EE102 - Practice Midterm Solutions

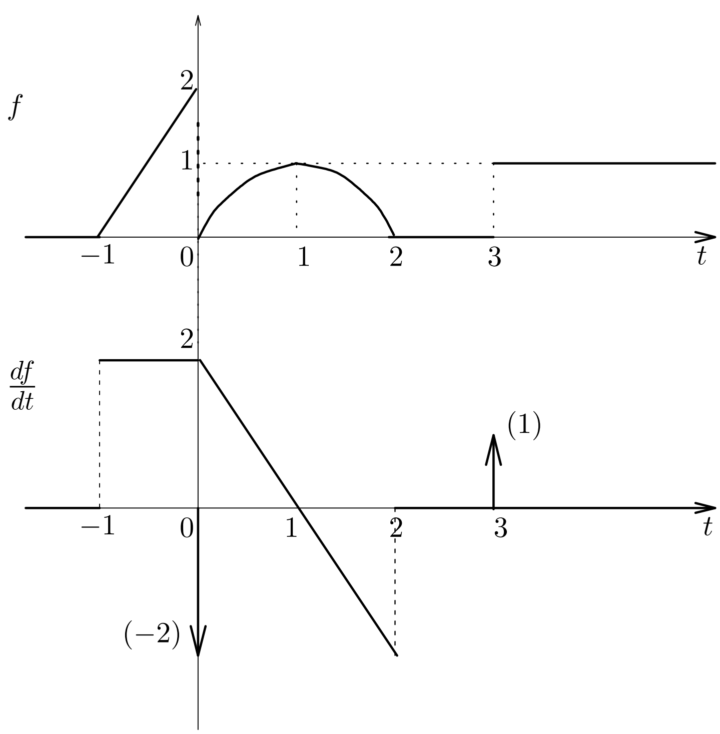
Problem 1 [15 pts]

For the function

$$f(t) = 2(t+1)[u(t+1) - u(t)] + (2t - t^2)u(t)u(2-t) + u(t-3).$$

Sketch $f(t)$ and $\frac{df}{dt}$, and give an analytic formula for the latter in its simplest form.

Solution



$$\frac{df}{dt} = 2[u(t+1) - u(t)] + (2-2t)[u(t) - u(t-2)] - 2\delta(t) + \delta(t-3).$$

Problem 2 [15 pts]

Given a linear, time-invariant system with impulse response function

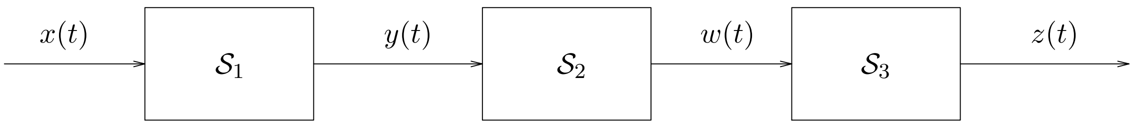
$$h(t) = u(t)e^{-t},$$

find the response to $x(t) = u(-t)e^{t}$.

Solution

We must perform the convolution between the given $h(t)$ and $x(t)$. Note that since $x(t)$ is nonzero for negative time, Laplace transforms as defined in this course do **not** apply.

$$\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} h(t-\sigma)x(\sigma)d\sigma \\
&= \int_{-\infty}^{\infty} u(t-\sigma)e^{-(t-\sigma)}u(-\sigma)e^{\sigma}d\sigma \\
&= e^{-t}\int_{-\infty}^{0} u(t-\sigma)e^{2\sigma}d\sigma \\
&= \begin{cases} e^{-t}\int_{-\infty}^{t} e^{2\sigma}d\sigma & \text{if } t<0 \\ e^{-t}\int_{-\infty}^{0} e^{2\sigma}d\sigma & \text{if } t>0 \end{cases} \\
&= \begin{cases} e^{-t}\cdot\frac{1}{2}e^{2\sigma}\Big|_{-\infty}^{t} & \text{if } t<0 \\ e^{-t}\cdot\frac{1}{2}e^{2\sigma}\Big|_{-\infty}^{0} & \text{if } t>0 \end{cases} \\
&= \begin{cases} \frac{1}{2}e^{t} & \text{if } t<0 \\ \frac{1}{2}e^{-t} & \text{if } t>0 \end{cases} \\
&= \frac{1}{2}e^{t}u(-t) + \frac{1}{2}e^{-t}u(t) \\
&= \boxed{\tfrac{1}{2}e^{-|t|}}
\end{aligned}$$



Problem 3 [20 pts]

Consider the cascade interconnection of the figure, where $\mathcal{S}_1$ and $\mathcal{S}_3$ are LTI, causal systems, and $\mathcal{S}_2$ is defined by the relationship

$$w(t) = e^t y(t).$$

(a) Is $\mathcal{S}_2$ LTI, causal?

