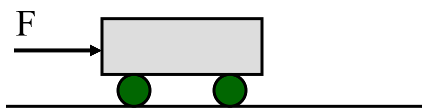
Lecture 3.The Dirac delta function

Motivation: Pushing a cart, initially at rest.



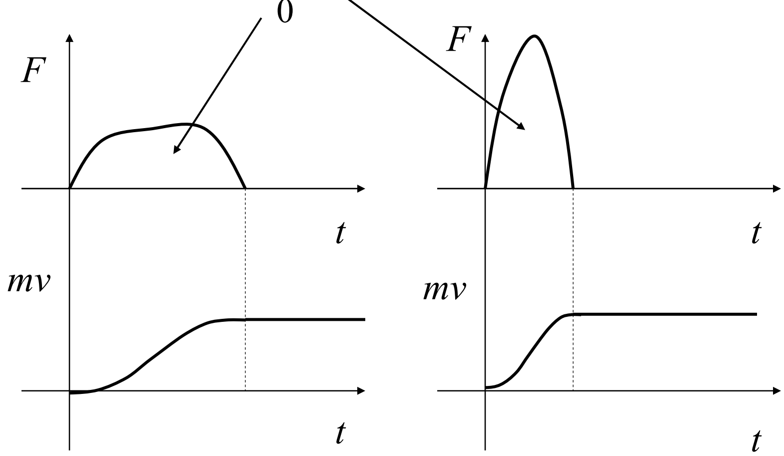
$$F = m\frac{dv}{dt}$$

$$\int_0^{t_0} F(t)dt = m\int_0^{t_0} \frac{dv}{dt}dt = m\left[v(t_0) - v(0)\right] = mv\left(t_0\right)$$

Applied *impulse*

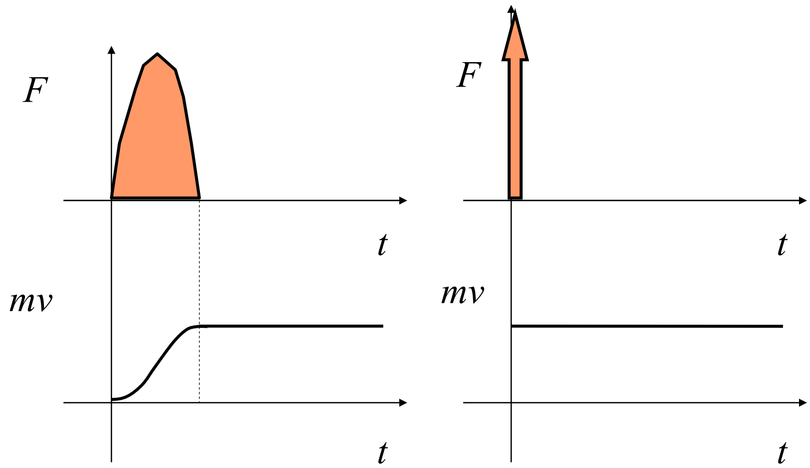
Acquired *momentum*

$$I = \int_0^{t_0} F(t)dt = mv\left(t_0\right)$$



Same final momentum, shorter time.

In the limit of short time, we idealize this as an instantaneous, infinitely large force.



Dirac's delta function models this kind of force.

Dirac delta function $\delta(t)$

- This "unit impulse" function is defined by the conditions:

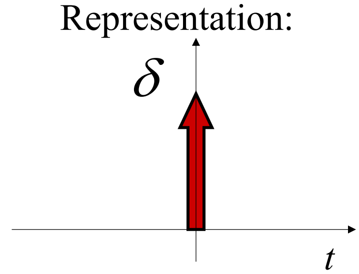
$$\delta(t) = 0, \; for \;\; t \neq 0.$$

$$\delta(0) = +\infty$$

$$\int_{-\infty}^{+\infty} \delta(t)\,dt = 1$$

Normalization

Representation:



Q: Is this mathematically rigorous, using standard calculus?

