Probability Theory: HW2

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Exercise 3.8

Let X be the count of aces and Y be for num of kings.

	X = 0	X=1	X=2
Y = 0	$\binom{44}{2}/\binom{52}{2}$	$\binom{4}{1}\binom{44}{1}/\binom{52}{2}$	$\binom{4}{2}/\binom{52}{2}$
Y = 1	$\binom{4}{1}\binom{44}{1}/\binom{52}{2}$	$\binom{4}{1}\binom{4}{1}/\binom{52}{2}$	0
Y = 2	$\binom{4}{2}/\binom{52}{2}$	0	0

Exercise 3.25

We have X, Y are independent r.v which means that we have $\mathbb{P}(X = x \text{ and } Y = y) = \mathbb{P}(X = x)\mathbb{P}(Y = y)$. Now we have $g(X)$ and $h(Y)$ which are r.v we need to show that g, h are independent as well. Consider,

$$\begin{aligned}\mathbb{P}(g(X) = x' \text{ and } h(Y) = y') &= \mathbb{P}(X \in g^{-1}(x') \text{ and } Y \in h^{-1}(y')) \\ &= \mathbb{P}(X \in g^{-1}(x'))\mathbb{P}(Y \in h^{-1}(y')) \text{ as } X, Y \text{ are independent} \\ &= \mathbb{P}(g(X) = x')\mathbb{P}(h(Y) = y')\end{aligned}$$

Hence we show that $g(X)$ and $g(Y)$ are also independent.

Exercise 3.42

N is the number of events $A_1, \ldots, A_n$ that occur. Now let 1_{A_i} be the indicator function for event A_i which is 1 if the event occurs i.e. $\omega \in A_i$ and 0 if $\omega \notin A_i$. So we have $N = 1_{A_1} + \cdots + 1_{A_n}$. So,

$$\begin{aligned}E(N) &= E(1_{A_1} + \cdots + 1_{A_n}) \\ &= E(1_{A_1}) + \cdots + E(1_{A_n})\end{aligned}$$

Now expected value of $E(1_{A_i}) = \mathbb{P}(A_i) \cdot 1 + (1 - \mathbb{P}(A_i)) \cdot 0$ where $\mathbb{P}(A_i)$ is the probability the $\omega \in A_i$ or the event occurring. So $(1_{A_i}) = \mathbb{P}(A_i)$. So we have,

$$\begin{aligned}E(N) &= P(A_1) + \cdots + P(A_n) \\ &= \sum_{i=1}^{n} \mathbb{P}(A_i)\end{aligned}$$

Problem 4

We have $X_1, X_2, \ldots, X_n$ are independent discrete random variable and we have $\mathbb{P}(X_i = k) = \frac{1}{N}$ where k goes from $1, \ldots, N$

We have $U_n = \min\{X_1, \ldots, X_n\}$. So $\mathbb{P}(U_n = k)$ is the probability that the least value that any X_i takes is k. So none of the X_i can take values smaller than k. We have $\mathbb{P}(X_i \geq k) = \sum_{n=k}^{N} \mathbb{P}(X_i = k) = (N - k + 1)/N$. Now as each X is independent we have probability that each of them are larger than equal to k is, $((N - k + 1)/N)^n$. So we have,

$$\begin{aligned}\mathbb{P}(U_n \geq k) &= \left(\frac{N - k + 1}{N}\right)^n \\ \mathbb{P}(U_n \geq k + 1) &= \left(\frac{N - k}{N}\right)^n\end{aligned}$$

Which gives us $\mathbb{P}(U_n = k) = \mathbb{P}(U_n \geq k) = \mathbb{P}(U_n \geq k + 1) = \left(\frac{N-k+1}{N}\right)^n - \left(\frac{N-k}{N}\right)^n$

Now similarly we have V_n is the max so $V_n = k$ means that no X_i takes on a value larger than k. Hence at least one X_i takes on the value k and the rest can take on other values. So first we have $\mathbb{P}(U_n \leq k) = (k)/N$ so,

$$\mathbb{P}(V_n \leq k) = \left(\frac{k}{N}\right)^n$$
$$\mathbb{P}(V_n \leq k-1) = \left(\frac{k-1}{N}\right)^n$$

And for $\mathbb{P}(V_n = k)$ we have probability that it's smaller than k minus probability that it's smaller than $k-1$ which would be the case where it's equal to k so,

$$\mathbb{P}(V_n = k) = \left(\frac{k}{N}\right)^n - \left(\frac{k-1}{N}\right)^n$$

Problem 7

We have $X_1, X_2, \dots$ are discrete random variables with mean μ. We have N a r.v whic his independent of the X_i. We have,

$$E(X_1 + \cdots + X_N) = \sum_n E((X_1 + \cdots + X_N) \mid N = n)\mathbb{P}(N = n)$$

Now given $N = n$ we have,

$$\begin{aligned} E(X_1 + \cdots + X_n) &= E(X_1) + \cdots + E(X_n) \\ &= \mu n \end{aligned}$$

So we get,

$$\begin{aligned} E(X_1 + \cdots + X_N) &= \sum_n \mu n \mathbb{P}(N = n) \\ &= \mu \sum_n n\mathbb{P}(N = n) \\ &= \mu(N) \end{aligned}$$

Exercise 4.18

We have X a random variable with $G_X(s)$ and k a positive integer. We have $Y = kX$ and $Z = Y + k$. We need to find the probability generating function of both.

