EE 102 **Homework #1 Solutions** **Fall 2001**

Professor Paganini

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1. Review of integration

   (a) We use integration by parts and get

   $$\int_0^{\pi} t\cos(t)dt = \underbrace{[t\sin(t)]_0^{\pi}}_{0} - \int_0^{\pi} \sin(t)dt = \cos(\pi) - \cos(0) = \underline{-2}.$$

   For the next integral we apply integration by parts.

   $$\begin{aligned}\int_0^{\pi} t^2\sin(t)dt &= \left[-t^2\cos(t)\right]_0^{\pi} - 2\int_0^{\pi} t(-\cos(t))dt \\ &= \pi^2 + 2\left[[t\sin(t)]_0^{\pi} - \int_0^{\pi}\sin(t)dt\right] \\ &= \pi^2 + 2\left[\cos(\pi) - \cos(0)\right] = \underline{\pi^2 - 4}\end{aligned}$$

   (b) With substitution $t - \tau = \sigma$, $d\tau = -d\sigma$ we get

   $$A(t) = \int_0^t f(t-\tau)d\tau = \int_t^0 f(\sigma)(-d\sigma) = \int_0^t f(\sigma)d\sigma.$$

   This can be rewritten with a factor of 1 inserted and partially integrated as

   $$A(t) = \int_0^t 1\cdot f(\sigma)\ d\sigma = [\sigma f(\sigma)]_0^t - \int_0^t \sigma f'(\sigma)d\sigma.$$

   Since the equation $A(t)$ is a function of t only (σ and τ are just integration variables and are exchangeable), we rewrite $A(t)$ as

   $$A(t) = tf(t) - \int_0^t \tau f'(\tau)d\tau,$$

   which is what we had to prove.

   (c) We integrate over the following four regions separately, considering for the previous region in our results.

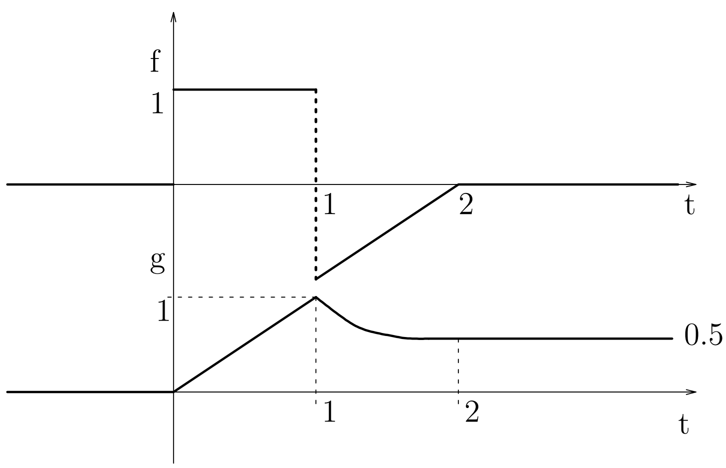
   * $\mathbf{t < 0}$, $\underline{g(t) = 0}$.
   * $\mathbf{0 \leq t \leq 1}$, $f(t) = 1$, $\underline{g(t)} = \int_0^t 1\ d\sigma \underline{= t}$ The previous result is always 0 so nothing has to be added.

* $\mathbf{1 \le t \le 2}$, $f(t) = t - 2$,

$$\underline{g(t)} = g(1) + \int_1^t (\sigma - 2)\ d\sigma = 1 + 0.5t^2 - 2t - 0.5 + 2 \underline{= 0.5t^2 - 2t + 2.5}.$$

  Be aware that the result of the previous region at the boundary $t = 1$, $g(1)$, has to be added.

* $\mathbf{2 \le t}$, $f(t) = 0$, $\underline{g(t)} = g(2) \underline{= 0.5}$ The result is the previous result at $t = 2$ since nothing is added in region 4.



2. Review of complex numbers

   (a) Get real and imaginary parts

      (1) One full rotation (2π) of a vector (phasor) in the complex plane does not modify the vector and we get $e^{i\phi} = e^{i(\phi + k2\pi)}$ where k is an *integer* value. The problem can be seen as 6 full rotations plus a three quart rotation. The rotation is clockwise because the sign of the exponent is negative.

$$e^{-i\frac{27}{2}\pi} = e^{-i(6+\frac{3}{4})2\pi} = e^{-i\frac{3}{4}2\pi} = e^{-i3\frac{\pi}{2}} = \underline{i}$$

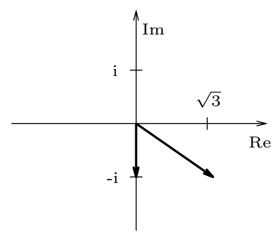
      (2) With $i = \sqrt{-1}$ and $i^2 = -1$ the problem can be written as

$$(i)^{i^6} = i^{(i^6)} = i^{((i^2)^3)} = i^{((-1)^3)} = i^{-1} = \frac{1}{i} = \frac{1 \cdot i}{i \cdot i} = \frac{i}{-1} = \underline{-i}.$$

      Another way to get the solution would use $i = e^{i\frac{\pi}{2}} = e^{-i3\frac{\pi}{2}}$ which gives the same result

$$i^{(i^6)} = i^{e^{i6\frac{\pi}{2}}} = i^{-1} = \underline{-i}.$$

(b) Get exponential form $|z|e^{i\phi}$



(1) From the figure it can be seen that $\phi = -\frac{\pi}{6}$ and the length of the vector $|z| = 2$.

