Real Analysis

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Chapter 1

Introduction

1.1 Logic and proofs

Types of proofs,

1. Direct proof
2. Argument by contradiction
3. Induction
4. Contrapositive (we show$\neg B \Rightarrow \neg A$)

Theorem 1.1. $a = b \Leftrightarrow \forall \varepsilon > 0, |a - b| < \varepsilon$

Proof. 1. To show, $a = b \Rightarrow \forall \varepsilon > 0, |a - b| < \varepsilon$.
Suppose $a = b$ so $|a - b| = 0$. We have $\forall \varepsilon > 0$ so $|a - b| = 0 < \varepsilon$
2. To show, $\forall \varepsilon > 0, |a - b| < \varepsilon \Rightarrow a = b$
Now assume this is not true, or that $a \neq b$ so $a - b \neq 0$ this means that there is a non-zero number k such that $|a - b| = \varepsilon_0$. Now take $\varepsilon = \frac{\varepsilon_0}{2}$. This gives us, $|a - b| = \varepsilon_0 > \varepsilon$ which contradicts the statement. Hence our assumption is false and we prove the results. $\square$

Example (Induction). $x_1 = 1$ and $x_{n+1} = \frac{1}{2}x_n + 1, \forall n \in \mathbb{Z}$. Show $x_{n+1} \geq x_n \forall n \in \mathbb{N}$
Define $S = \{n \in \mathbb{N}, s.t. x_{n+1} \geq x_n\}$ clearly, $S \subseteq N$.
$x_1 = 1$ and $x_2 = \frac{x_1}{2} + 1 = 1.5$. This gives us $x_2 > x_1$ so $1 \in S$
Suppose $n \in S$ and $x_{n+1} \geq x_n$. Note that,

$$x_{n+2} = \frac{1}{2}x_{n+1} + 1$$
$$x_{n+1} = \frac{1}{2}x_n + 1$$

Then $x_{n+2} = \frac{1}{2}x_{n+1} + 1 \geq \frac{1}{2}x_n + 1 = x_{n+1}$ or $x_{n+2} \geq x_{n+1}$ which means $n + 1 \in S$. So by induction we have $S = N$ and $x_{n+1} \geq x_n, \forall n \in \mathbb{N}$ $\diamond$

1.2 Real Numbers

Number systems,

1. Natural numbers $\mathbb{N}$

   $1, 2, 3, \ldots$

   Can't do subtraction

2. Integers $\mathbb{Z}$

   $\ldots, -3, -2, -1, 0, 1, 2, 3 \ldots$

   Can't do division

3. Rationals $\mathbb{R}$

   $\{\frac{p}{q}$ where $p, q \in \mathbb{Z}$ but $q \neq 0\}$

   Now we have $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{R}$

   But other numbers are still not captured,

   Example. $\sqrt{2}$ is not defined in $\mathbb{R}$. However if we define $x_1 = 2$, $x_{n+1} = \frac{1}{2}(x_n + \frac{2}{x_n})$. We know $x_{n+1} \in \mathbb{R}, \forall n \in \mathbb{N}$ (we can then show that $x_n \to \sqrt{2}$). ◇

   Theorem 1.2. $\sqrt{2}$ is not rational

   Proof. Argue by contradiction □

4. Real numbers $\mathbb{R}$

   We will define $\mathbb{R}$ as $\mathbb{Q}$ with the gaps filled in.

Definition (Axiom of completeness)**.** Every non-empty subset of $\mathbb{R}$ that is bounded above has a least upper bound called the supremum.

Let $S \subseteq \mathbb{R}$ and S is bounded above. If there is $u \in \mathbb{R}$ such that $s \leq u, \forall s \in S$ then S is bounded above by u (*Similar for bounded below*)

Definition (Least upper bound or supremum)**.** We say $u \in \mathbb{R}$ is the least upper bound for S if,

1. If u is an upper bound for S
2. $u \leq v$ for any other upperbound v of S.

Similar for greatest lower bound or infimum

Example. $S = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ where $S \subseteq R$ and $S \neq \phi$. Here S is bounded above by $1.1, 1.2, 2, 3, 4, \ldots$ By AoC, $\sup S$ exists (in this case is 1). Similarly S is bounded below as well and can also be shown that $\inf S = 0$ ◇

Note. Here, $1 \in S$ but $0 \notin S$. So sup or inf may or may not be in the set.

Definition. $S \subseteq \mathbb{R}$ then we say a real numbers $m \in S$ is a maximum if $\forall s \in S$ we have

$$s \leq m$$

Similar for minimum

Note. Following are true,

1. $m \in S$

2. m might not exist, consider,

$$S = [0, 1)$$

This does not have a maximum element but $\sup S = 1$

It does have a minimum element which is also equal to the infinium, $\inf S = 0$

Note. Following are true of AoC,

1. AoC doesn't hold for $\mathbb{Q}$
2. AoC will be basic to take limits.

Example. Consider $\phi \neq A \subseteq R$, and is bounded above. Let $c \in \mathbb{R}$. Define

$$A + c = \{a + c, a \in A\}$$

We show that $\sup(A) + c = \sup(A + c)$ ◇

Proof. Denote $s = \sup A$, so we have $s \geq a, \forall a \in A$.
1. To show $s + c$ is an upperbound. Above definition gives us, $s + c \geq a + c, \forall a \in A$. By definition we have $s + c$ is an upperbound of $A + c$.
2. To show $s + c$ is the smallest upperbound of $A + c$. Let b be an arbitrary upper bound of $A + c$. So $a + c \leq b, \forall a \in A$. Therefore $a \leq b - c, \forall a \in A$ where $b - c$ is an upperbound of A. But s is the least upper bound which means that $s \leq b - c$ or that $s + c \leq b$. So we showed that b must be greater than or equal to $s + c$. Hence, $s + c$ is the least upper bound. So $s + c = \sup(A + c)$ □

Lemma 1.3. Assume $s \in \mathbb{R}$ is an upperbound for a set $A \subseteq R$ and $A \neq \phi$. Then $s = \sup(A)$ if and only if $\forall \varepsilon, \exists a \in A, s.t\, a > s - \varepsilon$

Proof. (1) $\Rightarrow$ (2)
Assume $s = \sup(A)$, given $\varepsilon > 0$ we have $s - \varepsilon < a$. So $s - \varepsilon$ cannot be an upper bound of A. This means that $\exists a \in A$ such that $a > s - \varepsilon$.
(2) $\Rightarrow$ (1)
We have s such that $s - \varepsilon < a$ for some $a \in A$ and $\forall \varepsilon$. We need to show that s is the least upperbound. Let b be an arbitrary upperbound. Suppose $b < s$ so we have $\varepsilon = s - b > 0$ and $b = s - \varepsilon$ however we have some $a \in A$ such that $a > s - \varepsilon = b$ so $a > b$ which makes b not an upperbound and hence breaks our assumption. So $s \leq b$ □

1.3 Consequences of Completeness

Theorem 1.4 (Nested Interval property)**.** For any $n \in \mathbb{N}$, assume that we are given interval $I_n = [a_n, b_n] = \{x \in \mathbb{R} | a_n \leq x \leq b_n\}$ where $a_n \leq b_n$
Assume that $I_n \supseteq I_{n+1}, \forall n \in \mathbb{N}$ such that,

$$\ldots I_3 \subseteq I_2 \subseteq I_1$$

Then,

$$\bigcap_{n=1}^{\infty} I_n \neq \phi$$

Note. This means that for any $I_n, I_{n'}$ we have either $I_n \subseteq I_{n'}$ or $I_{n'} \subseteq I_n$

Proof. Take $A = \{a_n, n \in N\}$ we have $A \neq \phi$ and $A \subseteq \mathbb{R}$. A is bounded above as we have $a_1 \leq a_2 \ldots a_n \leq \ldots$ and $b_1 \geq b_2 \ldots\ b_n \geq \ldots$
So for every n, $a_n \leq b_n \leq b_1$. So b_1 is an upperbound for A. By AoC we have $\sup(A) = x \in \mathbb{R}$ exist.
Now we show that $x \in I_n, \forall n$.
Note that $\forall n, b_n$ is an upper bound for A.
$\forall m \in \mathbb{N}, a_m \in A$ and if,
$m \geq n$ then $a_m \leq b_m \leq b_n$
$m < n$ then $a_m \leq a_n < b_n$
As $\sup A = x$ then we have $x \leq b_n$ and as x is an upperbound of A we have, $a_n \leq x$ for all $n \in N$. So $x \in I_n$ hence proving the above statement. □

1.4 Density of $\mathbb{Q}$ in $\mathbb{R}$

Theorem 1.5 (Archimedean properties)**.** The following are true,

1. Given any $x \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $n > x$
2. Given any $y \in \mathbb{R}, y > 0$, $\exists n \in \mathbb{N}$ s..t $y > \frac{1}{n}$

Proof. (2) folows by (1) by setting $x = \frac{1}{y}$.
For (1) lets assume that there is no n for some $x \in \mathbb{R}$ that satisfies the condition. So $\exists x_0 \in \mathbb{R}$ such that $\forall n \in \mathbb{N}$, $n \leq x_0$. So N is bounded above by x_0. So by AoC let $\alpha = \sup N$. Now, $\alpha - 1$ is not an upperbound for $\mathbb{N}$. So $\exists n_0 \in \mathbb{N}$ such that $n > \alpha - 1$. So $\alpha < n_0 + 1 \in \mathbb{N}$. This is a contradiction, so (1) holds. □

Theorem 1.6. $\forall a, b \in R$ (a < b), $\exists r \in \mathbb{Q}$ such that $a < r < b$.