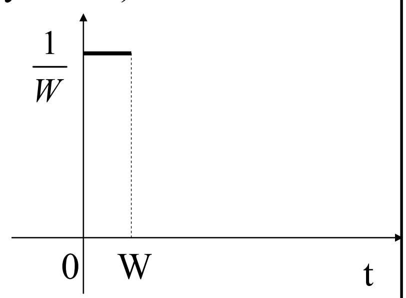
A: Not quite. Standard functions do not take infinity as a value, and that integral would not be valid.

How to make sense of it then?

- What physicists like Dirac (early 1900s) meant is something like this:

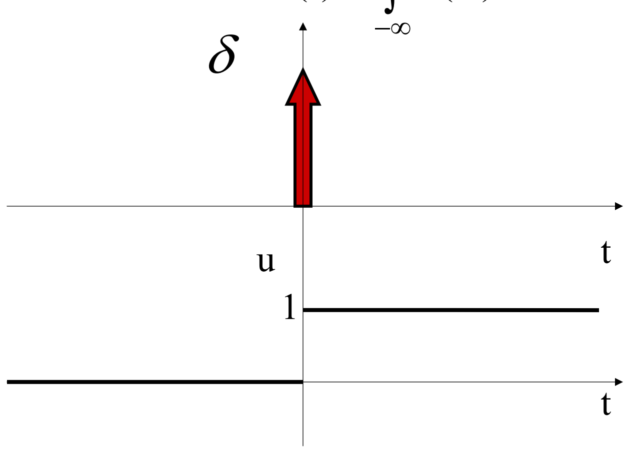
$$\lim_{W \to 0+} p_W(t), \text{ where}$$

$$p_W(t) = \begin{cases} \dfrac{1}{W} & \text{if } t \in [0, W] \\ 0 & \text{otherwise} \end{cases}$$



- Mathematical difficulty: in standard calculus, this family of functions would not converge.
- Later on (1950s) mathematicians developed the “theory of distributions” to make this precise. This generalization of calculus is beyond our scope, but we will still learn how to use it for practical problems.

Calculate: $u(t) = \int_{-\infty}^{t} \delta(\sigma) d\sigma$

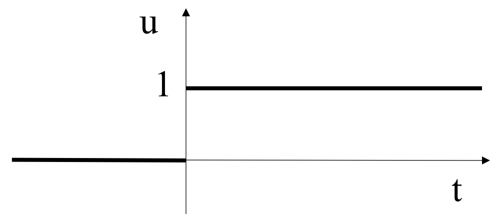


$$u(t) = \begin{cases} 0 & if \quad t < 0 \\ 1 & if \quad t > 0 \end{cases}$$

Remark: value at $t = 0$ is not well defined, we adopt one by convention.

Unit step function

$$u(t) = \begin{cases} 0 & if \quad t < 0 \\ 1 & if \quad t \geq 0 \end{cases}$$

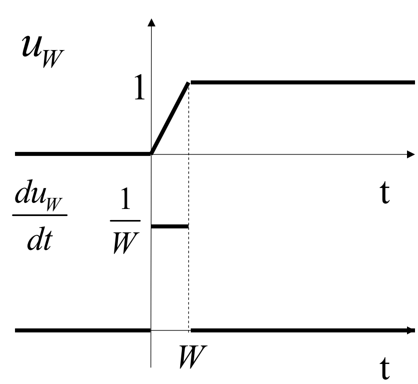


$$\frac{du}{dt} = \delta(t)$$

- In standard calculus, *u*(*t*) is only differentiable (and has zero derivative) for nonzero *t*.
- *u*(*t*) is discontinuous at *t* = 0. Value at point of discontinuity is arbitrarily chosen.

