MATH 4320 HW06-8

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Problem 2

(a). We know $\sinh(z) = \frac{e^z - e^{-z}}{2}$ and $\cosh(z) = \frac{e^z + e^{-z}}{2}$

So,

$$\begin{aligned}
2\sinh(z)\cosh(z) &= 2\frac{e^z - e^{-z}}{2}\frac{e^z + e^{-z}}{2} \\
&= 2\frac{e^{2z} + 1 - 1 - e^{-z}}{4} \\
&= \frac{e^{2z} - e^{-2z}}{2} \\
&= \sinh(2z)
\end{aligned}$$

(b). $\sin(2z) = 2\sin(z)\cos(z)$

We know that $-i\sinh(iz) = \sin(z)$ and $\cosh(iz) = \cos(z)$. Let $iz = z'$ then,

$$\sinh(z') = \frac{-\sin(\frac{z'}{i})}{i}$$

$$\cosh(z') = \cos(\frac{z'}{i})$$

So,

$$2\sinh(z')\cosh(z') = -2\frac{\sin(\frac{z'}{i})}{i}\cos(\frac{z'}{i})$$

$$\begin{aligned}
&= -2\sin\frac{\frac{2z'}{i}}{i} \\
&= \sinh(2z')
\end{aligned}$$

Problem 6

(a). $|\cosh z|^2 = \sinh^2 x + \cos^2 y$

This means that

$$\sinh^2 x \leq |\cosh z|^2$$

$$|\sinh x| \leq |\cosh z|$$

Now we need to show that $|\cosh z| \leq |\cosh x|$. We know that,

$$|\cosh z| = |\cosh x \cos y + i \sinh x \sin y|$$

$$\cosh^2 z = \cosh^2 x \cos^2 y - \sinh^2 x \sin^2 y$$

$$\cosh^2 z + \sinh^2 x \sin^2 y = \cosh^2 x \cos^2 y$$

So,

$$\cosh^2 y \leq \cosh^2 x \cos^2 y$$

But we know that $\cos^2 y \leq 1$ so,

$$\cos^2 y \leq \cosh^2 y$$

or
$$|\cosh z| \leq |\cosh x|$$

So we've shown that
$$|\sinh x| \leq |\cosh z| \leq |\cosh x|$$

(b). $|\sinh x| \leq |\cosh z| \leq \cosh x$

Problem 14

We have $\cosh^2 x - \sinh^2 x = 1$

(a).

$$\begin{aligned}
\cosh^2 z - \sinh^2 z &= (\cosh x \cos y + i \sinh x \sin y)^2 - (\sinh x \cos y + i \cosh x \sin y)^2 \\
&= (\cosh^2 x \cos^2 y - \sinh^2 x \sin^2 y + 2i \cosh x \cos y \sinh x \sin y) \\
&- (\sinh^2 x \cos^2 y - \cosh^2 x \sin^2 y + 2i \cosh x \cos y \sinh x \sin y) \\
&= (\cosh^2 x \cos^2 y - \sinh^2 x \sin^2 y) - (\sinh^2 x \cos^2 y - \cosh^2 x \sin^2 y) \\
&= (\cos^2 y(\cosh^2 x - \sinh^2 x)) + (\sin^2 y(\cosh^2 x - \sinh^2 x)) \\
&= (\cos^2 y) + (\sin^2 y) \\
&= 1
\end{aligned}$$

(b). We have $\sinh x + \cosh x = e^x$

$$\begin{aligned}
\sinh z + \cosh z &= \sinh x \cos y + i \cosh x \sin y + \cosh x \cos y + i \sinh x \sin y \\
&= \cos y(\sinh x + \cosh x) + i \sin y(\sinh x + \cosh x) \\
&= \cos y(e^x) + i(e^x) \\
&= e^x(\cos y + i \sin y) \\
&= e^x e^{iy} = e^{x+iy} \\
&= e^z
\end{aligned}$$

