Lecture 7

- Properties of the Laplace transform.
- Inverse Laplace transform of a rational, strictly proper function.

$$F(s) = \int_{0-}^{\infty} e^{-st} f(t)\,dt = \mathcal{L}\left[f(t)\right],$$

defined for $s \in \text{D.O.C.}$

Properties of the Laplace Transform

$$F(s) = \int_{0-}^{\infty} e^{-st} f(t)\,dt = \mathcal{L}\left[f(t)\right]$$

1) Linearity. $\mathcal{L}\left[\alpha f(t) + \beta g(t)\right] = \alpha \mathcal{L}\left[f(t)\right] + \beta \mathcal{L}\left[g(t)\right]$

Follows directly from linearity of the integral.

DOC is the intersection of DOC's of f and g.

Example:

$$\mathcal{L}\left[e^{-t}u(t) + e^{2t}u(t)\right] = \frac{1}{(s+1)} + \frac{1}{(s-2)}. \quad \text{DOC: } \operatorname{Re}\left[s\right] > 2$$

Another example:

$$\mathcal{L}\left[\cos(\omega t)\,u(t)\right] = \mathcal{L}\left[\left(\frac{e^{i\omega t}+e^{-i\omega t}}{2}\right)u(t)\right]$$

$$= \frac{1}{2}\mathcal{L}\left[e^{i\omega t}u(t)\right] + \frac{1}{2}\mathcal{L}\left[e^{-i\omega t}u(t)\right]$$

$$= \frac{1}{2(s-i\omega)} + \frac{1}{2(s+i\omega)}$$

$$= \frac{(s+i\omega)+(s-i\omega)}{2(s-i\omega)(s+i\omega)}$$

$$= \frac{s}{(s^2+\omega^2)}. \qquad \text{DOC: } \operatorname{Re}[s] > 0$$

Properties of the Laplace Transform

Let $f(t) \xrightarrow{\mathcal{L}} F(s)$

2) $e^{-at} f(t) \xrightarrow{\mathcal{L}} F(s+a)$

3) $\dfrac{df}{dt} \xrightarrow{\mathcal{L}} s\,F(s) - f(0-)$

4) $\displaystyle\int_{0-}^{t} f(\sigma)\,d\sigma \xrightarrow{\mathcal{L}} \frac{F(s)}{s}$

5) $t\,f(t) \xrightarrow{\mathcal{L}} -\dfrac{dF}{ds}$

6) $u(t-\tau)\,f(t-\tau) \xrightarrow{\mathcal{L}} e^{-\tau s}F(s)$

Property 2) $e^{-at} f(t) \xrightarrow{\mathcal{L}} F(s+a)$

Proof: let $f(t) \xrightarrow{\mathcal{L}} F(s) = \int_{0-}^{\infty} e^{-st} f(t)\, dt$

$$e^{-at} f(t) \xrightarrow{\mathcal{L}} \int_{0-}^{\infty} e^{-st} e^{-at} f(t)\, dt$$

$$= \int_{0-}^{\infty} e^{-(s+a)t} f(t)\, dt = F(s+a)$$

Example: $\mathcal{L}[u(t)\cos(t)] = \dfrac{s}{s^2+1}$

$$\Rightarrow \mathcal{L}\left[u(t)e^{-t}\cos(t)\right] = \frac{s+1}{(s+1)^2+1}$$

Property 3) $\dfrac{df}{dt} \xrightarrow{\mathcal{L}} s\,F(s) - f(0-)$

Proof: $\dfrac{df}{dt} \xrightarrow{\mathcal{L}} \int_{0-}^{\infty} e^{-st} \dfrac{df}{dt}\, dt \underbrace{=}_{\text{parts}} e^{-st} f(t)\Big|_{0-}^{\infty} - \int_{0-}^{\infty} (-s)e^{-st} f(t)dt$

$$\underbrace{=}_{\text{Inside DOC of } f} -f(0-) + s\int_{0-}^{\infty} e^{-st} f(t)dt = s\,F(s) - f(0-)$$

Note: DOC for $\dfrac{df}{dt}$ includes the DOC of $f(t)$.

