

Linear Algebra HW06

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3C

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Proof. Consider a case where $\text{range } T = W$. In this case let w_1, \dots, w_m be a basis for W which also would be a basis for $\text{range } T$, we can find v_1, \dots, v_n such that,

$$Tv_1 = w_1, \dots, Tv_m = w_m, \dots, Tv_n = 0$$

So with respect to the basis w_1, \dots, w_m in the first column we can have all zeroes except for the first row first element.

Now if $\dim \text{range } T < \dim W$. We can find a v_1 such that $T(v_1) = 0$. Now the matrix for this will have all in the first column as zero (as none of them dependent on w_1) \square

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We know that $T(v_1) = A_{1,1}w_1 + \dots + A_{n,1}w_m$. We need to show there exists a basis of W such that all values except for possibly $A_{1,1}$ is zero.

Consider the case when $T(v_1) = 0$ then we have $A_{1,1} = \dots = A_{n,1} = 0$ for any arbitrary basis of W .

If $T(v_1) \neq 0$ then consider all of $A_{2,1}, \dots, A_{n,1} = 0$ except for $A_{1,1}$ (the element in the first column and row). In this case consider,

$$T(v_1) = w_1$$

where w_1 is an arbitrary vector in W . Now we can extend w_1 to a basis of W . So in both cases we have a basis of W , w_1, \dots, w_n such that only possibly the first row first column element is zero.

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Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and let $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

We get,

$$AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

and

$$BA = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

So $AB \neq BA$

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Proof. We need to show $(AB)_{j..} = A_{j..}B$

We have,

$$\begin{aligned}(AB)_{j,.} &= \left(\sum_{k=1}^n A_{j,k} B_{k,1}, \dots, \sum_{k=1}^n A_{j,k} B_{k,p}, \right) \\ &= (A_{j,1}, \dots, A_{j,n}) B \\ &= A_{j,.} B\end{aligned}$$

□

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We know that

$$\begin{aligned}(AA)_{j,k} &= \sum_{r=1}^n A_{j,r} A_{r,k} \\ (A(AA))_{j,k} &= \sum_{r=1}^n A_{j,r} (AA)_{r,k} \\ (A^3)_{j,k} &= \sum_{p=1}^n A_{j,p} \left(\sum_{x=1}^n A_{p,x} A_{x,k} \right) \\ (A^3)_{j,k} &= \sum_{p=1}^n \sum_{x=1}^n A_{j,p} A_{p,x} A_{x,k}\end{aligned}$$

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Proof. \Rightarrow

Take A with rank 1 then we can decompose it to matrices R and C s.t.,

$$R = (m \times 1), C = (1 \times n)$$

Let $R = (c_1, \dots, c_m)^T$ and $C = (d_1, \dots, d_n)$

So we have

$$\begin{aligned}A_{jk} &= (RC)_{jk} = \sum_{n=1}^1 R_{m,1} C_{1,n} \\ &= c_j d_k\end{aligned}$$

\Leftarrow

If $A_{j,k} = c_j d_k$ then we can write A in terms of two matrices R times C such that $R = (c_1, \dots, c_m)^T$ and $C = (d_1, \dots, d_n)$

□

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Proof. We need to show, $a \Leftrightarrow b$

$a \Rightarrow b$.

If T is invertible we know that it is injective and surjective. If T is surjective

then we know that $\text{null } T = \{0\}$. So we have,

$$\dim \text{range}(T) = \dim(V)$$

To show that $T(v_1), \dots, T(v_n)$ is a basis of V we need to show that it is linearly independent and spans V .

To show linear independence we need to show that $c_1, \dots, c_n = 0$ if,

$$c_1 T(v_1) + \dots + c_n T(v_n) = 0$$

This is equal to,

$$T(c_1 v_1 + \dots + c_n v_n) = 0$$

We know that the $\text{null } T$ is $\{0\}$ so we have,

$$c_1 v_1 + \dots + c_n v_n = 0$$

We know that v_1, \dots, v_n is a basis for V . So it is linearly independent which means that, $c_1, \dots, c_n = 0$. So we have $T(v_1), \dots, T(v_n)$ is a linearly independent set.

The length of our list is the same as the length of the basis. Which means that we have linearly independent set of vectors are also a basis for V .

Another way of showing spanning is taking any $v \in V$ we can write it as $a_1 v_1 + \dots + a_n v_n = v$. We can apply T on both sides and show that v can be represented as a linear combination of $T(v_k)$

$$a \Leftarrow b$$

If Tv_1, \dots, Tv_n is a basis for V .

We need to show that T is injective and surjective. First consider an arbitrary $v \in V$ such that $T(v) = 0$. We know that $v = a_1 v_1 + \dots + a_n v_n$ where v_1, \dots, v_n is a basis for V . So we get,

$$T(a_1 v_1 + \dots + a_n v_n) = 0$$

$$a_1 T(v_1) + \dots + a_n T(v_n) = 0$$

We know that Tv_1, \dots, Tv_n is a basis which means that its linearly independent. So we have, $a_1, \dots, a_n = 0$. But if $a_1, \dots, a_n = 0$ then we have $v = a_1 v_1 + \dots + a_n v_n = 0$. Which means that for any $T(v) = 0$ means that $v = 0$. This means that it is injective.

We already know that if we have $V \rightarrow V$ such that both the dimensions are same then injective means that its surjective which means that it is invertible.

We also can show that any $w \in V$ can be written as $T(v) = w$ which means that it is surjective.

Take a $w \in V$ so $Tw = T(a_1 v_1 + \dots + a_n v_n) = a_1 T(v_1) + \dots + a_n T(v_n)$. We know that $T(v_1), \dots, T(v_n)$ is a basis which means that any vector $v \in V$ can be represented as a linear combination of $T(v_1), \dots, T(v_n)$. So we showed that any vector $v \in V$ can be represented by a $w \in V$ such that $T(w) = v$ which means that its surjective.