Problem 2

$$\begin{aligned}
\sin z &= 2 \\
z &= \sin^{-1}(2) \\
&= -i \log[2i + (1-4)^{\frac{1}{2}}] \\
&= -i \log[i(2+\sqrt{3})] \\
&= -i(\ln[2+\sqrt{3}] + i(\frac{\pi}{2} + 2n\pi)) \\
&= (\frac{\pi}{2} + 2n\pi) - i \ln[2+\sqrt{3}]
\end{aligned}$$

Problem 2

(a). $\int_0^1 1 + it^2 \, dt$

$$
\begin{aligned}
&= \int_0^1 1 - t^2 + 2it \, dt \\
&= (t - \frac{t^3}{3} + it^2)]_0^1 \\
&= (1 - \frac{1}{3} + i) - (0) \\
&= \frac{2}{3} + i
\end{aligned}
$$

(b). $\int_1^2 \frac{1}{t} - i^2 \, dt$

$$
\begin{aligned}
&= \int_1^2 \frac{1}{t^2} - 1 - \frac{2i}{t} \, dt \\
&= \frac{-1}{t} - t - 2i\ln(t)]_1^2 \\
&= (-\frac{1}{2} - 2 - 2i\ln(2)) - (-1 - 1) \\
&= (-\frac{1}{2} - 2 - i\ln(4) + 2) \\
&= -\frac{1}{2} - i\ln 4
\end{aligned}
$$

(c). $\int_0^{\frac{\pi}{6}} e^{i2t} \, dt$

$$
\begin{aligned}
&= \int_0^{\frac{\pi}{6}} e^{2it} \, dt \\
&= \frac{e^{2it}}{2i}]_0^{\frac{\pi}{6}} \\
&= (e^{i\frac{\pi}{3}} \frac{1}{2i}) - (\frac{1}{2i}) \\
&= \frac{1}{2i}(e^{i\frac{\pi}{3}} - 1) \\
&= \frac{1}{2i}(\cos(\frac{\pi}{3}) + i\sin(\frac{\pi}{3}) - 1) \\
&= \frac{1}{2i}(\frac{1}{2} + i\frac{\sqrt{3}}{2} - 1) \\
&= \frac{1}{2i}(-\frac{1}{2} + i\frac{\sqrt{3}}{2}) \\
&= \frac{\sqrt{(3)}}{4} + \frac{i}{4}
\end{aligned}
$$

(d). $\int_0^\infty e^{-zt}\, dt$

$$
\begin{aligned}
&= \int_0^\infty e^{-zt}\, dt \\
&= \frac{e^{-zt}}{-z}]_0^\infty \\
&= (\frac{-1}{z})(\frac{1}{e^{z\infty}} - \frac{1}{e^0}) \\
&= (-\frac{1}{z})(-1) = \frac{1}{z}
\end{aligned}
$$

Problem 3

$$
\begin{aligned}
&\int_0^{2\pi} e^{im\theta} e^{-in\theta}\, d\theta \\
&= \int_0^{2\pi} e^{\theta(im-in)}\, d\theta \\
&= \frac{e^{i\theta(m-n)}}{i(m-n)}]_0^{2\pi} \\
&= \frac{1}{i(m-n)}(e^{i2\pi(m-n)} - e^0)
\end{aligned}
$$

We know that $e^{0+2n\pi} = e^0 = 1$

So if $m \neq n$ we have,

$$= \frac{1}{i(m-n)}(1-1) = 0$$

If $m = n$ then we have,

$$
\begin{aligned}
&\int_0^{2\pi} e^{i\theta(0)}\, d\theta \\
&= 2\pi
\end{aligned}
$$

Problem 2

We have $C : |z| = 2$ where $Re(z)$ is positive we have,

$$z = z(\theta) = 2e^{i\theta}, \left(-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \right)$$

and

$$z = Z(y) = \sqrt{4-y^2} + iy, (-2 \leq y \leq 2)$$

We need to show that $Z(y) = z[\phi(y)]$ where,

$$\phi(y) = \arctan \frac{y}{\sqrt{4-y^2}}, \left(\frac{\pi}{2} < \arctan t < \frac{\pi}{2} \right)$$

