LECTURE 14

LECTURE NOTES: FEBRUARY 26, 2003

OUTLINE

REVIEW

- Capacitor Equivalent Circuits
- Inductor Equivalent Circuits
- R-L and R-C Natural Response
- R-L and R-C Step Response

OPERATIONAL R-C CIRCUITS

- Operational Amplifiers may be used to implement Integrator and Differentiator functions. These are required for several important signal generation and signal processing problems.
- Lets consider the Operational Integrator:

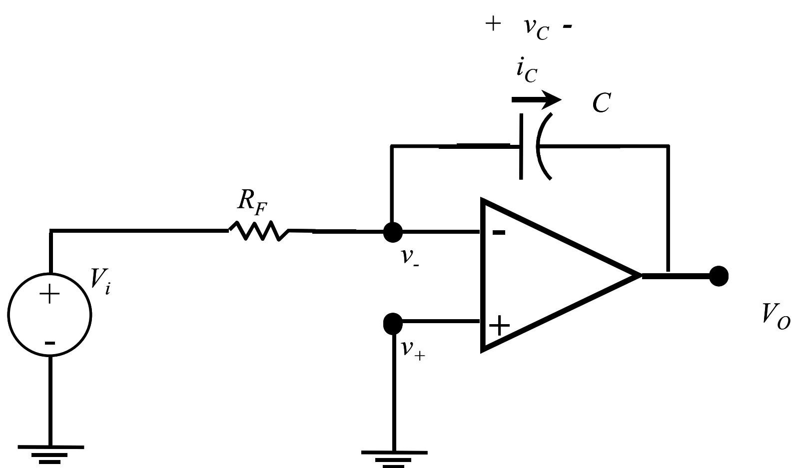

Figure 1. The Operational Integrator

- Lets consider this to be an ideal operational amplifier. Then, also, lets apply Node Voltage analysis with a node at the Inverting input.
- First, we have the Node Voltage equation.

$$\frac{v_i - v_-}{R_F} - i_C = 0$$

- And, the capacitor displacement current is

$$i_C = C\frac{dv_C}{dt} = C\frac{d}{dt}(v_- - v_O)$$

- But, for this ideal op amp

$$v_+ = v_= = 0$$

- So,

$$\frac{v_i}{R_F} - C\frac{d}{dt}(-v_O) = 0$$

- Now, we may integrate

$$\int_{t_O}^{t} d\tau \frac{d}{d\tau}\left(v_O(\tau)\right) = -\frac{1}{R_F C}\int_{t_O}^{t} v_i(\tau)d\tau$$

- And finally, we have an integral equation relating output to input voltage.

$$v_O(t) = v_O(\tau) - \frac{1}{R_F C}\int_{t_O}^{t} v_i(\tau)d\tau$$

- Lets consider the Operational Differentiator:

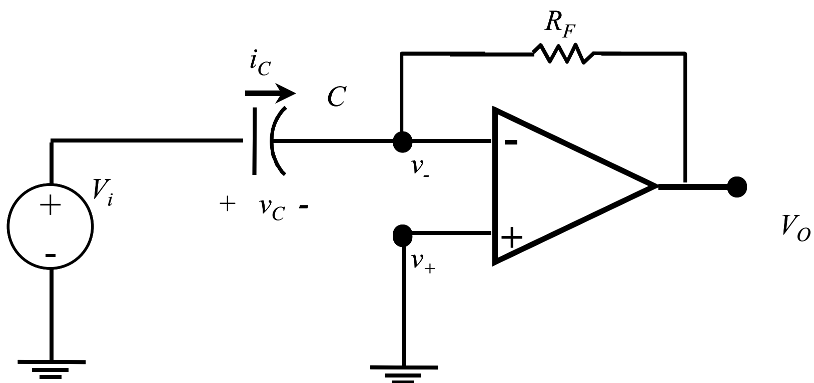

Figure 2. The Operational Differentiator

- Lets consider this to be an ideal operational amplifier. Then, also, lets apply Node Voltage analysis with a node at the Inverting input, as above.
- First, we have the Node Voltage equation.

$$\frac{v_- - v_O}{R_F} - i_C = 0$$

- And, the capacitor displacement current is

$$i_C = C\frac{dv_C}{dt} = C\frac{d}{dt}\left(v_i - v_-\right)$$

- But, for this ideal op amp

$$v_+ = v_= = 0$$

- And finally, we have a differential equation relating output to input voltage.

$$v_O = -R_F C\frac{d}{dt}\left(v_i\right)$$

- Circuit simulation of these devices provides interesting results.

NATURAL RESPONSE OF PARALLEL RLC CIRCUITS

- In our first introduction to reactive circuit elements, we have explored RC and RL circuits.

- These circuits are described by first order differential equations.

- They exhibit exponential response characteristics characterized by a time constant.

- The introduction of L and C components together will be discussed next. This combination of energy storage elements produces may useful and interesting behaviors that we will investigate now.

- To explore circuits with resistance, capacitance, and inductance, we will again explore natural and step response for both parallel and series combination circuits.

