Lecture 13

- Response of an LTI system to a periodic input.
- Application to rectifier example.
- Mean square approximation.
- Convergence in mean square.

Response of an LTI system to a periodic input.

LTI, causal and stable system (all poles of $H(s)$ in $\text{Re}[s] < 0$).

Recall: for a sinusoidal input $x(t) = e^{i\omega t}$, $y(t) = H(i\omega)\, e^{i\omega t}$

Now apply a periodic input of period T, $x(t) = \sum_{n=-\infty}^{+\infty} X_n\, e^{in\omega_0 t}$.

By linearity, the output is $y(t) = \sum_{n=-\infty}^{+\infty} H(in\omega_0) X_n\, e^{in\omega_0 t}$.

In particular, $y(t) = y(t+T)$ so the output is also periodic, and the above gives a Fourier expansion for the output.

Remark: this is for $x(t)$ defined on $t \in (-\infty, \infty)$.
If instead we start at $t = 0$, we also get a transient term.

$$x(t)=\sum_{n=-\infty}^{+\infty} X_n e^{in\omega_0 t} \longrightarrow \boxed{H(s)} \longrightarrow y(t)=\sum_{n=-\infty}^{+\infty} \underbrace{H(in\omega_0)X_n}_{Y_n} e^{in\omega_0 t}$$

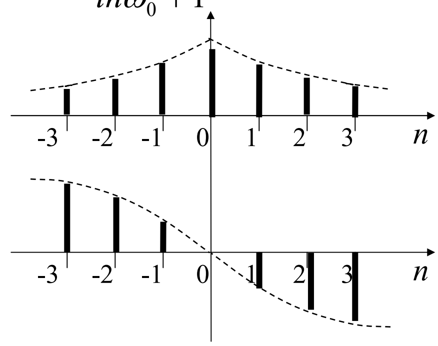
$$Y_n = H(in\omega_0)X_n \Rightarrow \begin{cases} |Y_n| = |H(in\omega_0)||X_n| \\ \theta_{Y_n} = \theta_{H(in\omega_0)} + \theta_{X_n} \end{cases}$$

Multiply magnitudes, add phases.

Example: $H(s)=\dfrac{1}{s+1} \Rightarrow H(in\omega_0)=\dfrac{1}{in\omega_0+1}$

$$|H(in\omega_0)| = \frac{1}{\sqrt{n^2\omega_0^2+1}}$$

$$\theta_{H(in\omega_0)} = -\text{Arctan}(n\omega_0).$$



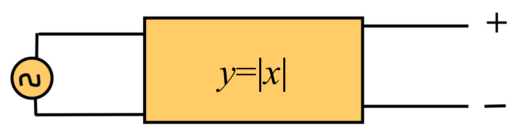
Attenuate higher frequencies: "lowpass filter".

Back to rectifier example

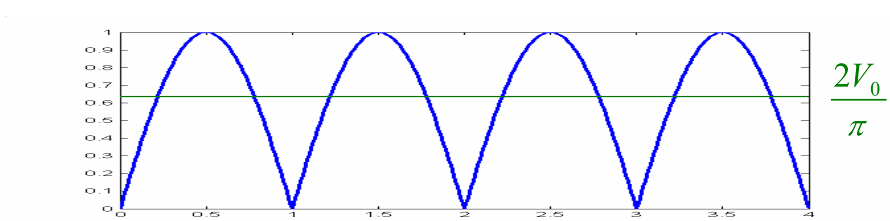
AC

DC

$x(t) = V_0 \sin(\omega_0 t)$



$y(t)$



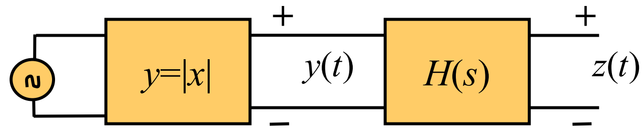
$\dfrac{2V_0}{\pi}$

$$y(t) = V_0|\sin(\omega_0 t)| = \underbrace{\frac{2V_0}{\pi}}_{\text{DC}} + \underbrace{\sum_{n\neq 0} \frac{2V_0}{\pi(1-4n^2)} e^{in\omega_{0y} t}}_{\text{AC COMPONENTS}}$$

Idea: add a lowpass filter in cascade to attenuate the AC components, and get a "purer" DC source.