$$\alpha = \sqrt{3} - i = \underline{2e^{-i\frac{\pi}{6}}}$$

(2) From the figure it can be seen that the vector has no real part. Its length is $|z| = 1$ and its phase is $\phi = -\frac{\pi}{2}$ which gives

$$\beta = -i = 1 \cdot e^{-i\frac{\pi}{2}} = \underline{e^{-i\frac{\pi}{2}}}.$$

(c) The complex conjugate of a number can be found in two ways. Either (i) negate its phase $\phi \to -\phi$, or (ii) negate its imaginary part $\mathrm{Im} \to -\mathrm{Im}$. We get

$$\frac{\alpha^3}{\overline{\beta}} = \frac{2^3 e^{-3i\frac{\pi}{6}}}{e^{i\frac{\pi}{2}}} = 8e^{-3i\frac{\pi}{6}} e^{-i\frac{\pi}{2}} = 8e^{-i\pi} = \underline{-8}.$$

(d) The equation can be written as $z^6 = 27$ and $z = 27^{\frac{1}{6}}$. In order to get all possible 6 results we use $27 = 27e^{i2k\pi}$ where k is any integer

$$z = \left[27e^{i2k\pi}\right]^{\frac{1}{6}} = \sqrt{3}e^{ik\frac{\pi}{3}}.$$

For $k = 0, 1, 2, 3, 4, 5$ we get the result as a set

$$\underline{z \in \left\{\sqrt{3}, \frac{\sqrt{3}}{2} + i\frac{3}{2}, -\frac{\sqrt{3}}{2} + i\frac{3}{2}, -\sqrt{3}, -\frac{\sqrt{3}}{2} - i\frac{3}{2}, \frac{\sqrt{3}}{2} - i\frac{3}{2}\right\}}.$$

3. Differential Equations

$$\frac{dy(t)}{dt} + y(t) = \frac{dx(t)}{dt} - 2x(t)$$

The left and right hand side of the equation can be rewritten as

$$e^{-t}\frac{d}{dt}\left(e^{t}y(t)\right) = e^{2t}\frac{d}{dt}\left(e^{-2t}x(t)\right).$$

Multiplying by e^t and integrating from 0 to t yields

$$e^t y(t) - e^0 y(0) = \int_0^t e^{3\sigma} \frac{d}{d\sigma} \left(e^{-2\sigma} x(\sigma) \right) d\sigma.$$

Doing integration by parts for the right hand side using $u(t) = e^{3t}$ and $v(t) = e^{-2t}x(t)$ gives

$$e^t y(t) = e^t x(t) - 3 \int_0^t e^{3\sigma} e^{-2\sigma} x(\sigma) d\sigma.$$

The final result is

$$\underline{y(t) = x(t) - 3 \int_0^t e^{-(t-\sigma)} x(\sigma) d\sigma.}$$

4. System descriptions

   $x(t)$ is the input signal and the corresponding output of the system is defined as $y(t) = T[x(t)]$. For proof of system linearity the input signal is written as a linear combination $x(t) = \alpha x_1(t) + \beta x_2(t)$ and

   $$T[\alpha x_1(t) + \beta x_2(t)] = \alpha y_1(t) + \beta y_2(t)$$

   has to be true. For time invariance

   $$T[x(t - \tau)] = y(t - \tau)$$

   has to be true. Causality means that the output of the system $y(t)$ is not dependent on future values of the input $x(t)$.

   (a) $y(t) = x(t + 1) - 3$
   Not linear: $T[\alpha x_1(t)+\beta x_2(t)] = \alpha x_1(t+1)+\beta x_2(t+1)-3 \neq \alpha y_1(t)+\beta y_2(t) = \alpha(x_1(t + 1) - 3) + \beta(x_2(t + 1) - 3)$.
   Time invariant: $T[x(t - \tau)] = x(t - \tau + 1) - 3 = y(t - \tau)$.
   Not causal: y at time t depends on value of x at time $t + 1$, i.e. in the future.

   (b) $y(t) = e^t x(t)$
   Linear: $T[\alpha x_1(t) + \beta x_2(t)] = e^t(\alpha x_1(t) + \beta x_2(t)) = \alpha e^t x_1(t) + \beta e^t x_2(t) = \alpha y_1(t) + \beta y_2(t)$.
   Time variant: $T[x(t - \tau)] = e^t x(t - \tau) \neq y(t - \tau) = e^{(t-\tau)} x(t - \tau)$.
   Causal and memoryless: $y(t)$ depends only on x at current time.

   (c) $y(t) = \int_t^\infty x(\sigma) d\sigma$
   Linear: $T[\alpha x_1(t) + \beta x_2(t)] = \int_t^\infty (\alpha x_1(\sigma) + \beta x_2(\sigma)) d\sigma = \alpha \int_t^\infty x_1(\sigma) d\sigma + \beta \int_t^\infty x_2(\sigma) d\sigma = \alpha y_1(t) + \beta y_2(t) = \alpha \int_t^\infty x_1(\sigma) d\sigma + \beta \int_t^\infty x_2(\sigma) d\sigma$.
   Time invariant: $T[x(t - \tau)] = \int_t^\infty x(\sigma - \tau) d\sigma = \int_{(t-\tau)}^\infty x(\rho) d\rho = y(t - \tau)$.
   Not causal: To find $y(t)$ you need values of x in the entire future (t, ∞).