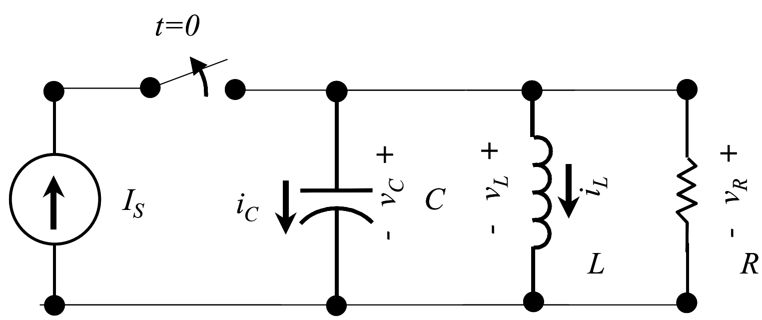

Figure 3. Our first RLC circuit for determining natural response of Parallel RLC circuits.

- First, we will examine natural response of the circuit above. We will assume that the switch above, opens at t=0.

- First, since this is a parallel circuit system. We can write

$$v_L = v_C = v_R = v$$

- Also, KCL requires that:

$$-i_L - i_C - i_R = 0$$

- We can also write down the current voltage relationships:

$$i_R = \frac{v}{R}$$

$$i_C = C\frac{dv}{dt}$$

$$i_L(t) = i_L(t_O) + \frac{1}{L}\int_{t_O}^{t} v(\tau)d\tau$$

- Now, for our problem, we will first assume:

$$t_O = 0$$

- Also, at t = 0,

$$i_L(0) = I_S$$

- So, we can now substitute:

$$\frac{v}{R} + C\frac{dv}{dt} + \frac{1}{L}\int_{0}^{t} v(\tau)d\tau + I_S = 0$$

- Differentiating,

$$C\frac{d^2v}{dt^2} + \frac{1}{R}\frac{dv}{dt} + \frac{1}{L}v(t) = 0$$

- And manipulating,

$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{1}{LC}v(t) = 0$$

- Now, to begin for this problem, lets first solve this with the assumed solution of:

$$v(t) = Ae^{St}$$

- We will later need to revise this solution for more general applications. However, we find with this solution.

$$S^2Ae^{St} + \frac{1}{RC}SAe^{St} + \frac{1}{LC}Ae^{St} = 0$$

- And, removing the common factor, we have a quadratic equation in the exponent, S.

$$S^2 + \frac{1}{RC}S + \frac{1}{LC} = 0$$

- The roots are

$$S_{1,2} = \frac{-\frac{1}{RC} \pm \sqrt{\left(\frac{1}{RC}\right)^2 - \frac{4}{LC}}}{2}$$

- Or

$$S_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_O{}^2}$$

- Where we define the damping coefficient to be

$$\alpha \equiv \frac{1}{2RC}$$

- And a resonance frequency in radians/sec

$$\omega_O{}^2 \equiv \frac{1}{LC}$$

- And a resonance frequency in Hz.

$$\omega_O \equiv 2\pi f_O$$

- So, our differential equation has become,

$$S^2 + 2\alpha S + \omega_O{}^2 = 0$$

- This can also be writtent

$$S^2 + \frac{2\omega_O}{Q}S + \omega_O{}^2 = 0$$

- Where,

$$Q \equiv 2\omega_O RC = \frac{\omega_O}{\alpha}$$

- As a brief aside, note the similarity of our circuit expression to a mechanical damped, harmonic oscillator:

$$m\frac{d^2x}{dt^2}+b\frac{dx}{dt}+kx=0$$

- We will return to this when we discuss oscillators

- Now note that both values for S are solutions, so their sum must also be a solution. Thus, the voltage response is.

$$v(t)=A_1e^{S_1t}+A_2e^{S_2t}$$

- Substituting, we can show the above sum is a solution with:

$$A_1e^{S_1t}\left(S_1^{\ 2}+\frac{\omega_O}{Q}S_1+\omega_O^{\ 2}\right)+A_2e^{S_2t}\left(S_2^{\ 2}+\frac{\omega_O}{Q}S_2+\omega_O^{\ 2}\right)=0$$

PARALLEL RLC OVERDAMPED BEHAVIOR

- Now, we may explore three operating limits for this circuit.

- The first will be referred to as "overdamped" behavior. The meaning of this term will become clear later.

- Here, Q < 1 and

$$\omega_O<\alpha$$

- We can solve for voltage and current in circuits of this type by first solving the Node Voltage equation and determining the initial conditions for voltage and current.

- We also find the values of ω_O and α. If these follow the inequality above, we will proceed as below with these equations describing the overdamped case.

- Now, for this case, lets examine voltage response. This will be

$$v(t)=A_1e^{S_1t}+A_2e^{S_2t}$$

- But, at t = 0, the capacitor voltage is zero and must remain so for times infinitesimally near t = 0. Specifically

$$v(t=0)=0$$

- And

$$i_L(t=0)=i_S$$

- Also, we can write down an equation for the coefficients

$$v(t=0)=A_1+A_2=0$$

- And an additional equation.

