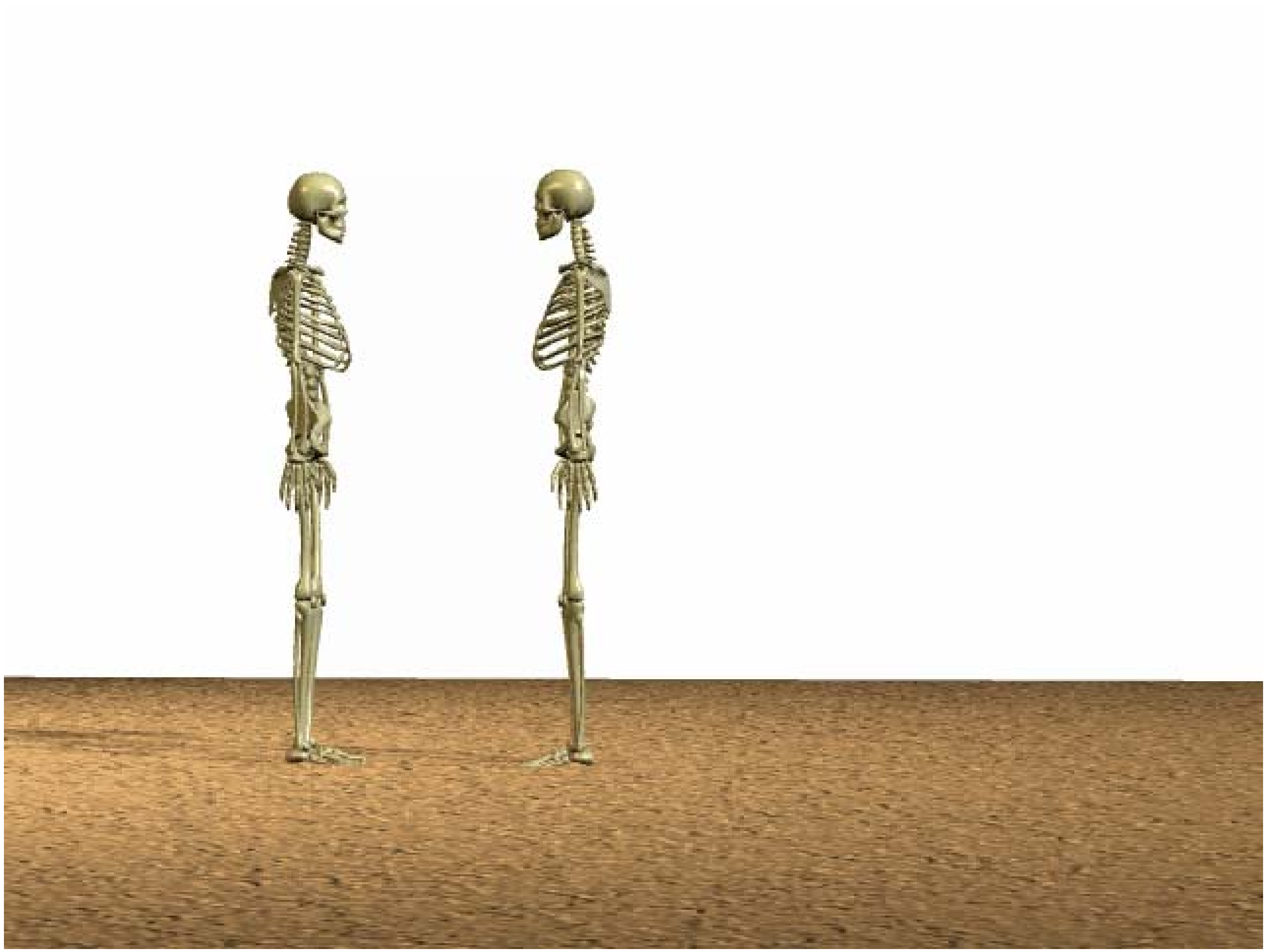
Affine Transformations in 3D



Affine Transformations in 3D

General form

$$\begin{pmatrix} Q_x \\ Q_y \\ Q_z \\ 1 \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} P_x \\ P_y \\ P_z \\ 1 \end{pmatrix}$$

Elementary 3D Affine Transformations

Translation

$$\begin{pmatrix} Q_x \\ Q_y \\ Q_z \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & T_x \\ 0 & 1 & 0 & T_y \\ 0 & 0 & 1 & T_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} P_x \\ P_y \\ P_z \\ 1 \end{pmatrix}$$

Scale Around the Origin

$$\begin{pmatrix} Q_x \\ Q_y \\ Q_z \\ 1 \end{pmatrix} = \begin{pmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} P_x \\ P_y \\ P_z \\ 1 \end{pmatrix}$$

Shear around the origin

Along x-axis

$$\begin{pmatrix} Q_x \\ Q_y \\ Q_z \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & a & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} P_x \\ P_y \\ P_z \\ 1 \end{pmatrix}$$

3 DRotation

Various representations

Decomposition into axis rotations (x-roll, y-roll, z-roll)

CCW positive assumption

Reminder 2D z-rotation

$$Q_x = cos\theta P_x - sin\theta P_y$$

$$Q_y = sin\theta P_x + cos\theta P_y$$

$$\begin{pmatrix} Q_x \\ Q_y \\ 1 \end{pmatrix} = \begin{pmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} P_x \\ P_y \\ 1 \end{pmatrix}$$

Z-roll

$$\boxed{\begin{aligned} Q_x &= cos\theta P_x - sin\theta P_y \\ Q_y &= sin\theta P_x + cos\theta P_y \end{aligned}}$$

$$Q_z = P_z$$

$$R_z(\theta) = \begin{pmatrix} cos(\theta) & -sin(\theta) & 0 & 0 \\ sin(\theta) & cos(\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & & 1 \end{pmatrix}$$

X-roll

Cyclic indexing

$$x \rightarrow \boxed{y \rightarrow z \rightarrow x} \rightarrow y$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \rightarrow \begin{bmatrix} x \\ y \\ z \\ x \\ y \end{bmatrix}$$

$$Q_y = cos\theta P_y - sin\theta P_z$$
$$Q_z = sin\theta P_y + cos\theta P_z$$
$$Q_x = P_x$$

$$R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & cos(\theta) & -sin(\theta) & 0 \\ 0 & sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & & 1 \end{pmatrix}$$

Y-roll

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \rightarrow \begin{bmatrix} x \\ y \\ \boxed{\begin{matrix} z \\ x \\ y \end{matrix}} \end{bmatrix}$$

$$Q_z = cos\theta P_z - sin\theta P_x$$

$$Q_x = sin\theta P_z + cos\theta P_x$$

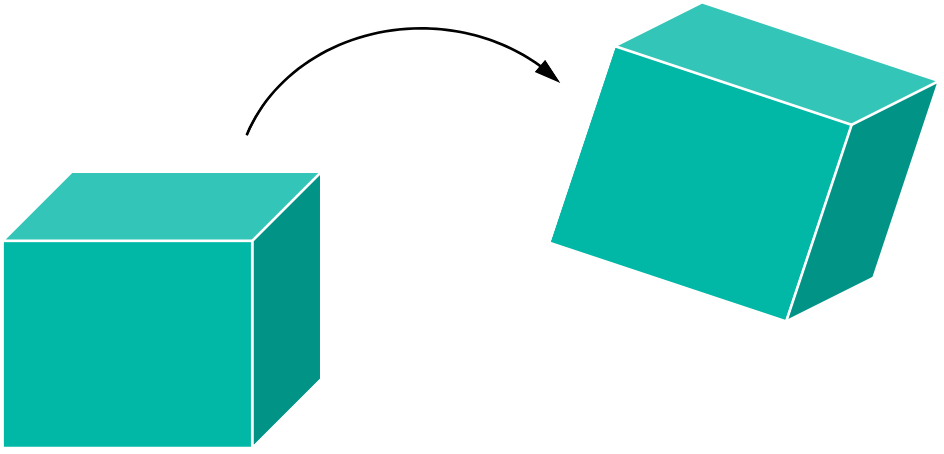
$$Q_y = P_y$$

$$R_y(\theta) = \begin{pmatrix} cos(\theta) & 0 & sin(\theta) & 0 \\ 0 & 1 & 0 & 0 \\ -sin(\theta) & 0 & cos(\theta) & 0 \\ 0 & 0 & & 1 \end{pmatrix}$$

Rigid body transformations

Translations and rotations

Preserve angles and distances



Composition of 3D Affine Transformations

The composition of affine transformations is an affine transformation.

