Lecture 14: Fourier Transforms.

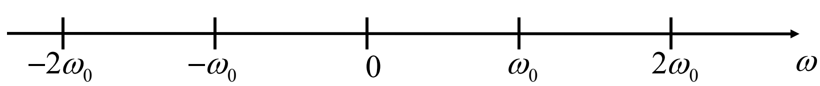
Review of methods for LTI systems

- Time domain methods (differential equations, convolutions). Apply to all cases, but may be cumbersome to compute.
- Laplace transforms. For systems starting at time t=0; useful to study transients, instability.
- Fourier series. For studying stable systems in steady state (after transients die down). Restricted to periodic inputs. Key idea: superposition of sinusoids.
- Question: can we extend this steady-state analysis via sinusoids to non-periodic inputs?
- Answer: Fourier transforms.

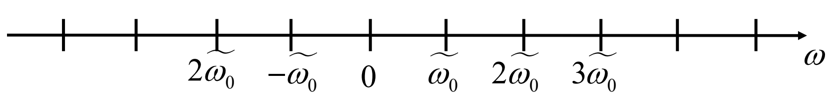
Fourier series as the period increases.

Consider the series $f(t) = \sum_{n=-\infty}^{+\infty} F_n e^{in\omega_0 t}$, for period $T = \dfrac{2\pi}{\omega_0}$

The frequencies involved in the expansion are $\{n\omega_0\}_{n=-\infty}^{+\infty}$.



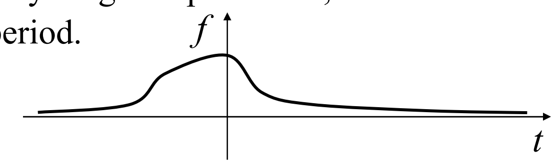
Now consider a function with twice the period, $\widetilde{T} = 2T$. Then $\widetilde{\omega_0} = \dfrac{\omega_0}{2}$. The frequencies involved in the expansion are $\{n\widetilde{\omega_0}\}_{n=-\infty}^{+\infty}$



As the period gets longer, we get a more dense set of frequencies. In the limit: include all frequencies.

Fourier Transform

We wish to extend the Fourier series concept to a non-periodic *f(t)*. Intuitively: *f(t)* takes infinitely long to repeat itself, so we think of it as a function of infinite period.



This suggests replacing

$$F_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\,e^{-in\omega_0 t}dt \text{ by an integral where } T \to \infty$$

Definition: The Fourier transform of time-domain function $f(t),\ t \in (-\infty, +\infty)$ is the function of frequency $\omega \in (-\infty, +\infty)$ defined by the integral $F(i\omega) = \mathcal{F}[f(t)] = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt$, assuming it converges.

$$F(i\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt$$

Example I: $f(t) = e^{-|t|}$.

$$F(i\omega) = \int_{-\infty}^{\infty} e^{-|t|}e^{-i\omega t}\,dt = \int_{-\infty}^{0} e^{t}e^{-i\omega t}\,dt + \int_{0}^{\infty} e^{-t}e^{-i\omega t}\,dt$$

$$= \int_{-\infty}^{0} e^{(1-i\omega)t}\,dt + \int_{0}^{\infty} e^{-(1+i\omega)t}\,dt \quad = \left.\frac{e^{(1-i\omega)t}}{1-i\omega}\right|_{-\infty}^{0} + \left.\frac{e^{-(1+i\omega)t}}{-(1+i\omega)}\right|_{0}^{\infty}$$

$$= \frac{1}{1-i\omega} + \frac{1}{1+i\omega} = \frac{(1+i\omega)+(1-i\omega)}{(1-i\omega)(1+i\omega)} = \boxed{\frac{2}{1+\omega^2}}$$

Example II: $f(t) = e^{-t}u(t)$. $\quad F(i\omega) = \int_{0}^{\infty} e^{-t}e^{-i\omega t}\,dt = \dfrac{1}{1+i\omega}$

Note: in case II, the Fourier transform $F(i\omega)$ coincides with the Laplace transform $F(s)$ evaluated at $s = i\omega$.

