MATH 4320 HW11-13

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Problem 1

(a). We have the expansion of $\frac{1}{z+z^2}$ as,

$$\sum_{n=0}^{\infty}(-1)^n z^{n-1}$$

$$= \frac{1}{z} + \sum_{n=0}^{\infty}(-1)^{n+1} z^n$$

So we have our residue as 1

(b). We have expansion of $z\cos(\frac{1}{z})$ as,

$$z\sum_{n=0}^{\infty}(-1)^n \frac{(\frac{1}{z})^{2n}}{2n!} = \sum_{n=0}^{\infty}(-1)^n \frac{1}{z^{2n-1}2n!}$$

We have power of z as 1 when $n = 1$ so coefficient is $\frac{-1^1}{2!} = -\frac{1}{2}$

So the residue is $-\frac{1}{2}$

(c). We have $\frac{z-\sin z}{z}$. We can first rewrite this as $1 - \frac{\sin z}{z}$. The expansion of which is,

$$1 - \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n}}{(2n+1)!}$$

We see for all values of n the highest power of z is greater than equal to 0. Hence the residue term is 0.

(d). We have $\frac{\cot z}{z^4}$ We can write this as,

$$\frac{\cos z}{z^4 \sin z}$$

$$= \frac{1}{z^4}\frac{1 - \frac{z^2}{2!} + \dots}{z - \frac{z^3}{3!} + \dots}$$

$$= \frac{1}{z^5}\frac{1 - \frac{z^2}{2!} + \dots}{1 - \frac{z^2}{3!} + \dots}$$

If we have $w = \frac{z^2}{3!}$

$$= \frac{1}{z^5}\frac{1 - \frac{z^2}{2!} + \dots}{1 - w}$$

And as $|w| < 1$ we have,

$$= \frac{1}{z^5}(1 - \frac{z^2}{2!} + \dots)(1 + w + w^2 + \dots)$$

So the coefficient of the $\frac{1}{z}$ term would be $\frac{1}{(3!)^2} - \frac{1}{5!} + \frac{1}{4!} - \frac{1}{2!3!}$

$$= -\frac{1}{45}$$

(e). We have $\frac{\sinh z}{z^4(1-z^2)}$

We can expand $\sinh z$ as,

$$\sum_{n=0}^{\infty}\frac{z^{2n+1}}{(2n+1)!} = (z + \frac{z^3}{3!} + \dots)$$

And we can expand $\frac{1}{z^4(1-z^2)}$ as,

$$\begin{aligned}\frac{1}{z^4}\sum_{n=0}^{\infty} z^{2n} &= \sum_{n=0}^{\infty} z^{2n-4} \\ &= \frac{1}{z^4}+\frac{1}{z^2}+\dots\end{aligned}$$

The product of both will be,

$$\left(z+\frac{z^3}{3!}+\dots\right)\left(\frac{1}{z^4}+\frac{1}{z^2}+\dots\right)$$

So the coefficient of $\frac{1}{z}$ in this product is,

$$1+\frac{1}{3!}=1+\frac{1}{6}=\frac{7}{6}$$

Problem 3

We have,

$$\int_C \frac{4z-5}{z(z-1)}$$

Now using residue at infinity we know this integral is,

$$= 2\pi i Res_{z=0}(\frac{1}{z^2}f(\frac{1}{z}))$$

$$\begin{aligned}Res_{z=0}\left(\frac{1}{z^2}f\left(\frac{1}{z}\right)\right) &= \frac{1}{z^2}\frac{\frac{4}{z}-5}{\frac{1}{z}(\frac{1}{z}-1)} \\ &= \frac{4}{z(1-z)}-\frac{5}{1-z} \\ &= 4\sum_{n=0}^{\infty} z^{n-1}-5\sum_{n=0}^{\infty} z^n\end{aligned}$$

So coefficient of the $\frac{1}{z}$ term is when $n=0$ where we have,

$$\frac{4}{z}$$

which is 4.

