Probability Theory: HW2

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Problem 2.10

We need to show that the indicator function 1_E is a discrete random variable. First we need to show that $1_E(\Omega)$ is countable.

We have,

$$1_E(\omega) = \begin{cases} 1 & \text{if } \omega \in E \\ 0 & \text{if } \omega \notin E \end{cases}$$

So for any $\omega \in \Omega$ we have either $1_E(\omega) = 1$ or $1_E(\omega) = 0$ hence we have $X(\Omega) = \{0, 1\}$ which is countable.

Now we need to show that $\forall a \in \mathbb{R}$ we have, $\{\omega : X(\omega) = a\} \in \mathscr{F}$. We see for $a = 1$ we have $\{\omega : X(\omega) = 1\} = E$ and we assume that $E \in \mathscr{F}$. Similarly we have for $a = 0$ that $\{\omega : X(\omega) = 0\} = \{\omega : \omega \notin E\} = E^c$. Using the properties of $\mathscr{F}$ we know that $E^c \in \mathscr{F}$. Lastly if $a \neq 1, 0$ we have $\{\omega : X(\omega) \neq 1, 0\} = \phi$ and we know that $\phi \in \mathscr{F}$.

Hence, we show that the indicator function is a discrete random variable.

Problem 2.11

1. $U(\omega) = \omega$

   First we check if $U(\Omega)$ is countable. As $U(\omega) = \omega$ we have,

   $$U(\Omega) = U(\{1, \ldots, 6\}) = \{U(\omega) : \omega \in \Omega\} = \{1, 2, 3, 4, 5, 6\}$$

   which is a countable subset of $\mathbb{R}$. Now we check if for any $a \in \mathbb{R}$ it's preimage is in the family of events.

   Take $a = 1$ we have $\{\omega : X(\omega) = 1\} = \{1\} \subset \Omega$. However, we see that $\{1\} \notin \mathscr{F}$ which means that it fails the condition and hence U is not a discrete random variable.

2. $V(\omega) = \begin{cases} 1 & \text{if } \omega \text{ is even} \\ 0 & \text{if } \omega \text{ is odd} \end{cases}$

   We see that V maps all $\omega \in \Omega$ to either $0, 1$. Hence, we have,

   $$V(\Omega) = V(\{1, \ldots, 6\}) = \{V(\omega) : \omega \in \Omega\}$$

   Now, as $\omega \in \Omega$ can either be even or odd we have, $\{V(\omega) : \omega \in \Omega\} = \{0, 1\}$ which is a countable subset of $\mathbb{R}$.

   Now, consider any $a \in \mathbb{R}$ we need to see if it's preimage is in the family of events. For $a = 1$ we have $\{\omega \in \Omega : X(\omega) = 1\} = \{\omega : \omega \text{ is even}\} = \{2, 4, 6\}$ and we see that $\{2, 4, 6\} \in \mathscr{F}$. Similarly for $a = 0$ we have $\{\omega \in \Omega : X(\omega) = 0\} = \{\omega : \omega \text{ is odd}\} = \{1, 3, 5\}$ and we see that $\{1, 3, 5\} \in \mathscr{F}$. And lastly for $a \neq 1, 0$ we have $\{\omega : X(\omega) \neq 1, 0\} = \phi \in \mathscr{F}$. Hence, V satisfies both conditions making it a discrete random variable.

3. $U(\omega) = \omega^2$ First we check if $W(\Omega)$ is countable. We have,

   $$W(\Omega) = \{W(\omega) : \omega \in \Omega\} = \{1^2, 2^2, \ldots, 6^2\} = \{1, 4, 9, 16, 25, 36\}$$

   which is a countable subset of $\mathbb{R}$.

   Now, consider any $a \in \mathbb{R}$ and we check the preimage of a. Take $a = 1$ we have $\{\omega : X(\omega) = 1\} = \{1\}$. But we see that $\{1\} \notin \mathscr{F}$. Hence W is not a discrete random variable.

Problem 2.24

We have X a discrete random variable having geometric distribution. Which means that,

$$\mathbb{P}(X = k) = p(1-p)^{k-1}$$

We need to find $\mathbb{P}(X > k)$ or $\mathbb{P}(X = k+1) + \mathbb{P}(X = k+2) + \dots$ which is $\sum_{n=1}^{\infty} \mathbb{P}(X = k+n)$ as X is geometric we have,

$$\begin{aligned}\sum_{n=1}^{\infty} \mathbb{P}(X = k+n) &= \sum_{n=1}^{\infty} p(1-p)^{k+n-1} \\ &= p\sum_{n=1}^{\infty}(1-p)^{k+n-1} \\ &= p((1-p)^k + (1-p)^{k+1} + \dots)\end{aligned}$$

Now using sum of geometric series we have,

$$\begin{aligned}\mathbb{P}(X > k) &= p((1-p)^k + (1-p)^{k+1} + \dots) \\ &= p\frac{(1-p)^k}{p} \\ &= (1-p)^k\end{aligned}$$

Problem 4

We need value of c and α such that,

$$p(k) = \begin{cases} ck^{\alpha} & \text{for } k = 1, 2, \dots \\ 0 & \text{otherwise} \end{cases}$$

is a mass function.

A mass function is defined as $p(x) = \mathbb{P}(X = x)$ if X is a discrete random variable. So we have $\mathbb{P}(X = k) = ck^{\alpha}$ if $k = 1, 2, 3, \dots$ else $\mathbb{P}(X = k) = 0$. So we need $\sum_{k=1}^{\infty} \mathbb{P}(X = n) = ck^{\alpha} = 1$. So we have,

$$\sum_{k=1}^{\infty} ck^{\alpha} = 1$$
$$c\sum_{k=1}^{\infty} k^{\alpha} = 1$$

Now the summation only converges if $\alpha < -1$. Assume it converges to m then we can define $c = \frac{1}{m}$.