Proof. It suffices to find $m, n \in Z$ such that,

$$a < \frac{m}{n} < b$$

Step 1: find n.
Note that $b - a > 0$ and $b - a \in R$. By (2) we have n s.t. $b - a > \frac{1}{n}$. Now fix such an n.
Step 2. find m for the fixed n.
Without loss of generality take $na > 0$ and by (1), $m_0 \in \mathbb{N}$ s.t. $M_0 > na$. Then consider a finite set $\{0, 1, \ldots, M_0\}$. Now take k in this set and compare with na. Take m to be the smallest one such that $m > na$.
So we have $m > na \geq m - 1$
Step 3: Check if m, n work,
We have,

$$m > na \geq m - 1$$
$$\frac{m}{n} > a \text{ and } \frac{m}{n} \leq a + \frac{1}{n}$$

But we have $b - a > \frac{1}{n}$ so $b > a + \frac{1}{n}$ which gives us,

$$a < \frac{m}{n} \leq a + \frac{1}{n} < b$$

$\square$

Theorem 1.7. $\exists s \in R$ such that $s^2 = 2$

Proof. Let $A = \{x > 0, x \in \mathbb{R}, s.t. x^2 < 2\}$. Clearly $A \subseteq \mathbb{R}$ and is nonempty. We have A is bounded above as 2 is an upper bound.
By AoC $\sup A \in \mathbb{R}$ exists and set $s = \sup A$. Claim $s^2 = 2$.
Now we will prove this by contradiction by showing it cannot be the case that $s^2 < 2$ or $s^2 > 2$.
Now assume that $s^2 < 2$ and let $0 < \delta = 2 - s^2$. We will show that there is some $\varepsilon > 0$ such that $(s+\varepsilon)^2 < 2$ (i.e. s cannot be a supremum)

Scratchwork
We want to find ε to satisfy the $(s+\varepsilon)^2 < 2$, for this we work backwards.

$$\begin{aligned}(s+\varepsilon)^2 &< 2\\ s^2 + \varepsilon^2 + 2s\varepsilon &< 2\end{aligned}$$

We have $s < s^2 < 2$ (as s is definitely greater than 1). So $2s < 4$ to get,

$$s^2 + \varepsilon^2 + 2s\varepsilon < s^2 + \varepsilon^2 + 4\varepsilon < 2$$

Now let's assume that $\varepsilon < 1$ as if ≥ 1 works then trivially $\varepsilon < 1$ works as well. So we have $\varepsilon^2 < \varepsilon$ so,

$$\begin{aligned}s^2 + 5\varepsilon &< 2\\ 5\varepsilon &< 2 - s^2\\ \varepsilon &< \frac{\delta}{5}\end{aligned}$$

Now we can take $\varepsilon = \min\{\frac{\delta}{10}, 1\}$.

If we take $\varepsilon = \min\{1, \frac{\delta}{10}\}$ then we have,

$$\begin{aligned}(s+\varepsilon)^2 &= s^2 + \varepsilon^2 + 2s\varepsilon\\ (s+\varepsilon)^2 &\leq s^2 + \varepsilon^2 + 2s\varepsilon \leq s^2 + \delta < 2\end{aligned}$$

$\square$

Exercise. Show $s^2 > 2$ is impossible. We similarly show that we can find an ε such that $(s-\varepsilon)^2 > 2$

1.5 Cardinality

Definition. We say that two sets A and B have the same cardinality if there is a bijective function $f : A \to B$. We write $A \sim B$

Remark. Types,

1. We say A is finite if $A \sim \{1, 2, \ldots, n\}$ for some $n \in N$
2. We say A is countable (countably infinite) then $A \sim N$

3. An infinite set that is not countable is called unountable.

Example. $E = \{2, 4, \dots\}$, we show $E \sim N$.
Take $f : N \to E$ defined as $f(n) = 2n$. ◇

Example. $N \sim Z$
Take $f : N \to Z$ s.t. $f(n) = \frac{n-1}{2}$ if n is odd else $-\frac{n}{2}$. ◇

Example. $(-1, 1) \sim \mathbb{R}$
Take $f(x) = \frac{x}{x^2-1}$ ◇

Theorem 1.8. Following are true,

1. $\mathbb{Q}$ is countable
2. $\mathbb{R}$ is uncountable

Proof. For $\mathbb{Q}$ define $A_1 = \{0\}$ and

$$A_n = \{\pm\frac{p}{q} : p + q = n, p, q \in \mathbb{N} \text{ and } p, q \text{ coprime}\}$$

Note that A_n is finite and $\forall x \in \mathbb{Q}$ we can find a unique $n \in \mathbb{N}$ s.t. $x \in A_n$
Now map elements of $A_0, A_1, \dots$ iterative with $1, 2, 3, \dots$. So by construction, any element from A_n will be listed. So there is a bijection between $\mathbb{Q}$ and $\mathbb{N}$ since $\mathbb{Q} = \bigcup_{n-1}^{\infty} A_n$ and A_n's are disjoint, so $\mathbb{Q} \sim \mathbb{N}$
Now we show the reals are not countable. Assume that $\mathbb{R}$ is countable and suppose there is a bijective funciton $f : \mathbb{N} \to \mathbb{R}$. Let $x_1 = f(1), x_2 = f(2), \dots$
Then $\mathbb{R} = \{x_1, x_2, \dots, \}$. Let I be a closed interval $I_1 \subseteq \mathbb{R}$ s.t. $x_1 \notin I_1$ and similarly find $I_2 \subseteq I_1$ such that $x_2 \notin I_2$. Similarly define for all n such that $I_{n+1} \subseteq I_n$ closed interval such that $x_n \notin I_{n+1}$.
Since $\forall n_0 \in N$ we have $x_{n_0} \notin I_{n_0}$ so $xn_0 \notin \bigcap_{n=1}^{\infty} I_n$. But $R = \{x_1, x_2, \dots\}$ so we have $\bigcap_{n=1}^{\infty} I_n = \phi$. However, this is a contradiction with the nested interval property. □

Theorem 1.9. Following are true,

1. $A \subseteq B$ if B is countable them A is either countable or finite.
2. If A_n is countable $\forall n \in \mathbb{N}$ then $\bigcup_{n=1}^{\infty} A_n$ is countable.

1.6 Cantor's Theorem

Cantor's diagonal argument

Theorem 1.10. The open interval $(0, 1)$ is uncountable.

Proof. Argue by contradiction that $f : \mathbb{N} \to (0, 1)$ is a bijection. So $\forall m \in \mathbb{Z}$ we have,

$$f(m) = 0.a_{m1}a_{m2} \dots a_{mn} \dots$$

$\forall m, \in \mathbb{N}$ where a_{mn} is the nth digit of $f(m)$, and $a_{mm} \in \{0, 1, \dots, 9\}$

So we have,

$$\begin{array}{cccccc} 1 & f(1) & a_{11} & a_{12} & a_{13} & \dots \\ 2 & f(2) & a_{21} & a_{22} & a_{23} & \dots \\ 3 & f(3) & a_{31} & a_{32} & a_{33} & \dots \\ 4 & f(4) & a_{41} & a_{42} & a_{43} & \dots \\ \vdots & & & & & \end{array}$$

Set $r = 0.b_1 b_2 \dots b_n \dots$ where,

$$b_n = \begin{cases} 2 & a_{nn} \neq 2 \\ 3 & a_{nn} = 2 \end{cases}$$

We show that $r \neq f(m), \forall m \in \mathbb{N}$. Consider $f(1)$ we have, either, $a_{11} = 2$ or $a_{11} \neq 2$. In the first case we have $r_{11} = 3$ second case we have $r_{11} = 2$. So in both cases the first digit is different. Now for an arbitrary $f(m)$ we have the m'th digit is different which means that for any $f(m)$ it cannot be true that $f(m) = r$ as the $m'th$ digit is different.

Clearly $r \in (0,1)$ so there must be some m such that $f(m) = r$. Hence, a contradiction. So our assumption that $(0,1)$ is countable is wrong which must mean that $(0,1)$ is not countable.

□

Remark. We already showed that $(-1,1) \sim \mathbb{R}$, so it is enough to show from here that $(0,1) \sim (-1,1)$

Definition. Consider A is a set. The power set of A, $P(A)$ is the collection of all subsets of A.

Theorem 1.11 (Cantor's Theorem)**.** Given any non-empty set A, there does not exist a function f s.t.,

$$f : A \to P(A)$$

is onto.

