Complex Analysis

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Contents

Chapter 1

Complex Numbers

1.12 Regions in the Complex Plane

Definition 1.1 (Epsilon neighborhood)**.** An epsilon neighborhood around a point z_0 is the set of all z such that,

$$|z - z_0| < \varepsilon$$

Definition 1.2 (Deleted neighborhood)**.** A deleted neighborhood around a point z_0 is the set of all z such that,

$$0 < |z - z_0| < \varepsilon$$

Remark. A deleted neighborhood is essentially an epsilon neighborhood but does not include the point z_0

Definition 1.3 (Interior point)**.** z_0 is an interior point when there exists a neighborhood of z_0 that contains only points of S

Definition 1.4 (Exeterior point)**.** z_0 is an exterior point when there exists a neighborhood of z_0 that contains no points of S

Definition 1.5 (Boundary point)**.** z_0 is a boundary point otherwise, i.e. all of the neighborhoods of z_0 contains a point in S and a point not in S

Definition 1.6 (Open set)**.** S is an open set if $\forall z \in S, \exists \varepsilon$ s.t. $B_\varepsilon(z) \subset S$

Remark. We can also say that an open set does not contain any of its boundary points.

Definition 1.7 (Closed set)**.** A set is closed if it doesn't contain its boundary points.

Definition 1.8 (Connected Set)**.** An open set is connected if z_1, z_2 can be joined by a polygonal line, consisting of finite number of line segments, joined end to end.

Definition 1.9 (domain)**.** A non empty open set that is connected is called a domain

Definition 1.10 (region). A domain together with some, none, or all of its boundary points is referred to as a region

Definition 1.11 (accumulation point). An accumulation point or limit point of a set S is z_0 if, each deleted neighborhood of z_0 contains at least one point of S

Remark. A closed set contains all of its accumulation points, but the opposite may not be true.

Remark. Every boundary point is not an accumulation point.

Example. Consider the set, $S = 5 \cup (0, 1)$
Here, the boundary points are $5, 0$ and 1 because they ε-neighborhood defined around these points contains both inerior points and exterior points.
However 5 is not an accumulation point because the deleted-neighborhood does not contain any interior points (as it removes 5). $\diamond$

Chapter 2

Analytic functions

2.1 13. Functions and Mappings

A translation translate a complex number to another location preserving direction and magnitude.

Example. $f(z) = z_0 + z$ ◇

A rotation rotates the complex number changing magnitude or direction.

Example. $f(z) = z_0 z$ This function rotates z by multiplying it with z_0. We can see this when representing it in euler notation as follows,

$$z_0 z = r r_0 e^{i(\theta+\theta_0)}.$$

◇

Example. $f(z) = z^2$

$$z = re^{i\theta}$$

$$z^2 = r^2 e^{2i\theta}$$

So magnitude is squared and angle is doubled ◇

A reflection will reflect z along the x axis.

Example. $f(z) = \bar{z}$ reflects z along the x axis. ◇

An analytic function is a differentiable function in the complex space.

$$f(z) = w.$$

$$f(x + iy) = u + iv.$$

$$= u(x, y) + iv(x, y).$$

$$u(z) = iv(z).$$

2.2 15. Limits

If a function f is defined at all points z in some deleted neighborhood of point z_0. Then, $f(z)$ has a limit w_0 as z approaches z_0, or

$$\lim_{z \to z_0} f(z) = w_0.$$

Essentially this means that the point $w = f(z)$ can be made arbitrary close to w_0 if we choose a point z close enouhg to z_0 but distinct from it (deleted neighborhood).

Definition 2.1 (Limit)**.** The limit of a function $f(z)$ as z goes to z_0 is w_0 if, $\forall \varepsilon > 0, \exists \delta > 0, s.t.$

$$|f(z) - w_0| < \varepsilon \text{ whenever, } 0 < |z - z_0| < \delta.$$

Remark. Essentially this menas that for every ε-neighborhood, $|f(z) - w_0| < \varepsilon$ there is a deleted-neighborhood, $0 < |z - z_0| < \delta$ of z_0 such that every point z in it has an image w in the ε-neighborhood

Remark. All points in the deleted-neighborhood are to be considered but their images need not fill up the ε-neighborhood

Theorem 2.2. When a limit of a function $f(z)$ exists at a point z_0, it is unique.

Proof. Suppose,

$$\lim_{z \to z_0} f(z) = w_0 \text{ and } \lim_{z \to z_0} f(z) = w_1.$$

This means that,

$$|f(z) - w_0| < \varepsilon \text{ when } 0 < |z - z_0| < \delta_0.$$

$$|f(z) - w_1| < \varepsilon \text{ when } 0 < |z - z_1| < \delta_1.$$

So,

$$|f(z) - w_0| + |f(z) - w_1| < 2\varepsilon.$$

We know that,

$$w_1 - w_0 = (f(z) - w_0) - (f(z) - w_1) \leq |f(z) - w_0| - |f(z) - w_1|$$

So,

$$w_1 - w_0 < 2\varepsilon, \text{where } \varepsilon \text{ can be chosen arbitrary small.}$$

Hence,

$$w_1 - w_0 = 0, \text{or, } w_1 = w_0.$$

□

Example. Show that, $f(z) = \frac{i\bar{z}}{2}$ in the open disk $|z| < 1$, then

$$\lim_{z \to 1} f(z) = \frac{i}{2}$$

$$\left| f(z) - \frac{i}{2} \right| = \left| \frac{i\bar{z}}{2} - \frac{i}{2} \right| = \frac{|z - 1|}{2}.$$

Hence, for any z and ε,

$$\left| f(z) - \frac{i}{2} \right| < \varepsilon \text{ when } 0 < |z - 1| < 2\varepsilon.$$

◇

Example. $f(z) = \frac{z}{\bar{z}}$ The limit,

$$\lim_{z \to 0} f(z).$$

does not exist.
Assume that it exists, that implies that by letting the point $z = (x, y)$ we can approach the point, $(0, 0)$ in any manner and we would get the same limit.
Now if we approach the point from the $x-$axis where $z = (x, 0)$ we get,

$$\lim_{x \to 0} f((x, 0)) = \frac{x + 0i}{x - 0i} = 1.$$

But if we approach it from the $y-$ axis where, $z = (0, y)$ we get,

$$\lim_{y\to 0} f((0,y)) = \frac{0+iy}{0-iy} = -1.$$

But we know that the limit should be unique, hence this implies that the limit does not exist. $\diamond$

2.3 19. Derivatives

Theorem 2.3. If a function $f(z)$ is continuous and non-zero at a point z_0 then, there exists a neighborhood where, $f(z) \neq 0$ throughout.

Proof. We know that $f(z)$ is continuous which means that, $\varepsilon > 0, \exists\delta$ such that,

$$|f(z) - f(z_0)| < \varepsilon, \text{ when } 0 < |z - z_0| < \delta.$$

But if we take, $\varepsilon = \frac{f(z_0)}{2}$ then we have,

$$|f(z) - f(z_0)| < \frac{|f(z_0)|}{2}.$$

However, if $f(z) = 0$ for this neighborhood then we have,

$$|f(z_0)| < \frac{|f(z_0)|}{2}.$$

which is a contradiction.

$\square$

Theorem 2.4. f is continuous on R which is closed and bounded, $\exists M > 0$, real $|f(z)| \leq M, \forall z \in R$ equality holds for at least one z.

Definition 2.5 (Derivative). f is differntiable at z_0 when $f'(z_0)$ exists where,

$$f'(z_0) = \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

Remark. Can also solve,

$$\lim_{z_0\to 0} \frac{f(z+z_0) - f(z)}{z_0}$$

.

Example. Find derivative of, $f(z) = \frac{1}{z}$

$$\lim_{z_0\to 0} \left(\frac{1}{z+z_0} - \frac{1}{z}\right)\frac{1}{z_0}$$

$$\lim_{z_0\to 0} \frac{z - z - z_0}{z(z+z_0)}\frac{1}{z_0}$$

$$\lim_{z_0\to 0} \frac{-1}{z(z+z_0)}$$

$$= \frac{-1}{z^2}$$

$\diamond$

Example. $f(z) = \bar{z}$

$$\lim_{z_0 \to 0} \frac{z \bar{+} z_0 - \bar{z}}{z_0}$$

Go from x and y axis.
From x,

$$\lim_{x_0 \to 0} \frac{\bar{z} + x_0 - \bar{z}}{x_0} = 1.$$

Similarly if we go from y we get -1, so the derivative doesn't exist.

