Intro to Proofs: HW10

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Problem 37

Proof. We have $H = \{2^k : k \in \mathbb{Z}\}$ we need to show its a subgroup of Q^*. First we know that $H \subseteq Q^*$ because $2^k \in Q^*, k \in \mathbb{Z}$. Now we need to show that H is a group under multiplication.
1. Closed under the operation. Consider an arbitrary element $x \in H = 2^{k_1}$ and consider $y \in H = 2^{k_2}$. Now we show that $x \cdot y \in H$. We have,

$$x \cdot y = 2^{k_1} \cdot 2^{k_2} = 2^{k_1+k_2}$$

As $k_1+k_2 \in \mathbb{Z}$ we have $2^{k_1+k_2} \in H$. Hence H is closed under multiplication.
2. Identity element. We need to show there exists $e \in H$ such that for any $g \in H$ we have $g \cdot e = g$. Consider $k = 0$ we have $1 = 2^0 \in Z$. Now we have for any $g \in H = 2^{k'}$,

$$g \cdot e = 2^{k'} \cdot 2^0 = 2^{k'} \cdot 1 = 2^{k'} = g$$

Hence we have defined an identity element $e = 2^0 = 1$
3. Inverse element. We need to show that for any $g \in H, \exists g' \in H$ such that $g \cdot g' = e$. Consider $g \in H, g = 2^k$. Now for any g let,

$$g' = 2^{-k} \in H \text{ as } -k \in \mathbb{Z}$$

So we have,
$$g \cdot g' = 2^k \cdot 2^{-k} = 2^{k-k} = 2^0 = 1 = e$$

Hence we found g' for any g such that $g \cdot g' = e$ which is our inverse.
4. Associatively. We need to show that,

$$2^{k_1} \cdot (2^{k_2} \cdot 2^{k_3}) = (2^{k_1} \cdot 2^{k_2}) \cdot 2^{k_3}$$

We have,

$$\begin{aligned} 2^{k_1} \cdot (2^{k_2} \cdot 2^{k_3}) &= 2^{k_1} \cdot (2^{k_2+k_3}) \\ &= 2^{k_1+(k_2+k_3)} \\ &= 2^{(k_1+k_2)+k_3} \\ &= 2^{k_1+k_2} \cdot 2^{k_3} \\ &= (2^{k_1} \cdot 2^{k_2}) \cdot 2^{k_3} \end{aligned}$$

Hence we show associativity.

□

Problem 43

Proof. It is enough to show that for any $x, y \in SL_2(\mathbb{Z})$ that $xy^{-1} \in SL_2(\mathbb{Z})$. So first consider $x, y \in SL_2(\mathbb{Z})$.

Let,

$$x = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and

$$y = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

We have,

$$y^{-1} = \begin{bmatrix} h & -f \\ -g & e \end{bmatrix}$$

Such that,

$$xy^{-1} = \begin{bmatrix} ah - bg & -af + be \\ hc - gd & -fc + de \end{bmatrix}$$

We know each element in xy^{-1} are integers because we're only multiplying and adding integers together. Now we need to show that $det(xy^{-1}) = 1$
We know,

$$det(xy^{-1}) = det(x)det(y^{-1})$$

But we also known that $det(y^{-1}) = \frac{1}{det(y} = \frac{1}{1} = 1$ and that $det(x) = 1$. So we get,

$$det(xy^{-1}) = 1 \cdot 1 = 1$$

So we have $xy^{-1} \in SL_2(\mathbb{Z})$

□

Problem 45

Proof. Consider two subgroups $H, F \subseteq G$. We need to show that $H \cap F$ is a subgroup of G as well. First we have $H \cap F \subseteq G$. Now we need to show its a group.
It is enough to show that for any $x, y \in H \cap F$ that $xy^{-1} \in H \cap F$. First we know for any $x, y \in H \cap F$ that means that $x, y \in H$ and $x, y \in F$. But we know that H and F are subgroups. Hence if $x, y \in H$ that means that $xy^{-1} \in H$. Similarly if $x, y \in F$ we have $xy^{-1} \in F$. So now we have, $xy^{-1} \in F$ and $xy^{-1} \in H$. This means that $xy^{-1} \in H \cap F$. So we have shown that $H \cap F$ is a subgroup. □

Problem 46

Proof. Consider the group Z and the subgroup $A = \{2^k : k \in \mathbb{Z}\}$ and $B = \{3^k : k \in \mathbb{Z}\}$. Both these are groups because they are orbits. Now consider their union $A \cup B$.
Take $x \in A, x = 2$ and $y \in B, y = 3$. We see that $xy = 2 \cdot 3 = 6 \notin A \cup B$ because $\nexists k \in Z$ such that $2^k = 6$ or $3^k = 6$. Hence it is not closed under the operation. Therefore is not a group. □

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Problem 52

Proof. Consider S_3 which is group of all permutations on three elements.

$$S_3 = \{e, (12), (13), (23), (123), (132)\}$$

this group is non-abelion because $(12)(23) \neq (23)(12)$. Hoover consider the subgroup,

$$\{e, (12)\}$$

Now this group is ableian because the elements commute. Hence we disprove the claim. $\square$