Linear Alebgra HW05

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3A

Problem 1

Proof. We know for a linear map, $T(u+v) = T(u) + T(v)$ and $T(\lambda v) = \lambda T(v)$
First we look at additivity,
Consider an arbitrary $u = (x_1, y_1, z_1)$ and $v = (x_2, y_2, z_2)$. So we have,

$$T(u+v) = T((x_1+x_2),(y_1+y_2),(z_1+z_2))$$

$$= (2(x_1+x_2)-4(y_1+y_2)+3(z_1+z_2)+b, 6(x_1+x_2)+c(x_1+x_2)(y_1+y_2)(z_1+z_2))$$

We need the above to be equal to,

$$T(u)+T(v) = (2x_1-4y_1+3z_1+b, 6x_1+cx_1y_1z_1)+(2x_2-4y_2+3z_2+b, 6x_2+cx_2y_2z_2)$$

$$= (2(x_1+x_2)-4(y_1+y_2)+3(z_1+z_2)+2b, 6(x_1+x_2)+c(x_1y_1z_1+x_2y_2z_2)$$

Comparing each of the terms we have,

$$2(x_1+x_2)-4(y_1+y_2)+3(z_1+z_2)+2b = 2(x_1+x_2)-4(y_1+y_2)+3(z_1+z_2)+b$$

$$2b = b$$

$$b = 0$$

Similarly comparing the second term we have,

$$6(x_1+x_2)+c(x_1+x_2)(y_1+y_2)(z_1+z_2) = 6(x_1+x_2)+c(x_1y_1z_1+x_2y_2z_2)$$

$$c(x_1+x_2)(y_1+y_2)(z_1+z_2) = c(x_1y_1z_1+x_2y_2z_2)$$

$$c((x_1+x_2)(y_1+y_2)(z_1+z_2)-(x_1y_1z_1+x_2y_2z_2)) = 0$$

For this to be true for any x, y, z we need $c = 0$. Hence for additvity we need $b = c = 0$
Now we check if $T(kv) = kT(v)$. Consider $v = (x, y, z)$. Then we have

$$T(kv) = T(kx, ky, kz) = (2kx - k4y + 3kz + b, 6kx + k^3cxyz)$$

We need this to be equal to

$$kT(v) = k(2x-4y+3z+b, 6x+cxyz) = (2kx-4ky+3kz+bk, 6kx+kcxyz)$$

Comparing the terms we have,

$$2kx - 4ky + 3kz + bk = 2kx - 4ky + 3kz + b$$

$$bk = b$$

$$b = 0$$

$6kx + kcxyz = 6kx + k^3cxyz$

$$c = k^2c$$

$$c = 0$$

So we have $b = c = 0$

$\square$

Problem 2

Proof. Similar to (1) but take $p_1 = a_1 + b_1x$ and $p_2 = a_2 + b_2x$ $\square$

Problem 3

Proof. Consider the standard basis $e_1, \dots, e_n$ of F^n. That is

$$\begin{aligned} e_1 &= (1, \dots, 0) \\ e_2 &= (0, 1, \dots, 0) \\ &\dots \\ e_n &= (0, \dots, 1) \end{aligned}$$

We have

$$\begin{aligned} T(x_1, \dots, x_n) &= T(x_1(1, \dots, 0), x_2(0, 1, \dots, 0), \dots x_n(0, \dots, 1)) \\ &= T(x_1e_1, \dots, x_ne_n) \\ &= x_1T(e_1) + \dots + x_nT(e_n) \end{aligned}$$

Let T map e_1 to $(A_{11}, \dots, A_{m1})$ and e_n to $(A_{1n}, \dots, A_{mn})$
So we have,

$$\begin{aligned} &= x_1(A_{11}, \dots, A_{m1}) + \dots x_n(A_{1n}, \dots, A_{mn}) \\ &= (x_1A_{11} + \dots + x_nA_{1n}, \dots, x_1A_{m1} + \dots + x_nA_{mn}) \end{aligned}$$

$\square$

Problem 4

Proof. Let us assume the contrary that $v_1, \dots, v_m$ is linearly dependent. This means that $\exists, a_1, \dots, a_m$ not all zero such that,

$$a_1v_1 + \dots + a_mv_m = 0$$

Now let us apply the lienar map on this vector and we get,

$$T(a_1v_1 + \dots + a_mv_m) = T(0) = 0$$

$$a_1T(v_1) + \dots + a_mT(v_m) = 0$$

Here we see that $\exists$ scalars $a_1, \ldots, a_m$ not all zero such that the linear combination of $T(v_1), \ldots, T(v_m)$ is equal to zero. This means that the list of vectors are linearly dependent. However we know that the list is linearly independent. Hence our assumption must be wrong and $v_1, \ldots, v_m$ are actually linearly independent. $\square$