$\diamond$

If we have a function $f(z) = u(x, y) + iv(x, y)$ then,

$$z_0 = x_0 + iy_0.$$

$$\Delta z = \Delta x + i\Delta y.$$

We have to show the following exist,

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}.$$

$$= \frac{u(x_0 + \Delta x, y_0 + \Delta y) + iv(x_0 + \Delta x, y_0 + \Delta y) - (u(x_0, y_0) + iv(x_0, y_0))}{\Delta x + i\Delta y}.$$

Horizontally, $\Delta y = 0$.
So,

$$\frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} + \frac{i(v(x_0 + \Delta x, y_0) - v(x_0, y_0))}{\Delta x}.$$

$$= u_x + iv_x.$$

Similary, if we go vertically, $\Delta x = 0$ and we get,

$$= v_y - iu_y.$$

Theorem 2.6. If, $f(z) = u + iv, f'(z)$ exists at, $z_0 = x_0 + iy_0$. Then, u_x, u_y, v_x, v_y eixsts at (x_0, y_0) and must satisify the Cauchy-Reimann equation.

$$f'(z_0) = u_x + iv_x \text{ at } (x_0, y_0).$$

Theorem 2.7. $f(z) = u(x, y)+iv(x, y)$ defined throughout the ε-neighborhood of $z_0 = x_0+iy_0$,
(a) u_x, u_y, v_x, v_y exists everywhere in the neighborhood
(b) u_x, u_y, v_x, v_y continuous at (x_0, y_0) and satisfy the Cauchy-Remainn equations

$$u_x = v_y, u_y = -v_x \text{ at } (x_0, y_0)$$

Then $f'(z_0)$ exists and,

$$f'(z_0) = u_x + iv_x \text{ at } (x_0, y_0).$$

Proof. We need to show,

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}.$$

$$= \lim_{\Delta z \to 0} \frac{u(x_0 + \Delta x, y_0 + \Delta y) + iv(x_0 + \Delta x, y_0 + \Delta y) - (u(x_0, y_0) + iv(x_0, y_0))}{\Delta z}.$$

Using taylor expansion we know,

$$f(x+\Delta x) = f(x) + \Delta x f'(x) + \frac{(\Delta x)^2}{2} f''(x).$$

$$u(x_0+\Delta x, y_0+\Delta y) - u(x_0, y_0) =$$

$$= u(x_0,y_0) + \Delta x u_x(x_0,y_0) + \frac{(\Delta x)^2}{2} u_{xx}(x_0,y_0) + \Delta y u_y(x_0,y_0) + \frac{(\Delta y)^2}{2} u_{yy}(x_0,y_0).$$

We can write the limit as,

$$\frac{u_x(x_0,y_0)\Delta x + u_y(x_0,y_0)\Delta y + \varepsilon_1(\Delta x^2) + \varepsilon_2(\Delta y^2)}{\Delta z} +$$

$$i \frac{v_x(x_0,y_0)\Delta x + v_y(x_0,y_0)\Delta y + \varepsilon_3(\Delta x) + \varepsilon_4(\Delta y)}{\Delta z}.$$

We know $u_x(x_0,y_0) = v_y(x_0,y_0)$ and $u_y(x_0,y_0) = -v_x(x_0,y_0)$, so,

$$\frac{u_x(x_0,y_0)\Delta x + -v_x(x_0,y_0)\Delta y + \varepsilon_1(\Delta x^2) + \varepsilon_2(\Delta y^2)}{\Delta z} +$$

$$i \frac{v_x(x_0,y_0)\Delta x + u_x(x_0,y_0)\Delta y + \varepsilon_3(\Delta x) + \varepsilon_4(\Delta y)}{\Delta z}.$$

$$= \frac{u_x(x_0,y_0)(\Delta x + i\Delta y) + u_y(x_0,y_0)(\Delta y - i\Delta x) + \varepsilon_4}{\Delta z}.$$

and $\Delta z = \Delta x + i\Delta y$

$$= \frac{u_x(x_0,y_0)(\Delta x + i\Delta y) + u_y(x_0,y_0)(\Delta y - i\Delta x) + \varepsilon_4}{\Delta x + i\Delta y}.$$

□

Definition 2.8 (Analytic function). A function f is analytic in an open set S, if f has derivative everywhere in S. It is analytic at a point z_0 if it is analytic in some neighborhood of z_0

Remark. Analytic functino has to be on an open set.

Remark. For it to be analytic at z_0 derivative should exist in the neighborhood of z_0 (not just the point z_0)

Example. $f(z) = (|z|)^2 = \sqrt{x^2+y^2}^2$

$$u = x^2 + y^2, v = 0$$

$$u_x = 2x, u_y = 2y.$$

$$v_x = 0, v_y = 0.$$

So the Cauchy-Reimann equation is only satisfied at $(0,0)$
$f'(0) = 0$ and it exists. ◇

Remark. $f(z) = |z|^2$ is not analytic anywhere. So even if the derivative exists at $z = 0$. The function is not analytic at $z = 0$ (or at any point)
Because, (1). $f'(z)$ exists at $z = 0$
(2). u_x, u_y, v_x, v_y exists $\not\Rightarrow f'(z)$
(3). $f(z)$ is continuous $\not\Rightarrow f'(z)$
Essentially it only exists for $z = 0$ and not in the neighborhood around it.

Definition 2.9 (Entire function)**.** A function f is analytic at each point in the entire plane.

Definition 2.10 (Singular point)**.** z_0 is a singular point if f fails to be analytic at z_0 but is analytic at some point in every neighborhood at z_0

Example. $f(z) = 2 + 3z^2 + z^3$
Is analytic everywhere so it is an entire function ◇

Example. $f(z) = \frac{1}{z}$
Is analytic at all non-zero, but $z = 0$ is a sigular point ◇

Example. $f(z) = |z|^2 = x^2 + y^2$
Is not analytic, no singular points either.

◇

2.4 Harmonic Function

Definition 2.11 (Harmonic function)**.** A real valued functino of $H(x, y)$ is said to be harmonic if in a given domain of the x, y plane, it has a continuous partial derivative of the first and second order $(H_x, H_y, H_{xx}, H_{yy}, H_{xy})$ and satisfies,

$$H_{xx}(x, y) + H_{yy}(x, y) = 0 \text{ Laplace equation.}$$

Theorem 2.12. If $f = u(x, y) + i(x, y)$ is analytic in a domain D, then u, v are harmonic in D

Theorem 2.13. If $f'(z) = 0$ everywhere in D then $f(z)$ is a constant in D.

Proof. Consider $f(z) = u(x, y) + iv(x, y)$ given that

$$f'(z) = u_x + iv_x = 0$$

Using Cauchy-Reimann equation we have, $u_y, v_y = 0$. So all of the first order derivative s are equal to 0 in D.
$U(x, y)$ is constant along any line L, extending from p to p'. Let the vector from p to p' be u. So we have,

$$\frac{du}{ds} = (\text{grad } u)u$$
$$\text{grad } u = u_x i + u_y j = 0$$

So u is a constant (a) on L. Similarly for $v = b$

$$f(z) = a + bi$$

□

Lemma 2.14. Suppose,
(a). $f(z)$ is analytic throughout D
(b). $f(z) = 0$ at each point at the domain or line segment containing D
Then $f(z) \equiv 0$ in D

Chapter 3

Elementary Functions

3.1 Exponential Function

The exponential function is e^z. But we can write this as

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(cos(y) + isin(y))$$

We can also write,

$$e^z = \rho e^{i\phi} \text{ where } \rho = |e^x| \text{ and } \phi = y$$

For a function, $e^{z_1}e^{z_2}$ we can write,

$$e^{z_1}e^{z_2} = e^{x_1+iy_1}e^{x_2+iy_2}$$

$$= e^{x_1+x_2}e^{i(y_1+y_2)}.$$

$$= e^{z_1+z_2}.$$

The derivative if e^z is an entire function

$$\frac{d}{dx}e^z = e^z \text{which is an entire function.}$$

$$e^{z+2} = e^z + e^2 = e^z$$

3.2 Log Function

The log function is $f(z) = log(z) = w = u + iv$. We know

$$e^w = z = e^{u+iv} = e^u e^{iv}.$$

We see that $r = e^u$ and $\theta = v + 2n\pi$

$$r = e^u \Rightarrow ln(r) = u$$

Similarly,

$$\theta = v + 2n\pi.$$

So we have,

$$f(z) = \log(z) = \ln|z| + i\arg(z).$$

and the principal direction is,

$$f(z) = \log(z) = \ln|z| + i\theta, \quad -\pi < \theta < \pi.$$

Some properties are,
(1).$e^{\log z} = z, (z \neq 0)$

(2).$|e^z| = e^x$
(3). $\log(e^z) = \ln|e^z| + i\arg(e^z)$

$$= \ln|e^x| + i(y + 2n\pi), n = 0, \pm 1, \pm 2.$$
$$= \ln e^x + iy + i2n\pi.$$
$$= z + 2n\pi.$$

Branches

The principal branch is
$$\log z = \ln r + i\theta \text{ where } r > 0, -\pi < \theta < \pi.$$

A branch cut is a portion of a line or curve that is introduced in order to deifne a branch F of a multiple-valued function f.
Points on the branch cut for F are singular points of F and any point that is common to all branches of f are called branch points.