Step function as a limit.

$$u_W(t) = \begin{cases} 0 \;\; for \;\; t < 0 \\ \dfrac{t}{W} \;\; for \;\; 0 \leq t \leq W \\ 1 \;\; for \;\; t > W \end{cases}$$



$$\lim_{W \to 0+} u_W(t) = u(t), \quad t \neq 0$$

$$\frac{du_W}{dt} = p_W(t),$$

$$\lim_{W \to 0+} p_W(t) = \delta(t)$$

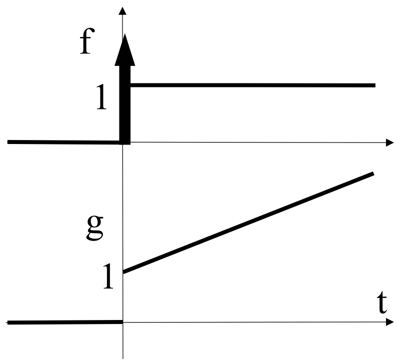
Remark: Physical variables (e.g. the car momentum) are not really discontinuous, they are more like $u_W(t)$.
The step function *u*(*t*) is a convenient idealization.

Example: integrate $f(t) = \delta(t) + u(t)$

$$g(t) = \int_{-\infty}^{t} f(\sigma)\,d\sigma = \int_{-\infty}^{t} \delta(\sigma) + u(\sigma)\,d\sigma = u(t) + \int_{-\infty}^{t} u(\sigma)\,d\sigma$$

Last term: $\displaystyle\int_{-\infty}^{t} u(\sigma)\,d\sigma = \begin{Bmatrix} 0 \ \ if \ t < 0 \\ t \ \ if \ \ t > 0 \end{Bmatrix} = t\,u(t)$

Step function used to compact formula.



$$\Rightarrow g(t) = u(t)\left(1 + t\right)$$

$$\frac{dg}{dt} = \delta(t) + u(t)$$

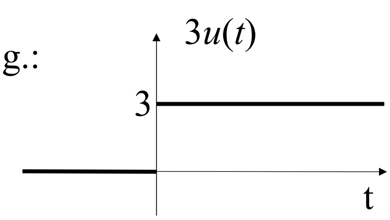
Due to jump

Standard derivative away from t = 0.

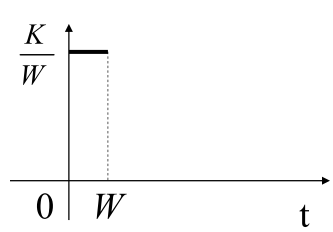
Scaled steps and deltas

$$K\,u(t) = \begin{cases} 0 \ \ if \ \ t < 0 \\ K \ \ if \ \ t \geq 0 \end{cases}$$

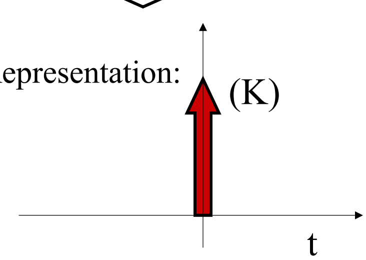
e.g.:



$$\frac{dKu}{dt} = K\,\delta(t) = \lim_{W \to 0+} \underbrace{K\,p_W(t)}$$



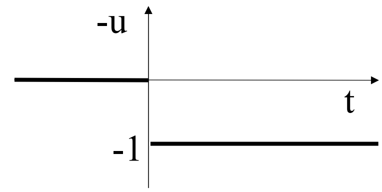
Representation:



Conceptually the same, different amount of impulse applied

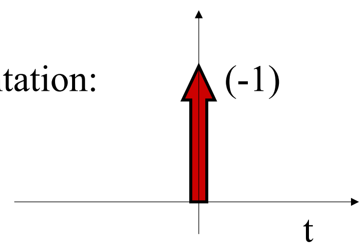
Negative steps and deltas

$$-u(t)=\begin{cases}0 & if \quad t<0\\ -1 & if \quad t\geq 0\end{cases}$$

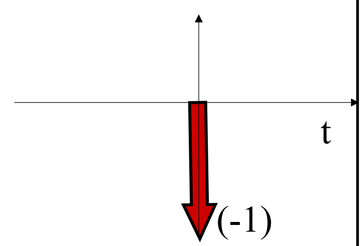


$$\frac{d\left(-u\right)}{dt}=-\delta(t)$$

Representation:

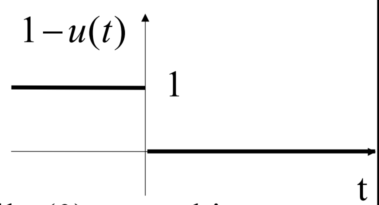


or



Reverse step and its derivative

$$1-u(t)\ =\ u(-t)$$



Strictly speaking, the two expressions differ at t = 0. But recall u(0) was arbitrary. There is no distinction between the two in this calculus.

$$\frac{d\left(1-u(t)\right)}{dt}=-\delta(t)$$ Differentiating the left-hand side

$$\frac{d\left(u(-t)\right)}{dt}=-\delta(-t)$$ Using right-hand side, composition rule $$\frac{dg\left(f(t)\right)}{dt}=g'\left(f(t)\right)\frac{df}{dt}$$

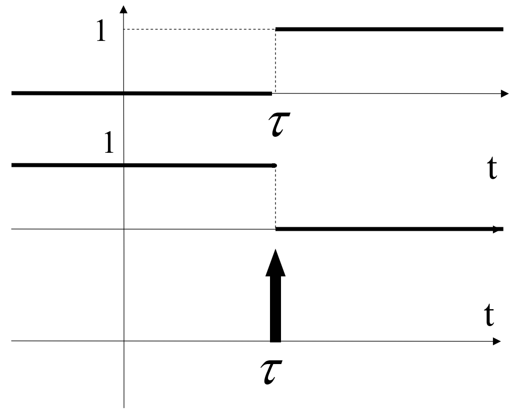
$$\Rightarrow \delta(t)=\delta(-t)$$ Delta is an even function

Translations and flips of steps and deltas

$$u(t-\tau)=\begin{cases}0 & if \quad t<\tau\\ 1 & if \quad t\geq\tau\end{cases}$$

$$u(\tau-t)=\begin{cases}1 & if \quad t\leq\tau\\ 0 & if \quad t>\tau\end{cases}$$

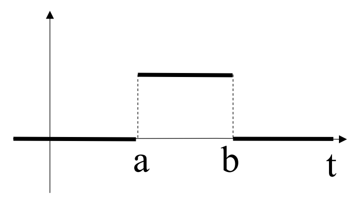
$$\delta(t-\tau)=\delta(\tau-t)$$



A pulse function:

$$u(t-a)-u(t-b)=u(t-a)\,u(b-t)$$

$$=\begin{cases}1 \; if \; t\in[a,b]\\ 0 \; otherwise\end{cases}$$



Example:

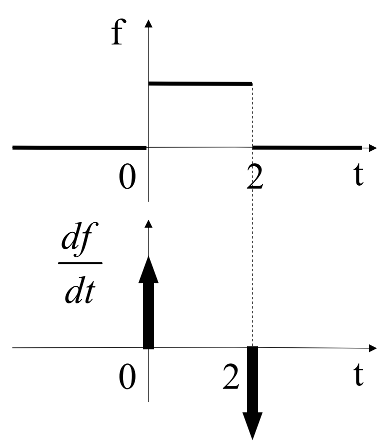
$$f(t)=u(t)-u(t-2)$$
$$=u(t)\,u(2-t)$$

Using the first expression,

$$\frac{df}{dt}=\delta(t)-\delta\left(t-2\right)$$

Using the second,

$$\frac{df}{dt}=\delta(t)\,u(2-t)+u(t)\,\delta\left(2-t\right)(-1)$$
$$=\delta(t)\,u(2-t)-u(t)\,\delta\left(t-2\right)$$

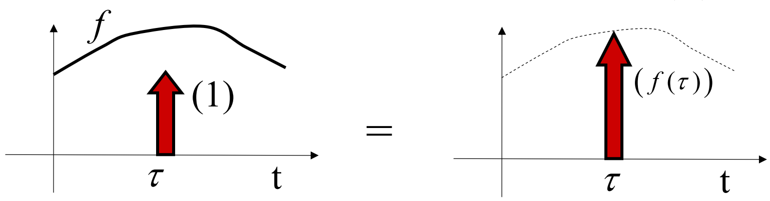


Different answer?

