Real Analysis: HW10

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November 16, 2025

Exercise 6.2.3
(a). For g_n we see that if $x \in [0,1)$ then $x^n \to 0$, if $x = 1$ then $x^n \to x$ and if $x \in (1, \infty)$ then $x^n \to \infty$. So considering these three cases we have, $(g_n) \to g$ where,

$$g(x) = \begin{cases} \text{x}, & x \in [0,1) \\ \frac{1}{2}, & x = 1 \\ 0, & x \in (1, \infty) \end{cases}$$

For case 1 note, we have $\left|\frac{x}{1+x^n} - x\right| < |\frac{x^{n+1}}{x^n+1}| < |x^{n+1}|$ and as we can make x^n arbitrarily small we can also make x^{n+1} and hence for any ε we can find N to make it arbitrarily smaller than ε. For case 2, we have $g_n(x) = \frac{1}{1+1^n} = \frac{1}{2}$ which just the sequence of the constant which obviously converges to the same constant. For case 3 we have $\left|\frac{x}{1+x^n} - 0\right| < \left|\frac{x}{x^n}\right| = |\frac{1}{x^{n-1}}|$. Note this goes to 0 as $n \to \infty$ and hence the limit goes to zero.

Note as $n \to \infty$ we have $\frac{1}{n} \to \infty$. So, $\forall \varepsilon > 0$ we have N s.t if $n > N$ then $|\frac{1}{n}| < \varepsilon$. However note that $h_n(x)$ takes on the value nx only if $0 \le x < \frac{1}{n}$. So we have for any $\varepsilon > 0$ some N for which we get $0 \le x < \frac{1}{n} < \varepsilon$ but this implies that $x = 0$. So the rest of the cases we have $x > \frac{1}{n}$ for arbitrarily large n and hence in that case we have the constant function 1 which converges to 1. This give us,

$$h_n(x) = \begin{cases} 1 & \text{if } x \in (0, \infty) \\ 0 & \text{if } x = 0 \end{cases}$$

(b) First note that for any n we have $g_n(x)$ is a continuous function (using algebraic continuity tearoom) and note that $g(x)$ is a piece wise function which is not continuous (for instance has a jump at $x = 1$ as limit from left is 1 but function is equal to $\frac{1}{2}$ at $x = 1$). Now, if the convergence were to be uniform on $[0, \infty)$ then that would imply that $g(x)$ will retain the continuity of $g_n(x)$, however we know $g(x)$ is not continuous, hence the convergence cannot be uniform.

Similarly we see that $h_n(x)$ is continuous (at point $x = \frac{1}{n}$ note limit from leftside is also just 1 as we have $nx = 1$) but $h(x)$ is not as limit at 0 is 1 but the function is equal to 0 at that point. Hence convergence cannot be uniform as h would also have to be uniform which it is not.

(c). For $g_n(x)$ consider the set $[0, k)$ where $k \in (0,1)$. In this set we have $g_n(x)$ is continuous and the convergence to $g(x) = x$ is uniform.

For all $x \in [0, k)$ we see,

$$\begin{aligned}\left|\frac{x}{1+x^n} - x\right| &= \left|\frac{x^{n+1}}{x^n+1}\right| \\ &\leq \left|\frac{x^{n+1}}{1}\right| \\ &\leq \left|t^{n+1}\right| < |t^n|\end{aligned}$$

Now note that we can choose n arbitarrily large to make $t^n < \varepsilon$ for any ε. More specifically choose $N = \frac{\log \varepsilon}{\log t} = log_t \varepsilon$ and for $n > N$ we get $\left|\frac{x}{1+x^n} - x\right| \leq |t^n| < \varepsilon$.

This gives us uniform convergence to $g(x) = x$ which is also continuous in $(0, k)$.

For $h_n(x)$ choose the domain as $(1, \infty)$. Note that for any choice of $n > 1$ we have $\frac{1}{n} < 1$ which means as $x \in (1, \infty)$ we have $x > \frac{1}{n}$ or $h_n(x) = 1$. Now trivially this is just constant for every n and hence uniformly convergence for any $N > 1$ as if we take $h(x) = 1$ then we have $|h_n(x) - h(x)| = 0$ for all $n > 1$.

Exercise 6.2.8 We have (g_n) a sequence of continuous functions that converge uniformly to g on a compact set K. Note the following,

$$\begin{aligned}\left|\frac{1}{g_n} - \frac{1}{g}\right| &= \left|\frac{g_n - g}{g_n g}\right| \\ &= |g_n - g|\left|\frac{1}{g_n g}\right|\end{aligned}$$

First note that (g_n) converges uniformly to g and as all g_n is continuous we have g is continuous. As K is compact we also have $g(K)$ is compact and hence g is bounded in K which means it attains a max and min. Let the min be m which means $g(x) \geq m$ for all $x \in K$ or that $\frac{1}{g(x)} \leq m$ for all $x \in K$.