Hence our integral evaluates to $2\pi i\cdot 4=8\pi i$

Problem 6

We have f is analytic throughout the finite plane except for a finite number of singular points. So consider a contour C that includes all our finite number of singular points. So we know the integral around this contour is,

$$\frac{1}{2\pi i}\int_C f(z)dz = Res_{z=z_1}+\dots+Res_{z=z_n}$$

Now because there are no singular points outside this contour we also know that,

$$\frac{1}{2\pi i}\int_C f(z) = -Res_{z=\infty}$$

So Putting the two together we have,

$$Res_{z=z_1} + \cdots + Res_{z=z_n} + Res_{z=\infty} = 0$$

Problem 2

(a). We have,

$$\frac{1-\cosh z}{z^3}$$

whose series expansion is,

$$\begin{aligned}
&\frac{1}{z^3} - \sum_{n=0}^{\infty} \frac{z^{2n-3}}{(2n)!} \\
&= \frac{1}{z^3} - \frac{1}{z^3} - \sum_{n=1}^{\infty} \frac{z^{2n-3}}{(2n)!} \\
&= -\sum_{n=0}^{\infty} \frac{z^{2n-1}}{(2n+2)!} \\
&= -\frac{1}{2z} - \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+4)!}
\end{aligned}$$

So we see we have a pole of order 1 and residue is $B = -\frac{1}{2}$

(b). We have,

$$\frac{1-e^{2z}}{z^4}$$

Expansion is,

$$\begin{aligned}
&= \frac{1}{z^4} - \frac{1}{z^4}\sum_{n=0}^{\infty} 2^n z^n \frac{1}{n!} \\
&= \frac{1}{z^4} - \sum_{n=0}^{\infty} 2^n z^{n-4} \frac{1}{n!} \\
&= \frac{1}{z^4} - \frac{1}{z^4} - \sum_{n=1}^{\infty} 2^n z^{n-4} \frac{1}{n!} \\
&= -\sum_{n=0}^{\infty} 2^{n+1} z^{n-3} \frac{1}{(n+1)!}
\end{aligned}$$

So our pole is of order 3 as the highest power of $\frac{1}{z}$ is 3 when $n = 0$. And coefficient of $\frac{1}{z}$ is when $n = 2$ where we have,

$$-2^3 \frac{1}{z}\frac{1}{3!} = -\frac{4}{3}\frac{1}{z}$$

So residue is $-\frac{4}{3}$

(c). We have

$$\frac{e^{2z}}{(z-1)^2}$$

Expansion around $z = 1$ is,

$$\begin{aligned}
e^{2(z-1+1}\frac{1}{(z-1)^2} &= e^2 e^{2(z-1)}\frac{1}{(z-1)^2} \\
&= e^2 \sum_{n=0}^{\infty} 2^n (z-1)^{n-2}\frac{1}{n!}
\end{aligned}$$

So when $n = 0$ we have highest power of $\frac{1}{z}$ as $m = 2$. Hence pole is of order 2. And when $n = 1$ we have coefficient as $2e^2$ which is our residue.

Problem 2

(a). We have an isolated singular point at $= -1$. So our residue is

$$-1^{\frac{1}{4}} = e^{\frac{\pi}{4}} = \cos(\frac{\pi}{4}) + i\sin(\frac{\pi}{4})$$

