Linear Algebra 5D

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5D

Problem 1

Proof. (a). We have $T^4 - I = 0$. Factorizing we get,

$$(T^2 + I)(T^2 - I) = (T + Ii)(T - Ii)(T + I)(T - I) = 0$$

We see that the eigenvalues are distinct which means that it is diagonizable.
(b). We have $T^4 - T = 0$. Let the polynomial be,

$$z(z^3 - 1) = z(z - 1)(z^2 + z + 1) = 0$$

We see again the roots are distinct which means that T is diagonalizable.
(c). We have $T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. The only eigenvalue is 0 but $T^2 = T^4$ □

Problem 2

Proof. If A is a diagonal matrix with respect to some basis of V that means that the basis $v_1, \ldots, v_n$ are eigenvectors of T with associated eigenvalues. We know that $v_1, \ldots, v_n$ are linearly independent and span V. Assume that λ_i appear n_i times and $n_i \neq \dim(E(\lambda_i, T))$. Now as $n_1 + \cdots + n_m = n$ given there are m distinct eigenvalues. That means that there exists some j such that $n_j > E(\lambda_j, T)$. So we have n_j linearly independent vectors associated with λ_j which means that they are all in $E(\lambda_j, T)$. But this means that $\dim(E\lambda_j, T) >= n_i$ which is a contradiction. □

Problem 3

Proof. We need to show that $V = nullT \oplus rangeT$.
If T is diagonalizble that means that we can write,

$$V = E(0, T) \oplus E(\lambda_1, T) \cdots \oplus E(\lambda_n, T)$$

Such that any $v \in V = u + v_1 + \cdots + v_n$.
We know that $E(0, T) = nullT$ so we have,

$$V = nullT \oplus E(\lambda_1, T) \cdots \oplus E(\lambda_n, T)$$

Let $U = E(\lambda_1, T) \cdots \oplus E(\lambda_n, T)$. Now we need to show that range $T = U$.
Let $Tv \in rangeT$. So there is $v \in V, v = u + v_1 + \cdots + v_n$ s.t. $Tv \in rangeT$.
So we have $Tv = T(u) + Tv_1 + \cdots + Tv_n = Tv_1 + \cdots + Tv_n =_1 v_1 + \cdots + \lambda_n v_n \in U$. SO we have range $T \subseteq U$.
Now consider $v_1 + \cdots + v_n \in U$. We need to show there is some $v \in V$ such that $Tv = v_1 + \cdots + v_n$. Consider $v =_1^{-1} v_1 + \cdots + \lambda_n^{-1} v_n$. We see that $Tv = v_1 + \cdots + v_n \in rangeT$. Hence $U \subseteq rangeT$.
This shows that range $T = U$

□

Problem 4

Proof. $a \Rightarrow b$ by definition.
$b \Rightarrow c$
We have $V = nullT + rangeT$. We also know that $dimV = dimnullT + dimrangeT = dimnullT + dimrangeT - dimnullT \cap rangeT \Rightarrow dim(nullt \cap rangeT) = 0 \Rightarrow nullT \cap rangeT = \{0\}$ $\square$

Problem 6

Proof. We have $E(8, T) = 4$. Assume the contrary that $T - 2I$ and $T - 6I$ are not-invertible. This means that $\dim(E(2, T)) \geq 1$ and $\dim(E(6, T)) \geq 1$. But that means $\dim V = 4 + 1 + 1 = 6 \neq 5$. Which is not true. $\square$

Problem 7

Proof. If λ is an eigenvalue of T that means,

$$Tv = \lambda v$$

$$T^{-1}Tv = \lambda T^{-1}v$$

$$T^{-1}v = \frac{1}{\lambda}v$$

which makes $\frac{1}{\lambda}$ an eigenvalue of T^{-1} such that for every $v \in E(\lambda, T), v \in E(\lambda^{-1}, T^{-1})$ $\square$

Problem 8

Proof. So we have

$$\dim V \geq \dim E(0, T) + \cdots + \dim E(\lambda_m, T)$$

But we know that $nullT = E(0, T)$ so,

$$rangeT \geq \dim E(\lambda_1, T) \cdots + \dim E(\lambda_m, T)$$

$\square$

Problem 9

Proof. We are given that R and S have three eigenvalue. Let it be defined as,

$$Ru_1 = 2u_1, Ru_2 = 6u_2, Ru_3 = 7u_3$$

and,

$$Tv_1 = 2v_1, Tv_2 = 6v_2, Tv_3 = 7v_3$$

Now we need to define S as follows,

$$Su_1 = v_1, Su_2 = v_2, Su_3 = v_3$$

So we have $S^{-1}TSu_1 = S^{-1}Tv_1 = S^{-1}2v_1 = 2u_1$ $\square$

Problem 11

Proof. Consider T is defined as,

$$\begin{bmatrix} 6 & 1 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

$\square$

Problem 14

Proof. (a). Consider $T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$
(b). Assume T is diagonalizable, that means that the diagonal entires of matrix of T are $\lambda_1, \ldots, \lambda_n$. So T^k will be $\lambda_1^k, \ldots, \lambda_n^k$. Hence T^k is also diagonalizable.
Now assume T^k is diagonalizable. Which means that it has a minimal polynomial $p(z) = (z - \lambda_1) \ldots (z - \lambda_m)$ where $\lambda_1, \ldots, \lambda_m$ are the distinct roots. As T^k is invertible these roots are non-zero. Now consider the 6 th root of any z. And we construct,

$$q(z) = (z^k - \lambda_1) \ldots (z^k - \lambda_n)$$

Now for each $(z^k - \lambda_1)$ we can write this as a product of $(z - u_1) \ldots (z - u_k)$ such that each $u_1, \ldots, u_k$ is distinct.
Now all this means that the mnimal polynomial of T has distinct factors which makes it diagonlizable.

$\square$

Problem 15

Proof. $a \Rightarrow b$
If T is diagonalizable then that means that the minimal polynomial of T has distinct roots. So there is no λ such that p is a polynomial multiple of $(z - \lambda)^2$
$b \Rightarrow c$ Assume they have zeroes in common which means that,

$$p(z) = (z - \lambda)q(z)$$

and,

$$p'(z) = (z - \lambda)r(z)$$

Differentiating first one we have,

$$p'(z) = (z - \lambda)q'(z) + q(z) = (z - \lambda)r(z)$$

Evaluating at $z = \lambda$ we get,

$$q(z) = 0$$

which means that λ is a zero of q or,

$$q(z) = (z - \lambda)s(z)$$

So $p(z) = (z - \lambda)^2 s(z)$
but p has distinct zeroes so contradiction.
$c \Rightarrow d$ Let us assume that is not the case, so $\exists q$ such that,

$$p = kq$$

and,

$$p' = k'q$$

So this means that p and q share the same zeroes and p' and q share the same zeores which means that p and p' share the same zeores which contradicts our previous conclusion.

$\square$