Linear Alebgra 3C

Aamod Varma

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3C

Problem 1

Proof. Let us assume the contrary that the matrix of T can be less than $\dim \text{range } T$ non-zero entries. Now consider a basis of W as $w_1, \ldots, w_n$ such that $w_1, \ldots, w_k$ spans range T. Now because we have only $\dim \text{range } T - 1$ non-zero entries in our matrix we have $r < k$ non-zero entries.
So we would have a maximum of r columns with non-zero entries in our matrix. By definition now,

$$Tv_k = A_{1k}w_1 + \cdots + A_{nk}w_n$$

We also know that the definition of linear maps maps a vector in the basis of V to a vector in W. So,

$$Tv_1 = w_1, \ldots, Tv_m = w_m$$

However because we have a maximum of r columns that are non-zero, we can only map any v to a linear combination of $w_1, \ldots, w_r$. But this means that $\dim range = r$ as any $v \in V$ is mapped to r linearly independent vectors. But this contradicts our assumption that $\dim range = k$. Hence our assumption must be wrong. □

Problem 2

Proof. Consider range T is spanned by w_1. Now let us extend this basis to $w_1, \ldots, w_n$. Let us rewrite this basis as,

$$w_1 - w_2, w_2 - w_3, \ldots, w_{n-1} - w_n, w_n$$

Let us show this is a basis first,

$$a_1(w_1 - w_2) + \cdots + a_{n-1}(w_{n-1} - w_n) + a_n(w_n) = 0$$
$$a_1w_1 + (a_2 - a_1)w_2 + \cdots + (a_n - a_{n-1})w_n = 0$$

We nkow that $w_1, \ldots, w_n$ is lin independent. Hence $a_1 = a_2 - a_1 = \cdots = 0$.
If $a_1 = 0$ the $a_2 - 0 = 0$and so on which means $a_1 = \cdots = a_n = 0$.
Hence the sum of the vectors in our basis all equal to w_1. Let this basis be as follows,

$$w'_1 = w_1 - w_2, \ldots, w'_n = w_n$$

Now we knwo that our range is 1 which means $\exists v \in V$ such that $Tv_1 = w_1$. Let us extend this basis to $v_1, \ldots, v_n$. Now let us modify this basis as follows,

$$v_1, v_2 + v_1, \ldots, v_n + v_1$$

First we show this is a basis,

$$a_1v_1 + a_2(v_2 + v_1) + \cdots + a_n(v_n + v_1) = 0$$
$$v_1(a_1 + \cdots + a_n) + a_2v_2 + \cdots + a_nv_n = 0$$

$v_1, \ldots, v_n$ is linearly independent hence, $a_2, \ldots, a_n = 0$ and $a_1 + \cdots + a_n = 0$ so $a_1 + 0 = 0$ and $a_1 = 0$. Hence our new basis is constructed as $v_1' = v_1, v_k' = v_k + v_1$.
We nkow that

$$Tv_1' = w_1$$
$$Tv_k' = T(v_k) + T(v_1') \text{ for } 1 < k \leq n$$

Now because range $T = 1$ either $T(v_k) \in$ range $T \Rightarrow T(v_k) = \lambda_k w_1$ or $T(v_k) = 0 \Rightarrow T(v_k') = T(v_1') = w_1$. If $T(v_k) \in$ range T. Let us rewrite v_k' as $v_k' = \frac{v_k + v}{\lambda_k + 1}$.
Essentially we constructed a basis of V such that $v_1', \ldots, v_n'$ such that for Tv_k' it is equal to w_1 or in other words,

$$Tv_1' = w_1$$
$$\cdots$$
$$Tv_n' = w_1$$

But we rewrite w_1 as sum of basis of W so we have,

$$Tv_1' = w_1' + \cdots + w_n'$$
$$\cdots$$
$$Tv_n' = w_1' + \cdots + w_n'$$

Now by how matrices are defined we have constructed a basis of V and W such that all entires of our matrix is just 1s.

$\square$

Problem 3

Proof. (a). We know that the matrix of T is defined as,

$$T(v_k) = A_{1k}w_1 + \cdots + A_{nk}w_n$$

and that of S is defined as,

$$S(v_k) = B_{1k}w_1 + \cdots + B_{nk}w_n$$

And for $(S + T)$ let it be,

$$(S + T)v_k = C_{1k}w_1 + \cdots + C_{nk}w_n$$

We know that $(S+T)v = Sv + Tv$. So for the basis of V given we have,

$$\begin{aligned}(S+T)v_k &= Sv_k + Tv_k \\ &= (A_{1k} + B_{1k})w_1 + \cdots + (A_{nk} + B_{nk})w_n\end{aligned}$$

So we have $C_{j,k} = A_{jk} + B_{jk}$. Which essentially means $M(S+T) = M(S) + M(T)$