Representation of the Fourier transform

$F(i\omega)$ is a complex function of the real variable ω. The "i" in the argument of $F(i\omega)$ plays no role, and could be omitted. We only put it in for compatibility with Laplace transforms, as in example II above (more on this later).

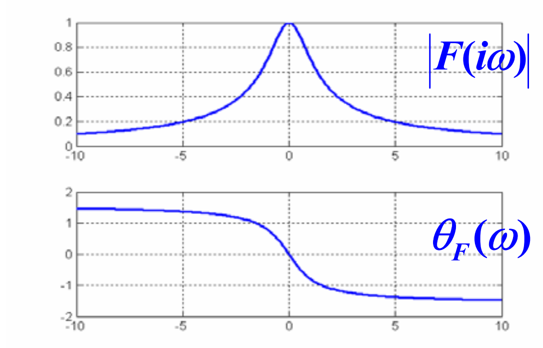
Magnitude and phase representation, $F(i\omega)=|F(i\omega)|e^{i\theta_F(\omega)}$

Example:

$$F(i\omega)=\frac{1}{1+i\omega}$$

$$|F(i\omega)|=\frac{1}{\sqrt{\omega^2+1}}$$

$$\theta_F(\omega)=-\text{Arctan}(\omega)$$



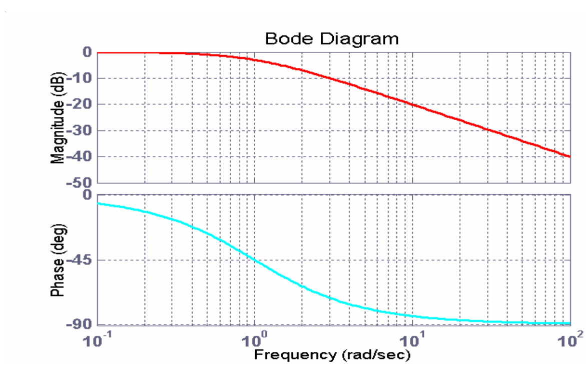
Bode Plots

Represent the frequencies $\omega>0$ in a logarithmic scale (i.e. proportional to $\log(\omega)$.).

Also plot the magnitude in "decibels (dB)": $20\log|F(i\omega)|$

$20\log|F(i\omega)|$

$\theta_F(\omega)$



Widely used, but we will not emphasize them in this course.

The inverse Fourier transform

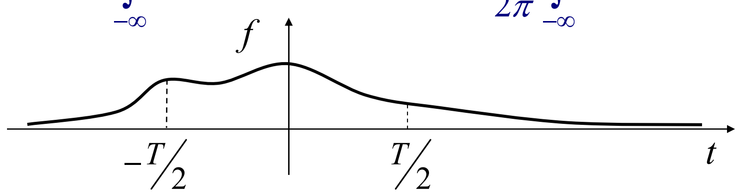
In Fourier series, we can reconstruct the function from the Fourier coefficients via $f(t) = \sum_{n=-\infty}^{+\infty} F_n e^{in\omega_0 t}$.

Similarly, for Fourier transforms we can reconstruct the function using the inverse formula

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(i\omega) e^{i\omega t}\, d\omega$$

A complete derivation is mathematically involved, but we can sketch a proof based on Fourier series case, letting the period go to infinity.

Proof: $F(i\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, dt \Rightarrow f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(i\omega) e^{i\omega t}\, d\omega$



Let $F_T(i\omega) = \int_{-T/2}^{T/2} f(t) e^{-i\omega t}\, dt$. Then $F(i\omega) = \lim_{T \to +\infty} F_T(i\omega)$

Considered on the interval $t \in \left[-\frac{T}{2}, \frac{T}{2} \right]$, $f(t)$ (if "well-behaved")

has a Fourier series expansion $f(t) = \sum_{n=-\infty}^{+\infty} F_n e^{in\omega_0 t}$, where

$$F_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-in\omega_0 t}\, dt = \frac{1}{T} F_T(in\omega_0) = \frac{\omega_0}{2\pi} F_T(in\omega_0)$$