$$i_C(t=0)=C\frac{d}{dt}v_C=C\frac{d}{dt}v=C(S_1A_1+S_2A_2)$$

- Also, at t = 0,

$$i_L(t=0)=i_S$$

$$i_R(t=0)=\frac{v_R(0)}{R}=\frac{v(0)}{R}=0$$

- Finally, with KCL evaluated at t = 0

$$i_C+i_L+i_R=0$$

$$i_C+i_S+\frac{v(0)}{R}=C(S_1A_1+S_2A_2)+i_S+\frac{v(0)}{R}=0$$

- So,

$$i_S=-C(S_1A_1+S_2A_2)$$

- Gathering the key results we have a set of equations we may use for solving for Parallel RLC natural response,

$$v(t)=A_1e^{S_1t}+A_2e^{S_2t}$$

$$A_1+A_2=0$$

$$i_S = -C(S_1 A_1 + S_2 A_2)$$

SUMMARY OF PARALLEL RLC OVERDAMPED BEHAVIOR

- Gathering the key results we have a set of equations we may use for solving for Parallel RLC natural response,

$$v(t) = A_1 e^{S_1 t} + A_2 e^{S_2 t}$$

$$A_1 + A_2 = 0$$

$$i_S = -C(S_1 A_1 + S_2 A_2)$$

PARALLEL RLC OVERDAMPED BEHAVIOR DEMONSTATION

- Now, in the PSpice demonstration C = 0.159 μF and L = 0.159μH. And

$$\alpha \equiv \frac{1}{2RC}$$

- And also

$$\omega_O \equiv \frac{1}{\sqrt{LC}} \cong 2\pi(1MHz)$$

- With these values, we further have:

$$Q \equiv 2\omega_O RC = 2R$$

- Lets consider R = 0.01, or Q = 0.02
- This is an overdamped condition. The response is plotted below.

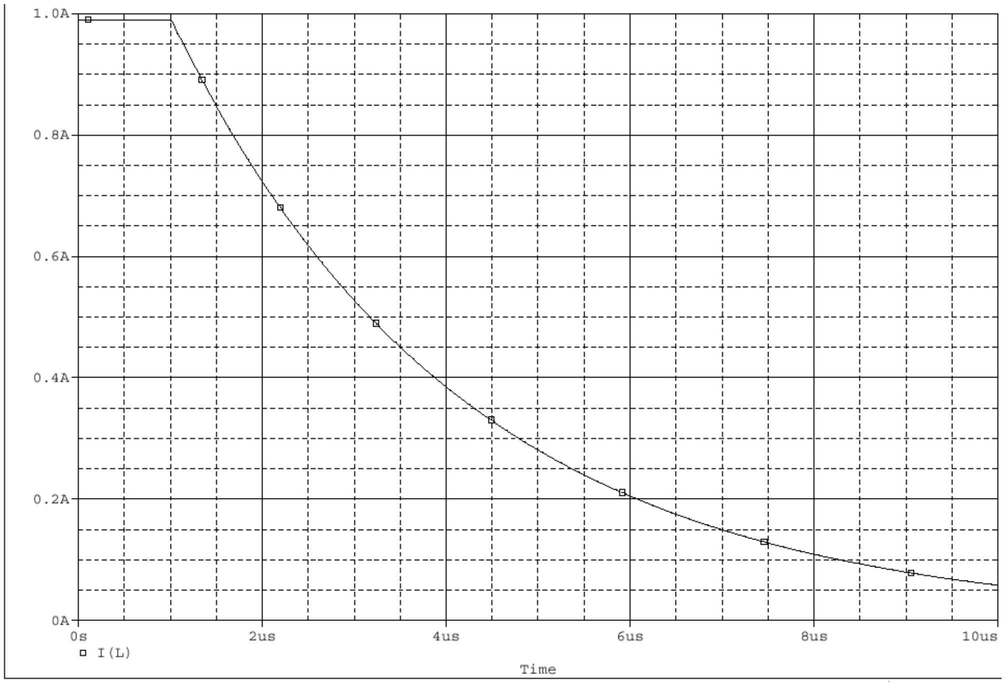

Figure 4. Inductor Current Response for Circuit of Figure 1 with R = *0.01, corresponding to Q =0.02*

- Lets also consider smaller and larger values of R, with R < 0.5Ω
- Lets examine our circuit voltage.
- Finally, lets also examine Capacitor current.

PARALLEL RLC CRITICALLY DAMPED CIRCUIT

- An important limit of operation is the critically damped limit.
- This represents the most rapid "settling" of a system to its "final" value. Lets examine this with PSpice.
- Achieving critical damping is frequently a system design requirement.
- Here, Q = 1 and $\omega_O = \alpha$
- Our differential equations show new behavior in this limit.
- We have to pursue an alternative solution

- First, we will return to our original equation.

$$\frac{d^2v}{dt^2} + 2\alpha\frac{dv}{dt} + \omega_O{}^2 v(t) = 0$$

- It has roots of

$$S_1 = -\alpha + \sqrt{\alpha^2 - \omega_O{}^2}$$

$$S_2 = -\alpha - \sqrt{\alpha^2 - \omega_O{}^2}$$

- Now, for convenience, we write

$$-2\alpha = S_1 + S_2$$

$$\omega_O{}^2 = S_1 S_2$$

- This allows us to rewrite our differential equation

$$\frac{d^2v}{dt^2} - (S_1 + S_2)\frac{dv}{dt} + S_1 S_2 v(t) = 0$$

- Now, we can rewrite this as well,

$$\frac{d}{dt}\left(\frac{d}{dt}v - S_2 v\right) - S_1\left(\frac{d}{dt}v - S_2 v\right) = 0$$

