

Intro to Proofs

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Contents

Real Numbers

Definition 0.1 (Properties of real numbers). Properties of \mathbb{R} are
 (d). \exists an order on \mathbb{R} which means $\forall x, y \in \mathbb{R}, x < y$ or $x > y$, or $x = y$
 Ordering follows the following properties,
 (1). $x < y, y < z \Rightarrow x < z$ (transitivity)
 (2). $x < y \Rightarrow x + z < y + z, \forall z \in \mathbb{R}$
 (3). $x < y, z > 0 \Rightarrow xz < yz$

Theorem 0.2. $xy = 0 \Leftrightarrow x = 0$ or $y = 0$

Proof. \Leftarrow Without loss of generality take, $x = 0$ Then we get,

$$0y.$$

We can write this as,

$$(0 + 0)y = 0y + 0y.$$

So,

$$0y = 0y + 0y.$$

Or, m

\Rightarrow

Assume the contrary that, $x \neq 0$ and $y \neq 0$ We have, $xy = 0$. Without loss of generality we take the multiplicative inverse of x so,

$$\frac{xy}{x} = \frac{0}{x}.$$

We showed that $0(k) = 0$ so $y = 0$

Which contradicts our assumption, hence our assumption must be wrong and $x = 0$ or $y = 0$

□

Theorem 0.3. $(-)(-x) = -x$

Proof. We start with $(-1)x$ and add x to both sides so,

$$(-1)x + x = x(1 - 1) = 0x = 0.$$

So we showed that $(-1)x$ is the additive identity of x .
We know that the additive identity is unique for any x .
Therefore, $(-1)x = -x$ □

Theorem 0.4. $\forall x < y, z < 0$

$$xz > yz.$$

Proof. If $z < 0$ then that means $z = -k$ for some $k > 0$.

We can write $x < y$ as $x - y < 0$

Now if we multiply both sides by k we get,

$$k(x - y) < 0$$

Now if $k(x - y) = z'$ we can say that $z' < 0 \Rightarrow -z' > 0$

Or that

$$(-1)k(x - y) > 0$$

$$z(x - y) > 0$$

$$xz > yz$$

□

Theorem 0.5. $\forall x \in \mathbb{R}$ if $x \neq 0$ then $x^2 > 0$

Theorem 0.6. $x^2 = -(-x^2)$

Case 1, $x > 0$:

$$x > 0$$

$$x \times x > x$$

$$x \times x > 0x$$

$$x^2 > 0$$

Case 2, $x < 0$:

$$x < 0$$

Then the additive inverse $(-x) > 0$

$$(-x)(-x) > (-x)0$$

$$(-)(-1)x^2 > 0$$

$-(-1) = 1$ as 1 is the additive inverse of -1

$$x^2 > 0$$

Example. $\forall a, b > 0$

$$\frac{a+b}{2} \geq \sqrt{ab}$$

◇

Proof.

$$0 \leq (\sqrt{a} - \sqrt{b})^2 = a - 2\sqrt{ab} + b.$$

$$2\sqrt{ab} \leq a + b$$

$$\sqrt{ab} \leq \frac{a+b}{2}$$

□

Example. $x^2 - x + 1$

◇

Theorem 0.7. $\forall x, y \in \mathbb{R}$ we have,

$$|x| \geq x \text{ and } |x+y| \leq |x| + |y|.$$

Proof. We use proof by cases.

□

Proof related to Sets

Theorem 0.8.

$$A \cup B \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A).$$

Proof. We need to show that,

$$A \cup B \setminus (A \cap B) \subseteq (A \setminus B) \cup (B \setminus A).$$

and,

$$(A \setminus B) \cup (B \setminus A) \subseteq A \cup B \setminus (A \cap B).$$

□

Theorem 0.9. $A \subseteq B \Leftrightarrow A \cup B = B$

Proof. \Rightarrow Take $\forall x \in A \cup B$, so either

Case 1, $x \in A$:

We know that by definition if, $A \subseteq B$ then for $x \in A, x \in B$ so $x \in B$

Case 2, $x \in B$: If $x \in B$ then we don't need to go further.

So we get $\forall x \in A \cup B, x \in B$

\Leftarrow

$\forall x \in A \Rightarrow x \in A \cup B = B$

So, $x \in B$ which means that, $A \subseteq B$

□

Disproofs

If we need to show existence, $\exists x.P(x)$. We can show using,

1. Direct constructions
2. Indirectly (contradiction). For instance we can show that, $\forall x, \neg P(x)$ is false

Example. $\exists a, b, c \in R - Q$ s.t. $a^{bc} \in Q$

◇

Example. Pigeonhole principle

Suppose there are m balls in n boxes, $m > n \geq 1$ then, \exists a box where there are at least, $\frac{m}{n} + 1$ balls

◇

Proof. Assume pigeonhole is false.