Example.
$$\frac{d}{dz}\log z = \frac{1}{z}, \text{ where } |z| > 0$$
The branches can be $\alpha < \arg z < \alpha + 2\pi$ ◇

Property. $\log z_1 z_2 = \log z_1 + \log z_2$

Proof.
$$\log z_1 z_2 = \ln(r_1 r_2) + i(\theta_1 + \theta_2 + 2n\pi)$$
$$= \log z_1 z_2 = \ln(r_1) + \ln(r_2) + i(\theta_1 + \theta_2 + 2n\pi)$$
$$= \log z_1 z_2 = \ln(r_1) + i(\theta_1 + 2n\pi) + \ln(r_2) + i(\theta_2 + 2n\pi)$$
$$= \log z_1 z_2 = \ln(r_1) + i(\theta_1 + 2n\pi) + \ln(r_2) + i(\theta_2 + 2n\pi)$$
$$= \log z_1 z_2 = \log z_1 + \log z_2$$
□

Property. $\log|z_1 z_2| = \log|z_1| + \log|z_2|$

Property. $\log(\frac{z_1}{z_2)} = \log(z_1) - \log(z_2)$

Property. $z^n = e^{n\log(z)}$

3.3 Power Function

We have a complex number c and we have $f(z) = z^c$. By definition we have $z^c = e^{c\log z}$
The derivative is $\frac{d}{dz}f(z) = \frac{d}{dz}(z^c)$

$$\frac{d}{dz}e^{c\log z} = e^{c\log z}\frac{d}{dz}c\log z = e^{c\log z}\frac{c}{z}$$

But we can write $\frac{e^{c\log z}c}{e^{\log z}} = ce^{(c-1)\log z} = cz^{c-1}$. The principal value of $z^c = e^{cLogz}$
If the function is $f(z) = c^z$ then we have

$$\frac{d}{dz}c^z = \frac{d}{z}e^{z\log c} = e^{z\log c}\frac{d}{dz}z\log c = e^{z\log c}\log c = c^z\log c$$

3.4 Trignometric Function

We know that $e^{i\theta} = \cos\theta + i\sin\theta$ and $e^{-i\theta} = \cos\theta - i\sin\theta$. So we can write,

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

and,

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

We have $\frac{d}{dz}\sin z = \cos z$ and $\frac{d}{dz}\cos z = -\sin z$

Property. $\sin(-z) = -\sin(z)$ and $\cos(-z) = \cos(z)$

Property. $\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$

Property. $\sin(2z) = 2\sin(z)\cos(z)$

Property. $\sin(z + \frac{\pi}{2}) = \cos(z)$

Consider the hyperbolic sin and cos functions,

$$\sinh z = \frac{e^z - e^{-z}}{2}, \cosh z = \frac{e^z + e^{-z}}{2}$$

We can write $\sin z = \sin(x + iy)$. Now expanding this we get,

$$\sin(x)\cos(iy) + \cos(x)\sin(iy) = \sin(x)\cosh(y) + i\cos x \sinh(y)$$

And we have,

$$|\sin z|^2 = \sin^2 x + \sinh^2 y$$
$$|\cos z|^2 = \cos^2 x + \cosh^2 y$$

3.5 Inverse Trignometric Functions

The function is $w = f(z) = \sin^{-1} z$. So we have

$$\sin(w) = z = \frac{e^{iw} - e^{-iw}}{2}$$

We know $2iz = (e^{iw} - e^{-iw}) \times e^{iw}$,

$$2ize^{iw} = e^{2iw} - e^0$$
$$e^{iw2} - 2ize^{iw} - 1 = 0$$

Solving this we get,

$$e^{iw} = iz \pm (1 - z^2)^{\frac{1}{2}}$$

Chapter 4

Integrals

Consider $f(z) = f(x+iy) = u(x,y) + iv(x,y)$. We can write this as,

$$w(t) = u(t) + iv(t)$$
$$w'(t) = u'(t) + iv'(t)$$

Example. $\frac{d}{dt}(w(t))^2 = \frac{d}{dt}(u+iv)^2$

$$\begin{aligned} &= \frac{d}{dt}(u^2 - v^2 + 2uvi) \\ &= 2uu' - 2vv' + i(2u'v + 2uv') \\ &= 2(u+iv)(u'+iv') \\ &= 2w(t)w'(t) \end{aligned}$$

◇

4.1 Definite Integrals

The integral of $w(t)$ with respect to t is,

$$\int_a^b w(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt$$

Exercise. Find c such that,

$$\int_a^b w(t)dt = w(c)(b-a) \text{ where } w(t) = e^{it}, a = 0, b = 2\pi$$

Solution. We have,

$$\int_0^{2\pi} e^{it} = \int_0^{2\pi} (\cos(t) + i\sin(t) = [\sin(t) - i\cos(t)]_0^{2\pi} = 0$$

Generally for arbitrary a and b we can show that, ...

Remark. In this case t is moving from 0 to 2π. But because we are in the complex plane it represents a loop.

Remark. Note that the mean value theorem for integrals does not carry over for complex integrals.

The mean value theorem for integrals in the real sense means that $\exists c$ such that,

$$\int_a^b w(t)dt = w(c)(b-a)$$

However take $w(t) = e^{it}$ and choose $b = 2\pi$ and $a = 0$. The right hand side will always be non-zero while the left hand size will be 0.

4.2 Contour

In the real case we have functions that are defined on intervals of the real line. However complex inputs lie in the d plane so instead of an interval we define it using curves.

Definition 4.1 (contour). We have $z(t) = x(t) = iy(t)$ is a contour if,
(1) C is simple arc or Jordan arc, it does not cross itself.

$$z(t_1) \neq z(t_2), t_{12}$$

(2) $z(a) = z(b)$; C simple closed curve.
It is positively oriented if the direction is anticlockwise

Example. $x = \begin{cases} x + ix, 0 \leq x \leq 1 \\ x + i, 1 \leq x \leq 2 \end{cases}$ ◇

Example. $z = re^{i\theta}, 0 \leq \theta \leq 2\pi$ ◇

Example. $z = re^{i3\theta}, 0 \leq \theta \leq 2\pi$
Not a simple arc ◇

Example. $\int_C w(z)dz = \int_{C_1} f[z(x)]z'(x)dx + \int_{C_2} f[z(x)]z'(x)dx$
Here C is the contour from example (1). ◇

Definition 4.2 (differentiable arc). If $z'(t) = x'(t) + y'(t)i$ which is continuous on $a \leq t \leq b$ then, $C : z(t)$ is a differential arc and

$$\begin{aligned} L &= \int_a^b |z'(t)|dt = \int_a^b \sqrt{|x'(t)|^2 + |y'(t)|^2} \text{ length.} \\ &= \int_a^b |z'(t)|dt = \int_\alpha^\beta |z'(\phi(\tau))|\phi'(\tau)d\tau \\ T &= \frac{z'(t)}{|z'(t)|} \text{ tangent vector} \end{aligned}$$

Definition 4.3 (smooth arc). An arc is smooth if its derivative $z'(t)$ is cont. on the closed interval $a \leq t \leq b$ and non-zero throughout the open interval $a < t < b$

4.3 Contour Integral

Consider the integer,

$$\int_C f(z)dz \text{ or } \int_{z_1}^{z_2} f(z)dz$$

We can parametric in terms of t as,

$$\int_C f(z)dz = \int_a^b f(z(t))z'(t)dt$$

$\int_{-C} f(z)dz$ represents going backwards from the curve.
An integral along a given curve C can be written as a sum of integrals of curves within it,

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz$$

Example. $\int_{C_1} \frac{dz}{z}$ where C_1 is the upper semicircle and $\int_{C_2} \frac{dz}{z}$ is the lower semicircle.

◇

Solution. For C_1

$$z = re^{i\theta}, r = 1, 0 \leq \theta \leq \pi$$

And for C_2 we have,

$$z = re^{i\theta}, r = 1, \pi \leq \theta \leq 2\pi$$

$$dz = ire^{i\theta} d\theta$$

For C_1 we have,

$$\int_{C_1} \frac{dz}{z} = \int_0^{\pi} \frac{1}{e^{i\theta}} ie^{i\theta} d\theta$$
$$= [i\theta]_0^{\pi} = i\pi$$

Similarly for C_2,

$$\int_{C_2} \frac{dz}{z} = -\int_{\pi}^{2\pi} \frac{1}{e^{i\theta}} ie^{i\theta} d\theta$$
$$= -[i\theta]_{\pi}^{2\pi} = [i\pi - i2\pi] = -i\pi$$

We see that it is not path independent.

