EE 102 # Homework #1 Solutions **Winter 2004**

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1. Review of integration

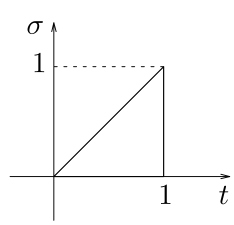
   (a) For the first integral, we define $u = t$ and $dv = e^{at} \Rightarrow v = \frac{1}{a}e^{at}$ to use the integration by parts. So we have:

$$\begin{aligned}
\int_0^1 te^{at}dt &= \left[\frac{1}{a}te^{at}\right]_0^1 - \int_0^1 \frac{1}{a}e^{at}dt \\
&= (\frac{1}{a}e^a - 0) - \left[\frac{1}{a^2}e^{at}\right]_0^1 \\
&= \frac{1}{a}e^a - \frac{1}{a^2}e^a + \frac{1}{a^2} \\
&= e^a(\frac{1}{a} - \frac{1}{a^2}) + \frac{1}{a^2}.
\end{aligned}$$

   The second integral can be solved by integration by parts if you define $u = \sigma$ and $dv = \cos(t-\sigma)d\sigma \Rightarrow v = -\sin(t-\sigma)$. Therefore we have:

$$\begin{aligned}
\int_0^t \sigma\cos(t-\sigma)d\sigma &= [-\sigma\sin(t-\sigma)]_0^t - \int_0^t -\sin(t-\sigma)d\sigma \\
&= 0 + [\cos(t-\sigma)]_0^t \\
&= 1 - \cos(t).
\end{aligned}$$

   (b) One way is to switch the order of integration in the double integral. For this, one must be careful to specify the right domain of integration: when t varies from 0 to 1 and σ varies from 0 to t, this generates the triangle depicted in the figure.



   To generate the same triangle with the opposite order of variables, σ must vary from 0 to 1 and t from σ to 1. So:

$$\begin{aligned}\text{LHS} &= \int_0^1 \left(\int_\sigma^1 f(\sigma)dt\right) d\sigma \\ &= \int_0^1 \left(f(\sigma)\int_\sigma^1 dt\right) d\sigma \\ &= \int_0^1 (1-\sigma)f(\sigma)d\sigma \\ &= \text{RHS.}\end{aligned}$$

Another way would be to integrate by parts, defining

$$g(t) := \int_0^t f(\sigma)d\sigma.$$

Note that $\frac{dg}{dt} = f(t)$. Then we have

$$\begin{aligned}\text{LHS} &= \int_0^1 g(t)dt \\ &= [tg(t)]_0^1 - \int_0^1 tf(t)dt \\ &= g(1) - \int_0^1 tf(t)dt \\ &= \int_0^1 f(\sigma)d\sigma - \int_0^1 \sigma f(\sigma)d\sigma \qquad \text{(dummy variable renamed)} \\ &= \int_0^1 (1-\sigma)f(\sigma)d\sigma \\ &= \text{RHS.}\end{aligned}$$

(c) We distinguish five different regions as following:

$$f(t) = \begin{cases} 0 & t \le 0 \\ t & 0 < t \le 1 \\ -t+2 & 1 < t \le 3 \\ t-4 & 3 < t \le 4 \\ 0 & 4 < t \end{cases}$$

Now we start integrating the function, region by region:

R1 $g(t) = \int_{-\infty}^t 0 d\sigma = 0$

R2 $g(t) = g(0) + \int_0^t \sigma d\sigma = 0 + \left[\frac{\sigma^2}{2}\right]_0^t = \frac{t^2}{2}$