□

Proof. \Leftarrow We know that T is invertible so its injective and surjective. We know for every $u \in U$, $T(u) = S(u)$.

We need to show S is injective. First consider u_1, u_2 so we need to show $S(u_1) = S(u_2) \Rightarrow u_1 = u_2$.

If $Su_1 = Su_2$ then we can say $T(u_1) = T(u_2)$. However we know that T is injective so this means that $u_1 = u_2$. Hence we show that S has to be injective.

\Rightarrow

We have S is injective and maps a subspace of V , U onto V . We need to show that there exists a linear map T from V to itself such that it is an invertible linear map.

First consider the basis of U as u_1, \dots, u_k . We can extend the basis from this to,

$$u_1, \dots, u_k, v_{k+1}, \dots, v_n$$

Let us define our linear map T such that $T(u) = S(u)$ if $u \in U$ or in other words if $u = a_1u_1 + \dots + a_ku_k$. And define $T(v) = v$ if $v \in V - U$

We need to show $T(v'_1) = T(v'_2) \Rightarrow v'_1 = v'_2$. Let $v'_1 = a_1u_1 + \dots + a_nu_n$ and $v'_2 = b_1u_1 + \dots + b_nv_n$. So we have,

$$\begin{aligned} T(v'_1) &= T(v'_2) \\ T(a_1u_1 + \dots + a_nv_n) &= T(b_1u_1 + \dots + b_nv_n) \\ T(a_1u_1) + \dots + a_nT(v_n) &= T(b_1u_1) + \dots + b_nT(v_n) \\ (a_1 - b_1)T(u_1) + \dots + (a_n - b_n)T(v_n) &= 0 \end{aligned}$$

TO DO LATER

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If T is surjective then there exists a map $S : W \rightarrow V$, TS is the identity map. Or that $T(S(v)) = v$ for $v \in V$.

Now let $U = \text{range}(S)$ we need to show that $T|_U$ is injective and surjective.

1. Injective.

Consider a $u \in \text{null } T|_U$. So

$$T_U(u) = 0$$

$$T_U(S(w)) = w = 0$$

But

$$S(w) = u$$

and $w = 0$

which means that $u = 0$.

2. Surjective

For any $w \in W$ we have $v = S(w) \in U$ such that $T(S(w)) = w \in W$.

So we show that $v = S(w)$ for any $w \in W$
Hence we show that it is isomorphic

□

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⇒

We have ST is invertible. Lets assume the contrary that either S is not invertible or T is not invertible.

1. S is not invertible. Means S is not surjective. We know that ST is invertible which means that $\forall v \in V \exists v' \text{ s.t. } STv' = v$. Now let $Tv' = v''$. This means that $\forall v \in V, \exists v'' \in V \text{ s.t. } S(v'') = v$. But this makes S surjective which contradicts our assumption.

2. T is not invertible means that T is not injective or surjective. We know that ST is injective and surjective. If T is not injective then $\exists v \in V \text{ s.t. } T(v) = 0$. Now this means $S(T(v)) = S(0) = 0$ or that $STv = 0$ for some $v \neq 0$. But this makes ST not injective and hence not invertible which contradicts our assumption.

So by proof by contradiction our assumption must be wrong and both S and T are invertible.

⇐

We have S and T are invertible. We need to show that ST is injective and surjective.

1. Injective

We need to show that $\text{null } ST = \{0\}$. Consider $v \in \text{null } ST$ such that $STv = 0$. We know that S is injective hence $Tv = 0$. But T is injective as well. Hence $v = 0$. So $\text{null } ST = \{0\}$. Hence ST is injective

2. Surjective

We need to show that $\forall v \in V, \exists v' \in V \text{ s.t. } STv' = v$. We know that S is surjective this means that for any $v \in V, \exists v'' \text{ s.t. } S(v'') = v$. We also know that T is surjective this means that for any $v'' \in V \exists v' \in V \text{ s.t. } T(v') = v''$. So we have $\exists v' \text{ s.t. } S(T(v')) = v$ or ST is surjective.

Hence we show that ST is invertible.

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We have $STU = I$ This means that ST and U are both invertible which means that S and T are also invertible. So $\exists S^{-1}$ and U^{-1} .

$$\begin{aligned} STU &= I \\ TU &= S^{-1} \\ U &= T^{-1}S^{-1} \\ US &= T^{-1}S^{-1}S \\ T^{-1} &= US \end{aligned}$$

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If Tv_1, \dots, Tv_m spans V that means that $m \geq \dim V$.

1. $m = \dim V$

Assume v_1, \dots, v_m is not a lin ind set of vectors. Then, $\exists a_1, \dots, a_m$ some not zero such that,

$$a_1v_1 + \dots + a_mv_m = 0$$

.

So we get,

$$a_1(Tv_1) + \dots + a_mT(v_m) = 0$$

where some a not zero.

But this implies that Tv_1, \dots, Tv_m list is not lin independent which is contradictory.

Hence we have v_1, \dots, v_m is a lin independent set of vectors with dimension same as basis of V which makes v_1, \dots, v_m a basis of V or spanning.

2. If $m > \dim V$.

Then we can remove vectors from our list to make it spanning and lin independent and spanning, consider the smaller list as follows,

$$Tv_1, \dots, Tv_k$$

Using similar reasoning we know that

$$v_1, \dots, v_k$$

has to be linearly independent as well. However in this case we have lin independent vectors whose number is the same as that of a basis which makes it a basis as well. Hence our list of vectors v_1, \dots, v_k is spanning. So we can continue extending this basis to v_1, \dots, v_m and still have it spanning.