Reaction-Diffusion Equation

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Abstract
The reaction-diffusion equation is a mathematical model that describes the dynamics of spatial patterns in systems where two or more substances interact and diffuse.

1 Introduction

1.1 Background

It is commonly used to study pattern formation in various scientific fields, including biology, chemistry, physics, and materials science. The basic form of the reaction-diffusion equation is given by:

$$.\frac{du}{dt} = D_u \nabla^2 u + f(u, v) \tag{1}$$

$$.\frac{dv}{dt} = D_v \nabla^2 v + g(u, v) \tag{2}$$

Here, u and v are the concentrations of two interacting substances, t is time,∇^2 is the Laplacian operator representing spatial diffusion, and D_u and D_v are the diffusion coefficients for u and v, respectively.

The functions $f(u, v)$ and $g(u, v)$ represent the reaction terms that describe how the substances interact. These functions often incorporate parameters that influence the behavior of the system, and they can exhibit various types of nonlinearities, such as the Lotka-Volterra or Gray-Scott models.

The reaction-diffusion equation is known for its ability to generate complex spatial patterns, such as spots, stripes, and spirals, through the interplay of diffusion and reaction processes. It has been used to model various phenomena, including chemical reactions, biological pattern formation, and the dynamics of excitable media.

Some well-known examples of reaction-diffusion systems include the Turing pattern, which is characterized by the spontaneous formation of patterns from homogeneous initial conditions, and the Belousov-Zhabotinsky reaction, a chemical system that exhibits oscillatory and wave-like behavior.

1.2 Turing Pattern Formation

Pattern formation is a phenomena seen throughout nature, where an initially homogenous system evolves to form structures with one or more wavelengths, often in steady state. Examples range from reasonably recognizable patterns such as stripes or spots on animals to more abstract patterns like the branching patterns of leaves.

The Schnakenberg system exhibits interesting behavior such as the formation of spatial patterns, including spots, stripes, and waves, depending on the values of the parameters , as well as the boundary conditions and initial conditions. This system is often used as a simplified model for studying biological pattern formation, chemical reaction networks, and other self-organizing systems.

2 Methods

- The fixed points of Turing pattern reaction is determined when $f(u, v) = 0$ and $g(u, v) = 0$.

$$f(u, v) = a - u - u^2 v = 0 \tag{3}$$

$$g(u, v) = b - u^2 v = 0 \tag{4}$$

$$u = a + b, v = \frac{b}{(a+b)^2}$$

- The stability of the turing pattern reaction is determined from its Jacobian matrix.

$$J = \begin{bmatrix} f_u & f_v \\ g_u & g_v \end{bmatrix} = \begin{bmatrix} -1 - 2uv & -u^2 \\ -2uv & -u^2 \end{bmatrix}$$

Evaluating Jacobian at $u = a + b, v = \dfrac{b}{(a+b)^2}$.

$$J = \begin{bmatrix} -1 - \dfrac{2b}{(a+b)} & -(a+b)^2 \\ \\ -\dfrac{2b}{(a+b)} & -(a+b)^2 \end{bmatrix}$$

The stability of non linear system can be determined by comparing τ and Δ with the below diagram

$$\tau = -1 - \frac{2b}{(a+b)} - (a+b)^2$$

$$\Delta = [(-1 - \frac{2b}{(a+b)}) * -(a+b)^2] - [-(a+b)^2 * -\frac{2b}{(a+b)})]$$

