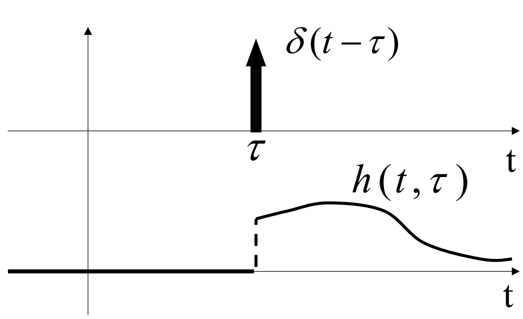
Lecture 4

- Impulse response of a linear system.
- Special cases of the impulse response:
  - Causal systems.
  - Time invariant systems
- Approximating signals by a train of impulses.
- Input-output relation of a linear system.

The impulse response of a linear system

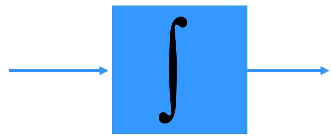
$$y(t) = T\left[x(t)\right]$$

Apply the input $\delta(t-\tau)$ to the system. Denote the corresponding output by $h(t,\tau)$. This function of the variables t,τ is called the system impulse response.

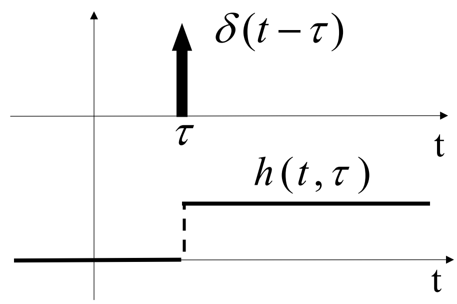


Impulse response function $h(t,\tau)$

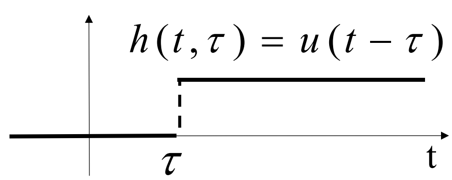
Example: integrator

$x(t)$  $y(t) = \int_{-\infty}^{t} x(\sigma)\,d\sigma$

$$h(t,\tau) = T\left[\delta(t-\tau)\right]$$
$$= \int_{-\infty}^{t} \delta(\sigma-\tau)\,d\sigma$$
$$= u(t-\tau)$$



Integrator:



Note that in this example:

a) $h(t,\tau)$ depends only on the difference $t-\tau$

b) $h(t,\tau) = 0$ for $t<\tau$

This is not a coincidence. In fact:

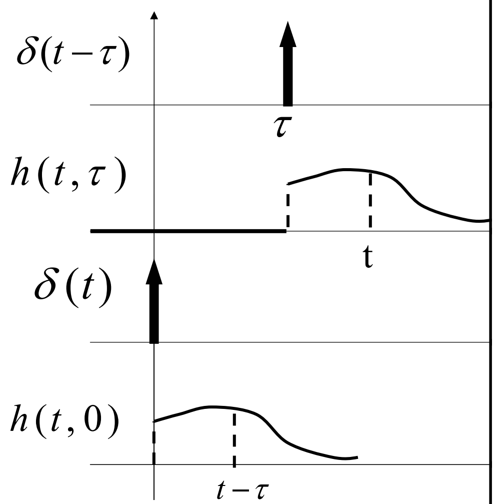
- a) is a general property of **time invariant** systems
- b) is a general property of **causal** systems.

Impulse response for LTI systems

Property:

$$h(t,\tau) = h(t-\tau,0)$$

This follows directly from time invariance



Notation:

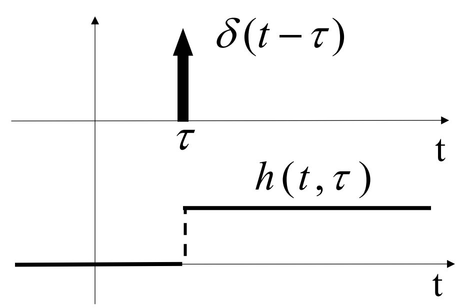
we often write $h(t-\tau)$ instead of $h(t-\tau,0)$

$h(t)$ is the response to $\delta(t)$

Impulse response for causal systems

Property:

$$h(t,\tau) = 0 \quad \text{for} \quad t < \tau$$

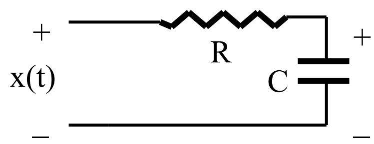


This follows directly from causality: the system cannot anticipate that the delta function is coming, so it cannot respond before $t = \tau$

More formally: let $x_1(t) = \delta(t-\tau)$, $x_2(t) \equiv 0$. Since they coincide for $t < \tau$, and the system is causal, we must have $y_1(t) = y_2(t)$ for $t < \tau$. But $y_2 = T[0] = 0$ because of linearity. So $y_1(t) = h(t,\tau) = 0$ for $t < \tau$.

