Lecture 10: Sinusoidal response

Response of LTI systems to complex exponentials.

LTI, causal system, with impulse response $h(t)$, and transfer function $H(s) = \mathcal{L}\left[h(t)\right]$, DOC: $\text{Re}[s] > \alpha$.

Input-output relation: $y(t) = \int_{0-}^{\infty} h(\sigma)x(t-\sigma)d\sigma$

Now consider $x(t) = e^{s_0 t}$, $t \in (-\infty, +\infty)$, $\text{Re}[s_0] > \alpha$.

To determine the output, we do not use Laplace since $x(t)$ is not zero for $t < 0$.

$$y(t) = \int_{0-}^{\infty} h(\sigma)x(t-\sigma)d\sigma = \int_{0-}^{\infty} h(\sigma)e^{s_0(t-\sigma)}d\sigma$$

$$= e^{s_0 t}\int_{0-}^{\infty} h(\sigma)e^{-s_0\sigma}d\sigma = H(s_0)e^{s_0 t}$$

The output function is equal to the input function times some constant complex number (a complex "gain").

In linear algebra terms, $e^{s_0 t}$ is an "eigenfunction" of the system, with "eigenvalue" $H(s_0)$.

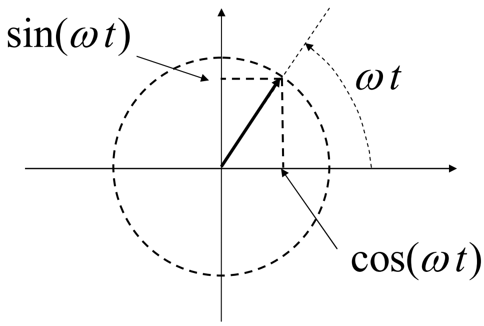
Example: $H(s) = \dfrac{1}{s(s+2)(s-1)}$.

$x(t) = e^{2t}$, $t \in (-\infty, +\infty) \rightarrow y(t) = H(2)e^{2t} = \dfrac{1}{8}e^{2t}$

Complex exponentials and sinusoids

For the case $s_0 = i\omega$, $x(t) = e^{i\omega t}$ is a complex number of magnitude 1 and phase ωt. We represent this as a vector in the complex plane, rotating at angular velocity ω.

"Phasor" diagram



The horizontal and vertical projections of this vector are sinusoidal functions: $\mathrm{Re}[e^{i\omega t}] = \cos(\omega t)$, $\mathrm{Im}[e^{i\omega t}] = \sin(\omega t)$.

Response of an LTI system to a sinusoid.

LTI, causal system, with impulse response $h(t)$.

Assume the DOC of $H(s) = \mathcal{L}\left[h(t)\right]$, includes $\mathrm{Re}[s] \geq 0$. i.e. all poles of $H(s)$ are in $\mathrm{Re}[s] < 0$.

Such systems are called "stable".

e.g., for a rational $H(s) = \dfrac{P(s)}{(s-a_1)\cdots(s-a_n)}$, $\mathrm{Re}[a_i] < 0$.

$h(t)$ is a sum of terms of the form $t^n e^{a_i t}, \rightarrow \lim_{t\to+\infty} h(t) = 0$.

The impulse response "dies down" with time.

Response of an LTI system to a sinusoid (cont.)

$x(t) = e^{i\omega t}$

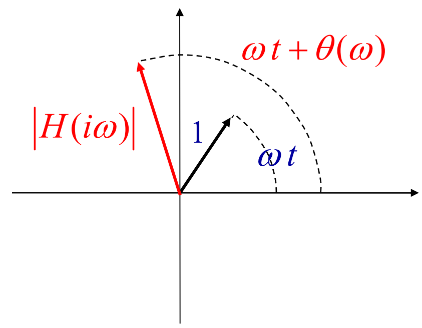
$t \in (-\infty, +\infty)$

Since $i\omega \in \text{DOC}$, $y(t) = H(i\omega)\, e^{i\omega t}$ as before.