Theorem 4.4. Suppose a function $f(z)$ is cont. in D the following statements are equivalent,
1. $f(z)$ has an antiderivative $F(z)$ throughout D.
2. Any contours entirely in D all have the same value,

$$\int_{z_1}^{z_2} f(z)dz = F(z)]_{z_1}^{z_2} = F(z_2) - F(z_1)$$

3. $\int_C f(z) = 0$, C closed contours entirely in D

4.4 Branch Cuts

Example. $z = 3e^{i\theta}, (0 \leq \theta \leq \pi)$ ◇

Lemma 4.5. If $w(t)$ is piecewise cont. then,

$$\left| \int_a^b w(t)dt \right| \leq \int_a^b |w(t)|dt$$

Proof. Let,

$$\int_a^b w(t)dt = re^{i\theta}$$

$r = \int_a^b e^{-i\theta} w(t)dt$. Both sides of this equations are real.

$$r = \int_a^b Re[e^{i\theta} w(t)]dt$$

But,

$$Re[e^{i\theta} w(t)] \leq |e^{-i\theta} w(t)| = |e^{-i\theta}||w(t)| \leq |w(t)|$$

So,

$$r \leq \int_a^b |w(t)|dt$$

Or,

$$\left|\int_a^b w(t)dt\right| \leq \int_a^b |w(t)|dt$$

□

Theorem 4.6. C has a length L and $f(z)$ is piecewise cont. on C and let $|f(z)| \leq M$ then,

$$\left|\int_C f(z)dz\right| \leq ML$$

Proof. $\int_C f(z)dz = \int_a^b f(z(t))z'(t)dt$. So we have,

$$\left|\int_C f(z)dz\right| \leq \int_a^b |f(z(t))z'(t)|dt \leq \int_a^b |Mz'(t)|dt = ML$$

□

Theorem 4.7. $f(z)$ is cont. over D then,
(a). $f(z)$ has antiderivative $F(z)$ throughout D
(b). $\int_{z_1}^{z_2} f(z)dz = F(z_2) - F(z_1)$
The antiderivative is independent to the path.
(c). $\int_C f(z)$ where c is a closed contour entirely in D

Proof. 1. (a) $\Rightarrow$ (b). We have, $c : z = z(t), z_1 = z(a), z_2 = z(b)$

$$\frac{d}{dt}[F[z(t)]] = F'[z(t)]z'(t) = f(z)z'(t)$$

So taking,

$$\int_C f(z)dz = \int_a^b f[z(t)]z'(t)dt = F[z(t)]$$

$$= F[z(t)]_a^b = F(z_2) - F(z_1)$$

2. (b) $\Rightarrow$ (c)

$$\int_{C_1} f(z)dz = F(z_2) - F(z_1)$$

$$\int_{C_2} f(z)dz = F(z_2) - F(z_1)$$

So,

$$\int_{C_1} f(z)dz = F(z_2) - F(z_1) = \int_{C_2} f(z)dz = F(z_2) - F(z_1)$$

$$\int_{C_1} f(z)dz = F(z_2) - F(z_1) - \int_{C_2} f(z)dz = F(z_2) - F(z_1) = 0$$

$C = C_1 - C_2$: a closed contour in D

$$\int_C f(z)dz = 0$$

3. (c) $\Rightarrow$ (a)
We define

$$F(z) = \int_{z_1}^{z_2} f(s)ds$$

$$\lim_{\Delta z \to 0} \frac{F(z+\Delta z) - F(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{1}{\Delta z}\left[\int_{z_0}^{z+\Delta z} f(s)ds - \int_{z_0}^{z} f(z)ds\right]$$

$$= \lim_{\Delta z \to 0} \frac{1}{\Delta z} \int_{z}^{z+\Delta z} f(s)ds$$

Remark.

$$\int_{z}^{z+\Delta z} ds = s]_z^{z+\Delta z} = \Delta z$$

Remark.

$$f(z) = \frac{1}{\Delta z} \int_{z}^{z+\Delta z} f(z)ds$$

$$\frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z) = \frac{1}{\Delta z} \int_{z}^{z+\Delta z} [f(s) - f(z)]ds$$

By cont. of $f(z), \forall \varepsilon, \exists \delta$,

$$|f(s) - f(z)| < \varepsilon \text{ whenever } |s - z| < \delta$$

□

4.5 Cauchy-Goursat Theorem

We have $\int_c f(z)dz = \int_a^b f(z(t))z'(t)dt$. We want to show that $\int_C f(z) = 0$ if C is a closed contour. Here,

$$f(z) = u(x,y) + iv(x,y)$$

$$z(t) = x(t) + iy(t)$$
$$z'(t) = (x'(t) + iy'(t))dt$$

$$\int_C f(z)dz = \int_a^b (u+iv)(x'+iy')dt$$
$$= \int_a^b (ux' - vy')dt + i\int_a^b (vx' + uy')dt$$
$$= \int_C udx - vdy + i\int_C vdx + udy$$

$$\text{Greens theorem says that,} \int_C Pdx + Qdy = \int\int_R (Qx - Py)dA$$

$$= \int\int_R (-vx - uy)dA + \int\int_R (ux - vy)dA$$

We know that if f is analytic then it satisfies the cauchy reimann equatoin such that,

$$u_x = v_y, u_y = -v_x$$

$$= \int \int_R (0) dA + \int \int_R (0) dA$$
$$= 0$$

So for any closed curve f such that f is analytic and f' is cont. the integral over the countour is 0.

Theorem 4.8. If a function f is analytic at all points interior to and on a simple closed contour C, then

$$\int_C f(z)dz = 0$$

Proof. We need to show that $\forall \varepsilon$,

$$\left| \int_c f(z)dz - 0 \right| < \varepsilon$$

For any region R consisitng of points interior to our contour R can be covered with a finite number of square san dpartial squares such that in each one there is a fixed oint z_j for which,

$$\left| \frac{f(z) - f(z_j)}{z - z_j} - f'(z_j) \right| < \delta_j(z)$$

We can say,

$$\int_C f(z)dz = \sum_{j=1}^{n} \int_{c_j} f(z)dz$$

We have,
(i). C_i is defined
(ii) $\delta_j(z) = \frac{f(z)-f(z_j)}{z-z_j} - f'(z_j)$ when $z \neq z_j$

$$\lim_{z \to z_j} \delta_j(z) = 0$$

We approximate,

$$f(z) = f(z_j) + (z - z_j)f'(z_j) + (z - z_j)\delta_j(z)$$

So,

$$\int_{C_j} f(z)dz = [f(z_j) - z_j f'(z_j)] \int_{C_j} dz$$
$$+ \int_{C_j} f'(z_j)zdz + \int_{Cj} (z - z_j)\delta_j(z)dz$$

The first and secodn term is equal to 0. So we can rewrite it as,

$$= \int_{C_j} (z - z_j)\delta_j(z)dz$$

(iii).

$$\left| \int_C f(z)dz \right| \leq \sum_{j=1}^{n} \left| \int_{C_j} (z - z_j)\delta_j(z)dz \right|$$

$$\left| \int_C f(z)dz \right| \leq \sum \left| \int_{C_j} f(z)dz \right|$$

$$f(z) = f(z_j) - z_j f'(z_j) + f'(z_j)z + (z - z_j)\delta_j(z)$$

The integral of the first three terms cancel out so,

$$\int_C f(z) = \int_C (z - z_j)\delta_j(z)dz$$

$$\leq \sum \left| \int_{C_j} (z - z_j)\delta_j(z)dz \right|$$

$$|z - z_j| \leq \sqrt{2}S_j$$

where S_j is the length of the square. So,

$$|(z - z_j)\delta_j(z)| < \sqrt{2}S_j\varepsilon$$

$$\int_{C_j} |(z - z_j)S_j(z)| < \sqrt{2}S_j\varepsilon 4S_j$$

$$\int_{C_j} |(z - z_j)\delta_j(z)| < \sqrt{2}S_j\varepsilon(4S_j + L_j)$$

So,

$$\sum \left|(z - z_j)\delta_j(z)dz\right| < (4\sqrt{2}S^2 + \sqrt{2}SL)\varepsilon$$

□

Simply Connnected Domain

Theorem 4.9. If f is analytic throughout a simply connected domain D, then,

$$\int_C f(z) = 0$$

for every closed contour lying in D

Exercise. C is in the open disk $|z| < 2$, compute,

$$\int_C \frac{\sin z}{(z^2 + 9)^5} dz$$

Solution.

Exercise. Givne $C : |z| = 1$,

$$\int_C ze^{-z}dz =$$

Theorem 4.10. Suppose,
(a). C is a closed contour in the counterclockwise direction
(b). $C_k(k = 1, 2, \ldots, n)$ are simply closed contours that are interior to C and are disjoint from each other.

$$\int_C f(z)dz + \sum_{k=1}^{n} \int_{C_j} f(z)dz = 0$$

Proof. We disect the contour such that it goes around our closed interior contours. This means that we have,

$$\int_{C*} f(z)dz = 0$$

We can rewrite this as

$$\int_{C+L_1+\frac{1}{2}C_1+L_2+C_2-L_2+\frac{1}{2}C_1-L_1} f(z)dz$$

$$= \int_C f(z) + \sum_{i=1}^{n} \int_{C_i} f(z)dz = 0$$

□

4.6 Multiply Connected Domain

Corollary 4.11. If C_1, C_2 are positively oriented simply closed contours where C_1 is interior to C_2. If f is analytic in the closed region containing C_1, C_2 and all the points in between then,

$$\int_(C_1) f(z)\, dz = \int_(C_2) f(z)\, dz$$

Proof. We construct a new contour between C_1 and C_2 where,

$$\int_{C^*} f(z)\, dz = \int_{C_2 - C_1} f(z)\, dz = 0$$

So,

$$\int_{C_2} f(z)\, dz - \int_{C_1} f(z)\, dz = 0$$

□

Example. C is any positively oriented simple closed contour which is surrounding the origin. ◇
The function $f(z) = \frac{1}{z}$ has a singularity at the origin. So f is analytic unto zero. Because of the principal of deformation of path we can deform any contour surrounding the singularity to a unit circle around with fixed radius preserving the integral. So we have,

$$\int_0^{2\pi} \frac{1}{e^{i\theta}} ie^{i\theta} d\theta$$

$$= 2\pi i$$

Theorem 4.12 (Cauchy Integral Formula). If f is analytic everywhere inside and on the simple closed contour C (positively oriented). If z_0 is interior to Z then,

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0}\, dz$$

Remark. If our integral has a singularity we can easily compute it using the "function" outside of it that doesn't have the singularity.