So we have $f(t) = \sum_{n=-\infty}^{+\infty} \frac{\omega_0}{2\pi} F_T(in\omega_0) e^{in\omega_0 t}$, for $t \in \left[-\frac{T}{2}, \frac{T}{2}\right]$

$$\Rightarrow f(t) - \frac{1}{2\pi}\int_{-\infty}^{\infty} F(i\omega) e^{i\omega t}\, d\omega = \underbrace{\sum_{n=-\infty}^{+\infty} \frac{\omega_0}{2\pi}\left[F_T(in\omega_0) - F(in\omega_0)\right] e^{in\omega_0 t}}_{\text{(I)}}$$

$$+ \underbrace{\sum_{n=-\infty}^{+\infty} \frac{\omega_0}{2\pi} F(in\omega_0) e^{in\omega_0 t} - \frac{1}{2\pi}\int_{-\infty}^{\infty} F(i\omega) e^{i\omega t}\, d\omega}_{\text{(II)}}$$

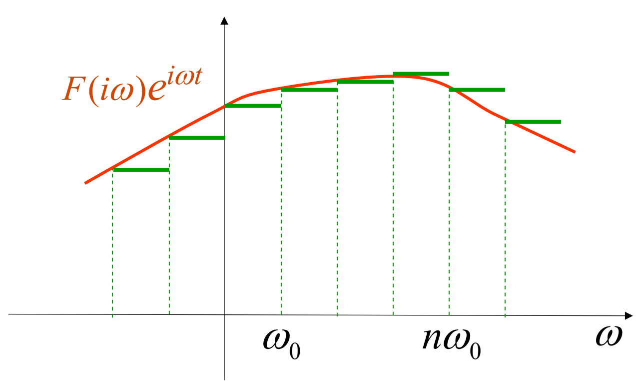
As $T \to \infty$:

- Term (I) goes to zero since $\lim_{T\to+\infty} F_T(in\omega_0) = F(in\omega_0)$
- Term (II) goes to zero as shown in the next slide.

So the left-hand side must be zero: $f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(i\omega) e^{i\omega t}\, d\omega$

Also, as $T \to \infty$ all t's will be included in the interval.

Term (II): $\frac{1}{2\pi}\sum_{n=-\infty}^{+\infty} F(in\omega_0) e^{in\omega_0 t}\omega_0 - \frac{1}{2\pi}\int_{-\infty}^{\infty} F(i\omega) e^{i\omega t}\, d\omega$



The sum on the left is a staircase approximation to the integral on the right. As $T \to \infty$, $\omega_0 \to 0$ and the approximation becomes exact, provided the integral exists. So the difference tends to zero.

Recap

Fourier Series
(T-periodic functions)

$$F_n = \frac{1}{T}\int_0^T f(t)\,e^{-in\omega_0 t}\,dt$$

$$f(t) = \sum_{n=-\infty}^{+\infty} F_n\,e^{in\omega_0 t}$$

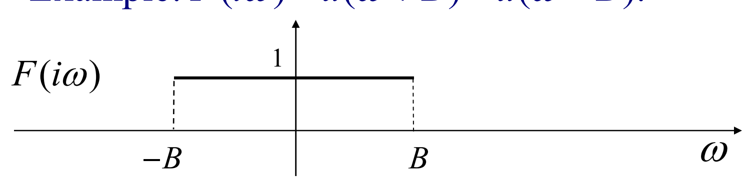
Fourier Transforms

$$F(i\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(i\omega)e^{i\omega t}\,d\omega$$

Note that the Fourier transform and its inverse are formally almost identical: except for a sign change and the factor 2π, t and ω could be interchanged. This is called duality.

Example: $F(i\omega) = u(\omega + B) - u(\omega - B)$.



