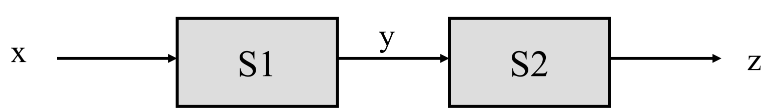
# Lecture 6

- Cascade of linear time-varying systems
- Step response of an LTI system
- LTI systems and differentiation.
- Laplace transforms: definition and basic examples.

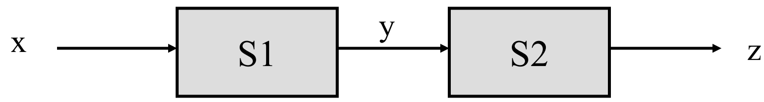
## Cascade of linear time-varying systems




S1, S2 have impulse response functions $h_1(t,\sigma)$, $h_2(t,\sigma)$

$$
\begin{aligned}
z(t) &= \int_{-\infty}^{\infty} h_2(t,\sigma)\, y(\sigma)\, d\sigma \\
&= \int_{-\infty}^{\infty} h_2(t,\sigma) \left[ \int_{-\infty}^{\infty} h_1(\sigma,\tau)\, x(\tau)\, d\tau \right] d\sigma \\
&\underbrace{=}_{\substack{\text{Change order} \\ \text{of integration}}} \int_{-\infty}^{\infty} \underbrace{\left[ \int_{-\infty}^{\infty} h_2(t,\sigma)\, h_1(\sigma,\tau)\, d\sigma \right]}_{h_{1,2}(t,\tau)} x(\tau)\, d\tau
\end{aligned}
$$

# Cascade of linear time-varying systems




S1, S2 have impulse response functions $h_1(t,\sigma)$, $h_2(t,\sigma)$

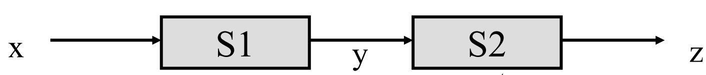
$$h_{1,2}(t,\tau) = \int_{-\infty}^{\infty} h_2(t,\sigma)h_1(\sigma,\tau)d\sigma$$

Impulse response of the cascade.

Note: in general, $h_{1,2} \neq h_{2,1}$.

LTV systems do not commute

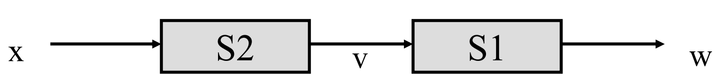
# Example: LTV systems do not commute




$$y(t) = x(t)\cos(t), \qquad z(t) = \int_{-\infty}^{t} y(\sigma)d\sigma$$

Applying $x(t) = \delta(t) \Rightarrow y(t) = \delta(t)\cos(t) = \delta(t) \Rightarrow z(t) = u(t)$

Therefore $h_{1,2}(t,0) = u(t)$




$$v(t) = \int_{-\infty}^{t} x(\sigma)d\sigma, \qquad w(t) = v(t)\cos(t).$$

Applying $x(t) = \delta(t) \Rightarrow v(t) = u(t) \Rightarrow w(t) = u(t)\cos(t)$

Therefore $h_{2,1}(t,0) = u(t)\cos(t) \neq h_{1,2}(t,0)$

# The step response of an LTI system


$$y(t) = T\left[x(t)\right]$$

The step response is defined as: $g(t) = T\left[u(t)\right]$

We assume the system is time invariant, with impulse response $h(t)$. Then:

$$g(t) = h * u = u * h = \int_{-\infty}^{\infty} u(t-\sigma)h(\sigma)d\sigma = \int_{-\infty}^{t} h(\sigma)d\sigma$$

$$g(t) = \int_{-\infty}^{t} h(\sigma)d\sigma, \qquad \frac{dg}{dt} = h$$

# LTI systems and differentiation

$$\frac{dg}{dt} = h \quad \text{means that} \quad \frac{d}{dt}T[u(t)] = T\left[\frac{du}{dt}\right]$$

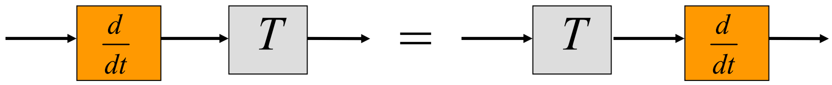
More generally, $\dfrac{d}{dt}T[x(t)] = T\left[\dfrac{dx}{dt}\right]$ for any $x(t)$