A basic property of delta

Let $f(t)$ be a standard function.

- If $f(t)$ is continuous at $t = 0, \Rightarrow f(t)\delta(t) = f(0)\delta(t)$
- If $f(t)$ continuous at $t = \tau,\ f(t)\delta(t-\tau) = f(\tau)\delta(t-\tau)$



Apply to the previous example:

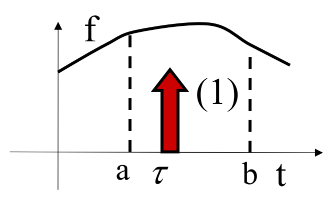
$$\frac{df}{dt} = \delta(t)u(2-t) - u(t)\delta(t-2) = \delta(t)\underbrace{u(2)}_{1} - \underbrace{u(2)}_{1}\delta(t-2)$$

$$= \delta(t) - \delta(t-2)$$ Consistent with previous answer

Consequence of our basic property

$$\int_{-\infty}^{+\infty} f(t)\delta(t)\,dt = \int_{-\infty}^{+\infty} f(0)\delta(t)\,dt = f(0)\int_{-\infty}^{+\infty} \delta(t)\,dt = f(0)$$

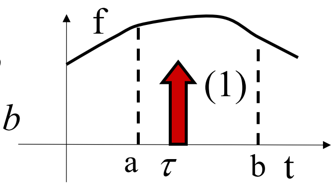
Similarly, $\displaystyle\int_{-\infty}^{+\infty} f(t)\delta(t-\tau)\,dt = f(\tau)$



$$\int_{a}^{b} f(t)\delta(t-\tau)\,dt = \int_{-\infty}^{+\infty} \left[u(t-a) - u(t-b)\right] f(t)\delta(t-\tau)\,dt$$

$$= f(\tau)\left[u(\tau-a) - u(\tau-b)\right] = \begin{cases} f(\tau) & if\ \ a < \tau < b \\ 0 & if\ \ \tau < a\ \ or\ \ \tau > b \end{cases}$$

$$\int_a^b f(t)\delta(t-\tau)\,dt = \begin{cases} f(\tau) & if\ a<\tau<b \\ 0 & if\ \tau<a \ \ or\ \ \tau>b \end{cases}$$

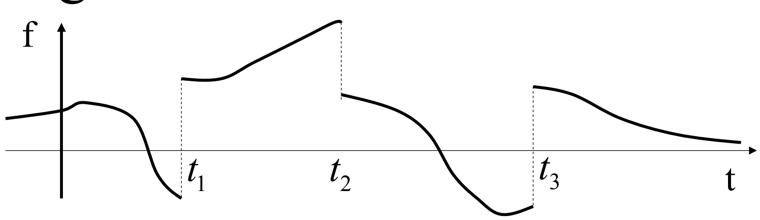


What if τ falls exactly in one of the limits of integration? In that case, for the integral to be well defined we must specify whether that point is included or not, as follows:

$$\int_{a-}^{b} f(t)\delta(t-a)\,dt = f(a), \qquad \int_{a+}^{b} f(t)\delta(t-a)\,dt = 0.$$

$$\int_{a}^{b+} f(t)\delta(t-b)\,dt = f(b), \qquad \int_{a}^{b-} f(t)\delta(t-b)\,dt = 0,$$

A general rule on differentiation



Let $f(t)$ have a standard derivative $g(t)$ except at a finite number of points $t_1, t_2, \ldots, t_n$. Then in this calculus,

$$\frac{df}{dt} = g(t) + \sum_{k=1}^{n} J_k\,\delta(t-t_k),$$ where the "jumps" are

$$J_k := f(t_k+) - f(t_k-) = \lim_{t\to t_k+} f(t) - \lim_{t\to t_k-} f(t)$$

and we assume the limits exist.