Chapter 1: Set Theory

1. **Cardinality** of a set X is $|X|$: Size of the set, or the number of elements in it.
 - $|\phi| = 0$ where $\phi = \{\}$ - no elements in it. $\phi \neq \{\phi\}$
 - $|\{\phi\}| = 1$

2. Set Builder notation: $X = \{expression : rule\}$
 - $E = \{n \in Z : n = 2k, k \in N\}$

3. **Intervals**: Closed: [a,b] - $a \leq x \leq b$; Open: (a,b) - $a < x < b$, Left-open: (a,b] - $a < x \leq b$; Right open: [a,b) - $a \leq x < b$; Infinite: (a,∞].

1.1 Cartesian Product

1. $A \times B = \{(a, b) : a \in A, b \in B\}$
2. $|A \times B| = mn$ if $|A| = m, |B| = n$
3. $A^n = A \times A \times \ldots \times A = \{(x_1, \ldots, x_n) : x_1, \ldots, x_n \in A\}$

1.3 Subsets

1. $\phi \subseteq A, \forall A$ as a set but $\phi \in A$ is not always true.
2. $N \subseteq Z \subseteq Q \subseteq R$
3. If $|A| = n$ then it has 2^n subsets.
 - $|P(A)| = 2^n$

1.4 Powersets

1. $P(A) = \{X : X \subseteq A\}$
2. $|P(A)| = 2^{|A|}$

Complement: $\bar{A} = U - A$ where U is a universal set.

Division Algorithm: Given $a, b, b > 0, \exists$ unique q, r s.t. $a = qb+r$ and $0 \leq r < b$

Chapter 2: Logic

1. A **statement** is a sentence that is either deinitely true or definitely false.
 - $5 = 2$ is a false statement
 - $2 \in Z$ is a true statement
 - The interger x is even is **not** a statement - open sentence

2.2 And, Or, Not

1. AND: $\wedge$, OR: $\vee$, NOT: $\neg A$
2. $P \wedge Q$: $TT : T, TF : F, FT : F, FF : F$
3. $P \vee Q : TT : T, TF : T, FT : T, FF : F$
4. $\neg P : T : F, F : T$

2.3 Conditionals

1. $P \Rightarrow Q : TT : T, TF : F, FT : T, FF : T$. Only false if true implies false.
2. $P \iff Q : P \Rightarrow Q \wedge Q \Rightarrow P$. Only true if both true/false
3. $P \iff Q \equiv (P \wedge Q) \vee (\neg P \wedge \neg Q)$

Demorgans

1. $\neg(P \wedge Q) = (\neg P) \vee (\neg Q)$
2. $\neg(P \vee Q) = (\neg P) \wedge (\neg Q)$

Other laws: Contrapositive - $(\neg Q) \Rightarrow (\neg P)$; Commutative - $P \wedge Q = Q \wedge P$; Distributive - $P \wedge (Q \vee R) = (P \wedge Q) \vee (P \wedge R)$

2.7 Quantifiers

1. For all (Universal quantifier): $\forall$, There exists (Existential quantifier): $\exists$
2. Every integer that is not odd is even is $\forall n \in Z, \neg$(n is odd) $\Rightarrow$ (n is even)
3. Order is relevant. $\forall x \in R, \exists y \in R, y^3 = x$ is true but $\exists y \in R, \forall x \in R, y^3 = x$ is obviously false.

2.10 Negations

1. $\neg(P \wedge Q) = \neg P \vee \neg Q$
2. $\neg(\forall x \in N, P(x)) = \exists x \in N, \neg P(x)$
3. $\neg(\forall x \in R, \exists y \in R, y^3 = x) = \exists x \in R, \forall y \in R, y^3 \neq x$

Chapter 4: Direct Proofs

1. Even: $n = 2a$, Odd: $n = 2a + 1$
2. Two integers are both even - same parity, both odd - opposite parity
3. $a|b$ if $b = ac, c \in Z$ - b is a multiple of a
4. n is composite $\iff n = ab, 1 < a, b < m$
5. $gcd(a, b)$ largest integer that divides a and b.

Chapter 5: Contrapositive Proof

1. $P \Rightarrow Q \equiv \neg Q \Rightarrow \neg P$ **5.2 Congruence**
2. $a \equiv b \pmod{n}$ if $n|(a - b)$
 - $9 \equiv 1 \pmod{4}$ as $4|9 - 1$
 - basically means a and b have the same remainder when divided by n

Chapter 6: Proof by contradiction

1. $P \equiv (\neg P) \Rightarrow (C \wedge \neg C)$
 - to prove P we can just assume $\neg P$ and come to a known false statement
2. To show $P \Rightarrow Q$. We suppose P and $\neg Q$ and prove a false statement

Chapter 7: Non-Conditional Statements

1. $P \iff Q \equiv P \Rightarrow Q \wedge Q \Rightarrow P$

Existance/Uniqueness Proof

1. To prove $x, R(x)$: We just need one example.

Chapter 8: Sets

1. To show $A \subseteq B$ we take $x \in A$ and show $x \in B$. Or $x \notin B$ and show $x \notin A$
2. To show $A = B$ we show $A \subseteq B$ and $B \subseteq A$

Perfect Numbers

1. 6 is a perfect number beacuse $6 = 1 + 2 + 3$

Chapter 9: Disproof

1. To disprove P : Prove $\neg P$
2. Disprove $P(x) \Rightarrow Q(x)$: We only need to give one example where this isn't true.
3. To disprove $\exists x \in S, P(x)$: We need to show $\forall x \in S, \neg P(x)$
4. Disprove P with contradiction: Assume P, deduce a contradiction.

Chapter 10: Induction

Proposition: $S_1, S_2, \ldots$ are all true.
Proof. (1). Prove that S_1 is true.
(2). Given $k \geq 1$, prove that $S_k \Rightarrow S_{k+1}$ is true.
By inductino this shows that all S_n is true.

Strong Induction

(1) Prove S_1
(2) Given $k \geq 1$, prove $S_1 \wedge S_2 \ldots \wedge S_k \Rightarrow S_{k+1}$

Choosing stuff

1. $\binom{n}{m} = \frac{n!}{m!(n-m)!}$
2. $\binom{n+1}{m} = \binom{n}{k-1} + \binom{n}{k}$ or $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

Bezout's Identity: If $a, b \in Z$ with $gcd(a, b) = d$ then $\exists x, y \in Z$ s.t $ax + by = d$

Modular Arithmetic: $m^2 + 3n^2 \equiv 2 \pmod{4}$