We are given that $\theta = \phi(y) = \arctan \frac{y}{\sqrt{4-y^2}}$. We can write this as,

$$\tan\theta = \frac{y}{\sqrt{4-y^2}}$$

Using the property $sec^2\theta - tan^2\theta = 1$ we can say that,

$$\sec\theta = \frac{2}{\sqrt{4-y^2}}$$

or that,

$$\cos\theta = \frac{\sqrt{4-y^2}}{2} \text{ cos is always positive in this region}$$

and similarly,

$$\sin\theta = \frac{y}{2} \text{ here y goes from -2 to 2}$$

So we have,

$$\begin{aligned} z &= 2e^{i\theta} \\ &= 2(\cos\theta + i\sin\theta) \\ &= 2(\frac{\sqrt{4-y^2}}{2} + i\frac{y}{2}) \\ &= \sqrt{4-y^2} + iy, \Big(-2 \le y \le 2 \Big) \end{aligned}$$

We have,

$$\tan\phi(y) = \frac{y}{\sqrt{4-y^2}}$$

$$\frac{d}{dy}\tan\phi(y) = \sec^2(\phi)\frac{d}{dy}\phi = \frac{\sqrt{4-y^2} + \frac{y^2}{\sqrt{4-y^2}}}{4-y^2}$$

$$\phi'(y) = \frac{1}{sec^2\phi}\frac{4}{\sqrt{4-y^2}} > 0$$

As both the terms are greater than zero.

Problem 6

(a). The arc intersects the real axis when $y(x) = 0$. So when $0 < x \le 1$ we have,

$$y(x) = x^3\sin(\pi/x)$$

We need this to be equal to zero.

So either $x^3 = 0$ or $\sin(\pi/x) = 0$. However $x^3 = 0 \Rightarrow x = 0$ however $x \ne 0$ so $\sin(\pi/x) = 0$. We know that $\sin(\theta) = 0$ when $\theta = n\pi, n = 0, 1, 2...$ So we have $n\pi = \frac{\pi}{x}$,

$$x = \frac{\pi}{n\pi} = \frac{1}{n}$$

When $y(x) = 0$ we knwo that $z = x$ so when $z = \frac{1}{n}$ we have $z = x + 0 = x$ where $n = 1, 2, ..$

(b). For C to be a smooth arc we need to show that it is continous over the domain $[0, 1]$. Or that $y'(x)$ is defined and exists in this region.

$$y(x) = x^3 \sin(\frac{\pi}{x}), 0 < x \leq 1$$
$$y'(x) = 3x^2(\sin\frac{\pi}{x}) - x\cos(\frac{\pi}{x})\pi$$

So we know that $y'(x)$ exists and is continuous and non-zero when $x \in (0, 1)$ and is 0 when $x = 1$.

Now we need to show continuity of y at $x = 0$. Or that,

$$\lim_{x\to 0} y(x) = y(0) = 0$$

Using the epsion-delta definition we need to show that, $\forall\varepsilon, \exists\delta$ s.t.

$$|x^3 \sin(\frac{\pi}{x}) - 0| < \varepsilon \text{ for some } |x - 0| < \delta$$

We know that $|x^3 \sin(\pi/x)| \leq |x^3|$ and we can make x arbitrarily small such that $|x^3| < \varepsilon$

So if we choose $\delta = \varepsilon^{\frac{1}{3}}$ we have $|x^3| < \varepsilon, \forall\varepsilon$ which means that

$$|x^3 \sin(\pi/x)| < \varepsilon, \forall\varepsilon$$

This shows that y is cont. at $x = 0$.

Now we need to show that $y'(x)$ exists and is equal to 0. Or that,

Similarly we can show that $\lim_{x\to 0} y'(x) = 0$ by taking $\delta = (\frac{\varepsilon}{3})^{\frac{1}{2}}$ as we can bound $|3x^2(\sin\pi/x) - x\cos(\pi/x)\pi| < 3x^2$ because $x\cos(\frac{\pi}{x})\pi$ is always positive as x tends to 0 from the positive real side.

