Chapter 11: Relations

Relation: $R \subseteq A \times A$. $(x, y) \in R$ is xRy

Reflexive: xRx. Symmetric: $xRy \Rightarrow yRx$.

Transitive: $xRy, yRz \Rightarrow xRz$.

Equivalence Relation: Reflexive, Symmetric and Transitive

Equivalence class containing a is the subset $\{x \in A : xRa\}$ of A consisting of all elements of A that relate to a. Denoted by $[a]$

$$[a] = \{x \in A : xRa\}$$

$$[a] = [b] \iff aRb$$

Partition: Non-empty subsets of A such that union of all subsets equal A and intersection of any is ϕ

Integer Modulo n: For $n \in N$ equivalence classes of the relation $\pmod{n}$ are $[0], \dots, [n-1]$ The integers modulo n is the set $Z_n\{[0], [1], \dots, [n-1]\}$. Following hold,

$$[a] + [b] = [a + b], [a] \cdot [b] = [ab]$$

If we have $[a][b] = [0]$ and the classes are for integer mod n. If n is prime then either $[a] = [0]$ or $[b] = [0]$. If n is composite then a or b could be its factors.

Chapter 12: Functions

Function: $f : A \to B$ is a relation $f \subseteq A \times B$ s.t. $\forall a \in A$ there is exactly one ordered pair $(a, b) \in f$ or $f(a) = b$.

- A is domain
- B is co domain
- $\{f(a) : a \in A\}$ is range

A function $f : A \to B$ is,

- injective: $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ OR $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$
- surjective: $\forall y \in B, \exists x \in A, f(x) = y$
- bijective: injective and surjective

Pigeonhole Principle Given $f : A \to B$

- If $|A| > |B|$ then f is not injective.
- If $|A| < |B|$ then f is not surjective.

Examples:

1. Show if $a \in, \exists k, l$ s.t. $10 | a^k - a^l$

$A = N$ and $B = \{0, \dots, 9\}$ and the function $f : A \to B$ such that it maps $k \in A$ to the last digit of a^k which will be in B.

2. Show any set of 7 integers contain pair of integers whose sum or difference is divisible by 10.

$A = \{a_1, \dots, a_7\}$ and $B = \{\{0\}, \{1, 9\}, \{2, 8\}, \{3, 7\}, \{4, 6\}, \{5\}\}$. Let $f : A \to B$ such that it maps any of the numbers to the set in B that contain its last digit.

Composition

If $f : A \to B$ and $g : B \to C$ then, $g \circ f : A \to C$ is $g(f(x))$

- $(h \circ g) \circ f = h \circ (g \circ f)$

Properties, let $f : A \to B$, $g : B \to C$ consider $g \circ f$

1. Show f is injective if $g \circ f$ is injective.

Assume f is not injective, so $\exists a_1 \neq a_2$ s.t. $f(a_1) = f(a_2)$. Now, $g(f(a_1)) = g(f(a_2))$ but as $g \circ f$ is injective this implies $a_1 = a_2$ which contradicts our assumption.

2. Show g is surjective if $g \circ f$ is surjective.

Definition implies that $\forall c \in C, \exists a \in A$ s.t. $g(f(a)) = c$. Let $f(a) = b \in B$. So we have $\forall c \in C, \exists b \in B, g(b) = c \Rightarrow g$ is surjective.

3. Show f, g is bijective $\Rightarrow g \circ f$ is bijective.

(a). Injectivity: Consider $g(f(a_1)) = g(f(a_2))$. As g is injective we have $f(a_1) = f(a_2)$, as f is injective we have $a_1 = a_2 \Rightarrow g \circ f$ is injective.

(b). Surjectivity: As g is surjective $\forall c \in C, \exists b \in B$ s.t. $g(b) = c$. As f is surjective $\forall b \in B, \exists a \in A$ s.t. $f(a) = b$. So we have $\forall c \in C, \exists a \in A$ s.t. $g(f(a)) = c$ **Inverse**

$f : A \to B$ is bijective $\iff f^{-1}$ is a function from $B \to A$

If $A \to B$ is bijective the inverse is $f^{-1} : B \to A$ such that $f \circ f^{-1} = i_A$ and $f^{-1} \circ f = i_A$

Image and Preimage

If $f : A \to B$ then,

- If $X \subseteq A$ image of X is $f(X) = \{f(x) : x \in X\} \subseteq B$
- If $Y \subseteq B$ preimage of Y is $f^{-1}(Y) = \{x : f(x) \in Y\} \subseteq A$

Examples $(f : A \to B)$,

1. $f(f^{-1}(Y)) \neq X$ in general

If f is not injective the pre-image could contain a_1 and a_2 but X could only contain a_1.

2. f is injective $\iff X = f^{-1}(f(X)), \forall X \subseteq A$ To show backward direction assume f is not injective and construct X such that it is not true.

Chapter 14: Cardinality

$|A| = |B| \iff \exists f : A \to B$ and f is bijective.

1. $|N| = |Z|$: $f : N \to Z$, $f(n) = \frac{(-1)^n(2n-1)+1}{4}$

2. $|N| \neq |R|$: We can show by using diagonal table.

3. $|(0, \infty)| = |(0, 1)|$: Let $f(x) = \frac{x}{x+1}$

4. $|R| = |(0, 1)|$: $|R| = |(0, \infty)|$ by $g(x) = 2^x$ then we use (3.)

5. $|Q| = |N|$. We show $|Q^+| = |N|$ and $|Q^{-1}| = |N|$ and use union.

Countable And Uncountable Sets

$|N| \neq |R|$ as there is not bijection from $N \to R$

A is **countably infinite** if $|N| = |A|$ or if there is a bijection from N to A else its uncountable.

A set A is countable infinite $\iff$ its elements can be arranged in an infinite list $a_1, a_2, \dots$

Eg.

We can show that Q is countable by plotting a 2x2 graph of all rationals and drawing a snake path from the top left which represents the list $a_1, a_2, \dots$

If A and B are countably infinite then so is $A \times B$. True because we can draw 2x2 "matrix" and draw snake from top left indicating each of the elements.

If A and B are countably infinite, their union in countable infinite.

Power Set: $|A| < |P(A)|$

We show this by constructing a $B = \{x \in A : x \notin f(x)\}$. There is no $x \in A$ that belongs to this set hence there cannot be a surjection so no bijection. Implies $|A| < |P(A)|$ as we have an injection but no bijection.

Functions to use for bijections in uncountable sets

1. $\ln(x)$ maps from R^+ to R as if its smaller than 1 its negative.
2. $e^x, 2^x$ maps from R to R^+
3. $\frac{kx}{x+1}$ maps from $(0, \infty)$ to $(0, k)$.
4. Diagonal argument can be used for uncountable sets (set of infinite sequences, reals, etc)
5. $R \to R \times R$. Injection from R to $R \times R$ we have $f(x) = (x, 0)$. For injection from $R \times R$ to R we can interleave the decimals for a, b in $(a, b) \in R \times R$ to create a new decimal for R.
6. $[0, 1) \to (0, 1)$. $f(x) = \frac{1}{4} + \frac{1}{2}x$ from $[0, 1)$ to $(0, 1)$
7. $R \times R = \{(x, y) : aconditiononx, y\}$. Easy to show injection to right to left. For left to right we can use $\frac{1}{1+x}$ and map to a square than can fit in our contour.

Theorems

1. A is countable if we can list the elements of A as $a_1, a_2, \dots$
2. A and B are countable then $A \times B$ are countable.
3. An infinite subset of a countably infinite subset is countably infinite.