$$= \frac{1+i}{\sqrt{2}}$$

(b). We have,

$$\frac{Log(z)}{(z^2+1)^2}$$

which can be written as,

$$\frac{Log(z)}{(z+i)^2(z-i)^2}$$

If $\phi(z) = \frac{Log(z)}{(z+i)^2}$ we have,

$$\frac{\phi(z)}{(z-i)^2}$$

As $z = i$ is an isolated singular point we have,

$$\frac{\phi^{2-1}(z)}{(2-1)!}$$

$$\phi'(z) = \frac{(z+i)^2\frac{1}{z} - Log(z)2(z+i)}{(z+i)^4}$$

And

$$\phi'(i) = (-\frac{4}{i} - i\frac{\pi}{2}4i)\frac{1}{16}$$

$$= (4i + 2\pi)\frac{1}{16}$$

$$= \frac{2i+\pi}{8}$$

(c). We have

$$\frac{z^{\frac{1}{2}}}{(z^2+1)^2}$$

We can write this as,

$$\frac{z^{\frac{1}{2}}}{(z+i)^2(z-i)^2}$$

We have,

$$\phi(z) = \frac{z^{1/2}}{(z+i)^2}$$

So

$$\phi'(z) = \frac{(z+i)^2\frac{z^{-1/2}}{2} - z^{1/2}2(z+i)}{(z+i)^4}$$

So

$$\begin{aligned}
\phi'(i) &= (-\frac{2}{i^{1/2}} - i^{1/2}4i)(\frac{1}{16}) \\
&= (i2i^{1/2} - i^{1/2}4i)(\frac{1}{16}) \\
&= (i^{3/2} - i^{3/2}2)(\frac{1}{8}) \\
&= (i^{3/2} - i^{3/2}2)(\frac{1}{8}) \\
&= -i^{3/2}(\frac{1}{8}) \\
&= (\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}})(\frac{1}{8}) \\
&= \frac{1-i}{8\sqrt{2}}
\end{aligned}$$

Problem 4

We need to find,

$$\int_C \frac{3z^3+2}{(z-1)(z^2+9)}\, dz$$

(a). Inside our contour $|z-2| \leq 2$ we have only one singularity. Hence the integral will evaluate $2\pi i Res_{z=1} f(z)$.

So we have $\phi(z) = \frac{3^3+2}{(z^2+9)}$

Our pole is of order 1 so we have,

$$\phi(1) = \frac{5}{10}$$

And our integral is $2\pi i \frac{1}{2} = \pi i$

(b). Inside our contour $|z| = 4$ we have three singularities hence the integral is sum of all three residues at that point. So we have,

$$2\pi i(Res_1 f(z) + Res_{3i} f(z) + Res_{-3i} f(z))$$

$$Res_1 f(z) = \frac{1}{2}$$

$$Res_{3i} f(z) = \frac{3(3i)^3+2}{(3i-1)(6i)}$$

$$Res_{-3i} f(z) = \frac{-3(3i)^3+2}{(3i+1)(6i)}$$

The sum times $2\pi i$ evaluates to,

$$6\pi i$$

Problem 7

(a).

$$f(z) = \frac{(3z+2)^2}{z(z-1)(2z+5)}$$

So using residue at infinity it is enough to find,

$$2\pi i Res_{z=0} \frac{(\frac{3}{z}+2)^2}{\frac{1}{z}(\frac{1}{z}-1)(2\frac{1}{z}+5)}$$

Which is,

$$2\pi i Res_{z=0} \frac{(3+2z)^2}{z(1-z)(2+5z)}$$

$$= 2\pi i(\frac{9}{2}) = 9\pi i$$

(b). We have,

$$f(z) = \frac{z^3 e^{1/z}}{1+z^3}$$

Our integral would be equivalent to,

$$2\pi i Res_{z=0}(\frac{1}{z^2} f(\frac{1}{z}))$$

$$= 2\pi i Res_{z=0}(\frac{e^z}{z^2(z^3+1)})$$

We have $\phi(z) = \frac{e^z}{z^3+1}$ and $m = 1$. So we have,

$$\phi'(0) = 1$$

Hence our integral is $2\pi i \cdot 1 = 2\pi i$

Problem 3

(a). We have $Res_{z=\frac{\pi i}{2}} \frac{\sinh z}{z^2 \cosh z}$

Using theorem we have $Res_{z=\frac{\pi i}{2}} \frac{\sinh z}{z^2 \cosh z} = \frac{\sinh \pi i/2}{(\pi i/2)^2(-\sinh(\pi i/2))+\cosh(\pi i/2)2(\pi i/2)}$

$$= -\frac{4}{\pi^2}$$

(b). We have $Res_{z=\pi i} \frac{e^{zt}}{\sinh z} + Res_{z=-\pi i} \frac{e^{zt}}{\sinh z}$