Problem 5

We need to show that $\mathbb{P}(X > m+n \mid X > m) = \mathbb{P}(X > n)$. In geometric distribution we know that $\mathbb{P}(X = k) = p(1-p)^{k-1}$ so we have $\mathbb{P}(X > n) = (1-p)^n$. Similarly we get,

$$\mathbb{P}(X > m+n \mid X > m) = \frac{\mathbb{P}((X > m) \cap (X > m+n))}{\mathbb{P}(X > m)}$$

Now if $X > m$ and $X > m+n$ as the first is included in the second it is equivalent to $X > m+n$ so we have,

$$\begin{aligned}
\mathbb{P}(X > m+n \mid X > m) &= \frac{\mathbb{P}((X > m) \cap (X > m+n))}{\mathbb{P}(X > m)} \\
&= \frac{\mathbb{P}(X > m+n)}{\mathbb{P}(X > m)} \\
&= \frac{(1-p)^{m+n}}{(1-p)^m} \\
&= (1-p)^{m+n-m} = (1-p)^n \\
&= \mathbb{P}(X > n)
\end{aligned}$$

For the lack of memory property we need $\mathbb{P}(X > m+n) = \mathbb{P}(X > n)\mathbb{P}(X > m)$.

It is enough to show that if $\mathbb{P}(X > m+n) = \mathbb{P}(X > n)\mathbb{P}(X > m)$ is true then the distribution is geometric. Let us define a function $f : \mathbb{R} \to [0,1]$ as $f(k) = \mathbb{P}(X > k)$ so we have $f(m+n) = f(m)f(n)$.

Now take $m = 0$ for some n we have $f(0+n) = f(0)f(n)$ which means that $f(0) = 1$ or $f(n) = 0$ for all n. If $f(n) = 1$ for all n then it's trivially memory less and hence geometric. If $f(0) = 1$ then consider $f(1) = f(1+0) = f(1)f(0)$ so $f(1) = p$ for some $p \in [0,1]$ now by induction we can show that for any k we have $f(k) = f(k-1+1) = f(k-1)f(1) = p^k$. So we have showed that we need $f(k) = \mathbb{P}(X > k) = p^k$. But this means that $\mathbb{P}(X = k) = \mathbb{P}(X > k-1) - \mathbb{P}(X > k) = p^{k-1} - p = p^{k-1}(1-p)$ which is the distribution for the geometric r.v.

Problem 7

Coupon-collecting problem. There are c different types of coupon, and each coupon obtained is equally likely to be any one of the c types. Find the probability that the first n coupons which you collect do not form a complete set, and deduce an expression for the mean number of coupons you will need to collect before you have a complete set.
We have c types of coupons with each coupon equally likely as the others. We need to find probability of first n coupons do not form a complete set.

Let us begin by defining a discrete random variable $X : \Omega \to \mathbb{R}$ defined as the number of coupons collected before getting a complete set. So we have $\mathbb{P}(X = n)$ is the probability that we get the competition of c coupons in the $n'th$ draw. So we need to find $\mathbb{P}(X > n)$ as that is the probability that we get a complete set only if we take more than n coupons. So we have,

$$\mathbb{P}(X < c) = 0$$

For $X = k \geq c$ we need the first $k-1$ coupons to NOT have the c coupons. We have c choices for the $k'th$ draw to be any coupon, so now we count the ways the remaining $c-1$ cards are distributed such that all of them are assigned a spot among the $k-1$ spots at least once. This is equivalent to the surjections from $k-1$ onto $c-1$. Now surjections from m to n is,

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)^m$$

So we have surjections from $k-1$ to $c-1$ as,

$$\sum_{n=0}^{c-1} (-1)^n \binom{c-1}{n} (c-1-n)^{k-1}$$

So we have,

$$\mathbb{P}(X = k) = \frac{c}{c^k} \sum_{n=0}^{c-1} (-1)^n \binom{c-1}{n} (c-1-n)^{k-1} \quad \text{if } k \geq c$$

Now we need $\mathbb{P}(X > n)$ as that is the probability that we have all c coupons being collected after only collecting greater than k coupons being collected. For this we can sum of the probability from $k+1$ to ∞ to get,

$$\mathbb{P}(X > n) = \sum_{k=n+1}^{\infty} \mathbb{P}(X = k) = \sum_{k=n+1}^{\infty} \frac{c}{c^k} \sum_{m=0}^{c-1} (-1)^m \binom{c-1}{m} (c-1-m)^{k-1}$$

Now for mean number of coupons we need $\mathbb{E}[X]$. First write $X = A_1 + \cdots + A_n$ where A_i defines the number of draws it takes to get the $i'th$ coupon given we got $i-1$ coupons. So here X is the sum of draws it takes to get each of the coupons. Now we find $E[A_i]$, first we know that the probabiltiy of getting an $i'th$ new coupon given $i-1$ coupons is $\frac{c-i+1}{c}$ so this gives us $E(A_i) = \frac{c}{c-i+1}$ or that,

$$\begin{aligned} E(X) &= E(A_1 + \cdots + A_n) \\ &= \frac{c}{c} + \frac{c}{c-1} + \frac{c}{c-2} + \dots \\ &= c \sum_{i}^{c} \frac{1}{i} \end{aligned}$$