Any 3D affine transformation can be performed as a series of elementary affine transformations.

Composite 3D Rotation around origin

$$R = R_z(\theta_3)R_y(\theta_2)R_x(\theta_1)$$

The order is important !!

It is often convenient to use other representations for 3D rotations….

Rotation around an arbitrary axis

Euler's theorem: Any rotation or sequence of rotations around a point is equivalent to a single rotation around an axis that passes through the point.

What does the matrix look like?

Rotation around an arbitrary axis

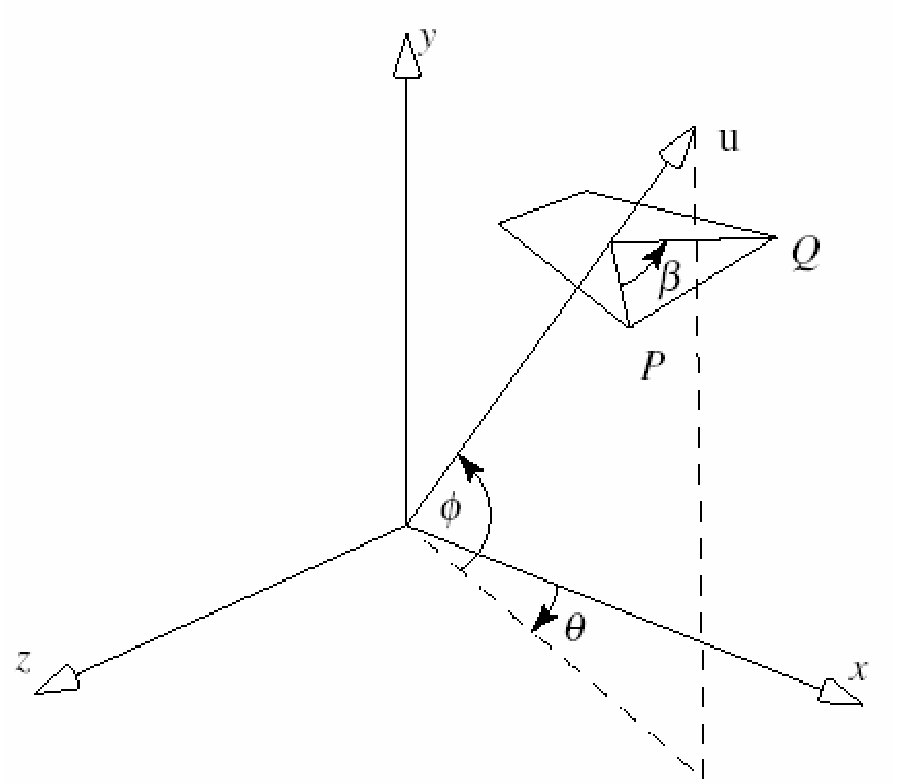
Axis: **u**

Point: P

Angle: β

Method:

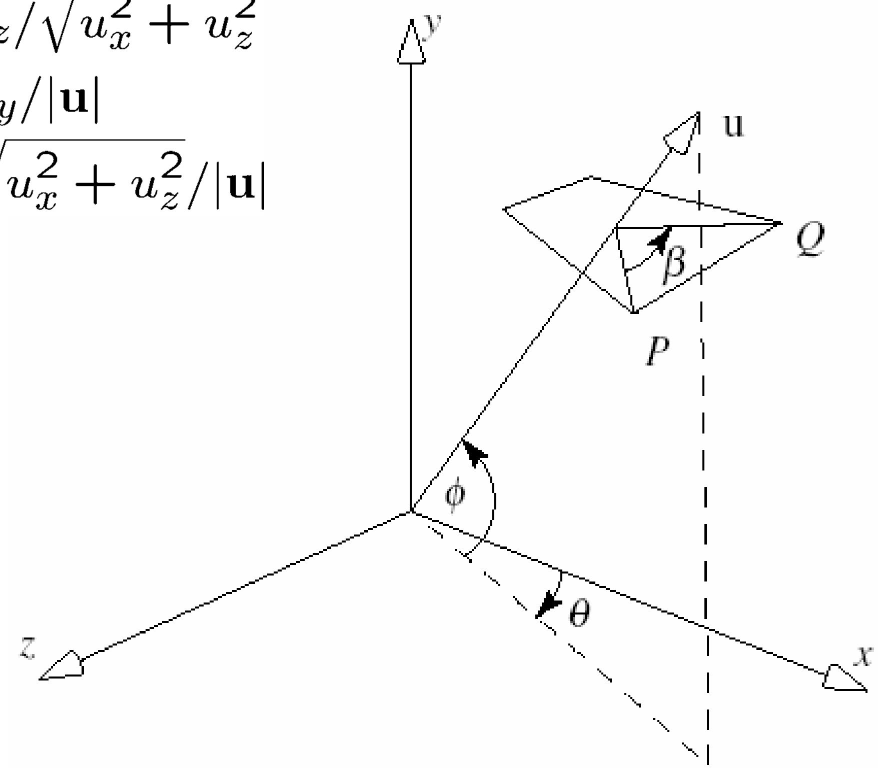
1. *Two rotations to align* ***u*** *with x-axis*
2. *Do x-roll by β*
3. *Undo the alignment*



Derivation

1. $R_z(-\phi)R_y(\theta)$
2. $R_x(\beta)$
3. $R_y(-\theta)R_z(\phi)$

$$cos(\theta) = u_x/\sqrt{u_x^2 + u_z^2}$$
$$sin(\theta) = u_z/\sqrt{u_x^2 + u_z^2}$$
$$sin(\phi) = u_y/|\mathbf{u}|$$
$$cos(\phi) = \sqrt{u_x^2 + u_z^2}/|\mathbf{u}|$$



Altogether:

$$\boxed{R_y(-\theta)R_z(\phi)\, R_x(\beta)\, R_z(-\phi)R_y(\theta)}$$

We can add translation too if the axis is not through the origin

Properties of affine transformations

1. *Preservation of affine combinations of points.*
2. *Preservation of lines and planes.*
3. *Preservation of parallelism of lines and planes.*
4. *Relative ratios are preserved*
5. *Affine transformations are composed of elementary ones.*

Affine Combinations of Points

$$W = a_1 P_1 + a_2 P_2$$
$$T(W) = T(a_1 P_1 + a_2 P_2) = a_1 T(P_1) + a_2 T(P_2)$$

Proof: from linearity of matrix multiplication

$$MW = M(a_1 P_1 + a_2 P_2) = a_1 M P_1 + a_2 M P_2$$

Preservations of Lines and Planes

$$L(t) = (1-t)P_1 + tP_2$$
$$T(L) = (1-t)T(P_1) + tT(P_2)$$

$$Pl(t) = (1-s-t)P_1 + tP_2 + sP_3$$
$$T(L) = (1-s-t)T(P_1) + tT(P_2) + sT(P_3)$$

Proof: Direct consequence of previous property.

Preservation of Parallelism

$$L(t) = P + t\mathbf{u}$$

$$ML = M(P + t\mathbf{u}) = MP + M(t\mathbf{u}) \rightarrow$$
$$ML = MP + t(M\mathbf{u})$$

$M\mathbf{u}$ independent of P.

Similarly for planes.

