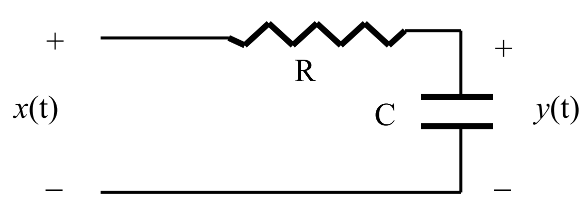
Lecture 2. Properties of Systems

- Linearity
- Time invariance
- Causality
- Memory.

Recall: RC circuit example



$$\frac{dy}{dt} + \alpha\, y = \alpha x, \qquad \alpha = \frac{1}{RC}$$

Assuming $y(0) = 0$, we have the input-output relationship

$$\boxed{y(t) = \int_0^t \alpha\, e^{-\alpha(t-\sigma)} x(\sigma)\, d\sigma}$$

$$y(t) = T\left[x(t)\right]$$

Properties of Input-Output Systems

$$y(t) = T\left[x(t)\right]$$

Linearity. The system is linear if

$$T[x_1(t)+x_2(t)] = T[x_1(t)] + T[x_2(t)]$$

$$T[k\,x(t)] = k\,T[x(t)] \qquad \text{for any } k, x_1, x_2.$$

Alternatively, if

$$T[k_1x_1(t)+k_2x_2(t)] = k_1T[x_1(t)] + k_2T[x_2(t)]$$

for any k_1, k_2, x_1, x_2.

Linearity of the RC circuit example.

$$y(t) = T\left[x(t)\right] = \int_0^t \alpha\, e^{-\alpha(t-\sigma)} x(\sigma)\, d\sigma.$$

$$T[k_1x_1(t)+k_2x_2(t)] = \int_0^t \alpha\, e^{-\alpha(t-\sigma)} [k_1x_1(\sigma)+k_2x_2(\sigma)]\, d\sigma$$

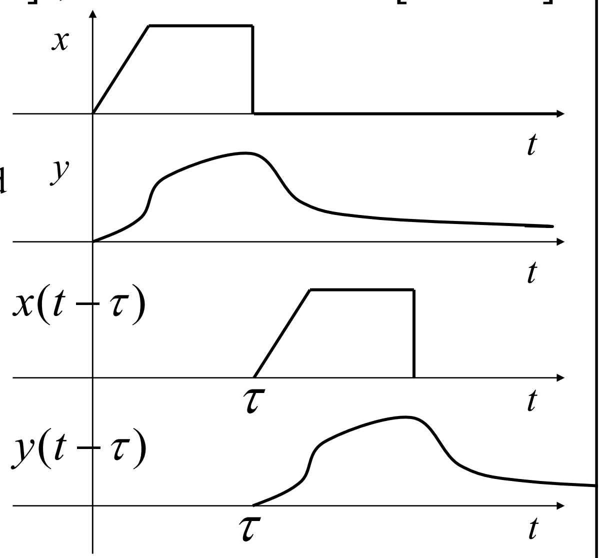
$$= k_1\int_0^t \alpha\, e^{-\alpha(t-\sigma)} x_1(\sigma)\, d\sigma + k_2\int_0^t \alpha\, e^{-\alpha(t-\sigma)} x_2(\sigma)\, d\sigma$$

$$= k_1T[x_1(t)] + k_2T[x_2(t)] \quad \Rightarrow \quad \text{LINEAR.}$$

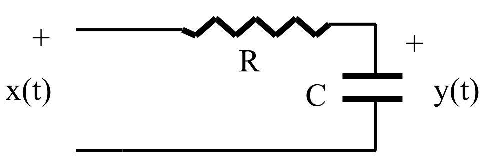
Time Invariance Property:

If $y(t) = T\left[x(t)\right]$, then $y(t-\tau) = T\left[x(t-\tau)\right]$

In words, a system is T.I. when: given an input-output pair, if we apply a delayed version of the input, the new output is the delayed version of the original output.



Time invariance of RC circuit



Seems intuitive based on physical grounds

Let's prove it using the formula $y(t) = \int_0^t \alpha\, e^{-\alpha(t-\sigma)} x(\sigma) d\sigma$

Assume all signals are zero for t < 0.

