Lecture 5: Convolutions and Applications

Recall: Input-output relation of a linear system

$$y(t) = T\left[x(t)\right]$$

Let the impulse response function be $h(t,\tau) = T\left[\delta(t-\tau)\right]$.

For a given input $x(t)$, the corresponding output is

$$y(t) = T\left[x(t)\right] = \int_{-\infty}^{\infty} h(t,\sigma)x(\sigma)d\sigma$$

SUPERPOSITION INTEGRAL

For the time invariant case: $h(t,\tau) = h(t-\tau)$

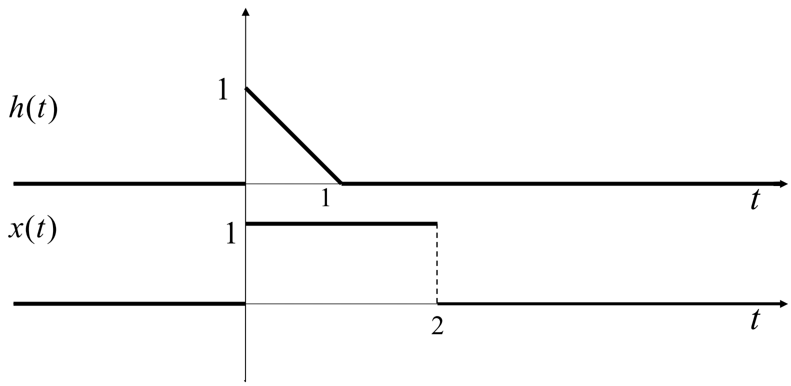
$$\Rightarrow \quad y(t) = \int_{-\infty}^{\infty} h(t-\sigma)x(\sigma)d\sigma$$

This operation is called the **convolution** of the functions $h(t)$ and $x(t)$. Notation:

$$y(t) = h(t) * x(t) \quad \text{or} \quad y = h * x$$

Given two functions of one variable, the convolution operation returns another function of one variable.

Example: Given $h(t)$, $x(t)$ below, find $y(t) = h(t) * x(t)$.



$$y(t) = \int_{-\infty}^{\infty} h(t-\sigma)x(\sigma)d\sigma$$

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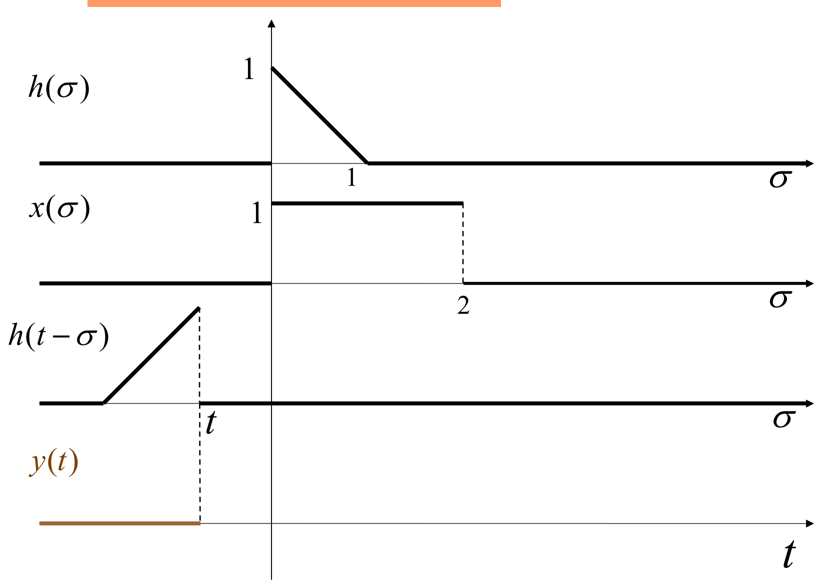
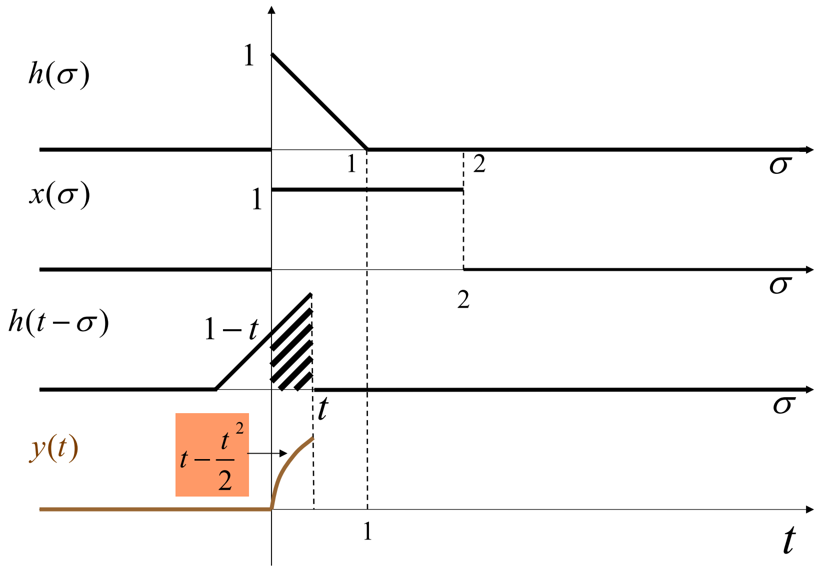
$t < 0$