Now to bound g_n. We know g_n converges uniformly to g. So for any ε for large enough N we have for $n > N$ that $|g_n - g| < \varepsilon$ which means $g - \varepsilon < g_n < g + \varepsilon$. But we have $m \leq g$ so $m - \varepsilon \leq g - \varepsilon < g_n$ or that $\frac{1}{g_n} < \frac{1}{m-\varepsilon}$. Now if we choose ε small enough so that $m - \varepsilon$ is positive then we have a positive upper bound for g_n or that $\left|\frac{1}{g_n}\right| < \frac{1}{m-\varepsilon}$ this gives us,

$$\begin{aligned}\left|\frac{1}{g_n} - \frac{1}{g}\right| &= |g_n - g|\left|\frac{1}{g_n g}\right| \\ &< |g_n - g|\frac{1}{m(m-\varepsilon)}\end{aligned}$$

Now let $M = \frac{1}{m(m-\varepsilon)}$ and as g_n is uniformly continuous to g choose N large enough so that we have $|g_n - g| < \frac{1}{M}\varepsilon$. This gives us,

$$\left|\frac{1}{g_n} - \frac{1}{g}\right| \leq |g_n - g|M \leq \varepsilon\frac{M}{M} = \varepsilon$$

And hence we get uniform convergence as N is independent of x.

Exercise 6.2.9 (a). We have (f_n) and (g_n) are uniformly convergent to f and g. Now note by definition this means we have $\forall \varepsilon > 0$ a N_1 such that if $n > N_1$ then we get,

$$|f_n - f| < \frac{\varepsilon}{2} \quad \forall x$$

and similarly a N_2 such that if $n > N_2$ then we get,

$$|g_n - g| < \frac{\varepsilon}{2} \quad \forall x$$

Now choose $N = \max\{N_1 N_2\}$ and we have both,

$$|f_n - f| < \frac{\varepsilon}{2} \text{ and } |g_n - g| < \frac{\varepsilon}{2}$$

Note this gives us,

$$\begin{aligned} |(f_n + g_n) - (f + g)| &= |(f_n - f) + (g_n + g)| \\ &\leq |f_n - f| + |g_n - g| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

Hence we found $\forall \varepsilon >)$ an $N = \max\{N_1, N_2\}$ independent of x such that if $n > N$ then we have $|(f_n + g_n) - (f + g)| < \varepsilon$ which by definition means we have uniform convergence to $f + g$.

(b). Consider the function $f_n = x^2$ which is trivially uniformly convergent to $f = x^2$ on $[1, \infty)$. Now consider $g_n = \frac{1}{n(1+x)}$ which also is uniform convergent to 0 on $[1, \infty)$. Now consider the product and we have $(f_n g_n) = \frac{x^2}{n(1+x)}$. Now note that does not converge uniformly to 0 on $[1, \infty)$ as it would depend on the value of x now as we have $\left|\frac{x^2}{n(1+x)}\right| \leq \left|\frac{x^2}{nx}\right| = \left|\frac{x}{n}\right| < \varepsilon$ so we need $n > \frac{x}{\varepsilon}$.

(c). Consider the case we have $M > 0$ such that $|f_n| \leq M$ and $|g_n| \leq M$ for all $n \in N$, now note the following,

$$\begin{aligned} |f_n g_n - fg| &= |f_n g_n - f_n g + f_n g - fg| \\ &\leq |f_n g_n - f_n g| + |f_n g - fg| \\ &\leq |f_n| \, |g_n - g| + |g \, ||f_n - f| \end{aligned}$$

With the additional constraint now note that we get $|f_n| \leq M$ and $g \leq M$ which gives us,

$$|f_n g_n - fg| \leq M \, |g_n - g| + M \, |f_n - f|$$

Now as f_n and g_n are uniformly convergent to f and g similar to in (a) we can find for all $\varepsilon > 0$ an N such that if $n > N$ then we get,

$$|f_n - f| < \frac{\varepsilon}{2M} \text{ and } |g_n - g| < \frac{\varepsilon}{2M}$$

So putting the two together we get,

$$
\begin{aligned}
|f_n g_n - fg| &\leq M\,|g_n - g| + M\,|f_n - f| \\
&= M\frac{\varepsilon}{2M} + M\frac{\varepsilon}{2M} \\
&= \varepsilon
\end{aligned}
$$

Hence we show that $f_n g_n$ is uniformly convergent to fg.