Then, there are at most $\frac{m}{n}$ balls in each box.

In case 1 where $\frac{m}{n} \notin N \Rightarrow$ total balls $\leq n[\frac{m}{n}] = \frac{nm}{n} = m$ which is a contradiction.

In case 2 where $\frac{m}{n} \in N$ there are at most $\frac{m}{n} - 1$ balls in each box. So total number of balls are $\frac{nm}{n} - n = m - n$ which is contradictory.

□

To disprove $\forall x.P(x)$ we can show that, $\exists \neg P(x)$

Example. 100 can't be written as the sum of two even integers and an odd integer.

◇

Proof. Suppose it's false $\Rightarrow \exists a, b, c \in Z$ s.t. $2|a, 2|b, 2 \nmid c$ and $100 = a + b + c$

But, $2|a, 2|b \Rightarrow 2|a + b$ but $2 \nmid c \Rightarrow 2 \nmid (a + b) + c = 100$

So we get, $2 \nmid 100$ which is a contradiction.

Which means that the original statement is true.

□

Example. \nexists the smallest positive real number

The smallest positive real number is defined as $x \in R$ s.t. $x > 0$ and $\forall y > 0, x \leq y$

◇

Proof. Let's assume it is true which means that $\exists x \in R$ s.t. $x > 0$ and $\forall y > 0, x \leq y$

We know that $x > 0 \Rightarrow \frac{x}{2} > 0$

So if we set $y = \frac{x}{2}$ then we get

$$x \leq \frac{x}{2}.$$

Which is a contradiction.

Hence it cannot be the case that there exists the smallest positive number. \square

Example. $\nexists f(x)$: a polynomial with integer coefficients s.t. $\forall n, f(n)$ is prime \diamond

Proof. Consider the general form of a polynomial,

$$f(x) = a_1x^n + \dots + a_n$$

Case 1: $a_n = 0$

If $a_n = 0$ then for any $x > 1$ we can take x common and get

$$f(x) = x(a_1x^{n-1} + \dots + a_{n-1})$$

So we get a factor $x \neq 1$

Case 2: $a_n = 1$

In this case we can just plug $x = 0$ and we get $f(x)$ is neither prime or composite

Case 3: $a_n > 1$??? \square

Example. Let $f(x) = x^3 + 2x - 5$ then \exists unique $x_0 \in [1, 2]$ s.t. $f(x_0) = 0$ \diamond

Proof. Using intermediate value theorem.

$$f(1) = -2$$

$$f(2) = 7$$

So because $-2 < 0 < 7$ we know that there must exist an $x_0 \in [1, 2]$ s.t. this is the case.

To show unique we need to show its strictly increasing. Or in other words, we need to show for every $x_1, x_2 \in [1, 2], x_1 \leq x_2 \Rightarrow f(x_1) \leq f(x_2)$

So we need to show that,

$$x_1^3 + 2x_1 - 5 \leq x_2^3 + 2x_2 - 5$$

$$x_1^3 + 2x_1 \leq x_2^3 + 2x_2$$

$$(x_1^3 - x_2^3) + 2(x_1 - x_2) \leq 0$$

It is enough to show that both $x_1^3 - x_2^3$ and $x_1 - x_2$ are smaller than or equal to 0.

$$x_1^3 - x_2^3 = (x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2) = k(x_1 - x_2) \leq 0$$

Similarly,

$$2(x_1 - x_2) \leq 0 \text{ a } x_1 - x_2 \leq 0$$

So we have, $x_1^3 - x_2^3 + 2(x_1 - x_2) \leq 0$

Which tells us that our function is strictly increasing which implies that we only have a unique $x_0 \in [1, 2]$ \square

Mathematical Induction

Theorem 8.10 (Properties of Natural Numbers). (a). $1 \in N$
 (b). $\forall k \in N, \exists k + 1 \in N$
 (c). $\forall k \in N - \{1\}, \exists n \in N, \text{ s.t. } k = n + 1 \in N$
 (d). Needs to be well-ordered.

An ordered set S is well-ordered if,

$$\forall A \in S \text{ s.t. } A \neq \phi, \exists x = \min A$$

$$\text{Or, } \exists x \in A \text{ s.t. } y \in A, x \leq y$$

Example. \mathbb{Q} is not well-ordered as it does not have a minimum ◇

Example. \mathbb{Z} is not well-ordered as it does not have a minimum ◇

Axiom 8.11. N is well-ordered

Theorem 8.12. If $A \subseteq S$ and S is a well-ordered set then A is well-ordered.