General form

Rotation, Scaling,
Shear

Translation

$$\begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Inverse of Rotations

Pure rotation only, no scaling or shear.

$$M = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix}$$

$$M^{-1} = M^T$$

Advanced concepts

Generalized shears

Decomposition of 2D AT:

2D : $M = T\ Sh\ S\ R$

3D: $M = T\ S\ R\ Sh_1\ Sh_2$

Rotations in 3D

Gimbal lock

Quaternions

Exponential maps

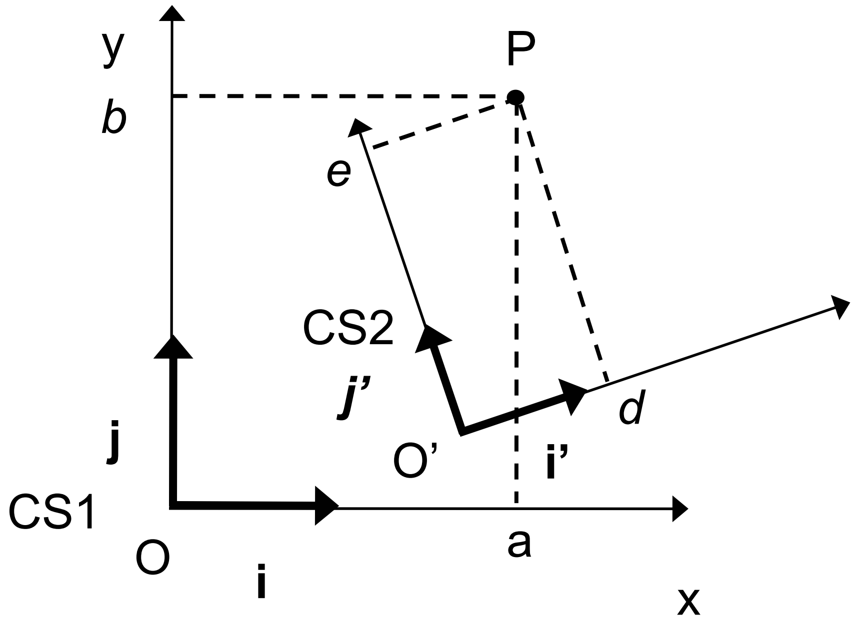
Transformations of Coordinate systems

Coordinate systems consist of vectors and an origin, therefore we can transform them just like points and vectors.

Alternative way to think of transformations

Transforming CS1 into CS2

What is the relationship between P in CS2 and P in CS1 if CS2 = T(CS1)?



$$CS1 : P = (a, b, c, 1)^T$$
$$CS2 : P = (d, e, f, 1)^T$$

$$O' = T(O),$$
$$\mathbf{i}' = T(\mathbf{i}),$$
$$\mathbf{j}' = T(\mathbf{j}),$$
$$\mathbf{k}' = T(\mathbf{k})$$

Derivation

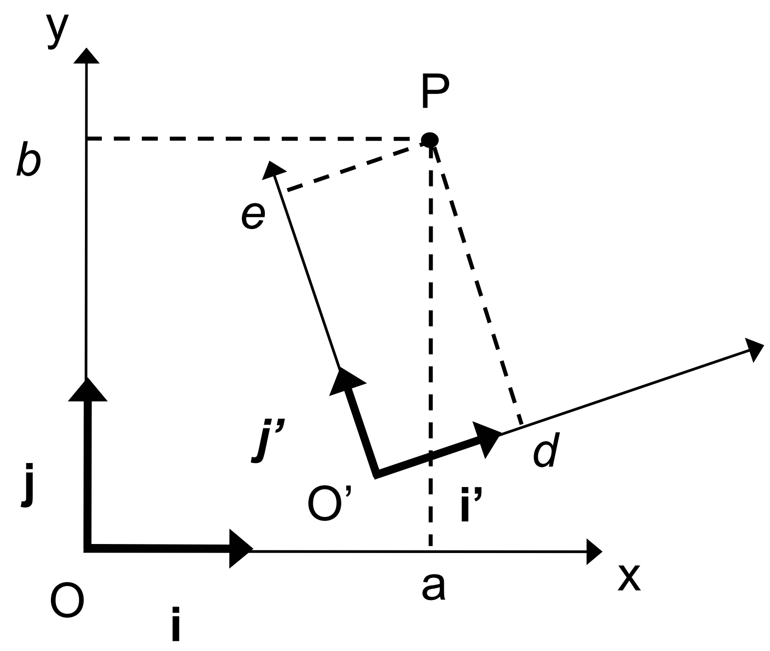
$$
\begin{aligned}
P_{CS1} &= d\mathbf{i}' + e\mathbf{j}' + f\mathbf{k}' + \mathbf{O}' \\
P_{CS1} &= dT(\mathbf{i}) + eT(\mathbf{j}) + fT(\mathbf{k}) + T(\mathbf{O}) \\
&= d(\mathbf{M}\mathbf{i}) + e(\mathbf{M}\mathbf{j}) + f(\mathbf{M}\mathbf{k}) + \mathbf{M}\mathbf{O} \\
&= d(\mathbf{M}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}) + e(\mathbf{M}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}) + f(\mathbf{M}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}) \\
&= \mathbf{M}\begin{bmatrix} d \\ 0 \\ 0 \end{bmatrix} + \mathbf{M}\begin{bmatrix} 0 \\ e \\ 0 \end{bmatrix} + \mathbf{M}\begin{bmatrix} 0 \\ 0 \\ f \end{bmatrix} \\
&= \mathbf{M}(\begin{bmatrix} d \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ e \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ f \end{bmatrix}) \\
&= \mathbf{M}\begin{bmatrix} d \\ e \\ f \end{bmatrix}
\end{aligned}
$$

P in CS1 vs P in CS2

Proof in pages 245,246 of [Hill]

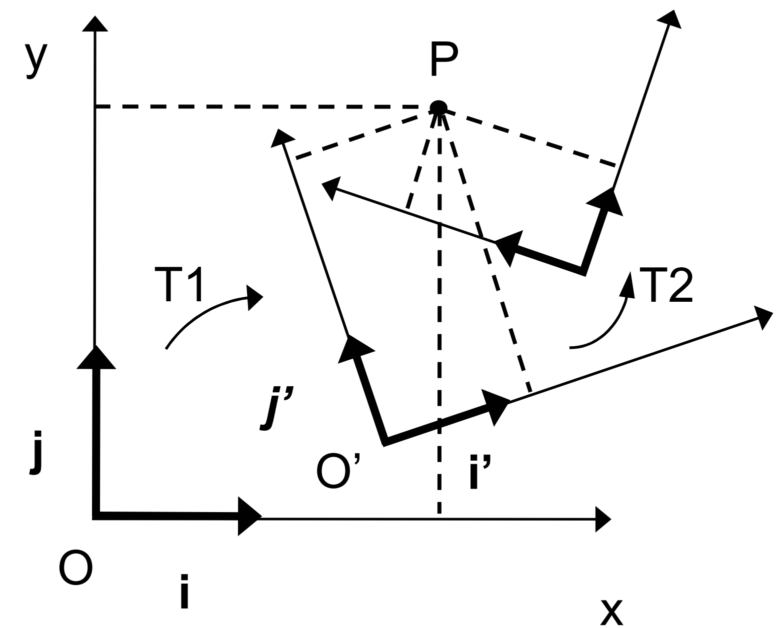
$$P_{CS1} = \mathrm{M} P_{CS2}$$

$$\begin{pmatrix} a \\ b \\ c \\ 1 \end{pmatrix} = \mathrm{M} \begin{pmatrix} d \\ e \\ f \\ 1 \end{pmatrix}$$