Problem 2

We need to find $\int_C f(z)dz$

(a). $f(z) = z - 1$ where $z = 1 + e^{i\theta}, (\pi \leq \theta \leq 2\pi)$

So $dz = ie^{i\theta}d\theta$ and $f(z)dz = e^{i\theta}ie^{i\theta}d\theta$

We get,

$$\int_{\pi}^{2\pi} ie^{2i\theta}\, d\theta$$
$$= \frac{1}{2}[e^{2i\theta}]_{\pi}^{2\pi}$$
$$= \frac{1}{2}0 = 0$$

(b). $z = x, (0 \leq x \leq 2)$. We have $dz = dx$ so,

$$\int_0^2 x - 1dx$$

$$= [\frac{x^2}{2} - x]_0^2 = (0 - 0) = 0$$

Problem 6

So we have C : semicircle $z = e^{i\theta}$ and $f(z)$ is the principal branch e^{iLogz} $dz = ie^{i\theta}d\theta$ so we get,

$$\int_0^\pi ie^{i(Log(e^{i\theta})+\theta)}\,d\theta$$

$$= \int_0^\pi ie^{i(i\theta+\theta)}\,d\theta$$

$$= \int_0^\pi ie^{\theta(i-1)}\,d\theta$$

$$= \frac{ie^{\theta(i-1)}}{i-1}]_0^\pi$$

$$= -\frac{1}{2}(i-1)(e^{\pi i - \pi} - 1)$$

$$= -\frac{1}{2}(1-i)(e^{-\pi} + 1)$$

Problem 11

(a). $z = 2e^{i\theta}$ so $dz = 2ie^{i\theta}d\theta$. Given $f(z) = \bar{z}$

If $z = 2e^{i\theta}$ then $\bar{z} = 2e^{-i\theta}$. So we have,

$$\int_C \bar{z}\,dz$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2e^{-i\theta}2ie^{i\theta}\,d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-i\theta}4ie^{i\theta}\,d\theta$$

$$= [2i\theta]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= 4\pi i$$

(b). $z = \sqrt{4-y^2} + iy$ So $\bar{z} = \sqrt{4-y^2} - iy$ and

$$dz = (-\frac{y}{\sqrt{4-y^2}} + i)dy$$

So we get,

$$\int_{-2}^2 (\sqrt{4-y^2} - iy)(-\frac{y}{\sqrt{4-y^2}} + i)\,dy$$

$$\int_{-2}^{2} -y + i\sqrt{4-y^2} + \frac{iy^2}{\sqrt{4-y^2}} + y \, dy$$
$$\int_{-2}^{2} i\sqrt{4-y^2} + \frac{iy^2}{\sqrt{4-y^2}} \, dy$$
$$\int_{-2}^{2} \frac{4i}{\sqrt{4-y^2}} \, dy$$

Taking $y = 2\sin(\theta)$ and parameterizing it from $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ and taking $dy = 2\cos(\theta)$ we have,

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{4}{2\cos\theta} \, d\theta$$
$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{4i}{2\cos\theta} 2\cos\theta \, d\theta$$
$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4i \, d\theta$$
$$= 4\pi i$$

Problem 1

(a). We know that,

$$\left| \int_C \frac{z+4}{z^3-1} dz \right| \leq \int_C \left| \frac{z+4}{z^3-1} \right| dz \leq \int_C |M| dz$$

Where M bounds our function.

We know that $|z+4| \leq |z| + |4| = 6$ and $||z^3| - |1|| \leq |z^3 - 1| < |z^3| + |1|$. So $7 \leq |z^3 - 1| \leq 9$. So we can write,

$$\left| \frac{z+4}{z^3-1} \right| \leq \frac{6}{7}$$

Which means that the integral is bounded by,

$$\int_C |M| = \int_C \left| \frac{6}{7} \right| = \frac{6\pi}{7}$$

(b). We know that,

$$\left| \int \frac{dz}{z^2-1} \right| \leq \int \left| \frac{1}{z^2-1} \right| dz \leq \int_C |M| dz$$

Where M bounds our function.