(b) We are told that

- The impulse response of $\mathcal{S}_3$ is $h_3(t) = \delta(t) - u(t)$.
- Applying the input $x(t) = e^{-t}u(t)$, the overall output is $z(t) = tu(t)$.

Find the impulse response $h_1(t)$ of the first system.

Solution

(a) $\mathcal{S}_2$ is **linear** since

$$T[\alpha y_1 + \beta y_2] = e^t[\alpha y_1(t) + \beta y_2(t)] = \alpha e^t y_1(t) + \beta e^t y_2(t) = \alpha T[y_1] + \beta T[y_2].$$

It is **time-varying** since

$$T[y(t-\tau)] = e^t y(t-\tau) \neq e^{(t-\tau)} y(t-\tau) = w(t-\tau).$$

It is memoryless and **causal** since the output only depends on the input at the current time.

(b) Since $\mathcal{S}_2$ is LTV, transfer furctions do not apply here. But we can apply them to the other blocks. For instance, $\mathcal{S}_3$ has transfer function

$$H_3(s) = 1 - \frac{1}{s} = \frac{s-1}{s};$$

with the given output $z(t) = tu(t)$ we have

$$Z(s) = \frac{1}{s^2} = H_3(s)W(s) = \frac{s-1}{s}W(s) \implies W(s) = \frac{1}{s(s-1)} = \frac{1}{s-1} - \frac{1}{s}.$$

This implies that $w(t) = u(t)[e^t - 1]$, and therefore

$$y(t) = e^{-t} w(t) = u(t)[1 - e^{-t}] \quad \Longrightarrow \quad Y(s) = \frac{1}{s} - \frac{1}{s+1} = \frac{1}{s(s+1)}.$$

Now applying transfer functions to $\mathcal{S}_1$, we have

$$H_1(s) = \frac{Y(s)}{X(s)} = \frac{\frac{1}{s(s+1)}}{\frac{1}{s+1}} = \frac{1}{s},$$

and therefore $\boxed{h_1(t) = u(t).}$

Another way to approach this problem, using only Laplace, is to invoke the Laplace properties to characterize system $\mathcal{S}_2$ by the relationship (**not** a transfer function!)

$$W(s) = Y(s-1).$$

Since $Y(s) = H_1(s)X(s)$, and $Z(s) = H_3(s)W(s)$, we find that

$$Z(s) = H_3(s)Y(s-1) = H_3(s)H_1(s-1)X(s-1).$$

From here we can solve for

$$H_1(s-1) = \frac{Z(s)}{H_3(s)X(s-1)},$$

or, replacing s by $s+1$,

$$H_1(s) = \frac{Z(s+1)}{H_3(s+1)X(s)} = \frac{\frac{1}{(s+1)^2}}{\frac{s}{s+1} \cdot \frac{1}{s+1}} = \frac{1}{s},$$

leading to the same answer we had before.

Problem 4 [25 pts]

Consider the system described by the input-output relationship $y(t) = |x(t)|$.

a) Is the system (i) linear? (ii) time invariant? (iii) causal?

b) We apply the input $x(t) = u(t)\sin(t)$; sketch $y(t)$ and also the difference $z(t) = y(t) - y(t - \pi)$.

c) Find the Laplace transform $Y(s)$ for the output $y(t)$ in part b), and its DOC.
Hint: It may help to work with $z(t)$, and express it in terms of $x(t)$ and $x(t - \pi)$.

Solution

(a) The system is **nonlinear** since

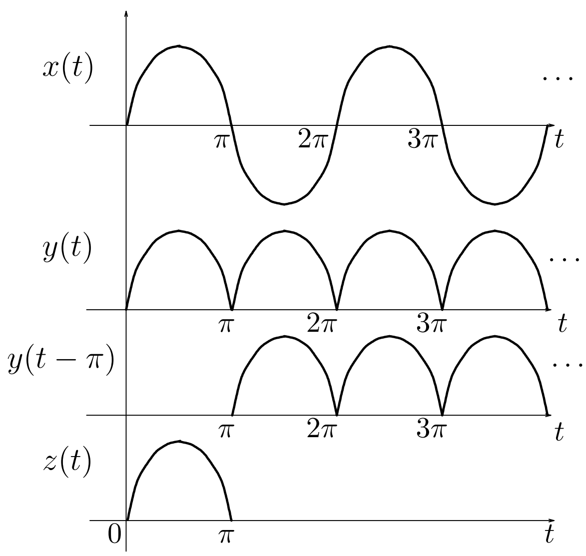
$$T[-x(t)] = |-x(t)| = |x(t)| \neq -T[x(t)].$$

It is **time-invariant** since

$$T[x(t-\tau)] = |x(t-\tau)| = y(t-\tau).$$

It is memoryless and **causal** since the output only depends on the input at the current time.

(b) $y(t) = u(t)|\sin(t)|$. Plots are given below.