Successive transformations of CS

CS1 → CS2 → CS3

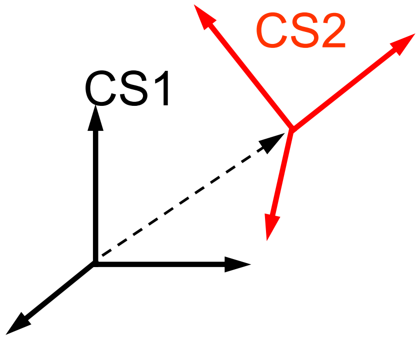


Working backwards:

$$P_{CS2} = M_2 P_{CS3} \rightarrow \begin{pmatrix} d \\ e \\ f \\ 1 \end{pmatrix} = M_2 \begin{pmatrix} g \\ h \\ m \\ 1 \end{pmatrix}$$

$$P_{CS1} = M_1 P_{CS2} \rightarrow \begin{pmatrix} a \\ b \\ c \\ 1 \end{pmatrix} = M_1 \begin{pmatrix} d \\ e \\ f \\ 1 \end{pmatrix} = M_1 M_2 \begin{pmatrix} g \\ h \\ m \\ 1 \end{pmatrix}$$

Transformations as a change of basis



We know the basis vectors and we know that

$$P_{CS1} = MP_{CS2}$$

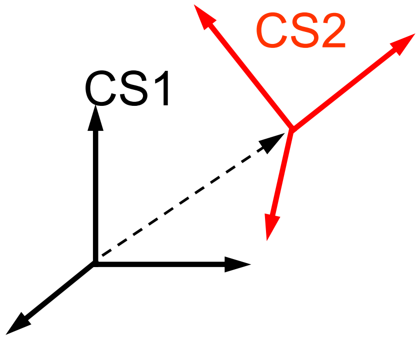
What is M with respect to the basis vectors?

$$P_{CS2} = a\mathbf{i}_{CS2} + b\mathbf{j}_{CS2} + c\mathbf{k}_{CS2} + O_{CS2} = a\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$P_{CS1} = a\mathbf{i}_{CS1} + b\mathbf{j}_{CS1} + c\mathbf{k}_{CS1} + O_{CS1} = a\begin{bmatrix} i_x \\ i_y \\ i_z \end{bmatrix} + b\begin{bmatrix} j_x \\ j_y \\ j_z \end{bmatrix} + c\begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix} + \begin{bmatrix} O_x \\ O_y \\ O_z \end{bmatrix}$$

$$P_{CS1} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} i_x & j_x & k_x & O_x \\ i_y & j_y & k_y & O_y \\ i_z & j_z & k_z & O_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ 1 \end{bmatrix} = MP_{CS2}$$

Transformations as a change of basis



$$P_{CS1} = MP_{CS2}$$

$$P_{CS1} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} i_x & j_x & k_x & O_x \\ i_y & j_y & k_y & O_y \\ i_z & j_z & k_z & O_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ 1 \end{bmatrix} = MP_{CS2}$$

Rule of thumb

Transforming a point P:

Transformations: T1,T2,T3

Matrix: M = M3 x M2 x M1

Point transformed by: MP

Succesive transformations happen with respect to the same CS

Transforming a CS

Transformations: T1, T2, T3

Matrix: M = M1 x M2 x M3

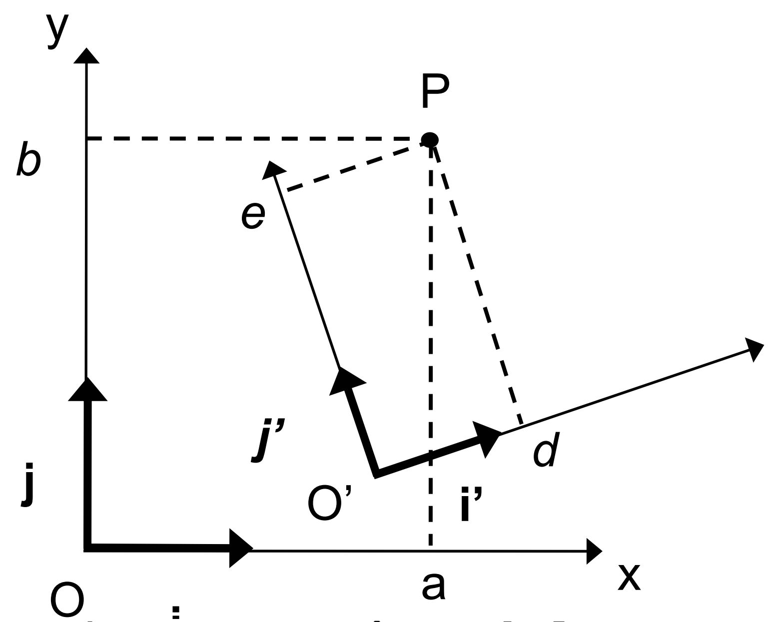
A point has original coordinates MP

Each transformations happens with respect to the new CS.

A helpful way to think about transformations

Input-Output

Output ← M ← Input:



M takes P in CS2 and produces Pin CS1

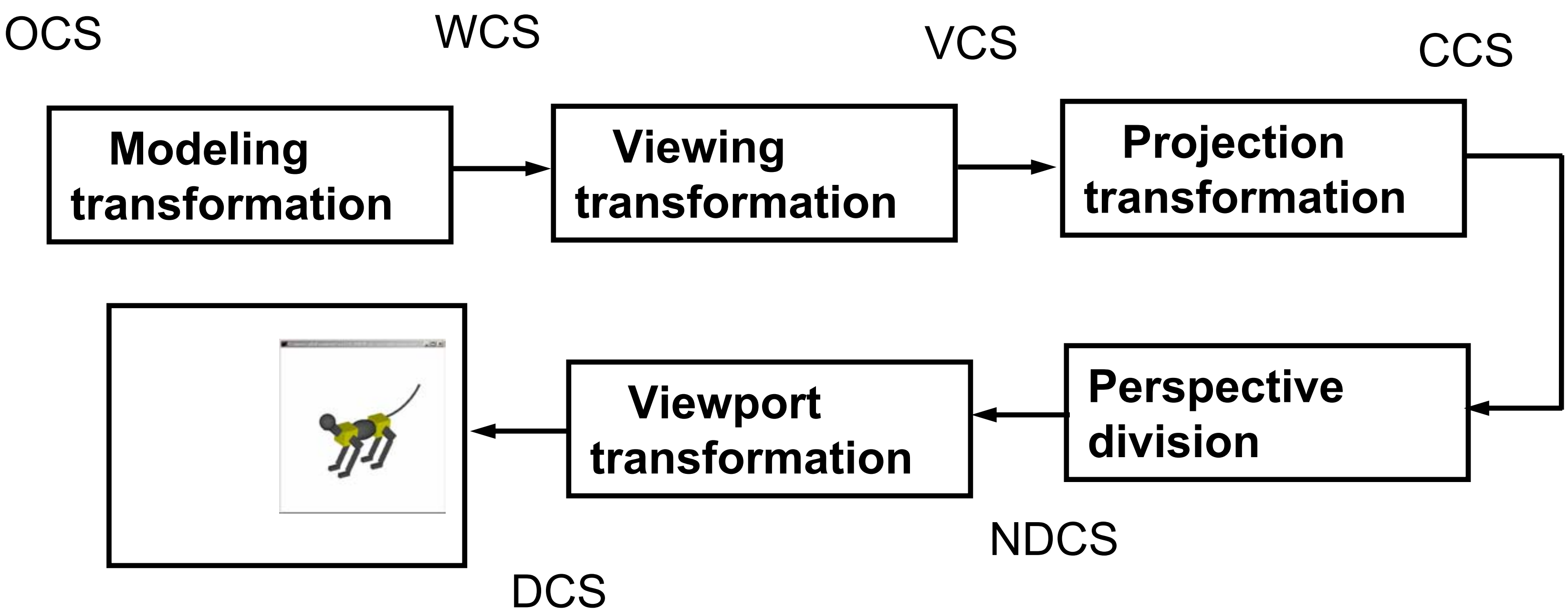
$$P_{CS1} = M_{CS1 \leftarrow CS2} P_{CS2}$$

$$P_{CS1} = {}_{CS1}M_{CS2} P_{CS2}$$

Rule of thumb

To find the transformation matrix that transforms P from CSA coordinates to CSB coordinates, we find the sequence of transormations that aling CSB to CSA accumulating matrices from left to right.