Proof. Let $B \subseteq A$ and $B \neq \phi \Rightarrow B \subseteq S$
 So B has a min x which means that A is well-ordered by definition. □

Example. $[1, \infty)$ is not well-ordered because a subset $(1, \infty)$ does not have a min ◇

Theorem 8.13. $\forall a \in \mathbb{Z}, d \in \mathbb{N}, \exists q, r \in \mathbb{Z} \times \{0, 1, \dots, d-1\} \text{ s.t.}$

$$a = dq + r$$

Proof. Let $S = \{a - nd : n \in \mathbb{Z}, a - nd \in \mathbb{N}\}$
 First we can see that S is non-empty as we can take

$$n = -|a| - 1 \Rightarrow a - nd > 0$$

Now because this is a subset of \mathbb{N} it follows the well-ordering principle implying that $\min S = a - nd = m$

$m \in S \Rightarrow \exists l \in \mathbb{Z} \text{ s.t. } m = a - ld$

Case 1: If $m > d$ then

$$a - (l+1)d > 0$$

$$a - (l+1)d \in S \Rightarrow a - (l+1)d < m$$

Which is a contradiction. This means that $m \not> d$

Case 2: $m = d$

Let $q = l+1, r = 0$

$$m = d \Rightarrow a - ld = d \Rightarrow a - (l+1)d = 0$$

Case 3: $0 < m < d$

Let $q = l, r = m \Rightarrow a = dq + r$

Now to show uniqueness,

Suppose, $(q, r), (q', r') \in \mathbb{Z} \times \{0, 1, \dots, d-1\}$ and

$$a = qd + r = q'd + r'$$

We have,

$$(q - q')d = r' - r$$

$$0 - (d-1) \leq r' - r \leq d-1$$

And,

$$d|r' - r \Rightarrow r' - r = 0$$

□

Definition 8.14. Let $a, b \in \mathbb{N}, d = \text{GCD}(a, b) \in \mathbb{N}$ if

(a). $d|a$ and $d|b$ and

(b). If $d' \in \mathbb{N}$ s.t. $d'|a$ and $d'|b$ then $d \geq d'$

Theorem 8.15. $\forall a, b \in \mathbb{N}, \exists p, q \in \mathbb{Z} \text{ s.t. } \text{GCD}(a, b) = ap + bq$

Proof. Let $S = \{a_m + b_n : m, n \in \mathbb{Z}, a_m + b_n \subseteq \mathbb{N}\}$

We know S is non-empty as $m, n = 1$ makes it $a + b > 0$ as $a, b \in \mathbb{N}$

So by well-ordering principle we know that $\exists \min S = d$ and $p, q \in \mathbb{Z} \text{ s.t.}$

$$d = ap + bq$$

If $d' \in \mathbb{N}$ s.t. $d'|a$ and $d'|b \Rightarrow d'|ap + bq = d$

So, $d \in \mathbb{N} \Rightarrow d \geq d'$

$$d \in \mathbb{N} \Rightarrow \exists m \in \mathbb{Z}, r \in \{0, \dots, d-1\} \text{ s.t. } a = md + r$$

Which means $r = a - md = a - m(ap + bq) = a(1 - mp) + b(-mq)$

$r < d$ but $d = \min S \Rightarrow r \notin S \Rightarrow r = 0$

So $a = md$ so $d|a$. Similarly, $d|b$

This means d is the greatest common divisor.

□

Theorem 8.16 (Induction principle). Suppose $k \in \mathbb{N}$, $S \subseteq \mathbb{N}$ satisfy,

(a). $k \in S$

(b). if $n \in S$ then $n + 1 \in S$

then $\{k, k + 1, \dots\} \subseteq S$

Proof. Let $A = \{n \in \mathbb{N}, n \geq k : n \notin S\}$

Suppose $A \neq \emptyset \Rightarrow n_0 = \min A$ exists

Which means $n_0 \geq k$ but $k \notin A$ due to (a). So, $n_0 > k \Rightarrow n_0 - 1 \geq k$ and

$n_0 - 1 \notin A$ as $n_0 = \min A$

(b). and $n_0 - 1 \notin A \Rightarrow n_0 \notin A$ contradiction which implies that $A = \emptyset$

□

Corollary 8.17. If a statement $P(n)$, $n \in \mathbb{N}$ satisfies

(a) $P(k)$ is true

(b) $P(n) \Rightarrow P(n + 1)$

Then $P(n)$ is true for all $n \geq k$