We know that $||z^2| - |1|| \leq z^2 - 1$ so, $4 - 1 \leq z^2 - 1$ which means that

$$\frac{1}{z^2-1} \leq \frac{1}{3}$$

Where $|M| = \frac{1}{3}$ so we have, our integral is bounded above by,

$$\int_C \left| \frac{1}{3} \right| = \frac{\pi}{3}$$

Problem 4

$$\left| \int_C \frac{2z^2-1}{z^4+5z^2+4} dz \right| \leq \int_C \left| \frac{2z^2-1}{z^4+5z^2+4} \right| dz \leq \int_C |M| dz$$

Where M is the upper bouned for our function.

First we know that $|2z^2-1| \leq |2z^2| + |1| = 2R^2+1$ and that

$$||z^4+5z^2| - |4|| \leq z^4+5z^2+4$$
$$|||z^4| - |5z^2|| - |4|| \leq z^4+5z^2+4$$
$$R^4 - 5^{R^2} - 4 \leq z^4+5z^2+4$$
$$(R^2-1)(R^2-4) \leq z^4+5z^2+4$$

So we have $|M| = \frac{2R^2+1}{(R^2-1)(R^2-4)}$

And as $\int_C dz = \pi R$ we can upperbound our integerarl by,

$$\frac{\pi R(2R^2+1)}{(R^2-1)(R^2-4)}$$

Problem 4

Our contour is any integral that extends from $z=-3$ to $z=3$. Which means that $r=3$ and $\pi \leq \theta \leq 2\pi$ We have

$$z = 3e^{i\theta}$$
$$f(z) = z^{\frac{1}{2}} = \sqrt{3}e^{i\frac{\theta}{2}}$$
$$dz = 3ie^{i\theta}$$

So,

$$\int_{\pi}^{2\pi} \frac{3}{2}\sqrt{3}ie^{i\frac{\theta}{2}}e^{i\theta}\, d\theta$$
$$= 2\sqrt{3}e^{i\frac{3\theta}{2}}]_{\pi}^{2\pi}$$
$$= 2\sqrt{3}((-1+0-)(0-i))$$
$$= 2\sqrt{3}(-1+i)$$

If we had gone around $C_2 - C_1$ our bounds would have been 0 to 2π which would result in a value of $-4\sqrt{3}$

Problem 2

(a). All we have to show is that $f(z)$ is analytic throughotut the region between C_1 and C_2.

We know that $\frac{1}{3z^2+1}$ is analytic except at $z = i\frac{1}{\sqrt{3}}$. We know that this lies outside the region (inside the square). So now using the priciple of deformation we know that the integral is same for C and C_2.

(b). We have $\sin(\frac{z}{2}) = 0$ so $\frac{z}{2} = n\pi$ or $z = 2n\pi$. For $n = 0$ it lies outside our region. If $n > 0, z$ lies outside our contour as well. So we are able to deform C_2 to C_1.

(c). We have singularity only when $z = 0$ which is outisde our region. Hence we can deform our regino and preserve the integral.

Problem 5

We are given that a function f is entire. We see that the contour C_3 and C_1 are positive oriented and define a simple closed curve. According to Cauchy-Goursat theorem we know that,

$$\int_{C_2+C_3} f(z) = 0$$

if f is entire.

So,

$$\int_{C_2} f(z)dz + \int_{C_3} f(z)dz = 0$$

$$\int_{C_2} f(z)dz = -\int_{C_3} f(z)dz$$

Similarly we see that C_1 is negatively oriented while C_3 is positive oriented. So the contour $C_3 - C_2$ is a closed contour positive ortiented. Which means that,

$$\int_{C_3-C_2} f(z) = 0$$

$$\int_{C_3} f(z)dz - \int_{C_1} f(z)dz = 0$$

$$\int_{C_3} f(z)dz = \int_{C_1} f(z)dz$$

So adding both sides we get,

$$\int_{C_1} f(z)dz + \int_{C_2} f(z)dz = 0$$

or,

$$\int_{C_1+C_2} f(z)dz = 0$$

$$\int_{C} f(z)dz = 0$$