EE102 – SYSTEMS & SIGNALS

Winter Quarter, 2004.

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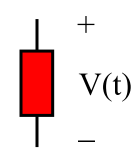
Lecture 1. Intro to Signals & Systems

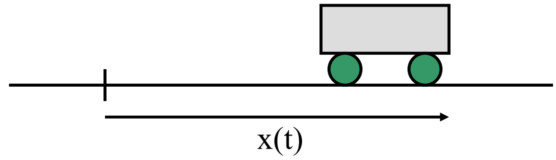
Signal: Function that describes the evolution of a variable with time.

Examples:

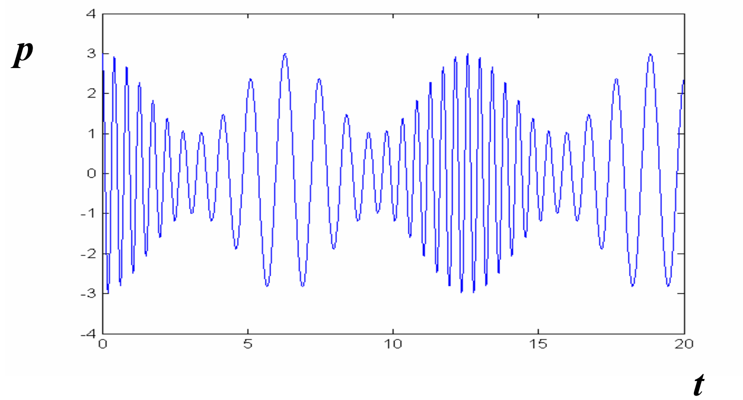
–Voltage across an electrical component.



– Position of a moving object.



Sound = pressure of the air outside your ear



- Information lies in the time evolution.
- The signal can be converted to and from other domains: electrical (in a stereo), electro-chemical (in your brain).
- What matters is the mathematical structure.

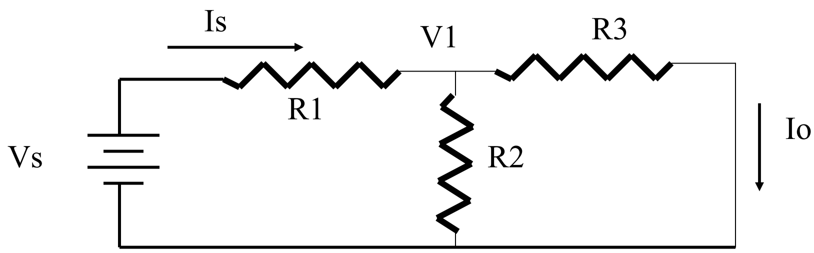
Signal examples (cont):

- Population of a species over time (decades)
- Daily value of the Nasdaq

Time can be continuous (a real number) or discrete (an integer). This course focuses on continuous time.

System: component that establishes a relationship between signals

- Example: circuit



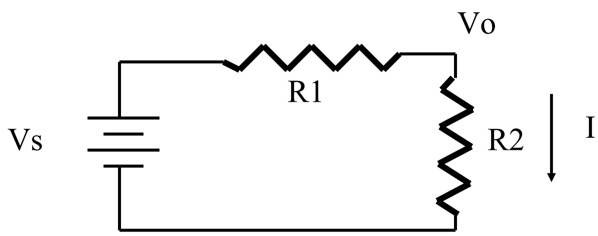
Relationship between voltages and currents.

Systems: examples

- Car: Relationship between signals:
  - Throttle/brake position
  - Motor speed
  - Fuel concentration in chamber
  - Vehicle speed.
  - …
- Ecosystem: relates populations, …
- The economy: relates GDP, inflation, interest rates, stock prices,…
- The universe…

Math needed to study signals & systems?

Example 1: static system

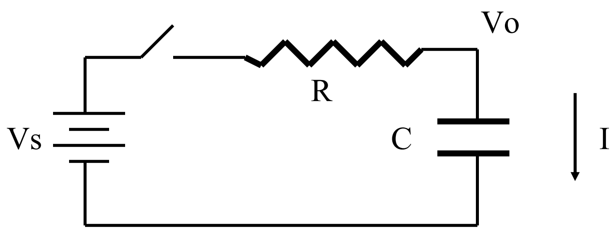


$$I = \frac{V_s}{R_1 + R_2}$$

$$V_0 = \frac{V_s R_2}{R_1 + R_2}$$

- Not much math there…
- Time does not enter in a fundamental way.