Proof. We can rewrite the integral as,

$$\int_C \frac{f(z)}{z - z_0}\, dz = \int_{C_p} \frac{f(z)}{z - z_0}\, dz$$

where C_p is the unit circle constructed around z_0 by deforming our contour.

$$
\begin{aligned}
&= \int_{C_p} \frac{f(z) - f(z_0)}{z - z_0} + \frac{f(z_0)}{z - z_0}\, dz \\
&= \int_{C_p} \frac{f(z) - f(z_0)}{z - z_0} + f(z_0) \int_{C_p} \frac{1}{z - z_0}\, dz \\
&= \int_{C_p} \frac{f(z) - f(z_0)}{z - z_0} + f(z_0) 2\pi i\ dz
\end{aligned}
$$

Because f is analytic as f is cont. Around z_0 so, $\forall \varepsilon, \exists \delta$ s.t.,

$$|f(z) - f(z_0)| < \varepsilon \text{ when } |z - z_0| < \delta$$

So,

$$\left| \int_{C_p} \frac{f(z) - f(z_0)}{z - z_0}\, dz \right| < \frac{\varepsilon}{\rho} 2\pi\rho = 2\pi\varepsilon$$

□

Theorem 4.13. Let f be analytic inside and on a simply closed contour C taken in a positive direction and z_0 inside C we have,

$$f^n(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{z - z_0}^{n+1}\, dz$$

Proof. We need to show that $f'(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^2}\, dz$. Using the Cauchy integral formula for each term in the numerator we have,

$$
\begin{aligned}
lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} &= \frac{1}{\Delta z} \frac{1}{2\pi i} \int_C \frac{f(s)}{s - (z + \Delta z)} - \frac{f(s)}{s - z} \\
&= \frac{1}{\Delta z} \frac{1}{2\pi i} \int_C f(s) \frac{s - z - (s - (z + \Delta z))}{(s - (z + \Delta z)(s - z)}\, dz \\
&= \frac{1}{2\pi i} \int_C \frac{f(s)}{(s - (z + \Delta z))(s - z)}\, dz \\
&= \frac{1}{2\pi i} \int_C f(s) \left(\frac{1}{(s - z)^2} + \frac{\Delta z}{(s - z - \Delta z)(s - z)^2} \right) dz
\end{aligned}
$$

Now we have,

$$\lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} = \frac{1}{2\pi i} \int_C f(s) \frac{1}{(s - z)^2}\, dz$$

$$f'(z) = \frac{1}{2\pi i} \int_C f(s) \frac{1}{(s - z)^2}\, dz$$

□

Exercise. $C : |z| = 1, \theta : 0 \to 2\pi$ Find,

$$\int_C \frac{\cos(z)}{(z^2 + 9)z}\, dz$$

$= \frac{1}{9} 2\pi i$

Exercise.

$$\int_C \frac{e^{2z}}{z^4}\, dz$$

Exercise.

$$\int_C \frac{dz}{z^{n+1}}$$

We have,

$$f(z_0)^{(n)} = \frac{n!}{2n\pi} \int_C \frac{dz}{z^{n+1}}$$

So $\int_C \frac{dz}{z^{n+1}} = 0$

Exercise.

$$\frac{z}{2z+1}\, dz$$

$= -\pi i$

Exercise.

$$\frac{e^{-z}}{(z-5)(z+2)}$$

$= 0$ because the doesn't exist a singularity within the contour hence the integral is 0 as the function is analytic within.

4.7 Consequences

Theorem 4.14. If f is analytic at a point then the derivative of all orders are analytic in that point.

Proof. If f is analytic at z_0 then $\exists \varepsilon$ s.t., $|z - z_0| < \frac{\varepsilon}{2}$ we have

$$f''(z) = \frac{1}{\pi i} \int_C \frac{f(s)}{(s-z)^3} dz$$

This is defined for all z in our neighborhood which means that our $f'(z)$ is analytic within the contour where

$$|z - z_0| < \frac{\varepsilon}{2}$$

$\square$

Theorem 4.15. If f is cont. on domain D and if $\int_C f(z)dz = 0$ for every contour $C \in D$. Then f is analytic throughout D.

Theorem 4.16. Suppose f is analytic inside and on C which is centered at z_0 with radius R. Then if $M_R \geq |f(z)|$ on C_R then

$$f^{(n)}(z_0) \leq \frac{n! M_R}{R^n}$$

Proof.

$$\begin{aligned} |f^{(n)}(z_0)| &= \left| \frac{n!}{2\pi i} \int \frac{f(z)}{(z-z_0)^{n+1}} dz \right| \\ &\leq \left| \frac{n!}{2\pi i} \int \frac{M_R}{R} dz \right| \end{aligned}$$

□

4.8 Liouvillie's theorem and fundamental theorem of algebra

Theorem 4.17. Let f be entire and bounded in the complex plane then $f(z)$ is constant throughout the plane.

Proof.

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int \frac{f(s)}{s-z_0}ds$$
$$|f'(z_0)| \leq \frac{M_R}{R}$$

As R goes to ∞ we see that $f'(z_0)$ goes to 0. □

Theorem 4.18. Any polynomial, $p(z) = a_0 + a_1 z + \cdots + a_n z^n$ has at least one zero.

Proof. Assume there is no zero, we have

$$\left|\frac{1}{p(z)}\right| < \frac{2}{|a_n|R^n}, |z| > R$$

□

4.9 Maximum modulus Principle

Lemma 4.19. $|f(z)| \leq |f(z_0)|$, $|z - z_0| < \varepsilon$ f is analytic. Then,

$$f(z) \equiv f(z_0)$$

Theorem 4.20. If f is analytic and not constant in D then $f(z)$ has no maximum in D

Chapter 5

Series

5.1 Convergence of Sequences

If $\lim_{n\to\infty} z_n = z$ then we know that $\forall \varepsilon > 0$, $\exists n_0 > 0$ s.t.

$$|z_n - z| < \varepsilon$$

when $n > n_0$

> **Theorem 5.1.** $z_n = x_n + iy_n, z = x + iy$ Then
>
> $$\lim_{n\to\infty} z_n = z$$
>
> $$\Leftrightarrow$$
>
> $$\lim_{n\to\infty} x_n = x \text{ and } \lim_{n\to\infty} y_n = y$$

So $\sum_{n=1}^{\infty} z_n$ converges to S,

$$S_N = \sum_{n=1}^{N} z_n \to S \text{ as } N \to\infty$$

> **Theorem 5.2.** $z_n = x_n + iy_n$, $S = X + iY$ then
>
> $$\sum_{n=1}^{\infty} z_n = S$$
>
> $$\Leftrightarrow$$
>
> $$\sum_{n=1}^{\infty} x_n = X \text{ and } \sum_{n=1}^{\infty} y_n = Y$$

5.2 Convergence of Series

An infinite series,

$$\sum_{n=1}^{\infty} z_n = z_1 + \cdots + z_n + \ldots$$

of complex numbers converges to the sum S if the sequence,

$$S_n = \sum_{n=1}^{N} z_n = z_1 + \cdots + a_N \quad (N = 1, 2, \ldots)$$

of partial sums converges to S ; we write

$$\sum_{n=1}^{\infty} z_n = S$$

Theorem 5.3. If $z_n = x_n + i_n$ and $S = X + iY$ then,

$$\sum_{n=1}^{\infty} z_n = S$$

$$\Leftrightarrow$$

$$\sum_{n=1}^{\infty} x_n = X, \sum_{n=1}^{\infty} y_n = Y$$

Remark. We can write,

$$\sum_{n=1}^{\infty} x_n + iy_n = \sum_{n=1}^{\infty} x_n + i \sum_{n=1}^{\infty} y_n$$

Corollary 5.4. If complex number series converges then $z_n \to 0$ when $n \to \infty$

Proof. To prove this we can write $z_n = x_n + iy_n$ and using property of convergence of series from the reals. □

Remark. This also means that the terms of convergent series are bounded. So $\exists M$ such that $|z_n| \leq M$ for each positive integer n.

Corollary 5.5. The absolute convergence of a series of complex numbers implies the convergence of series.

The remainder is $\rho_n = S - S_N = \sum_{n=N+1}^{\infty} z_n$

Remark. A series converges to $S \Leftrightarrow$ the sequence of remainders tends to zero.

5.3 Absolute and Uniform Convergence

Theorem 5.6. If a power series,

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

converges when $z = z_1$ then it is absolutely convergent at each point z in the open disk $|z - z_0| < R_1$ where $R_1 = |z_1 - z_0|$

Proof. Consider it converges when $z = z_1$. This means that,

$$\sum_{n=0}^{\infty} a_n (z_1 - z_0)^n$$

converges which implies that each term is bounded by some value M.