- And, with this we can define a variable,

$$X \equiv \left(\frac{d}{dt}v - S_2 v\right)$$

- So, the second order differential equation becomes,

$$\frac{d}{dt}X - S_1 X = 0$$

- But, the solution for this is simple,

$$X = Ae^{S_1 t}$$

- so, we can write down

$$\left(\frac{d}{dt}v - S_2 v\right) = Ae^{S_1 t}$$

- Now, we can multiply through in the above by

$$e^{-S_2 t}$$

- This gives,

$$\left(e^{-S_2 t}\frac{d}{dt}v - e^{-S_2 t}S_2 v\right) = Ae^{(S_1 - S_2)t}$$

- Finally, we recognize that

$$\frac{d}{dt}(e^{-S_2 t}v) = -S_2 e^{-S_2 t}v + e^{-S_2 t}\frac{dv}{dt}$$

- So, we can write, noting the equivalence with the term in parentheses above,

$$-\frac{d}{dt}e^{-S_2 t}v = Ae^{(S_1 - S_2)t}$$

- Finally, we can now integrate this equation

$$\int_0^t d\tau \frac{d}{d\tau}e^{-S_2 t}v(\tau) = -\int_0^t d\tau A e^{(S_1 - S_2)\tau}$$

- Now, lets consider the example where $S_1 \neq S_2$. Then,

$$e^{-S_2 t}v(t) - A_O = -\frac{Ae^{(S_1 - S_2)t}}{S_1 - S_2}$$

- But, if $S_1 = S_2$ we can not apply this. So, instead we return to our first order differential equation,

$$-\frac{d}{dt}e^{-S_2 t}v = A$$

- and we integrate this

$$\int_0^t d\tau \frac{d}{d\tau} e^{-S_2 t} v(\tau) = -\int_0^t d\tau A$$

- so, this provides

$$e^{-S_2 t} v(t) - A_O = -At$$

- or

$$e^{-S_2 t} v(t) = A_O - At$$

- Summarizing, for $S_1 \neq S_2$

$$e^{-S_2 t} v(t) = A_O - \frac{Ae^{(S_1 - S_2)t}}{S_1 - S_2}$$

- If $S_1 = S_2$

$$e^{-S_2 t} v(t) = A_O - At$$

- Now, we can compute voltage for these two examples,

$$v(t) = A_O e^{S_2 t} - \frac{A}{S_1 - S_2} e^{S_1 t}$$

$$v(t) = A_O e^{S_2 t} - Ate^{S_1 t}$$

- And, we can define constants. For $S_1 = S_2$

$$D_2 = A_O$$

$$D_1 = -A$$

- And define for $S_1 \neq S_2$

$$A_2 = A_O$$

$$A_1 = -A/(S_1+S_2)$$

- So, for the overdamped and underdamped, cases where, $S_1 \neq S_2$

$$v(t) = A_1 e^{S_1 t} + A_2 e^{S_2 t}$$

- And for the critically damped case, where $S_1 = S_2$
- Then, $S_1 = S_2 = -\alpha$
- And

$$v(t) = (D_1 + D_2 t)e^{-\alpha t}$$

- Lets now examine the critically damped example in more detail

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SUMMARY OF PARALLEL RLC CRITICALLY DAMPED BEHAVIOR

- Here, Q = 1 and

$$\omega_O = \alpha$$

- We can solve for voltage and current in circuits of this type by first solving the Node Voltage equation and determining the initial conditions for voltage and current.
- We also find the values of ω_O and α. If these follow the equality above, we will proceed as below with these equations describing the critically damped case.
- Now, for this case, lets examine voltage response. This will be

$$v(t) = (D_1 + D_2 t)e^{-\alpha t}$$

- But, at t = 0, the capacitor voltage is zero and must remain so for times infinitesimally near t = 0. Specifically

$$v(t=0) = 0 = D_1$$

- and

$$i_C(t=0) = C\frac{d}{dt}v(t=0) = C\left(-\alpha D_1 + D_2\right)$$

- and

$$i_S = -C\left(-\alpha D_1 + D_2\right)$$

- We can use these equations to solve for D_1 and D_2.

- For example,

$$i_S = -C\left(-\alpha D_1 + D_2\right)$$

- and

$$v(t) = i_S C t e^{-\alpha t}$$

- Lets examine this behavior with PSpice.

