Linear Alebgra HW07

Aamod Varma

October 17, 2024

16

We consider the standard basis for both $F^{n,1}$ and $F^{m,1}$ such that for any $x \in F^{n,1}$ we have $M(x) = x$ similarly, $Tx \in F^{m,1}$ so $M(Tx) = Tx$. We can say,

$$\begin{aligned} Tx &= M(Tx) \\ &= M(T)M(x) \\ &= M(T)x \end{aligned}$$

Let $M(T)$ be a matrix A so we get,

$$Tx = Ax, \forall x \in F^{n,1}$$

20

Consider the map $T : P(R) \to P(R)$. Defined by $T(p(x)) = (x^2 + x)p''(x) + 2xp'(x) + p(3)$. Now $T(p(x))$ has the same degree as $p(x)$. And we also see that if $p(x) \neq 0$ then $T(p(x)) \neq 0$ which means that T is injective which means that it is surjective as dim is same.
Now this shows that for any $q(x) \in P(R)$ we can find $p(x)$ s.t. $T(p(x)) = q(x)$

21

We can rephrase the questino as,
(a) $Ax = 0$
(b) $Ax = c$
Where A is a matrix and x and c are vectors as follows, $x = (x_1, \dots, x_n)^T$ and $c = (c_1, \dots, c_n)^T$
Now in (a) we can see Ax as a linear map T from $F^{n,1}$ to itself. So we have $Tx = 0$ mean that $x = 0$. THis means that T is injective. Similarly in (b) we nkow that for any $c \in F^{n,1}$ $\exists x \in F^{n,1}$ s.t. $Tx = c$ which means that T is surjective. We know that injective and surjective are equivalent as T maps from $F^{n,1}$ to itself.

5A

Problem 1

Proof. (a). $U \subseteq \text{null } T$ we need to show $\forall u \in U$ $T(u) \in \text{null } T$.
Consider any $u \in \text{null } T$ we know that $T(u) = 0$ by definition. We know that $0 \in \text{null } T$ becuase $T(0) = 0$ which means that $T(u) \in \text{null } T$. Hence we show that for any $u \in U$ $T(u) \in U$. Which implies that U is an invariant under T.

(b). We have $\text{range } T \subseteq U$ this means that for any $u \in V$, $T(u) \in \text{range } T \Rightarrow T(u) \in U$. So consider any $u \in U$ so we know that $T(u) \in \text{range } T \Rightarrow T(u) \in U$. Which shows us that U is invariant under T.

□

Problem 2

Proof. We need to show that $V_1+\cdots+V_m$ is invariant under T if $V_1,\ldots,V_n$ are invariant under T.
So we need to show that if $v \in V_1+\cdots+V_m$ then $T(v) \in V_1+\cdots+V_m$. If $v \in V_1+\cdots+V_m$ that means that we can write v as

$$v = v_1+\cdots+v_m, \text{ where } v_1 \in V_1,\ldots,v_m \in V_m$$

Now $T(v) = T(v_1+\cdots+v_m) = T(v_1)+\cdots+T(v_m)$. But we know that $v_1 \in V_1,\ldots,v_m \in V_m$ which means that $T(v_1) \in V_1,\ldots,T(v_m) \in V_m$ as we know that $V_1,\ldots,V_m$ are invariant subspaces. So now let us write

$$T(v_1) = v_1',\ldots,T(v_m) = v_m'$$

such that $v_1' \in V_1,\ldots,v_m' \in V_m$
So we have,

$$\begin{aligned} T(v) &= T(v_1)+\cdots+T(v_m) \\ &= v_1'+\cdots+v_m' \end{aligned}$$

So we have writen any $T(v)$ as $v_1'+\cdots+v_m'$ such that $v_1' \in V_1,\ldots,v_m' \in V_m$ which means that $T(v) \in V_1+\cdots+V_m$. Hence by definition this makes our subspace $V_1+\cdots+V_m$ invariant under T.

□

Problem 3

Proof. Let $V_1,\ldots,V_m$ represent every collection of subspaces that are invariant under T. We need to show that $v \in V_1\cap\cdots\cap V_m \Rightarrow T(v) \in V_1\cap\cdots\cap V_m$.
Consider an arbitrary $v \in V_1\cap\cdots\cap V_m$. Now this means that

$$v \in V_1,\ldots,v \in V_m$$

But because $V_1,\ldots,V_m$ are all invariant under T this means that

$$T(v) \in V_1,\ldots,T(v) \in V_m$$

Now by definition this means that $T(v) \in V_1\cap\cdots\cap V_m$. Which makes the subspace $V_1\cap\cdots\cap V_m$ invariant under T. □

Problem 4

Proof. Let us assume the contrary that $U \neq \{0\}$ and $U \neq V$. We know that $0 < \dim U < \dim V$. Let $\dim U = k$ and $\dim V = n$. So consider a basis for U as,

$$u_1,\ldots,u_k$$

Now let us define an operator on U such that

$$T(u_1) = v, \ldots, T(u_k) = v$$

where $v \in V - U \neq \phi$ (for instance let $v = v_m$)
So we constructed a map on U such that $T(U) = v \notin U$.

