MATH 4320 HW09-10

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Problem 2

(a). We have C is the circle $|z-i|=2$ in the positive sense. We have our integral,

$$\int_C \frac{1}{z^2+4}dz$$

First we can rewrite this as,

$$\int_C \frac{1}{(z+2i)(z-2i)}$$

So we see that the singularity is it $-2i$ and $2i$. However $z=-2i$ lies outside our contour C. So let $f(z)=\frac{1}{z+2i}$ and we write it is,

$$\int_C \frac{f(z)}{z-2i}$$

Using theorem we know this is equivalent to $2\pi i f(z')$ where z' is the singularity point which is at $z=2i$ in this case. So we have,

$$= 2\pi i \frac{1}{2i+2i}$$

$$= 2\pi i \frac{1}{4i} = \frac{\pi}{2}$$

(b). We have $\frac{1}{(z^2+4)^2}$. Let us rewrite this as,

$$\frac{1}{((z+2i)(z-2i))^2} = \frac{1}{(z+2i)^2(z-2i)^2}$$

We already know that $-2i$ lies outside our contour so let $f(z)=\frac{1}{(z+2i)^2}$. And we get,

$$\int_C \frac{f(z)}{(z-2i)^2}$$

We have,

$$f^{(n)}z = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-z')^{n+1}}dz$$

So in our case we have $n=1$ so,

$$f'(z) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-2i)^2}$$

So our integral is,

$$f'(z)2\pi i \text{ where } z=2i$$

We know $f(z)=\frac{1}{(z+2i)^2}$ so $f'(z)=-\frac{2}{(z+2i)^3}$, so our integral is,

$$-\frac{2}{(z+2i)^3}2\pi i = -\frac{2}{(4i)^3}2\pi i$$

$$= -4\pi i \frac{1}{-64i}$$

$$= \frac{\pi}{16}$$

Problem 4

Consider the case when z is inside the the contour. This means that there is a singularity at $s = z$. We nkow using the cauchy goursat extension that,

$$f^n(z) = \frac{n!}{2\pi i}\int_C \frac{f(s)}{(s-z)^{n+1}}ds$$

We see that our term with singuarity is in the denominator and hence we can take $f(s) = s^3 + 2s$. So let us first rewrite our itnegral as,

$$\int_C \frac{f(s)}{(s-z)^3}ds$$

In our case we have $n = 2$ so we have,

$$f''(s) = \frac{2!}{2\pi i}g(z)$$

We have $f(s) = s^3 + 2s$ so $f'(s) = 3s^2 + 2$ and $f''(s) = 6s$.

So,

$$\begin{aligned} 6z &= \frac{2}{2\pi i}g(z) \\ g(z) &= 6\pi i z \end{aligned}$$

Now when z is outside the contour we see that our functino is all analytic inside our contour. So as the contour is closed we know that the integral will be zero.

Problem 6

We need to show the functino is analytic at each point z interior to C which means that we need to show the existence of the derivative at any neighborhood of each of the points in our contour.

We have,

$$g(z) = \frac{1}{2\pi i}\int_C \frac{f(s)}{s-z}ds$$

Using the definitino of the derivative we have,

$$\begin{aligned} g'(z) &= \lim_{h\to 0}\frac{g(z+h)-g(z)}{h} \\ &= \lim_{h\to 0}\frac{1}{2\pi i h}\int_C \frac{f(s)}{s-(z+h)} + \frac{f(s)}{(s-z)}\, ds \\ &= \lim_{h\to 0}\frac{1}{2\pi i}\int_C \frac{f(s)}{(s-z)(s-(z+h))}\, ds \\ &= \lim_{h\to 0}\frac{1}{2\pi i}\int_C \frac{f(s)}{(s-z)^2}ds + \int_C \frac{hf(s)}{(s-z-h)(s-z)^2}\, ds \end{aligned}$$

The right hand integral goes to zero as $h \to 0$

So we have,

$$g'(z) = \frac{1}{2\pi i}\int_C \frac{f(s)\, ds}{(s-z)^2}$$

At all points z within our contour

Problem 1

We have $f(z)$ is entire and we have $u(x, y) \leq u_0$ for all (x, y). We need to show $u(x, y)$ is constant throughout the plane.

We have $g(z) = e^{f(z)}$. We write $f(z) = u + iv$ where both u and v are functinos on x and y. So we get,

$$|g(z)| = |e^{u+iv}| = |e^u e^{iv}| = |e^u||e^{iv}|$$

We know that $|e^{iv}| = 1$ so we get,

$$|e^u||e^{iv}| = |e^u| \cdot 1 \leq |e^{u_0}|$$

So we've shown that the functino $g(z)$ is bounded. Now because it is entire then it must be constant according to Liouvillie's theorem. For that to be true we need $f(z)$ to be constant hence $u(x, y) = Re(f(z))$ must be constant.