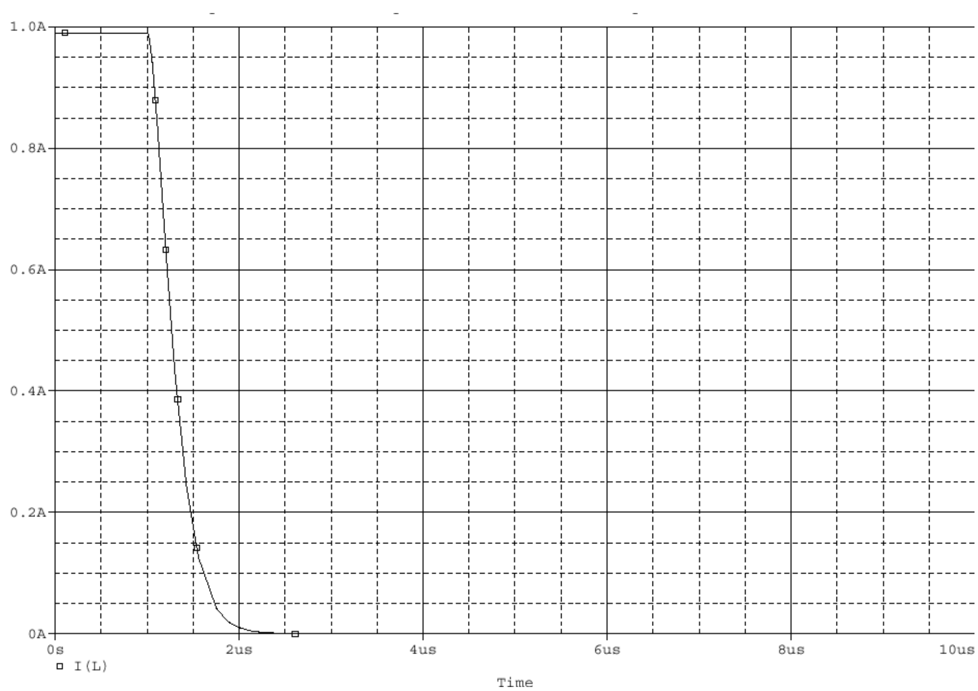


Figure 5. Inductor Current Response for Critically Damped Condition for the circuit of Figure 1 with R = *0.5Ω*

- Lets examine other values, is there a value that offers faster settling?

PARALLEL RLC UNDERDAMPED CIRCUIT BEHAVIOR

- A particularly interesting and important limit is the underdamped operating condition.

$$Q > 1$$

$$\omega_O > \alpha$$

- We have the roots to our equation

$$S_1 = -\alpha + \sqrt{\alpha^2 - \omega_O{}^2}$$

$$S_2 = -\alpha - \sqrt{\alpha^2 - \omega_O{}^2}$$

- Now, the argument of the square root is negative definite! Thus, $S_{1,2}$ contain imaginary components.

$$S_1 = -\alpha + j\omega_d$$

$$S_2 = -\alpha - j\omega_d$$

- With

$$\omega_d \equiv \sqrt{\omega_O{}^2 - \alpha^2}$$

- This is referred to as the damped resonance frequency.
- For high Q, and small α

$$\omega_d \cong \omega_O$$

- Now, our circuit voltage will be

$$v(t) = A_1 e^{S_1 t} + A_2 e^{S_2 t}$$

- And this can be written,

$$v(t) = A_1 e^{-\alpha t} e^{j\omega_d t} + A_2 e^{-\alpha t} e^{-j\omega_d t}$$

- Or,

$$v(t) = e^{-\alpha t}\left(A_1 e^{j\omega_d t} + A_2 e^{-j\omega_d t}\right)$$

- This exposes both a damping factor and an oscillatory factor.
- Now, we use the Euler formula:

$$e^{j\theta} = \cos\theta + j\sin\theta$$

- We can proceed to manipulate our voltage equation.

$$v(t) = e^{-\alpha t}\left[\left(A_1 + A_2\right)\cos(\omega_d t) + \left(A_1 - A_2\right)\sin(\omega_d t)\right]$$

- Finally, we can establish a new set of constants,

$$v(t) = e^{-\alpha t}\left[B_1 \cos(\omega_d t) + B_2 \sin(\omega_d t)\right]$$

SUMMARY OF PARALLEL RLC UNDERDAMPED CIRCUIT BEHAVIOR

- Here, Q > 1 and

$$\omega_O > \alpha$$

- We can solve for voltage and current in circuits of this type by first solving the Node Voltage equation and determining the initial conditions for voltage and current.

- We also find the values of ω_O and α. If these follow the equality above, we will proceed as below with these equations describing the underdamped case.

- Now, for this case, lets examine voltage response. This will be

$$v(t) = e^{-\alpha t}\left[B_1 \cos(\omega_d t) + B_2 \sin(\omega_d t)\right]$$

- But, at t = 0, the capacitor voltage is zero and must remain so for times infinitesimally near t = 0. Specifically

$$v(t = 0) = B_1$$

- and

$$i_C(t = 0) = C\frac{d}{dt}v(t = 0)$$

- And substituting,

$$i_C(t = 0) = -\alpha C e^{-\alpha t}\left[B_1 \cos(\omega_d t) + B_2 \sin(\omega_d t)\right] - Ce^{-\alpha t}\left[B_1\omega_d \sin(\omega_d t) - B_2\omega_d \cos(\omega_d t)\right]\Big|_{t=0}$$

- Evaluating at t = 0

$$i_C(t = 0) = C\frac{d}{dt}v(t = 0) = C\left(-\alpha B_1 + \omega_d B_2\right)$$

- and

$$i_S = -C\left(-\alpha B_1 + \omega_d B_2\right)$$

- We can use these equations to solve for B_1 and B_2.
- Lets consider a PSpice Example

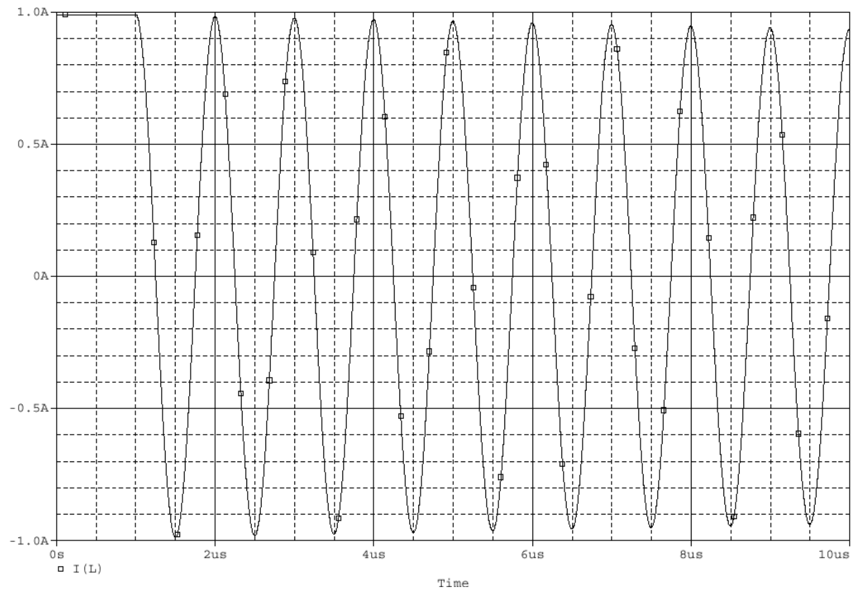


Figure 6. Inductor Current Response for Underdamped condition for circuit of Figure 1 with R *= 500 and Q = 1000*

