Linear Algebra 5C

Aamod Varma

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5C

Problem 1

Proof. If $F = C$ then T has an upper triangular matrix regardless.
If $F = R$ then consider $T(x, y) = (-y, x)$. We have $T^2 + I = 0$. So the minimal polynomial of T does not have any real eigenvalues.
However $T' = T^2$ has the upper triangular matrix $-I$ □

Problem 2

Proof. (a). We have $(A + B)_{jk} = A_{jk} + B_{jk}$
As both A and B are upper triangular matrices we know that $A_{jk} = B_{jk} = 0$ for $j > k$. Hence $(A+B)_{jk} = 0$ for $j > k$. And $(A+B)_{kk} = A_{kk}+B_{kk} = \alpha_k + \beta_k$
(b). We have,

$$(AB)_{jk} = \sum_{n=1}^{m} A_{jn}B_{nk}$$

First consider if $j > k$. We have,

$$\sum_{n=1}^{k} A_{jn}B_{nk} + \sum_{n=k+1}^{m} A_{jn}B_{nk}$$

In the first sum we have $j > k$ and $k > n$ so $j > n$ which means that $A_{jn} = 0$ so the sum goes to zero. In the second sum we have $n > k$ so $B_{nk} = 0$ so that goes to zero. Hence the sum is always 0 for any choice of j, k if $j > k$.
This shows that AB is an upper triangular matrix.
Now if $j = k$ we have,

$$(AB)_{kk} = \sum_{n=1}^{m} A_{kn}B_{nk} = A_{kk}B_{kk} = \alpha_k\beta_k$$

□

Problem 3

Proof. (1). We know if T is an upper triangular matrix then the minimal polynomial of T can be written as $(z - \lambda_1) \dots (z - \lambda_n)$. We also know that if T is invertible then its minimal polynomial is,

$$(z - \frac{1}{\lambda_1}) \dots (z - \lambda_n)$$

Because it is of this form we can create an upper triangular matrix with the reciprocals on the diagonal.
(2). The existance of an upper triangular matrix for T implies that for the

basis $v_1, \dots, v_n$ we can write,

$$\begin{aligned} Tv_1 &= \lambda_1 v_1 \\ Tv_2 &= a_1 v_1 + \lambda_2 v_2 \\ &\dots \\ Tv_n &= b_1 v_1 + \dots + \lambda_n v_n \end{aligned}$$

Now let us apply T^{-1} on each side and we get,

$$\begin{aligned} v_1 &= \lambda_1 T^{-1} v_1 \\ v_2 &= a_1 T^{-1} v_1 + \lambda_2 T^{-1} v_2 \\ &\dots \\ v_n &= b_1 T^{-1} v_1 + \dots + T^{-1} \lambda_n v_n \end{aligned}$$

Rearranging the term we get,

$$\begin{aligned} T^{-1} v_1 &= \frac{v_1}{\lambda_1} \\ T^{-1} v_2 &= \frac{v_2}{\lambda_2} - \frac{a_1}{\lambda_2} T^{-1} v_1 \\ &\dots \\ T^{-1} v_n &= \frac{v_n}{\lambda_n} - \frac{b_1}{\lambda_n} T^{-1} v_1 + \dots \end{aligned}$$

Going from the beginning we have $T^{-1}v_1 \in span(v_1), T^{-1}v_2 \in span(v_1, v_2)$ and going forward we get $T^{-1}v_k \in span(v_1, \dots, v_{k-1})$. This makes our matrix upper triangular.
We see that for any $k \in \{1, \dots, n\}$ that the term before v_k for $T^{-1}v_k$ is $\frac{1}{\lambda_k}$.
Hence our diagonal is $\frac{1}{\lambda_k}$ for the v_k.

□

Problem 6

Proof. If $F = C$ that means that there exists an upper triangular matrix with respect to some basis of V. Let this basis be $v_1, \dots, v_n$ where $n = \dim V$.
Now this means that for any k, $span(v_1, \dots, v_k)$ is invariant under T as $T(v_k) \in span(v_1, \dots, v_k), T(v_{k-1}) \in span(v_1, \dots, v_{k-1}), \dots$ □

Problem 7a

Proof. (a). Consider the list $(v, Tv, \dots, T^{\dim V} v)$. As the dimension is $\dim V{+}1$ there is some smallest k such that $T^{k+1} \in span(v, \dots, T^k)$. Which makes $U = span(v, \dots, T^k)$ invariant under T. Let p_v be the minimal polynomial of $T_{|U}$ so we see that,

$$p_v(T)v = p_v(T_{|U})v = 0$$

We know that the degree cannot be smaller than k so it is of least degree. □

Problem 8b

Proof. We have $T^2v + 2Tv + 2v = 0$
So the minimal polynomial is either $z^2+2z+2 = 0$ or a polynomial multiple of it whose roots are $-1+i$ or $-1-i$.
Which means that eigenvalues of T are the same so it must appear on the diagonal of A. □

Problem 9

Proof. Now let T be the linear map associated with B. Because $F = C$ there exists some basis of V in which there is a linear map C which is upper triangular. Let this basis be $v_1, \dots, v_n$. If B is a matrix defined on the basis $u_1, \dots, u_n$. Then we can define $A = M(T, (v_1, \dots, v_n), (u_1, \dots, u_n))$ such that $A^{-1}BA = M(T, (v_1, \dots, v_n), (v_1, \dots, v_n)) = C$. □

Problem 10

Proof. $a \Rightarrow b$
If the matrix is lower triangular then we can say that,

$$\begin{aligned} Tv_1 &= a_1v_1, \dots, a_nv_n \\ &\dots \\ T(v_n) &= b_nv_n \end{aligned}$$

So we see that for any j we have $Tv_j \in span(v_j, \dots, v_n)$ but $span(v_j, \dots, v_n) \subset span(v_k, \dots, v_n)$. So for any $v \in span(v_k, \dots, v_n)$ we have $Tv \in span(v_1, \dots, v_n)$ which makes the span invariant.
$b \Rightarrow c$ If the span is invariant then it follows.
$c \Rightarrow b$ If c is true that means that we can write $Tv_1, \dots, Tv_k$ in the way we wrote above which means we can make a lower triangular matrix. □