Graphics Pipeline



Translation in OpenGL

glTranslate3f(GLfloat x, GLfloat y, GLfloat z) ;

glTranslate3d(GLdouble x, GLdouble y, GLdouble z);

$$\begin{pmatrix} 1 & 0 & 0 & T_x \\ 0 & 1 & 0 & T_y \\ 0 & 0 & 1 & T_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Scaling in OpenGL

glScalef(GLfloat sx, GLfloat sy, GLfloat sz) ;

glScaled(GLdouble sx, GLdouble sy, GLdouble sz) ;

$$\begin{pmatrix} sx & 0 & 0 & 0 \\ 0 & sy & 0 & 0 \\ 0 & 0 & sz & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Rotation in OpenGL

glRotatef(GLfloat angle, GLfloat x, GLfloat y, GLfloat z) ;

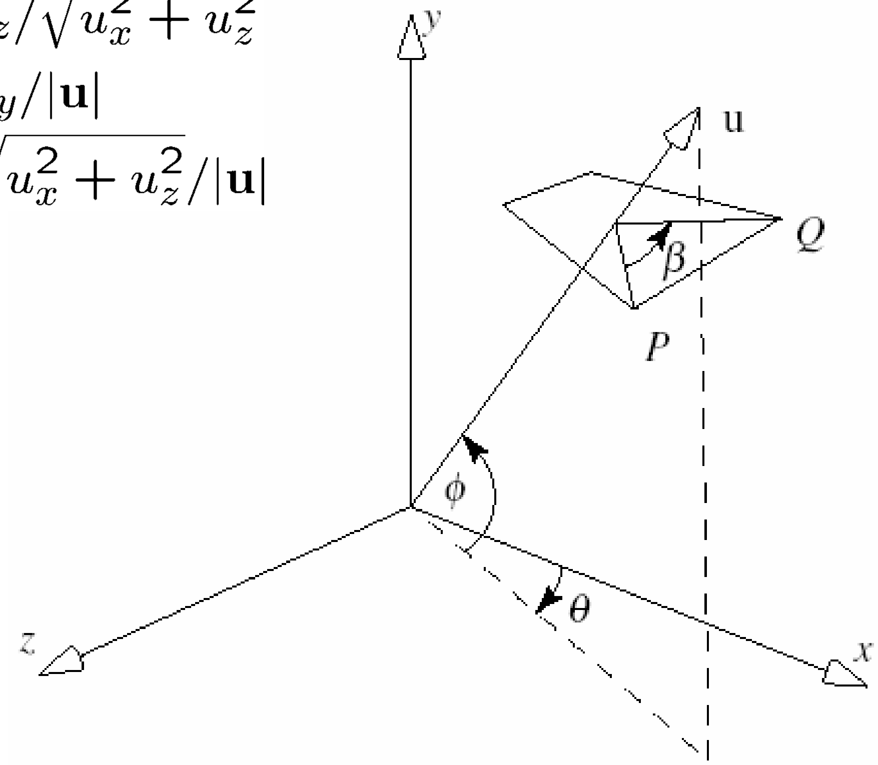
glRotated(GLdouble angle, GLdouble ux, GLdouble uy, GLdouble uz) ;

(Matrix in the next slide)

Matrix created

1. $R_z(-\phi)R_y(\theta)$
2. $R_x(\beta)$
3. $R_y(-\theta)R_z(\phi)$

$$cos(\theta) = u_x/\sqrt{u_x^2 + u_z^2}$$
$$sin(\theta) = u_z/\sqrt{u_x^2 + u_z^2}$$
$$sin(\phi) = u_y/|\mathbf{u}|$$
$$cos(\phi) = \sqrt{u_x^2 + u_z^2}/|\mathbf{u}|$$



Altogether:

$$\boxed{R_y(-\theta)R_z(\phi)\ R_x(\beta)\ R_z(-\phi)R_y(\theta)}$$

We can add translation too if the axis is not through the origin

Composition of transformations in OpenGL

Successively transforming the coordinate system

$M = M1\ M2\ M3\ \ldots.\ Mn$

$Pwolrd = M\ Pobj$

OpengGL Modelview Matrix

Each transformation post multiplies the current modelview matrix CM

```
glMatrixMode(GL_MODELVIEW) ;
glLoadIdentity() ;              // CM = I
glRotatef(45, 0,0,1) ;          // CM = I*Rz(45) ;
glTranslatef(1,1,1) ;           // CM = CM*T(1,1,1) =
                                //    = I*Rz(45) *T(1,1,1)
glScale(2,1,1) ;                // CM = CM *S(2,1,1) = I*Rz*T*S
```

Arbitrary matrices

Arbitrary affine (or not) transformations

glLoadMatrixf(GLfloat *M) ; // CM = M

glLoadMatrixd(GLdouble *M) ; // CM = M

glMultMatrixf(GLfloat *M) ; // CM = CM*M

glMultMatrixd(GLfloat *M) ; // CM = CM*M

Tricky Point

There are no multi-dimensional arrays in c.

Column-major order vs. row-major order.

OpenGL uses column major order that is:

$$\textbf{float} \quad m[16] = \boxed{a0, a1, a2, a3} \ldots, a15;$$

$$\textbf{becomes}:$$

$$\begin{bmatrix} \boxed{\begin{matrix} a0 \\ a1 \\ a2 \\ a3 \end{matrix}} & \begin{matrix} a4 \\ a5 \\ a6 \\ a7 \end{matrix} & \begin{matrix} a8 \\ a9 \\ a10 \\ a11 \end{matrix} & \begin{matrix} a12 \\ a13 \\ a14 \\ a15 \end{matrix} \end{bmatrix}$$

Feedback

```
GLdouble m[16] ; glGetDoublev(GL_MODELVIEW_MATRIX,m) ;

GLfloat m[16] ; glGetFloatv(GL_MODELVIEW_MATRIX,m) ;
```

Matrix Stack

Why a stack?

- Reuse of transformations
- Control the effect of transformations
- Hierarchical structures

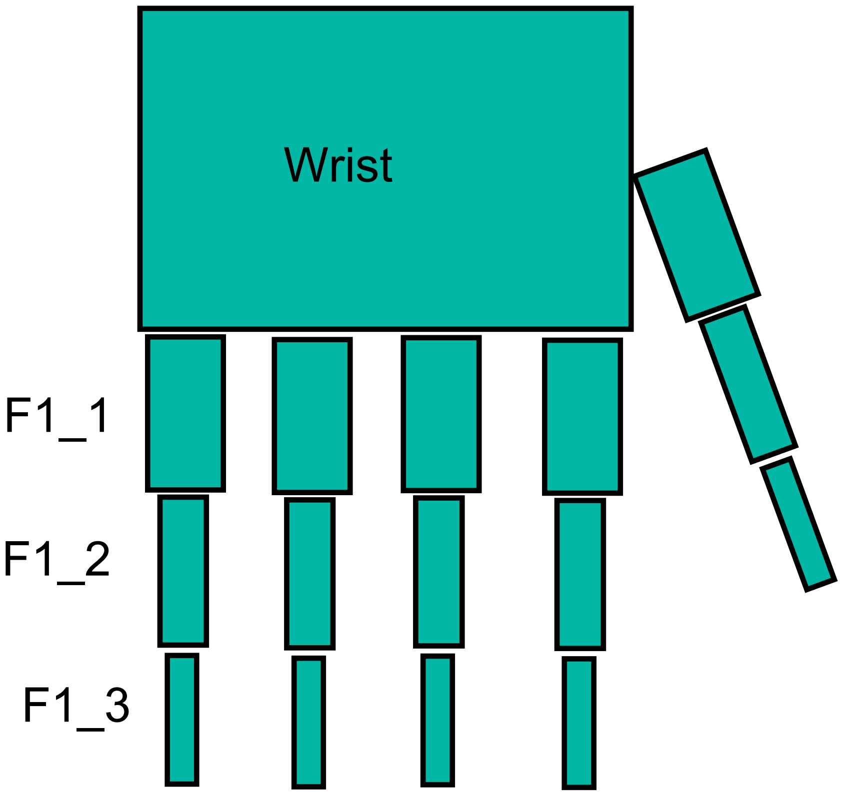
Manipulating the stack

- glPushMatrix() ;
- glPopMatrix() ;