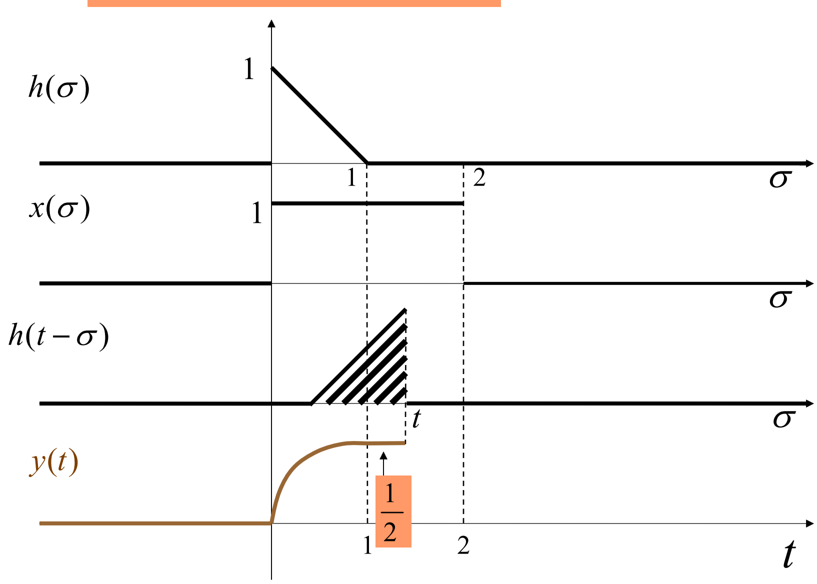
$$y(t) = \int_{-\infty}^{\infty} h(t-\sigma)x(\sigma)d\sigma$$

$0 \le t < 1$



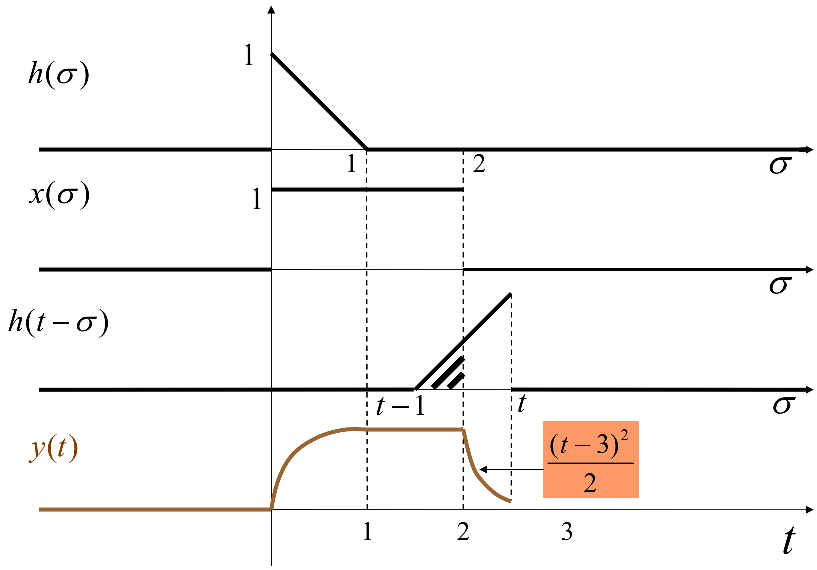
$$y(t) = \int_{-\infty}^{\infty} h(t-\sigma)x(\sigma)d\sigma$$