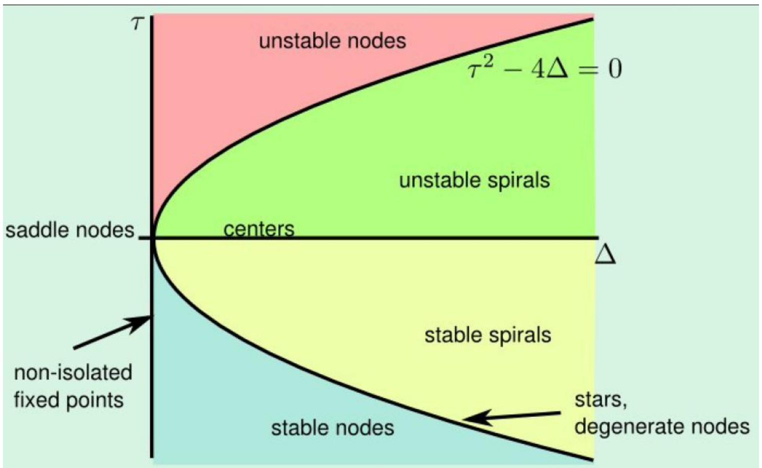

Figure 1: This graph illustrates the classification of fixed points according to τ vs Δ

- Existence and uniqueness

Suppose that $f(x)$ and $f^1(x)$ are continuous on an open interval R of the x-axis, and suppose that x_0 is a point in R. Then the initial value problem has a solution $x(t)$ on some time interval $(-\tau, \tau)$ about t = 0, and the solution is unique.
This theorem says that if f(x) is smooth enough, then solutions exist and are unique.

3 Results

Turing pattern reaction diffusion equation

$$\frac{du}{dt} = D_u \nabla^2 u + a - u - u^2 v \tag{5}$$

$$\frac{dv}{dt} = D_v \nabla^2 v + b - u^2 v \tag{6}$$

In the following below cases, i am changing the parameters to analyse the patterns formed