Rectifier plus filter

$x(t) = V_0 \sin(\omega_0 t)$



For example, choose $H(s) = \dfrac{\alpha}{s+\alpha}$ (this can be implemented by an RC circuit with $\alpha = \dfrac{1}{RC}$, as in Hwk #5).

$$z(t) = \sum_{n=-\infty}^{+\infty} H(in\omega_{0y}) Y_n\, e^{in\omega_{0y}t} = \overbrace{H(0)Y_0}^{\text{DC}} + \overbrace{\sum_{n\neq 0} H(in\omega_{0y})\, Y_n\, e^{in\omega_{0y}t}}^{\text{AC COMPONENTS}}$$

Now $H(0) = 1$, so the DC component is unchanged.

$$H(in\omega_{0y}) = \frac{\alpha}{in\omega_{0y} + \alpha} = \frac{1}{i\dfrac{n\omega_{0y}}{\alpha} + 1} \Rightarrow \left| H(in\omega_{0y}) \right| = \frac{1}{\sqrt{\left(\dfrac{n\omega_{0y}}{\alpha}\right)^2 + 1}}$$

AC component is attenuated.

Recall: $\underbrace{\dfrac{1}{T}\displaystyle\int_0^T |y(t)|^2\, dt}_{\text{TOTAL POWER}} = \underbrace{|Y_0|^2}_{\substack{\text{DC POWER} \\ P_{DC}(y)}} + \underbrace{\sum_{n\neq 0} |Y_n|^2}_{\substack{\text{AC POWER} \\ P_{AC}(y)}}$ (Parseval)

For the rectifier, $P_{AC}(y)$ represents 19% of the total power.
Say we want to reduce it by a factor of 5 by filtering.

$$\frac{1}{T}\int_0^T |z(t)|^2\, dt = |Z_0|^2 + \sum_{n\neq 0} |Z_n|^2 = \underbrace{|H(0)Y_0|^2}_{P_{DC}(z)} + \underbrace{\sum_{n\neq 0} |H(in\omega_{0y})Y_n|^2}_{P_{AC}(z)}$$

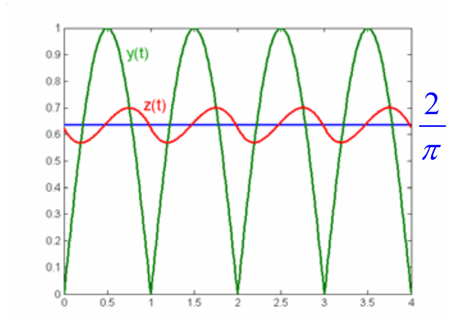
$P_{DC}(z) = P_{DC}(y)$. We want $P_{AC}(z) \le \dfrac{P_{AC}(y)}{5}$.

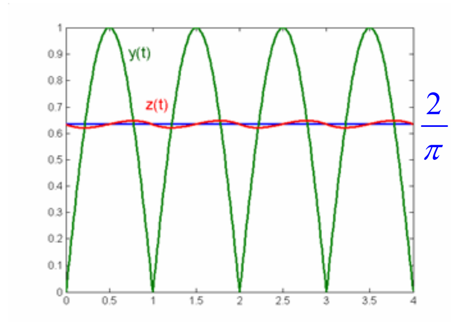
One design: impose $\left| H(in\omega_{0y}) \right|^2 = \dfrac{1}{\left(\dfrac{n\omega_{0y}}{\alpha}\right)^2 + 1} \le \dfrac{1}{5}$ for all $n \neq 0$

$$\Rightarrow \left(\frac{n\omega_{0y}}{\alpha}\right)^2 \ge 4 \;\; \forall\; n \neq 0, \text{ most restrictive for } n = \pm 1 : \boxed{\frac{\omega_{0y}}{\alpha} \ge 2}.$$

For the RC circuit implementation, $\boxed{\mathrm{RC}\,\omega_{0y} \ge 2}$

What does $z(t)$ look like?