Using theorem,

$$Res_{z=\pi i} \frac{e^{zt}}{\sinh z} = \frac{e^{(\pi i)t}}{\cosh(\pi i)} = \frac{\cos(\pi t) + i\sin(\pi t)}{-1}$$

$$Res_{z=-\pi i} \frac{e^{zt}}{\sinh z} = \frac{e^{(-\pi i)t}}{\cosh(-\pi i)} = \frac{\cos(-\pi t) + i\sin(-\pi t)}{-1}$$

So their sum is,

$$-2\cos(\pi t)$$

Problem 6

First we need to find,

$$\int_{C_N} \frac{dz}{z^2 \sin z}$$

We know this integral would be,

$$2\pi i \sum_{n=1}^{K} Res_{z=z_n} f(z)$$

Where z_n are the singularities of our function f within our domain. So we have $f(z) = \frac{1}{z^2 \sin z}$

So our singularities are when $z = 0, z = \pm\pi, z = \pm 3\pi, \ldots$. So our integral would be,

Or in other words we have $z = 0, z = (n)\pi$ for $n \in \mathbb{N}$

$$2\pi i \sum_{n=1}^{N} Res_{z=z_n} f(z)$$

If $p(z) = 1$ and $q(z) = z^2 \sin z$ such that $f(z) = p(z)/q(z)$ We know if $q(z) \neq 0$ residue would be,

$$\frac{p(z_n)}{q'(z_n)} = \frac{1}{(z_n^2 \cos z_n + 2z_n \sin z_n)}$$

First if $z_n = 0$ we can find residue using the Taylor expansion,

We have,

$$\begin{aligned}
\frac{1}{z^2 \sin z} &= \frac{1}{z^2(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots)} \\
&= \frac{1}{z^3(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \ldots)} \\
&= \frac{1}{z^3(1 - (\frac{z^2}{3!} - \frac{z^4}{5!} - \ldots))}
\end{aligned}$$

If $w = (\frac{z^2}{3!} - \frac{z^4}{5!} - \ldots)$ we have

$$= \frac{1}{z^3(1 - w)}$$

And for $w \leq 1$ we have,

$$\begin{aligned}
\frac{1}{z^3(1 - w)} &= \frac{1}{z^3} \sum_{n=0}^{\infty} w^n \\
&= \frac{1}{z^3} \sum_{n=0}^{\infty} (\frac{z^2}{3!} - \frac{z^4}{5!} + \ldots)^n
\end{aligned}$$

We need the coefficient for $\frac{1}{z}$. In our case the that only happens in the first element of the sequence on the right so we have which is,

$$\frac{1}{z^3} + (\frac{z^2}{3!} + \ldots)$$

$$\frac{1}{3!z} + \ldots$$

So we have our residue as $\frac{1}{3!} = \frac{1}{6}$

Now as for all the other singularities we see that $q'(z_n) \neq 0$ we have,

$$\frac{p(z_n)}{q'(z_n)} = \frac{1}{z_n^2 \cos z_n + 2z_n \sin z_n}$$

We have our singularities as,

$$n\pi \text{ for } n \in \mathbb{N}$$

So we have

$$\begin{aligned}
Res_{z=z_n} &= \frac{1}{(n)^2\pi^2 \cos n\pi + 0 \cdot 2z_n} \\
&= \frac{1}{n^2\pi^2 \cos n\pi} \\
&= \frac{1}{n^2\pi^2(-1)^n} \\
&= \frac{(-1)^n}{n^2\pi^2}
\end{aligned}$$

Now because $\frac{(-1)^n}{n^2\pi^2}$ is an even function, we have $f(-n) = f(n)$. Hence,

$$\sum_{n=-N}^{N} \frac{(-1)^n}{n^2\pi^2} = 2\sum_{n=1}^{N} \frac{(-1)^n}{n^2\pi^2}$$

So the sum of all our residues is,

$$\frac{1}{6} + 2\sum_{n=1}^{N} \frac{(-1)^n}{n^2\pi^2}$$

So our integral is,

$$\int_{C_N} \frac{dz}{z^2 \sin z} = 2\pi i \left[\frac{1}{6} + 2\sum_{n=1}^{N} \frac{(-1)^n}{n^2\pi^2}\right]$$