$1 \le t < 2$



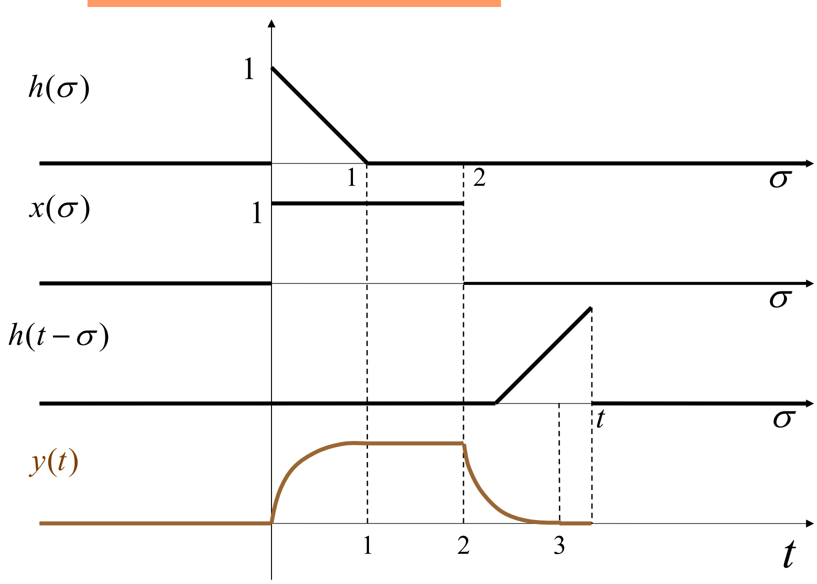
$$y(t) = \int_{-\infty}^{\infty} h(t-\sigma)x(\sigma)d\sigma$$

$2 \le t < 3$



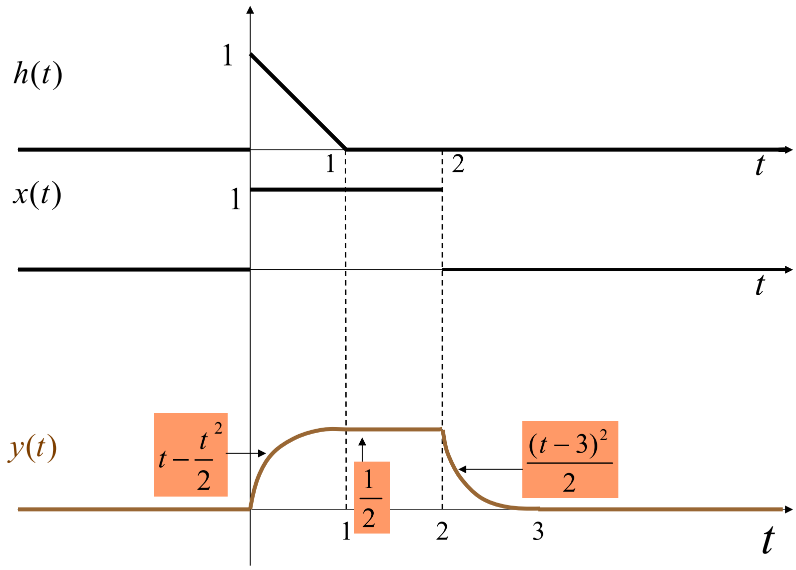
$$y(t) = \int_{-\infty}^{\infty} h(t-\sigma)x(\sigma)d\sigma$$

$3 \le t$



$$y(t) = \int_{-\infty}^{\infty} h(t-\sigma)x(\sigma)d\sigma$$

Summary



Properties of convolution.

$$[f * g](t) = \int_{-\infty}^{\infty} f(t-\sigma)g(\sigma)d\sigma$$

1) Associative: $(f * g) * h = f * (g * h)$

2) Commutative: $f * g = g * f$

3) Distributive: $f * (g + h) = f * g + f * h$

4) Unit of convolution: $f * \delta = f$

Algebraic properties of a product!

Properties of convolution → Proof

1) Associative: later

2) Commutative:

$$[f * g](t) = \int_{-\infty}^{\infty} f(t-\sigma) g(\sigma) d\sigma$$

$$\underbrace{=}_{t-\sigma=\tau} \int_{\infty}^{-\infty} f(\tau) g(t-\tau)(-d\tau)$$

$$= \int_{-\infty}^{\infty} g(t-\tau) f(\tau) d\tau = [g * f](t)$$

Properties of convolution → Proof

3) Distributive:

$$[f * (g+h)](t) = \int_{-\infty}^{\infty} f(t-\sigma)\big(g(\sigma)+h(\sigma)\big) d\sigma$$

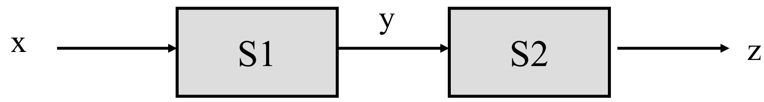
$$= \int_{-\infty}^{\infty} f(t-\sigma) g(\sigma) d\sigma + \int_{-\infty}^{\infty} f(t-\sigma) h(\sigma) d\sigma$$

$$= [f * g](t) + [f * h](t)$$

4) Unit of convolution: the Dirac delta function.

$$[f * \delta](t) = \int_{-\infty}^{\infty} f(t-\sigma) \delta(\sigma) d\sigma = f(t)$$

Impulse response of a cascaded system



Consider two LTI systems, S1 and S2.