Example 2: dynamical system



$$I = C\frac{dV_0}{dt}$$

$$V_s - V_0 = R\,I$$

- Switch closes at t=0. For t >= 0, we have the ordinary differential equation (ODE)

$$V_s = V_0 + R\,C\frac{dV_0}{dt}$$

Solution: $V_0 = V_s\left(1 - e^{-\frac{t}{RC}}\right)$ Time is essential here.

Dynamic, differential equation models appear in many systems

- Mechanical system, e.g. the mass-spring system $$m\frac{d^2x}{dt^2}+k\,x=0$$
- Chemical reactions
- Population dynamics
- Economic models

The issue of complexity

- Consider modeling the dynamic behavior of
  - An IC with millions of transistors
  - A biological organism
- "Reductionist" method: zoom in a component, write for it a differential equation model, then combine these into an overall model.
- Difficulty: solving those ODE's is impossible; even numerical simulation is prohibitive.
- Even harder: **design** the differential equation (e.g., the circuit) so that it has a desired solution.

The “black box” concept

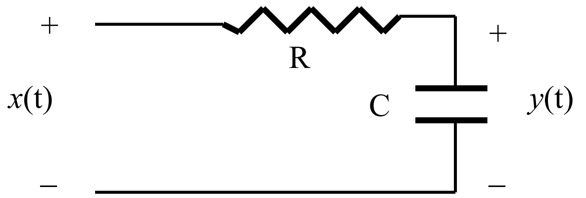
- Idea: describe a portion of a system by a input-output (cause-effect) relationship.
- Derive a mathematical model of this relationship. This can involve ODEs, or other methods we will study. Make reasonable approximations.
- Interconnect these boxes to describe or design a more complex system.

Definition: Input-Output System

- The input function x(t) belongs to a space X, and can be freely manipulated from outside.
- The output function y(t) varies in a space Y, and is uniquely determined by the input function.
- The relationship between input and output is described by a transformation T between X and Y. Notation:

$$y(t) = T\left[x(t)\right] \quad \text{or} \quad y(\bullet) = T\left[x(\bullet)\right]$$

Example: RC circuit as an input-output system



$$x = y + RC\frac{dy}{dt}$$

- We assume here that time starts at t=0, $y(0) = 0$
- To represent the mapping from x to y explicitly, we must solve the differential equation

$$\frac{dy}{dt} + \alpha\, y = \alpha x, \qquad y(0) = 0,$$

Here $\alpha = \dfrac{1}{RC}$

Solution of $\frac{dy}{dt} + \alpha\, y = \alpha x, \quad y(0) = 0.$

- This is a linear ODE, with constant coefficients, and non-homogeneous (nonzero right hand side).
- Let us review first how to solve the ***homogeneous*** equation $\qquad \dfrac{dy}{dt} + \alpha\, y = 0$

Solution by "separation of variables":

$$\frac{dy}{dt} = -\alpha\, y \;\Rightarrow\; \frac{dy}{y} = -\alpha\, dt$$

Indefinite integral: $\Rightarrow \; \log(y) = -\alpha\, t + \underbrace{K}_{\text{constant}}$

$$y = e^{-\alpha t} e^{K} \;\Rightarrow\; \boxed{y = Ce^{-\alpha t}}$$

Solution of $\frac{dy}{dt} + \alpha\, y = \alpha x, \quad y(0) = 0.$

To solve the non-homogeneous equation, one method is to ``vary the constant" in the homogeneous solution.