Example: $f(t) = e^{2t} \rightarrow F(s) = \dfrac{1}{s-2}$

$$\frac{df}{dt} = 2e^{2t} \rightarrow \mathcal{L}\left[\frac{df}{dt}\right] = \frac{2}{s-2} = s\frac{1}{s-2} - 1 = s\cdot\mathcal{L}[f(t)] - \overbrace{f(0-)}^{1}$$

Property 3) $\frac{df}{dt} \xrightarrow{\mathcal{L}} sF(s) - f(0-)$

Example (cont.): If instead we write $f(t) = e^{2t}u(t)$,

the transform is still $F(s) = \dfrac{1}{s-2}$

Now $f(0-) = 0$, so $\mathcal{L}\left[\dfrac{df}{dt}\right] = \dfrac{s}{s-2}$. Different!

However now $\mathcal{L}\left[\dfrac{df}{dt}\right] = \mathcal{L}\left[2e^{2t}u(t) + \delta(t)\right] = \dfrac{2}{s-2} + 1 = \dfrac{s}{s-2}$,

so it is still consistent with the property.

Property 4) $g(t) = \int_{0-}^{t} f(\sigma)d\sigma \xrightarrow{\mathcal{L}} G(s) = \frac{F(s)}{s}$

Proof: since $\dfrac{dg}{dt} = f(t)$, $\underbrace{\Rightarrow}_{\text{Property 3)}}$ $F(s) = sG(s) - g(0-) \underbrace{=}_{g(0-)=0} sG(s)$

Example: let $f_0(t) = u(t) \xrightarrow{\mathcal{L}} F_0(s) = \dfrac{1}{s}$

Now let $f_1(t) = t\,u(t) = \displaystyle\int_0^t f_0(\sigma)d\sigma \xrightarrow{\mathcal{L}} F_1(s) \underbrace{=}_{4)} \dfrac{F_0(s)}{s} = \dfrac{1}{s^2}$

$$f_2(t) = t^2 u(t) = 2\int_0^t f_1(\sigma)d\sigma \xrightarrow{\mathcal{L}} F_2(s) = \frac{2F_1(s)}{s} = \frac{2}{s^3}$$

Similarly,

$$f_n(t) = t^n u(t) = n\int_0^t f_{n-1}(\sigma)d\sigma \xrightarrow{\mathcal{L}} F_n(s) = \frac{nF_{n-1}(s)}{s} = \frac{n!}{s^{n+1}}$$

Property 5) $t\,f(t) \xrightarrow{\mathcal{L}} -\dfrac{dF}{ds}$

Proof: we differentiate $F(s)=\int_{0-}^{\infty} e^{-st} f(t)\,dt$ with respect to s,

taking the derivative inside the integral. This is valid inside the DOC.

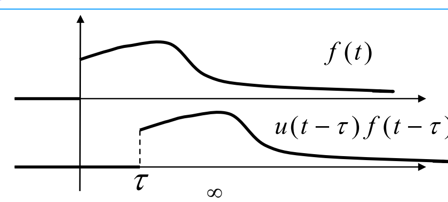
$$\Rightarrow \frac{dF}{ds}=\int_{0-}^{\infty}\frac{d}{ds}e^{-st}f(t)\,dt=\int_{0-}^{\infty}e^{-st}(-t)f(t)\,dt=-\mathcal{L}\left[t\,f(t)\right]$$

Back to previous example: $f_0(t)=u(t)\xrightarrow{\mathcal{L}}F_0(s)=\dfrac{1}{s}$

$$f_1(t)=t\,u(t)\xrightarrow{\mathcal{L}}F_1(s)\underbrace{=}_{5)}-\frac{dF_0}{ds}=-\left(-\frac{1}{s^2}\right)=\frac{1}{s^2}$$

6) Delay property: $u(t-\tau)f(t-\tau) \xrightarrow{\mathcal{L}} e^{-\tau s}F(s)$

Careful: the delayed function must be set to zero for $t<\tau$.