Proof. If A is finite and has n elements, then $P(A)$ has 2^n elements. Easy to see you cannot have an onto mapping.
If A is infinite, let's assume that there is $f : A \to P(A)$ such that f is onto. As f is onto, $\forall B \subseteq A, B \in P(A)$ we can find a s.t.$f(a) = B$.

Define $B = \{a \in: s.t.\ a \notin f(a)\} \subseteq A$. So $B \in P(A)$. Since f is onto we can find a' such that $f(a') = B$. So we have either,

1. $a' \in B$: Then $a' \notin f(a')$ by definition. But $f(a') = B$ so $a \in f(a')$. A contradiction.
2. $a' \notin B$: If $a' \notin B$ then by definition of B we have $a' \in f(a')$ but $f(a') = B$ which means that $a' \in B$. A contradiction.

In both cases we have a contradiction, which means our assumption must be wrong and there must not exist an $f : A \to P(A)$ that is onto.

□

Remark. There is no onto map then the is no bijection, so $A \not\sim P(A)$ for any A.

Chapter 2

Sequences and Series

2.1 Sequences

Definition (Sequences)**.** A sequence is a function whose domain is $\mathbb{N}$ or $\{0\} \cup \mathbb{N}$.

Remark. Common notations are $\{a_n\}_{n=1}^{\infty}, (a_n), \{a_n\}$

Example. $\{\frac{n+1}{n}\}_{n-1}^{\infty}$ ◇

Definition. A sequence (a_n) converges to $a \in R$ if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n > N$, $|a_n - a| < \varepsilon$. We write,

$$\lim_{n\to\infty} a_n = a$$

Remark. The choice of N depends on ε

Example. $\{\frac{1}{n}\}_{n=1}^{\infty}$ then $\lim_{n\to\infty} \frac{1}{n} = 0$.
Let $a_n = \frac{1}{n}$ and $a = 0$ we need $\forall\varepsilon > 0, \exists N, s.t. \forall n > N$, $|\frac{1}{n} - 0| = |\frac{1}{n}| < \varepsilon$. So we need $n > \frac{1}{\varepsilon}$. So for any $\varepsilon > 0$ choose $N \in \mathbb{N}$ s.t. $N > \frac{1}{\varepsilon}$. Then $\forall n > N$ we have $|a_n - a| = |\frac{1}{n}| = \frac{1}{n} < \frac{1}{N} < \varepsilon$. So by definition we have,

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

◇

Notation (Epsilon neighborhood of a)**.** $V_\varepsilon(a) = \{x \in \mathbb{R}, |x - a| < \varepsilon\}$

Definition (Topological definition of convergence)**.** We say that a is the limit of a sequence $\{a_n\}$ if $\forall\varepsilon > 0$, $V_\varepsilon(a)$ contains all but finitely many element of $\{a_n\}$

Remark. This means that the epsilon neighborhood of the limit doesn't contain only finite element of the sequence. In this case those finite elements are the elements before N.

Definition. A sequence $\{a_n\}$ that does not converge is said to be divergent.

Theorem 2.1. The limit of a sequence when it exists, must be unique.

Proof. Assume it is not unique and that $\lim_{n\to\infty} a_n = b_1$ and $\lim_{n\to\infty} a_n = b_2$ and that $b_1 \neq b_2$. Now we have, Take $N = \max(N_1, N_2)$. So $\forall n > N$,

$$|a_n - b_1| < \varepsilon \text{ and } |a_n - b_2| < \varepsilon$$

If we have $\varepsilon = \frac{|b_1 - b_2|}{3}$. We have,

$$|b_1 - b_2| = |b_1 - a_n + a_n - b_2| \leq |b_1 - a_n| + |a_n - b_2| < 2 = 2\frac{|b_1 - b_2|}{3}$$

Which is a contradiction. So $b_1 = b_2$ □

Remark. To analyze the limit of a sequence,

1. Identify the limit (sometimes given)
2. $\forall \varepsilon > 0$
3. Find N which always depends on ε (in scratch paper)
4. Set N from (3)
5. Show N works

Example. Show $\lim_{n\to\infty} \frac{n+1}{n} = 1$ ⋄

Proof.

$$|\frac{n+1}{n} - 1| < \varepsilon$$
$$|\frac{1}{n}| < \varepsilon$$
$$N > \frac{1}{\varepsilon} \text{ will work}$$

$\forall \varepsilon > 0$ take $N \in \mathbb{N}$ such that $N > \frac{1}{\varepsilon}$. So $\forall n > N$ we have,

$$|\frac{n+1}{n} - 1| = |\frac{1}{n}| = \frac{1}{n} < \frac{1}{N} < \varepsilon$$

Which means that 1 is the limit. □

Example. To find $\lim_{n\to\infty} \frac{1+\sqrt{n}}{\sqrt{n}}$

We see that $\frac{1+\sqrt{n}}{\sqrt{n}} = \frac{1}{\sqrt{n}} + 1$ so the limit goes to 1 as $n \to \infty$.

We need $\forall \varepsilon > 0$ exists N s.t $n > N$ we have,

$$\left|\frac{1+\sqrt{n}}{\sqrt{n}} - 1\right| < \varepsilon$$
$$\left|\frac{1}{\sqrt{n}}\right| < \varepsilon$$
$$\frac{1}{\varepsilon} < \sqrt{n}$$
$$\frac{1}{\varepsilon^2} < n$$

If we take $N > \frac{1}{\varepsilon^2}$ then $\forall n > N$,

$$\left|\frac{1+\sqrt{n}}{\sqrt{n}} - 1\right| = \frac{1}{\sqrt{n}} < \frac{1}{\sqrt{N}} < \varepsilon$$

So the limit is 1. $\diamond$

Example.

$$\lim_{n\to\infty} \frac{2n+1}{5n+1} = \frac{2}{5}$$

$\diamond$

Proof. For $\forall\varepsilon > 0$,

Scratchwork.

$\forall\varepsilon > 0$ we want N s.t.,

$$\begin{aligned}\left|\frac{2n+1}{5n+1} - \frac{2}{5}\right| &< \varepsilon \\ \left|\frac{10n+5-(10n+2)}{5(5n+1)}\right| &< \varepsilon \\ \frac{3}{5}\frac{1}{5n+1} &< \varepsilon\end{aligned}$$

clearly suffices to require $\frac{1}{5n} < \varepsilon$ since,

$$\frac{3}{5}\frac{1}{5n+1} < \frac{1}{5n+1} < \frac{1}{5n}$$

So we have $\frac{1}{5n} < \varepsilon$ which means $n > \frac{1}{5\varepsilon}$

Take $N > \frac{1}{5\varepsilon}$ then $\forall n > N$ we have,

$$\begin{aligned}\left|\frac{2n+1}{5n+1} - \frac{2}{5}\right| = \frac{10n+5-(10n+2)}{5(5n+1)} &< \frac{1}{5n+1} \\ &< \frac{1}{5n} \\ &< \frac{1}{5N} \\ &< \varepsilon\end{aligned}$$

So,

$$\lim_{n\to\infty} \frac{2n+1}{5n+1} = \frac{2}{5}$$

$\square$

Definition. A sequence is said to be bounded if $\exists M$ s.t. $|a_n| \leq M, \forall n \in \mathbb{N}$. Can also be written as,

$$\sup |a_n| \leq M$$

Theorem 2.2. Every convergent sequence is bounded.

Proof. Let (a_n) be a convergent sequence then,

$$\lim_{n\to\infty} a_n = a$$

Take $\varepsilon = 1$ we can find a N such that $\forall n > N$ we have,

$$|a_n - a| < 1$$

Now for $n > N$ we have $|a_n| = |a_n - a + a| \leq |a_n - a| + |a| < |a| + 1$

Set $M = \max\{|a_1|, \ldots, |a_N|, |a| + 1\}$. Then $\forall n \in \mathbb{N}$, $|a_n| \leq M$ and hence the sequence is bounded by M

$\square$

Theorem 2.3. If $\lim_{n\to\infty} a_n = a, \lim_{n\to\infty} b_n = b$ then,

1. $\lim_{n\to\infty} ca_n = ca, \forall c \in \mathbb{R}$
2. $\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n = a + b$
3. $\lim_{n\to\infty}(a_n b_n) = (\lim_{n\to\infty} a_n)(\lim_{n\to\infty} b_n) = ab$
4. If $b \neq 0$ then, $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} = \frac{a}{b}$
5. $a_n \geq 0$ then $a \geq 0$
6. $a_n \geq c$ then $a \geq c$
7. $a_n \leq b_n$ then $b \geq a$

Remark. If $a_n > c, \forall n \in \mathbb{N}$ we know $a \geq c$
Example. If $c = 0, a_n = \frac{1}{n}$ although $a_n > c$ we can't say that $a > c$ as $a = 0 \geq c = 0$ $\diamond$

Proof. (1) Two cases, $c = 0$ or $c \neq 0$. If $c = 0$ then its trivial. Now if $c \neq 0$. Since $\lim_{n\to\infty} a_n = a$ we have $\forall \varepsilon > 0$ exists N_c such that $\forall n > N$,

$$|a_n - a| < \frac{\varepsilon}{|c|}$$
$$|c||a_n - a| < \varepsilon$$
$$|ca_n - ca| < \varepsilon$$

So $\lim_{n\to\infty} ca_n = ca$
(2) We have $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$ so there is some N_1, N_2 s.t. $\forall \varepsilon > 0$ if $n > \max\{N_1, N_2\}$ then,

$$|a_n - a| < \frac{\varepsilon}{2}$$
$$|b_n - b| < \frac{\varepsilon}{2}$$

Now we have $|(a_n + b_n) - (a + b)| = |(a_n - a) + (b_n - b)| \leq |a_n - a| + |b_n - b| \leq \varepsilon$
So by definition we now have,

$$\lim_{n\to\infty}(a_n + b_n) = a + b$$

(5) If $a_n \geq 0$ then $a \geq 0$.