$\omega_{0y} = 2\pi, \alpha = 1.$

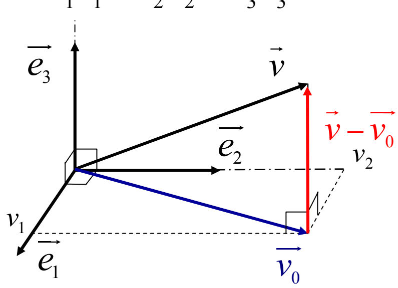
$\omega_{0y} = 2\pi, \alpha = 0.2$

As α becomes smaller, so does $|H(in\omega_{0y})| = \dfrac{1}{\sqrt{\left(\dfrac{n\omega_{0y}}{\alpha}\right)^2 + 1}}$

and the AC gets more attenuated.

Mean square approximation

Back to vectors. Question: what is the best approximation of $\vec{v} = v_1\vec{e_1} + v_2\vec{e_2} + v_3\vec{e_3}$ as a linear combination of only $\vec{e_1}$ and $\vec{e_2}$?



In other words, find α_1, α_2 such that $\vec{v_0} = \alpha_1\vec{e_1} + \alpha_2\vec{e_2}$ minimizes $|\vec{v} - \vec{v_0}|$.

Answer: it should be such that $\vec{v} - \vec{v_0}$ is perpendicular to the $\vec{e_1}, \vec{e_2}$ plane. In particular,

$$\left\langle \vec{v} - \vec{v_0}, \vec{e_n} \right\rangle = 0,\ n = 1,2 \ \Rightarrow v_n = \left\langle \vec{v}, \vec{e_n} \right\rangle = \left\langle \vec{v_0}, \vec{e_n} \right\rangle = \alpha_n,\ n = 1,2$$

So $\vec{v_0} = v_1\vec{e_1} + v_2\vec{e_2}$ and the approximation error is $|\vec{v} - \vec{v_0}| = |v_3|$

In particular, $|\vec{v}|^2 = |\vec{v_0}|^2 + |\vec{v} - \vec{v_0}|^2$ (Pythagoras' Theorem).

Extension to Fourier

Question: what is the best approximation of $f(t) = \sum_{n=-\infty}^{+\infty} F_n\, e^{in\omega_0 t}$ as a linear combination of $\left\{e^{in\omega_0 t}\right\}_{n=-N}^{N}$ only? In other words, find $\alpha_{-N},\ldots,\alpha_0,\ldots,\alpha_N$ such that $\widehat{f_N}(t) = \sum_{n=-N}^{+N} \alpha_n\, e^{in\omega_0 t}$ minimizes $\left\|f - \widehat{f_N}\right\|_{\mathrm{RMS}}^2 = \frac{1}{T}\int_0^T \left|f(t) - \widehat{f_N}(t)\right|^2 dt$. This is called mean square approximation.

The answer is analogous to the vector case: $\alpha_n = F_n,\ -N \le n \le N$.

In other words, the best approximation $\widehat{f_N}(t) = \sum_{n=-N}^{+N} F_n\, e^{in\omega_0 t}$ is just the truncation of the Fourier series.

Proof: Starting with $\widehat{f_N}(t) = \sum_{n=-N}^{N} \alpha_n\, e^{in\omega_0 t}$ we see that

$$f(t) - \widehat{f_N}(t) = \sum_{n=-N}^{N} \left(F_n - \alpha_n\right) e^{in\omega_0 t} + \sum_{|n|>N} F_n\, e^{in\omega_0 t}$$

The above is a Fourier expansion for $f(t) - \widehat{f_N}(t)$, so we can use Parseval to conclude that

$$\left\|f - \widehat{f_N}\right\|_{\mathrm{RMS}}^2 = \frac{1}{T}\int_0^T \left|f(t) - \widehat{f_N}(t)\right|^2 dt = \sum_{n=-N}^{+N} \left|F_n - \alpha_n\right|^2 + \sum_{|n|>N} \left|F_n\right|^2$$

Since all terms are positive the minimizer is $\alpha_n = F_n$, for $-N \le n \le N$.