Proof: let $y(t) = T\left[x(t)\right]$. Since $T$ is LTI we have

$$\frac{y(t+\tau) - y(t)}{\tau} = \frac{T\left[x(t+\tau)\right] - T\left[x(t)\right]}{\tau} = T\left[\frac{x(t+\tau) - x(t)}{\tau}\right].$$

Taking limit as $\tau \to 0$, we have $\dfrac{dy}{dt} = T\left[\dfrac{dx}{dt}\right]$.

$$\frac{d}{dt}T[x(t)] = T\left[\frac{dx}{dt}\right], \quad \text{for } T \text{ linear time invariant.}$$




Another way to see it: LTI systems commute,

and the "differentiator" $\dfrac{d}{dt}$ is also LTI:

$$\frac{d}{dt}\left[x(t-\tau)\right] = \frac{dx}{dt}(t-\tau)$$

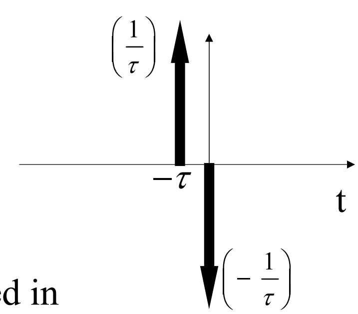
Q: What is the impulse response of the differentiator?

A: $\dfrac{d\delta}{dt}$. But what is this?

## The derivative of delta.

A natural definition would be $\dfrac{d\delta}{dt} = \lim\limits_{\tau \to 0} \dfrac{\delta(t+\tau) - \delta(t)}{\tau}$.

This is the limit of a pair of opposing impulses, of increasing magnitude and becoming close together in time.




Strange object, rarely encountered in physical models, or in the rest of this course.

One example: an electric **dipole:** a pair of positive and negative electric charges, becoming close together in *space*.

# Laplace Transforms

- Time-domain tools for studying systems: differential equations and convolutions.
- We want a more convenient analytical tool.
- Idea: transform time-domain functions to functions in another domain.

$$f(t) \xrightarrow{\mathcal{L}} F(s)$$

- This mapping should be such that the system operations become simpler.

# Laplace Transform – Definition

Given a time-domain function $f(t)$, its Laplace transform is the function of the complex variable $s$

defined by $F(s) = \int_0^\infty e^{-st} f(t)\, dt.$

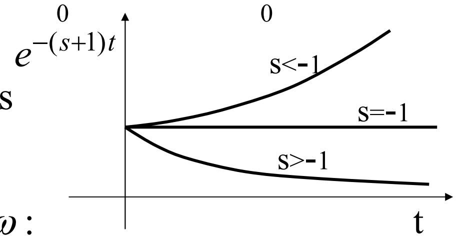
Remarks:

- The integral will be defined only for $s$ in a region of the complex plane (more on this later).
- The Laplace transform maps one function *f(t)* to the other *F(s)*. We denote this by

$$F(s) = \mathcal{L}\left[ f(t) \right]$$

Example: $f(t)=e^{-t}$; $\quad F(s)=\int_0^\infty e^{-st}e^{-t}dt=\int_0^\infty e^{-(s+1)t}dt$

Assume first that $s\in\mathbb{R}$.
Then the integral converges only for $s>-1$.




Now for complex $s=\alpha+i\omega$:

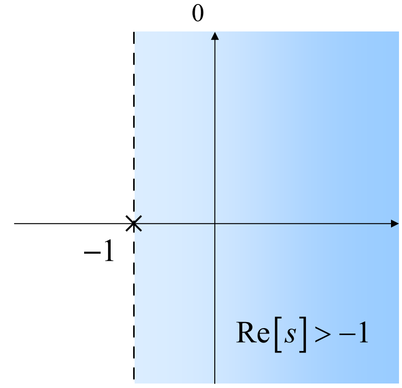
$$\left|e^{-(s+1)t}\right|=\left|e^{-(\alpha+1)t}e^{-i\omega t}\right|=e^{-(\alpha+1)t}$$

$$\int_0^\infty\left|e^{-(s+1)t}\right|dt \quad \text{converges for} \quad \alpha=\mathrm{Re}[s]>-1.$$

$\int_0^\infty e^{-(s+1)t}dt$ is absolutely convergent for $\mathrm{Re}[s]>-1$.

Example: $f(t)=e^{-t}$; $\quad F(s)=\int_0^\infty e^{-st}e^{-t}dt=\int_0^\infty e^{-(s+1)t}dt$

$\mathrm{Re}[s]>-1$ is called the domain of convergence or region of convergence (DOC or ROC) of the Laplace transform $\mathcal{L}\left[e^{-t}\right]$.