Assume to the contrary that $a < 0$, now take $\varepsilon = \frac{|a|}{2}$ so according to our definition we have some N such that if $n > N$ then,

$$|a_n - a| < \varepsilon = \frac{|a|}{2} = -\frac{a}{2}$$

Now we have $-\varepsilon < a_n - a < \varepsilon$ or that $a_n < \varepsilon + a = a - \frac{a}{2} = \frac{a}{2} < 0$ as $a < 0$. But this is a contradiction as $a_n \geq 0$. Hence, we have $a \geq 0$

(6). $a_n \geq c$ then $a \geq c$. Set $x_n = a_n - c$ claim $\lim_{n\to\infty} x_n = a - c$ which is true from (2). Then by (5) we have $x_n \geq 0$ so $\lim_{n\to\infty} x_n = a - c > 0$ so $a > c$.

(7) Let $x_m = b_n - a_m$ then $\lim_{n\to\infty} x_n = b - a$ by (1), (2). Now we use (5) as $x_n \geq 0$ so $\lim_{n\to\infty} x_n = b - a \geq 0$ so $b \geq 0$. ☐

Example. $x_n \leq y_n \leq z_n$. Suppose $\lim_{n\to\infty} x_n = l$ and $\lim_{n\to\infty} z_n = l$ then, $\lim_{n\to\infty} y_n = l$ ◇

Proof. We need to first show that y_n is convergent. We have $x_n \leq y_n \leq z_n$ so, $x_n - l \leq y_n - l \leq z_n - l$. If $y_n - l \geq 0$. then $|y_n - l| \geq |z_n - l|$ else $|y_n - l| \leq |x_n - l|$. So in either case we have $|y_n - l| \leq \max\{|z_n - l|, |x_n - l|\}$

Since, $\lim_{n\to\infty} x_n = l$ and $\lim_{n\to\infty} z_n = l$. Then for $\forall\varepsilon$ we can find N_1, N_2 such that $\forall n > N_1, |x_n - l| < \varepsilon$ and $\forall n > N_2$, $|z_n - l| < \varepsilon$.

Now take $N = \max\{N_1, N_2\}$ then $\forall n > N$ we have $|y_n - l| \leq \max\{|x_n - l|, |z_n - l|\} < \varepsilon$. Which means that $\lim_{n\to\infty} y_n = y$ and y is convergent to l.

☐

2.2 The Monotone Convergence Theorem

Definition. A seq $\{a_n\}$ is increasing if $a_n \leq a_{n+1}, \forall n \in \mathbb{N}$ and is decreasing if $a_n \geq a_{n+1}, \forall n \in \mathbb{N}$. A sequence is *monotone* if it is either increase or decreasing.

Theorem 2.4 (M.C.T)**.** If a sequence is *monotone and bounded*, then it converges.

Proof. Let (a_n) be increasing (same proof for decreasing) and let $A = \{a_n, n \in \mathbb{N}\}$. Clearly $A \neq \phi$ and A is bounded. So, using axiom of completeness we have $s = \sup A$ exists. Now, we claim that,

$$\lim_{n\to\infty} a_n = s$$

For $\forall\varepsilon > 0$, we can find N such that,

$$s - \varepsilon < a_N \leq s$$

as $s - \varepsilon$ will not be an upper bound anymore. Now $\forall n \geq N$ since $\{a_n\}$ is increaseing i.e. $s - \varepsilon < a_N \leq a_n \leq s < s + \varepsilon$. So,

$$|a_n - s| < \varepsilon$$

Therefore $\lim_{n\to\infty} a_n = s$ ☐

2.3 Subsequences and Bolzano-Weierstrass Theorem

Definition (Subsequences)**.** Let $\{a_n\}$ be a sequence. Let

$$n_1 < n_2 < n_3 < \dots < n_k < n_{k+1} < \dots$$

be an increasing seq of natural numbers. Then,

$$\{a_{nk}\} = \{a_{n1}, a_{n2}, \dots\}$$

is called a subsequence of $\{a_n\}$

Example. $(a_n) = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots)$ so here,

1. $(\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \dots)$ is a subsequence.
2. $(\frac{1}{10}, \frac{1}{100}, \frac{1}{100}, \dots)$ is a subsequence.
3. $(\frac{1}{10}, \frac{1}{5}, \frac{1}{100}, \dots)$ is NOT a subsequence.
4. $(\frac{1}{3}, \frac{1}{3}, \frac{1}{5}, \frac{1}{5}, \dots)$ is NOT a subsequence.

$\diamond$

Theorem 2.5. Subsequence of a convergent seq converges to the same limit of the original seq.

Proof. Suppose $\{a_n\}$ converges to a so we have,

$$\lim_{n\to\infty} a_n = a$$

Now let $\{a_{nk}\}$ be a subsequence of (a_n). Now, $\forall \varepsilon > 0$ we have N such that for $n > N$,

$$|a_n - a| < \varepsilon$$

Now we see that $n_k \geq k$ so for any $k > N$ we have $n_k \geq k > N$ so ,

$$|a_{nk} - a| < \varepsilon$$

Therefore $\lim_{k\to\infty} a_{nk} = a$ $\square$

Example. Let $0 < b < 1$ and consider $\{b^n\}$ so we have,

$$1 > b > b^2 > b^3 > \dots$$

Note that $0 \leq b^n \leq 1$. So $\{b^n\}$ is decreasing and bounded which means it converges.

Now consider the subsequence $\{b^{2n}\}$ we know this converges to the same limit as the original sequence. Now we write $b^{2n} = b^n \cdot b^n$ so,

$$\lim_{n\to\infty} b^{2n} = \lim_{n\to\infty} b^n b^n = \lim_{n\to\infty} b^n \lim_{n\to\infty} b^n$$

So we have $l = l^2$ or $l = 0, 1$. But we can't have $l = 1$ as l is an upperbound and $1 > b > b^2 > \dots$. So we have $l = 0$. $\diamond$

Remark. This theorem also means that if any two subsequences converge to different values then it means that the main sequence diverges. We can show this by contradiction as if main were to converge the all subsequence converges to the same limit and hence they can't be different.

Example. Take $\{a_n\}$ where $a_n = (-1)^N$. We have $a_{2n} = 1 = (-1)^{2n} = 1$ and $a_{2n+1} = -1$. So,

$$\{a_{2n}\} = \lim_{n\to\infty} a_{2n} = 1$$
$$\{a_{2n+1}\} = \lim_{n\to\infty} a_{2n+1} = -1$$

We have two subsequence that converge to diff limits and hence means that $\{a_n\}$ is not convergent. $\diamond$

Example. Take $\{a_n\}$ where $a_n = \begin{cases} 1 & n \text{ is prime} \\ 0 & \text{otherwise} \end{cases}$. Here, $\{a_n\}$ diverges as the two subsequences converge to different values. $\diamond$

Example. Take $(1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, \frac{1}{5}, -\frac{1}{5}, -\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, -\frac{1}{5}, -\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, -\frac{1}{5}, -\frac{1}{5}, -\frac{1}{5}, -\frac{1}{5}, \dots)$. So here,

$$\left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \dots\right) \to \frac{1}{5}$$
$$\left(\frac{-1}{5}, \frac{-1}{5}, \frac{-1}{5}, \dots\right) \to \frac{-1}{5}$$

and hence $\{a_n\}$ doesn't converge. $\diamond$

Theorem 2.6 (Bolzano-Weierstrass). Every bounded sequence contains a convergent subsequence.

Proof. Let $\{a_n\}$ be a bounded sequence. So $|a_n| \leq M, \forall n \in \mathbb{N}$. Now, set $I_0 = [-M, M]$ where $|I_0| = 2M$. Then bisect I_0 into $[-M, 0], [0, M]$. At least one of them contains infinitely many elements of (a_n). Out of the two half intervals let I_1 be the one for which this is the case. Now a_{n_1} be some term in the sequence (a_n) satisfying $a_n \in I_1$.

Now bisect I_1 into two same-size subintervals and similar to above choose the half with infinitely many elements and denote I_2 and pick an $a_{n_2} \in I_2$ and $n_2 > n_1$. We repeat this process, i.e. suppose we find I_k and a_{nk} we bisect I_k to two halfs and choose the one containing infinitely many elements and denote I_{k+1} and choose $a_{n_{k+1}} \in I_{k+1}$.