(d) $y(t) = \begin{cases} x(t) & \text{if } x(t) > 0 \\ 0 & \text{else} \end{cases}$

Not linear. To make it simple we just consider one of the linearity conditions which is $T[\alpha x(t)] = \alpha y(t)$. If $x(t)$ is multiplied by a *negative* α then it can be seen that $x(t)$ switches its sign and $y(t)$ takes a totally different value. Example: $x(t) = 3$, which gives $y(t) = 3$. Now take $\alpha = -1$, $T[\alpha x(t)] = 0 \neq \alpha y(t) = -3$.

Time invariant: $T[x(t-\tau)] = \begin{cases} x(t-\tau) & \text{if } x(t-\tau) > 0 \\ 0 & \text{else} \end{cases} = y(t-\tau)$.

Causal and memoryless: $y(t)$ depends only on x at current time.

EE 102 # Homework #2 Solutions

Fall 2001 **Professor Paganini**

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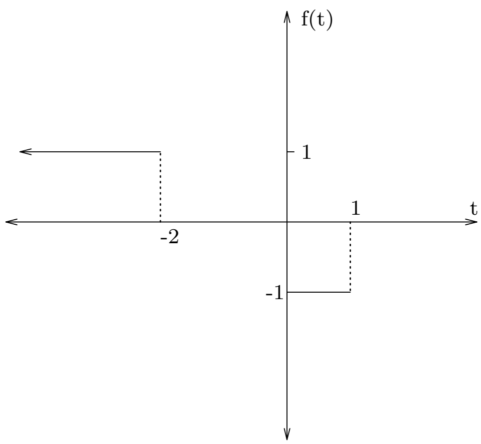
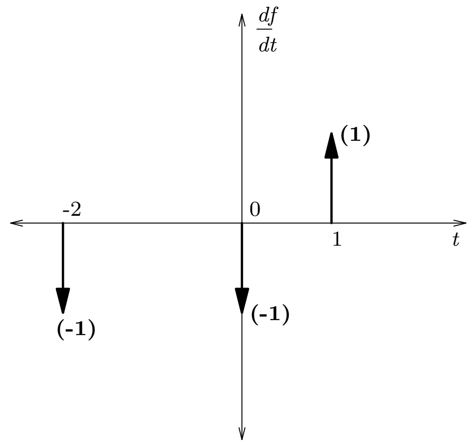
1. (a) $f(t) = 1 - u(t+2) - u(t) + u(t-1)$.

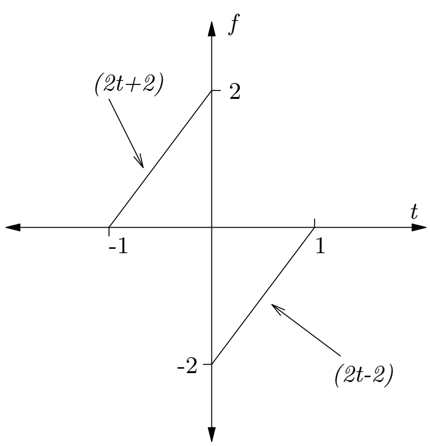
   $\frac{df}{dt}(t) = -\delta(t+2) - \delta(t) + \delta(t-1)$.

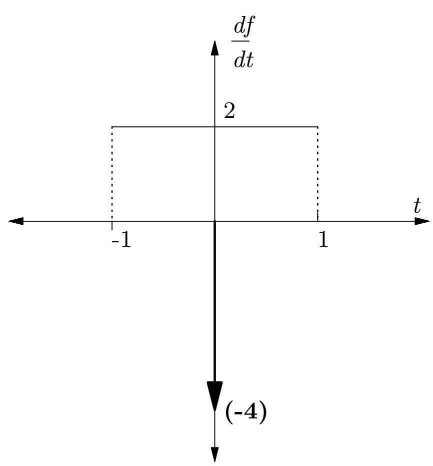




   (b) $f(t) = (2t+2)[u(t+1) - u(t)] + (2t-2)[u(t) - u(t-1)]$.

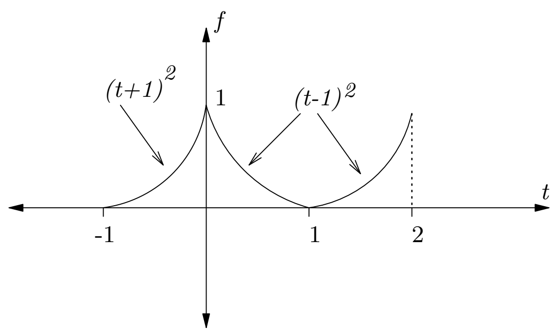
   $\frac{df}{dt}(t) = 2[u(t+1) - u(t-1)] - 4\delta(t)$.

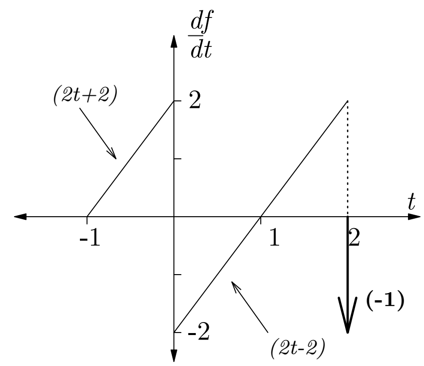




(c) $f(t) = (t+1)^2[u(t+1) - u(t)] + (t-1)^2[u(t) - u(t-2)]$.