(b). We have $M(T)$ defined as follows,

$$\begin{aligned}(T)v_k &= A_{11}w_n + \cdots + A_{nk}w_n \\ \lambda \times (T)v_k &= (\lambda A_{11})w_1 + \cdots + (\lambda A_{nk}w_n)\end{aligned}$$

Or in other words the matrix $\lambda M(T)$ is defined as,

$$\lambda \times (T)v_k = (\lambda A_{11})w_1 + \cdots + (\lambda A_{nk}w_n)$$

Now consider the matrix of λT we have,

$$\begin{aligned}(T)v_k &= A_{11}w_n + \cdots + A_{nk}w_n \\ (\lambda T)v_k &= (\lambda) \times Tv_k \\ &= (\lambda A_{11})w_1 + \cdots + (\lambda A_{nk}w_n)\end{aligned}$$

Hence as each element of $\lambda M(T)$ is the same as that of $M(\lambda T)$ we have $M(\lambda T) = \lambda M(T)$

$\square$

Problem 4

Proof. Consider a basis of P_3 as $p_1, \ldots, p_4$ and a basis of P_2 as $1 + x + x^2$. By definitino of linear map we have,

$$\begin{aligned}Dp_1 &= 1 \\ Dp_2 &= x \\ Dp_3 &= x^2 \\ Dp_4 &= 0\end{aligned}$$

So we have $p_1 = x, p_2 = x^2, p_3 = x^3, p_4 = 1$ as a basis of P_3

$\square$

Problem 5

Proof. Let $\dim \operatorname{range} T = r$ and consdier a basis for range T as $w_1, \ldots, w_r$. Let us nwo extedn this basis to $w_1, \ldots, w_m$. Now as $w_1, \ldots, w_r \in \operatorname{range} T$ we nkow $\exists v_1, \ldots, v_r$ such that $Tv_1 = w_1, \ldots, Tv_r = w_r$. It is easy to show that $v_1, \ldots, v_r$ is linearly independent.
Now extend our $v_1, \ldots, v_r$ to a basis of V as $v_1, \ldots, v_n$. So we have,

$$\begin{aligned}
Tv_1 &= w_1 \\
&\ldots \\
Tv_r &= w_r \\
&\ldots \\
Tv_n &= 0
\end{aligned}$$

Based on how we define our matrix we can write,

$$\begin{aligned}
Tv_1 &= 1w_1 + \cdots + 0w_r + \cdots + 0w_m \\
&\ldots \\
Tv_r &= 0w_1 + \cdots + 1w_r + \cdots + 0w_m \\
&\ldots \\
Tv_n &= 0w_1 + \cdots + 0w_r + \cdots + 0w_m
\end{aligned}$$

Hence for only row k column k we have all ones and rest are all zeores.

□

Problem 6

Proof. First consider if $Tv_1 = 0$. In this case we have $Tv_1 = 0w_1 + \cdots + 0w_n$ where $w_1, \ldots, w_n$ is a basis of W. So we have all zeroes in the first column. Now consider if $Tv_1 = w_1 \neq 0$. Let us now extend this basis to $w_1, \ldots, w_n$. And we have $Tv_1 = 1w_1 + \cdots + 0w_n$. So we have all 0s except for a 1 in the first column first row. □

Problem 8

Proof. We need to show $(AB)_{j,.} = A_{j,.}B$
First we have,

$$AB_{j,k} = \sum_{r=1}^{n} A_{j,r}B_{r,k}\$$$

So,

$$\begin{aligned}
AB_{j,.} &= (\sum_{r=1}^{r=n} A_{j,r}B_{r,1}, \dots, \sum_{r=1}^{r=n} A_{j,r}B_{r,n}) \\
&= (A_{j,1}, \dots, A_{j,n})B \\
&= (A_{j,.}B_{.,1}, \dots, A_{j,.}B_{.,n}) \\
&= A_{j,.}B
\end{aligned}$$

□

Problem 9

Proof. We have,

$$\begin{aligned}
(aB)_{1,k} &= \sum_{r=1}^{n} A_{1,n}B_{n,k} \\
(aB) &= (\sum_{r=1}^{n} A_{1,n}B_{n,1}, \dots, \sum_{r=1}^{n} A_{1,n}B_{n,1})
\end{aligned}$$

Now we have,

$$\begin{aligned}
a_1B_{1,.} + \dots + a_nB_{n,.} &= a_1B_{11} + \dots + a_nB_{n,1}, \dots, a_1B_{n1} + \dots + a_nB_{nn} \\
&= (\sum_{r=1}^{n} a_nB_{n,1}, \dots, \sum_{r=1}^{n} a_nB_{n,1})
\end{aligned}$$

Hence we have our equality. □

Problem 10

Proof. Let $A = (1,0;0,0)$ and $B = (0,1;0,0)$. We have AB = (0,1;0,0) and $BA = (0,0;0,0)$ □

