points interior to and on a simple closed contour C, then

$$\int_C f(z)dz = 0$$

Proof. We need to show that $\forall\varepsilon$,

$$\left|\int_c f(z)dz - 0\right| < \varepsilon$$

For any region R consisitng of points interior to our contour R can be covered with a finite number of square san dpartial squares such that in each one there is a fixed oint z_j for which,

$$\left|\frac{f(z)-f(z_j)}{z-z_j} - f'(z_j)\right| < \delta_j(z)$$

We can say,
$$\int_C f(z)dz = \sum_{j=1}^{n} \int_{c_j} f(z)dz$$
We have,
(i). C_i is defined
(ii) $\delta_j(z) = \frac{f(z)-f(z_j)}{z-z_j} - f'(z_j)$ when $z \neq z_j$
$$\lim_{z \to z_j} \delta_j(z) = 0$$
We approximate,
$$f(z) = f(z_j) + (z - z_j)f'(z_j) + (z - z_j)\delta_j(z)$$
So,
$$\int_{C_j} f(z)dz = [f(z_j) - z_j f'(z_j)] \int_{C_j} dz$$
$$+ \int_{C_j} f'(z_j)zdz + \int_{Cj} (z - z_j)\delta_j(z)dz$$
The first and secodn term is equal to 0. So we can rewrite it as,
$$= \int_{C_j} (z - z_j)\delta_j(z)dz$$
(iii).
$$\left| \int_C f(z)dz \right| \leq \sum_{j=1}^{n} \left| \int_{C_j} (z - z_j)\delta_j(z)dz \right|$$
$$\left| \int_C f(z)dz \right| \leq \sum \left| \int_{C_j} f(z)dz \right|$$
$$f(z) = f(z_j) - z_j f'(z_j) + f'(z_j)z + (z - z_j)\delta_j(z)$$
The integral of the first three terms cancel out so,
$$\int_C f(z) = \int_C (z - z_j)\delta_j(z)dz$$
$$\leq \sum \left| \int_{C_j} (z - z_j)\delta_j(z)dz \right|$$
$$|z - z_j| \leq \sqrt{2}S_j$$
where S_j is the length of the square. So,
$$|(z - z_j)\delta_j(z)| < \sqrt{2}S_j\varepsilon$$
$$\int_{C_j} |(z - z_j)S_j(z)| < \sqrt{2}S_j\varepsilon 4S_j$$

$$\int_{C_j} |(z-z_j)\delta_j(z)| < \sqrt{2}S_j\varepsilon(4S_j + L_j)$$

So,

$$\sum \left|(z-z_j)\delta_j(z)dz\right| < (4\sqrt{2}S^2 + \sqrt{2}SL)\varepsilon$$

□

Simply Connnected Domain

Theorem 4.7. If f is analytic throughout a simply connected domain D, then,

$$\int_C f(z) = 0$$

for every closed contour lying in D

Exercise. C is in the open disk $|z| < 2$, compute,

$$\int_C \frac{\sin z}{(z^2+9)^5} dz$$

Solution.

Exercise. Givne $C : |z| = 1$,

$$\int_C ze^{-z} dz =$$

Theorem 4.8. Suppose,
(a). C is a closed contour in the counterclockwise direction
(b). $C_k(k = 1, 2, \ldots, n)$ are simply closed contours that are interior to C and are disjoint from each other.

$$\int_C f(z)dz + \sum_{k=1}^{n} \int_{C_j} f(z)dz = 0$$

Proof. We disect the contour such that it goes around our closed interior contours. This means that we have,

$$\int_{C*} f(z)dz = 0$$

We can rewrite this as

$$\int_{C+L_1+\frac{1}{2}C_1+L_2+C_2-L_2+\frac{1}{2}C_1-L_1} f(z)dz$$

$$= \int_C f(z) + \sum_{i=1}^{n} \int_{C_i} f(z)dz = 0$$

□

4.6 Multiply Connected Domain

Corollary 4.9. If C_1, C_2 are positively oriented simply closed contours where C_1 is inteiror to C_2. If f is analytic in the closed region containing C_1, C_2 and all the points in between then,

$$\int_(C_1) f(z)\, dz = \int_(C_2) f(z)\, dz$$

Proof. We construct a new contour between C_1 and C_2 where,

$$\int_{C^*} f(z)\, dz = \int_{C_2 - C_1} f(z)\, dz = 0$$

So,

$$\int_{C_2} f(z)\, dz - \int_{C_1} f(z)\, dz = 0$$

□

Example. C is any positively oriented simple closed contour which is surrounding the origin. ◇