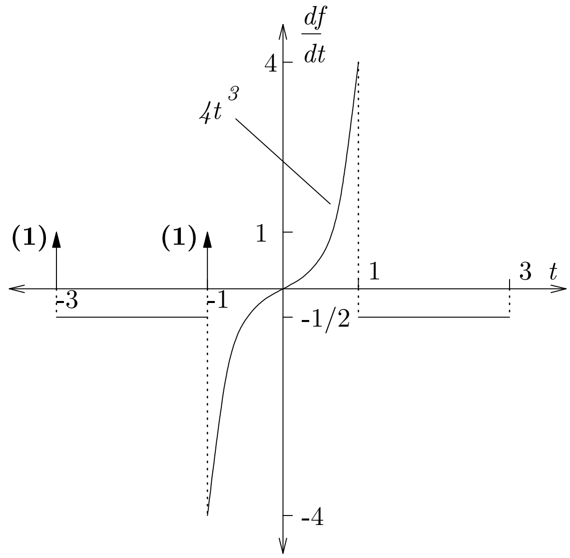
$\frac{df}{dt}(t) = 2(t+1)[u(t+1) - u(t)] + 2(t-1)[u(t) - u(t-2)] - \delta(t-2)$.





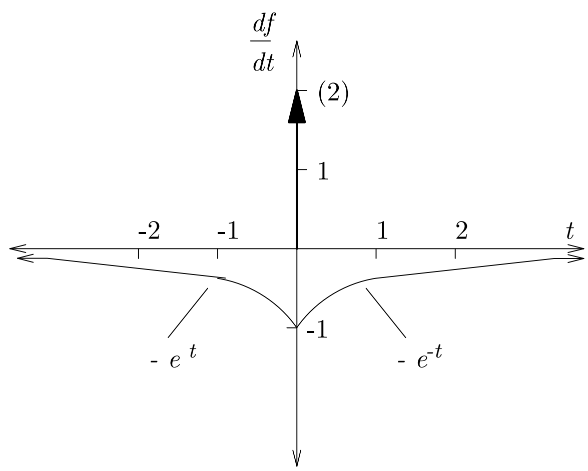
2. (a)

$$
\begin{aligned}
f(t) &= (-\frac{t}{2} - \frac{1}{2})[u(t+3) - u(t+1)] + t^4[u(t+1) - u(t-1)] \\
&\quad + (-\frac{t}{2} + \frac{3}{2})[u(t-1) - u(t-3)]. \\
\frac{df}{dt}(t) &= \delta(t+3) - \frac{1}{2}[u(t+3) - u(t+1)] + \delta(t+1) + 4t^3[u(t+1) - u(t-1)] \\
&\quad + (-\frac{1}{2})[u(t-1) - u(t-3)].
\end{aligned}
$$



(b)

$f(t) = -e^{t}u(-t) + e^{-t}u(t).$
$\frac{df}{dt}(t) = -e^{t}u(-t) + 2\delta(t) - e^{-t}u(t).$



3. (a) $\int_{-\infty}^{\infty} e^{\sin(\pi t)}\ \delta(t+\frac{1}{2})\ dt = e^{\sin(\pi(-\frac{1}{2}))} = e^{-1} = 1/e$

(b) $\int_{-\infty}^{3} e^{t^2-3t-4}\delta(t-4)\ dt = 0$ since $4 \notin (-\infty, 3]$.

(c) $\int_{a-}^{\infty} \cos(t)\delta(t-a)\ dt = \cos(a)$ since the impulse at $t = a$ is included in the domain of integration.

4. (a)

$$y(t) = T[x(t)] = \int_{-\infty}^{t} \cos(t+\sigma)x(\sigma-1)d\sigma$$

By definition,

$$h(t,\tau) = T[\delta(t-\tau)] = \int_{-\infty}^{t} \cos(t+\sigma)\delta(\sigma-\tau-1)d\sigma = \cos(t+\tau+1)u(t-\tau-1)$$

(b) The system is not time-invariant since $h(t,\tau)$ is not a function of $(t-\tau)$ alone. The system is causal since $h(t,\tau) = 0$ for $(t-\tau) < 0$.

5. As solved in Homework # 1,

$$y(t) = T[x(t)] = x(t) - 3\int_{0-}^{t} e^{-(t-\sigma)}x(\sigma)d\sigma, \quad t \geq 0$$

This system is linear and time-invariant.

By definition,

$$h(t) = T[\delta(t)] = \delta(t) - 3\int_{0-}^{t} e^{-(t-\sigma)}\delta(\sigma)d\sigma = \delta(t) - 3e^{-t}u(t).$$

