Lecture 16

- Relationship between Fourier and Laplace transforms
- Filtering:
  - Lowpass, highpass and bandpass filters.
  - Ideal filters and real approximations.
- Application: amplitude demodulation.

Fourier and Laplace

Laplace:

- Applies to large classes of functions, including those that "blow up" exponentially as time $\rightarrow \infty$:

$$F(s) = \int_0^\infty f(t)e^{-st}\,dt \text{ converges for } \mathrm{Re}[s] > \alpha \text{ when } |f(t)| \leq Ce^{\alpha t}$$

- Considers only positive time.

Fourier:

- More restrictive convergence.
  One sufficient condition is integrability, i.e. $\int_{-\infty}^{\infty} |f(t)|\,dt < \infty.$

  Generalizes to sinusoids, but not to increasing exponentials.
- Includes negative time.

Examples:

1) $f(t) = u(t)e^t$. $\quad F(s) = \mathcal{L}[f(t)] = \dfrac{1}{s-1}$, DOC $\mathrm{Re}[s] > 1$.

$\mathcal{F}[f(t)]$ is not defined.

2) $f(t) = e^{-|t|}$.

Laplace transform would apply only to the $t > 0$ portion.

$\mathcal{F}[f(t)]$ well defined, and considers all time.

3) $f(t) = e^{-t}u(t)$.

$\mathcal{L}[f(t)] = \dfrac{1}{s+1}$, DOC $\mathrm{Re}[s] > -1$. $\qquad \mathcal{F}[f(t)] = \dfrac{1}{i\omega + 1}$

Rule: if $\begin{cases} f(t) = 0 \text{ for } t < 0. \\ \text{DOC of } F(s) = \mathcal{L}[f(t)] \text{ contains } \mathrm{Re}[s] \geq 0. \end{cases}$

Then the Fourier transform exists, and $\mathcal{F}[f(t)] = F(s)|_{s=i\omega}$.

What happens if the DOC of $F(s) = \mathcal{L}[f(t)]$ is $\mathrm{Re}[s] > 0$?

The Fourier transform exists, but need not be equal to $F(s)|_{s=i\omega}$

Example: $f(t) = u(t)$ $\qquad F(s) = \dfrac{1}{s}$, DOC $\mathrm{Re}[s] > 0$.

$U(i\omega) = \mathcal{F}[f(t)] = \int_0^\infty e^{-i\omega t}\, dt$ does not converge absolutely,

but it can be defined in a generalized sense. What do we get?

One approach: use the derivative property

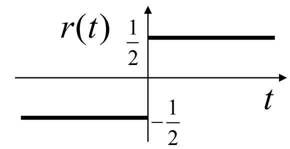
$$1 = \mathcal{F}[\delta(t)] = \mathcal{F}\left[\frac{du}{dt}\right] = i\omega U(i\omega) \Rightarrow U(i\omega) \overset{?}{=} \frac{1}{i\omega}.$$

This would correspond to setting $s = i\omega$ in the Laplace transform

Problem: the same approach would apply to $\mathcal{F}[C + u(t)]$

for any constant C, since $\dfrac{d}{dt}(C + u(t)) = \delta(t)$

For example, let $r(t) = -\frac{1}{2} + u(t), \quad \frac{dr}{dt} = \delta(t)$



$$1 = \mathcal{F}\left[\frac{dr}{dt}\right] = i\omega\, R(i\omega) \overset{?}{\Rightarrow} R(i\omega) = \frac{1}{i\omega}.$$

Now $R(i\omega) = \mathcal{F}\left[-\frac{1}{2} + u(t)\right] = -\frac{1}{2}\mathcal{F}[1] + \mathcal{F}[u(t)] = -\pi\,\delta(\omega) + U(i\omega)$

So both $R(i\omega)$ and $U(i\omega)$ can't be equal to $\frac{1}{i\omega}$. Which one is right?