$\square$

Problem 5

We have the matrix as, $\begin{bmatrix} 0 & -3 \\ 1 & 0 \end{bmatrix}$. So eigen values are $\pm i\sqrt{3}$.
However this exist outside R hence within our vector space we don't have an eigenvalue.

Problem 6

We have the matrix as, $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. So eigen values are ± 1.
If eigven value is 1 then eigenvector is $(1, 1)$ and if its -1 then eignvector is $(1, -1)$

Problem 7

We have the matrix as, $\begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix}$. So eigenvalues are 0 and 5. Eigenvector for 0 is $(1, 0, 0)$ and for 5 is $(0, 0, 1)$

Problem 8

We have $P^2 = P$. So consider an arbitrary v which is an eigenvector of λ. So we have,

$$Pv = \lambda v$$

But we know $P(P(v)) = P(v)$. So $P(\lambda v) = P(v)$

$$\lambda P(v) = P(v)$$

Now this is ture if either $\lambda = 1$ or $P(v) = 0$. If $P(v) = 0$ that means $v \in T$ or that $\lambda = 0$ is an eigvenvalue of T.

Problem 9

We have our basis of $P(R)$ as $1 + x$. Which can be written as $(1, 0), (0, 1)x$.
So the matrix of our linear map is $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. So $\lambda = 0$ which gives us the eigenvector as $(1, 0)$

Problem 10

$P_4(R)$ is spanned by $1+x+x^2+x^3+x^4$. Which can be written as $(1,0,0,0,0),\ldots,(0,0,0,0,1)$. Now to define our linear map we need to see where our standard basis will map to. We have,

$$T(1) = 0$$
$$T(x) = x$$
$$T(x^2) = 2x^2$$
$$T(x^3) = 3x^3$$
$$T(x^4) = 4x^4$$

So the matrix of our linear map will be,

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

It is clear that our eigenvalues are $0,1,2,3,4$. And its eigenvector are $(1,0,0,0),\ldots,(0,0,0,1)$

Problem 12

Proof. We know that $V = U \oplus W$. So any $v \in V$ can be written as

$$v = u + w, u \in U, w \in W$$

Consider a basis for U as $u_1,\ldots,u_n$ and a basis for W as $w_1,\ldots,w_m$.
So we have all v can be written as $a_1u_1+\cdots+a_nu_n+b_1w_1+\cdots+b_mv_m$. Because $U \cap W = \{0\}$ we know that no $w_k \in \mathrm{U} or in other words our list of vectors \mathrm{u}_1,\ldots,u_n,w_1,\ldots,w_m$ is linearly indepndent and is a basis for V.
Now given a basis of V we can define a linear map $P : V \to V$ as follows,

$$P(u_1) = u_1,\ldots,P(u_n) = u_n$$

$$P(w_1) = 0,\ldots,P(w_m) = 0$$

Hence we have defined a linear map as we have assigned vectors in V for our basis of V.
Now for any $P(u+w)$ let $u = a_1u_1+\cdots+a_nu_n$ and $w = b_1w_1+\cdots+b_mw_m$

$$\begin{aligned} P(u+w) &= P(u) + P(w) \\ &= P(a_1u_1 + \cdots + a_nu_n) + P(b_1w_1 + \cdots + b_mw_m) \\ &= a_1P(u_1) + \cdots + a_nP(u_n) + b_1P(w_1) + \cdots + b_mP(w_m) \\ &= a_1u_1 + \cdots + a_nu_n + 0 \\ &= u \end{aligned}$$

Hence we define P such that $P(u+w) = u$ □

Problem 13

Proof. (1). We need to show that if $Tv = \lambda v$ then $S^{-1}TSv = \lambda v$ givne that S is invertible.
Consider that $S^{-1}TSv = v'$. Becuase S is invertibel we can apply S on both sides and get,

$$T(S(v)) = S(v')$$

.
But we assume that T has an eigenvalue λ. Now because S is invertible in V there exists some v such that $S(v)$ is an eigenvector of T. For that v we have $T(S(v)) = \lambda S(v)$. So we get,

$$\begin{aligned} T(S(v)) &= S(v') \\ \lambda S(v) &= S(v') \\ S(v') &= S(\lambda v) \\ v' &= \lambda v \end{aligned}$$

So for some v such that $S(v)$ is an eigen vector of T we have v is an eigenvector of $S^{-1}TS$ such that $\lambda$is the eigenvalue associated with the vector.
Hence we show that both T and $S^{-1}TS$ have the same eigenvalues.
(b). If v is an eigenvector of T then $v' = S^{-1}v$ is an eigenvector of $S^{-1}TS$.
□

Problem 19

Proof. Assume λ is an eigven value, this means that

$$\lambda z_1 = 0, \lambda z_2 = z_2, \ldots$$

But if $\lambda z_1 = 0$ then $z_1 = 0 \Rightarrow z_2 = 0 \Rightarrow z_3 = 0 \ldots$
Hence the only possiblblity is $(0, \ldots)$ however this isn't a valid consideratino for an eigenvector. So $\nexists \lambda$ such that it is an eivenvalue. Or in other words for any lambda there doesn't exist a eigenvector.