EE 102 **Homework #3 Solutions** Fall 2001

Professor Paganini

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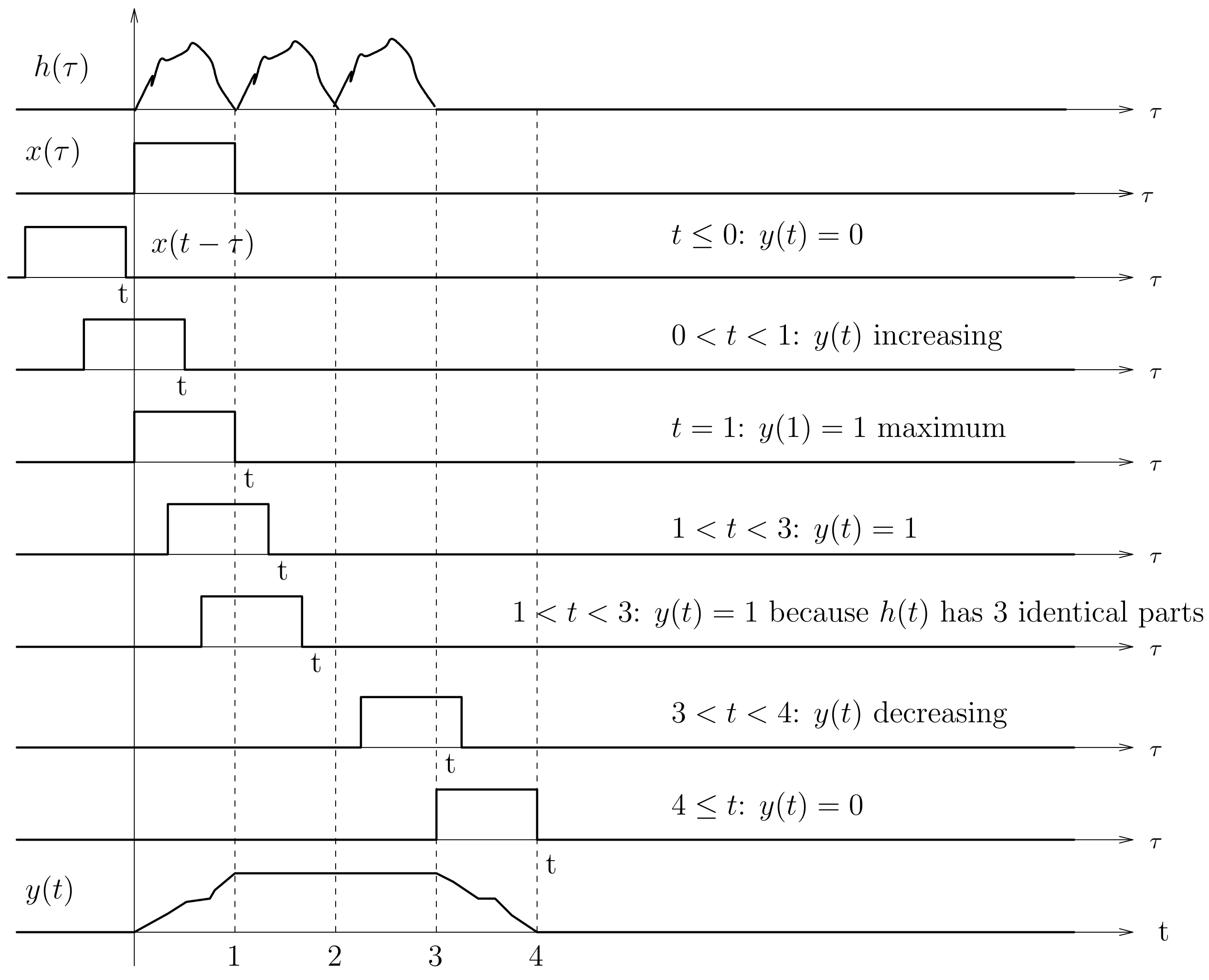
1. The convolution with the input $x(t) = u(t) - u(t-1)$ is depicted in the following figure.

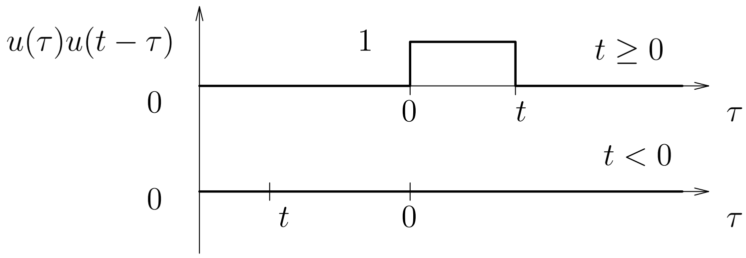


The convolution $y(t) = \int_{-\infty}^{\infty} h(t-\tau)x(\tau)d\tau$ is rewritten as $y(t) = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau$ in order to get a simpler graphical convolution, where $x(\tau)$ is shifted by t.

2. For the following convolutions the evaluation of the product

$$u(\tau)u(t-\tau) = \begin{cases} 1 & \text{if } t \geq 0, \text{ for } 0 \leq \tau \leq t \\ 0 & \text{if } t < 0, \text{ for all } \tau \end{cases}$$

is helpful. $u(\tau)u(t-\tau)$ is drawn as a function of τ in the next figure.



(a)

$$u * f = \int_{-\infty}^{\infty} e^{-\tau} u(\tau)u(t-\tau)d\tau = \begin{cases} \int_0^t e^{-\tau} d\tau & \text{if } t \geq 0 \\ 0 & \text{else} \end{cases} = \underline{u(t)\left(1 - e^{-t}\right)}$$

(b)

$$f * f = \int_{-\infty}^{\infty} e^{-\tau} e^{-(t-\tau)} u(\tau)u(t-\tau)d\tau$$

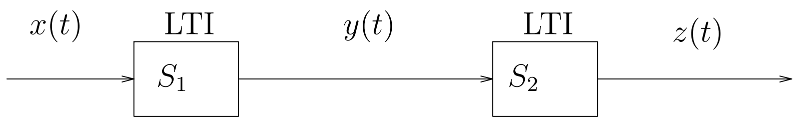
$$= e^{-t} \int_{-\infty}^{\infty} u(\tau)u(t-\tau)d\tau = \begin{cases} e^{-t} \int_0^t d\tau & \text{if } t \geq 0 \\ 0 & \text{else} \end{cases} = \underline{u(t)\ t\ e^{-t}}$$