RC circuit example

$$\alpha = \frac{1}{RC}$$



$$y(t) = \int_0^t \alpha\, e^{-\alpha(t-\sigma)} x(\sigma)\, d\sigma,$$

LTI, causal system. So the impulse response function is $h(t-\tau)$, where $h(t)$ is the response to $x(t) = \delta(t)$. To find it, write

$$h(t) = \int_{0-}^{t} \alpha e^{-\alpha(t-\sigma)} \delta(\sigma)\, d\sigma = \alpha e^{-\alpha t} \text{ for } t \geq 0.$$

Note that to avoid ambiguities, we start the integral in $0-$. This means the circuit initial conditions are zero before the impulse is applied. Also, by causality $h(t) = 0$ for $t < 0$. In summary:

$$h(t) = \alpha e^{-\alpha t} u(t)$$

Another example: $y(t) = \int_{-\infty}^{t} (\sigma+1)^2 x(\sigma)\, d\sigma,$

Apply the input $x(t) = \delta(t-\tau)$. The output is

$$y(t) = \int_{-\infty}^{t} (\sigma+1)^2 \delta(\sigma-\tau)\, d\sigma = \begin{cases} (\tau+1)^2 & \text{if } \tau < t \\ 0 & \text{if } \tau > t \end{cases}$$

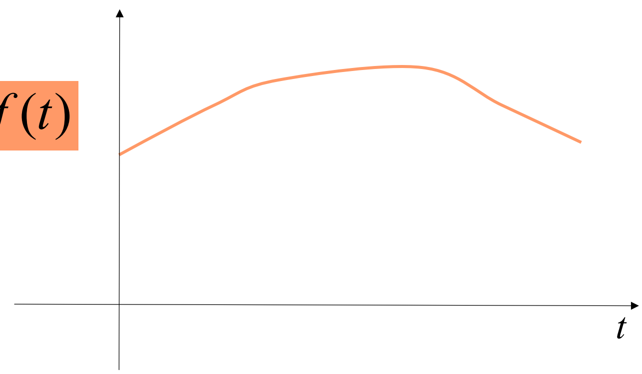
$$h(t,\tau) = (\tau+1)^2 u(t-\tau)$$ Time varying, causal.

Another way to compute the integral is to add a step function and extend the limit of integration to $+\infty$:

$$y(t) = \int_{-\infty}^{t} (\sigma+1)^2 \delta(\sigma-\tau)\, d\sigma = \int_{-\infty}^{+\infty} (\sigma+1)^2 u(t-\sigma)\delta(\sigma-\tau)\, d\sigma$$
$$= (\tau+1)^2 u(t-\tau)$$

Why is the impulse response useful?

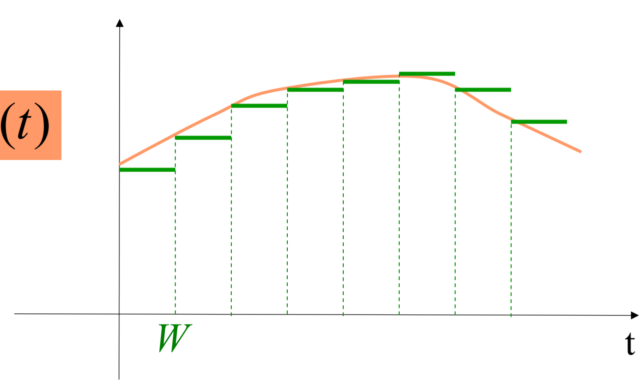
A function $f(t)$



Idea: we can use impulses to approximate other functions

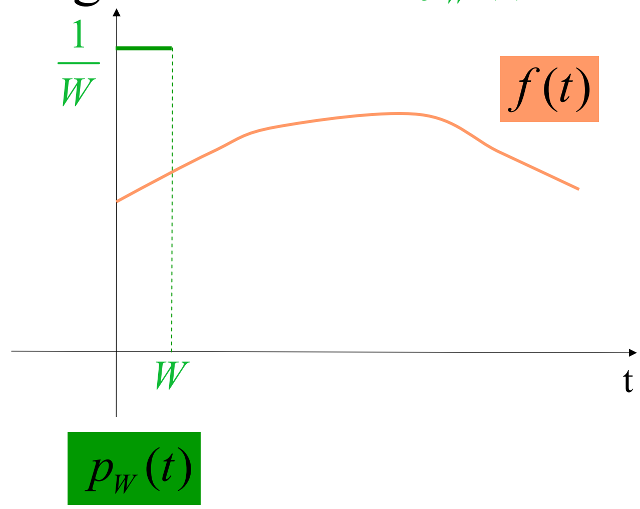
First, a "staircase approximation" $f_W(t)$

