Probability Theory: HW5

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Exercise 6.36

We have,

$$f(x,y,z) = \begin{cases} 8xyz & \text{if } 0 < \text{x,y,z} < 1, \\ 0 & \text{otherwise} \end{cases}$$

To find if X, Y, Z are independent, we can find their marginal. So we have,

$$\begin{aligned} f_X(x) &= \int_0^1 \int_0^1 8xyz \; dy \; dz \\ &= \int_0^1 [8xz\frac{y^2}{2}]_0^1 \; dz \\ &= \int_0^1 4xz \; dz \\ &= \int_0^1 4x\frac{z^2}{2}]_0^1 \; dz \\ &= \int_0^1 4x\frac{1}{2} \; dz \\ &= 2x \end{aligned}$$

Similarly we get $f_Y(x) = 2y$ and $f_Z(z) = 2z$. Now note that we have $f(x,y,z) = f_X(x)f_Y(y)f_Z(z)$ which means that the r.v are independent.

Now to find $\mathbb{P}(X > Y)$. Here Z can take on any value so we have,

$$\begin{aligned} P(X > Y) &= \int_0^1 \int_0^1 \int_y^1 8xyz \; dx \; dy \; dz \\ &= \int_0^1 \int_0^1 4yz[x^2]_y^1 \; dy \; dz \\ &= \int_0^1 \int_0^1 4yz[1 - y^2] \; dy \; dz \\ &= \int_0^1 \int_0^1 4yz - 4y^3z \; dy \; dz \\ &= \int_0^1 [2y^2z - y^4z]_0^1 \; dz \\ &= \int_0^1 [2z - z] \; dz \\ &= \int_0^1 [z] \; dz \\ &= \frac{z^2}{2}]_0^1 \\ &= \frac{1}{2} \end{aligned}$$

Now note that $\mathbb{P}(X > Y)$ and $\mathbb{P}(Y > Z)$ will be the same value as X, Y, Z all have the same distribution and pdf so it is also equal to $\frac{1}{2}$.

Exercise 6.45

We have, X, Y and the joint density,

$$f(x,y) = \begin{cases} \frac{1}{2}(x+y)e^{-x-y} & \text{if x, y} > 0 \\ 0 & \text{otherwise} \end{cases}$$

Now to find the density function of $X+Y$. We can use the convolution formula to achieve this. So the pdf of $X+Y$ evaluated at z would be when $y = z - x$ and we integrate x from 0 to z as both $x, y > 0$ so $z - x > 0, x < z$. This gives us,

$$\begin{aligned} f_{X+Y}(z) &= \int_0^z \frac{1}{2}(x + z - x)e^{-x-z+x}\ dx \\ &= \int_0^z \frac{1}{2}(z)e^{-z}\ dx \\ &= \frac{1}{2}(z)e^{-z}[x]_0^z \\ &= \frac{1}{2}z^2e^{-z} \qquad \text{for } z > 0 \end{aligned}$$

Exercise 6.55

We have,

$$f(x,y) = \begin{cases} \frac{1}{4}e^{-\frac{1}{2}(x+y)} & \text{if x,y} > 0 \\ 0 & \text{otherwise} \end{cases}$$

First we need to joint density of $U = \frac{1}{2}(X - Y)$ and $V = Y$. We rewrite this as functions of U and V to get, $Y = V$ and $X = 2U + V$. Now the jacobinan of this is $2 \cdot 1 = 2$. So the joint of U, V is,

$$\begin{aligned} f_{U,V}(u,v) &= 2f(2u + v, v) \\ &= 2\frac{1}{4}e^{-\frac{1}{2}(2u+v+v)} \\ &= \frac{1}{2}e^{-u-v} \text{ for } u, v \in A \end{aligned}$$

Now to find the density of U. We have,

$$f_U(u) = \int_{-\infty}^{\infty} \frac{1}{2}e^{-u-v}\ dv$$

Now if $u \geq 0$ the lower limit is 0 and if $u < 0$, note that $X \geq 0$ which means $2u + v \geq 0$ or that $v \geq -2u$ so lower limit is $-2u$. For both we have,

$$\begin{aligned} f_U(u) &= \int_0^{\infty} \frac{1}{2}e^{-u-v}\ dv \\ &= \int_0^{\infty} \frac{1}{2}e^{-u}e^{-v}\ dv \\ &= \frac{1}{2}e^{-u}[-e^{-v}]_0^{\infty} \\ &= \frac{1}{2}e^{-u} \end{aligned}$$

Now for $u < 0$ we have,

$$\begin{aligned}
f_U(u) &= \int_{-2u}^{\infty} \frac{1}{2} e^{-u-v}\ dv \\
&= \int_{-2u}^{\infty} \frac{1}{2} e^{-u} e^{-v}\ dv \\
&= \frac{1}{2} e^{-u} [-e^{-v}]_{-2u}^{\infty} \\
&= \frac{1}{2} e^{-u} e^{2u} \\
&= \frac{1}{2} e^{u} \\
&= \frac{1}{2} e^{-|u|}
\end{aligned}$$

