Intro to Proofs: HW05

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9.4

Proof. If the polynomial is prime $\forall n \in N$ then $n^2 + 17n + 17$ has only 1 and itself as its factors.
However let us take $n = 17$ we have, $17^2+17^2+17 = 17(17+17+1) = 17\times35$
We see that the polynomial has factors 17 and 35 in this case (also note that 35 can be furhter factored). So this shows us that $n^2+17n+17$ is not prime $\forall n \in \mathbb{N}$
So the statement is false. $\square$

9.9

Proof. Let $A = \{1, 2\}$ and $B = \{2\}$. With this we have $A - B = \{1\}$.

$$P(A) = \{\{1\}, \{2\}, \{1, 2\}, \phi\}$$

$$P(B) = \{\{2\}, \phi\}$$

$$P(A) - P(B) = \{\{1\}, \{1, 2\}\}$$

$$P(A - B) = \{\{1\}, \phi\}$$

It is easy to see that $P(A) - P(B) \not\subseteq P(A - B)$
So the statement is false. $\square$

9.23

Proof. We have $x^3 < y^3$. We can write this as,

$$x^3 - y^3 < 0$$

$$(x - y)(x^2 + xy + y^2) < 0$$

We know that $x^2 + xy + y^2$ is always positive as we can write it as, $(x + \frac{y}{2})^2 + \frac{3y^2}{4}$ so we can divide both sides by it.

$$(x - y) < 0 \Rightarrow x < y$$

$\square$

9.34

Proof. We use a counter example to disprove this statement.
Consider $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$ where $A \cup B = \{1, 2, 3, 4, 5\}$
We can take $X = \{2, 3, 4\}$. We see that $X \subseteq A \cup B$. However it is not true that either $X \subseteq A$ as $4 \notin A$ and not true that $X \subseteq B$ as $2 \notin B$

$\square$

10.2

Proof. First let us consider $n = 1$, we have,

$$1^2 = \frac{1(1+1)(2+1)}{6} = \frac{6}{6} = 1$$

Now let us assume the statement is true for an arbitrary $n = k$, we have,

$$1^2 + 2^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}$$

We need to show just using this result that the statement holds true for $n = k+1$ or that,

$$1^2 + \cdots + k^2 + (k+1)^2 = \frac{(k+1)(k+2)(2(k+1)+1)}{6} = \frac{(k+1)(2k^2+7k+6)}{6}$$

We can replace the first k terms in the left with the formula above and we get,

$$\frac{k(k+1)(2k+1)}{6} + (k+1)^2 = \frac{k(k+1)(2k+1) + 6(k+1)^2}{6}$$

$$\frac{(k+1)(k(2k+1) + 6(k+1))}{6} = \frac{(k+1)(2k^2+7k+6)}{6}$$

We see that this is the same formula for the sum of $k+1$ terms that we had to show.
Hence we showed that if the staement holds true for $n = k$ then it will also hold true for $n = k+1$ and by inductino we can say that the statemetn is true for every positive integer n.

$\square$

10.8

Proof. First let us consider the case when $n = 1$. We get,

$$\frac{1}{2!} = 1 - \frac{1}{(n+1)!}$$

The left hand side evaluates to $\frac{1}{2!} = \frac{1}{2}$. And the right hand side evaluates to,

$$1 - \frac{1}{2!} = 1 - \frac{1}{2} = \frac{1}{2}$$

So we have $\frac{1}{2!} = 1 - \frac{1}{(1+1)!} = \frac{1}{2}$ so the equality holds for $n = 1$.
Now let us assume the equality holds for an arbitreary $n = k$, we get,

$$\frac{1}{2!} + \cdots + \frac{k}{(k+1)!} = 1 - \frac{1}{(k+1)!}$$

We need to show that the following equality is true given this,

$$\frac{1}{2!} + \cdots + \frac{k}{(k+1)!} + \frac{k+1}{(k+2)!} = 1 - \frac{1}{(k+2)!}$$

Using the righthand side from the equality we assumed for $n = k$ we have the lefthand side of what we want to prove as,

$$1 - \frac{1}{(k+1)!} + \frac{k+1}{(k+2)!}$$

$$= 1 + \frac{k+1}{(k+2)!} - \frac{k+2}{(k+1)!} = 1 - \frac{1}{(k+2)!}$$

Which is the righthand side of what we want to prove.
Hence by inductino we showed that the statement holds for all $n \in N$

□

10.14

Proof. We need to show $5|2^n a \Rightarrow 5|a, \forall n \in N$ If $n = 1$ we see that

$$5|2^n a = 5|a \text{ as } 2^n = 1$$

For an arbitrary $n = k$ we have,

$$5|2^k a \Rightarrow 5|a$$

We have to show that given this, the statemetn holds for $n = k + 1$ or,

$$5|2^{k+1} a \Rightarrow 5|a$$

If the statement holds for k then we know that $2^k a = 5m_0 \Rightarrow a = 5n_0$ for some $m \in Z$
Taking the lefthand side of what we have to prove we see,

$$2^{k+1} a = 5m_1$$

First we start with $2^k a = 5m_0$. Multiplifying both sides by 2 we get,

$$2^{k+1} a = 10m_0 = 5(2m_0) = 5m_1$$

However notice that we have the same a that we already assumed is divisbile in the case where $n = k$. Hence we show that even in the case where $n = k + 1$ a is still divisbile by 5.