Proof:

$$\mathcal{L}\left[u(t-\tau)f(t-\tau)\right]=\int_{0-}^{\infty}e^{-st}u(t-\tau)f(t-\tau)\,dt=\int_{\tau-}^{\infty}e^{-st}f(t-\tau)\,dt$$

$$\underbrace{=}_{\sigma=t-\tau}\int_{0-}^{\infty}e^{-s(\sigma+\tau)}f(\sigma)\,d\sigma=e^{-s\tau}\int_{0-}^{\infty}e^{-s\sigma}f(\sigma)\,d\sigma=e^{-s\tau}F(s)$$

Example: $\mathcal{L}\left[e^{t}\right]=\dfrac{1}{s-1}\Rightarrow\mathcal{L}\left[u(t-1)e^{t-1}\right]=\dfrac{e^{-s}}{s-1}$

But $\mathcal{L}\left[e^{t-1}\right]=\mathcal{L}\left[e^{t}e^{-1}\right]=\dfrac{e^{-1}}{s-1}$, different answer! No confusion if

we include $u(t)$ in the original transform, writing $\mathcal{L}\left[u(t)e^{t}\right]=\dfrac{1}{s-1}$

Inverse Laplace Transform

Given $F(s)$, find $f(t)$

Concentrate on the case of rational functions

$$F(s) = \frac{P(s)}{Q(s)} = \frac{p_m s^m + \cdots + p_1 s + p_0}{q_n s^n + \cdots + q_1 s + q_0}$$

(that is, a ratio of two polynomials).

Without loss of generality, we assume $q_n = 1$.

We factor $Q(s)$ in terms of its roots $a_1, a_2, \cdots, a_n$.

$$Q(s) = s^n + \cdots + q_1 s + q_0 = (s - a_1)(s - a_2)\cdots(s - a_n)$$

I. First, the strictly proper case

This means that $m = \deg(P(s)) < \deg(Q(s)) = n.$

The inverse transform is found by :

1) Decomposing $F(s) = \dfrac{P(s)}{Q(s)}$ in partial fractions.

2) Using the Laplace table.

Case 1: $Q(s)$ has simple roots $a_1, a_2, \cdots, a_n$ (mutually distinct).

Then there always exist constants $A_1, A_2, \cdots, A_n$ that satisfy the Partial Fraction Expansion

$$F(s) = \frac{P(s)}{(s - a_1)(s - a_2)\cdots(s - a_n)} = \frac{A_1}{s - a_1} + \frac{A_2}{s - a_2} + \cdots + \frac{A_n}{s - a_n}$$

Then $f(t) = \mathcal{L}^{-1}\left[F(s)\right] = \left(A_1 e^{a_1 t} + A_2 e^{a_2 t} + \cdots + A_n e^{a_n t}\right) u(t)$

Example: $F(s)=\dfrac{1}{s(s+1)}=\dfrac{A_1}{s}+\dfrac{A_2}{s+1}$

We must find the constants A_1, A_2. They are guaranteed to exist, so any method that provides enough equations to determine them will do. One (usually longer) way:

- Common denominator: $\dfrac{1}{s(s+1)}=\dfrac{A_1(s+1)+A_2 s}{s(s+1)}$
- Identify numerator coefficients: $\begin{cases} A_1+A_2=0 \\ A_1=1 \end{cases}$
- Solve equations: $A_1=1, \quad A_2=-1.$

$$F(s)=\frac{1}{s(s+1)}=\frac{1}{s}-\frac{1}{s+1} \rightarrow f(t)=u(t)\left[1-e^{-t}\right]$$

Example: $F(s)=\dfrac{1}{s(s+1)}=\dfrac{A_1}{s}+\dfrac{A_2}{s+1}$

A more efficient way:

- Multiply both sides by s: $\boxed{\dfrac{1}{s+1}=A_1+\dfrac{A_2 s}{s+1}}$
- Now evaluate (or take limit) at $s=0$. $\rightarrow \boxed{1=A_1.}$

Automating this step: to find A_1 (coefficient for s):

"Cover" the factor s in $F(s)$: $\dfrac{1}{(s+1)}$

Evaluate the rest at "covered" root $s=0$. $\dfrac{1}{(\underbrace{s}_{0}+1)}=1$

To find A_2: cover $(s+1)$ factor, evaluate at $s=-1$.

$$\frac{1}{\underbrace{s}_{-1}}=-1$$

General case, simple roots.

$$\frac{P(s)}{(s-a_1)(s-a_2)\cdots(s-a_n)}=\frac{A_1}{s-a_1}+\frac{A_2}{s-a_2}+\cdots+\frac{A_n}{s-a_n}$$

Brute force method gives n equations, n unknowns.