Function $f(t)$

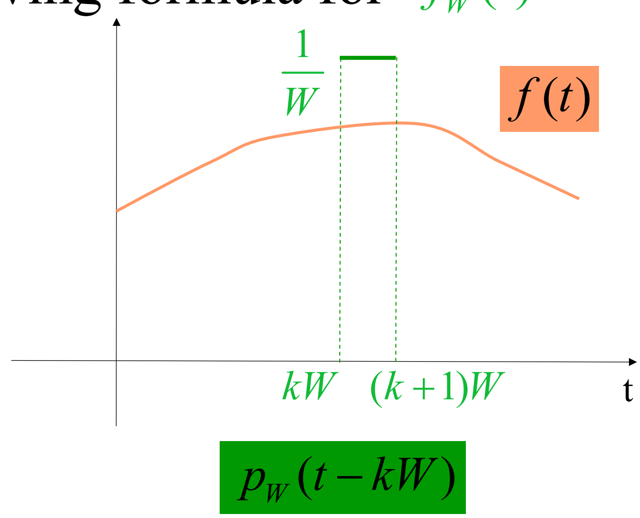


$$f_W(t) = \sum_k f(kW) \cdot W \cdot p_W(t - kW)$$

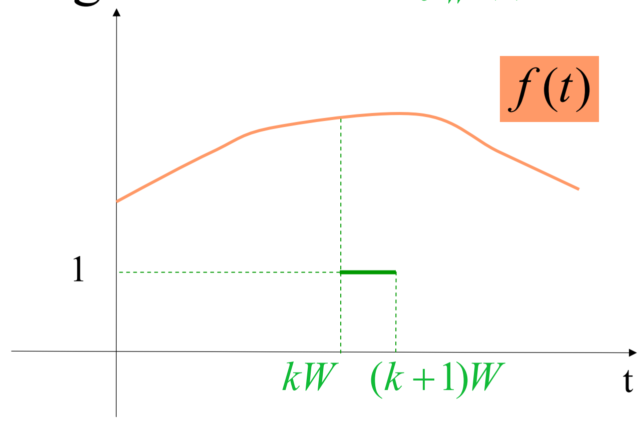
Deriving formula for $f_W(t)$



Deriving formula for $f_W(t)$

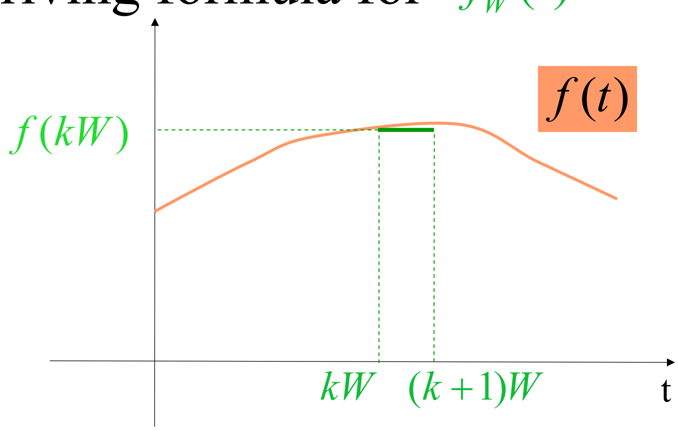


Deriving formula for $f_W(t)$



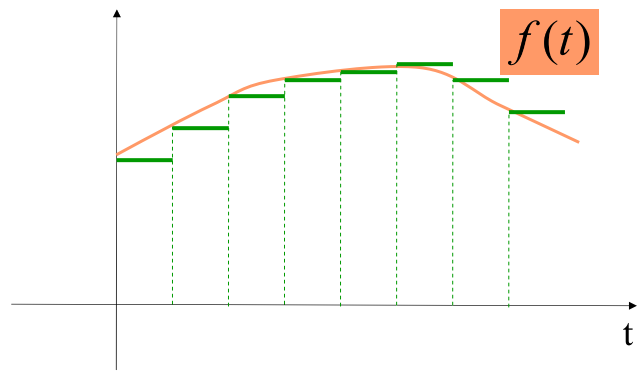
$W \cdot p_W(t - kW)$

Deriving formula for $f_W(t)$



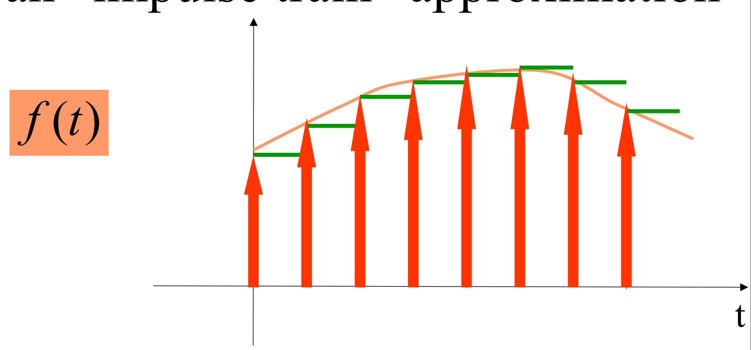
$f(kW) \cdot W \cdot p_W(t - kW)$

Deriving formula for $f_W(t)$



$$f_W(t) = \sum_k f(kW) \cdot W \cdot p_W(t - kW)$$

Now, an "impulse train" approximation



"Staircase" approximation

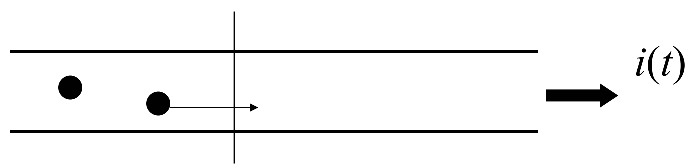
$$f_W(t) = \sum_k f(kW) \cdot W \cdot p_W(t - kW)$$

Replace pulse $p_W(t)$ by impulse $\delta(t)$

$$f_W^{\delta}(t) = \sum_k f(kW) \cdot W \cdot \delta(t - kW)$$

Are impulse trains physical?

- **In cart example:** applying an impulse train as a force is like doing a periodic "hammering" instead of a smooth push.
- **Electrical example**. The current $i(t)$ flowing through a section of a cable is made up of discrete electrons going through. So $i(t)$ is naturally modeled as an impulse train.



In the limit as step-size goes to 0

The expression $\sum_k f(kW)\cdot\delta(t-kW)\cdot W$

is like a Riemann sum for the integral

$$\int_{-\infty}^{\infty} f(\sigma)\cdot\delta(t-\sigma)\,d\sigma$$

The approximation is exact in the limit:

$$\lim_{W\to 0} f_W^{\delta}(t) = \boxed{\int_{-\infty}^{\infty} f(\sigma)\cdot\delta(t-\sigma)\,d\sigma = f(t)}$$

"Resolution" of a function as a superposition of delta's

Input-output relation of a linear system