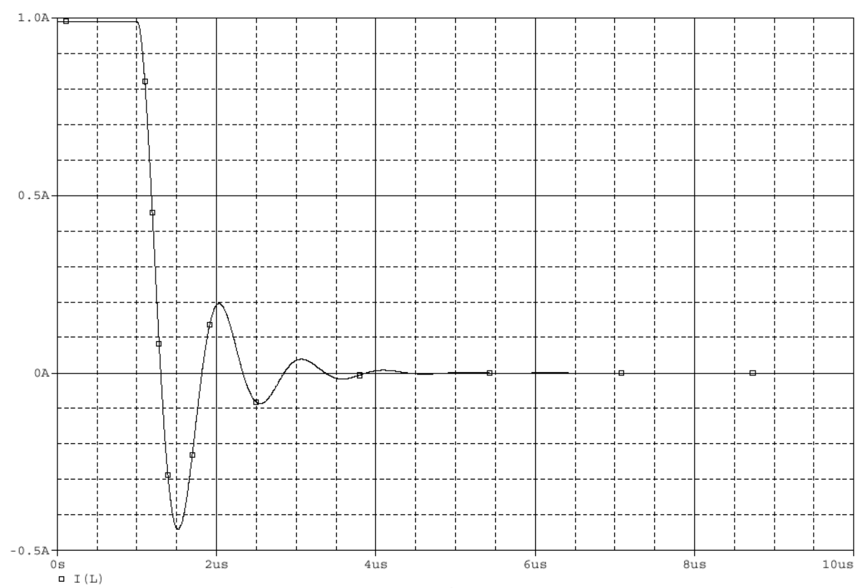


Figure 7. Inductor Current Response for Underdamped condition for circuit of Figure 1 with R *= 4 and Q = 8. Note that the resonance frequency is slightly less than for the example above.*

- What is the frequency?
- Lets examine role of α in damped frequency term. This is seen in Figure 4.
- How does damping of the oscillation depend on α?

OSCILLATOR SYSTEMS FOR PRECISION TIME AND FREQUENCY REFERENCE

- In the PSpice example, lets examine the sensitivity of resonance frequency to "parasitic" capacitance.
- Now, a broad range of products requires precision frequency and time standards. How are these obtained.
- Lets discuss

1) Oscillators
2) Frequency references
3) Time references
4) Precision oscillators:

RLC Circuit:	1 part in 10^2
Crystal Oscillator:	1 part in 10^7
Atomic Clock:	1 part in 10^{14}

- Lets also discuss the Quartz Crystal Oscillator and model this with PSpice.

STEP RESPONSE OF PARALLEL RLC CIRCUITS

- Now, a complete picture of the second-order RLC circuit systems requires that we examine step response as well.

- This is a small extension of our previous analysis.

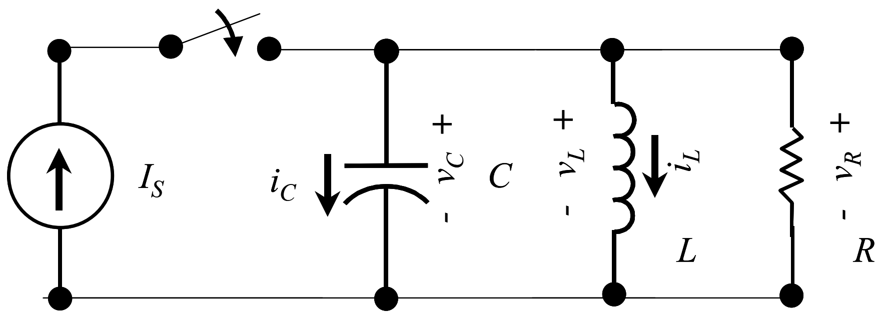


Figure 8. Parallel RLC Circuit designed for analysis of Step Response

- Consider the circuit above.

- At t = 0 as switch closes, we may apply KCL at the node joining the switch with the R, L, and C. This will provide:

$$-i_L - i_C - i_R + I_S = 0$$

- Now, we can write down the currents in terms of voltages,

$$i_L + C\frac{dv_C}{dt} + \frac{v_R}{R} = I_S$$

- But, also,

$$v_L = L\frac{di_L}{dt}$$

- So, since

$$v_L = v_C = v_R$$

- We can write

$$i_L + \frac{C}{L}\frac{d^2 i}{dt^2} + \frac{1}{R}\frac{di_L}{dt} = I_S$$

- Manipulating into a familiar form,

$$\frac{d^2 i_L}{dt^2} + \frac{1}{RC}\frac{di_L}{dt} + \frac{1}{LC} i_L = \frac{I_S}{LC}$$

- And we obtain finally,

$$\frac{d^2 i_L}{dt^2} + 2\alpha \frac{di_L}{dt} + \omega_O{}^2 i_L = \omega_O{}^2 I_S$$

- We may also solve for voltage across all elements and use this to derive resistor and capacitor currents.

