Linear Alebgra HW05

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Proof. We know for a linear map, $T(u+v) = T(u) + T(v)$ and $T(\lambda v) = \lambda T(v)$
First we look at additivity,
Consider an arbitrary $u = (x_1, y_1, z_1)$ and $v = (x_2, y_2, z_2)$. So we have,

$$T(u+v) = T((x_1+x_2), (y_1+y_2), (z_1+z_2))$$

$= (2(x_1+x_2)-4(y_1+y_2)+3(z_1+z_2)+b, 6(x_1+x_2)+c(x_1+x_2)(y_1+y_2)(z_1+z_2))$

We need the above to be equal to,

$$T(u)+T(v) = (2x_1-4y_1+3z_1+b, 6x_1+cx_1y_1z_1)+(2x_2-4y_2+3z_2+b, 6x_2+cx_2y_2z_2)$$

$= (2(x_1+x_2)-4(y_1+y_2)+3(z_1+z_2)+2b, 6(x_1+x_2)+c(x_1y_1z_1+x_2y_2z_2)$

Comparing each of the terms we have,

$$2(x_1+x_2)-4(y_1+y_2)+3(z_1+z_2)+2b = 2(x_1+x_2)-4(y_1+y_2)+3(z_1+z_2)+b$$

$$2b = b$$

$$b = 0$$

Similarly comparing the second term we have,

$$6(x_1+x_2)+c(x_1+x_2)(y_1+y_2)(z_1+z_2) = 6(x_1+x_2)+c(x_1y_1z_1+x_2y_2z_2)$$

$$c(x_1+x_2)(y_1+y_2)(z_1+z_2) = c(x_1y_1z_1+x_2y_2z_2)$$
$$c((x_1+x_2)(y_1+y_2)(z_1+z_2)-(x_1y_1z_1+x_2y_2z_2)) = 0$$

For this to be true for any x, y, z we need $c = 0$. Hence for additvity we need $b = c = 0$
Now we check if $T(kv) = kT(v)$. Consider $v = (x, y, z)$. Then we have

$$T(kv) = T(kx, ky, kz) = (2kx - k4y + 3kz + b, 6kx + k^3cxyz)$$

We need this to be equal to

$$kT(v) = k(2x-4y+3z+b, 6x+cxyz) = (2kx-4ky+3kz+bk, 6kx+kcxyz)$$

Comparing the terms we have,

$$2kx - 4ky + 3kz + bk = 2kx - 4ky + 3kz + b$$

$$bk = b$$

$$b = 0$$

$6kx + kcxyz = 6kx + k^3cxyz$

$$c = k^2c$$

$$c = 0$$

So we have $b = c = 0$

□

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Proof. 1. Associativity. We have $(T_1T_2)T_3 = T_1(T_2T_3)$
Consider the operation on a vector v so we have, $(T_1T_2)T_3v$ which is,

$$((T_1T_2)(T_3(v)) = T_1(T_2(T_3(v)))$$

Now looking at the right side we have, $T_1(T_2T_3) = T_1(T_2(T_3(v)))$. So we showed tha tthe LHS is equal to the RHS.
2. Identity. Consider a vector v we have,

$$TIv = T(I(v)) = T(v)$$

Now,
$$ITv = I(T(v)) = T(v) \text{ because } Iv = v, \forall v$$

3. Distributive Property
To show that,
$$(S_1 + S_2)T = S_1T + S_2T$$

Consider an abitrary vector v in the domain of T. We have,

$$(S_1 + S_2)Tv = (S_1 + S_2)(T(v))$$

By definitino of addition of linera maps we have,

$$= (S_1(T(v))) + (S_2(T(v)))$$

Simliary we have,

$$(S_1T + S_2T)v = S_1T(v) + S_2T(v) = S_1(T(v)) + S_2(T(v))$$

We see that the distributive property holds.
Now To show that $S(T_1 + T_2) = ST_1 + ST_2$. Consider v we have,

$$S(T_1 + T_2)v = S(T_1(v) + T_2(v)) = S(T_1(v)) + S(T_2(v))$$

And we have,

$$(ST_1 + ST_2)v = ST_1(v) + ST_2(v) = S(T_1(v)) + S(T_2(v))$$

We see that the property holds again.

□

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Proof. Let V be a one dimentional vector space. This means that the basis of V contains a single vector, let the basis be $\{v\}$. Now we are considering a linear map from V to itself.
So assume that the linear map T maps some v_0 in V to w_0. We need to show that $w_0 = \lambda v_0$ for some $\lambda \in F$. Because T maps V to itself we known that that $w_0 \in V$ for any w_0. If $w_0 \in V$ then wek now that it can be written as a linear compbination of its basis. As the basis only has one vector we can write $w_0 = \lambda_1 v$. Similarly as $v_0 \in V$ we can write $v_0 = \lambda_2 v$. So we have,

$$\frac{v_0}{\lambda_2} = v$$

$$w_0 = \lambda_1 \frac{v_0}{\lambda_2} = \lambda v_0$$

□

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Proof. Consider the function that maps any vecotor (x, y) to the $max(|x|, |y|)$. We can see that this satisfies homogeneity. For instance consider $(2, 6)$. Our function maps this to 6. Now consider $(2 \times 3, 6 \times 3)$ which is mapped to 18 which is 3×6 as we saw above.
Now consider two vector $(1, 0)$ and $(0, 4)$. Our function maps both these vectors to 1 and 4 respectively. However it maps its sum $(1, 4)$ to $4 \neq 4+1$. Hence it does not follow additvity. Hence not a linear space. □