Linear Alebgra 3D

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Problem 1

Proof. We are given that T^{-1} is the inverse of T which menas that $TT^{-1} = I$ and $T^{-1}T = I$. Now we know that if $AB = BA = I$ then B is the inverse of T. So similarly we see that $T$is the inverse of T^{-1}. Hence $(T^{-1})^{-1} = T$ □

Problem 2

Proof. We need to show that $(ST)^{-1} = T^{-1}S^{-1}$. If ST is invertiebl we can find M such that $STM = I$ and $MST = I$. Let $M = T^{-1}S^{-1}$
So we have,

$$\begin{aligned} STM &= ST(T^{-1}S^{-1}) \\ &= STT^{-1}S^{-1} \\ &= SS^{-1} \\ &= I \end{aligned}$$

And,

$$\begin{aligned} MST &= T^{-1}S^{-1}ST \\ &= T^{-1}T \\ &= I \end{aligned}$$

Hence ST is invertible and the inverse is $T^{-1}S^{-1}$

□

Problem 3

Proof. (a) $\Rightarrow$ (b) We know that T is invertiebl hence it is both injective and surjective. We can say that $Tv_1, \ldots, Tv_n$ spans range of T. But we nkow that range $T = V$ as it is injective. We also know that $\dim V = n = \dim \operatorname{range} T$.
Hence we have n vectors $Tv_1, \ldots, Tv_n$ that span V and because the number of vectors are equal to basis length the vectors themselve form a basis of V.
(b) $\Rightarrow$(c) (c) is just a specific case of (b)
(c) $\Rightarrow$ (a) For some basis $v_1, \ldots, v_n$ we know that $Tv_1, \ldots, Tv_n$ is a basis of V. So that means that for any $v \in V$, v is in the span of $Tv_1, \ldots, Tv_n$ or that it is in the range of T. This means that V is surjective. As it is a map onto itself and T is surjective this means T is also injective. Hence T is invertiebl. □

Problem 5

Proof. $\Leftarrow$
We have T is an invertiebl linear map such that $Tu = Su, \forall u \in U$ and we

need to show that S is injective.
For any $u_1, u_2 \in V$ we need to show that $Su_1 = Su_2 \Rightarrow u_1 = u_2$
We know that $Su_1 = Tu_1$ and $Su_2 = Tu_2$. So we have $Tu_1 = Tu_2$. But we know that T is invertibel hence it is also injective. So we have $u_1 = u_2$. Hence S is injective.
$\Rightarrow$ We know that S is injective from U to V, we need to construct a map T such that $Tu = Su$ but T is invertible (both injective and surjective).
First consider the basis of U as $u_1, \ldots, u_n$. We know S is injective hence null S $= \{0\}$. So we know that S can be written as,

$$Su_1 = v_1, \ldots, Su_n = v_n$$

Such that $v_1, \ldots, v_n$ span range S and as $\dim \operatorname{range} S = \dim U = n$. $v_1, \ldots, v_n$ are linearly independent.
Now let us first extend our basis of U to a basis of V as follows,$u_1, \ldots, u_m$ and extned our basis of range S to a basis of V as $v_1, \ldots, v_m$.
Let us define a linear map as follows,

$$\begin{aligned}
Tu_1 &= Su_1 = v_1 \\
&\ldots \\
Tu_n &= Su_n = v_n \\
Tu_k &= v_k \text{ for k > n}
\end{aligned}$$

Now based on our definition we have range T is spanned by $v_1, \ldots, v_n$ whichi spans V. Hence range T = V. Which both means that it is surjective and because null space is $\{0\}$ we also have injectivity. Hence T is invertible.
Based on our definitino $T(u) = S(u)$ is also true, as for any $u = a_1u_1 + \ldots a_nu_n$ we have, $S(u) = a_1Su_1 + \cdots + a_nSu_n = a_1Tu_1 + \cdots + a_nTu_n = T(a_1u_1 + \cdots + a_nu_n) = T(u)$ □

Problem 9

Proof. We know T is surjective we need to show there is a subspace U of V such that $T|_U$ is an isomophism of U onto W.
First because T is surjection we have range $T = W$. Consider $v_1, \ldots, v_n$ is a basis on V then we can say T is defined as,

$$Tv_1 = w_1, \ldots, Tv_n = w_n$$

Now because $w_1, \ldots, w_n$ span W let us reduce it to a basis of W, take this as $w_1, \ldots, w_k$ without loss of generality. Now consider the subspace spanned by $v_1, \ldots, v_k$. Now we need to show it is an isomophism or that it is injective and surjective.
Firstly we know it is surjective as it maps to $w_1, \ldots, w_k$ which we nkow span W. We need to show that it is injective.
Because U is spanned by n vectors and W is spanned by n vectors we nkow by rank nullity theorem that $\dim \operatorname{null} T = 0$ hence our map is injective.
□