So for any n, we have

$$a_n(z_1 - z_0)^n \leq M$$

Now consider $\rho = \frac{|z-z_0|}{|z_1-z_0|}$. We know that $\rho \leq 1$ because $|z - z_0| \leq |z_1 - z_0|$. Hence we have,

$$a_n(z_1 - z_0)^n \rho^n \leq M\rho^n$$
$$a_n(z - z_0)^n \leq M\rho^n$$

So we get that any term of form $a_n(z - z_0)^n$ is bounded by a number as well and as n increase because ρ is smaller than zero the terms go to zero. This means that our series,

$$\sum_{n=0}^{\infty} a_n(z - z_0)^n$$

also converges.
In other words because $M\rho^n$ converges and because each term in our series is smaller than this term our series converges by comparison test. □

Remark. The greatest circle is called our circle of convergence and is the largest circle possible. Because if there was a point outside the circle for which the series converges then using the proof above the circle compassing that point would be the circle of convergence.

Now consider the following terminology where the series is defined for $|z - z_0| < R$

$$S(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$$
$$S_N(z) = \sum_{n=0}^{N-1} a_n(z - z_0)^n$$
$$\rho_N(z) = S(z) - S_N(z)$$

Since the power series converges for any value of z when $|z - z_0| < R$ we have $\exists N_e$ such that $\forall \varepsilon$,

$$|\rho_N(z)| < \varepsilon \text{ whenever } N > N_e$$

When the choice of N_e is only dependent on ε and independent on z then convergence is said to be uniform in that region.

Theorem 5.7. If z_1 is a point inside $|z - z_0| = R$ of a series,

$$\sum_{n=0}^{\infty} a_n(z - z_0)^n$$

then that series must be uniformly convergent in the closed disk $|z - z_0| \leq R_1$ where $R_1 = |z_1 - z_0|$

Proof. So first consider the reminder term of our series with z and z_1 as follows,

$$\rho_N(z) = \lim_{n\to\infty} \sum_{n=N}^{m} a_n(z - z_0)^n$$
$$\sigma_N(z) = \lim_{n\to\infty} \sum_{n=N}^{m} a_n(z_1 - z_0)^n$$

Now because $|z - z_0| \leq |z_1 - z_0|$ we get,

$$\rho_N(z) \leq \sigma_N(z)$$

inside $|z - z_0| \leq R_1$ where R_1 is the circle $|z_1 - z_0|$
Since they are remainders of convergent series we have $\exists N_e$ such that for any ε,

$$\sigma_N < \varepsilon \text{ whenever } N > N_e$$

Now putting the two together we have,

$$\rho_N(z) < \varepsilon \text{ whenever } N > N_e$$

Now as N_e is independent of our value of z within our circle we have uniform convergence as the remainder goes to 0. □

5.4 Continuity of Sums of Power Series

Theorem 5.8. A power series,

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

represents a continuous functions $S(z)$ at each point inside its circle of convergence $|z - z_0| = R$
In other words we have,

$$|S(z) - S(z_1)| < \varepsilon \text{ whenever } |z - z_1| < \delta$$

Proof. We have,

$$S(z) = \rho_N(z) + S_N(z)$$
$$|S(z) - S(z_1)| = |S_N(z) - S_N(z_1) + \rho_N(z) - \rho_N(z_1)|$$
$$|S(z) - S(z_1)| \leq |S_N(z) - S_N(z_1)| + |\rho_N(z)| + |\rho_N(z_1)|$$

□

We have,

$$|\rho_N(z)| < \frac{\varepsilon}{3} \text{ when } N > N_e$$

And because $S_N(z)$ is a polynomial and is continuous we have,

$$|S_N(z) - S_N(z_1)| < \frac{\varepsilon}{3} \text{ when } ||z - z_1| < \delta$$

So putting everything together we have,

$$|S(z_) - S(z_1)| \leq \text{ whenever } |z - z_1| < \delta$$

5.5 Integration And Differentiation of Power Series

Theorem 5.9. Let C be any contour interior to the circle for convergence of the power series (1), and let g(z) be any function that is continuous on C. Then the series formed by multiplying each term of the power series by $g(z)$ can be integrated term by term over C,

$$\int_C g(z)S(z)dz = \sum_{n=0}^{\infty} a_n \int_C g(z)(z - z_0)^n dz$$

Proof. We have,

$$g(z)S(z) = \sum_{n=0}^{N-1} a_n g(z)(z-z_0)^n + g(z)\rho_N(z)$$

$$\int_C g(z)S(z) = \sum_{n=0}^{N-1} a_n \int_C g(z)(z-z_0)^n dz + \int_C g(z)\rho_N(z)dz$$

Now let M be the max of $g(z)$ on C and L be the length of C. We know $\exists N_e$ such that,

$$|\rho_N(z)| < \varepsilon \text{ whenever } N > N_e$$

So we have,

$$\left| \int_C g(z)\rho_N(z)dz \right| \leq M\varepsilon L$$

Which means,

$$\lim_{N\to\infty} \int_C g(z)\rho_N(z)dz = 0$$

Or,

$$\int_C g(z)S(z)dz = \lim_{N\to\infty} \sum_{n=0}^{N-1} g(z)(z-z_0)^n$$

□

Theorem 5.10. The power series can be differentiated term by term. Or,

$$S'(z) = \sum_{n=1}^{\infty} n a_n (z-z_0)^{n-1}$$

Proof. Let z be a point interior to circle of convergence of series. Let us define the function,

$$g(s) = \frac{1}{2\pi i} \frac{1}{(s-z)^2}$$

at each point s on C.
We have,

$$\int_C g(s)S(s)ds = \frac{1}{2\pi i} \int_C \frac{S(s)ds}{(s-z)^2} = S'(z)$$

$$\int_C g(s)S(s)ds = \sum_{n=0}^{\infty} a_n \int_C g(s)(s-z_0)^n$$

This can be reduced to,

$$= \frac{1}{2\pi i} \frac{(s-z_0)^n}{(s-z)^2} = \frac{d}{dz}(z-z_0)^n$$

So we have,

$$S'(z) = \frac{d}{dz} a_n (z-z_0)^n$$

□

5.6 Absolute and Uniform Convergence

Theorem 5.11. If a power series,

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n$$

converges when $z = z_1$ then it is absolutely convergent at each point z in the open disk $|z-z_0| < R_1$ where $R_1 = |z_1 - z_0|$

Proof. Consider it converges when $z = z_1$. This means that,

$$\sum_{n=0}^{\infty} a_n(z_1-z_0)^n$$

converges which implies that each term is bounded by some value M.
So for any n, we have

$$a_n(z_1-z_0)^n \leq M$$

Now consider $\rho = \frac{|z-z_0|}{|z_1-z_0|}$. We know that $\rho \leq 1$ because $|z-z_0| \leq |z_1-z_0|$. Hence we have,

$$\begin{aligned} a_n(z_1-z_0)^n\rho^n &\leq M\rho^n \\ a_n(z-z_0)^n &\leq M\rho^n \end{aligned}$$

So we get that any term of form $a_n(z-z_0)^n$ is bounded by a number as well and as n increase because ρ is smaller than zero the terms go to zero. This means that our series,

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n$$

also converges.
In other words because $M\rho^n$ converges and because each term in our series is smaller than this term our series converges by comparison test. □

Remark. The greatest circle is called our circle of convergence and is the largest circle possible. Because if there was a point outside the circle for which the series converges then using the proof above the circle compassing that point would be the circle of convergence.

Now consider the following terminology where the series is defined for $|z-z_0| < R$

$$\begin{aligned} S(z) &= \sum_{n=0}^{\infty} a_n(z-z_0)^n \\ S_N(z) &= \sum_{n=0}^{N-1} a_n(z-z_0)^n \\ \rho_N(z) &= S(z) - S_N(z) \end{aligned}$$

Since the power series converges for any value of z when $|z-z_0| < R$ we have $\exists N_e$ such that $\forall \varepsilon$,

$$|\rho_N(z)| < \varepsilon \text{ whenever } N > N_e$$

When the choice of N_e is only dependent on ε and independent on z then convergence is said to be uniform in that region.

Theorem 5.12. If z_1 is a point inside $|z - z_0| = R$ of a series,

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

then that series must be uniformly convergent in the closed disk $|z - z_0| \leq R_1$ where $R_1 = |z_1 - z_0|$

Proof. So first consider the reminder term of our series with z and z_1 as follows,

$$\rho_N(z) = \lim_{n \to \infty} \sum_{n=N}^{m} a_n (z - z_0)^n$$
$$\sigma_N(z) = \lim_{n \to \infty} \sum_{n=N}^{m} a_n (z_1 - z_0)^n$$

Now because $|z - z_0| \leq |z_1 - z_0|$ we get,

$$\rho_N(z) \leq \rho_N(z)$$

inside $|z - z_0| \leq R_1$ □

Chapter 6

Residues And Poles

6.1 Isolated Singular Points

A singular point z_0 is isolated if $\exists, 0 < |z - z_0| < \varepsilon$ of z_0 where f is analytic.