"Clever" method to find A_1: multiply by $(s-a_1)$,

$$\frac{P(s)}{(s-a_2)\cdots(s-a_n)}=A_1+\left(\frac{A_2}{s-a_2}+\cdots+\frac{A_n}{s-a_n}\right)(s-a_1),$$

Evaluate (or take limit) at $s=a_1$. $\rightarrow \dfrac{P(a_1)}{(a_1-a_2)\cdots(a_1-a_n)}=A_1$

Automate: to find A_k: cover $(s-a_k)$, evaluate at $s=a_k$

Case 2: multiple roots.

Assume $Q(s)$ has a multiple root a_i, repeated r_i times. That is, there is a factor $(s-a_i)^{r_i}$ in the denominator.

This is handled by including in the expansion for

$\dfrac{P(s)}{Q(s)}$ the terms $\dfrac{A_i^1}{s-a_i}+\dfrac{A_i^2}{(s-a_i)^2}+\cdots+\dfrac{A_i^{r_i}}{(s-a_i)^{r_i}}$

Example: $F(s)=\dfrac{1}{(s+2)(s+1)^2}=\dfrac{A}{s+2}+\dfrac{B}{s+1}+\dfrac{C}{(s+1)^2}$

$A=\dfrac{1}{(\underbrace{s}_{-2}+1)^2}=1$ obtained by usual method

Multiply by $(s+1)^2$, evaluate at $s=-1$:

$$\frac{1}{(\underbrace{s}_{-1}+2)}=\left(\frac{A}{s+2}+\frac{B}{s+1}\right)(s+1)^2+C \rightarrow \boxed{C=1.}$$

$$F(s)=\frac{1}{(s+2)(s+1)^2}=\frac{1}{s+2}+\frac{B}{s+1}+\frac{1}{(s+1)^2}$$

"Covering" method cannot be used to determine B. We need another equation. e.g., evaluate at $s=0$ (easy point):

$$F(0)=\frac{1}{2}=\frac{1}{2}+B+1\ \rightarrow B=-1$$

Another way: multiply by s, take limit as $s\rightarrow\infty$.

$$0=\lim_{s\to\infty}\frac{s}{(s+2)(s+1)^2}=\lim_{s\to\infty}\left(\frac{s}{s+2}+\frac{Bs}{s+1}+\frac{s}{(s+1)^2}\right)=1+B$$

Table+properties $\rightarrow\ \mathcal{L}^{-1}\left[\frac{1}{(s-a)^{n+1}}\right]=\frac{t^n}{n!}e^{at}u(t)$

$$\rightarrow f(t)=\mathcal{L}^{-1}\left[\frac{1}{s+2}-\frac{1}{s+1}+\frac{1}{(s+1)^2}\right]=\left(e^{-2t}-e^{-t}+t\,e^{-t}\right)u(t)$$

Case 3: Complex roots.

The preceding methods apply also to complex roots, simple or multiple, using complex operations.
Example:

$$F(s)=\frac{s}{(s^2+1)}=\frac{s}{(s+i)(s-i)}=\frac{A}{s+i}+\frac{B}{s-i}$$

$$A=\frac{\overbrace{s}^{-i}}{(\underbrace{s}_{-i}-i)}=\frac{1}{2};\qquad B=\frac{\overbrace{s}^{i}}{(\underbrace{s}_{i}+i)}=\frac{1}{2}.$$

$$\rightarrow f(t)=\frac{1}{2}e^{-it}u(t)+\frac{1}{2}e^{it}u(t)=\cos(t)u(t).$$

Already known from the table.