- Case 1
  Increasing the diffusion constant of u D_u

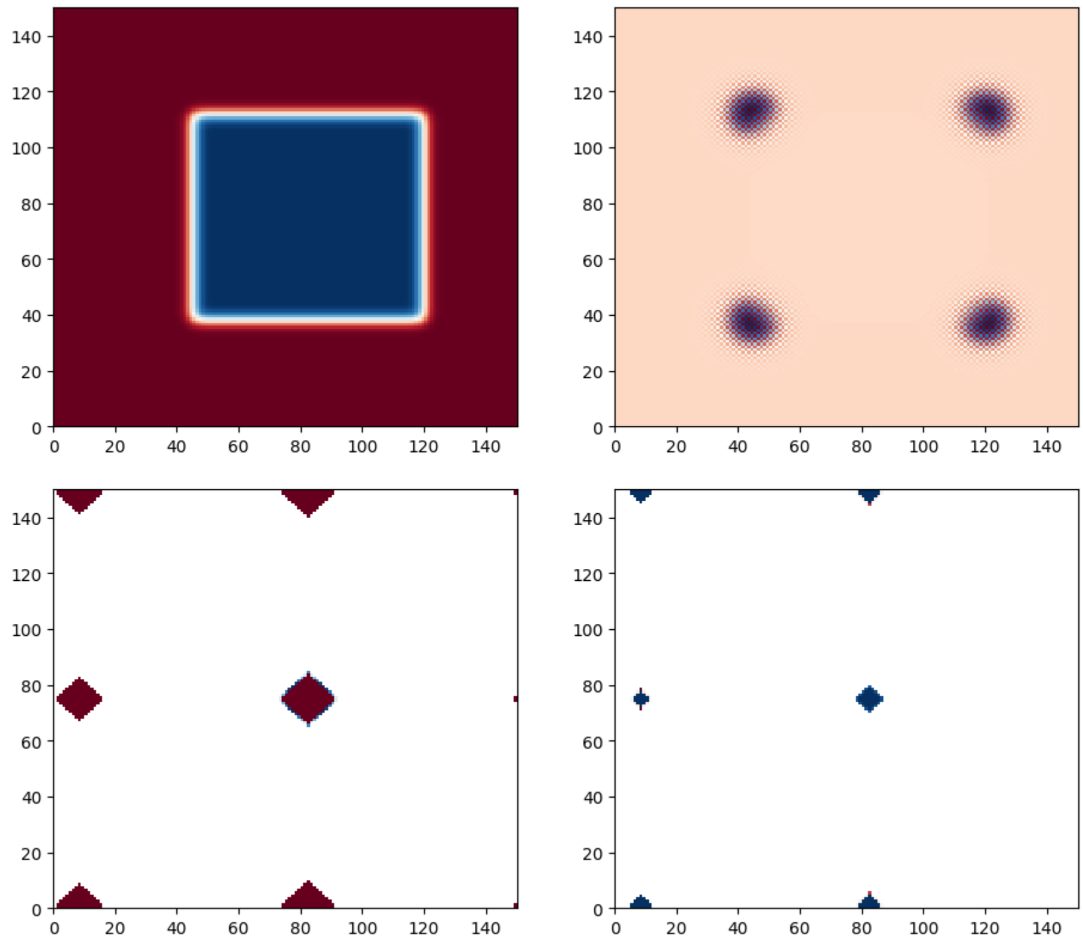

Figure 2: The plot of u when we change d_u to 0.2, 0.4,0.6,0.8 respectively

- Case 2
  Increasing the diffusion constant of v D_v

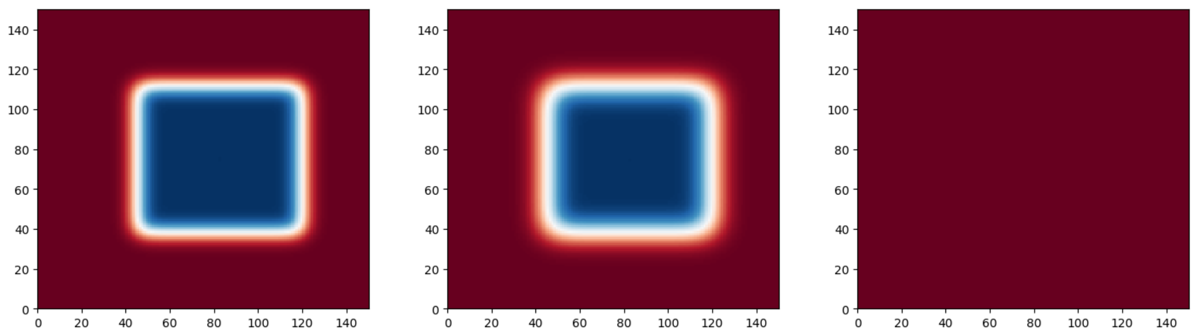

Figure 3: The plot of v when we change d_v to 0.25, 0.5,0.75 respectively

- Case 3
  "a" typically represents the rate of production of the u species

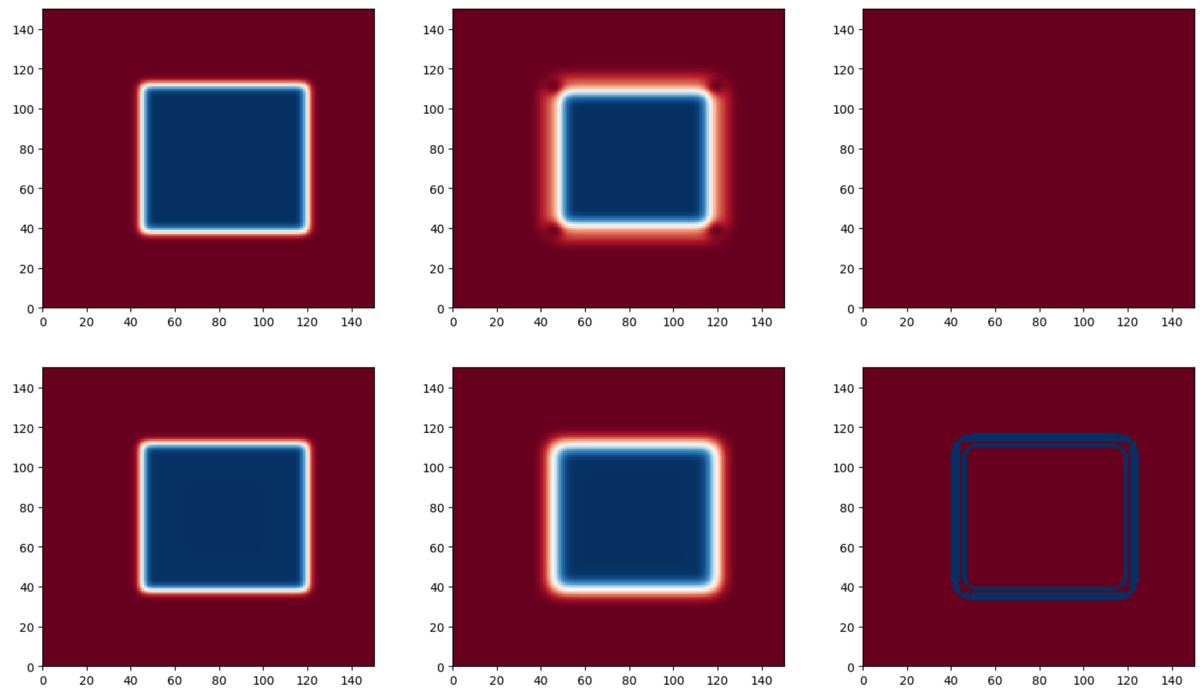

Figure 4: The plot of u and v when we change "a" to 0.2 & 0.6 respectively

- Case 4
  "b" represents the rate of consumption of u and v species