Example.

$$f(z) = \frac{z-1}{z^5(z^2+9)}$$

We have $z = 0$ and $z = \pm 3i$. $\diamond$

Example.

$$F(z) = Log(z) = \ln(r) + i\theta, -\pi < \theta < \pi$$

Here $r = -1$ is always going to be a singular point. So we can't draw an epsilon neighborhood around a singular points that is isolated. $\diamond$

Example.

$$f(z) = \frac{1}{\sin(\pi/2)}$$

We have $z_n = \frac{1}{n}$ is a singular point. We are able to take $\varepsilon = \frac{1}{2}\left|\frac{1}{n+1} - \frac{1}{n}\right|$
However $z = 0$ is not an isolated singular point because for any ε we can choose N such that $|\frac{1}{N}| < \varepsilon$ $\diamond$

Example. Isolated singular point at $z_0 = \infty$ is when $R_1 < |z| < \infty$ $\diamond$

6.2 Residues

When z_0 is an isolated singular point of f there is a positive number R such that f is analytic at each point z for which $0 < |z - z_0| < R$. So $f(z)$ has a Laurent series.

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

Where,

$$b_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{-n+1}}dz$$

So,

$$b_1 = \frac{1}{2\pi i}\int_C f(z)dz$$

This means that the contour integral of $f(z)$ around C is,

$$\int_C f(z)dz = 2\pi i b_1$$

Example.

$$\int_C \frac{e^8 - 1}{z^4} dz \qquad C : |z| = 1$$

First we take the Laurent series around $z = 0$. The Laurent series expansion of the function is,

$$\frac{1}{z^4}(\sum_{n=0}^{\infty} \frac{z^n}{n!} - 1) = \frac{1}{z^4} + \cdots + \frac{1}{3!z} + \cdots - 1$$

So we see that our b_1 term is equal to $\frac{1}{3!}$. Hence we have,

$$\int_C f(z) = 2\pi i \frac{1}{3!} = \frac{\pi i}{3}$$

◇

Example.

$$\int_C \frac{dz}{z(z-2)^5} \qquad C : |z - 2| = 1$$

First we see that R is around the singular point 2 hence we need the expansion around 2. We have,

$$\frac{1}{z(z-2)^5} = \frac{1}{2(z-2)^5} \sum_{n=0}^{\infty} (-1)^n \frac{(z-2)^n}{2^n}$$

Here b_1 is when $n = 4$. ◇

6.3 Cauchy Residue Theorem

$$\int_C f(z)dz = 2\pi i \sum_{k=1}^{n} Res_{z-z_k} f(z)$$

6.4 Residue at Infinity

Let f be analytic $R_1 < |z| < \infty$. Such that the point at ∞ is said to be an isolated point. Let $C_0 : |z| = R_0$ in the clockwise direction.
So,

$$\int_{C_0} f(z)dz = 2\pi i Res_{z=\infty}(f(z))$$

6.5 Three types of Isolated singular points

1. **Removable Singular Points:** When every b_n is zero so that,

   $$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

   Here z_0 is known as a removable singular point. So the residue of a removable singular point is always 0.

2. **Essential Singular Points:** If an infinite number of b_n are non-zero. Then z_0 is a essential singular point.

3. **Poles of Order m:** If the principal part of f at z_0 contains at least one nonzero term but the number of terms is finite. So $\exists m$ such that $b_{m+1} = \cdots = 0$

6.6 Residue at Poles

Theorem 6.1. If z_0 is an isolated singular point of f then the following are equivalent,

1. z_0 is a pole of order m of f
2. $f(z)$ can be written as,
$$f(z) = \frac{\phi(z)}{(z - z_0)^m}$$
where $\phi(z)$ is analytic and nonzero at z_0

Proof. We can write $f(z)$ as,

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{(z - z_0)} + \cdots + \frac{b_m}{(z - z_0)^m} \\ &= \frac{1}{(z - z_0)^m} \sum_{n=0}^{\infty} a_n (z - z_0)^{n+m} + b_1 (z - z_0)^{m-1} + \ldots \end{aligned}$$

We see that the summation that we have is analytic as it is just a Taylor expansion (hence a polynomial). So we have,

$$\phi(x) = \begin{cases} f(z)(z - z_0)^m & z \neq z_0 \\ b_m & z = z_0 \end{cases}$$

$\square$

Theorem 6.2. If (a) and (b) are true then,

$$Res_{z=z_0} f(z) = \phi(z_0) \text{ if } m = 1$$

$$Res_{z=z_0} f(z) = \frac{\phi^{(m+1)}(z_0)}{(m-1)!} \text{ if } m = 2, 3, \ldots$$

Proof. If $m = 1$ we have,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{(z - z_0)}$$

So

$$(z - z_0) f(z) = \phi(z) = b_1 \text{ if } z = z_0$$

Now if $m \neq 1$ we have,

$$f(z) = \frac{1}{(z - z_0)^m} \left(\sum_{n=0}^{\infty} a_n (z - z_0)^{n+m} + b_1 (z - z_0)^{m-1} + \cdots + b_m \right)$$

We can take the derivative $m - 1$ times to isolate b_1 which gets us,

$$\begin{aligned} \phi^{m-1}(z) &= \cdots + (m-1)! b_1 \\ \phi^{m-1}(z_0) &= 0 + (m-1)! b_1 \end{aligned}$$

So we have,

$$b_1 = \frac{\phi^{m-1}(z)}{(m-1)!}$$

□

Example.

$$f(z) = \frac{z+4}{z^2+1} = \frac{z+4}{(z-i)(z+i)}$$

So we have,

$$\phi(z) = \frac{z+4}{(z+i)}$$

Which means our residue,

$$Res_{z=i} f(z) = \phi(i)$$

◇

Example.

$$f(z) = \frac{1-\cos(z)}{z^3}$$

◇

6.7 Zeroes of Analytic Functions

Theorem 6.3. If f is analytic at z_0 then the following are equivalent,

1. f has a zero of order m at z_0
2. $f(z) = (z-z_0)^m g(z)$, $g(z)$ is analytic, $g(z_0) \neq 0$

Proof. $(a) \Rightarrow (b)$
We have

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n = a_0 + a_1(z-z_0) + \dots$$

Because $z = z_0$ is a zero we have,

$$f(z_0) = 0$$

□

Theorem 6.4. Suppose,

1. f is analytic at z_0
2. $f(z) = 0$ at each point z of a domain D or segment L containing z_0
   Then $f(z) \equiv 0$ in the neighborhood

Proof. □

6.8 Behavior of functions near Isolated Points

Theorem 6.5. If z_0 is a removable singular point of f, then f is bounded and analytic in some neighborhood $0 < |z - z_0| < \varepsilon$ of z_0

Theorem 6.6. If a function is bounded and analytic in some deleted neighborhood $0 < |z - z_0| < \varepsilon$ of z_0. If f is not analytic at z_0 then it has a removable singularity there.

Proof.

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} b_n (z - z_0)^{-n}$$

We have,

$$b_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz$$

Let M such that $|f(z)| < M$, $0 < |z - z_0| < \varepsilon$.
So

$$|b_n| \leq \frac{1}{2\pi} \frac{M}{\rho^{n+1}} 2\pi\rho = M\rho^n$$

Now as $\rho \to 0$ we have $|b_n| \to 0$ □

Theorem 6.7. If z_0 is an essential singularity of f and let w_0 be any complex number. Then, for any positive ε, the inequality,

$$|f(z) - w_0| < \varepsilon$$

is satisfied at some point z in each deleted neighborhood of $0 < |z - z_0| < \delta$ of z_0.

Proof. We have z_0 is an isolated singularity of f. There is a neighborhood $0 < |z - z_0| < \delta$ where f is analytic.
Assume the theorem is false so,

$$|f(z) - w_0| \geq \varepsilon \text{ when } 0 < |z - z_0| < \delta$$

Now define,

$$g(z) = \frac{1}{f(z) - w_0},\ 0 < |z - z_0| < \delta$$

which is bounded and analytic. Which also means that z_0 is a removable singularity of g. So Let g be defined at z_0 such that it is analytic. We have,

$$f(z) = \frac{1}{g(z)} + w_0$$

This means that f becomes analytic at z_0 when it is defined as,

$$f(z_0) = \frac{1}{g(z_0)} + w_0$$

But this means that z_0 is a removable singularity of f and not an essential one which is a contradiction.
If $g(z_0) = 0$ then g must have a zero of some finite order m at z_0 as $g(z)$ is not identically equal to zero in $|z - z_0| < \delta$. This means that f has a pole of order m at z_0 which is not an essential singularity hence a contradiction. □