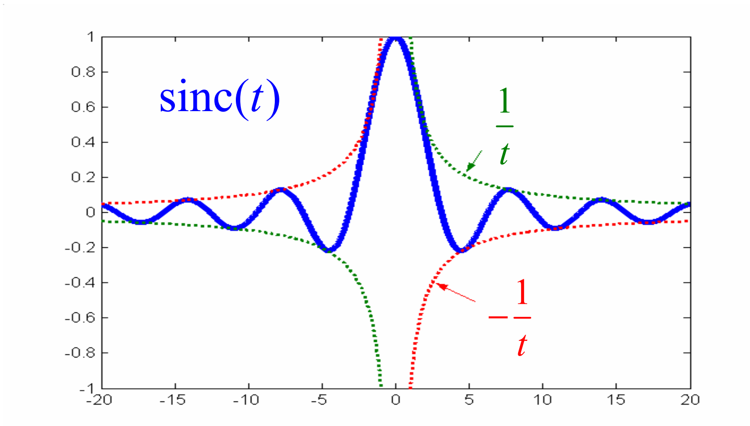
$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(i\omega)e^{i\omega t}\,d\omega = \frac{1}{2\pi}\int_{-B}^{B} e^{i\omega t}\,d\omega$$

$$= \frac{1}{2\pi}\left.\frac{e^{i\omega t}}{it}\right|_{\omega=-B}^{\omega=B} = \frac{1}{2\pi i t}\left(e^{iBt} - e^{-iBt}\right) = \boxed{\frac{\sin(Bt)}{\pi t}}$$

Notation: we define the function $\mathrm{sinc}(t) = \dfrac{\sin(t)}{t}$

Then we have $f(t) = \dfrac{B}{\pi}\mathrm{sinc}(\mathrm{B}t)$

Plot of $\operatorname{sinc}(t) = \dfrac{\sin(t)}{t}$



Note that $\operatorname{sinc}(0) = \lim_{t \to 0} \dfrac{\sin(t)}{t} = 1$

Example: $f(t) = \delta(t)$. $\quad F(i\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-i\omega t}\, dt = 1$ for all ω.

The inverse formula would say that $\delta(t) = \dfrac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t}\, d\omega$

Now the function $e^{i\omega t}$ has magnitude 1 for all ω.
Not an absolutely convergent integral!

We can interpret it as

$$\delta(t) = \lim_{B \to \infty} \frac{1}{2\pi} \int_{-B}^{B} e^{i\omega t}\, d\omega$$
$$= \lim_{B \to \infty} \frac{B}{\pi} \operatorname{sinc}(\mathrm{B}t)$$

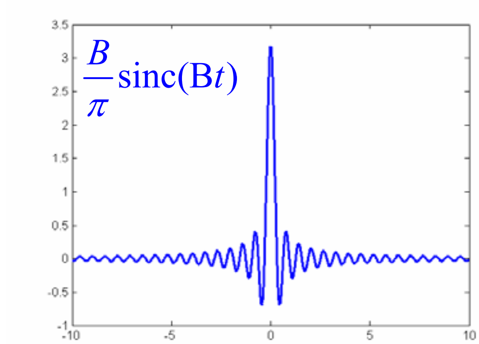
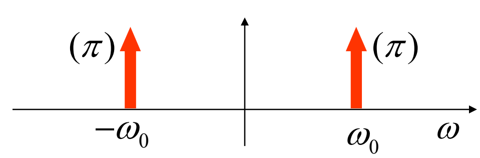


Table of Basic Fourier Transforms

$f(t)$	$F(i\omega)$
$\delta(t)$	1
1	$2\pi\,\delta(\omega)$
$\frac{B}{\pi}\mathrm{sinc}(\mathrm{B}t)$	$u(\omega+B)-u(\omega-B)$
$u(t+a)-u(t-a)$	$2a\,\mathrm{sinc}(a\omega)$
$\delta(t-\tau)$	$e^{-i\omega\tau}$
$e^{i\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$

They can be easily obtained applying the Fourier definition, or the inverse formula. Notice the time-frequency duality.

$f(t)$	$F(i\omega)$
$e^{i\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$
$\cos(\omega_0 t)=\dfrac{e^{i\omega_0 t}+e^{-i\omega_0 t}}{2}$	$\pi\left[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right]$



$\sin(\omega_0 t)=\dfrac{e^{i\omega_0 t}-e^{-i\omega_0 t}}{2i}$	$i\,\pi\left[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)\right]$