Problem 6

Our functino is $f(z) = e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos(y) + i\sin(y))$

Which means the funcitno we wnat to analyse that is $u(x, y) = Re(f(z))$ is

$$u(x, y) = e^x \cos(y)$$

We know that the maximum value of e^x in our domain is at $x = 1$ and the maximum value of $\cos(y)$ in our domain is at $y = 0$ as $\cos(0) = 1$. Hence the max value of our funcitno e^x is at $z = 1$.

The minimum value of e^x is equal to e at $x = 0$ and of $\cos(y)$ is when $y = \pi$ where $\cos(y) = -1$. Hence the minmum value of u will be when we have the max value of e^x and the min value of $\cos(y)$ which is at $1 + \pi i$

Problem 8

(a). We have $(z - z_0)(z^{k-1} + z^{k-2}z_0 + \cdots + z(z_0)^{k-2} + (z_0)^{k-1}$. Now let us expand this as follows,

$$(z^k + z^{k-1}z_0 + \cdots + z^2 z_0^{k-2} + z z_0^{k-1}) - (z_0 z^{k-1} + z^{k-2}z_0^2 + \cdots + z z_0^{k-1} + z_0^k)$$

We see that the middle terms cancel each other out leaving ounly the outer terms.

$$= z^k - z_0^k$$

(b). Now using this factorization We have,

$$P(z) = a_0 + a_1 z + \cdots + a_n z^n$$

and

$$P(z_0) = a_0 + a_1 z_0 + \cdots + a_n z_0^n$$

So,

$$P(z) - P(z_0) = a_1(z - z_0) + a_2(z^2 - z_0^2) + \cdots + a_n(z^n - z_0^n)$$

Using our facorizatino from above we have,

$$\begin{aligned} P(z) - P(z_0) &= a_1(z - z_0) + a_2(z - z_0)Q_2(z) + \cdots + a_n(z - z_0)Q_n(z) \\ &= (z - z_0)(a_1 Q_1(z) + \cdots + a_n Q_n(z)) \\ &= (z - z_0)Q(z) \end{aligned}$$

Problem 1

Using definitnio we need to show that for any choice of ε we can find a n_0 such that $\forall n > n_0$,

$$\left(\frac{1}{n^2} + i\right) - i < \varepsilon$$

We have $\frac{1}{n^2} < \varepsilon$ and,

$$\frac{1}{\varepsilon} < n^2$$

$$n > \frac{1}{\sqrt{\varepsilon}}$$

So for any $n_0 = n > \frac{1}{\sqrt{\varepsilon}}$ we have $|\frac{1}{n^2} + i - i| < \varepsilon$ which makes i the limit of the sequence.

Problem 3

We have $\lim_{n\to\infty} z_n = z$. Using the definitino we know that, for a given ε ,$\exists n_0$ such that $\forall n > n_0$

$$|z_n - z| < \varepsilon$$

Now we know that $||z_n| - |z|| \leq |z_n - z|$ which means that,

$$||z_n| - |z|| < \varepsilon$$

and there exists n_0 for this epsilon such that it is true $\forall n > n_0$. Hence we can say that,

$$\lim_{n\to\infty} |z_n| = |z|$$

Problem 7

We have $\sum_{n=1}^{\infty} z_n = S$. Let c be a complex number $x + iy$ and then we have,

$$\begin{aligned}\sum_{n=1}^{\infty} cz_n = \sum_{n=1}^{\infty}(x+iy)z_n &= \sum_{n=1}^{\infty} xz_n + i\sum_{n=1}^{\infty} yz_n \\ &= x\sum_{n=1}^{\infty} z_n + iy\sum_{n=1}^{\infty} z_n \\ &= xS + iyS \\ &= S(x+iy) = cS\end{aligned}$$

Problem 2

We need to find the taylor series of e^z we know

$$f(z) = \sum_{n=0}^{\infty} f^{(n)}(z_0)\frac{1}{n!}(z-z_0)^n$$

(a). In our case we have $f = e^z$ and $z_0 = 1$

We also know that $f^{(n)}z_0 = e^{z_0} = e$ for any value of n as $f'(z) = e^z$. Hence we have,

$$e^z = e\sum_{n=0}^{\infty} \frac{(z-1)^n}{n!}$$

(b). We know that

$$e^z = \sum_{n=z}^{\infty} \frac{z^n}{n!}$$

Now let us replace z with $z-1$ and we get,

$$e^{z-1} = \sum_{n=0}^{\infty} \frac{(z-1)^n}{n!}$$

$$\frac{e^z}{e} = \sum_{n=0}^{\infty} \frac{(z-1)^n}{n!}$$

$$e^z = e\sum_{n=0}^{\infty} \frac{(z-1)^n}{n!}$$

Problem 3

We have,

$$f(z) = \frac{z}{z^4+4} = \frac{z}{4} \cdot \frac{1}{1+\frac{z^4}{4}}$$

We know that

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$$

Let us replace z with $-\frac{z^4}{4}$ and we get,

$$\frac{1}{1+\frac{z^4}{4}} = \sum_{n=0}^{\infty} (\frac{-z^4}{4})^n$$

$$= \sum_{n=0}^{\infty} (\frac{-z^{4n}}{4^n})$$

$$\frac{z}{4} \cdot \frac{1}{1+\frac{z^4}{4}} = \sum_{n=0}^{\infty} (-1)^n \frac{z^{4n+1}}{2^{2n+2}}$$