□

Problem 20

Proof. (a). Consider an artbiary λ we need,

$$\lambda z_1 = z_2, \lambda z_2 = z_3, \ldots$$

For any arbitrary z_1 and λ we can define $z_n = \lambda z_{n-1}$.
(b). For an arbitary λ every eigenvector is of form,

$$(z_1, \lambda z_1, \lambda^2 z_1, \lambda^3 z_1, \ldots)$$

□

Problem 21

Proof. (a). $\Leftarrow$ We have T is invertible and λ is an eigenvalue of T. So $\exists v$ such that,

$$T(v) = \lambda v$$

Now because T is invertbile we can apply T^{-1} and we get,

$$\begin{aligned} T^{-1}Tv &= T^{-1}(\lambda v) \\ v &= T^{-1}(\lambda v) \\ v &= \lambda T^{-1}(v) \\ \frac{v}{\lambda} &= T^{-1}(v) \end{aligned}$$

So we've shown that $\frac{1}{\lambda}$ is an eigenvalue of T^{-1}
$\Rightarrow$ The argument is exactly the same as (a) because $T^{-1} = T'$ is also an invertbile linear map and just consider $\lambda' = \frac{1}{\lambda}$ as the eigvenvalue of this map.
(b). Let $v \in V$ such that $T(v) = \lambda v$. Then we have

$$\begin{aligned} T^{-1}Tv &= T^{-1}\lambda v \\ v &= \lambda T^{-1}v \\ \frac{v}{\lambda} &= T^{-1}v \end{aligned}$$

So we've shown that for any arbitary eigenvector v with eigenvalue λ v is also an eigenvector of T^{-1} with eigenvalue of $\frac{1}{\lambda}$

□

Problem 21

Proof. Consider two cases either $w = -u$ or $w \neq -u$.
If $w = -u$ then we have $T(u) = -3u$ and $T(w) = -3w$ which makes -3 a eigenvalue with eigenvector u.
Now if $w \neq -u$ then we have,

$$T(u) = 3w, T(w) = 3u$$

$$\begin{aligned} T(u) + T(w) &= 3(u + w) \\ T(u + w) &= 3(u + w) \end{aligned}$$

And because $u \neq w$ we know that $u + w \neq 0$ which is required for an eigenvector. So here we show that $u+w$ is an eigenvector and the associated eigenvalue for this is 3.

$\square$

Problem 25

Proof. Let u be an eigenvector such that $T(u) = \lambda_1 u$ and $T(w) = \lambda_2 w$
We are told that $T(u+w) = \lambda_3(u+w)$. So we have,

$$T(u) + T(w) = \lambda_3(u+w)$$

$$\lambda_1 u + \lambda_2 w = \lambda_3(u+w)$$
$$\lambda_1 u + \lambda_2 w = \lambda_3 u + \lambda_3 w$$

Now if u, w are linearly dependent (one is in the span of the other) then it is trivial to show that $\lambda_1 = \lambda_2$ and that $\lambda 3 = \lambda_1 = \lambda 2$.
Now if u, w are linearly independent this means that neither are in the span of each other, $\nexists k$ s.t. $u = kw$.
Hence the only solution to the equation $(\lambda_1 - \lambda_3)u + (\lambda_2 - \lambda 3)v = 0$ is if the coefficients are equal to 0. Or

$$\lambda_1 = \lambda_3$$

$$\lambda_2 = \lambda_3$$

But this then means that $\lambda_1 = \lambda_2$. So we show that in both cases $\lambda_1 = \lambda_2$
$\square$

Problem 28

Proof. First within the range of T we can construct T such that we have at most $\dim \text{range } T$ eigenvector curresponding to each subspace spanned by the basis of range T. That is, we can consturct $\dim \text{range } T$ invariant subspace from V to itself such that $T(v) \in \text{range } T$.
Now consider the case when range T doesn't span V then $\exists v$ such that $T(v) = 0$. Hence we have another eigven value 0 such that $T(v) = 0v$ and $v \in T$
So we have shown that we have atmost $\dim \text{range } T + 1$ eigenvalues. $\square$

Problem 30

Proof. For $(T-2I)(T-3I)(T-4I) = 0$ to be true we have either $(T-2I) = 0$ or $(T-3I) = 0$ or $(T-4I) = 0$.
Let us consider each case, 1. $(T-2I) = 0$ or

$$(T-2I)v = 0v = 0$$
$$Tv - 2v = 0$$
$$Tv = 2v$$

Which means that 2 is an eigenvalue or that $\lambda = 2$. We can use similar reasoning for (2) and (3) to conclude that either $\lambda = 2$ or 3 or 4. $\square$