EE 102 | # Homework #4 Solutions | **Winter 2004**

Professor Paganini

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1. (a)

$$\begin{aligned}
f(t) &= te^{-2t}\sin(3t) \\
&= e^{-2t}(t\sin(3t)) \\
\text{Let } g(t) &= t\sin(3t) \\
\text{From property (5), } G(s) &= \frac{-d}{ds}\left(\frac{3}{s^2+9}\right) = \frac{6s}{(s^2+9)^2}, \quad Re[s] > 0; \\
\text{From property (2), } F(s) &= G(s+2) = \frac{6(s+2)}{((s+2)^2+9)^2}, \quad Re[s] > -2; \\
\Rightarrow F(s) &= \frac{6(s+2)}{(s^2+4s+13)^2}, \quad Re[s] > -2.
\end{aligned}$$

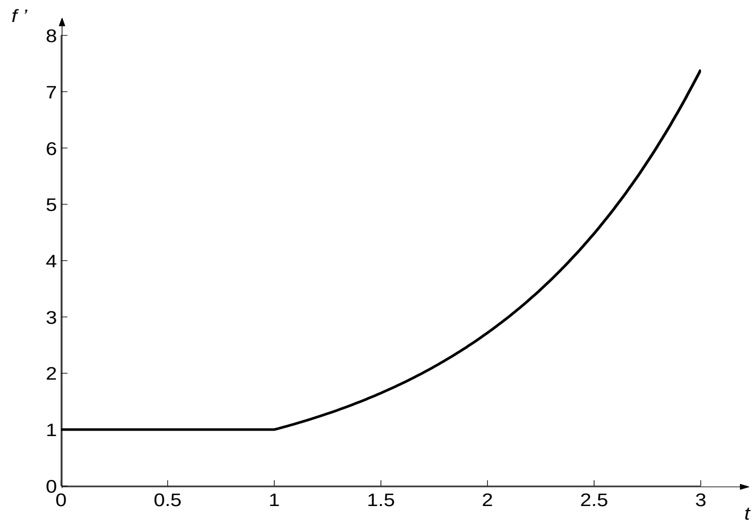
(b)

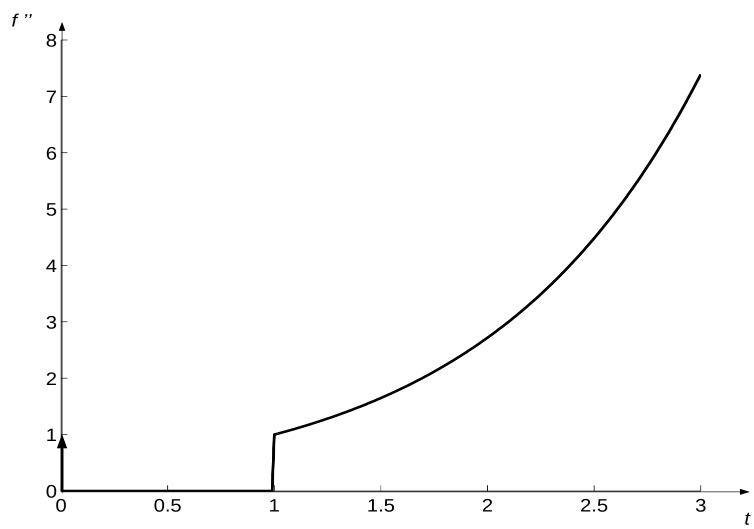
$$\begin{aligned}
f(t) &= \int_0^{t-1} \tau\sin(2\tau)d\tau \quad \text{for } t > 1, \quad \text{and 0 otherwise} \\
&= u(t-1)\int_{0^-}^{t-1} \tau\sin(2\tau)u(\tau)d\tau. \\
\text{Let } g(t) &= t\sin(2t)u(t) \\
\text{and } h(t) &= \int_{0^-}^{t} g(\tau)d\tau; \\
\text{thus } f(t) &= h(t-1)u(t-1). \\
\text{From property (5), } G(s) &= \frac{-d}{ds}\left(\frac{2}{s^2+4}\right) = \frac{4s}{(s^2+4)^2}\ , Re[s] > 0.
\end{aligned}$$

$$\begin{aligned}
\text{From property (4), } H(s) &= \frac{G(s)}{s} = \frac{4}{(s^2+4)^2}\ , Re[s] > 0. \\
\text{From property (6), } F(s) &= e^{-s}H(s) \\
\Rightarrow F(s) &= \frac{4e^{-s}}{(s^2+4)^2}\ , Re[s] > 0.
\end{aligned}$$

2. (a)

$$\begin{aligned} f(t) &= t[u(t) - u(t-1)] + e^{t-1}u(t-1) \\ \frac{df}{dt} &= u(t) - u(t-1) + e^{t-1}u(t-1) \\ \frac{d^2f}{dt^2} &= \delta(t) + e^{t-1}u(t-1) \end{aligned}$$





(b)

$$\begin{aligned} \frac{d^2 f}{dt^2} &= \delta(t) + e^{t-1}u(t-1) \\ \Rightarrow L\{\frac{d^2 f}{dt^2}\} &= 1 + \frac{e^{-s}}{s-1}\ , Re[s] > 1. \end{aligned}$$

Note that the initial conditions $\frac{df}{dt}(0-) = f(0-) = 0$, so we have

$$\frac{df}{dt} = \int_{0-}^{t} \frac{d^2 f}{dt^2} dt, \qquad f(t) = \int_{0^-}^{t} \frac{df}{dt} dt.$$

