Real Analysis: HW8

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Exercise 4.5.3 Let f be such a function for which for $x < y$ in $[a, b]$ we have for any L s.t. $f(x) \leq L \leq f(y)$ some c for which we get $f(c) = L$. We need to show that f is continuous on $[a, b]$.

Consider any arbitrary $c \in [a, b]$ now we'll show continuity at c. First consider the left side of c i.e. x for which we have $a \leq x < c$. Note that between $f(a)$ and $f(c)$ using the IVP (on $[a, c]$) for any value between them we can find an x satisfying it. So for the arbitrary ε consider the region $(f(a), f(c)) \cap (f(c) - \varepsilon, f(c))$ and note that we can find some x_1 in $[a, c)$ for which we have $f(x_1) = \max f(c) - \varepsilon, f(a)$ so for any value $x \in (x_1, c)$ we have $-\varepsilon < f(x) - f(c)$. Similarly for the case of $c < x \leq b$ we can find some x_2 such that we have for $x \in (c, x_2)$ we get $f(x) - f(c) < \varepsilon$. So choose $\delta = \min\{|x_1 - c|, |x_2 - c|\}$ then for any $|x - c| < \delta$ we get $|f(x) - f(c)| < \varepsilon$.

Exercise 4.5.7 We have f is continuous on $[0, 1]$ with range in there as well. To show there is some x for which we have $f(x) = x$. Let $g(x) = f(x) - x$. It is enough to show that $g(x)$ has a root.

Consider $x = 0$ we have $g(0) = f(0) - 0$ so the possible value of $g(0)$ are in $[0, 1]$. Now consider $g(1) = f(1) - 1$ we have $f(1) \in [0, 1]$ so $f(1) - 1 \in [-1, 0]$. Now consider the two cases,

Case 1: Either $g(0) = 0$ or $g(1) = 0$. For the first we get $g(0) = f(0) - 0 = 0$ or $f(0) = 0$ which gives us a fixed point 0. For the second we have $g(1) = f(1) - 1 = 0$ which means $f(1) = 1$ and we have a fixed point 1.

Case 2: We have both $g(0) \neq 0 \Rightarrow g(0) \in (0, 1]$ and $g(1) \neq 0 \Rightarrow g(1) \in [-1, 0)$. This means that $g(0)$ is necessarily negative and $g(1)$ positive. So using the intermediate value theorem as g is continuous as f and x are both continuous we have for any $L \in [g(0), g(1)]$ there is some x such that $g(x) = L$. But as $g(0)$ is negative and $g(1)$ is positive we have $0 \in [g(0), g(1)]$ so there must be some $x \in [0, 1]$ such that we have $g(x) = L = 0 = f(x) - x$ which gives us $f(x) = x$ for some $x \in [0, 1]$

Exercise 6.2.1 We have,

$$f_n(x) = \frac{nx}{1 + nx^2}$$

(a). Pointwise limit is $f(x) = \frac{1}{x}$. Consider any arbitrary $x \in (0, \infty)$. Now for any $\varepsilon > 0$ choose $N = \frac{1}{\varepsilon x^3}$. For any $n > N$ we have,

$$\begin{aligned}
\left|\frac{nx}{1+nx^2}-\frac{1}{x}\right| &= \left|\frac{nx^2}{x(1+nx^2)}-\frac{(1+nx^2)}{x(1+nx^2)}\right| \\
&= \left|\frac{1}{x+nx^3}\right| \\
&\leq \left|\frac{1}{nx^3}\right|
\end{aligned}$$

But we have $n > N = \frac{1}{\varepsilon x^3}$ which gives us $\frac{1}{nx^3} < \varepsilon$ so we have,

$$\left|\frac{nx}{1+nx^2}-\frac{1}{x}\right| \leq \left|\frac{1}{nx^3}\right| < \varepsilon$$

Which means for any x we have the sequence of numbers $f_n(x)$ converge to $f(x)$.