This means, to try a solution of the form $\boxed{y(t) = C(t)e^{-\alpha t}}$

$$\frac{dy}{dt} = \frac{dC}{dt}e^{-\alpha t} + \underbrace{C(t)\left[-\alpha e^{-\alpha t}\right]}_{-\alpha y}$$

$$\Rightarrow \frac{dy}{dt} + \alpha\, y = \frac{dC}{dt}e^{-\alpha t} = \alpha x \quad \Rightarrow \frac{dC}{dt} = \alpha e^{\alpha t} x(t)$$

Integrate to find $C(t) = C(0) + \int_0^t \alpha\, e^{\alpha\sigma} x(\sigma) d\sigma$

Some remarks on integration of a function $f(t)$:

- $F(t) = \int f(t)dt$ typically denotes a function whose derivative is $f(t)$ $\left(i.e.\ \frac{dF}{dt} = f(t) \right)$. There are, however, infinitely many such functions, that differ from each other by a constant. Example: $\int t\ dt = \frac{t^2}{2} + K$
- $F(t) = \int_0^t f(\sigma)d\sigma$ is used to denote a specific one of these integral functions: namely, the one satisfying $F(0) = 0$. Here we use the "dummy variable" σ (or any other name) to distinguish it from t, the limit of integration. Example: $\int_0^t \sigma\ d\sigma = \frac{t^2}{2}$. This notation will be used extensively in this course.
- The "definite integral" $\int_a^b f(t)dt$ or $\int_a^b f(\sigma)d\sigma$, with fixed limits of integration, denotes a number (area under the curve). Example: $\int_0^3 \sigma\, d\sigma = \frac{9}{2}$.

Solution of $\frac{dy}{dt} + \alpha y = \alpha x, \quad y(0) = 0.$

The dummy variable notation was used to go from $\frac{dC}{dt} = \alpha e^{\alpha t} x(t)$

to $\int_0^t \frac{dC}{d\sigma} d\sigma = \int_0^t \alpha e^{\alpha\sigma} x(\sigma) d\sigma$. But the first term is equal to

$C(t) - C(0)$, so we have $C(t) = C(0) + \int_0^t \alpha e^{\alpha\sigma} x(\sigma) d\sigma$.

Now $y(t) = C(t)e^{-\alpha t} = e^{-\alpha t} C(0) + \int_0^t \alpha e^{-\alpha(t-\sigma)} x(\sigma) d\sigma$

Using the initial condition $y(0) = 0$, we get

$$\boxed{y(t) = \int_0^t \alpha e^{-\alpha(t-\sigma)} x(\sigma) d\sigma}$$

This is an input-output representation of the form $y = T[x]$

Solution of $\frac{dy}{dt} + \alpha y = \alpha x, \quad y(0) = 0.$

Another (equivalent) method is the integrating factor: here we "guess" that the left hand side can be written as

$$\frac{dy}{dt} + \alpha y = e^{-\alpha t}\left[e^{\alpha t} \frac{dy}{dt} + \alpha e^{\alpha t} y \right] = e^{-\alpha t} \frac{d}{dt}\left[e^{\alpha t} y \right]$$

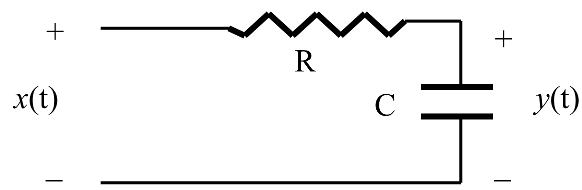
Plugging into the equation, we can solve for

$$\frac{d}{dt}\left[e^{\alpha t} y \right] = \alpha\, e^{\alpha t} x \Rightarrow e^{\alpha t} y(t) = y(0) + \int_0^t \alpha e^{\alpha\sigma} x(\sigma) d\sigma$$

For $y(0) = 0$, we get

$$\boxed{y(t) = \int_0^t \alpha e^{-\alpha(t-\sigma)} x(\sigma) d\sigma}$$

Recap: RC circuit example



$$\frac{dy}{dt} + \alpha y = \alpha x,$$

$$\alpha = \frac{1}{RC}$$

Assuming $y(0) = 0$, we have the input-output relationship

$$\boxed{y(t) = \int_0^t \alpha e^{-\alpha(t-\sigma)} x(\sigma) d\sigma}$$

$$y(t) = T\left[x(t)\right]$$