# Probability Theory

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# Chapter 1

# Introduction

**Example.** What is the probability that two people among $N$ people have the same birthday. ◇

**Example.** What is the probability that all people have different birthday
We have,

$$
\begin{aligned}
q_1 &= 1 \\
q_2 &= \big(1 - \frac{1}{365}\big) \\
q_3 &= q_3\big(1 - \frac{2}{365}\big) \\
&\vdots \\
q_n &= \prod_{i=1}^{n-1} \big(1 - \frac{i}{365}\big)
\end{aligned}
$$

We get $q_n = 0.14$ which gives us 0.86 for the previous example.

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**Note.** We assume certain assumptions like the following to make this work,

1. Uniformity
2. Independence

Here we have a probability model and deduced the probability of an event,
**Example.** Say there is a test for a disease,

1. P(positive | sick) = 1
2. P(positive | not sick) = 0.01

Need to find P(sick | positive) which would be P(positive | sick) P(sick) / P(positive)
We test everybody, we have Assume 100 S and 100 NS,
100 P from the S, 99 P from the NS
So we have 199 P of which only 100 S which gives around .5

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## 1.1 Probability Theory

Experiment whose outcome is not determined. We define the following,

1. $\Omega$ : Sample space, set of possible outcomes

**Example.** (a) Throw a die,
$\Omega = \{1, 2, 3, 4, 5, 6\} \to$ finite

(b) Flip a coin till heads,
$\Omega = \{1, 2, 3, \dots\} = \mathbb{N} \to$ countably infinite

(c) Time to wait till next bus arrival,
$\Omega = \mathbb{R}^+ \to$ uncountabaly infinite

$\diamond$

2. $F$ : Family of events, $A, B, \dots$
   Something that may or may not happen
   **Example.** (a) For a die we can ask,
   - Is the outcome even?
   - Is the outcome $\leq 3$?

   Here an event $A \subseteq \Omega$ and $|\Omega| = 6$ so $|2^\Omega| = 64$
   We have $F =$ family of events $= 2^\Omega$

   (b) Here we have,
   $\Omega = \mathbb{N}$ so $F = 2^{\mathbb{N}}$

   (c) In this case our sample space is $R^+ = (0, \infty)$. But we cannot take $2^{\mathbb{R}}$. So we axiomatically define $F$ as noted below. Under this definition $F$ is the smallest family that contains all open intervals of $R$

   $\diamond$

3. $P$ : How likely an event is

**Definition 1.1** (Axiomatic definition of $F$)**.** So here we define $F$ to be a family of events of $\Omega$ if,

1. not empty
2. if $A \in F \Rightarrow A^c \in F$ $(A^c = \Omega \setminus A)$
3. for any two $A, B \in F$ then $A \cup B \in F$
4. If $A_i$ for $i = 1, \dots, \infty$ are events, then $\bigcup_{i=1}^{\infty} A_i$ is an event

**Note.** Here, countable closure $\Rightarrow$ finite closure (proof just involves adding infinite $\phi$ to our finite sets $A_1, \dots, A_n$)

**Note.** Using this definition we have,

1. $A \in F \Rightarrow A^c \in F, \Rightarrow A \cup A^c = \Omega \in F$ and $\phi = \Omega^c \in F$
   So every event space has $\Omega, \phi$
2. $(A \cup B)^c = A^c \cap B^c \in F$ so,
   If $A_i, i = 1, 2, \dots$ are events then we have,
   $(\bigcap_{i=1}^{\infty} A_i)^c \in F = \bigcup_{i=1}^{\infty} A_i^c \in F$