Lecture 17

- Fourier transform of a periodic signal.
- The Sampling Theorem.

Fourier transform of a periodic signal

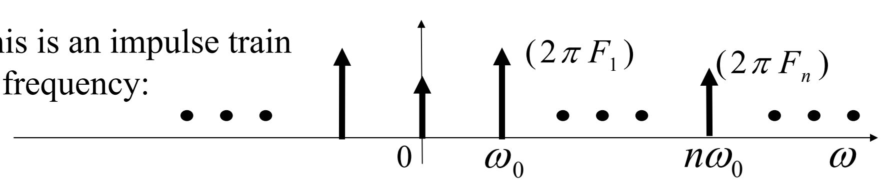
Before, we studied a periodic function by means of the Fourier series

$$f(t) = \sum_{n=-\infty}^{+\infty} F_n e^{in\omega_0 t}$$

Fourier transforms were motivated by extending this type of analysis to non-periodic functions. They do, however, also apply to the above case:

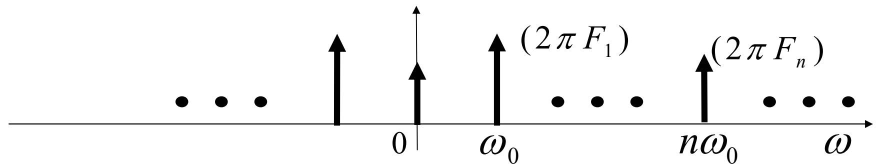
$$\mathcal{F}\left[\sum_{n=-\infty}^{+\infty} F_n e^{in\omega_0 t}\right] = \sum_{n=-\infty}^{+\infty} F_n \mathcal{F}\left[e^{in\omega_0 t}\right] = \boxed{\sum_{n=-\infty}^{+\infty} F_n 2\pi\delta(\omega - n\omega_0)}$$

This is an impulse train in frequency:



An impulse in the transform indicates a pure sinusoid at that frequency. Here, at all multiples of the fundamental frequency.

Periodic function of time $\Leftrightarrow$ Impulse train in frequency



The graph is analogous to the line spectra we used for Fourier series, replacing lines by deltas.

Dual property:

Impulse train in time $\Leftrightarrow$ Periodic function of frequency

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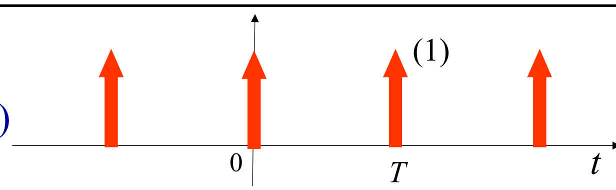
Example:

$$f(t) = \sum_{k=-\infty}^{+\infty} \delta(t - kT)$$

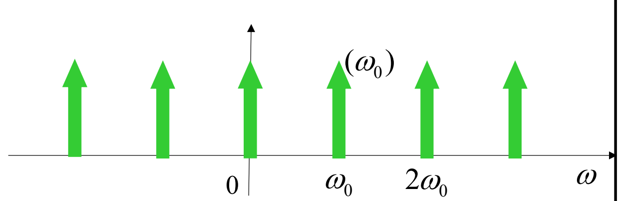


Periodic impulse train in time $\Leftrightarrow$ Periodic impulse train in frequency

The Fourier coefficients are $F_n = \dfrac{1}{T}\displaystyle\int_{-\frac{T}{2}}^{\frac{T}{2}} \delta(t) e^{-in\omega_0 t} dt = \dfrac{1}{T}$

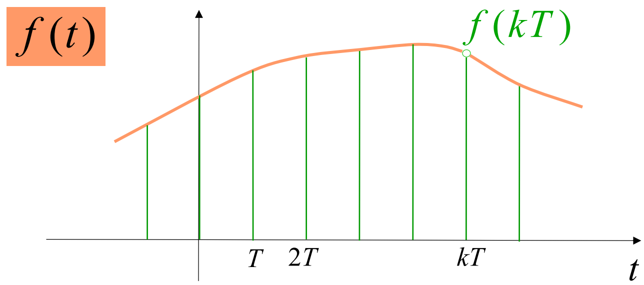
so the previous analysis gives $F(i\omega) = \displaystyle\sum_{n=-\infty}^{+\infty} \frac{2\pi}{T} \delta(\omega - n\omega_0)$

$$\boxed{\mathcal{F}\left[\sum_{k=-\infty}^{+\infty} \delta(t - kT)\right] = \sum_{n=-\infty}^{+\infty} \omega_0 \delta(\omega - n\omega_0)}$$



The Sampling Theorem

Given a signal $f(t)$, we take samples of it every T seconds, and generate a sequence $f(kT)$, k integer.