We found $\{a_{nk}\}$ a subsequence of $\{a_n\}$ and $a_{nk} \in I_k$ such that,

$$I_1 \supseteq I_2 \supseteq I_3 \dots$$

As I_k is a closed interval, by N.I.P, $\bigcap_{j=0}^{\infty} I_j \neq \phi$. Now let x be in this intersection. Then we claim the following, $\{x\} = \bigcap_{j=1}^{\infty} = I_j$ and $\lim_{k\to\infty} a_{nk} = x$

Note that $\bigcap_{j=1}^{\infty} I_j \subseteq I_k, \forall k$ so $|\bigcap_{j=1}^{\infty} I_j| \leq |I_k$ and,

$$\begin{aligned} |I_k| = \frac{1}{2}|I_{k-1}| + \dots &= (\frac{1}{2})^{k-1}|I_1| \\ &= (\frac{1}{2})^k |I_0| \\ &= (\frac{1}{2})^k (2M) \end{aligned}$$

as $k \to \infty, |I_k| = 0$ so we have,

$$|\bigcap_{j=1}^{\infty} I_j| = 0$$

$\forall \varepsilon > 0$ we want N for all $k > N$ that,

$$|a_{nk} - x| < \varepsilon$$

We see for any k we have $a_{nk} \in I_k$ and $x \in I_k$. So,

$$|a_{nk} - x| \leq |I_k| = \frac{M}{2^{k-1}}$$

So we just need to choose k such that $\frac{M}{2^{k-1}} < \varepsilon$ so we take $k > \log_2(\frac{M}{\varepsilon}) + 1$. So now we can choose N such that $N \in \mathbb{N}$ and $\frac{M}{2^{N-1}} < \varepsilon$

Now $\forall \varepsilon > 0$ take N such that $N \in \mathbb{N}$ and,

$$N > \log_2(\frac{M}{\varepsilon}) + 1$$

or
Now for any $k > N$ since $a_{nk} \in I_k, x \in I_k$ we have,

$$\begin{aligned} |a_{nk} - x| &\leq |I_k| \\ &= \frac{M}{2^{k-1}} < \frac{M}{2^{N-1}} < \varepsilon \end{aligned}$$

So we have $\lim_{k\to\infty} a_{nk} = x$

□

2.4 Cauchy Criterion

Definition (Cauchy Criterion). A sequence (a_n) is Cauchy if $\forall \varepsilon > 0, \exists N$ s.t. $|a_n - a_m| < \varepsilon, \forall n, m > N$.

Theorem 2.7. A sequence (a_n) is convergent if and only if it is Cauchy.

Proof. ($\Rightarrow$) Let (a_n) be a convergent sequence that converges to a. Then $\forall \varepsilon > 0, \exists N$ s.t. $|a_n - a| < \frac{\varepsilon}{2}, \forall n > N$. Now for $n, m > N$ we have,

$$\begin{aligned} |a_n - a_m| &= |(a_n - a) - (a_m - a)| \\ &\leq |a_n - a| + |a_m - a| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \end{aligned}$$

Which completes our proof. □

Lemma 2.8. Every Cauchy sequence is bounded

Proof. Take $\varepsilon = 1$, then we have N such that $|x_m - x_n| < 1$ for all $m, n > N$ so we can write,

$$|x_m - x_{N+1}| < 1, \quad \forall m > N.$$

So $|x_m| = |x_m - x_{N+1} + X_{N+1}| < 1 + |X_{N+1}|$. Now just pick $\max\{|x_1|, \ldots, |x_{N-1}|, |x_N| + 1\}$. □

Now we can prove the other direction of the theorem.

Proof. ($\Leftarrow$) Let (a_n) be a Cauchy sequence. By the lemma above, we know that (a_n) is bounded. So by the Bolzano-Weierstrass theorem, tells us that it contains a convergent subsequence. So consider a subsequence $\{x_{nk}\}$ s.t. $\lim_{k\to\infty} x_{nk} = x$.

Now, $\forall \varepsilon > 0$, we can find N_1 such that $\forall k > N_1$, $|x_{nk} - x| < \frac{\varepsilon}{2}$. And we have N_2 s.t. $\forall m, n > N_2$ we have $|x_n - x_m| < \frac{\varepsilon}{2}$. Now take $N = \max\{N_1, N_2\}$. We have,

$$|x_n - x| = |x_n - x_{nk} + x_{nk} - x| \leq |x_n - x_{nk}| + |x_{nk} - x|$$

If we take $k > N$, note that $n_k \geq k > N$, so $|x_{nk} - x| < \frac{\varepsilon}{2}$ and $|x_n - x_{nk}| < \frac{\varepsilon}{2}$. It follows that $|x_n - x| \leq \varepsilon$

So

$$\lim_{n\to\infty} x_n = x$$

□

Example. *M.C.T.* $\Rightarrow$ *N.I.P*
Consider your nested intervals and have $I_n = [a_n, b_n]$ then $\{a_n\}$ is an increasing sequence and it is bounded above (b_1) is an upper bound. So using M.C.T it converges to some x. ◇

Example. NIP $\Rightarrow$ AoC

◇

2.5 Series

Definition. Let $\{b_n\}$ be a sequence. An infinite series is formally given by,

$$\sum_{n=1}^{\infty} b_n = b_1 + b_2 + \dots$$

Definition. The partial sum $\{s_m\}$ is defined as,

$$s_m = \sum_{n=1}^{m} b_n$$

Remark. We say $\sum_{n=1}^{\infty} b_n$ converges to b if,

$$\lim_{m\to\infty} s_m = B$$

Example. We have,

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

so,

$$s_m = \sum_{n=1}^{m} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \cdots + \frac{1}{m^2}$$

We see that $s_{m+1} > s_m > 0$. We can do,

$$\begin{aligned}
s_m &= \sum_{n=1}^{m} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \cdots + \frac{1}{m^2} \\
&< 1 + \frac{1}{2} + \frac{1}{3 \cdot 2} + \frac{1}{3 \cdot 4} \cdots \frac{1}{m(m-1)} \\
&= 1 + \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{m-1} - \frac{1}{m}\right) \\
&= 2 - \frac{1}{m} < 2
\end{aligned}$$

So we have $s_{m+1} > s_m$ and $0 < s_m < 2$ for any m. So we have a bounded increasing sequence and by $M.C.T$ we have $\lim_{m\to\infty} s_m = s \in \mathbb{R}$ exists and,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = s$$

◇

Example. We have $\sum_{n=1}^{\infty} \frac{1}{n}$ (Harmonic series). We have,

$$\begin{aligned}
s_m &= \sum_{n=1}^{m} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m} \\
s_2 &= 1 + \frac{1}{2} \\
s_4 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 2
\end{aligned}$$

So we get,

$$\begin{aligned}
s_{2^k} &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots + \left(\frac{1}{2^{k-1}+1} + \cdots + \frac{1}{2^k}\right) \\
&> 1 + \frac{1}{2} + \left(\frac{1}{4}2\right) + \left(\frac{1}{8} \cdot 4\right) + \left(2^{k-1} \cdot \frac{1}{2^k}\right) \\
&= 1 + \frac{1}{2} + \frac{1}{2} + \cdots = 1 + \frac{k}{2}
\end{aligned}$$

So $s_{2^k} > 1 + \frac{k}{2}$ so $\{s_m\}$ diverges as $1 + \frac{k}{2}$ diverges. ◇

Proposition 2.9. $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$

2.5.1 Properties of Infinite series

Theorem 2.10. Assume $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$. Then,

1. $\sum_{n=1}^{\infty} ca_n = cA, \forall c \in \mathbb{R}$
2. $\sum_{n=1}^{\infty} (a_n + b_n) = A + B$

Proof. Set $s_m = \sum_{n=1}^{m} a_n, k_m = \sum_{n=1}^{m} b_n$. Let

$$t_m = \sum_{n=1}^{m} ca_n = c \sum_{n=1}^{m} a_n = cs_m$$

. Now as $\lim_{n\to\infty} s_m = A$ we have $\lim_{m\to\infty} cs_m = cA$ so $\lim_{m\to\infty} t_m = \lim_{m\to\infty} cs_m = c\lim_{m\to\infty} s_m = cA$

For 2, define $U_m = \sum_{n=1}^{m}(a_n + b_n) = \sum_{n=1}^{m} a_m + \sum_{n=1}^{m} b_m$. Now we have, $\lim_{m\to\infty} = \lim_{m\to\infty}(s_m + k_m) = \lim_{m\to\infty} s_m + \lim_{m\to\infty} k_m = A + B$ □

Theorem 2.11 (Cauchy criterion for series). $\sum_{n=1}^{\infty} a_n$ converge if and only if given $\varepsilon > 0, \exists N, s.t. \forall n > m > N$ we have,

$$|a_{m+1} + \cdots + a_n| < \varepsilon$$

Proof. Define $s_n = \sum_{k=1}^{n} a_k$ and $s_m = \sum_{k=1}^{m} a_k$. So for $n > m$ we have,

$$s_n - s_m = a_{m+1} + \cdots + a_n$$

By the cauchy criterion applied to $\{s_n\}$, we know $\{s_n\}$ converges if and only if $\forall \varepsilon > 0, \exists N s.t. m, n > N$ and $|s_m - s_n| < \varepsilon$. This is equivalent to,

$$|a_{m+1} + \cdots + a_n| < \varepsilon$$

□

Corollary 2.12. If $\sum_{n=1}^{\infty} a_n$ converges then $\lim_{n\to\infty} a_n = 0$.