The function $f(z) = \frac{1}{z}$ has a singularity at the origin. So f is analytic upto zero. Because of the principal of deformation of path we can deform any contour surrounding the singularity to a unit circle around with fixed radius preserving the integral. So we have,

$$\int_0^{2\pi} \frac{1}{e^{i\theta}} i e^{i\theta} d\theta$$

$$= 2\pi i$$

Theorem 4.10 (Cauchy Integral Formula)**.** If f is analytic everywhere inside and on the simple closed contour C (positively oriented). If z_0 is interior to Z then,

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0}\, dz$$

Remark. If our integral has a singularity we can easily compute it using the "function" outside of it that doesn't have the singularity.

Proof. We can rewrite the integral as,

$$\int_C \frac{f(z)}{z - z_0}\, dz = \int_{C_p} \frac{f(z)}{z - z_0}\, dz$$

where C_p is the unit circle constructed around z_0 by deforming our contour.

$$
\begin{aligned}
&= \int_{C_p} \frac{f(z)-f(z_0)}{z-z_0} + \frac{f(z_0)}{z-z_0}\, dz \\
&= \int_{C_p} \frac{f(z)-f(z_0)}{z-z_0} + f(z_0)\int_{C_p} \frac{1}{z-z_0}\, dz \\
&= \int_{C_p} \frac{f(z)-f(z_0)}{z-z_0} + f(z_0)2\pi i \;\; dz
\end{aligned}
$$

Because f is analytic as f is cont. around z_0 so, $\forall\varepsilon, \exists\delta$ s.t.,

$$|f(z)-f(z_0)| < \varepsilon \text{ when } |z-z_0| < \delta$$

So,

$$\left| \int_{C_p} \frac{f(z)-f(z_0)}{z-z_0}\, dz \right| < \frac{\varepsilon}{\rho} 2\pi\rho = 2\pi\varepsilon$$

□

Theorem 4.11. Let f be analytic inside and on a simply closed contour C taken in a positive direction and z_0 inside C we have,

$$f^n(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{z-z_0}^{n+1}\, dz$$

Proof. We need to show that $f'(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^2}\, dz$. Using the cauchy integral formula for each term in the numerator we have,

$$
\begin{aligned}
lim_{\Delta z\to 0}\frac{f(z+\Delta z)-f(z)}{\Delta z} &= \frac{1}{\Delta z}\frac{1}{2\pi i}\int_C \frac{f(s)}{s-(z+\Delta z)} - \frac{f(s)}{s-z} \\
&= \frac{1}{\Delta z}\frac{1}{2\pi i}\int_C f(s)\frac{s-z-(s-(z+\Delta z))}{(s-(z+\Delta z)(s-z)}\, dz \\
&= \frac{1}{2\pi i}\int_C \frac{f(s)}{(s-(z+\Delta z))(s-z)}\, dz \\
&= \frac{1}{2\pi i}\int_C f(s)\left(\frac{1}{(s-z)^2} + \frac{\Delta z}{(s-z-\Delta z)(s-z)^2}\right) dz
\end{aligned}
$$

Now we have,

$$\lim_{\Delta z\to 0}\frac{f(z+\Delta z)-f(z)}{\Delta z} = \frac{1}{2\pi i}\int_C f(s)\frac{1}{(s-z)^2}\, dz$$

□

Exercise. $C : |z| = 1, \theta : 0 \to 2\pi$ Find,

$$\int_C \frac{\cos(z)}{(z^2+9)z}\, dz$$

$= \frac{1}{9} 2\pi i$

Exercise.

$$\int_C \frac{e^{2z}}{z^4}\, dz$$

Exercise.

$$\int_C \frac{dz}{z^{n+1}}$$

Exercise.

$$\frac{z}{2z+1}\, dz$$

$= -\pi i$

Exercise.

$$\frac{e^{-z}}{(z-5)(z+2)}$$

$= 0$ because the doesn't exist a singularity within the conutour hence the integral is 0 as the functino is analytic within.