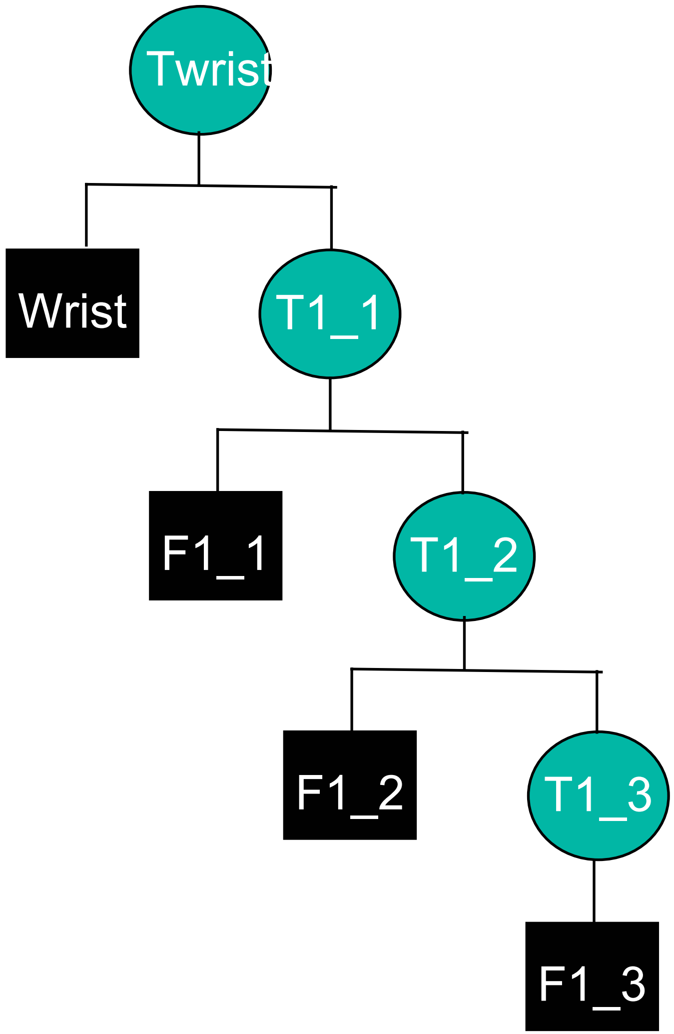
Example

Wrist and 5 fingers

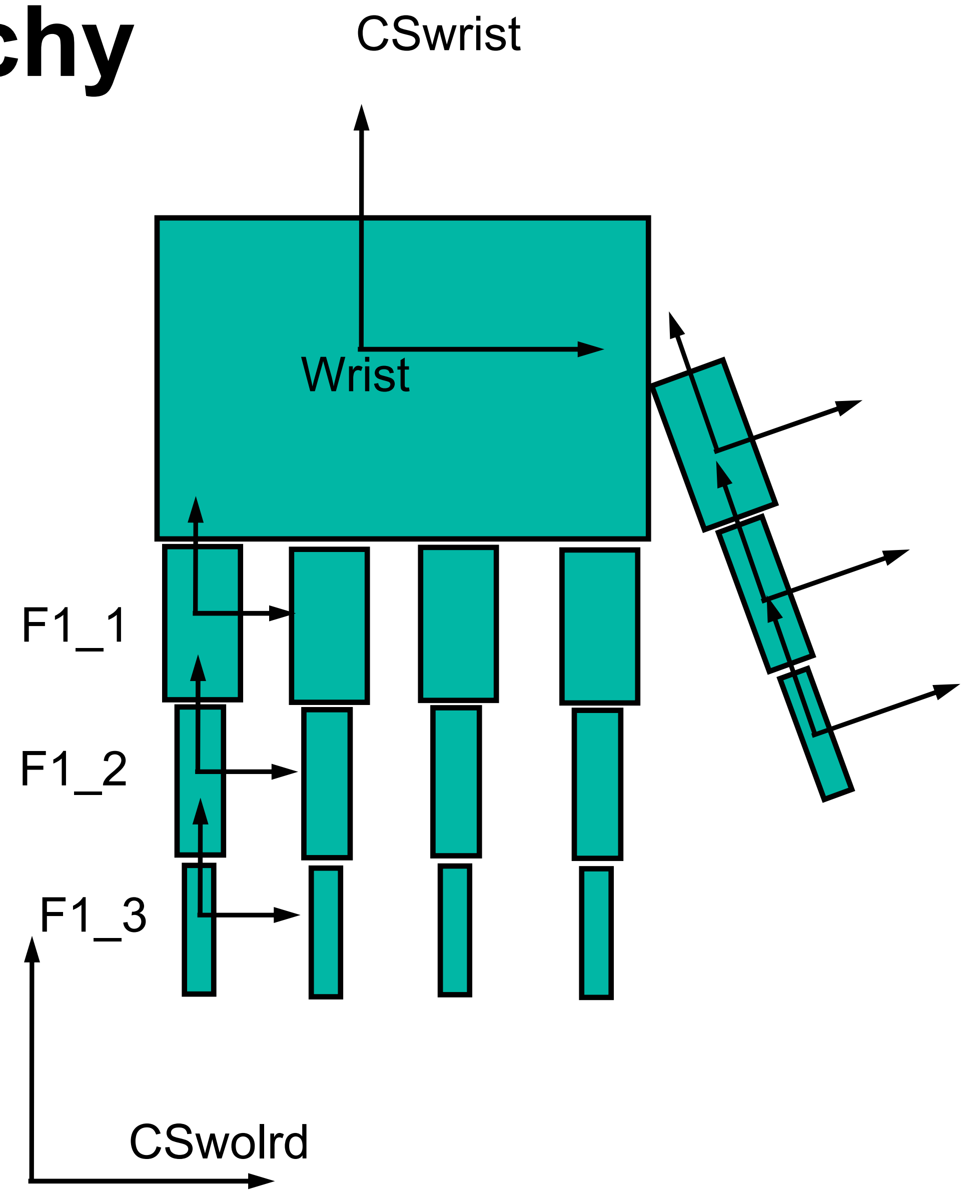
We want the figures to stay attached to the wrist as the wrist moves.