Proof. Take $m = n - 1$ from the previous statement so we have $\forall \varepsilon > 0$ exists N such that, n $>$ m $>$ N$s.t.$,$|s_n - s_m| = |a_n| < \varepsilon so \lim_{n\to\infty} a_n = 0$ □

Corollary 2.13. If $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{n=1}^{\infty} a_n$ diverges.

Remark. $\lim_{n\to\infty} a_n = 0$ does not imply that $\sum_{n=1}^{\infty} a_n$ converges - for instance $\frac{1}{n}$.

2.5.2 Comparison Test

Theorem 2.14. If (a_k) and (b_k) are sequence s.t. $0 \leq a_k \leq b_k, \forall n \in \mathbb{N}$. Then,

1. $\sum_{n=1}^{\infty} b_n$ converge means that $\sum_{n=1}^{\infty} a_n$ converges.
2. $\sum_{n=1}^{\infty} a_n$ diverges means that $\sum_{n=1}^{\infty} b_n$ diverges.

Proof. (1) Let $k_n = \sum_{n=1}^{m} b_n$ and $t_n = \sum_{n=1}^{m} a_n$. We know that $t_m \leq k_n$ for all m. So,

$$\lim_{n\to\infty} k_m \text{ exists}$$

So k_n is bounded and we have t_m is increasing so MCT says that $\{t_m\}$ converges so $\sum_{n=1}^{\infty} a_n$ converges.

We also see that for any m, n we have $a_{m+1} + \cdots + a_n < b_{m+1} + \cdots + b_n$ so we can say,

$$|a_{m+1} + \cdots + a_n| \leq |b_{m+1} + \cdots + b_n|$$

we apply cauchy criterion twice to show a_n converges. □

2.5.3 Absolute convergence

Theorem 2.15 (Absolute convergence test). If $\sum_{n=1}^{\infty} |a_n|$ converges then,

$$\sum_{n=1}^{\infty} a_n \text{ converges}$$

Proof. We have $\forall m < n$ that,

$$|a_{m+1} + \cdots + a_n| \leq |a_{m+1}| + \cdots + |a_n|$$

Now use Cauchy criterion again. $\square$

Theorem 2.16 (Cauchy condensation test). Suppose $\{b_n\}$ is a decreasing sequence and $b_n \geq 0$ then,

$$\sum_{n=1}^{\infty} b_n \text{ converges if and only if}$$

$$\sum_{n=0}^{\infty} 2^n b_{2^n} = b_1 + 2b_2 + 4b_4 + \dots \text{ converges}$$

Proposition 2.15. $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$

Proof. We can take $b_n = \frac{1}{n^p}$ and use the above theorem to get,

$$b_{2^n} = 2^{-np}$$
$$2^n b_{2^n} = 2^{-n(p-1)}$$

To check $\sum_{n=0}^{\infty} 2^n b_{2^n} = \sum_{n=0}^{\infty} 2^{-n(p-1)}$. We denote $J = 2^{(1-p)} = 2^{-(p-1)}$. So,

$$\sum_{n=0}^{\infty} 2^{-n(p-1)} = \sum_{n=0}^{\infty} J^n$$

If $p = 1$ then $J = 1$ clearly $\sum_{n=0}^{\infty} J^n$ diverges.

Now if $p \neq 1$ then $J \neq 1$. To check $\sum_{n=0}^{\infty} J^n$ we look at $\sum_{n=0}^{m} J^n$. Notice that,

$$\sum_{n=0}^{n} J^n = \frac{J^{m+1} - 1}{J - 1}$$

and as $m \to \infty$ we have,

$$\sum_{n=0}^{\infty} J^n \text{ converge if and only if } |J| < 1$$

As $J = 2^{(1-p)}$ we have $J < 1 \Leftrightarrow P > 1$ and hence satisfies the above convergence. $\square$

Chapter 3

Topology of $\mathbb{R}$

Definition. We have $a \in \mathbb{R}$ and $\varepsilon > 0$ then the ε neighborhood of a is defined as,

$$V_\varepsilon(a) = \{x \in \mathbb{R} \mid (x - a) < \varepsilon\}$$

Definition. A set $O \subseteq R$ is open if $\forall a \in O, \exists \varepsilon > 0$ s.t. $V_\varepsilon(a) \subseteq O$

Example. Some examples are,

1. $\mathbb{R}$ is open
2. $(c, d) = \{x \in \mathbb{R} < c < x < d\}$
   We can choose $e = \frac{1}{2}\min\{a - c, d - a\}$ then we easily have $V_\varepsilon(a) \subseteq (c, d)$

◇

Theorem 3.1. (i). Union of arbitrary collection of open sets is open
(ii). Intersection of a finite collection of open sets is open.

Proof. (i). Let $\{O_k : k \in N\}$ be a collection of arbitrary open sets where $O = \bigcup_k O_k$. Now consider some $o \in O$. As O is the union by definition it means there is some k for which $o \in O_k$. Now by assumption O_k is open so we have $V_\varepsilon(o) \subset O_k \subset O$. So We have $V_\varepsilon(o) \subset O$ for any $o \in O$ which makes O open.

(ii). Let $O_1, \ldots, O_n$ be finite open sets. Take $o \in O_1 \cap \cdots \cap O_n$ by definition we have $o \in O_1 \ldots o \in O_n$ each being open which means we have $V_{\varepsilon_1}(o) \subset O_1, \ldots, V_{\varepsilon_n}(o) \subset O_n$. Now choose $\min\{\varepsilon_1, \ldots, \varepsilon_n\}$ say ε_k to get $V_k(o)$ which is necessarily contained in all the other neighborhoods and hence $V_{\varepsilon_k}(o) \subset O_1, \ldots, O_n$ or $V_{\varepsilon_k}(o) \in O_1 \cap \cdots \cap O_n$. □

Definition (Limit Point). x is a limit point of A if $\forall \varepsilon > 0$ we have,

$$\{V_\varepsilon(x) \cap \ A\} \setminus \{x\} \neq \phi$$

Note. So the deleted neighborhood of x has to be a subset of A.

Note. This doesn't say whether x is in A or not.

Theorem 3.2. A point x is a limit point of A if and only if $x = \lim a_n$ for some sequence (a_n) contained in A such that $a_n \neq x$ for all $n \in \mathbb{N}$

Proof. ($\Rightarrow$) Assume x is a limit point. By definition the deleted neighborhood for any ε is in A. Now choose a sequence of ε such that we have $\varepsilon = \frac{1}{n}$. And as we have $(V_\varepsilon \cap A)\{x\} \subset A$ we can choose some a_n from this. Hence we now have a sequence (a_n). Now we see that for an arbitrary ε we can choose N such that $N > \frac{1}{\varepsilon}$ which will make it so that we have for any a_n, $|x - a_n| < \varepsilon$ which means x is the limit of the sequence.

($\Leftarrow$) We have (a_n) a sequence which converges to x and we need to show that x is a limit point of A. By definition of convergence for any ε we can find an N such that for $n > N$ the $\varepsilon-$neighborhood of x contains some a_n. This gives us $a_N \in V_\varepsilon(x)$ and we have $a_n \neq x$ and $a_n \in A$ so the intersection of the neighborhood with A aside from x is non-empty making it a limit point. $\square$

Definition. A point $a \in A$ is an isolated point if it is not a limit point.

Note. Isolated point is necessarily in A but limit point need not be (ends of an open interval)

Example. For $[1, 2] \cup \{5\}$ we have 5 as an isolated point and every other as a limit point. $\diamond$

Definition. A set $F \subseteq R$ is closed if it contains its limit points.

Theorem 3.3. A set $F \subseteq \mathbb{R}$ is closed if and only if every Cauchy sequence contained in F has a limit that is also an element of F.

Proof. ($\Rightarrow$) Assume F is closed, so it contains it's limit points. Now consider a Cauchy sequence of F, (a_n). As it's Cauchy it converges say $\lim a_n = x$. Now x is the limit of a sequence contained in F making it a limit point. F is closed and contains it's limit points so $x \in F$.