(c)

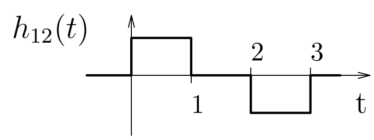
$$u * u = \int_{-\infty}^{\infty} u(\tau)u(t-\tau)d\tau = \begin{cases} \int_0^t d\tau = t & \text{if } t \geq 0 \\ 0 & \text{else} \end{cases} = \underline{u(t)t}$$

3. Cascade of Linear time-invariant (LTI) systems

   As seen in class, an LTI system commutes with differentiation: that is, if we apply the derivative of a certain input, we get the derivative of the output. We use this property to derive the necessary information about the systems, as summarized in the figure below. In it we use the notation $r(t) := tu(t)$ (the "ramp" function).



line 1	$\delta(t)$	$\longrightarrow$	$u(t) - u(t-2)$		
line 2			$tu(t) = r(t)$	$\longrightarrow$	$r(t) - r(t-1)$
line 3			$\frac{dr(t)}{dt} = u(t)$	$\longrightarrow$	$\frac{dr(t)}{dt} - \frac{r(t-1)}{dt} = u(t) - u(t-1)$
line 4			$u(t-2)$	$\longrightarrow$	$u(t-2) - u(t-3)$
line 5	$\delta(t)$	$\longrightarrow$	$u(t) - u(t-2)$	$\longrightarrow$	$u(t) - u(t-1) - u(t-2) + u(t-3)$



   In this picture the input of the whole system, $x(t)$, is represented in the leftmost column. In the middle column $y(t)$, the input to the second subsystem S_2 can be seen and the column on the rightmost side represents the output, $z(t)$, of the system.

   The first and second lines of formulas are given in the problem, where the given graph of $z(t)$ could be found as $r(t) - r(t-1)$.

   Formula line 3 makes use of the fact that the derivative of an input gives the derivative of the output. Line 4 follows by shifting, using time invariance. Line 5, which is under the horizontal bar gives the result of the linear combination of lines 3 and 4. The impulse response of the whole system is $h_{12} = u(u) - u(t-1) - u(t-2) + u(t-3)$ and is plotted.

EE 102 # Homework #4 Solutions

Fall 2001 **Professor Paganini**

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1. (a)

$$f(t) = u(t-2)e^{2t} = \begin{cases} 0 & \text{if } t < 2 \\ e^{-2t} & \text{if } t > 2. \end{cases}$$

$$\begin{aligned} F(s) &= \int_0^\infty u(t-2)e^{2t}e^{-st}dt = \int_2^\infty e^{-(s-2)t}dt \\ &= \left[-\frac{e^{(s-2)t}}{(s-2)}\right]_{t=2}^{t\to\infty} = \frac{e^4 e^{-2s}}{(s-2)} \text{with DOC } \mathrm{Re}[s] > 2. \end{aligned}$$

(b)

$$\begin{aligned} f(t) &= u(t) - u(t-1) + u(t-2) - u(t-3) \\ &= \begin{cases} 1 & \text{if } 0 < t < 1 \text{ or } 2 < t < 3 \\ 0 & \text{Otherwise.} \end{cases} \end{aligned}$$

$$\begin{aligned} F(s) &= \int_0^1 e^{-st}dt + \int_2^3 e^{-st}dt \\ &= \begin{cases} \left[-\frac{e^{-st}}{s}\right]_{t=0}^{t=1} + \left[-\frac{e^{-st}}{s}\right]_{t=2}^{t=3} & \text{if } s \neq 0 \\ 2 & \text{if } s = 0 \end{cases} \\ &= \begin{cases} (1 - e^{-s} + e^{-2s} - e^{-3s})/s & \text{if } s \neq 0 \\ 2 & \text{if } s = 0 \end{cases} \\ &= \begin{cases} (1 - e^{-s})(1 + e^{-2s})/s & \text{if } s \neq 0 \\ 2 & \text{if } s = 0 \end{cases} \end{aligned}$$

The DOC is the entire s-plane.

2. (a) $\mathbf{L}\left[e^{-at}u(t)\right] = \frac{1}{(s+a)}$ with DOC $\mathrm{Re}[s] > \mathrm{Re}[a]$.

$$f(t) = e^t u(t) + e^{-2t}u(t)$$

$$F(s) = \frac{1}{s-1} + \frac{1}{s+2}$$

with DOC $\mathrm{Re}[s] > 1$. Uses linearity.