$$y(t) = T\left[x(t)\right]$$

We know the system impulse response function $h(t,\tau) = T\left[\delta(t-\tau)\right]$. We want to use it to find the response to any input $x(t)$. Strategy:

1. Approximate $x(t)$ by a train of impulses.
2. Use linearity to obtain the output corresponding to this approximation.
3. Take the limit as W goes to zero.

Given: $h(t,\tau) = T\left[\delta(t-\tau)\right]$, and an input $x(t)$.

1) Write $x_W^{\delta}(t) = \sum_k x(kW)\cdot\delta(t-kW)\cdot W$

2) Using linearity, the corresponding output is

$$T\left[x_W^{\delta}(t)\right] = \sum_k x(kW)\cdot T\left[\delta(t-kW)\right]\cdot W$$

$$= \sum_k h(t,kW)\cdot x(kW)\cdot W$$

3) Taking limit as $W \to 0$, we obtain

$$T\left[x(t)\right] = \lim_{W\to 0}\sum_k h(t,kW)\cdot x(kW)\cdot W = \int_{-\infty}^{\infty} h(t,\sigma)x(\sigma)d\sigma$$

Input-output relation of a linear system

$$y(t) = T\left[x(t)\right]$$

Let the impulse response function be

$$h(t,\tau) = T\left[\delta(t-\tau)\right].$$

For a given input $x(t)$, the corresponding output is

$$y(t) = T\left[x(t)\right] = \int_{-\infty}^{\infty} h(t,\sigma)x(\sigma)d\sigma$$

SUPERPOSITION INTEGRAL

Back to earlier example:

System defined by $y(t) = \int_{-\infty}^{t} (\sigma+1)^2 x(\sigma)d\sigma$

We found before that $h(t,\tau) = (\tau+1)^2 u(t-\tau)$

The superposition integral gives

$$y(t) = \int_{-\infty}^{\infty} h(t,\sigma)x(\sigma)d\sigma = \int_{-\infty}^{\infty} (\sigma+1)^2 u(t-\sigma)x(\sigma)d\sigma$$

$$= \int_{-\infty}^{t} (\sigma+1)^2 x(\sigma)d\sigma$$

Recover original definition. Having the impulse response function is equivalent to having the complete definition.