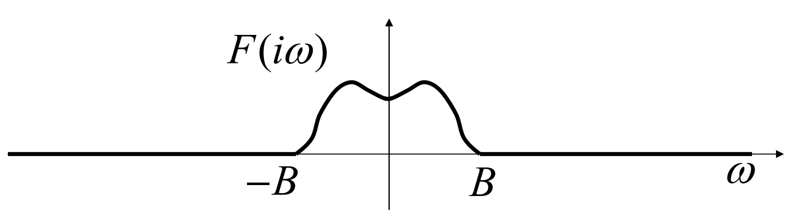


Q: Can we recover *f(t)* from its samples?

A: In general, no. No way of knowing what happened between sample times.

We narrow it down to a special class of functions.

Definition: A signal $f(t)$ is said to be band-limited to $[-B, B]$ if $F(i\omega) = 0$ for $|\omega| > B$.

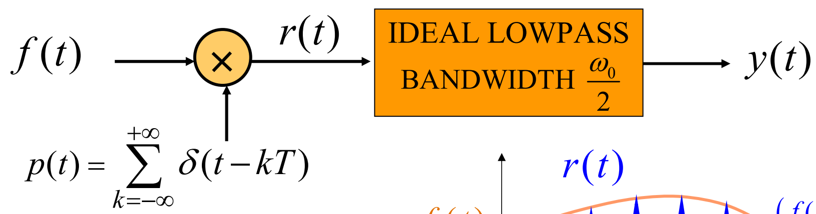


Sampling Theorem:

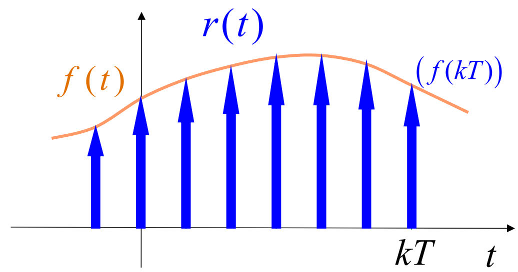
Assume $f(t)$ is band-limited to $[-B, B]$. Let $\omega_0 = \dfrac{2\pi}{T} > 2B$.

Then $f(t)$ can be uniquely recovered from its samples $f(kT)$

Proof: consider the following interconnection of systems

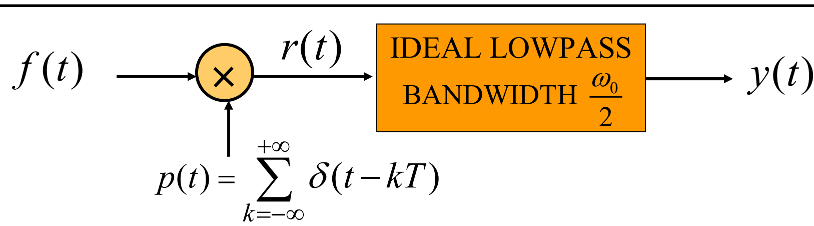


$$r(t)=\sum_{k=-\infty}^{+\infty} f(t)\delta(t-kT)$$
$$=\sum_{k=-\infty}^{+\infty} f(kT)\delta(t-kT)$$



The impulse train *r(t)* depends only on the samples *f(kT)*.

We will show that the output *y(t)* reconstructs the input *f(t):*
This means we have determined *f(t)* by its samples.



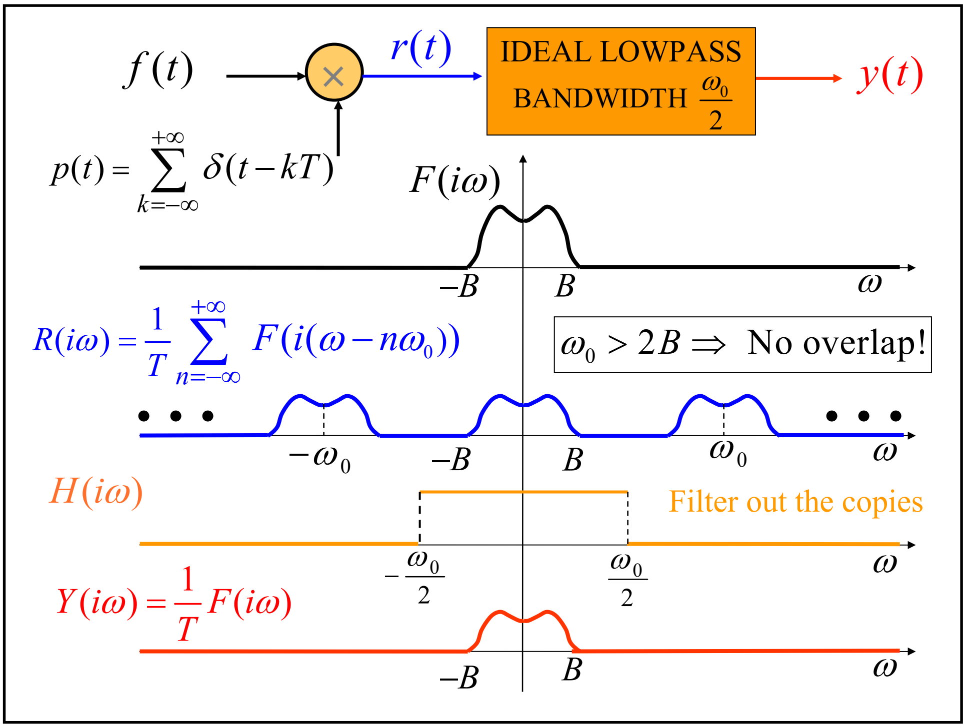
Going to the frequency domain:

$$R(i\omega)=\mathcal{F}\left[r(t)\right]=\mathcal{F}\left[f(t)p(t)\right]=\frac{1}{2\pi}F(i\omega)*P(i\omega)$$
$$=\frac{1}{2\pi}F(i\omega)*\left[\sum_{n=-\infty}^{+\infty}\omega_0\delta(\omega-n\omega_0)\right]=\frac{\omega_0}{2\pi}\sum_{n=-\infty}^{+\infty}F(i\omega)*\delta(\omega-n\omega_0)$$

Now,

$$F(i\omega)*\delta(\omega-n\omega_0)=\int_{-\infty}^{\infty}F(i\lambda)\delta(\omega-\lambda-n\omega_0)d\lambda=F\left(i(\omega-n\omega_0)\right)$$

Also, $\frac{\omega_0}{2\pi}=\frac{1}{T}$. Therefore, $\boxed{R(i\omega)=\frac{1}{T}\sum_{n=-\infty}^{+\infty}F(i(\omega-n\omega_0))}$



$$Y(i\omega)=\frac{1}{T}F(i\omega)\Rightarrow y(t)=\frac{1}{T}f(t).$$

Signal was recovered from its samples, so the Theorem is proved.

We can also get an explicit time-domain formula.

$$r(t)=\sum_{k=-\infty}^{+\infty}f(kT)\delta(t-kT)\Rightarrow y(t)=\sum_{k=-\infty}^{+\infty}f(kT)h(t-kT)$$

where $h(t)=\dfrac{\sin\left(\frac{\omega_0 t}{2}\right)}{\pi t}=\dfrac{\sin\left(\frac{\omega_0 t}{2}\right)}{\frac{\omega_0 T}{2}t}=\dfrac{1}{T}\operatorname{sinc}\left(\dfrac{\omega_0 t}{2}\right)$

is the impulse response of the ideal lowpass filter.

$$f(t)=Ty(t)=\sum_{k=-\infty}^{+\infty}f(kT)\,T\,h(t-kT)=\sum_{k=-\infty}^{+\infty}f(kT)\operatorname{sinc}\left(\frac{\omega_0}{2}(t-kT)\right)$$

$$\boxed{f(t)=\sum_{k=-\infty}^{+\infty}f(kT)\operatorname{sinc}\left(\frac{\omega_0}{2}(t-kT)\right)}$$

Formula that interpolates a band-limited function from its samples

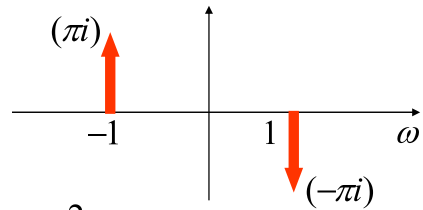
Aliasing

What happens if we don't sample fast enough ($\omega_0 \leq 2B$)? In the previous argument, the various "copies" of $F(i\omega)$ present in $R(i\omega)$ will overlap, and the lowpass filter cannot tell them apart. This is called aliasing. Here $f(t)$ cannot be recovered from its samples.

Example: $f(t) = \sin(t)$.

$F(i\omega) = i\pi\left[\delta(\omega+1) - \delta(\omega-1)\right]$,

band-limited to $[-1,1]$.



We sample it with period $T = \pi \Rightarrow \omega_0 = 2$

This is equal, but not greater than twice the signal bandwidth

The samples are $f(kT) = \sin(k\pi) = 0$ for all k!

Clearly, we cannot recover $f(t)$ from these samples,

we cannot distinguish it from the zero function.