Probelm 5

Proof. We need to show addivitiy and homogenity.
(1). Additivity
We need to show for any $T_1, T_2 \in L(V, W)$ that $T_1 + T_2 \in L(V, W)$. In other words we need to show that $T_1 + T_2$ is also a linear map.
Consider $v_1, v_2 \in V$ we have

$$(T_1 + T_2)(v_1 + v_2) = T_1(v_1 + v_2) + T_2(v_1 + v_2)$$

$$= T_1(v_1) + T_2(v_1) + T_1(v_2) + T_2(v_2)$$
$$= (T_1 + T_2)(v_1) + (T_1 + T_2)(v_2)$$

Hence we show that $T_1 + T_2$ is additive.
Now consider $v_1 \in V$ we have $(T_1 + T_2)(\lambda v_1)$ we get,

$$= T_1(\lambda v_1) + T_2(\lambda v_1)$$

$$= \lambda T_1(v_1) + \lambda T_2(v_1)$$
$$= \lambda(T_1 v_1 + T_2 v_1)$$
$$= \lambda(T_1 + T_2)v_1$$

Which means that it is homogenous.
Hence we show that $(T_1 + T_2) \in L(V, W)$ or that $L(V, W)$ is additive.
(2). Homogenous
Consider $T \in L(V, W)$ we need to show that λT is a linear map as well.
First we show that λT is additive. Consider v_1, v_2,we have,

$$(\lambda T)(v_1 + v_2) = \lambda(T)(v_1 + v_2)$$

$$= \lambda(T v_1 + T v_2)$$
$$= \lambda T v_1 + \lambda T v_2$$

Which shows that λT is additive.
Now we check homogenous, consider $v \in V$ and $k \in F$ we have,

$$(\lambda T)(kv) = \lambda(T)(kv)$$

$$= \lambda k T(v)$$
$$= k(\lambda T)v$$

Hence we show that λT is homogenous. This makes λT a linear map.
Therefore we show that $L(V, W)$ is a vector space.

$\square$

Problem 6

Proof. 1. Associativity. We have $(T_1T_2)T_3 = T_1(T_2T_3)$
Consider the operation on a vector v so we have, $(T_1T_2)T_3v$ which is,

$$((T_1T_2)(T_3(v)) = T_1(T_2(T_3(v)))$$

Now looking at the right side we have, $T_1(T_2T_3) = T_1(T_2(T_3(v)))$. So we showed tha tthe LHS is equal to the RHS.
2. Identity. Consider a vector v we have,

$$TIv = T(I(v)) = T(v)$$

Now,

$$ITv = I(T(v)) = T(v) \text{ because } Iv = v, \forall v$$

3. Distributive Property
To show that,

$$(S_1 + S_2)T = S_1T + S_2T$$

Consider an abitrary vector v in the domain of T. We have,

$$(S_1 + S_2)Tv = (S_1 + S_2)(T(v))$$

By definitino of addition of linera maps we have,

$$= (S_1(T(v))) + (S_2(T(v)))$$

Simliary we have,

$$(S_1T + S_2T)v = S_1T(v) + S_2T(v) = S_1(T(v)) + S_2(T(v))$$

We see that the distributive property holds.
Now To show that $S(T_1 + T_2) = ST_1 + ST_2$. Consider v we have,

$$S(T_1 + T_2)v = S(T_1(v) + T_2(v)) = S(T_1(v)) + S(T_2(v))$$

And we have,

$$(ST_1 + ST_2)v = ST_1(v) + ST_2(v) = S(T_1(v)) + S(T_2(v))$$

We see that the property holds again.

□

Problem 7

Proof. As T is a linear map from V to itself and V is one dimentional say with basis $\{v'\}$. Then T is defined as

$$T(v') = \lambda v'$$

for some λ

Now we know $\forall v \in V$ we can write v as a linear combination of the basis of V, or

$$v = kv'$$

for some $k \in F$.
We have,

$$T(v') = \lambda v'$$
$$kT(v') = k\lambda v'$$
$$T(kv') = \lambda(kv')$$
$$T(v) = \lambda(v)$$

□

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Proof. Consider the function that maps any vecotor (x, y) to the $max(|x|, |y|)$. We can see that this satisfies homogeneity. For instance consider $(2, 6)$. Our function maps this to 6. Now consider $(2 \times 3, 6 \times 3)$ which is mapped to 18 which is 3×6 as we saw above.
Now consider two vector $(1, 0)$ and $(0, 4)$. Our function maps both these vectors to 1 and 4 respectively. However it maps its sum $(1, 4)$ to $4 \neq 4+1$. Hence it does not follow additvity. Hence not a linear space. □

Problem 9

Proof. Consider the functino that maps any complex number $x + iy$ to x. First we show this functino is linear.
Consider two complex number $x_1 + iy_1$ and $x_2 + iy_2$. We have

$$f(z_1 + z_2) = f((x_1 + x_2) + i(y_1 + y_2)$$
$$= x_1 + x_2$$
$$= f(x_1 + iy_1) + f(x_2 + iy_2)$$
$$= f(z_1) + f(z_2)$$

Now we show it is not homogenous.
Consider $\lambda = i$ then we have,

$$f(\lambda z_1) = f(-y_1 + ix_1)$$
$$= -y_1$$

however we know that $\lambda(fz_1) = ix_1 \neq -y_1$
Hence it is not homogenous.