Introduce the notation $h(t) = \alpha\, e^{-\alpha t}$

$$y(t) = \int_0^t h(t-\sigma)\, x(\sigma) d\sigma$$

$$y(t) = T\left[x(t)\right] = \int_0^t h(t-\sigma)\,x(\sigma)\,d\sigma$$

Now, apply the delayed input $\tilde{x}(t) = x\left(t-\tau\right)$

$$T\left[\tilde{x}(t)\right] = \int_0^t h(t-\sigma)\tilde{x}(\sigma)d\sigma = \int_0^t h(t-\sigma)x(\sigma-\tau)d\sigma$$

$u = \sigma - \tau$

$du = d\sigma$

$x\left(u\right) = 0$ for $u < 0$

$$= \int_{-\tau}^{t-\tau} h(t-\tau-u)x(u)du = \int_0^{t-\tau} h(t-\tau-u)x(u)du$$

Rename dummy variable

$$= \int_0^{t-\tau} h(t-\tau-\sigma)x(\sigma)d\sigma = y\left(t-\tau\right)$$

Proof works for any *h(t)*!

Another example: $y(t) = t\,x(t)$

Linear? Yes, easy to show.

Time invariant? No:

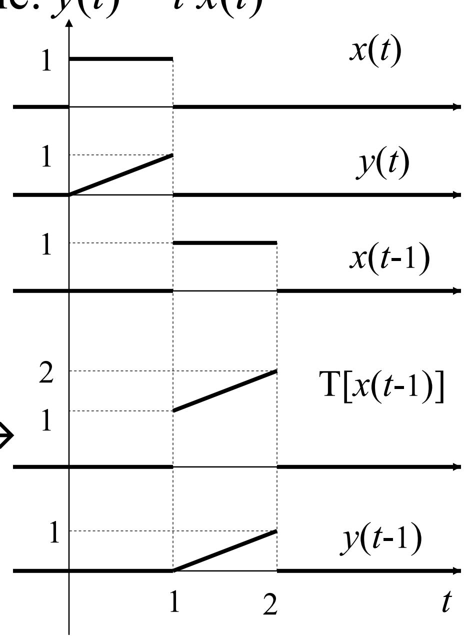
$$T[x(t-\tau)] = t\,x(t-\tau)$$

$$y(t-\tau) = (t-\tau)\,x(t-\tau)$$

They are different!

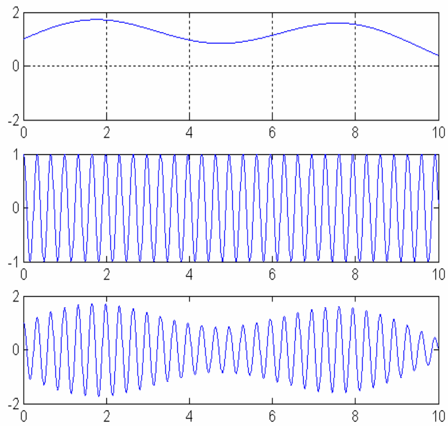
Compare for a particular $x(t)$ →

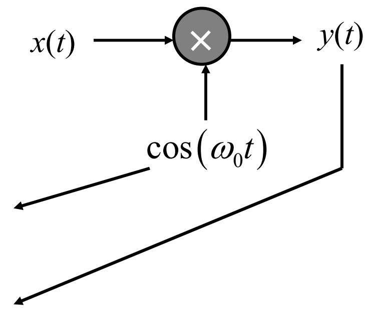
Time-varying system



Example: amplitude modulation

$$y(t) = x(t)\cos(\omega_0 t)$$





Used in AM radio!

Modulator $\quad y(t) = x(t)\cos(\omega_0 t)$

Again, this is a linear system.

Is it time invariant?

$$T\left[x(t-\tau)\right] = x(t-\tau)\cos(\omega_0 t)$$

$$y(t-\tau) = x(t-\tau)\cos\left(\omega_0 (t-\tau)\right)$$

Only equal if $\quad \omega_0 \tau = 2k\pi$

Therefore, it is a time varying system

Notation: LTI = linear, time invariant
LTV= linear, time varying

Causality and memory

- A system is causal if $y(t_0)$ depends only on $x(t)$, $t \le t_0$.
  (present output only depends on past and present inputs)
- A system is memoryless if $y(t_0)$ depends only on $x(t_0)$.
  (present output only depends on present input).
- Causal, not memoryless: we say it has memory.

Examples: Delay system $y(t) = x(t-\tau),\ \tau > 0$
is causal, and has memory.

Backward shift system $y(t) = x(t+\tau),\ \tau > 0$
is non-causal: output anticipates the input.

Non-causal systems are not physically realizable

Recap: properties of main examples

EXAMPLE	RC Circuit	Modulator
$y = T[x]$	$y(t) = \int_0^t \alpha e^{-\alpha(t-\sigma)} x(\sigma) d\sigma$	$y(t) = x(t)\cos(\omega_0 t)$
Linear?	Y	Y
Time Invariant?	Y	N
Causal?	Y	Y
Memoryless?	N	Y