Also, $\left\|f - \widehat{f_N}\right\|_{\mathrm{RMS}}^2 = \sum_{|n|>N} \left|F_n\right|^2$, which we denote by $\overline{\varepsilon_N^2}$. This is the mean square error when approximating $f(t)$ up to the N-th harmonic.

We can also generalize other two properties of the vector case:

1) Since $\left\langle f, e^{in\omega_0 t} \right\rangle = F_n = \left\langle \widehat{f_N}, e^{in\omega_0 t} \right\rangle$ for $-N \le n \le N$,

we have $\left\langle f - \widehat{f_N}, e^{in\omega_0 t} \right\rangle = 0, \quad -N \le n \le N.$

In an abstract sense, $f - \widehat{f_N} \perp e^{in\omega_0 t}$, the error function $f - \widehat{f_N}$ is orthogonal to the basis functions involved in the approximation. This is called the "orthogonality principle".

2) Generalization of Pythagoras' Theorem:

$$\|f\|_{\text{RMS}}^2 = \sum_{n=-\infty}^{\infty} |F_n|^2 = \sum_{n=-N}^{N} |F_n|^2 + \sum_{|n|>N} |F_n|^2 = \left\|\widehat{f_N}\right\|_{\text{RMS}}^2 + \left\|f - \widehat{f_N}\right\|_{\text{RMS}}^2$$

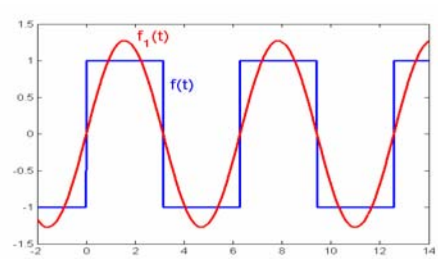
In particular, the mean square error can be computed by

$$\overline{\varepsilon_N^2} = \sum_{|n|>N} |F_n|^2 = \|f\|_{\text{RMS}}^2 - \left\|\widehat{f_N}\right\|_{\text{RMS}}^2 = \frac{1}{T}\int_0^T |f(t)|^2\, dt - \sum_{n=-N}^{N} |F_n|^2$$

$$\boxed{\overline{\varepsilon_N^2} = \sum_{|n|>N} |F_n|^2 = \frac{1}{T}\int_0^T |f(t)|^2\, dt - \sum_{n=-N}^{N} |F_n|^2}$$

Example: Square Wave $f(t) = \sum_{\substack{n=-\infty \\ n \text{ odd}}}^{+\infty} \frac{2}{in\pi} e^{in\omega_0 t}$

$$\widehat{f_1}(t) = \sum_{\substack{n=-1 \\ n \text{ odd}}}^{1} \frac{2}{in\pi} e^{in\omega_0 t} = \frac{4}{\pi}\sin(\omega_0 t)$$



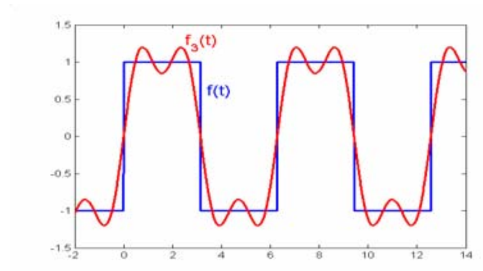
$$\overline{\varepsilon_1^2} = \sum_{\substack{|n|>1 \\ n \text{ odd}}} \left|\frac{2}{in\pi}\right|^2 = \underbrace{\frac{1}{T}\int_0^T |f(t)|^2\, dt}_{1} - \sum_{\substack{n=-1 \\ n \text{ odd}}}^{1} \left|\frac{2}{in\pi}\right|^2 = 1 - \frac{4}{\pi^2} - \frac{4}{\pi^2} \approx 0.19$$

$$\boxed{\overline{\varepsilon_N^2} = \sum_{|n|>N} |F_n|^2 = \frac{1}{T}\int_0^T |f(t)|^2\,dt - \sum_{n=-N}^{N} |F_n|^2}$$