Now we are given that the integral goes to 0 as $N \to \infty$ this means that,

$$\begin{aligned}
2\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2\pi^2} &= -\frac{1}{6} \\
\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2\pi^2} &= -\frac{1}{12} \\
\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} &= -\frac{\pi^2}{12} \\
\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} &= \frac{\pi^2}{12}
\end{aligned}$$

Problem 5

We have,

$$\int_0^{\infty} \frac{x^2 dx}{(x^2+1)(x^2+4)}$$

First we have the function is even which means that $f(-x) = f(x)$ hence,

$$\int_{-\infty}^{\infty} f(x)dx = 2\int_{0}^{\infty} f(x)dx$$

First consider the complex valued function,

$$f(z) = \frac{z^2}{(z^2+1)(z^2+4)}$$

Consider the positively oriented semicircle and we have,

$$\int_C f(z)dz = \int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz$$

So first integral we see our singularities which are $z = i$ and $z = 2i$. So our integral is the sum of residues at these two points.

$$f(z) = \frac{z^2}{(z+i)(z-i)(z+2i)(z-2i)}$$

So $Res_{z=i} f(z) = \frac{i^2}{2i(3i(-i))} = -\frac{1}{6i}$

And $Res_{z=2i} f(z) = \frac{4i^2}{(3i)(i)(4i)} = -\frac{4}{-12i} = \frac{1}{3i}$

Sum is $\frac{1}{6i}$ and integral is $2\pi i \frac{1}{6i} = \frac{\pi}{3}$

Now we can also show the,

$$\int_{C_R} f(z)dz$$

goes to 0 as we can bound $|f(z)| \leq \frac{R^2}{(R^2-1)(R^2-4)}$

So we see the power of the denominator is greater than numerator hence as $R \to \infty$ the integral goes to zero.

Hence we have,

$$\int_C f(z)dz = \int_{-\infty}^{\infty} f(x)dx = \frac{\pi}{3}$$

$$2\int_0^{\infty} = \frac{\pi}{3}$$

$$\int_0^{\infty} = \frac{\pi}{6}$$

Problem 4

$$\int_{-\infty}^{\infty} \frac{x\sin(ax)}{x^4+4}\,dx$$

We can take

$$f(z) = \frac{ze^{iaz}}{z^4+4}$$

If we consider the positively oriented semicircle from $-R$ to R we have,

$$\int_C f(z)dz = \int_{C_R} f(z)dz + \int_{-\infty}^{\infty} f(x)\,dx$$

Our singularities is when $z^4 = -4$. So we have,

$$\begin{aligned} z^4 &= -4 \\ z^4 &= 4e^{-(\pi+2n\pi)i} \\ z &= (4e^{-(\pi+2n\pi)i})^{1/4} \\ z &= (4e^{-(\pi/4+\pi n/2)i}) \end{aligned}$$

So the singularities within our contour are when $n = 1, 2$ which are,

$$z = i + 1, z = i - 1$$

Now using theorem the residues are as follows,

$$Res_{z=i+1} f(z) = \frac{p(i+1)}{q'(i+1)} = \frac{(i+1)e^{ia(i+1)}}{4(i+1)^3}$$

$$\frac{(i+1)e^{ia(i+1)}}{4(i+1)^3} = \frac{e^{ia(i+1)}}{4(i+1)^2}$$
$$\frac{e^{ia}e^{-a}}{8i}$$

Similarly we have,

$$Res_{z=i-1} f(z) = \frac{p(i-1)}{q'(i-1)} = \frac{(i-1)e^{ia(i-1)}}{4(i-1)^3}$$

$$\frac{(i-1)e^{ia(i-1)}}{4(i-1)^3} = \frac{e^{ia(i-1)}}{4(i-1)^2}$$
$$\frac{e^{-ia}e^{-a}}{-8i}$$