- First, recognize that

$$i_L = v_O + \int_0^t \frac{1}{L} v_L(\tau) d\tau$$

- So, we can rewrite our KCL-derived equation, using also that $v(0) = v_O = 0$

$$\int_0^t \frac{1}{L} v_L(\tau) d\tau + \frac{v}{R} + C\frac{d}{dt} v = I_S$$

- Differentiating

$$\frac{1}{L} v + \frac{1}{R}\frac{d}{dt} v + C\frac{d^2}{dt^2} v = 0$$

- And manipulating

$$\frac{d^2}{dt^2} v + \frac{1}{RC}\frac{d}{dt} v + \frac{1}{LC} v = 0$$

- This again yields our familiar natural response result.

$$\frac{d^2}{dt^2} v + 2\alpha \frac{d}{dt} v + \omega_O{}^2 v = 0$$

- The voltage solution for the step-driven circuit system has the same form as for the natural response.

- For Q < 1

$$v(t) = A_1 e^{S_1 t} + A_2 e^{S_2 t}$$

- For Q = 1

$$v(t) = (D_1 + D_2 t)e^{-\alpha t}$$

- For Q >1

$$v(t) = e^{-\alpha t}\left[B_1 \cos(\omega_d t) + B_2 \sin(\omega_d t)\right]$$

- Now, to obtain the current dependence, we will use our differential equation in current and voltage:

$$i_L + C\frac{dv_C}{dt} + \frac{v_R}{R} = I_S$$

- With this, and our expressions for *v,* above, we derive the currents for each condition.
- First,

$$i_L = I_S - C\frac{dv_C}{dt} - \frac{v_R}{R}$$

- So, for Q < 1, we can substitute v(t) and get

$$i_L = I_S - \left(CA_1 S_1 + \frac{1}{R}\right)e^{S_1 t} - \left(CA_2 S_2 + \frac{1}{R}\right)e^{S_2 t}$$

- And we see that we can write this:

$$i_L = I_S + A_1' e^{S_1 t} + A_2' e^{S_2 t}$$

- Each of the current equations for each of the cases will take this form, as we will see next.

SUMMARY OF STEP RESPONSE OF PARALLEL RLC CIRCUITS

- Gathering results together for all the voltage and current equations.
- For Q < 1

$$v(t) = A_1 e^{S_1 t} + A_2 e^{S_2 t}$$

$$i_L = I_S + A_1' e^{S_1 t} + A_2' e^{S_2 t}$$

- For Q = 1

$$v(t) = (D_1 + D_2 t)e^{-\alpha t}$$

$$i_L = I_S + (D_1' + D_2' t)e^{-\alpha t}$$

- For Q >1

$$v(t) = e^{-\alpha t}\left[B_1 \cos(\omega_d t) + B_2 \sin(\omega_d t)\right]$$

$$i_L(t) = I_S + e^{-\alpha t}\left[B_1' \cos(\omega_d t) + B_2' \sin(\omega_d t)\right]$$

- Our problem solving procedure is similar to the previous cases.
- We can solve for voltage and current in circuits of this type by first solving the Node Voltage equation and determining the initial conditions for voltage and current.
- We also find the values of ω_O and α. If these follow the equality above, we will proceed as below with these equations describing the underdamped, overdamped, or critically damped case as appropriate.
- Now, for this case, lets examine voltage response. This will be

$$v(t) = A_1 e^{S_1 t} + A_2 e^{S_2 t}$$

- Also, at t = 0, the capacitor voltage is zero and must remain so for times infinitesimally near t = 0. Specifically

$$v(t=0) = L\frac{d}{dt} i_L(t=0) = L\left(S_1 A_1' + S_2 A_2'\right) = 0$$

- Also,

$$i_L(t=0) = I_S + A_1' + A_2'$$

- As for the above examples, we solve for A_1 and A_2.

- Lets also examine the RLC Parallel circuit Step Response with PSpice using the circuit below.

