Intro to Proofs: HW07

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11.5.3

$\cdot$	[0]	[1]	[2]	[3]
[0]	[0]	[0]	[0]	[0]
[1]	[0]	[1]	[2]	[3]
[2]	[0]	[2]	[0]	[2]
[3]	[0]	[3]	[2]	[1]

$+$	[0]	[1]	[2]	[3]
[0]	[0]	[1]	[2]	[3]
[1]	[1]	[2]	[3]	[0]
[2]	[2]	[3]	[0]	[1]
[3]	[3]	[0]	[1]	[2]

11.5.6

For $\mathbb{Z}_6$ this is not true as we can take $[a] = [2] \neq [0]$ and $[b] = [3] \neq [0]$ and we have,

$$[a] \cdot [b] = [2] \cdot [3] = [6] = [0]$$

However for $\mathbb{Z}_7$ we cannot find equivalence classes $[a], [b]$ such that one of them is not [0]. Becuase for $\mathbb{Z}_7$ either a is a multiple of 7 or it is not. If,
1. a is a multiple of 7 then $[a] = [0]$ by definition.
2. a is not a multiple of 7 then a and 7 are coprime which means that if $[a][b] = [a][c]$ then $[b] = [c]$. Here $[c] = 0$ which means that $[a][b] = [0] = [a][c]$. We have a is coprime with $n = 7$ which means that $[b] = [c] = [0]$.
Hence either $[a] = 0$ or $[b] = 0$

11.5.7

1. $[8] + [8] = [16] = [7]$
2. $[24] + [11] = [35] = [8]$
3. $[21] \cdot [15] = [315] = [0]$
4. $[8][8] = [64] = [1]$

12.1.1

Domain is $A = \{0, 1, 2, 3, 4\}$
Range is $\{2, 3, 4\}$
$f(2) = 4$ and $f(1) = 3$

12.1.4

$$
\begin{aligned}
f_1 &= \{(a,0),(b,0),(c,0)\} \\
f_2 &= \{(a,0),(b,0),(c,1)\} \\
f_3 &= \{(a,0),(b,1),(c,0)\} \\
f_4 &= \{(a,0),(b,1),(c,1)\} \\
f_5 &= \{(a,1),(b,0),(c,0)\} \\
f_6 &= \{(a,1),(b,0),(c,1)\} \\
f_7 &= \{(a,1),(b,1),(c,0)\} \\
f_8 &= \{(a,1),(b,1),(c,1)\}
\end{aligned}
$$

12.1.5

A relation that is not a functino is,

$$R = \{(a,d),(a,e),(b,d),(c,d),(d,d)\}$$

12.1.8

First we know that the set is a relation because it is a subset of $\mathbb{Z}\mathbb{Z}$ by definition. Now we need to show that $\forall x \in \mathbb{Z}$ there exists only one ordered pair of the form $(x,y) \in f$.
We know all the pairs in our set are such that $x + 3y = 4$. So for any x,

$$3y = 4 - x$$

$$y = \frac{4-x}{3}$$

However we see that for $y \in \mathbb{Z}$ we need $x \equiv 4 \pmod 3$. However we know that this isn't true $\forall x \in \mathbb{Z}$. For instance take $x = 2$ then there isn't a $y \in Z$ such that that $(x,y) \in f$.
Hence because we can't assign a $y \in \mathbb{Z}$ for all $x \in \mathbb{Z}$ f is not a funciton.

12.2.5

1. For injectivity we need to shwo that for any $y \in \mathbb{Z}$ if $f(x) = f(x')$ that means that $x = x'$.
Consider $f(n) = 2n + 1$ and $f(n') = 2n' + 1$. We have,

$$
\begin{aligned}
f(n) &= f(n') \\
2n + 1 &= 2n' + 1 \\
2n &= 2n' \\
n &= n'
\end{aligned}
$$

which means that it is injective.
2. For surjective we need to show that for all $y \in \mathbb{Z}$ there exists an $x \in \mathbb{Z}$ such that $f(x) = y$.

Consider an arbitrary $y \in Z$ we have

$$y = 2n + 1$$
$$y - 1 = 2n$$
$$n = \frac{y-1}{2}$$

We see that for $n \in \mathbb{Z}$ we need $2|y-1$. However this isn't true $\forall y \in \mathbb{Z}$. For instance take any $y = 2k$ then there doesn't exists an n such that $f(n) = y$
Hence f is not surjective.

