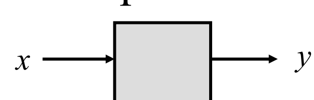
Lecture 9

- System defined by a differential equation.
- Transfer function of an LTI, causal system.
- Cascaded systems and other block diagram interconnections.

System defined by a differential equation



Assume all signals are zero for $t < 0$.

For $t \geq 0$, the input and output are related by

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_0 y(t) = b_m \frac{d^m x}{dt^m} + \cdots + b_1 \frac{dx}{dt} + b_0 x(t)$$

Using Laplace (zero initial conditions), we get

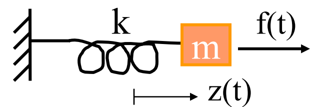
$$\left(s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0\right)Y(s) = \left(b_m s^m + \cdots + b_1 s + b_0\right)X(s)$$

$$Y(s) = \frac{\left(b_m s^m + \cdots + b_1 s + b_0\right)}{\left(s^n + \cdots + a_1 s + a_0\right)} X(s) = H(s)X(s)$$

$$Y(s) = H(s)X(s)$$

In the Laplace domain, the input and output are related by a simple multiplication by a certain function *H(s)*. This is a first example of a **transfer function** of an LTI system.

Example: mass-spring system



$$m\frac{d^2z}{dt^2} = f(t) - k\,z \;\longrightarrow\; m\frac{d^2z}{dt^2} + k\,z = f(t)$$

$$\left(ms^2 + k\right)Z(s) = F(s) \;\longrightarrow\; Z(s) = \frac{1}{\left(ms^2 + k\right)}F(s)$$

Transfer function of an LTI, causal system

Recall: $y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(t-\sigma)x(\sigma)d\sigma$

Theorem: Given an LTI, causal system, with impulse response $h(t)$; suppose the input satisfies $x(t) = 0$ for $t < 0$. Let $X(s) = \mathcal{L}[x(t)]$, $H(s) = \mathcal{L}[h(t)]$, $Y(s) = \mathcal{L}[y(t)]$. Then

$$Y(s) = H(s)X(s)$$

for any s in the DOC of both $H(s)$ and $X(s)$.

Proof:

Using causality, $h(t)=0$ for $t<0$. Therefore:

$$y(t)=\int_{-\infty}^{\infty} h(\sigma)x(t-\sigma)d\sigma = \int_{0-}^{\infty} h(\sigma)x(t-\sigma)d\sigma$$

$$Y(s)=\int_{0-}^{\infty} e^{-st}y(t)dt = \int_{0-}^{\infty} e^{-st}\left[\int_{0-}^{\infty} h(\sigma)x(t-\sigma)d\sigma\right]dt$$

$$\underbrace{=}_{\text{Change order of integration}} \int_{0-}^{\infty} h(\sigma)\underbrace{\left[\int_{0-}^{\infty} e^{-st}x(t-\sigma)dt\right]}_{\mathcal{L}\left[u(t-\sigma)x(t-\sigma)\right]}d\sigma$$

Note: by hypothesis, $x(t)=u(t)x(t)$. ($x(t)=0$ for $t<0$).

$$\underbrace{=}_{\text{Delay property}} \int_{0-}^{\infty} h(\sigma)e^{-s\sigma}X(s)d\sigma = H(s)X(s).$$

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Example:

$h(t)=\delta(t)-u(t)e^{-t}$; $x(t)=t\ u(t)$. Find $y(t)$.

$$H(s)=1-\frac{1}{s+1}=\frac{s}{s+1}; \qquad X(s)=\frac{1}{s^2}.$$

$$Y(s)=H(s)X(s)=\frac{s}{s+1}\cdot\frac{1}{s^2}=\frac{1}{s(s+1)}=\frac{1}{s}-\frac{1}{s+1}$$

$$\Rightarrow y(t)=u(t)\left[1-e^{-t}\right].$$

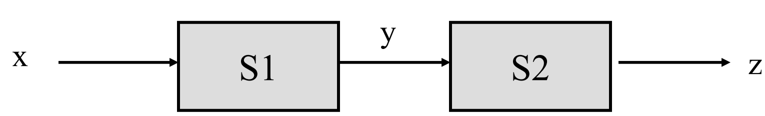
Easier than convolution!

Remarks:

- As defined in this course, the Laplace transform only considers times $t \geq 0$. "One-sided" transform.
- As such, it is used to study signals and LTI systems which start operating at a given time (for convenience, chosen to be 0).
- If a problem involves signals starting at $t = -\infty$, or non-causal systems, we do not apply the previous result with Laplace. At this point in the course, we can only approach it in the time domain, via convolutions.
- Later on, we will introduce Fourier transforms that can be used to study signals involving negative times.

Definition: The Laplace transform $H(s) = \mathcal{L}\left[h(t)\right]$ of the system impulse response function is called the transfer function (or system function) of the LTI, causal system.