Example: Square Wave $f(t) = \sum_{\substack{n=-\infty \\ n \text{ odd}}}^{+\infty} \frac{2}{in\pi} e^{in\omega_0 t}$

$$\widehat{f}_3(t) = \sum_{\substack{n=-3 \\ n \text{ odd}}}^{3} \frac{2}{in\pi} e^{in\omega_0 t}$$

$$= \frac{4}{\pi}\sin(\omega_0 t) + \frac{4}{3\pi}\sin(3\omega_0 t)$$



$$\overline{\varepsilon_3^2} = \underbrace{\frac{1}{T}\int_0^T |f(t)|^2\,dt}_{1} - \sum_{\substack{n=-3 \\ n \text{ odd}}}^{3} \left|\frac{2}{in\pi}\right|^2 = 1 - \frac{4}{\pi^2}\left(\frac{1}{3^2} + 1 + 1 + \frac{1}{3^2}\right) \approx 0.1$$

Convergence in mean square

Parseval: $\frac{1}{T}\int_0^T |f(t)|^2\,dt = \sum_{n=-\infty}^{+\infty} |F_n|^2 = \lim_{N\to+\infty} \sum_{n=-N}^{+N} |F_n|^2$.

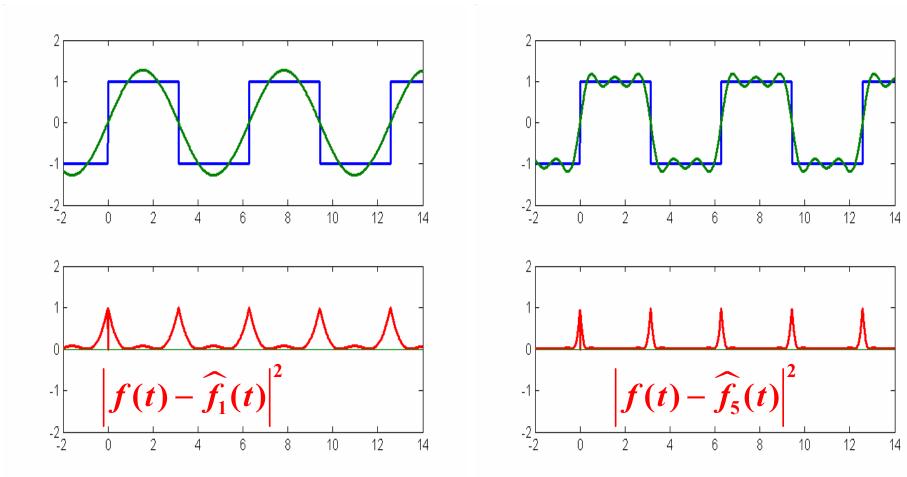
Provided $\frac{1}{T}\int_0^T |f(t)|^2\,dt$ is finite ($f(t)$ is "square integrable")

then $\frac{1}{T}\int_0^T |f(t)|^2\,dt - \sum_{n=-N}^{+N} |F_n|^2 = \sum_{|n|>N} |F_n|^2$ converges to zero

But this difference is exactly $\overline{\varepsilon_N^2} = \left\| f - \widehat{f}_N \right\|_{\text{RMS}}^2$. So we have $\lim_{N\to+\infty} \left\| f - \widehat{f}_N \right\|_{\text{RMS}}^2 = \lim_{N\to+\infty} \frac{1}{T}\int_0^T \left| f(t) - \widehat{f}_N(t) \right|^2 dt = 0$. We say that $\widehat{f}_N(t) = \sum_{n=-N}^{+N} F_n\, e^{in\omega_0 t}$ converges to $f(t)$ in the <u>mean square sense.</u>

This is another notion of convergence of Fourier series, different from pointwise convergence. Requirement: $f(t)$ square integrable

Mean square convergence for the square wave



As $N \to \infty$, the area under the error curve tends to zero.
This is mean-square convergence.