Problem 11

Proof. Let $B + C = X$ such that, First be have $(B+C)_{j,k} = B_{j,k} + C_{j,k} = X_{j,k}$
So we have,

$$A(B+C) = AX$$

$$
\begin{aligned}
(AX)_{j,k} &= \sum_{r=1}^{n} A_{j,r}X_{r,k} \\
&= \sum_{r=1}^{n} A_{j,r}(B_{r,k} + C_{r,k}) \\
&= \sum_{r=1}^{n} A_{j,r}(B_{r,k}) + A_{j,r}(C_{r,k}) \\
&= \sum_{r=1}^{n} A_{j,r}(B_{r,k}) + \sum_{r=1}^{n} A_{j,r}(C_{r,k}) \\
&= AB_{j,k} + AC_{j,k}
\end{aligned}
$$

Hence we have $A(B+C) = AB + AC$ $\square$

Problem 12

Proof. We know if T and S are linear map from U, V and V, W respectively then $M(ST) = M(S)M(T)$
Let $A = M(T), B = M(S), C = M(R)$ for a linear map T, S, R So we have,

$$
\begin{aligned}
(AB)C &= (M(T)M(S))M(R) \\
&= (M(TS))M(R) \\
&= M((TS)R) \\
&= M(T(SR)) \\
&= M(T)M(SR) \\
&= M(T)(M(S)M(R)) \\
&= A(BC)
\end{aligned}
$$

$\square$

Problem 13

Proof. We know that

$$(AA)_{j,k} = \sum_{r=1}^{n} A_{j,r}A_{r,k}$$

$$(A(AA))_{j,k} = \sum_{r=1}^{n} A_{j,r}(AA)_{r,k}$$

$$(A^3)_{j,k} = \sum_{p=1}^{n} A_{j,r}(\sum_{x=1}^{n} A_{r,x}A_{x,k})$$

$$(A^3)_{j,k} = \sum_{p=1}^{n}\sum_{x=1}^{n} A_{j,r}A_{r,x}A_{x,k}$$

□

Problem 14

Proof. To show that the functino is a linear map we need to show additvity and homogenity. Consider $A \in F^{m,n}$ and $B \in F^{m,n}$.
1. Additive.
We need to show that $(A+B)^t = A^t + B^t$. First we know that $(A+B)_{j,k} = A_{j,k} + B_{j,k}$

$$\begin{aligned}(A+B)^t_{j,k} &= (A+B)_{k,j} \\ &= A_{k,j} + B_{k,j} \\ &= A^t_{j,k} + B^t_{j,k}\end{aligned}$$

So we have $(A+B)^t = A^t + B^t$
2. Scalar multiplication.
We need to show that $(kA)^t = kA^t$
We have $(kA)_{j,k} = k(A_{j,k})$. So

$$(kA)^t_{j,k} = (kA)_{k,j}$$

$$= kA_{k,j}$$

So we have $(kA)^t = kA^t$

□

Problem 15

Proof. We have,

$$AC_{j,k} = \sum_{r=1}^{n} A_{j,n} C_{n,k}$$

So $AC^t_{j,k} = AC_{k,j}$ which is,

$$= \sum_{r=1}^{n} A_{k,r} C_{r,j}$$

Now $C^t_{j,k} = C_{k,j}$ and $A^t_{j,k} = A_{k,j}$. So

$$(C^t A^t)_{j,k} = \sum_{r=1}^{n} C_{r,j} A_{k,r}$$

$$= \sum_{r=1}^{n} A_{k,r} C_{r,j}$$

$$= (AC)^t_{j,k}$$

□

Problem 16

Proof. $\Leftarrow$ First we know that for any rank k matrix we can write it as a prodcut of RC such that $R = m \times k$ and $C = k \times n$ if A is a $m \times n$ matrix. So if A is a rnak 1 matrix we can write it as a $m \times 1$ times $1 \times n$ product of matrices.
Let $C = (c_1, \dots, c_m)^T$ and $R = (d_1, \dots, d_n)$
So we have $A_{j,k} = C_{j,1}R_{1,k} = c_j d_k$
$\Rightarrow$ We have $A_{j,k} = c_j d_k$. Let $C \in F^{m,1}$ where $C_{j,1} = c_j$. Now we have $A_{\cdot,k} = d_k C$. So each column of A is a scalar multiplcatino of C. Which means that our matrix A has rank 1. $\square$

Problem 17

Proof. (a) $\Rightarrow$ (b) We need to show that $A_{\cdot,1}, \dots, A_{\cdot,n}$ are linearly independent or that, if

$$\lambda_1 A_{\cdot,1} + \cdots + \lambda_n A_{\cdot,n} = 0$$

The only solution is all $\lambda = 0$
By definitino of matrix we have,

$$Tv_k = A_{1,k}u_1 + \cdots + A_{n,k}u_n$$

Consider $\lambda_k T v_k = \lambda_k A_{1,k} u_1 + \cdots + \lambda_k A_{n,k}$
Now as T is injective we have $\lambda_k T v_k = 0$. This can only be true if $\lambda_k = 0$. Which means our columns are linearly independent. $\square$