Answer: $R(i\omega) = \mathcal{F}\left[-\frac{1}{2} + u(t)\right] = \frac{1}{i\omega}$. Why? a purely imaginary transform must correspond to an odd function of time, such as $r(t)$

Correspondingly, $\boxed{U(i\omega) = \frac{1}{i\omega} + \pi\delta(\omega)}$ is the transform of the step

Note that $i\omega U(i\omega) = i\omega\left(\frac{1}{i\omega} + \pi\delta(\omega)\right) = 1 + \underbrace{i\omega\pi\,\delta(\omega)}_{0} = 1$

Problem with the previous derivation: dividing by $i\omega$ misses the δ

Application to proving the integration property: $\boxed{\mathcal{F}\left[\int_{-\infty}^{t} f(\sigma)d\sigma\right] = \frac{F(i\omega)}{i\omega} + \pi F(0)\delta(\omega)}$

Proof: Consider a system with impulse response $h(t) = f(t)$,

and step response $g(t) = \int_{-\infty}^{t} f(\sigma)d\sigma$. The main theorem says that

$$G(i\omega) = H(i\omega)U(i\omega) = F(i\omega)\left(\frac{1}{i\omega} + \pi\delta(\omega)\right) = \frac{F(i\omega)}{i\omega} + \pi F(0)\delta(\omega)$$

Another delicate transform (follows from $U(i\omega)$ by modulation property):

$$\boxed{\mathcal{F}[u(t)\cos(\omega_0 t)] = \frac{i\omega}{-\omega^2 + \omega_0^2} + \frac{\pi}{2}\delta(\omega - \omega_0) + \frac{\pi}{2}\delta(\omega + \omega_0)}$$

Setting $s = i\omega$ on the Laplace transform $\mathcal{L}[u(t)\cos(\omega_0 t)] = \frac{s}{s^2 + \omega_0^2}$ gives the first term only, again misses the δ's.

Filtering

Let $H(i\omega) = \mathcal{F}[h(t)]$ be the system frequency response function.

Then for any input $x(t)$ with a Fourier transform $X(i\omega)$,

$$Y(i\omega) = H(i\omega)\, X(i\omega)$$

Magnitudes are multiplied: $|Y(i\omega)| = |H(i\omega)||X(i\omega)|$

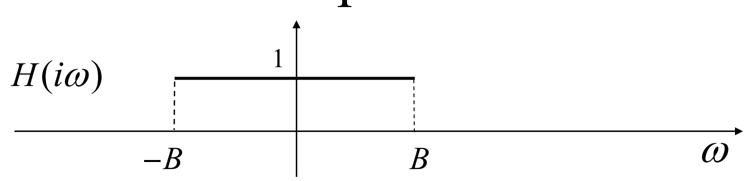
Amplify some frequencies, attenuate others.

In decibels (dB), the effect becomes additive:

$$20 \log|Y(i\omega)| = 20 \log|H(i\omega)| + 20 \log|X(i\omega)|$$

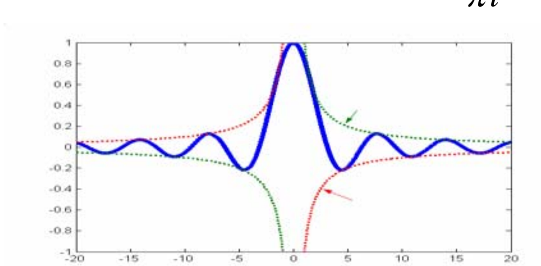
Phases are added: $\theta_{Y(i\omega)} = \theta_{H(i\omega)} + \theta_{X(i\omega)}$

Ideal lowpass filter



Is it physically realizable? Looking at the impulse response:

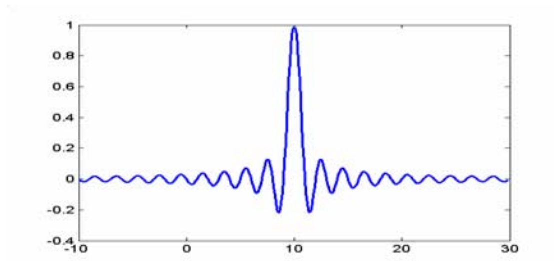
$$h(t) = \mathcal{F}^{-1}[H(i\omega)] = \frac{\sin(Bt)}{\pi t} = \frac{B}{\pi}\mathrm{sinc}(\mathrm{B}t)$$



$h(t)$ is not zero for $t < 0$. So the ideal lowpass filter is not causal!