(b). Note that the bounds are dependent on x, so the convergence is not uniform on $(0,\infty)$. For instance if there was a fixed bound n choosing $x = \frac{1}{n}$ we see that $|f_n(x) - f(x)| = \frac{1}{\frac{1}{n}(1+\frac{1}{n})} = \frac{n^2}{1+n} > \frac{n^2}{2n} = \frac{n}{2}$. Hence it cannot be uniformly convergent.
(c). It is not uniformly convergent for the same reason as (b), we can pick $x = \frac{1}{n}$ which will be in $(0,1)$.
(d). It is uniformly convergent. Note that we have,

$$\begin{aligned}
\left|\frac{nx}{1+nx^2}-\frac{1}{x}\right| &= \left|\frac{1}{x+nx^3}\right| \\
&\leq \left|\frac{1}{nx^3}\right|
\end{aligned}$$

But as $x \in (1,\infty)$ we have $\frac{1}{nx^3} < \frac{1}{n}$ so we get,

$$\left|\frac{nx}{1+nx^2}-\frac{1}{x}\right| \leq \left|\frac{1}{nx^3}\right| \leq \left|\frac{1}{n}\right| < \varepsilon$$

So for any choice of ε we can choose $N = \frac{1}{\varepsilon}$ such that for $n > N$ we get,

$$\left|\frac{nx}{1+nx^2}-\frac{1}{x}\right| < \varepsilon$$

and the choice of N is independent of x.

Exercise 6.2.2
(a). We have,

$$f_n(x) = \begin{cases} 1 & \text{if } x = 1, \frac{1}{2}, \dots, \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$$

Each f_n is continuous at 0 as for any choice of ε we can choose $\delta = \frac{1}{n+1}$ which gives us $|x| < \frac{1}{n+1}$ which means $f_n(x) = 0$ so $|f_n(x) - f(0)| = 0 < \varepsilon$ hence making it continuous.

We see that f is not continuous as for any δ we choose we can find n large enough such that $x = \frac{1}{n} < \delta$ but $f(x) = 1 \neq 0$ hence making it discontinue.

As f is discontinue we can say that f_n does not uniformly converge to f as if it did then f would also have to be continuous but it is not.
(b).

$$g_n(x) = \begin{cases} x & \text{if } x = 1, \frac{1}{2}, \dots, \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$$

Here, note that g_n is continuous at zero for the same reason as above, for all $x < \frac{1}{n}$ the function returns 0 and hence is equal to $f(0)$ as well.

Here f is continuous as for any ε choose $\delta = \varepsilon$ then for $x < \delta$ we have either x is of the form $\frac{1}{n}$ which gives us $|f(x) - f(0)| = |f(x)| = |x| < \delta = \varepsilon$ or it is not in that form which means $|f(x) - f(0)| = 0 < \varepsilon$ and hence it is continuous.

The convergence is uniform as we can choose $N = \frac{1}{\varepsilon}$ and for any $n > \frac{1}{\varepsilon}$ we get,

$$|g_n(x) - g(x)| < \varepsilon$$

(c).
Here each h_n is continuous at 0, and the limit h is continuous as for any ε. However note that in this case the convergence is not uniform as if we consider the sequence $x_n = \frac{1}{n}$ then we have $|h(x_n) - h_n(x_n)| = |1 - \frac{1}{n}|$ which cannot be made arbitrarily small as $\frac{1}{n}$ decreases as n increases.

Exercise 6.2.5
($\Rightarrow$) First assume that f_n converges to f uniformly, so we can find N large enough such that for any $\varepsilon > 0$ we have for all x that $|f_n(x) - f(x)| < \varepsilon/2$. Now note that for any $m, n > N$ we have,

$$\begin{aligned} |f_m(x) - f_n(x)| &= |f_m(x) - f(x) + f(x) - f_n(x) \\ &\leq |f_m(x) - f(x)| + |f_n(x) - f(x) \\ &\leq \varepsilon \end{aligned}$$

which is the forward direction.
($\Leftarrow$) Assume we have N such that for $m, n > N$ we get $|f_m(x) - f_n(x)| < \varepsilon$ for any $\varepsilon > 0$ for all x. Now consider some arbitrary x then using cauchy criteron for sequences we get that that the sequence of numbers $f_i(x)$ has a limit say $f(x) = L$. Now as this is true for arbitrary x we have pointwise convergence to some function f. Now we need to show that it is uniform convergence.

Now for $n \geq N$ we have,

$$|f_n(x) - f(x)| \leq |f_n(x) - f_m(x)| + |f_m(x) - f(x)|$$

Now because of cauchy critera we have $|f_n(x) - f_m(x)| < \varepsilon/2$. And now as f_i converges to f for any x we can choose m large enough such that we get $|f_m(x) - f(x)| < \varepsilon/2$ so this give sus,

$$|f_n(x) - f(x)| < \varepsilon$$

for all x for a fixed N