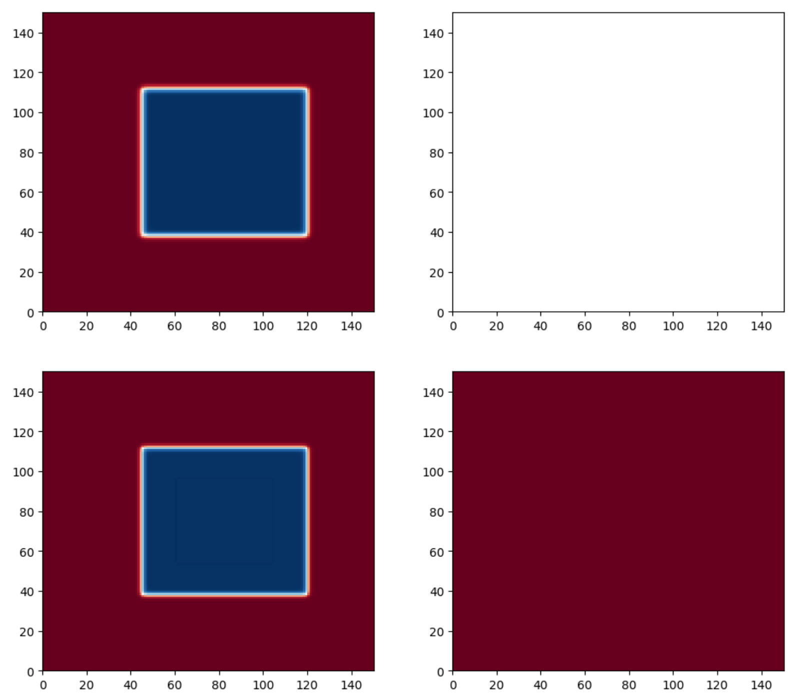

Figure 5: The plot of u and v when we change "b" to 0.2 & 0.6 respectively

4 Analysis

- Case 1
  As you increase the diffusion constant of u the color pattern diffuses eventually.

- Case 2
  As you increase the diffusion constant of v the color pattern diffuses gradually and red color dominates the pattern.

- Case 3
  when you alter the value for a-rate of production, not only does it change U but also V. Blue patterns diffuses and red dominates for u while the solid blue changes into red and the white border changes to blue .