Hierarchy



Twrist
Wrist
T1_1
F1_1
T1_2
F1_2
T1_3
F1_3



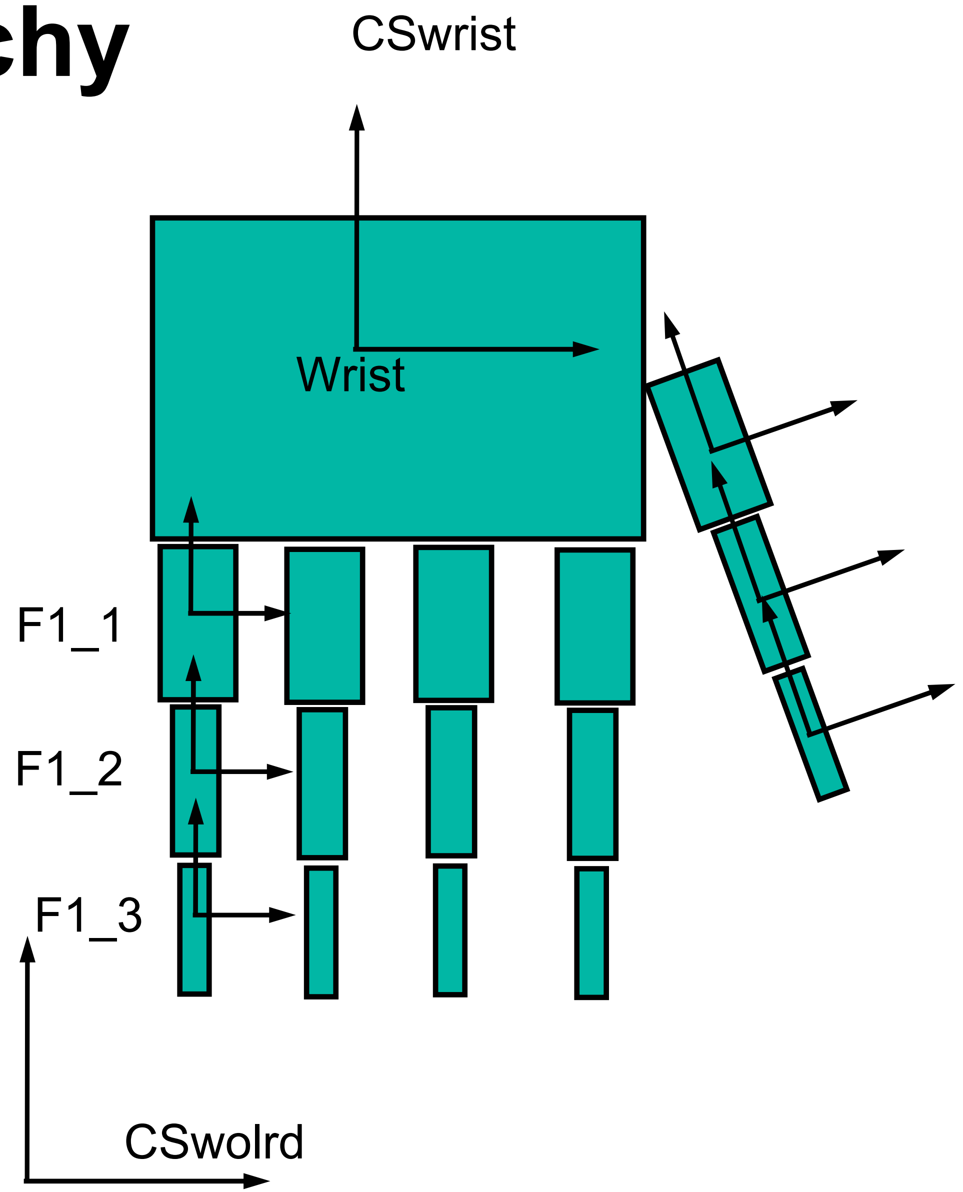
CSwrist
Wrist
F1_1
F1_2
F1_3
CSwolrd

Hierarchy

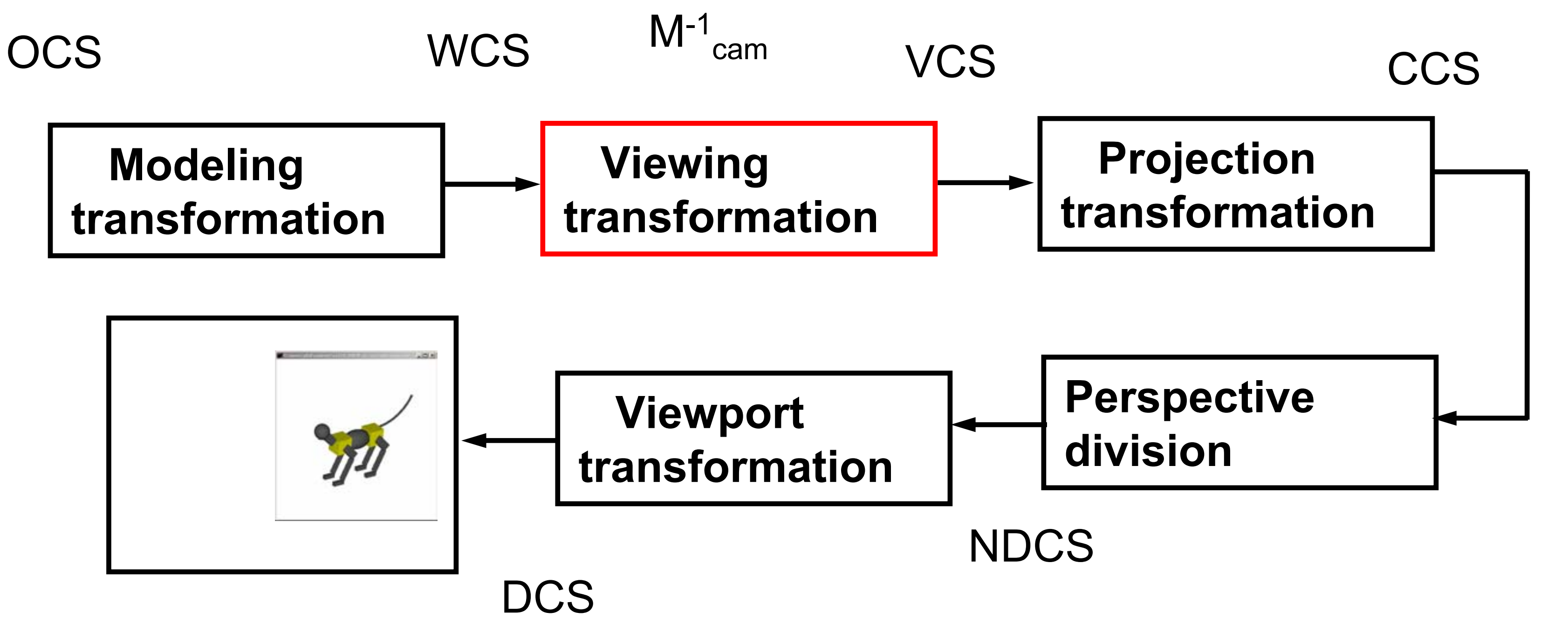
CSF1_1 = T1_1(CSwrist)

CSF1_2 = T1_2(CSF1_1)

CSF1_3 = T1_3(CSF1_2)

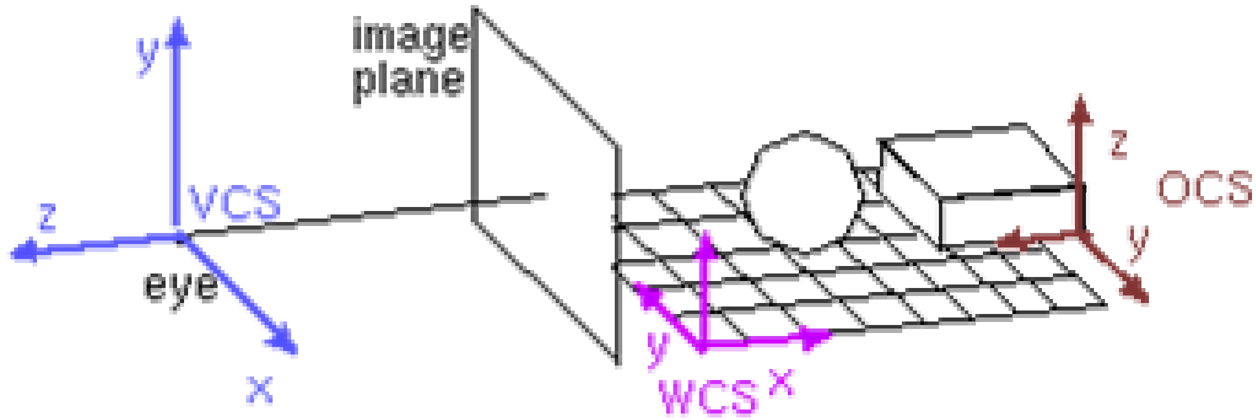


Graphics Pipeline



Camera transformation (Hill 358-366)