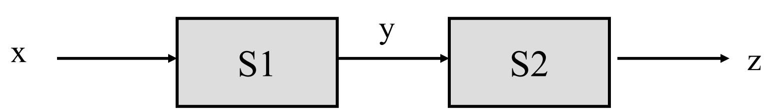
S1 has input $x(t)$, output $y(t)$, and impulse response $h_1(t)$.

S2 has input $\mathrm{y}(t)$, output $\mathrm{z}(t)$, and impulse response $h_2(t)$

The cascade has input $x(t)$, output $\mathrm{z}(t)$. It is easy to see, using the definitions, that it is also an LTI system.

We wish to find its impulse response $h_{1,2}(t)$.

Impulse response of a cascaded system



Apply an input $x(t) = \delta(t)$ to the cascade

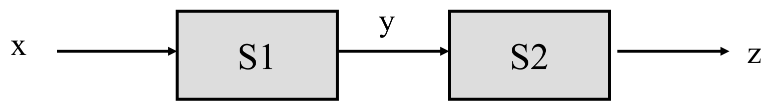
By definition, $\mathrm{y}(t) = h_1(t)$ (impulse response of S1)

Then for S2 we have $z = h_2 * y = h_2 * h_1$

Conclusion: The impulse response of the cascade is the **convolution** of the impulse responses of each stage.

$$h_{1,2} = h_2 * h_1$$

Associativity of convolution



S1,S2 LTI. We have seen that $h_{1,2} = h_2 * h_1$

Applying now a generic input $x(t)$, we have

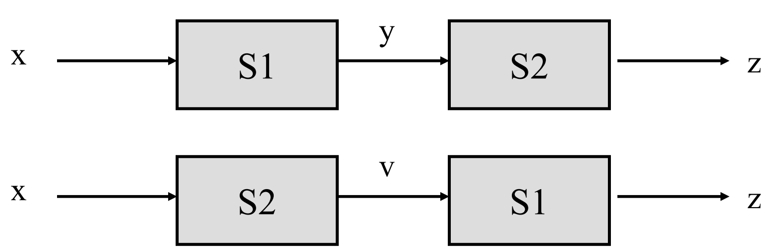
$$z = (h_2 * h_1) * x.$$

Looking at S1 with that input, we have $y = h_1 * x$

Now S2 gives $z = h_2 * y = h_2 * (h_1 * x)$.

Conclusion: $(h_2 * h_1) * x = h_2 * (h_1 * x)$.

A consequence of commutativity

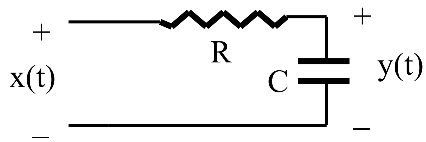


Since $h_2 * h_1 = h_1 * h_2$, the above cascades of LTI systems have the same impulse response.

Therefore, they are equivalent: LTI systems **commute.**

Note: they are equivalent only as mappings from x to z. The intermediate signals y and v will **not** be the same.

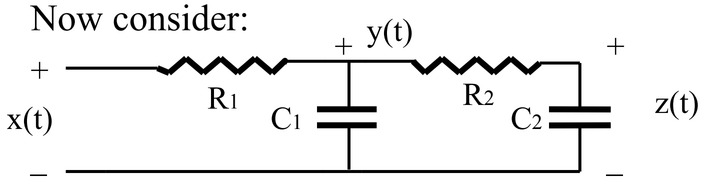
A cascaded circuit



Recall: $h(t) = \alpha e^{-\alpha t} u(t)$

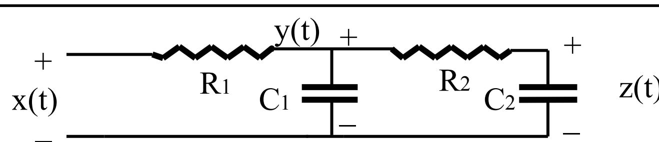
$$\alpha = \frac{1}{RC}$$

Now consider:



True or false: is the impulse response of this circuit equal to $h(t) = \left[\alpha_1 e^{-\alpha_1 t} u(t)\right] * \left[\alpha_2 e^{-\alpha_2 t} u(t)\right]$?

Answer: False!



Reason: when we derived model of the first stage, we assumed the same current went through R_1 and C_1. This is not true here, so $h_1(t) \neq \alpha_1 e^{-\alpha_1 t} u(t)$. It is true that $h_2(t) = \alpha_2 e^{-\alpha_2 t} u(t)$.

- **Q:** So what is the value of input-output system models if we can't break complex systems into cascades of simpler parts?
- **A:** We can, at least approximately, when some simplifying assumptions hold. e.g., when $R_2 >> R_1$ in the above circuit.
- Complex engineering systems are **designed** so that such approximations hold, and we can understand them.
- The decomposition strategy would not work for a "random" system, or one "designed" by nature. For example, complex biological or economic systems are much harder to study!