So sum of residues is

$$\begin{aligned} &= \frac{e^{ia}e^{-a}}{8i} + \frac{e^{-ia}e^{-a}}{-8i} \\ &= \frac{e^{-a}}{8i}(e^{ia} - e^{-ia}) \\ &= \frac{e^{-a}}{8i}(\cos(a) + i\sin(a) - \cos(a) + i\sin(a)) \\ &= \frac{e^{-a}}{8i}(2i\sin(a)) \\ &= \frac{e^{-a}}{4}\sin(a) \end{aligned}$$

So our integral is

$$\begin{aligned} &= 2\pi i\frac{e^{-a}}{4}\sin(a) \\ &= \frac{\pi i e^{-a}\sin(a)}{2} \end{aligned}$$

And we have,

$$Im(\int_C f(z)dz) = \frac{2^{-a}\sin(a)}{2}$$

So we have,

$$Im(\int_C f(z)dz) = \int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)dz$$

We know the right integral goes to zero when R goes to ∞ as we can write our function as,

$$f(z) = \frac{ze^{iaz}}{z^4+4} = \phi(z)e^{iaz}$$

And using Jordan lemma we have,

$$\lim_{R\to\infty} \int_{C_R} \phi(z)e^{iaz}dz = 0$$

So we get,

$$\int_{-\infty}^{\infty} \frac{x\sin(ax)}{x^4+4}\,dx = \frac{\pi}{2}e^{-a}\sin(a)$$

Problem 9

We have,

$$\int_{-\infty}^{\infty} \frac{x\sin x}{x^2+2x+2}\,dx$$

First lets construct our function as,

$$f(z) = \frac{ze^{iz}}{z^2+2z+2} = \frac{ze^{iz}}{(z+1)^2+1}$$

Considering the positively oriented contour lying in the upper half plane we have,

$$\int_C f(z)dz = \int_{-\infty}^{\infty} f(z)\,dz + \int_{C_R} f(z)dz$$

First to find $\int_C f(z)dz$ we look at the singularities within our domain. We have,

$$(z+1)^2 = -1 = e^{(-\pi+2n\pi)i}$$

$$z+1 = e^{(-\frac{\pi}{2}+n\pi)i}$$

$$z = e^{(-\frac{\pi}{2}+n\pi)i} - 1$$

So the only singularity in our domain is when $n = 1$ and we have $z = i - 1$.

So we have our residue as,

$$
\begin{aligned}
Res_{z=i-1}f(z) = \frac{p(i-1)}{q'(i-1)} &= \frac{ze^{iz}}{2(z+1)} \\
&= \frac{(i-1)e^{i(i-1)}}{2i} \\
&= \frac{(i-1)e^{-1-i}}{2i} \\
&= \frac{(i-1)e^{-1}e^{-i}}{2i} \\
&= \frac{(i-1)e^{-1}(\cos(1) - i\sin(1))}{2i} \\
&= \frac{e^{-1}(i\cos(1) + \sin(1) - \cos(1) + i\sin(1))}{2i} \\
&= \frac{e^{-1}}{2i}(i(\cos 1 + \sin 1) + \frac{e^{-1}}{2i}(\sin 1 - \cos 1)
\end{aligned}
$$

So our integral would be $2\pi i Res_{z=i-1}f(z)$,

$$= i\pi e^{-1}(\cos 1 + \sin 1) + \pi e^{-1}(\sin 1 - \cos 1)$$

Now because we only need the imaginary part as $Im f(z) = f(x)$ we have,

$$Im\int_C f(z)dz = \int_{-R}^{R} f(x)\,dx + Im\int_{C_R} f(z)dz$$

$$\pi e^{-1}(\cos 1 + \sin 1) = \int_{-R}^{R} f(x)\,dx + Im\int_{C_R} f(z)dz$$

However we know that j

$$\lim_{R\to\infty}\int_{C_R} f(z)dz = 0$$

as we can write $f(z) = \phi(z)e^{iz}$ so using jordans lemma we have the integral goes to zero.

Hence we get as $R \to \infty$,

$$\int_{-\infty}^{\infty} f(x)\,dx = \pi e^{-1}(\cos 1 + \sin 1)$$