- Case 4
  when you alter the value for b-rate of consumption of u and v species, not only does it change v but also u. The color pattern for u diffuses and become a blank plot, while the color pattern diffuses and red dominates

- Consider the case where $D_u = .15, D_v = .055, A = 0.02545, B = 0.062$
  Fixed point in the above condition is

  $$u_* = 0.02545 + 0.062 = 0.0875, v_* = 0.062/0.0875^2 = 8.09795918367$$

  The stability of non linear systems are

  $$\tau = -2.4248, \Delta = 1.839775$$

  since $\tau < 0$ its has saddle nodes

5 Conclusion

In conclusion, the Schnakenberg reaction-diffusion equation provides a simplified yet insightful model for understanding pattern formation and dynamics in reaction-diffusion systems. By considering the interactions between two chemical species u and v, governed by diffusion and reaction kinetics, the Schnakenberg model captures essential aspects of pattern formation observed in various biological and chemical systems. Through analysis of the Schnakenberg system, we can draw several conclusions

The Schnakenberg model demonstrates that simple local interactions between chemical species, coupled with diffusion processes, can lead to the emergence of complex spatial patterns. These patterns may include spots, stripes, or other organized structures, depending on the system parameters and initial conditions.
Stability analysis of the Schnakenberg system around its steady states provides insights into the conditions under which patterns form and persist. By examining the eigenvalues of the linearized system, we can determine the stability of steady states and predict their behavior under perturbations.
Overall, the Schnakenberg reaction-diffusion equation serves as a useful tool for studying

pattern formation phenomena in diverse natural systems, including biological morphogenesis, chemical reaction networks, and ecological dynamics. Its simplicity allows for analytical insights while capturing essential features of complex spatiotemporal dynamics, making it a valuable model in theoretical and computational biology, chemistry, and physics.

6 Acknowledgement

I had to use Artificial intelligence to paraphrase my passage especially my introduction and conlusion.

7 Appendix

```
#Author: Ashfaq

import numpy as np
import matplotlib.pyplot as plt
from matplotlib.animation import FuncAnimation

# functions

def creating_two_phenomenea(nx, ny):
 #initial conditions

    # these two lines are place holders
    u = np.ones((nx,ny))
    v = np.zeros((nx,ny))

    # grid
    x, y = np.meshgrid(np.linspace(0, 1, nx), np.linspace(0, 1, ny))

    mask = (0.30<x) & (x<0.60) & (0.30<y) & (y<0.60)

    u[mask] = 0
    v[mask] = 1

    return u, v


def periodic_bc(u):
    u[0, :] = u[-2, :]
    u[-1, :] = u[1, :]
    u[:, 0] = u[:, -2]
    u[:, -1] = u[:, 1]


def five_point_scheme(u):
```

```
    """
    Five point scheme
    """
    return ( u[ :-2, 1:-1] + u[1:-1, :-2] - 4*u[1:-1, 1:-1] + u[1:-1, 2:] + u[2:  , 1


def turing_pattern_formation(U, V, params):

    dt = params[0]
    d_u_coff = params[1]
    d_v_coff = params[2]
    A  = params[3]
    B  = params[4]

    u, v = U[1:-1,1:-1], V[1:-1,1:-1]

    Lu = five_point_scheme(U)
    Lv = five_point_scheme(V)

    uuv = u*u*v
    u += dt*( d_u_coff*Lu + A - u + uuv )
    v += dt*( d_v_coff*Lv + B  - uuv)

    periodic_bc(U)
    periodic_bc(V)




nu1, nu2 = .15, .055
A, B = 0.02545, 0.062

params = np.array([0.5, 0.15, 0.055, 0.02545, 0.062])

U, V = creating_two_phenomenea(50,50)

for t in range(8):
    turing_pattern_formation(U, V, params)

V_scaled = np.uint8(255*(V-V.min()) / (V.max()-V.min()))

plt.figure(figsize=(5,10))
plt.subplot(2,1,1)
plt.pcolormesh(U,cmap='RdBu')
plt.subplot(2,1,2)
plt.pcolormesh(V_scaled,cmap='RdBu')
```

```
plt.show()
```