Also, writing $H(i\omega) = |H(i\omega)|\, e^{i\theta(\omega)}$ we have

$$y(t) = |H(i\omega)|\, e^{i(\omega t + \theta(\omega))}$$

Gain in magnitude,
Add to phase.



Phasor diagram

Now, consider "real" sinusoids.

$$x(t) = \cos(\omega t) = \text{Re}[e^{i\omega t}] = \frac{e^{i\omega t} + e^{-i\omega t}}{2}$$

By linearity, $y(t) = \dfrac{H(i\omega)\, e^{i\omega t} + H(-i\omega)\, e^{-i\omega t}}{2}$

Suppose $h(t)$ is real-valued (as in any physical system). Then

$$H(-i\omega) = \int_{0-}^{\infty} h(t) e^{i\omega t} dt = \int_{0-}^{\infty} \overline{h(t) e^{-i\omega t}} dt = \overline{\int_{0-}^{\infty} h(t) e^{-i\omega t} dt} = \overline{H(i\omega)}$$

(bar denotes complex conjugate). Therefore,

$$y(t) = \frac{H(i\omega)\, e^{i\omega t} + \overline{H(i\omega)\, e^{i\omega t}}}{2} = \text{Re}\left[H(i\omega)\, e^{i\omega t} \right]$$

$$= \text{Re}\left[|H(i\omega)|\, e^{i(\omega t + \theta(\omega))} \right] = |H(i\omega)| \cos(\omega t + \theta(\omega))$$

Response of an LTI system to a sinusoid

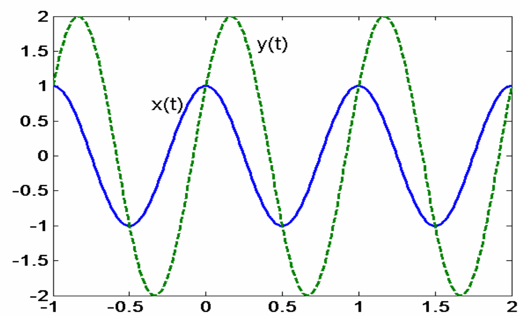
$$x(t) = \cos(\omega t) \longrightarrow \boxed{H(s)} \longrightarrow y(t) = |H(i\omega)| \cos\left(\omega t + \theta(\omega)\right)$$

Example:

$\omega = 2\pi$

$|H(i\omega)| = 2$

$\theta(\omega) = -\dfrac{\pi}{3}$



The output is another sinusoid, of the same frequency, with a different amplitude and phase.

Example: $H(s) = \dfrac{1}{s+1}$, apply the input $x(t) = \cos(t)$

$$\omega = 1 \rightarrow H(i\omega) = \frac{1}{i+1} = \frac{1}{\sqrt{2} \cdot e^{i\pi/4}} = \frac{1}{\sqrt{2}} e^{-i\pi/4}$$

$$|H(i\omega)| = \frac{1}{\sqrt{2}}, \qquad \theta(\omega) = -\frac{\pi}{4}$$

$$\boxed{y(t) = \frac{1}{\sqrt{2}} \cos\left(t - \frac{\pi}{4}\right)}$$

Also, we can write

$$y(t) = \frac{1}{\sqrt{2}} \left[\cos(t) \cos\left(\frac{\pi}{4}\right) + \sin(t) \sin\left(\frac{\pi}{4}\right) \right] = \frac{1}{2} \cos(t) + \frac{1}{2} \sin(t)$$

Note: sin and cos are the same function, except for a time shift. We use the term “sinusoid” for all such functions.

Example: $H(s) = \dfrac{1}{s+1}$, apply the input $x(t) = \cos(t)$

What if we attack this problem using Laplace?