(b)

$$f(t) = u(t-\pi)e^{(t-\pi)}\cos(t) = -e^{(t-\pi)}\cos(t-\pi)u(t-\pi)$$

since $\cos(t-\pi) = -\cos(t)$.

$$\mathbf{L}\left[e^t\cos(t)u(t)\right] = \frac{(s-1)}{(s-1)^2+1} \quad \text{with DOC Re}[s] > 1.$$

Hence, by property 6 (Delay property),

$$F(s) = -\frac{(s-1)e^{-s\pi}}{(s-1)^2+1} \quad \text{with DOC Re}[s] > 1.$$

(c)

$$f(t) = \int_0^t g(\sigma)d\sigma$$

where $g(t) = t^2e^{-t}$. By Property 4,

$$F(s) = \frac{G(s)}{s}$$

Now,

$$\mathbf{L}\left[e^{-t}\right] = \frac{1}{s+1}$$

with DOC Re$[s] > -1$. By Property 5,

$$\mathbf{L}\left[te^{-t}\right] = -\frac{d}{ds}\left(\frac{1}{s+1}\right) = \frac{1}{(s+1)^2}$$

with DOC Re$[s] > -1$. Using Property 5 again,

$$\mathbf{L}\left[t^2e^{-t}\right] = -\frac{d}{ds}\left(\frac{1}{(s+1)^2}\right) = \frac{2}{(s+1)^3} = G(s)$$

Hence

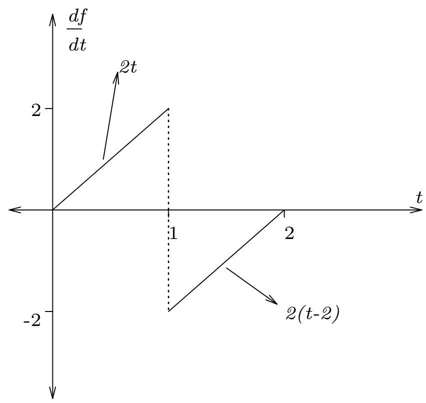
$$F(s) = \frac{2}{s(s+1)^3}$$

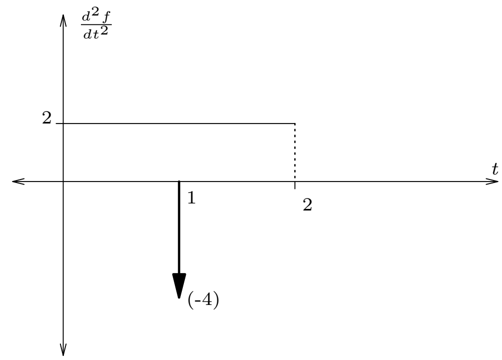
The DOC is now Re$[s] > 0$ (as also seen from the fact that f(t) is an increasing function of t and hence its integral doesn't exist).

3.

$$f(t) = t^2[u(t) - u(t-1)] + (t-2)^2[u(t-1) - u(t-2)]$$
$$f'(t) = 2t[u(t) - u(t-1)] + 2(t-2)[u(t-1) - u(t-2)]$$

$$\begin{aligned} f''(t) &= 2[u(t) - u(t-1)] - 4\delta(t-1) + 2[u(t-1) - u(t-2)] \\ &= 2[u(t) - u(t-2)] - 4\delta(t-1) \end{aligned}$$





$$\begin{aligned} \mathbf{L}[f''(t)] &= 2\mathbf{L}[u(t)] - 2\mathbf{L}[u(t-2)] - 4\mathbf{L}[\delta(t-1)] \\ &= \frac{2}{s} - \frac{2e^{-2s}}{s} - 4e^{-s} = \frac{2(1-e^{-2s})}{s} - 4e^{-s} \end{aligned}$$

Actually, the above equations are valid only for $s \neq 0$. For $s = 0$, using the definition,

$$\mathbf{L}[f''(t)] = \int_0^\infty (2[u(t) - u(t-2)] - 4\delta(t-1))dt = 0$$

Hence, the DOC is the entire complex plane. Now,

$$\mathbf{L}[f''(t)] = s\mathbf{L}[f'(t)] - f'(0-)$$

Since $f'(0-) = 0$ (note it's the limit from the *left*):

$$\mathbf{L}[f'(t)] \;=\; \left(\frac{1}{s}\right)\mathbf{L}[f''(t)] = \frac{2(1-e^{-2s})}{s^2} - \frac{4e^{-s}}{s}, \text{if } s \neq 0.$$

If $s = 0$,

$$\mathbf{L}[f'(t)] = \int_0^\infty f'(t)dt = 0$$

The DOC is again the entire complex plane.

Since $f(0-) = 0$,

$$\mathbf{L}[f(t)] = \left(\frac{1}{s}\right)\mathbf{L}[f'(t)] = \frac{2(1-e^{-2s})}{s^3} - \frac{4e^{-s}}{s^2}, \text{for } s \neq 0.$$

$$\text{For } s = 0, \quad \mathbf{L}[f(t)] = \int_0^\infty f(t)dt = \frac{2}{3}$$

The DOC is again the entire complex plane.

4. We will use the expansion of $F(s)$ into partial fractions to get f(t).

(a)

$$F(s) = \frac{s+11}{s^2-3s+4} = \frac{s+11}{(s+1)(s-4)} = \frac{A}{(s+1)} + \frac{B}{(s-4)}$$

$$A = \left[\frac{s+11}{s-4}\right]_{s=-1} = \frac{-1+11}{-1-4} = -2; \qquad B = \left[\frac{s+11}{s+1}\right]_{s=4} = \frac{4+11}{4+1} = 3$$

$$F(s) = \frac{-2}{(s+1)} + \frac{3}{(s-4)} \implies f(t) = (-2e^{-t} + 3e^{4t})u(t)$$

(b)

$$F(s) = \frac{4s+10}{s^3+6s^2+10s} = \frac{4s+10}{s[(s+3)^2+1]} = \frac{A}{s} + \frac{B(s+3)+C}{(s+3)^2+1}$$