So we have,

$$f_U(u) = \frac{1}{2} e^{-|u|} \qquad \text{for } u \in \mathbb{R}$$

Exercise 6.61

We have X, Y are independent r.v that are exponential with parameter λ. Now let $U = X$ and $V = X + Y$. Rewriting this we have $X = V - U$ and $Y = U$ the Jacobian of which is 1. So we have,

$$\begin{aligned}
f_{U,V} &= f(v-u, u) \\
&= f(v-u) f(u) \\
&= \lambda^2 e^{-\lambda(v-u+u)} \\
&= \lambda^2 e^{-\lambda v}
\end{aligned}$$

Now to find the conditional given $X+Y = a$ or $V = a$ we integrate over U first to get the marginal of V. Note that $U = Y$ so $u \geq 0$. Now as $X = a - Y$ the maximum value Y can take is a as X is also non-negative. So our lower bound is 0 and upper is a so we have,

$$\begin{aligned}
f_V(v) &= \int_0^v \lambda^2 e^{-\lambda v}\ du \\
&= \lambda^2 e^{-\lambda v} [u]_0^v \\
&= \lambda^2 e^{-\lambda v} v
\end{aligned}$$

So this gives us $f_{U|V}(u \mid v) = f_{X|X+Y} = \frac{f(u,v)}{f_V(v)} = \frac{1}{v}$. So here we have $v = a$ which gives us $f_{U|V}(u \mid a) = \frac{1}{a}$ which is uniform in the interval $(0, a)$. Hence knowing $X + Y$ gives us no useful information about the distribution of X in the interval $(0, a)$.

Exercise 6.70

We have X, Y are uniformly distributed on the uni disk so,

$$f_{X,Y}(x,y) = \begin{cases} \pi^{-1} & \text{if } x^2 + y^2 \leq 1, \\ 0 & \text{otherwise} \end{cases}$$

To find $E\sqrt{X^2+Y^2}$ we have,

$$E\sqrt{X^2+Y^2} = \int \int \sqrt{x^2+y^2}\pi^{-1}\ dx\ dy$$

Now changing to polar coordinates we have,

$$
\begin{aligned}
E\sqrt{X^2+Y^2} &= \int_0^{2\pi}\int_0^1 \sqrt{r^2} r\pi^{-1}\ dr\ d\theta \\
&= \int_0^{2\pi}\int_0^1 r^2\pi^{-1}\ dr\ d\theta \\
&= \int_0^{2\pi} [r^3]_0^1 \frac{1}{3}\pi^{-1}\ d\theta \\
&= \int_0^{2\pi} \frac{1}{3}\pi^{-1}\ d\theta \\
&= \frac{1}{3}\pi^{-1}2\pi \\
&= \frac{2}{3}
\end{aligned}
$$

For $E[X^2+Y^2]$ we have

$$
\begin{aligned}
E[X^2+Y^2] &= \int_0^{2\pi}\int_0^1 r^2 r\pi^{-1}\ dr\ d\theta \\
&= \int_0^{2\pi}\int_0^1 r^3\pi^{-1}\ dr\ d\theta \\
&= \int_0^{2\pi} [r^4]_0^1 \frac{1}{4}\pi^{-1}\ d\theta \\
&= \int_0^{2\pi} \frac{1}{4}\pi^{-1}\ d\theta \\
&= \frac{1}{4}\pi^{-1}2\pi \\
&= \frac{2}{4} = \frac{1}{2}
\end{aligned}
$$

Exercise 7.10

We have X is uniformly distributed on (a,b) and we need to find $E(X^k)$. If x is uniformly distributed then the density function is,

$$
f(x) = \begin{cases} \frac{1}{b-a} & \text{if } x \in [a,b] \\ 0 & \text{otherwise} \end{cases}
$$

So we have,

$$
\begin{aligned}
E(X^k) &= \int_a^b \frac{x^k}{b-a}dx \\
&= \frac{1}{b-a}\left[\frac{x^{k+1}}{k+1}\right]_a^b \\
&= \frac{1}{b-a}\left(\frac{b^{k+1}-a^{k+1}}{k+1}\right) \\
&= \frac{b^{k+1}-a^{k+1}}{(b-a)(k+1)}
\end{aligned}
$$