□

Problem 10

Proof. We show counter example. Assume $q = 1 + x$, $p_1 = x$ and $p_2 = 2x$. We have,

$$q(p_1) = q(x) = 1 + x$$

$$q(p_2) = q(2x) = 1 + 2x$$

$q(p_1 + p_2) = q(3x) = 1 + 3x$
It is easy to see that $1 + 3x \neq 2 + 3x$
Hence T is not additive and not linear. □

Problem 12

Proof. First consider the basis of U as $u_1, \ldots, u_n$. Now let us extend this basis to V as follows, $u_1, \ldots, u_n, v_{n+1}, \ldots, v_m$. We need to shwo that T is not a linear map.
We know that $T(v) = S(v)$ for any $v \in U$. So we have

$$T(u_1) = S(u_1) \neq 0$$

Now consider v_m we have

$$T(v_m) = 0$$

by definition.
Now consider the sum of these vectors and we have $T(u_1 + v_m)$. We know that $u_1 + v_m$ cannot be in U as it cannot be represented as a linear combination of $u_1, \ldots, u_n$ as v_m is linearl independent with $u_1, \ldots, u_n$. Hence $u_1 + v_m \in V$ but $\notin U$. Therefore by defintion we have $T(u_1 + v_m) = 0$.
However we know that $T(u_1) + T(v_m) = S(u_1)$. But $S(u_1) \neq 0$. Which shows us that T is not additive. Hence T is not a linear map. □

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Proof. First consider the basis of U as $u_1, \ldots, u_n$. Now let us extend this basis of U to span V and we have $v_{n+1}, \ldots, v_m$.
Let us define our linear map for our basis as follows,

$$T(u_1) = S(u_1), \ldots, T(u_n) = S(u_n)$$

and

$$T(v_{n+1}) = 0, \ldots, T(v_m) = 0$$

So we have defined as linear map such that for any $u \in U$ say $a_1u_1 + \cdots + a_nu_n$ we have,

$$T(u) = T(a_1u_1 + \cdots + a_nu_n)$$

$$T(u) = a_1(Tu_1) + \cdots + a_nT(u_n)$$

$$= a_1S(u_1) + \cdots + a_nS(u_n)$$

$$= S(a_1u_1 + \cdots + a_nu_n)$$

$$= S(u)$$

□

Problem 14

Proof. We have V is finite dim and W is infinite dim. We need to show that $L(V, W)$ is infinite dimentional or in other words there isn't a basis for $L(V, W)$. We see that a new linear map T is independent from other maps if the range T is distinct from those spaces.
So it is enough to show that there isn't an upperbound on the number of linearly independent lienar maps in L. Or we need to shwo that for any $n \in N$ we can consturct a linearly independent set of linear maps $T_1, \ldots, T_n$
We prove this by induction. First consider consider the base case T_1 that maps to any subspace of W. Now T_1 is linearly independent to itself.
Now let us assume it is true for an artbirary n. That is the list $T_1, \ldots, T_n$ is lienarly independent.

□

Problem 15

Proof. Let us assume the contrary that we can consturct a linear map T such that $Tv_k = w_k$ for any choice of $w_1, \ldots, w_m \in W$.
We know that $v_1, \ldots, v_m$ is linearly dependent. So $\exists v_k$ such that $v_k = a_1 v_1 + \ldots a_{k-1} v_{k-1}$
Now let us choose a choice of $w_1, \ldots, w_n$ as follows, $n \neq k, w_n = 0$ and if $n = k$ then w_n is any arbitary non-zero vector in W.
Based on our assumptino we can construct a map such that $Tv_k = w_k$ for any k so we have,

$$T(v_1) = w_1 = 0$$

$$\ldots$$

$$T(v_k) = w_k = w$$

$$\ldots$$

$$T(v_n) = w_n = 0$$

But we know that $T(v_k) = T(a_1 v_1 + \cdots + a_{k-1} v_{k-1})$. So because T is linear we have,

$$= a_1 T(v_1) + \cdots + a_{k-1} T(v_{k-1})$$

$$= a_1 0 + \ldots_{a\ k-1} 0$$

$$= 0$$

So we have $T(v_k) = 0$. But we just showed above that $T(v_k) = w$ such that $w \neq 0$. Hence we have a contradiction. So our assumption must be wrong and we cannot have a linear map that satifies $Tv_k = w_k$ for any choice of $w_1, \ldots, w_k$

□