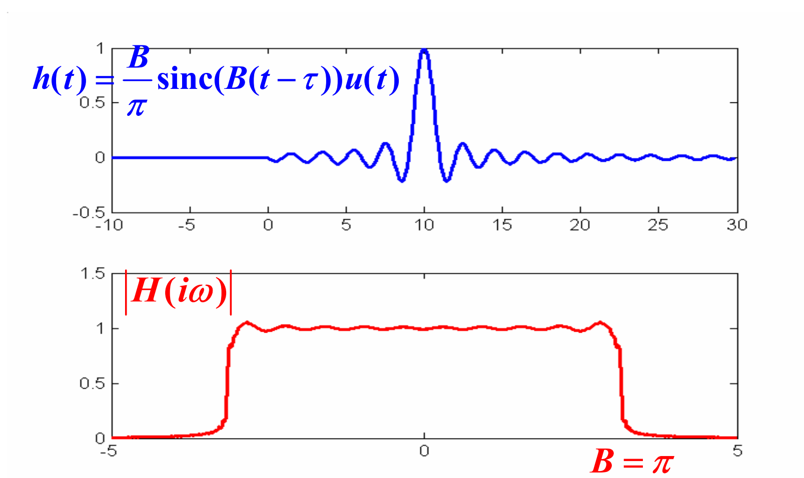
An ideal lowpass filter cannot be built, but we can build approximations to it. If we only care about the magnitude of $|H(i\omega)|$ (and not the phase), one way is to add a delay:

$$h(t) = \frac{B}{\pi}\text{sinc}(\text{B}(t-\tau)) \Rightarrow H(i\omega) = \begin{cases} e^{-i\omega\tau} \text{ for } |\omega| < B \\ 0 \text{ otherwise.} \end{cases}$$



This is still non-causal, but we approximate it by truncating $h(t)$ to positive time: $h(t) = \frac{B}{\pi}\text{sinc}(\text{B}(t-\tau))u(t)$

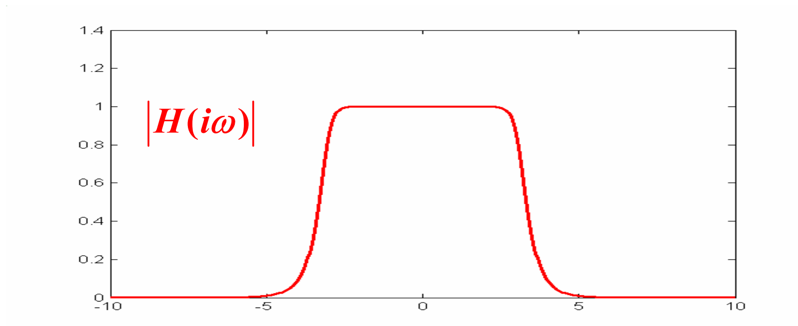
Causal approximation to ideal lowpass



This *h(t)* may be difficult to implement. Easier alternative?

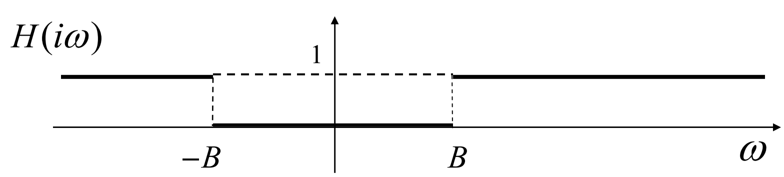
"Butterworth" filter of order 10.

This is a rational *H(s),* with denominator of order 10.