Inside this DOC, we compute

$$F(s)=\int_0^\infty e^{-(s+1)t}dt=\left.\frac{e^{-(s+1)t}}{-(s+1)}\right|_0^\infty \underbrace{=}_{\mathrm{Re}[s]>-1}\left(0-\frac{1}{-(s+1)}\right)=\frac{1}{(s+1)}$$

## Laplace Transform – Remarks

- We integrate in $t \in [0,+\infty)$. This is a "one-sided" Laplace transform. Values of $f(t)$ for negative time are irrelevant.
- Some books also define the "bilateral" Laplace transform, integrating on $(-\infty,+\infty)$. We will not use it in this course.
- Still, to emphasize onesided-ness we often write $\mathcal{L}\left[ f(t)u(t)\right]$. This reaffirms that only $[0,+\infty)$ counts. For instance,

$$\mathcal{L}\left[ e^{-t}u(t)\right] = \frac{1}{s+1}.$$

- The point $t=0$ is included in the integration; more precisely,

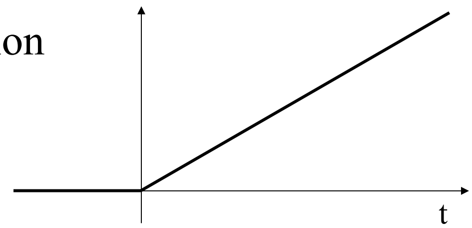
$$F(s) = \int_{0-}^{\infty} e^{-st} f(t)dt. \quad \text{Example: } \mathcal{L}\left[\delta(t)\right] = \int_{0-}^{\infty} e^{-st}\delta(t)dt = 1.$$

## Table of Basic Laplace Transforms

| $f(t)$ | $F(s)$ | DOC |
|---|---|---|
| $\delta(t)$ | $1$ | All $s \in \mathbb{C}$ |
| $e^{at}u(t)$ | $\frac{1}{s-a}$ | $\mathrm{Re}[s] > \mathrm{Re}[a]$ |
| $u(t)$ | $\frac{1}{s}$ | $\mathrm{Re}[s] > 0$ |
| $t^n u(t)$ | $\frac{n!}{s^{n+1}}$ | $\mathrm{Re}[s] > 0$ |
| $\cos(\omega t)u(t)$ | $\frac{s}{s^2+\omega^2}$ | $\mathrm{Re}[s] > 0$ |
| $\sin(\omega t)u(t)$ | $\frac{\omega}{s^2+\omega^2}$ | $\mathrm{Re}[s] > 0$ |

Example: ramp function

$$r(t) = t\,u(t)$$




$$R(s) = \int_0^\infty \underbrace{t}_{u} \underbrace{e^{-st}dt}_{dv} \underbrace{=}_{\text{parts}} \left.\frac{t\,e^{-st}}{-s}\right|_0^\infty - \int_0^\infty \frac{e^{-st}}{-s}dt$$

$$\underbrace{=}_{\mathrm{Re}[s]>0} \frac{1}{s}\int_0^\infty e^{-st}dt = \frac{1}{s}\cdot\left(\left.\frac{e^{-st}}{-s}\right|_0^\infty\right)$$

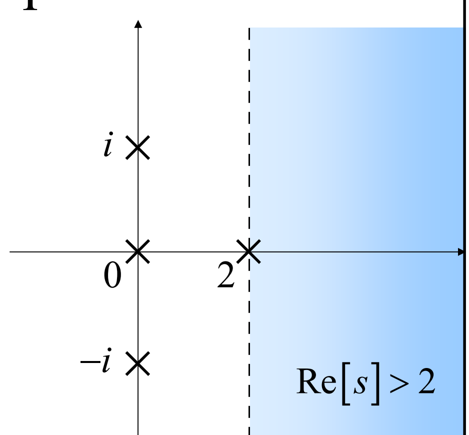
$$\underbrace{=}_{\mathrm{Re}[s]>0} \frac{1}{s}\cdot\left(0 - \frac{1}{-s}\right) = \frac{1}{s^2}$$

# DOC and poles.

A pole of a function $F(s)$ is a point $s_0 \in \mathbb{C}$ where $\lim\limits_{s\to s_0} F(s) = \infty$. For example,

$$F(s) = \frac{(s+7)}{s(s-2)(s^2+1)}$$

has poles at $0, 2, \pm i$.




DOC of the (one-sided) Laplace transform: to the right of the right-most pole (the one with greatest real part).