Problem 10

(a). First we know that,

$$\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$$

which means that

$$\frac{\sinh z}{z^2} = \sum_{n=0}^{\infty} \frac{z^{2n-1}}{(2n+1)!}$$

Let us take the first term out so we get,

$$= \frac{1}{z} + \sum_{n=1}^{\infty} \frac{z^{2n-1}}{(2n+1)!}$$

$$= \frac{1}{z} + \sum_{n=0}^{\infty} \frac{z^{2(n+1)-1}}{(2(n+1)+1)!}$$

$$= \frac{1}{z} + \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+3)!}$$

(b). We know,

$$\sin(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

So,

$$\begin{aligned}
\sin(z^2) &= \sum_{n=0}^{\infty} (-1)^n \frac{z^{4n+2}}{(2n+1)!} \\
\frac{\sin(z^2)}{z^4} &= \sum_{n=0}^{\infty} (-1)^n \frac{z^{4n-2}}{(2n+1)!} \\
&= \frac{1}{z^2} - \frac{z^2}{3!} + \frac{z^6}{5!} + \dots
\end{aligned}$$

Problem 2

$$f(z) = \frac{1}{1+z} = \frac{1}{z} \cdot \frac{1}{1+1/z}$$

We know,

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$$

So we have,

$$\frac{1}{1+1/z} = \sum_{n=0}^{\infty} (\frac{-1}{z})^n = \sum_{n=0}^{\infty} (-1)^n \frac{1}{z^n}$$

$$\begin{aligned}
f(z) &= \sum_{n=0}^{\infty} (-1)^n \frac{1}{z^{n+1}} \\
&= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{z^n}
\end{aligned}$$

Problem 4

(1).

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$$

$$\frac{1}{z^2(1-z)} = \sum_{n=0}^{\infty} z^{n-2} = \frac{1}{z^2} + \frac{1}{z} + \sum_{n=0}^{\infty} z^n$$

This would be useful in $0 < |z| < 1$

(2). We have,

$$f(z) = \frac{1}{z^2(1-z)}$$

we can rewrite this as,

$$f(z) = \frac{1/z^3}{1/z - 1} = -\frac{1/z^3}{1 - 1/z}$$

We have $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$ so,

$$\frac{1}{1-1/z} = \sum_{n=0}^{\infty} \frac{1}{z^n}$$

$$
\begin{aligned}
-\frac{1/z^3}{1-1/z} &= -\sum_{n=0}^{\infty} \frac{1}{z^{n+3}} \\
&= -\sum_{n=3}^{\infty} \frac{1}{z^n}
\end{aligned}
$$

which would be valid at $1 < |z| < \infty$

Problem 5

1. D_1 We have,

$$\frac{1}{2-z} = \frac{1}{2(1-\frac{z}{2})} = \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$$

And,

$$\frac{1}{z-1} = -\sum_{n=0}^{\infty} z^n$$

So,

$$f(z) = \sum_{n=0}^{\infty} (2^{-n-1} - 1) z^n$$

2. D_2 We have,

$$
\begin{aligned}
\frac{1}{z-1} = \frac{1}{z(1-1/z)} &= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} \\
&= \sum_{n=1}^{\infty} \frac{1}{z^n}
\end{aligned}
$$

And,

$$\frac{1}{2-z} = \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$$

So,

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} + \sum_{n=1}^{\infty} \frac{1}{z^n}$$

3. D_3

We have,

$$
\begin{aligned}
\frac{1}{z-1} = \frac{1}{z(1-1/z)} &= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} \\
&= \sum_{n=1}^{\infty} \frac{1}{z^n}
\end{aligned}
$$

Similarl,

$$
\begin{aligned}
\frac{1}{2-z} = \frac{1}{z(2/z-1)} &= -\frac{1}{z(1-2/z)} \\
&= -\sum_{n=1}^{\infty} \frac{2^{n-1}}{z^n}
\end{aligned}
$$

So we get,

$$f(z) = \sum_{n=1}^{\infty} \frac{1 - 2^{n-1}}{z^n}$$

Problem 2

We substitute z with $\frac{1}{1-z}$ and we have,

$$\frac{1}{(1 - (\frac{1}{1-z}))^2} = \sum_{n=0}^{\infty} \frac{n+1}{(1-z)^n}$$

$$\frac{(1-z)^2}{z^2} = \sum_{n=0}^{\infty} \frac{n+1}{(1-z)^n}$$

$$\begin{aligned} \frac{1}{z^2} &= \sum_{n=0}^{\infty} \frac{n+1}{(1-z)^{n+2}} \\ &= \sum_{n=2}^{\infty} \frac{(-1)^n(n-1)}{(z-1)^n} \end{aligned}$$