Transforms objects to camera coordinates



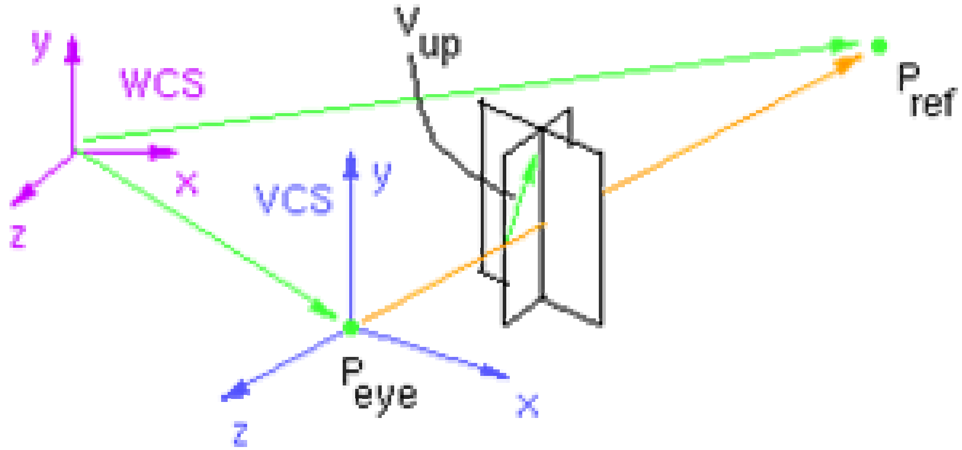
$$\left.\begin{aligned} P_{wcs} = M_{cam} P_{vcs} \rightarrow P_{vcs} &= M_{cam}^{-1} P_{wcs} \\ P_{wcs} &= M_{mod} P_{obj} \end{aligned}\right\} \rightarrow$$

$$P_{vcs} = M_{cam}^{-1} M_{mod} P_{obj}$$

Defining Mcam

Common way

Eye point
Reference point
Upvector



To build Mcam we need to define a camera coordinate system (origin, i, j, k)

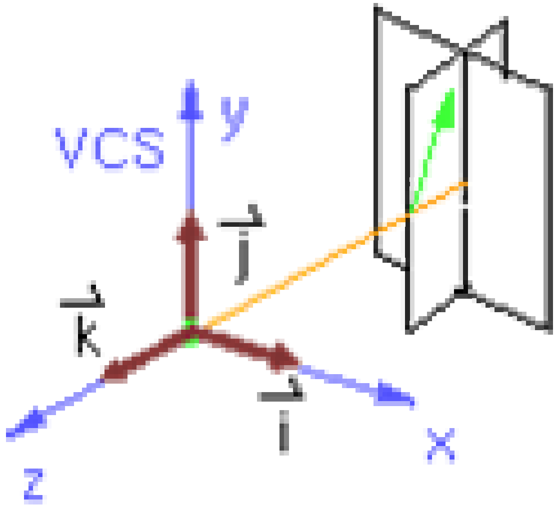
Camera Coordinate system

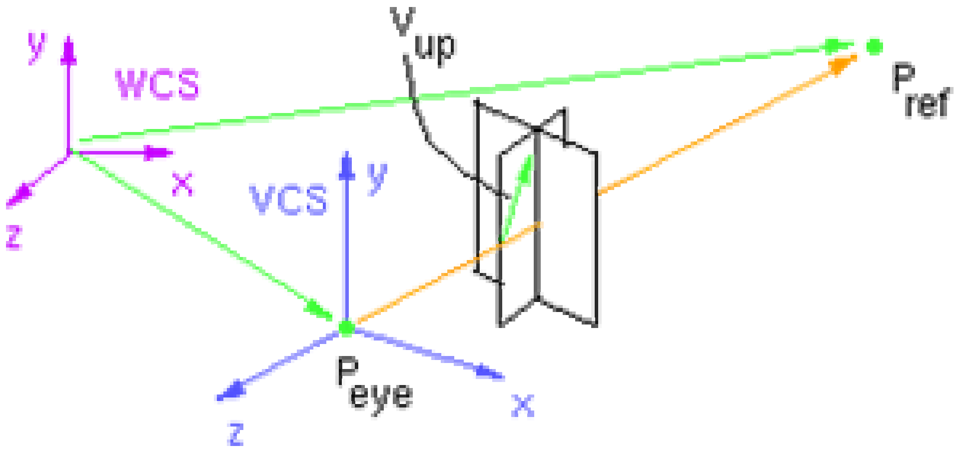
$$\mathbf{k} = \frac{P_{eye} - P_{ref}}{|P_{eye} - P_{ref}|}$$

$$\mathbf{I} = \mathbf{v}_{up} \times \mathbf{k}$$

$$\mathbf{i} = \frac{\mathbf{I}}{|\mathbf{I}|}$$

$$\mathbf{j} = \mathbf{k} \times \mathbf{i}$$

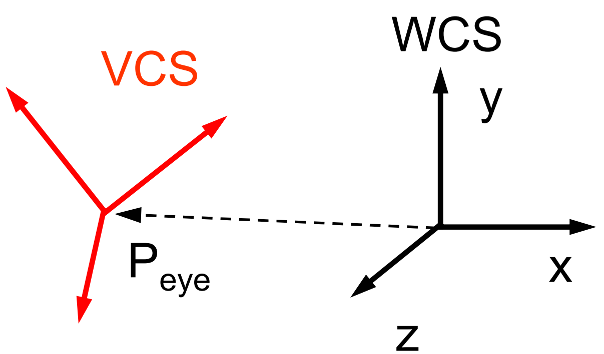




Building Mcam

Change of basis

Our reference system is WCS,
we know the camera parameters with respect to the world



Align WCS with VCS

$$M_{cam} = \begin{bmatrix} 1 & 0 & 0 & P_{eye,x} \\ 0 & 1 & 0 & P_{eye,y} \\ 0 & 0 & 1 & P_{eye,z} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} i_x & j_x & k_x & 0 \\ i_y & j_y & k_y & 0 \\ i_z & j_z & k_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P_{wcs} = M_{cam} P_{vcs}$$

Building Mcam inverse

Invert smart

$$M_{cam}^{-1} = \left(\begin{bmatrix} 1 & 0 & 0 & P_{eye,x} \\ 0 & 1 & 0 & P_{eye,y} \\ 0 & 0 & 1 & P_{eye,z} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} i_x & j_x & k_x & 0 \\ i_y & j_y & k_y & 0 \\ i_z & j_z & k_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right)^{-1}$$

$$= \left(\begin{bmatrix} i_x & j_x & k_x & 0 \\ i_y & j_y & k_y & 0 \\ i_z & j_z & k_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right)^{-1} \left(\begin{bmatrix} 1 & 0 & 0 & P_{eye,x} \\ 0 & 1 & 0 & P_{eye,y} \\ 0 & 0 & 1 & P_{eye,z} \\ 0 & 0 & 0 & 1 \end{bmatrix} \right)^{-1}$$

Building Mcam inverse

Invert smart

$$M_{cam}^{-1} = \left(\begin{bmatrix} i_x & j_x & k_x & 0 \\ i_y & j_y & k_y & 0 \\ i_z & j_z & k_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right)^{-1} \left(\begin{bmatrix} 1 & 0 & 0 & P_{eye,x} \\ 0 & 1 & 0 & P_{eye,y} \\ 0 & 0 & 1 & P_{eye,z} \\ 0 & 0 & 0 & 1 \end{bmatrix} \right)^{-1}$$

Transpose

$$= \begin{bmatrix} i_x & i_y & i_z & 0 \\ j_x & j_y & j_z & 0 \\ k_x & k_y & k_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -P_{eye,x} \\ 0 & 1 & 0 & -P_{eye,y} \\ 0 & 0 & 1 & -P_{eye,z} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P_{vcs} = M_{cam}^{-1} P_{wcs}$$

Camera in OpenGL

gluLookAt(ex,ey,ez,rx,ry,rz,ux,uy,uz)

The resulting matrix post-multiplies the modelview matrix

```
glMatrixMode(GL_MODELVIEW);
glLoadIdentity();
gluLookAt(ex,ey,ez,rx,ry,rz,ux,uy,uz);

// setup modelling
// transformations here
```

End of Modeling transformations

1. *Preservation of affine combinations of points.*
2. *Preservation of lines and planes.*
3. *Preservation of parallelism of lines and planes.*
4. *Relative ratios are preserved*
5. *Affine transformations are composed of elementary ones.*

Camera transformation as a change of basis.

OpenGL matrix stack.