This means we can obtain the transforms using the integration property (3):

$$L\{\frac{df}{dt}\} = \frac{1}{s} L\{\frac{d^2 f}{dt^2}\} = \frac{1}{s} + \frac{e^{-s}}{s(s-1)}; \qquad \text{DOC:}Re[s] > 1;$$
$$L\{f(t)\} = \frac{1}{s} L\{\frac{df}{dt}\} = \frac{1}{s^2} + \frac{e^{-s}}{s^2(s-1)} \qquad \text{DOC:}Re[s] > 1.$$

3. (a)

$$\begin{aligned} F(s) &= \frac{s+1}{s(s+2)} \\ &= \frac{1}{2}(\frac{1}{s} + \frac{1}{s+2}) \\ f(t) &= [\frac{1}{2}(1 + e^{-2t})]u(t). \end{aligned}$$

(b)

$$\begin{aligned} F(s) &= \frac{1}{(s+2)^2(s+1)} \\ &= \frac{-1}{(s+2)} + \frac{-1}{(s+2)^2} + \frac{1}{(s+1)} \\ f(t) &= [(-1-t)e^{-2t} + e^{-t}]u(t). \end{aligned}$$

(c)

$$\begin{aligned} F(s) &= \frac{3s}{(s^2+1)(s^2+4)} \\ &= \frac{s}{s^2+1} + \frac{-s}{s^2+4} \\ f(t) &= [\cos(t) - \cos(2t)]u(t). \end{aligned}$$

4. (a) To find the impulse response, take $x(t) = \delta(t)$ and solve the differential equation for $y(t) = h(t)$:

$$\frac{d^2h}{dt^2} + 5\frac{dh}{dt} + 6h(t) = \delta(t)$$

To solve this equation we take Laplace transforms on both sides. Since the initial conditions $h(0-)$ and $\frac{dh}{dt}(0-)$ are zero, we have from the derivative property

$$L\{\frac{dy}{dt}\} = sH(s), \qquad L\{\frac{d^2h}{dt^2}\} = s^2H(s).$$

So we get

$$\begin{aligned} s^2H(s) + 5sH(s) + 6H(s) &= 1 \\ \Longrightarrow H(s) &= \frac{1}{s^2+5s+6} \\ &= \frac{1}{(s+2)} + \frac{-1}{(s+3)} \\ \Longrightarrow h(t) &= [e^{-2t} - e^{-3t}]u(t). \end{aligned}$$

(b) We apply the same reasoning taking now $x(t) = u(t)$, therefore $X(s) = \frac{1}{s}$. The solution to the equation will now be $g(t)$. Using Laplace, we get

$$\begin{aligned} s^2G(s) + 5sG(s) + 6G(s) &= \frac{1}{s} \\ \Longrightarrow G(s) &= \frac{1}{s(s^2+5s+6)}. \end{aligned}$$

Note that we do not have to compute the inverse LT to compute the initial and final values of $g(t)$. Instead, we can apply I.V.T and F.V.T respectively on $G(s)$:

Applying the I.V.T:

$$\begin{aligned} \lim_{t\to 0^+} g(t) &= \lim_{s\to\infty} sG(s) \\ &= \lim_{s\to\infty} \frac{1}{(s^2+5s+6)} \\ \Rightarrow g(0^+) &= 0 \end{aligned}$$

Since all poles of G(s) are in $Re[s] < 0$ and at $s = 0$,
We can apply the F.V.T:

$$
\begin{aligned}
\lim_{t\to\infty} g(t) &= \lim_{s\to 0^+} sG(s) \\
&= \lim_{s\to 0^+} \frac{1}{(s^2+5s+6)} \\
\Rightarrow g(\infty) &= \frac{1}{6}
\end{aligned}
$$

5. (a) Since the switch is ON at $t = 0$,
$\rightarrow v(t) = Eu(t)$
The differential equation becomes:

$$
\begin{aligned}
L\frac{di}{dt} + Ri(t) &= Eu(t) \\
\text{Taking LT of both sides:} & \\
L(sI(s) - i(0^-)) + RI(s) &= \frac{E}{s} \\
\Rightarrow I(s) &= \frac{E}{L}\frac{1}{s(s+\frac{R}{L})} \\
&= \frac{E}{R}\left(\frac{1}{s} + \frac{-1}{s+\frac{R}{L}}\right) \\
\Rightarrow i(t) &= \frac{E}{R}[1 - e^{-\frac{R}{L}t}]u(t)
\end{aligned}
$$

(b) The initial current value for this period is the final value of the current of the previous period:
Let $i_1(t)$ and $i_2(t)$ denote the current in the first and second periods respectively. Also let $v_2(t)$ denote the voltage in the second period.

$$
\begin{aligned}
i_2(0^-) &= \lim_{t\to\infty} i_1(t) \\
&= \frac{E}{R} \\
\text{and } v_2(t) &= 0
\end{aligned}
$$

The system equation becomes:

$$
\begin{aligned}
L\frac{di_2}{dt} + Ri_2(t) &= v_2(t) \\
\text{Taking LT of both sides:} & \\
L(sI_2(s) - i_2(0^-)) + RI_2(s) &= 0 \\
LsI_2(s) - \frac{EL}{R} + RI_2(s) &= 0 \\
I_2(s) &= \frac{E}{R}\,(\frac{1}{s+\frac{R}{L}}) \\
\Rightarrow i_2(t) &= [\frac{E}{R}\,e^{-\frac{R}{L}t}]u(t)
\end{aligned}
$$

Where $t = 0$ is the time of switching off the source.