□

10.17

Proof. Let us first check for $n = 2$. We have,

$$\overline{A_1 \cap A_2} = \overline{A_1} \cup \overline{A_2}$$

which is true by using demoragans law.
Or in other words if $x \notin A_1 \cap A_2$ means that $x \notin A_1$ or $x \notin A_2$ which means that $x \in \overline{A_1} \cup \overline{A_2}$
Let us now assume it is true for an arbitrary $n = k$ we have,

$$\overline{A_1 \cap \cdots \cap A_k} = \overline{A_1} \cup \cdots \cup \overline{A_k}$$

we need to show that the following follows,

$$\overline{A_1 \cap \cdots \cap A_k \cap A_{k+1}} = \overline{A_1} \cup \cdots \cup \overline{A_k} \cup \overline{A_{k+1}}$$

Consider the intersectino of $A_1 \cap \cdots \cap A_k = A_0$. Now we have because of demorgans law (we also just showed it above for the case of n = 2),

$$= \overline{A_0 \cap A_{k_1}} = \overline{A_0} \cup \overline{A_{k+1}}$$

$$= \overline{A_1 \cap \cdots \cap A_k} \cup \overline{A_{k+1}}$$

Now from our assumption we can write this as,

$$= \overline{A_1} \cup \cdots \cup \overline{A_k} \cup \overline{A_{k+1}}$$

Which is the right hand side for the case of $n = k + 1$.
Hence by induction we have shown that the satement is true for all $n \geq 2$ □

10.21

Let us take the case for $n = 1$. We have,

$$\frac{1}{1} \leq 2 - \frac{1}{1} = 2$$

Which is obviously true.
Now let us assume the case for $n = k$. We get,

$$\frac{1}{1} + \cdots + \frac{1}{k^2} \leq 2 - \frac{1}{k}$$

We need to show that the case for $n = k + 1$ follows from this, or,

$$\frac{1}{1} + \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2} \leq 2 - \frac{1}{k+1}$$

Let us take our inequality for $n = k$ and add the term $\frac{1}{(k+1)^2}$ on both sides, we get,

$$\frac{1}{1} + \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2} \leq 2 - \frac{1}{k} + \frac{1}{(k+1)^2}$$

Now let us rearrange the terms in the right in this inequality, we have,

$$2-\left(\frac{1}{k}-\frac{1}{(k+1)^2}\right)=2-\left(\frac{k^2+k+1}{k(k+1)^2}\right)$$

It is obvious that,

$$2-\left(\frac{k^2+k+1}{k(k+1)^2}\right)=2-\left(\frac{k^2+k}{k(k+1)^2}\right)-\frac{1}{k(k+1)^2}\leq 2-\left(\frac{k^2+k}{k(k+1)^2}\right)=2-\frac{1}{(k+1)}$$

So we have shown from the case $n=k$ that,

$$\frac{1}{1}+\cdots+\frac{1}{(k+1)^2}\leq 2-\frac{1}{(k+1)}$$

which is the case for $n=k+1$. Hence by induction we can say that our inequality is true for al $n\in N$

10.30

Proof. Let us start witih the case for $n=1$. We have,

$$F_1=\frac{\left(\frac{1+\sqrt{5}}{2}\right)^1-\left(\frac{1-\sqrt{5}}{2}\right)^1}{\sqrt{5}}$$

$$=\frac{2\sqrt{5}}{\sqrt{5}}=2$$

which is true.
Now let us assume the case for $n=k$ this means that, $F_{n-2}+F_{n-1}=F_n$ or that,

$$F_k=\frac{\left(\frac{1+\sqrt{5}}{2}\right)^{k-2}-\left(\frac{1-\sqrt{5}}{2}\right)^{k-2}}{\sqrt{5}}+\frac{\left(\frac{1+\sqrt{5}}{2}\right)^{k-1}-\left(\frac{1-\sqrt{5}}{2}\right)^{k-1}}{\sqrt{5}}=\frac{\left(\frac{1+\sqrt{5}}{2}\right)^{k}-\left(\frac{1-\sqrt{5}}{2}\right)^{k}}{\sqrt{5}}$$