Write $X(s) = \mathcal{L}\left[\cos(t)\right] = \dfrac{s}{s^2+1}$

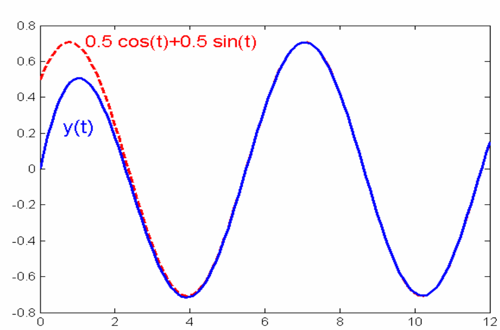
$$\rightarrow Y(s) = H(s)X(s) = \frac{s}{(s+1)(s^2+1)} = -\frac{1}{2}\frac{1}{s+1} + \frac{1}{2}\frac{s+1}{s^2+1}$$

$$\rightarrow y(t) = -\frac{1}{2}u(t)e^{-t} + \frac{1}{2}u(t)\left(\cos(t) + \sin(t)\right)$$

Different answer! In particular, there is an additional exponential term in the output.

Reason: Laplace is computing the response to $u(t)\cos(t)$, instead of $\cos(t)$ for $t \in (-\infty, +\infty)$.

$$y(t) = T\left[u(t)\cos(t)\right] = \underbrace{-\frac{1}{2}u(t)e^{-t}}_{\text{TRANSIENT TERM}} + \underbrace{\frac{1}{2}u(t)\left(\cos(t) + \sin(t)\right)}_{\text{STEADY STATE SOLUTION}}$$



As $t \rightarrow +\infty$ (in "steady state"), the "transient" exponential term dies down, and we cannot distinguish the response to $u(t)\cos(t)$ from the response to $\cos(t)$, $-\infty < t < \infty$.

General case: response of $H(s) = \dfrac{P(s)}{(s-a_1)\cdots(s-a_n)}$

$\mathrm{Re}[a_i] < 0$, to $x(t) = u(t)e^{i\omega t}$.

For simplicity (not essential), assume the poles a_i are distinct

$$Y(s) = H(s)X(s) = H(s)\frac{1}{s-i\omega} = \frac{P(s)}{(s-a_1)\cdots(s-a_n)(s-i\omega)}$$

$$= \sum_i \frac{A_i}{s-a_i} + \frac{B}{s-i\omega} \qquad B = H(\underbrace{s}_{i\omega}) = H(i\omega)$$

$$y(t) = \overbrace{\sum_i A_i e^{a_i t} u(t)}^{\text{TRANSIENT TERM}} + \overbrace{H(i\omega)\, e^{i\omega t} u(t)}^{\text{STEADY STATE OUTPUT}}$$

Similarly: response of $H(s) = \dfrac{P(s)}{(s-a_1)\cdots(s-a_n)}$

$\mathrm{Re}[a_i] < 0$, to $x(t) = u(t)\cos(\omega t) = \dfrac{1}{2}\left[e^{i\omega t} + e^{-i\omega t}\right]u(t)$

$$y(t) = \left[\sum_i A_i e^{a_i t} + \frac{1}{2}H(i\omega)\, e^{i\omega t} + \frac{1}{2}H(-i\omega)\, e^{-i\omega t}\right]u(t)$$

$$= \left[\overbrace{\sum_i A_i e^{a_i t}}^{\text{TRANSIENT}} + \overbrace{|H(i\omega)|\cos\big(\omega t + \theta(\omega)\big)}^{\text{STEADY STATE OUTPUT}}\right]u(t)$$

For the next few weeks of the course, we will study systems from the point of view of their steady state behavior, ignoring transients. Equivalently, we study signals over the time interval $(-\infty, \infty)$. The systems under study will be stable so that they reach steady state

Response to a sum of sinusoids

$$x(t) \longrightarrow \boxed{H(s)} \longrightarrow y(t)$$

$$x(t) = e^{i\omega_1 t} + e^{i\omega_2 t}, \quad t \in (-\infty, \infty).$$

By linearity, $y(t) = H(i\omega_1)e^{i\omega_1 t} + H(i\omega_2)e^{i\omega_2 t}$

Also, take $x(t) = \cos(\omega_1 t) + \cos(\omega_2 t + \varphi), \quad t \in (-\infty, +\infty)$.
Note that in addition to the different frequencies, we can include a difference φ in the phase.