R3

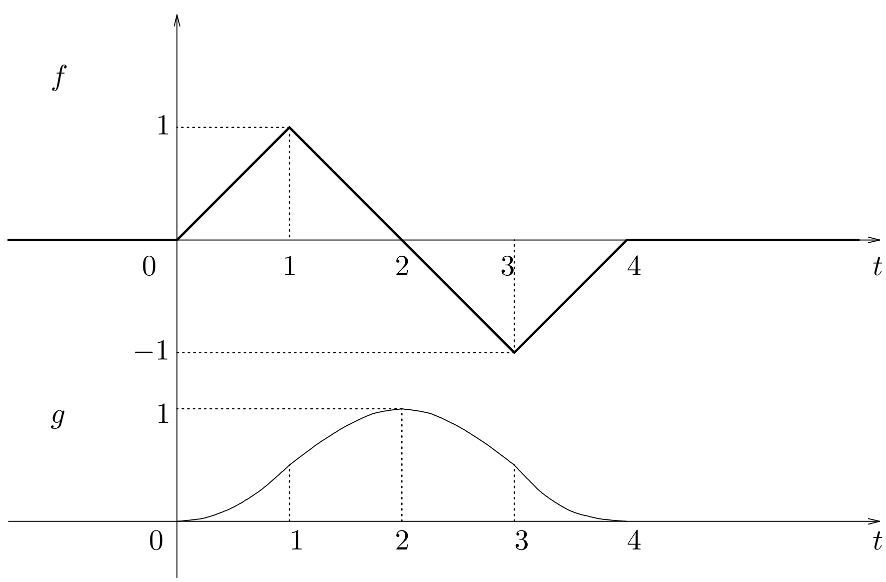
$$\begin{aligned} g(t) &= g(1) + \int_1^t (-\sigma + 2) d\sigma \\ &= \frac{1}{2} + \left[-\frac{\sigma^2}{2} + 2\sigma \right]_1^t \\ &= -\frac{t^2}{2} + 2t - 1 \end{aligned}$$

R4

$$\begin{aligned} g(t) &= g(3) + \int_3^t (\sigma - 4) d\sigma \\ &= (-\frac{9}{2} + 6 - 1) + \left[\frac{\sigma^2}{2} - 4\sigma \right]_3^t \\ &= \frac{t^2}{2} - 4t + 8 \end{aligned}$$

R5

$$\begin{aligned} g(t) &= g(4) + \int_4^t 0 d\sigma \\ &= g(4) + 0 \\ &= 0 \end{aligned}$$



2. Review of complex numbers

   (a) (1)

$$
\begin{aligned}
e^{-\frac{8}{3}\pi i} &= \cos(-\frac{8}{3}\pi) + i\sin(-\frac{8}{3}\pi) \\
&= -\frac{1}{2} - i\frac{\sqrt{3}}{2}
\end{aligned}
$$

   (2)

$$
(i)^{-20} = (\frac{1}{i})^{20} = (-i)^{20} = i^{20} = (-1)^{10} = 1
$$

   (b) (1) We have: $mag(-1-i) = \sqrt{1+1} = \sqrt{2}$ and $phase(-1-i) = Arctan(\frac{-1}{-1}) + \pi = \frac{5\pi}{4}$. So $\alpha = 2e^{i\frac{5\pi}{4}}$.

   (2) We have: $mag(1-i) = \sqrt{1+1} = \sqrt{2}$ and $phase(1-i) = Arctan(\frac{-1}{1}) = -\frac{\pi}{4}$. So $\beta = e^{-i\frac{\pi}{4}}$.

   (c)

$$
\begin{aligned}
\frac{\alpha^2}{\overline{\beta}} &= \frac{4e^{i\frac{5\pi}{2}}}{e^{i\frac{\pi}{4}}} \\
&= 4e^{i(\frac{5\pi}{2}-\frac{\pi}{4})} \\
&= 4e^{i\frac{9\pi}{4}} \\
&= 4e^{\frac{\pi}{4}} \\
&= 2\sqrt{2} + 2\sqrt{2}i
\end{aligned}
$$

   (d) $z^3 + 8 = 0 \Rightarrow z = \sqrt[3]{-8}$; So in fact the roots are the third roots of -8. Now noting that $-8 = 8e^{\pi i}$ and that $\sqrt[3]{8} = 2$ we conclude that the roots are $z_1 = 2e^{i\frac{\pi}{3}}, z_2 = 2e^{i\pi} = -2$ and $z_3 = 2e^{-i\frac{\pi}{3}}$.