We have

$$4s+10 = A[(s+3)^2+1] + s(Bs+3B+C)$$

Comparing coefficients of different powers of s on both sides, we get

$$\begin{aligned} 0 &= A+B \\ 4 &= 6A+3B+C \\ 10 &= 10A \end{aligned}$$

which are easily solved to give $A = 1$, $B = -1$, $C = 1$. So

$$F(s) = \frac{1}{s} + \frac{(-1)(s+3)+1}{(s+3)^2+1} \implies f(t) = u(t) + e^{-3t}[-\cos(t) + \sin(t)]u(t)$$

(c)

$$F(s) = \frac{2s^2 - s - 5}{(s-1)^2(s+3)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s+3}$$

$$B = \left[\frac{2s^2 - s - 5}{s+3}\right]_{s=1} = -1; \qquad C = \left[\frac{2s^2 - s - 5}{(s-1)^2}\right]_{s=-3} = 1$$

To find A, we let $s = 0$ in the partial fraction expansion and get

$$-\frac{5}{3} = -A + (-1) + \frac{1}{3} \implies A = 1$$

$$F(s) = \frac{1}{s-1} + \frac{(-1)}{(s-1)^2} + \frac{1}{s+3} \implies f(t) = e^t u(t) - te^t u(t) + e^{-3t} u(t)$$

5.

$$\begin{aligned} \mathcal{L}[f'(t)] &= s\mathcal{L}[f(t)] - f(0^-) = s[F(s)] \\ \mathcal{L}[f''(t)] &= s\mathcal{L}[f'(t)] - f'(0^-) = s^2[F(s)] \end{aligned}$$

Taking Laplace transform on both sides of the differential equation,

$$s^2 F(s) + \alpha s F(s) + F(s) = \frac{1}{s}$$

$$\implies \quad F(s) = \frac{1}{s(s^2 + \alpha s + 1)}$$

(a) By the Initial Value Theorem,

$$\lim_{t \to 0^+} f(t) = \lim_{s \to +\infty} [sF(s)] = \lim_{s \to +\infty} \frac{1}{(s^2 + \alpha s + 1)} = 0.$$

which is independent of α.

(b) The Final Value Theorem states that

$$\lim_{t \to +\infty} f(t) = \lim_{s \to 0^+} [sF(s)]$$

if the poles of $F(s)$ lie at 0 or strictly on the left half of the complex plane.

The poles of F(s) are at $s = 0$ and at

$$s_{1,2} = \frac{-\alpha \pm \sqrt{\alpha^2 - 4}}{2}.$$

$s_{1,2}$ are real for $\alpha \geq 2$ and $\alpha \leq -2$.
For $\alpha \geq 2$, $\sqrt{\alpha^2 - 4} < \alpha$, so $s_{1,2} < 0$.
For $\alpha \leq -2$, $\sqrt{\alpha^2 - 4} < -\alpha$, so $s_{1,2} > 0$.
Hence, the Final value theorem is valid if $\alpha \geq 2$ and not for $\alpha \leq -2$.

$s_{1,2}$ are complex for $-2 < \alpha < 2$.
For $0 < \alpha < 2$, $\mathrm{Re}[s_{1,2}] = -\frac{\alpha}{2} < 0$.
For $-2 < \alpha < 0$, $\mathrm{Re}[s_{1,2}] = -\frac{\alpha}{2} > 0$.
For $\alpha = 0$, $s_{1,2} = \pm i$. Hence, the Final value theorem is valid for $0 < \alpha < 2$ and not for $-2 < \alpha \leq 0$.

So, if $\alpha > 0$, the Final Value Theorem is valid, and

$$\lim_{t \to +\infty} f(t) \quad = \quad \lim_{s \to 0^+} \frac{1}{(s^2 + \alpha s + 1)} = 1$$

If $\alpha < 0$, then the poles satisfy $Re[s_{1,2}] > 0$. Using the partial fraction expansion

$$F(s) = \frac{A}{s} + \frac{B}{s - s_1} + \frac{C}{s - s_2}$$

we see that $f(t)$ will be of the form $f(t) = Au(t) + Be^{s_1 t}u(t) + Ce^{s_2 t}u(t)$. The magnitude of the complex exponential is $e^{Re[s_i]t}$ and goes to infinity as $t \to +\infty$. If $\alpha = 0$, then $f(t)$ contains $\sin(t)$ or $\cos(t)$, and does not have a limit as $t \to +\infty$.

(c) If $\alpha = 1$,

$$F(s) = \frac{1}{s(s^2 + s + 1)} = \frac{1}{s[(s + \frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2]} = \frac{A}{s} + \frac{Ms + N}{s^2 + s + 1}.$$

Multiply by s, evaluate at $s = 0$: $\Longrightarrow A = 1$.
Multiply by s, limit as $s \to \infty$: $\Longrightarrow 0 = A + M \quad \Longrightarrow M = -1$.
Evaluate at $s = -1$: $\Longrightarrow \; -1 = -A - M + N \quad \Longrightarrow N = -1$.
Use formula given in class ($\alpha = -\frac{1}{2}$, $\beta = \frac{\sqrt{3}}{2}$, $M = N = -1$), to get

$$F(s) = \frac{1}{s} + \frac{-s - 1}{(s + \frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} \quad \Longrightarrow f(t) = u(t) + e^{-\frac{t}{2}} \left[-\cos(\frac{\sqrt{3}}{2}t) - \frac{1}{\sqrt{3}}\sin(\frac{\sqrt{3}}{2}t) \right] u(t)$$