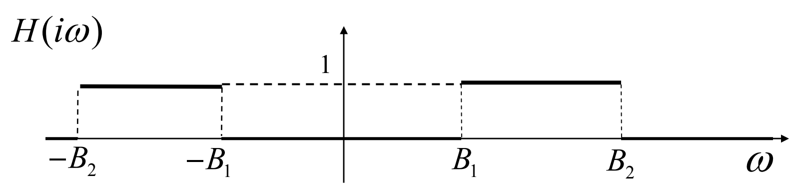
For more on filter design, see signal processing courses.
Here we will focus on ideal filters.

Ideal highpass filter

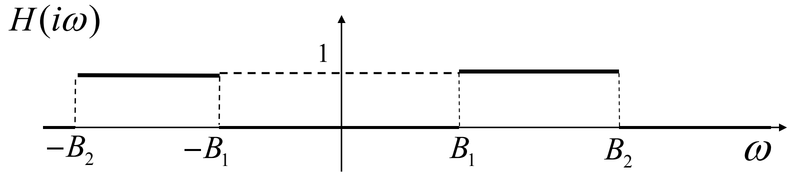


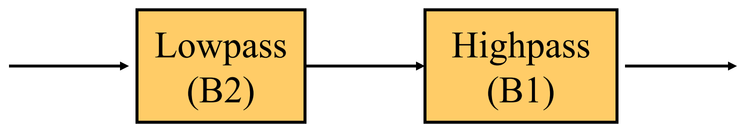
$$h(t) = \delta(t) - \frac{B}{\pi}\mathrm{sinc}(\mathrm{B}t)$$

Ideal bandpass filter

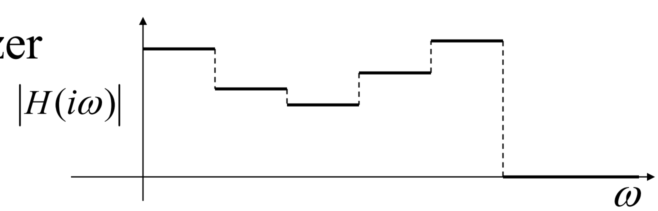


Bandpass as cascade of lowpass and highpass

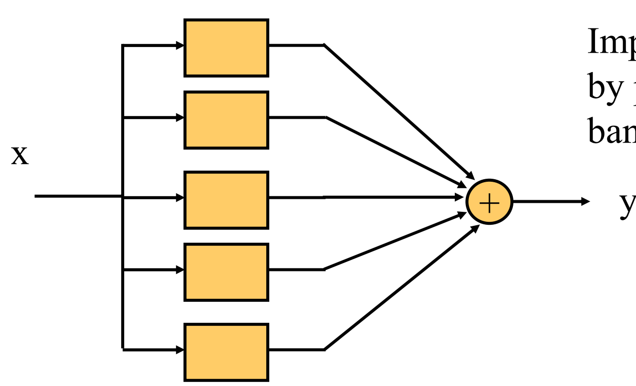




An equalizer



Adjustable gains for each frequency band.



Implementation by parallel bandpass filters.

Example: pure delay system: $y(t) = x(t-\tau)$

Impulse response function: $h(t) = \delta(t-\tau)$.

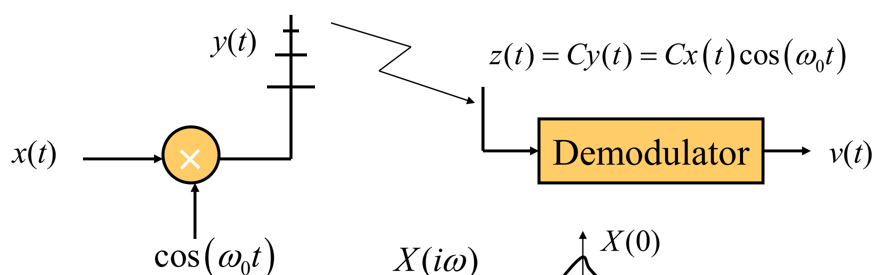
Frequency response function: $H(i\omega) = e^{-i\omega\tau}$