($\Leftarrow$) Consider any Cauchy sequence (a_n) in F such that the limit is in F. Now assume F is not closed so it doesn't contain it's limit points which means there is some sequence (a_n) contained in F for which we have $\lim a_n = x \notin F$. But (a_n) converges so it's a Cauchy sequence and by assumption it's limit must be in F so $x \in F$. A contradiction. $\square$

Closure

Definition. Given $A \subseteq R$, if L is the set of all limit points of A. Then the closure is defined to be $\overline{A} = A \cup L$

Theorem 3.4. For $A \subseteq R$, $\overline{A}$ is closed and is the smallest closed set containing A

Proof. First we show it's the smallest closed set containing A. Let B be a closed set such that $A \subseteq B$. Now as B is closed and $A \subseteq B$, B contains all the limit points of A so we have $L \subseteq B$ so we get $A \cup L \subseteq B$ or $\overline{A} \subseteq B$. Hence, $\overline{A}$ is contained in any closed set containing A and hence is the smallest.

If L is the set of limit points of A then by construction we have $\overline{A} = A \cup L$ so A contains all it's limit points. Now in addition we need to show that no new limit points are introduced by taking the union of A and L. We show two things,

(1) L is closed
(2) If x is the limit point of $A \cup L$ then it is of A as well.

(1) Consider any $x_n \in L$. We have for some sequence $\{a_{mn}\}$ in A that $\lim a_{mn} = x_n$. We need to show that L contains it's limit points. So consider a sequence in $L, (x_n)$ such that $\lim x_n = x$. So for $n > N$ and $m > M$ we have that,

$$|a_{mn} - x| < |a_{mn} - x_n| + |x_n - x| < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

Which makes x the limit of a sequence from A and hence a limit point of A and hence in L. So L contains all of it's limit points.

(2) Consider $x_n \in A \cup L$ such that $\lim x_n = x$. Now it's possible to find a sub sequence of x_n that is either all in A or all in L the limit of whom is still x. If that sub sequence belongs to only A then x is a limit points of A and is in L therefore $x \in A \cup L$. If it only belongs to L as L is closed it contains it's limit points and hence $x \in L$ so $x \in A \cup L$.

Hence, we complete the proof. □

Complement

$$A \subseteq R, A^c = \{x \in \mathbb{R} \mid x \notin A\}$$

Theorem 3.5. A set O is open if and only if O^c is closed and F is closed if and only if F^c is open.

Proof. ($\Rightarrow$) Assume O is open. Now consider O^c and take x a limit point of it. So for every $\varepsilon > 0$ the deleted neighborhood is in O^c so it cannot be in O but that must mean that x cannot be in O as O is open so x must have a $\varepsilon-$neighborhood. Hence $x \in O^c$ and O^c is closed.

($\Leftarrow$) □

Theorem 3.6. The intersection of an arbitrary collection of closed sets is closed.

Theorem 3.7. The union of a finite collection of closed sets is closed.

Definition (Compactness). A set $K \subseteq R$ is compact if every sequence in K has a subsequence that converges to a limit in K.

Definition. A set $A \subseteq R$ is bounded if $\exists M > 0$ such that $|x| \leq M, \forall x \in A$.

Theorem 3.8. A set $K \subseteq R$ is compact if and only if it is closed and bounded.

Proof. □

Theorem 3.9.

3.1 Open Covers

Definition. Let $A \subseteq \mathbb{R} \ldots$

Theorem 3.10 (Heine-Borel Theorem). Let K be a subset of $\mathbb{R}$. Then the following are equivalent,

1. K is compact
2. K is closed and bounded
3. Every open cover of K has a finite subcover.

3.2 Perfect sets and Connected sets

Definition. $P \subseteq \mathbb{R}$ is perfect if it is closed and contains no isolated points.

Remark. So all points in P are it's limit points.

Example. Closed interval $[a, b]$ where $a \neq b$ has no isolated points. ◇

Example. For $a < b < c$ we have $[a, b] \cup \{c\}$ is closed but not perfect. ◇

Definition. Two sets $A, b \subseteq \mathbb{R}$ are separated if $\overline{A} \cup B$ and $A \cup \overline{B}$ are empty. $E \subseteq \mathbb{R}$ is disconnected if it can be written as $A \cup B$ where A, B are nonempty separated sets.

Note. Here $\overline{A}$ is the closure of the set not the complement.

Chapter 4

Sequences and Series of Functions

4.1 Uniform Convergence of a Sequence of Functions

Pointwise Convergence

Definition. For each $n \in \mathbb{N}$ let f_n be defined on $A \subseteq R$, then the sequence (f_n) of functions converges pointwise on A to f if, forall $x \in$, the sequennce of real numbers $f_n(x)$ converges to $f(x)$.

Alternate definition is

Let (f_n) be a sequence of functions defined on $A \subseteq R$. Then, (f_n) converges pointwise on A to limit function f defined on A if $\forall \varepsilon > 0$ for $x \in A$ there exists $N \in \mathbb{N}$ such that we have $|f_n(x) - f(x)| < \varepsilon$ whenever $n \geq N$.

Note. When we consider pointwise convergence, as we can have different choices of N for different x it's hard to have the limit function behave properly to get continuity. TO fix this we introduce Uniform Convergence.

Uniform Convergence

Definition. Let (f_n) be a sequence of functions defined on $A \subseteq R$. Then, (f_n) converges uniformly on A to limit function f defined on A if $\forall \varepsilon > 0, \exists N$ such that we have $|f_n(x) - f(x)| < \varepsilon$ whenever $n \geq N$ for all $x \in A$.

Cauchy Criterion

Theorem 4.1. A sequence of function (f_n) on $A \subseteq \mathbb{R}$ converges uniformly on A if and only if for every $\varepsilon > 0$ we have $N \in \mathbb{N}$ such that $|f_n(x) - f_m(x)| < \varepsilon$ whenever $m, n > N$ and $x \in A$

Proof. ($\Rightarrow$) Assume that we have uniform convergence, so we have a fixed N such that for any $n > N$ we have,

$$|f_n(x) - f(x)| < \frac{\varepsilon}{2}$$

Now note we also have,

$$
\begin{aligned}
|f_m(x) - f_n(x)| &= |f_m(x) - f(x) + f(x) - f_n(x)| \\
&\leq |f_m(x) - f(x)| + |f(x) - f_n(x)| \\
&\leq |f_m(x) - f(x)| + |f(x) - f_n(x)|
\end{aligned}
$$

Now if we take $m, n > N$ we get,

$$|f_m(x) - f_n(x)| = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

($\Leftarrow$) Here we have a given N such that for any $m, n > N$ we get,

$$|f_m(x) - f_n(x)| < \varepsilon$$

Now note that we can write for some n, m that,

$$
\begin{aligned}
|f_n(x) - f(x)| &= |f_n(x) - f_m(x) + f_m(x) - f(x)| \\
&\leq |f_n(x) - f_m(x)| + |f_m(x) - f(x)| \\
&\leq \frac{\varepsilon}{2} + |f_m(x) - f(x)|
\end{aligned}
$$

Now for each x note that because of the cauchy criterion we have pointwise convergence which means we can choose m large enough (note that this doesn't change the value of our choice of N, i.e N is still independent of x) such that we have $|f_m(x) - f(x)| < \varepsilon/2$ hence we get,

$$|f_n(x) - f(x)| < \varepsilon$$

and uniform convergence. □

Continuity

Theorem 4.2. Continuous Limit Theorem Let (f_n) be a sequence of functions on $A \subseteq R$ that converges uniformly on A to f. If each f_n is continuous on A, then f is continuous at c.

Proof. We need to show that f is continuous at c or that $\forall \varepsilon > 0$ we have δ such that if $|x - c| < \delta$ then we have $|f(x) - f(c)| < \varepsilon$. Now note the following,

$$
\begin{aligned}
|f(x) - f(c)| &= |f(x) - f_n(x) + f_n(x) - f_n(c) + f_n(c) - f(c)| \\
&\leq |f(x) - f_n(x)| + |f_n(x) - f_n(c)| + |f_n(c) - f(c)|
\end{aligned}
$$

Now from uniform convergence we get for all ε an N such that if $n > N$

$$|f_n(x) - f(x)| < \frac{\varepsilon}{3} \text{ and } |f_n(c) - f(c)| < \frac{\varepsilon}{3}$$

And as we have each f_n is continuous we get δ such that if $|x - c| < \delta$ then $|f_n(x) - f_n(c)| < \varepsilon$. Putting all together we get,

$$|f(x) - f(c)| < \varepsilon$$

□

4.2 Series of Functions

Definition. If f_n and f are functions defined on A then,

$$\sum f_n(x) = f_1(x) + f_2(x) + \dots$$

converges pointwise on A to $f(x)$ if the sequence of partial sums $s_k(x)$ defined as,

$$s_k(x) = f_1(x) + \dots + f_k(x)$$

converges point wise to $f(x)$. Similarly the series converges uniformly if $s_k(x)$ converges uniformly to $f(x)$.

Note. Note that if each f_n are continuous then we have s_k is continuous as well and if we have uniform convergence then f is as well.

Theorem 4.3. Let f_n be continuous on $A \subseteq R$ and assume $\sum f_n$ converges uniformly to f. Then f is continuous on A.