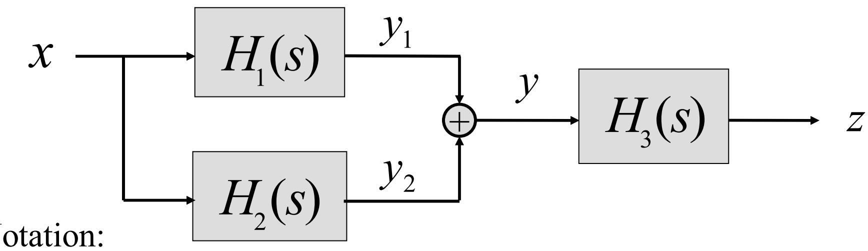
Transfer function of a cascaded system



$$z = h_2 * y = h_2 * h_1 * x$$

$$\rightarrow Z(s) = H_2(s)Y(s) = \underbrace{H_2(s)H_1(s)}_{H_{12}(s)}X(s)$$

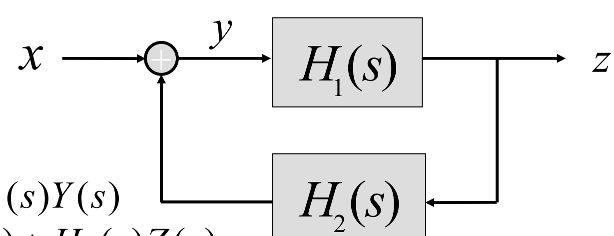
More generally, we can build “block-diagrams”



Notation:
$H(s)$ inside a box means an LTI system with this transfer function.
The "adder" block $\oplus$ represents $y(t) = y_1(t) + y_2(t)$.

$$\begin{aligned} Z(s) = H_3(s)Y(s) &= H_3(s)\left[Y_1(s) + Y_2(s)\right] \\ &= H_3(s)\left[H_1(s)X(s) + H_2(s)X(s)\right] \\ &= \underbrace{H_3(s)\left[H_1(s) + H_2(s)\right]}_{\substack{H(s),\text{ overall} \\ \text{transfer function.}}} X(s). \end{aligned}$$

Feedback interconnection



$$Z(s) = H_1(s)Y(s)$$
$$Y(s) = X(s) + H_2(s)Z(s)$$
$$\rightarrow Z(s) = H_1(s)X(s) + H_1(s)H_2(s)Z(s)$$
$$\rightarrow \left[1 - H_1(s)H_2(s)\right]Z(s) = H_1(s)X(s)$$

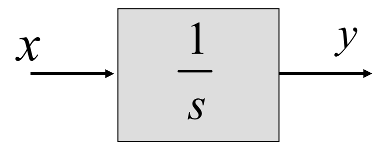
$$Z(s) = \underbrace{\frac{H_1(s)}{1 - H_1(s)H_2(s)}}_{\substack{H(s),\text{ overall} \\ \text{transfer function.}}} X(s).$$

Main lesson:
Use simple algebra to study complex systems.

Example: build a transfer function from simple blocks.

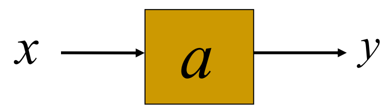
Integrator block: $y(t) = \int_{0-}^{t} x(\sigma)\,d\sigma$

In Laplace, $Y(s) = \frac{1}{s} X(s)$



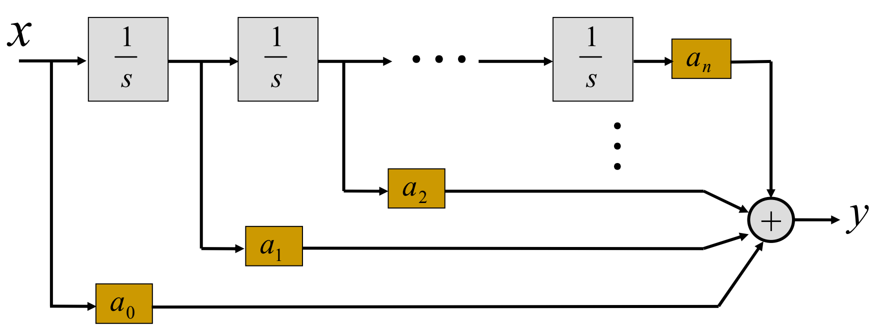
Amplifier: $y(t) = a\,x(t)$

In Laplace, $Y(s) = a\,X(s)$



There exist circuits (e.g., based on OP-AMPs) that approximately implement these basic functions.

Now, we use them to build a more complicated transfer function. An “analog computer”.



$$Y(s) = a_0 X(s) + a_1 \frac{1}{s} X(s) + a_2 \frac{1}{s^2} X(s) + \cdots + a_n \frac{1}{s^n} X(s)$$

$$= \left(a_0 + \frac{a_1}{s} + \cdots + \frac{a_n}{s^n} \right) X(s) = \underbrace{\frac{a_0 s^n + a_1 s^{n-1} + \cdots + a_n}{s^n}}_{H(s)} X(s)$$

We can build any numerator polynomial by choosing $a_0, \ldots, a_n$

Also, using feedback one can build different denominators.