We have $\mathbb{P}(Y = y) = \mathbb{P}(X = y/k)$. So if we have,

$$G_X(s) = u_0 + u_1 s + u_2 s^2 + \dots$$

So give $X = x$ as Y takes on the value kx. Here x is the exponent so we need the corresponding coefficient to be that of kx. This is equivalent to,

$$G_Y(s) = u_0 + u_1 s^k + u_2 s^{2k} + \dots$$

Which is $G_Y(s) = G_X(s^k)$

Now consider $Z = X + k$. Here for a given $Z = z$ the probability of this is the same as $X = z - k$. So given an exponent x for Z the coefficient / probability must be for $x + k$ and hence we have,

$$\begin{aligned}
G_X(s) &= u_0 + u_1 s^1 + u_2 s^2 + \dots \\
G_Z(s) &= u_0^k + u_1 s^{1+k} + u_2 s^{2+k} + \dots \\
G_Z(s) &= s^k (u_0 + u_1 s + u_2 s^2 + \dots) \\
G_Z(s) &= s^k G_X(s)
\end{aligned}$$

Exercise 4.41

We have X, Y independent r.v. X is a binomial distribution with m, p and Y is binomial with n, p.

For binomial we have $G_X(s) = (q + ps)^n$ and $G_Y(s) = (q + ps)^m$. So for $X + Y$ we have,

$$G_{X+Y}(s) = G_X(s) G_Y(s) = (q + ps)^m + (q + ps)^n = (q + ps)^{n+m}$$

which is also a binomial with parameters p and $m + n$.

Now we know the generating function of a bernoulli with parameter p is $G_{X_i}(s) = (q + ps)$. As we have n independent bernoullis say $X_1 + \cdots + X_n$ is,

$$\begin{aligned}
G_{X_1 + \cdots + X_n}(s) &= G_{X_1}(s) \dots G_{X_n}(s) \\
&= (q + PS)^n
\end{aligned}$$

Which we see is the binomial with parameter p and n.

Exercise 5.13

We have $Y = \max\{0, X\}$. We know that Y cannot take on values smaller than zero. And for greater than zero we have it as the same as X so we have,

$$F_Y(u) = \begin{cases} F_X(y) \text{ if } y \geq 0 \\ 0 \text{ if } y < 0 \end{cases}$$

Problem 5

(a) We have probaility of n flowers is $(1 - p)p^n$. Now the generating function for this is if we take $q = 1 - p$ then,

$$\begin{aligned}
G_X(s) &= q + qps + q(ps)^2 + \dots \\
&= q(1 + ps + (ps)^2 + \dots) \\
&= \frac{q}{1 - ps}
\end{aligned}$$

Now take Y to be 1 if we get a ripe fruit and 0 if we don't so we have,

$$G_Y(s) = \frac{1}{2} + \frac{1}{2} s$$

Now if Z is r.v. for num of ripe fruits we get,

$$\begin{aligned}
G_Z(s) &= G_X(G_Y(s)) \\
&= \frac{1 - p}{1 - p(\frac{1}{2} + \frac{1}{2s})} \\
&= 2 \frac{1 - p}{2 - p} \cdot \frac{1}{1 - \frac{ps}{2-p}}
\end{aligned}$$

We see the second term is a geometric series as well so we have probability that $Z = r$ is the coefficient of the s^r term which would be,

$$\mathbb{P}(Z = r) = \frac{2(1-p)}{2-p}\left(\frac{p}{2-p}\right)^r$$

(b) We need now the probability of having n flowers given that it produces r ripe fruits. Which is,

$$\mathbb{P}(X = n \mid Z = r) = \mathbb{P}(Z = r \mid X = n)\frac{\mathbb{P}(X = n)}{\mathbb{P}(Z = r)}$$

Now given n the probability of getting r ripe fruits is $\mathbb{P}(Z = r \mid X = n) = \binom{n}{r}\frac{1}{2^n}$. And we have $\mathbb{P}(X = n) = (1-p)p^n$ and $\mathbb{P}(Z = r) = \frac{2(1-p)}{2-p}\left(\frac{p}{2-p}\right)^r$. This gives

$$\begin{aligned}
\mathbb{P}(X = n \mid Z = r) &= \binom{n}{r}\frac{1}{2^n}\frac{(1-p)p^n}{\frac{2(1-p)}{2-p}\left(\frac{p}{2-p}\right)^r} \\
&= \binom{n}{r}\frac{p^{n-r}(2-p)^{r+1}}{2^{n+1}}
\end{aligned}$$