(c) Applying the Laplace delay property to the equation $y(t) - y(t-\pi) = z(t)$, we have

$$(1 - e^{-\pi s})Y(s) = Z(s) = \int_0^\pi \sin(t)e^{-st}dt.$$

One way would be to perform the integration, then solve for $Y(s)$. A simpler way, indicated by the hint, is to write

$$\begin{aligned} z(t) &= \sin(t)[u(t) - u(t-\pi)] \\ &= \sin(t)u(t) - \sin(t)u(t-\pi) \\ &= \sin(t)u(t) + \sin(t-\pi)u(t-\pi) \\ &= x(t) + x(t-\pi). \end{aligned}$$

Therefore

$$(1 - e^{-\pi s})Y(s) = Z(s) = (1 + e^{-\pi s})X(s) = (1 + e^{-\pi s})\frac{1}{s^2 + 1}.$$

This gives the final answer

$$Y(s) = \frac{1 + e^{-\pi s}}{(1 - e^{-\pi s})(s^2 + 1)}.$$

For the DOC, note that the denominator roots at $s = \pm i$ are not really poles, since they also make the numerator zero. However the roots of

$$1 - e^{-\pi s} = 0$$

are of the form $s = 2ki$, for k integer, and do not cancel with the numerator. These poles have zero real part (the easiest one is $s = 0$); therefore the DOC is $Re[s] > 0$.

Problem 5 [25 pts]

Consider the differential equation defined for $t \geq 0$,

$$\frac{d^2y(t)}{dt^2} + \frac{dy(t)}{dt} + y(t) = te^{-t}, \qquad y(0) = \alpha, \quad \frac{dy(t)}{dt}(0) = \beta.$$

(a) Find the Laplace transform $Y(s)$ as a function of α, β.

(b) Compute the initial and final values $\lim_{t\to 0+} y(t)$, $\lim_{t\to +\infty} y(t)$. Do they depend on α, β?

(c) Now take $\alpha = 0$, $\beta = 1$. Find the solution $y(t)$ for $t \geq 0$.

Solution

(a) Applying Laplace, and its derivative property to the differential equation, we have,

$$\left[s^2Y(s) - y(0)s - \frac{dy}{dt}(0)\right] + [sY(s) - y(0)] + Y(s) = \mathcal{L}[t\ e^{-t}] = \frac{1}{(s+1)^2}.$$

Substituting the initial conditions, and regrouping we have

$$(s^2 + s + 1)Y(s) = \alpha s + \alpha + \beta + \frac{1}{(s+1)^2},$$

therefore

$$Y(s) = \frac{(s+1)^2(\alpha s + \alpha + \beta) + 1}{(s+1)^2(s^2 + s + 1)}.$$

(b) Since $Y(s)$ is strictly proper, it has no singularities and we can apply the initial value theorem. Also, its poles are all in $Re(s) < 0$, so we can apply the final value theorem. We find

$$\lim_{t\to 0+} y(t) = \lim_{s\to +\infty} sY(s) = \alpha,$$
$$\lim_{t\to +\infty} y(t) = \lim_{s\to 0+} sY(s) = 0.$$

Only the initial value depends (is actually equal to) the initial condition.

(c) For $\alpha = 0, \beta = 1$, we get

$$Y(s) = \frac{s^2+2s+2}{(s+1)^2(s^2+s+1)} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{Ms+N}{s^2+s+1} \tag{1}$$

Multiply by $(s+1)^2$, limit at $s = -1$ gives $\boxed{B = 1.}$

$$\lim_{s\to +\infty} sY(s) = \boxed{0 = A + M.}$$

Evaluating at $s = 0$ gives

$$2 = A + B + N \implies \boxed{1 = A + N.}$$

For one more equation, we can for instance set $s = 1$:

$$\frac{5}{12} = \frac{A}{2} + \frac{1}{4} + \frac{M+N}{3}$$

Substituting M, N as a function of A from the boxed equations we find

$$\frac{5}{12} = \frac{A}{2} + \frac{1}{4} - \frac{A}{3} + \frac{1}{3} - \frac{A}{3},$$

which leads to $\boxed{A = 1}$ and from here to $\boxed{M = -1}$ $\boxed{N = 0}$. We now substitute back in (1), and use completion of squares for the denominator with complex roots,

$$\begin{aligned} Y(s) &= \frac{1}{s+1} + \frac{1}{(s+1)^2} - \frac{s}{(s+\frac{1}{2})^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \\ &= \frac{1}{s+1} + \frac{1}{(s+1)^2} - \frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2 + \left(\frac{\sqrt{3}}{2}\right)^2} + \frac{1}{\sqrt{3}} \frac{\frac{\sqrt{3}}{2}}{(s+\frac{1}{2})^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \end{aligned}$$

The inverse Laplace transform is then

$$y(t) = u(t)(1+t)e^{-t} + u(t)e^{-\frac{1}{2}t}\left[-\cos\left(\frac{\sqrt{3}}{2}t\right) + \frac{1}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}t\right)\right].$$