3. Differential equations:

   (a) Based on the method we have seen in class we have these series of equations:

$$
\begin{aligned}
\frac{dx}{dt} + x &= e^{-t}\frac{d(xe^t)}{dt} \Rightarrow \\
e^{-t}\frac{d(xe^t)}{dt} &= te^{-t} \Rightarrow \\
\frac{d(xe^t)}{dt} &= t \Rightarrow \\
x(t)e^t &= x(0) + \frac{t^2}{2} \Rightarrow \\
x(t) &= e^{-t}\frac{t^2}{2} + x(0)e^{-t} \\
&= e^{-t}(\frac{t^2}{2} + 3)
\end{aligned}
$$

(b) We know that $\frac{dy}{dt} + y = e^{-t}\frac{d(e^t y(t))}{dt}$. So

$$\begin{aligned}
e^{-t}\frac{d(e^t y(t))}{dt} &= \frac{dx}{dt} - x \Rightarrow \\
\frac{d(e^t y(t))}{dt} &= e^t\big(x'(t) - x(t)\big) \Rightarrow \text{(integrate from 0 to } t\text{:)} \\
[e^\sigma y(\sigma)]_0^t &= \int_0^t e^\sigma x'(\sigma)d\sigma - \int_0^t e^\sigma x(\sigma)d\sigma \Rightarrow \text{(integrating by parts:)} \\
e^t y(t) - 0 &= [e^\sigma x(\sigma)]_0^t - \int_0^t x(\sigma)e^\sigma d\sigma - \int_0^t e^\sigma x(\sigma)d\sigma \\
&= e^t x(t) - 0 - 2\int_0^t e^\sigma x(\sigma)d\sigma \Rightarrow \\
y(t) &= x(t) - 2e^{-t}\int_0^t e^\sigma x(\sigma)d\sigma.
\end{aligned}$$

4. System properties:

(a)

Nonlinear: $T[x - x] = 1 + \int_{-\infty}^{t+1} 0 d\sigma = 1$ violating the linearity definition $T[0] = T[x - x] = T[x] - T[x] = 0$.

Time Invariant: $T[x(t-\tau)] = 1+\int_{-\infty}^{t+1} x(\sigma-\tau)d\sigma = \int_{-\infty}^{t+1-\tau} x(u)du = y(t-\tau)$.

Noncausal: The output at time t, depends on the input at times after time t, e.g., on $x(t + 0.3)$.

(b)

Linear: $T[ax_1(t) + bx_2(t)] = ax_1(-t) + bx_2(-t) = aT[x_1] + bT[x_2]$.

Time Varying: Let $x_1(t) = t$. Then $y_1(t) = -t$. Now if we delay x_1 by 1 we get $x_2(t) = x_1(t - 1) = t - 1$. Now $T[x_2(t)] = -t - 1$ while the delayed version of y_1 is $y_1(t - 1) = -t + 1$.

Noncausal: $y(-2) = x(2)$. So this system is not causal.

(c)

Nonlinear: $T[2x(t)] = e^{2x(t)} \neq 2e^{x(t)} = 2y(t)$.

Time Invariant: $T[x(t - \tau)] = e^{x(t-\tau)} = y(t - \tau)$.

Causal: $y(t)$ only depends on the input at time t.

(d)

Linear: $T[ax_1 + bx_2] = (t^2 + 1)(ax_1(t - 1) + bx_2(t - 1)) = a(t^2 + 1)x_1(t - 1) + b(t^2 + 1)x_2(t - 1) = aT[x_1] + bT[x_2]$.

Time Varying: $T[x(t-\tau)] = (t^2+1)x(t-\tau-1) \neq ((t-\tau)^2+1)x(t-\tau-1) = y(t - \tau)$.

Causal: This system is obviously causal because y at time t only depends on the previous input at time $t - 1$.

(e)

Linear: $T[ax_1+bx_2] = \int_{t-2}^{t}(t-\sigma)(ax_1(\sigma)+bx_2(\sigma))d\sigma = a\int_{t-2}^{t}(t-\sigma)x_1(\sigma)d\sigma + b\int_{t-2}^{t}(t-\sigma)x_2(\sigma)d\sigma = aT[x_1] + bT[x_2]$.

Time Invariant: $y_1(t) = T[x(t)] = \int_{t-2}^{t}(t-\sigma)x(\sigma)d\sigma$. Now if we delay input by τ we will have: $y_2(t) = T[x(t-\tau)] = \int_{t-2}^{t}(t-\sigma)(x(\sigma-\tau)d\sigma$. Doing the substitution of variables $u = \sigma - \tau$, we have: $y_2(t) = \int_{t-\tau-2}^{t-\tau}(t-\tau-u)x(u)du = y_1(t-\tau)$.

Causal: x at present time, t, only depends on the input from time $t-2$ to present time.