$|H(i\omega)| = 1$ for all ω: this is an "allpass" filter

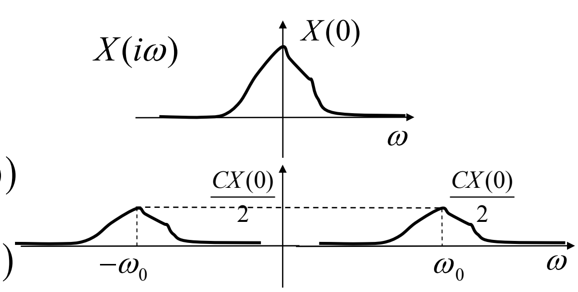
A rational allpass filter: $H(i\omega) = \dfrac{2-i\omega}{2+i\omega}$.

Both these systems do not affect the magnitude of the signals (i.e. the input and output Fourier transforms have the same magnitude).
They do, however, affect the phase.

Application: amplitude demodulation

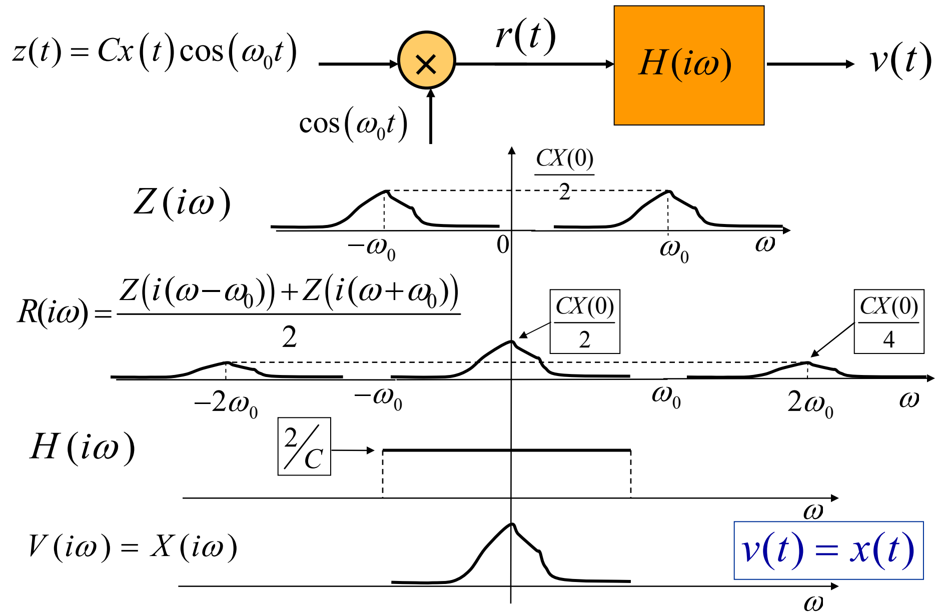


$$Z(i\omega) = \frac{C}{2}X\left(i(\omega-\omega_0)\right) + \frac{C}{2}X\left(i(\omega+\omega_0)\right)$$



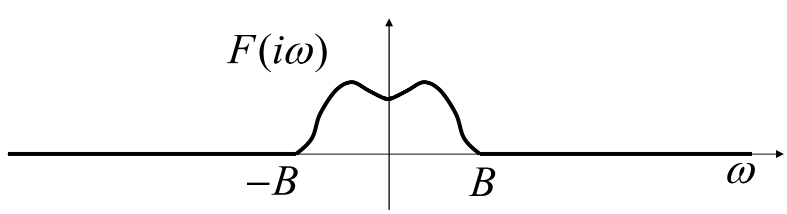
Q: What operation can we do to make the received signal v(t) equal to the message x(t)?

Solution: another modulation + lowpass filtering



A definition needed for homework 4(b)

Definition: A signal $f(t)$ is said to be band-limited to $[-B, B]$ if $F(i\omega) = 0$ for $|\omega| > B$.



More on these functions next time.