Exercise 7.36

We have $X_1, X_2, \dots$ is a sequence of uncorrelated random variables with variance σ^2 and $S_n = X_1 + \cdots + X_n$. We have assuming $m < n$ that,

$$\begin{aligned}
Var(S_n + S_m) &= Var(S_m) + Var(S_n) + 2Cov(s_m, s_n) \\
2Cov(S_m, S_n) &= Var(S_n + S_m) - (Var(S_m) + Var(S_n)) \\
&= Var(X_1 + \dots X_n + X_1 + \cdots + X_m) \\
&- (Var(X_1 + \dots X_m) + Var(X_1 + \cdots + X_n)) \\
&= Var(2X_1 + \cdots + 2X_m + X_{m+1} + \cdots + X_n) \\
&- (2Var(X_1) + \cdots + 2Var(X_m) + Var(X_{m+1}) + \cdots + Var(X_n)) \\
&= 4(Var(X_1) + \cdots + Var(X_m)) + Var(X_{m+1}) + \cdots + Var(X_m) \\
&- 2(Var(X_1) + \cdots + Var(X_m)) - (Var(X_{m+1}) + \cdots + Var(X_n)) \\
&= 2(Var(X_1) + \cdots + Var(X_m)) \\
&= 2(Var(X_1 + \cdots + X_m)) \\
&= 2Var(S_m)
\end{aligned}$$

So we get,

$$\begin{aligned}
2Cov(S_m, S_n) &= 2Var(S_m) \\
Cov(S_m, S_n) &= Var(S_m)
\end{aligned}$$

And we get,

$$\begin{aligned}
Var(S_m) &= Var(X_1) + \cdots + Var(X_m) \\
&= m\sigma^2 \quad \text{if } m < n
\end{aligned}$$

Problem 6

We have $X_1, X_2, \dots, X_n$ are independent r.v each with distribution function F and density f. $U = \min\{X_1, \dots, X_n\}$ and $V = \max\{X_1, \dots, X_n\}$. The joint cdf of U, V can be seen as,

$$F_{U,V}(u, v) = \mathbb{P}(U \leq u, V \leq v)$$

Now note that for $V \leq v$ we need each $X_n \leq v$ as V is the max so there cannot be a X_n with value greater than v. Now if we have each $X_n \leq v$. For U note that if we need $U > u$ then we need all $X_n > u$ so for $U \leq u$ we have for all X_n that $u \leq X_n \leq v$. So now our solution would be the probability that all $V \leq v$ minus that when $U \geq u$ and $V \leq v$ this is,

$$F_{U,V}(u, v) = (F(v))^n - (F(v) - F(u))^n$$

So,

$$\begin{aligned}
\frac{\partial}{\partial u}\frac{\partial}{\partial v} F_{U,V}(u, v) &= \frac{\partial}{\partial u}\frac{\partial}{\partial v}(F(v))^n - (F(v) - F(u))^n \\
&= \frac{\partial}{\partial u}[n(F(v))^{n-1} f(v) - n(F(v) - F(u))^{n-1} f(v)] \\
&= nf(v)\frac{\partial}{\partial u}[(F(v))^{n-1} - (F(v) - F(u))^{n-1}] \\
&= nf(v)[-(n-1)(F(v) - F(u))^{n-2}(-f(u))] \\
&= n(n-1)f(u)f(v)(F(v) - F(u))^{n-2}
\end{aligned}$$

Problem 20

We have X and Y are r.v with vector (X, Y) uniformly distributed in $R = \{(x, y) : 0 < y < x < 1\}$. We need to find joint probability of (X, Y) first.

Note that the area of the regions in $\int_0^1 \int_y^1 \, dx \, dy = \int_0^1 (1 - y) dy = \frac{1}{2}$. So the joint probability of X, Y is $\frac{1}{\frac{1}{2}} = 2$ for $0 < y < x < 1$. Now to find $\mathbb{P}(X + Y < 1)$. Note that as $x + y < 1$ we have $x < 1 - y$ but we also have $x > y$ so we get $y < x < 1 - y$. Now for bounds on y note that y goes from 0 to 1. But y can't be greater than $\frac{1}{2}$ as that would be we have $x > 1/2$ and $x + y > 1/2$ So we have,

$$\begin{aligned}
\mathbb{P}(X + Y < 1) &= \int_0^{\frac{1}{2}} \int_y^{1-y} 2 \, dx \, dy \\
&= \int_0^{\frac{1}{2}} 2[1 - y - y] \, dy \\
&= \int_0^{\frac{1}{2}} 2[1 - 2y] \, dy \\
&= \int_0^{\frac{1}{2}} 2 - 4y \, dy \\
&= [2y - 2y^2]_0^{\frac{1}{2}} \\
&= 1 - \frac{1}{2} = \frac{1}{2}
\end{aligned}$$

Now to find $f_X(x)$ and EX. We have the joint as 2 and to get f_X we need to integrate over y. As y goes from 0 to x we have,

$$\begin{aligned}
f_X(x) &= \int_0^x 2 \, dy \\
&= 2[y]_0^x \\
&= 2x
\end{aligned}$$

So the expectation of X is $\int_0^1 x \cdot 2x \, dx = \frac{2x^3}{3}]_0^1 = \frac{2}{3}$. Now we have $f_{Y|X}(y \mid x) = \frac{f(x,y)}{f_X(x)} = \frac{2}{2x} = \frac{1}{x}$. So $E(Y \mid X = x) = \int_0^x y \frac{1}{x} dy = \frac{y^2}{2x}]_0^x = \frac{x^2}{2x} = \frac{x}{2}$ for $0 < x < 1$