$$y(t) = |H(i\omega_1)|\cos\left(\omega_1 t + \theta(\omega_1)\right) + |H(i\omega_2)|\cos\left(\omega_2 t + \theta(\omega_2) + \varphi\right)$$

Moral: for LTI systems, it is easy to study the (steady-state) effect of applying input sinusoids, and their sums.
We will explore what signals can be represented in this way.

Periodic functions

A function $f(t)$ defined on $t \in (-\infty, +\infty)$ is called periodic if there exists $T > 0$ such that

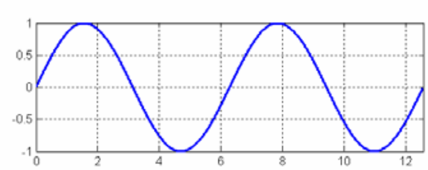
$$f(t+T) = f(t) \quad \forall\ t \in (-\infty, +\infty)$$

The smallest T satisfying the above is called the period of $f(t)$

Examples:

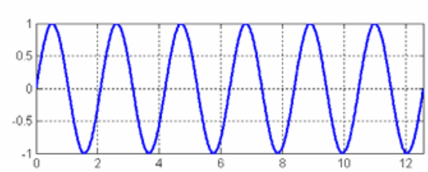
$f(t) = \sin(t)$.

Period is $T = 2\pi$



$f(t) = \sin(3t)$.

Period is $T = \dfrac{2\pi}{3}$.

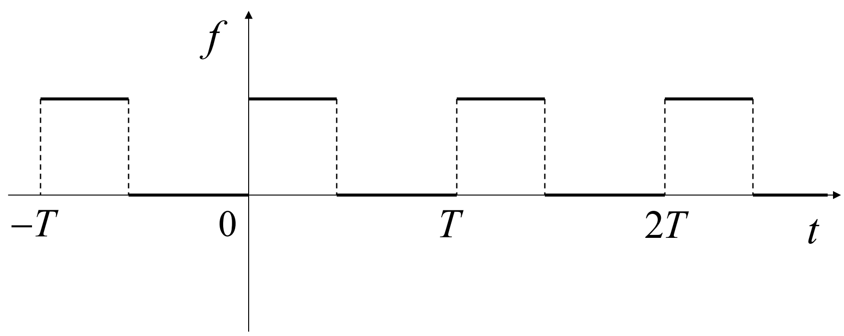


In general, a sinusoid $f(t) = \sin(\omega_0 t)$ is periodic,

with period $T = \dfrac{2\pi}{\omega_0}$.

Same thing for the functions $\cos(\omega_0 t)$, $\cos(\omega_0 t + \varphi)$, $e^{i\omega_0 t}$

Another periodic function: the "square wave"



Is a sum of sinusoids periodic?

$$f(t) = \underbrace{\cos(\omega_1 t)}_{f_1(t)} + \underbrace{\cos(\omega_2 t)}_{f_2(t)}$$

$f_1(t)$ has period $T_1 = \dfrac{2\pi}{\omega_1}$, $f_2(t)$ has period $T_2 = \dfrac{2\pi}{\omega_2}$,

So $f_1(t + T_1) = f_1(t)$, and $f_2(t + T_2) = f_2(t)$.

To get $f_1(t+T) + f_2(t+T) = f_1(t) + f_2(t)$ for some T,

T must be a common multiple of T_1, T_2. Examples:

- $\sin(2t) + \cos(3t)$ is periodic. The period is the minimum common multiple of $T_1 = \pi$, $T_2 = \dfrac{2\pi}{3}$. Namely, $T = 2\pi$.
- $\sin(t) + \cos(\pi t)$ is not periodic. $T_1 = 2\pi$, $T_2 = 2$ have no common multiple.