12.2.6

We have $f(m,n) = 3n - 4m$. To show injectivity we need to show for any $f(m,n) = f(m',n')$ we have $n = n', m = m'$.
If,

$$f(m,n) = f(m',n')$$
$$3n - 4m = 3n' - 4m'$$
$$3(n - n') = 4(m - m')$$

However now consider $n' = 1$ and $n = 5$ and $m = 4$ and $n' = 1$ and we have $12 = 12$.
So it is not injective.
Now we need to check surjectivity.
We need to show that for all $y \in \mathbb{Z}$, $\exists m, n$ such that $f(m,n) = 3n - 4m = y$.
Consider $y = 2k$ we have,

$$3n - 4m = 2k$$

We can choose $n = 2k$ and $m = k$
For $y = 2k + 1$ we have,

$$3n - 4m = 2k + 1$$

We cannot find m, n for all kwhich means that this isn't surjective.

12.2.10

$f : \mathbb{R} - \{1\} \to \mathbb{R} - \{1\}$ defined by $f(x) = (\frac{x+1}{x-1})^3$
1. Injectivity.
We need to show $f(x) = f(x') \Rightarrow x = x'$,

$$(\frac{x+1}{x-1})^3 = (\frac{x'+1}{x'-1})^3$$

Because we know that the terms are real we can say that,

$$\frac{x+1}{x-1} = \frac{x'+1}{x'-1}$$
$$xx' - x + x' - 1 = xx' + x - x' - 1$$
$$2x = 2x'$$
$$x = x'$$

So it is injective.
2. Bijective.
We have need to show for all y exists x such taht $f(x) = y$.
We have,

$$\begin{aligned}
y &= (\frac{x+1}{x-1})^3 \\
y^{\frac{1}{3}} &= \frac{x+1}{x-1} \\
y^{\frac{1}{3}}x - y^{\frac{1}{3}} &= x+1 \\
y^{\frac{1}{3}}x - x &= y^{\frac{1}{3}} + 1 \\
x(y^{\frac{1}{3}} - 1) &= y^{\frac{1}{3}} + 1 \\
x &= \frac{y^{\frac{1}{3}} + 1}{(y^{\frac{1}{3}} - 1)}
\end{aligned}$$

So for all y if $y \neq 1$ we have $x = \frac{y^{\frac{1}{3}}+1}{y^{\frac{1}{3}}-1}$ and we have,

$$\begin{aligned}
f(x) &= f(\frac{y^{\frac{1}{3}} + 1}{y^{\frac{1}{3}} - 1}) \\
&= \left(\frac{\frac{y^{\frac{1}{3}}+1+y^{\frac{1}{3}}-1}{y^{\frac{1}{3}}}}{\frac{y^{\frac{1}{3}}+1-y^{\frac{1}{3}}+1}{y^{\frac{1}{3}}}}\right)^3 \\
&= \left(\frac{y^{\frac{1}{3}} + 1 + y^{\frac{1}{3}} - 1}{y^{\frac{1}{3}} + 1 - y^{\frac{1}{3}} + 1}\right)^3 \\
&= \left(\frac{y^{\frac{1}{3}} + 1 + y^{\frac{1}{3}} - 1}{y^{\frac{1}{3}} + 1 - y^{\frac{1}{3}} + 1}\right)^3 \\
&= (\frac{2y^{\frac{1}{3}}}{2})^3 \\
&= (y^{\frac{1}{3}})^3 \\
&= y
\end{aligned}$$

Hence f is surjective.

12.2.14

We have $\theta(X) = \overline{X}$
1. Injectivity. We need to show $\theta(X) = \theta(X') \Rightarrow X = X'$
We have,

$$\begin{aligned}
\theta(X) &= \theta(X') \\
\overline{X} &= \overline{X'} \\
X &= X'
\end{aligned}$$

Hence θ is injective.
2. Surjectivity.
For all $Y \in P(X)$ we need to find $X \in P(X)$ such that $\theta(X) = Y$. We have,

$$\overline{X} = Y$$

$$X = \overline{Y}$$

So we ahve X such that $\theta(X) = Y$

$$\theta(X) = \theta(\overline{Y}) = \overline{\overline{Y}} = Y$$

Hence it is surjective.

12.2.16

Total number of functions are: 7^5
Number of injective functions are: $\frac{7!}{(7-5)!} = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3$
Total surjective functions are: 0
So total bijective functinos are : 0