Problem 11

Proof. $\Rightarrow$ We know that ST is invertieble. Assume to the contrary that S or T is not invertiebl.
(1). We have S is not invertiebl and $ST = S(T(v))$ is invertible this means that $\forall v \in V$ we can find $v' \in V$ such that $S(T(v')) = v$. So if $T(v') = v''$ then $S(v'') = v$ for any v. But this means that S is surjective. However based on our assumption we nkow S is not surjective. So this contradicts our assumption.
(2). Consider T is not invertible $\Rightarrow$ T is not injective. If T is not injective the $\exists v \in V$ such that $Tv = 0$. So we have $STv = S(0) = 0$ which means that $STv = 0$ for some non-zero v which makes ST not injective and not invertiebl. Hence a contradiction.
$\Leftarrow$ We can show $\exists M = T^{-1}S^{-1}$ such that $STM = I, MST = I$ $\square$

Problem 12

We have $STU = I$ this means that ST and U are both invertible which means that S and T are also invertibel. So $\exists S^{-1}$ and U^{-1}.

$$\begin{aligned} STU &= I \\ TU &= S^{-1} \\ U &= T^{-1}S^{-1} \\ US &= T^{-1}S^{-1}S \\ T^{-1} &= US \end{aligned}$$

Problem 13

Proof. Consider the backward shift operator $T((x_1, \dots)) = (x_2, \dots)$. It is not injective and hence not invertible. Let S be I and U be the forward shift operator. We have $STU = I$ but T is not invertible. $\square$

Problem 14

Proof. RST is a map from V to V hence RST is invertieble. This means that RST is injective. First lets show that T is invertible. Assume T is not invertiebl means that T is not injective or $\exists v \in V$ such that $Tv = 0$ but this means that $RSTv = 0$ but this is a contradictino as RST is invertieble. Now we know that T is invertible so range T is V.
Now assume S is not injective $\exists v \in$ range $T = V$ such that $S(v) = 0$ but this gives us a contradictino as well as RST is invertiebl. Hence S is injective.

$\square$

Problem 15

Proof. We know that $Tv_1, \dots, Tv_m$ span V. So, first let us reduce it to a basis of V as,

$$Tv_1, \dots, Tv_n$$

Now consider $v_1, \dots, v_n$. Let us assume $v_1, \dots, v_n$ is not linearly independent. This implies that exists not all zero coefficients $a_1, \dots, a_n$ such that,

$$a_1v_1, \dots + a_nv_n = 0$$

Now apply the operator T as follows,

$$T(a_1v_1 + \dots + a_nv_n) = 0$$

$$a_1(Tv_1) + \dots + a_nT(v_n) = 0$$

So we have not all zero coefficients such that linear combination of $Tv_1, \dots, Tv_n$ is 0. But this contrdicts our assumption that it is linearly independent. Hence our assumptino is wrong and $v_1, \dots, v_n$ is linearly independent.
Now we know that V is spanned by by n vectors $Tv_1, \dots, Tv_n$. So any list of n vectors that are linearly independent is a basis for V. Hence $v_1, \dots, v_n$ is a basis of V. We can further expand this basis $v_1, \dots, v_m$ and still have it spanning. □

Problem 16

Proof. We know that Tx in $F^{m,1}$ so $Tx = M(Tx)$,

$$= M(T)M(x)$$

$$Ax$$

□

Problem 20

Proof. □

Problem 21

Proof. Consider the operation as a linear map from $F^{n,1}$ to $F^{n,1}$. Let the map be given by T and the matrix is A.
(a). So we have, $Ax = 0 \Rightarrow x = 0$. This means that the null space is spanned only by the zero vector hence T is injective.
(b). We have for any $c \in F^{n,1}$,$\exists x \in F^{n,1}$ such that $Ax = c$. Now this means that our map is surjective.
As the linear map maps onto the same space we know that injectivity and surjectivity is bidirectional. Hence $(a) \Rightarrow (b)$ and $(b) \Rightarrow (a)$

$\square$