Proof. We have uniform convergence to f, which means we have that the sequence of partial sums s_k uniformly converges to f. Now note we have,

$$s_k = f_1 + \dots + f_k$$

But each f_i is conscious and by algebraic continuity theorem we have s_k the sum of conscious functions is continuous as well. Now we have a sequence of continuous functions s_k uniformly converges to f by theorem in the previous section we have f is continuous. $\square$

Theorem 4.4 (Cauchy Criterion for Uniform Convergence of Series)**.** A series $\sum f_n$ converges uniformly on A if and only if $\forall \varepsilon > 0$ there is $N \in \mathbb{N}$ such that,

$$|f_{m+1}(x) + \dots + f_n(x)| < \varepsilon$$

whenever $n > m \geq N$ and $x \in A$

Proof. ($\Rightarrow$) Assume we have uniform convergence. This means that we get $(s_k) \to f$ uniformly. Now using the Cauchy criterion for sequence of functions this means that we have N such that for $n > m > N$,

$$|s_n(x) - s_m(x)| < \varepsilon$$

But note $s_k(x) = \sum_{n=1}^{k} f_n(x)$ so we have, $s_n(x) - s_m(x) = |f_{m+1}(x) + \dots + f_n(x)|$ so,

$$|f_{m+1}(x) + \dots + f_n(x)| < \varepsilon$$

as desired.

($\Leftarrow$) Now assume we have for a given N that,

$$|f_{m+1}(x) + \dots + f_n(x)| < \varepsilon$$

for any choice of $n > m > N$. Similarly we rewrite this as $|s_m - s_n| < \varepsilon$ and note that we showed in the previous section that this implies uniform convergence. $\square$

Corollary 4.5 (Weierstrass M-Test)**.** For each $n \in \mathbb{N}$ let f_n be a function defined on A, and let $M_n > 0$ such that,

$$|f_n(x)| \leq M_n$$

for all $x \in$. If $\sum M_n$ converges then $\sum f_n$ converges uniformly on A

Proof. Note we have,

$$\begin{aligned} |s_k(x)| &= |f_1(x) + \cdots + f_k(x)| \\ &\leq |f_1(x)| + \cdots + |f_k(x)| \\ &\leq M_1 + \cdots + M_n \end{aligned}$$

Now this gives us $|s_n - s_m| \leq M_{m+1} + \cdots + M_n$
But by Cauchy criterion for series we have N such that for $n, m > N$ we have $\sum_{m+1}^{n} M_i < \varepsilon$ hence we get $|s_n - s_m| < \varepsilon$ and by above theorem we have f_n uniformly converges. □

4.3 Power Series

We have functions of the form,

$$f(x) = \sum_{n=0}^{\infty} = a_0 + a_1 x + \ldots$$

Theorem 4.6. If $\sum_{n=0}^{\infty} a_n x^n$ converges at some point $x_0 \in \mathbb{R}$, then it converges absolutely for any x satisfying $|x| < |x_0|$.

Proof. First note that if we have $\sum_{n=0}^{\infty} a_n x_0^n$ then that means we have $|a_n x_0^n|$ is bounded and infact goes to zero. So we can write $|a_n x_0^n| \leq M$ for some $M > 0$ for all $n \in \mathbb{N}$. Now take $x \in \mathbb{R}$ such that we have $|x| \leq |x_0|$ we have the following,

$$\begin{aligned} |a_n x^n| &\leq |a_n x_0^n| \left| \frac{x^n}{x_0^n} \right| \\ &\leq M \left| \frac{x^n}{x_0^n} \right| \end{aligned}$$

But note we have $|x| < |x_0|$ which implies that $\left| \frac{x}{x_0} \right| < 1$. So now if we consider the sum $\sum_{n=0}^{\infty} |a_n x^n|$ note for each term in the sequence we have that it's smaller than $M \left| \frac{x^n}{x_0^n} \right|$. But now we have $\sum_{n=0}^{\infty} M \left| \frac{x^n}{x_0^n} \right|$ is a convergent sequence as M is a constant and $\frac{x}{x_0}$ is smaller than 1 so the geometric series converges and hence by comparison test we have that $\sum_{n=0}^{\infty} |x_n x^n|$ converges as well for $|x| < |x_0|$ □

Note. This means that a given power series must either converge for just 0 or $\mathbb{R}$ or some bounded interval around 0 (we don't know if it's open or closed or etc).

Uniform Convergence

Theorem 4.7. If $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely at some point x_0, then it converges uniformly on $[-c, c]$ where $c = |x_0|$

Proof. Weierstrass M-Test tells us that if we have $|f_n(x)| \leq M_n$ for all $x \in A$ and if $\sum M_n$ converges then $\sum f_n$ converges uniformly on A. First note we have $|x| \leq |x_0|$ which gives us,

$$|a_n x^n| \leq |a_n x_0^n| = M_n$$

But if we have absolute convergence at x_0 this means we have $\sum M_n$ converges. So by Weierstrass M-Test we have $\sum a_n x^n$ converges in $[-c, c]$ where $c = |x_0|$. □

Remark. Using Theorem 4.1 and 4.2 we basically get that if $\sum_{n=0}^{\infty} ax^n$ converges at some point $x_0 \in \mathbb{R}$ then for any $0 < r < |x_0|$ we have uniform convergence on $[-r, r]$. But note does not mean we have uniform convergence on $[0, x_0)$.

Abels Theorem

Lemma 4.8. If we have $b_1 \geq b_2 \geq \cdots \geq 0$ and $\sum_{n=1}^{\infty} a_n$ is a series with bounded partial i.e. $|a_1 + \cdots + a_n| \leq A$ for all n then we have,

$$|a_1 b_1 + \cdots + a_n b_n n| \leq A b_1$$

Proof. Let $s_n = a_1 + \cdots + a_n$. Now we can rewrite this as,

$$\begin{aligned}
\left|\sum_{k=1}^{n} a_k b_k\right| &= \left|s_n bn + 1 + \sum_{k=1}^{n} s_k (b_k - b_{k+1})\right| \\
&\leq A b_{n+1} + \sum_{k=1}^{n} A(b_k - b_{k+1}) \\
&= A b_{n+1} + (A b_1 - A b_{n+1}) = A b_1
\end{aligned}$$

□

Note. Note that in the above lemma if we had $\sum_{n=1}^{\infty} |a_n|$ and A is the upper bound of the partial sums of the absolutes, then this is a lot more trivial as we can simply use triangle inequality.

Theorem 4.9 (Abel's Theorem)**.** Let $g(x) = \sum_{n=0}^{\infty} a_n x^n$ be a power series that converges to $x = R > 0$. Then the series uniformly converges on $[0, R]$.

Remark. We can say the same if $x = -R$

Remark. The main difference between Abel's Theorem and the conclusion from theorem 4.1 and 4.2 is that we can establish uniform convergence in the entire set $[0, R]$ not just a subinterval $[0, r]$ where $r < R$ if we have convergence at $x = R > 0$.

Proof. First we can write,

$$g(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} (a_n R^n) \left(\frac{x}{R}\right)^n$$

Now by Cauchy criterion for uniform convergence it is enough to show that for any $\varepsilon > 0$ we have an N such that if $n > m \geq N$ then we have,

$$\left|(a_{m+1} R^{m+1}) \left(\frac{x}{R}\right)^{m+1} + \cdots + (a_n R^n) \left(\frac{x}{R}\right)^n\right| < \varepsilon$$

Now clearly at $x = R$ we have convergence so we can use Cauchy criterion on that to get N

and n, m such that,

$$\left|a_{m+1}R^{m+1} + \cdots + a_n R^n\right| < \frac{\varepsilon}{2}$$

Now we use Abels lemma on the equation above this one to get,

$$\left|(a_{m+1}R^{m+1})\left(\frac{x}{R}\right)^{m+1} + \cdots + (a_n R^n)\left(\frac{x}{R}\right)^n\right| \leq \left|a_{m+1}R^{m+1} + \cdots + a_n R^n\right|\left(\frac{x}{R}\right)^{m+1}$$
$$\frac{\varepsilon}{2}\left(\frac{x}{R}\right)^{m+1}$$

But we have $\left(\frac{x}{R}\right) < 1$ so we easily get $\frac{\varepsilon}{2}\left(\frac{x}{R}\right)^{m+1} < \varepsilon$ hence completing the proof. $\square$

Theorem 4.10. If a power series converges on a set $A \subseteq R$ then it converges uniformyl on any compact set $K \subseteq A$

Proof. If K is compact then it contains it's maximum and minim say x_0 and x_1 which by assumption is also in A. Now if the series converges on A then it converges for both points x_1 and x_0. Now if x_1, x_0 are positive then by Abel's theorem we have convergence for $[0, x_0]$ and $[0, x_1]$ and hence $[x_0, x_1]$. If x_1, x_0 are negative then similarly we have uniform convergence on $[x_0, 0]$ and $[x_1, 0]$ and hence on $[x_0, x_1]$ as well. Lastly, assume x_0 is negative then we have convergence for $[x_0, 0]$ and for $[0, x_1]$ and hence also in the union $[x_0, x_1]$ by picking the larger N. $\square$