Theorem 6.8. If z_0 is a pole of f, then,

$$\lim_{z \to z_0} f(z) = \infty$$

Proof. If f has a pole of order m at z_0 then we have,

$$f(z) = \frac{\phi(z)}{(z - z_0)^m}$$

where $\phi(z)$ is analytic and non-zero at z_0.
So we have,

$$\lim_{z \to z_0} \frac{1}{f(z)} = \lim_{z \to z_0} (z - z_0)^m / \phi(z) = \frac{0}{\phi(z_0)} = 0$$

which means that $\lim_{z \to z_0} f(z) = \infty$ □

Chapter 7

Applications of Residues

7.1 Evaluation of Improper Integrals

We define the improper integral of f over the semi-infinite interval $0 \leq x < \infty$ as,

$$\int_0^\infty f(x)\,dx = \lim_{R\to\infty}\int_0^R f(x)\,dx$$

and,

$$\int_{-\infty}^\infty f(x)\,dx = \lim_{R_1\to\infty}\int_{-R_1}^0 f(x)\,dx + \lim_{R_2\to\infty}\int_0^{R_2} f(x)\,dx$$

This integral is also assigned the value Cauchy principal value,

$$\text{P.V.}\int_{-\infty}^\infty f(x)\,dx = \lim_{R\to\infty}\int_{-R}^R f(x)\,dx$$

If integral (2) converges the Cauchy principal value (3) exists. And that value is the number to which integral (2) converges as,

$$\lim_{R\to\infty}\int_{-R}^R f(x)\,dx = \lim_{R\to\infty}\left[\int_{-R}^0 f(x)\,dx + \int_0^R f(x)\,dx\right]$$

If $f(-x) = f(x)$ for all x then f is an even function and the Cauchy principle value exists. Consider the semicircle region in the complex plane, we have,

$$\int_C f(z)dz = \int_{-R}^R f(x)dx + \int_{C_R} f(z)dz$$

Example.

$$f(x) = \int_0^\infty \frac{1}{x^6+1}\,dx$$

First let our function be $f(z) = \frac{1}{z^6+1}$. Let our contour be the positively oriented semi-circle $-\infty$ to ∞ such that,

$$\int_C f(z) = \int_{-\infty}^\infty f(x)dx + \int_{C_R} f(z)$$

We have,

$$\int_C f(z) = 2\pi i \sum Res f(z)$$

Our f has isolated singular points so, the integral over the contour is the integral over a contour around each of the singular points which are,

$$z_1 = i, z_2 = \sqrt{3}/2 + \frac{i}{2}, z_3 = -\sqrt{3}/2 + \frac{i}{2}$$

So integral is,

$$2\pi i\left(\frac{1}{6i^5}+\frac{1}{6(\sqrt{3}/2+i/2)^5}+\frac{1}{-6(\sqrt{3}/2+i/2)^5}\right)$$

Now we have,

$$\left|\int_{C_R} f(z)dz\right| = \pi R\frac{1}{R^6-1}$$

As R goes to infinity then we have the above going to 0.

$\diamond$

7.2 Example

We have,

$$\int_0^\infty \frac{1}{x^6+1}\,dx$$

We consider,

$$f(z)=\frac{1}{z^6+1}$$

whose isolated singularities are the zeroes of z^6+1
If we consider the semicircle in the upper half plane there are three singularities,

$$c_0=e^{i\pi/6}, c_1=i, c_2=e^{i5\pi/6}$$

So we have,

$$\int_{-R}^{R} f(x)\,dx+\int_{C_R} f(z)\,dz = 2\pi i(B_0+B_1+B_2)$$

Where B_k is the residue of $f(z)$ at c_k
So we have,

$$\int_{-R}^{R} f(x)\,dx=\frac{2\pi}{3}-\int_{C_R} f(z)\,dz$$

Now we show that the integral on the right goes to zero when $R\to\infty$

7.3 Jordan's Lemma

Theorem 7.1. Suppose,
(a). $f(z)$ is analytic exterior to $|z|=R_0$
(b). C_r denotes a semicircle $z=Re^{i\theta}(0\le\theta\le\pi)$ where $R>R_0$
(c). For all points z on C_R, there is a positive constant M_R such that,

$$|f(z)|\le M_R \text{ and } \lim_{R\to\infty} M_R=0$$

Then for every positive constant a,

$$\lim_{R\to\infty}\int_{C_R} f(z)e^{iaz}dz=0$$

Remark. The proof is based on Jordan's Inequality,

$$\int_0^\pi e^{-R\sin\theta}d\theta<\frac{\pi}{R}$$

Consider,

$$y=\frac{2\theta}{\pi} \text{ and } y=\sin\theta$$

Proof.

$$\int_{C_R} f(z)e^{iaz} = \int_0^{\pi} f(Re^{i\theta})e^{iaRe^{i\theta}} Rie^{i\theta} d\theta$$

Now since $|f(Re^{i\theta})| \leq M_R$ and $|e^{iaRe^{i\theta}}| \leq e^{-aR\sin\theta}$
it follows that,

$$\left| \int_{C_R} f(z)e^{iaz}dz \right| \leq M_R R \int_0^{\pi} e^{-aR\sin\theta} d\theta < M_R \frac{\pi}{a}$$

□

Example.

$$\int_0^{\infty} \frac{\sin x}{x} dx$$

First let

$$f(z) = \frac{e^{iz}}{z}$$

. We have a singularity at $z = 0$ so let our contour be such that it jumps at $z = 0$.
So we have,

$$\int_{L_1} f(z) + \int_{L_2} f(z) + \int_{\rho} f(z) + \int_{R} f(z) = \int_{C} f(z) = 0$$

because there is no singularity in our contour.
We also have,

$$\lim_{R\to\infty} \int_{C_R} f(z) = 0$$

from Jordan's lemma

$$\int_{L_1} f(z) + \int_{L_2} f(z) + \int_{\rho} f(z) = 0$$

Now,

$$\lim_{\rho\to 0} \int_{C_\rho} f(z) = -B_0 \pi i$$

And the residue is 1 because $e^0 = 1$ so,

$$\lim_{\rho\to 0} \int_{C_\rho} f(z) = -\pi i$$

And we also have,

$$\int_{L_1} f(z) + \int_{L_2} f(z) = 2i \int_0^{\infty} \frac{\sin x}{x}$$

So

$$2i \int_0^{\infty} \frac{\sin x}{x} = \pi i$$

$$\int_0^{\infty} \frac{\sin x}{x} = \frac{\pi}{2}$$

◇

7.4 Definite Integrals involving Sines and Cosines

We can evaluate integrals like,

$$\int_0^{2\pi} F(\sin\theta, \cos\theta) d\theta$$

We take,

$$z = e^{i\theta} \qquad (0 \leq \theta \leq 2\pi)$$

to describe the contour. Then we have,

$$\frac{dz}{d\theta} = ie^{i\theta} = iz$$

which gives us,

$$\sin\theta = \frac{e^{i\theta} - e^{-\theta}}{2i} \qquad \text{and} \qquad \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

So our substitutions are,

$$\sin\theta = \frac{z - z^{-1}}{2i}, \qquad \cos\theta = \frac{z + z^{-1}}{2}, \qquad d\theta = \frac{dz}{iz}$$

Our integral is transformed to,

$$\int_c F\left(\frac{z - z^{-1}}{2i}, \frac{z + z^{-1}}{2}\right) \frac{dz}{iz}$$

Example.

$$\int_0^{2\pi} \frac{d\theta}{1 + a\sin(\theta)} \text{for} \qquad -1 < a < 1$$

Using the above substitution we have,

$$\int_C \frac{2/a}{z^2 + (2i/a)z - 1} dz$$

◇

7.5 Argument principle

Theorem 7.2. Let C be a positively oriented simple closed contour,
(a). $f(z)$ is meromorphic in the domain interior to C (analytic in D except for its poles)
(b). $f(z)$ is analytic and non-zero on C.
(c). z is number of zeroes and p is number of poles in C.
Then,

$$\frac{1}{2\pi i}\Delta c \arg f(z) = z - p$$

Proof. Let $z = z(t), (a \leq t \leq b)$ so we have,

$$\int_C \frac{f'(z)}{f(z)} dz = \int_a^b \frac{f'[z(t)]z'(t)}{f[z(t)]}$$

If our transformation is $w = f(z)$ we can express it as $w = \rho(t)e^{i\phi(t)}$
So we have,

$$f'[z(t)]z'(t) = \frac{d}{dt} f[z(t)] = \rho'(t)e^{i\phi(t)} + i\rho(t)e^{i\rho(t)}\rho'(t)$$

We get,

$$\int_C \frac{f'(z)}{f(z)} dz = \int_a^b \frac{\rho'(t)}{\rho(t)} dt + i\int_a^b \phi'(t) dt = \ln\rho(t)\Big]_a^b + i\rho(t)\Big]_a^b$$

But we have

$$\rho(b) = \rho(a) \qquad \textit{and} \qquad \phi(b) - \phi(a) = \Delta_C \arg f(z)$$

So we have,

$$\int_c \frac{f'(z)}{f(z)} dz = i\Delta_C \arg f(z)$$

□

Chapter 8

Mapping By Elementary Functions

8.1 Linear Transformations

8.2 $\mathbf{w = 1/z}$

$$\frac{1}{z} = \frac{\overline{z}}{z^2} = \frac{x - iy}{x^2 + y^2}$$

So we have,

$$u = \frac{x}{x^2 + y^2} \quad v = -\frac{y}{x^2 + y^2}$$