Now we need to show that it holds for $n=k+1$ or that,

$$F_{k+1}=\frac{\left(\frac{1+\sqrt{5}}{2}\right)^{k-1}-\left(\frac{1-\sqrt{5}}{2}\right)^{k-1}}{\sqrt{5}}+\frac{\left(\frac{1+\sqrt{5}}{2}\right)^{k}-\left(\frac{1-\sqrt{5}}{2}\right)^{k}}{\sqrt{5}}=\frac{\left(\frac{1+\sqrt{5}}{2}\right)^{k+1}-\left(\frac{1-\sqrt{5}}{2}\right)^{k+1}}{\sqrt{5}}$$

We have $F_{k+1}=F_k+F_{k-1}$. But we have an expression for F_k from our assumption. Putting that in here we get,

$$=\frac{\left(\frac{1+\sqrt{5}}{2}\right)^{k-2}-\left(\frac{1-\sqrt{5}}{2}\right)^{k-2}}{\sqrt{5}}+\frac{\left(\frac{1+\sqrt{5}}{2}\right)^{k-1}-\left(\frac{1-\sqrt{5}}{2}\right)^{k-1}}{\sqrt{5}}+\frac{\left(\frac{1+\sqrt{5}}{2}\right)^{k-1}-\left(\frac{1-\sqrt{5}}{2}\right)^{k-1}}{\sqrt{5}}$$

$$=\frac{\left(\frac{1+\sqrt{5}}{2}\right)^{k-2}-\left(\frac{1-\sqrt{5}}{2}\right)^{k-2}}{\sqrt{5}}+\frac{2\left(\frac{1+\sqrt{5}}{2}\right)^{k-1}-2\left(\frac{1-\sqrt{5}}{2}\right)^{k-1}}{\sqrt{5}}$$

$$= \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{k-2} - \left(\frac{1-\sqrt{5}}{2}\right)^{k-2} + 2\left(\frac{1+\sqrt{5}}{2}\right)^{k-1} - 2\left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\right)$$

$$= \frac{1}{\sqrt{5}}\left(\frac{\left(\frac{1+\sqrt{5}}{2}\right)^{k+1}}{\frac{(1+\sqrt{5})^3}{2^3}} + \frac{2\left(\frac{1+\sqrt{5}}{2}\right)^{k+1}}{\frac{(1+\sqrt{5})^2}{2^2}} - \frac{\left(\frac{1-\sqrt{5}}{2}\right)^{k+1}}{\frac{(1-\sqrt{5})^3}{2^3}} - \frac{2\left(\frac{1-\sqrt{5}}{2}\right)^{k+1}}{\frac{(1-\sqrt{5})^2}{2^2}}\right)$$

$$= \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{k+1}\left(\frac{1}{\left(\frac{1+\sqrt{5}}{2}\right)^3} + \frac{2}{\left(\frac{1+\sqrt{5}}{2}\right)^2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}\left(\frac{1}{\left(\frac{1-\sqrt{5}}{2}\right)^3} + \frac{2}{\left(\frac{1-\sqrt{5}}{2}\right)^2}\right)\right)$$

Simplfying this expression we have,

$$= \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{k+1}\left(\frac{8(2+\sqrt{5})}{(1+\sqrt{5})^3}\right) - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}\left(\frac{8(2-\sqrt{5})}{(1-\sqrt{5})^3}\right)\right)$$

$$= \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{k+1}\left(1\right) - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}\left(1\right)\right)$$

$$= \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}}{\sqrt{5}}$$

Which is what we had to show for the case of $n = k + 1$

$\square$

10.35

Let us take the case for $n = 2$ and $k = 1$. For the statement to be true we have $\binom{n}{k}$ is even.

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{2!}{1!(1!)} = \frac{2}{1} = 2 \text{ which is even}$$

Now let us perform induction on both n and k independently. First let us fix n and perform induction on k. Let $n = 2m$ for some integer $m \in N$. Assume the case for $k = 2k' + 1$ is true. This means that,

$$\binom{2m}{2k'+1} = \frac{(2m)!}{(2k'+1)!(2m-2k'-1)!} \text{ is even}$$

Now we need to show that it is also true for the next odd integer, $k' + 1$ or $k = 2k' + 3$

$$\binom{2m}{2k'+3} = \frac{(2m)!}{(2k'+3)!(2m-2k'-3)!}$$

We can write

$$(2k'+3)! = (2k'+1)!(2k'+2)(2k'+3)$$

and

$$(2m-2k'-3)! = \frac{(2m-2k'-1)!}{(2m-2k'-1)(2m-2k'-2)}$$

So we get,

$$\binom{2m}{2k'+3} = \frac{(2m)!}{(2k'+1)!(2m-2k'-1)!} \frac{(2m-2k'-1)(2m-2k'-2)}{(2k'+2)(2k'+3)}$$