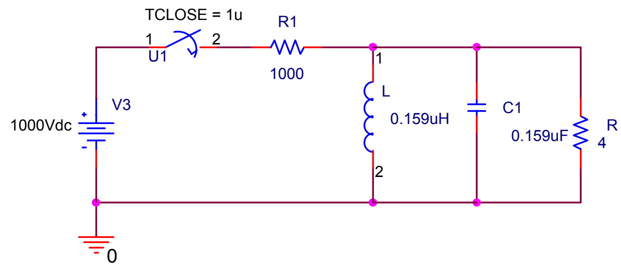


Figure 9. RLC Parallel Circuit for demonstration of Step Response

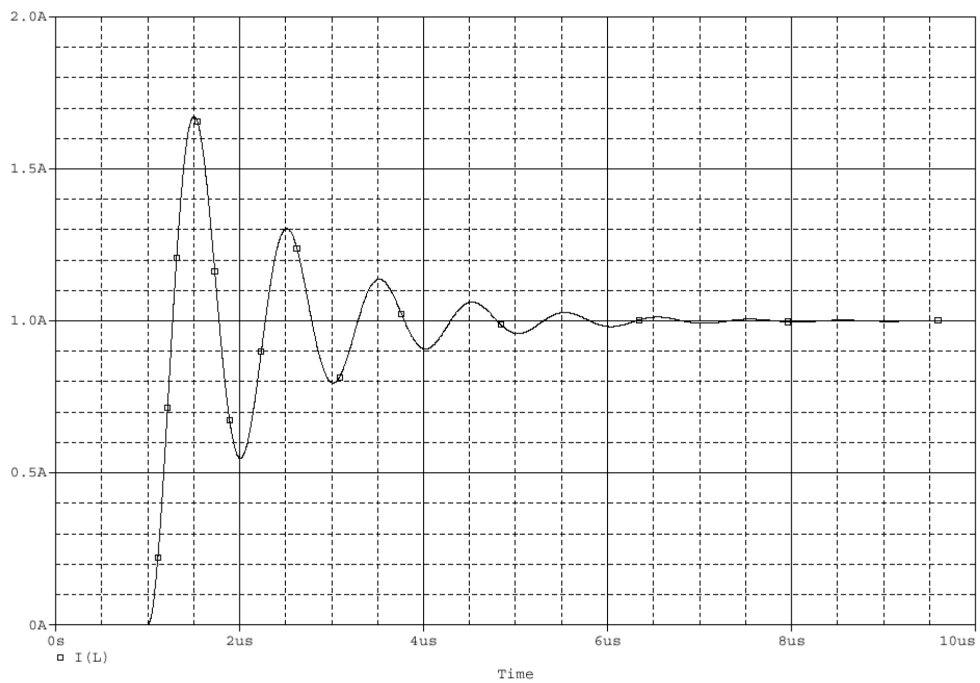


Figure 10. Transient Step response of the Circuit of Figure6 . Note that this corresponds to a Q value of 8

NATURAL RESPONSE OF SERIES RLC CIRCUITS

- Our next (and last) circuit system is the series RLC circuit.

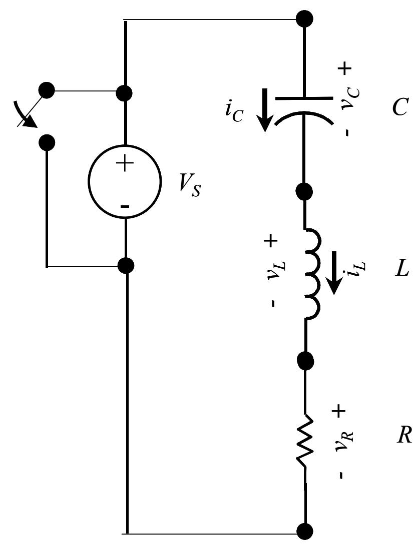


Figure 11. An RLC circuit for determining natural response of Series RLC circuits.

- First, we will examine natural response of the circuit above. We will assume that the switch above, closes at t=0, removing the voltage source.
- First, since this is a series circuit system. We can write that:

$$i_L = i_C = i_R = i$$

- and for t > 0

$$v_L + v_C + v_R = 0$$

- So, this allows us to write

$$v_c(t = o) + \frac{1}{C}\int_0^t i(\tau)d\tau + L\frac{di}{dt} + iR = 0$$

- Note also that for t = 0, all of the voltage drop is across the capacitory because in steady stage, capacitor current will be zero. Therefore, the voltage drops across inductor and resistor are zero. So,

$$-v_S + \frac{1}{C}\int_0^t i(\tau)d\tau + L\frac{di}{dt} + iR = 0$$

$$\frac{1}{C}\int_0^t i(\tau)d\tau + L\frac{di}{dt} + iR = v_S$$

- Differentiating, we have

$$\frac{d^2}{dt^2}i + \frac{R}{L}\frac{d}{dt}i + \frac{i}{C} = 0$$

$$\frac{d^2 i}{dt^2} + 2\alpha_S \frac{di}{dt} + \omega_O{}^2 i = 0$$

- with

$$\alpha_S = R/2L$$

- and

$$\omega_O{}^2 = 1/LC$$

- and

$$Q = \frac{2\omega_O L}{R} = \frac{\omega_O}{\alpha_S}$$

- We can solve this using the above approaches for parallel circuits.
- For the three circuit conditions we have.
- Overdamped Response: Q <1 and $\alpha_S > \omega_O$

$$i(t) = A_1 e^{S_1 t} + A_2 e^{S_2 t}$$

- Underdamped Response: Q >1 and $\alpha_S < \omega_O$

$$i(t) = e^{-\alpha_S t}\left[B_1 \cos(\omega_d t) + B_2 \sin(\omega_d t)\right]$$

- And, Critically damped Response: Q >1 and $\alpha_S < \omega_O$

$$i(t) = (D_1 + D_2 t)e^{-\alpha_S t}$$

- All of our results apply, here for the Series circuit, where we replace the damping coefficient by α_S and also replace the role of voltage by current.

SUMMARY OF PROBLEM SOLVING PROCEDURES FOR SERIES RLC CIRCUITS

- The problem solving procedures for series RLC circuits are similar to those of parallel circuits.
- We can solve for voltage and current in circuits of this type by first solving the Node Voltage equation and determining the initial conditions for voltage and current.
- We also find the values of ω_O and α. If these follow the equality above, we will proceed as below with these equations describing the underdamped, overdamped, or critically damped case as appropriate.
- Lets then list the defining equations for each of three operating cases:

OVERDAMPED SERIES RLC CIRCUIT

$Q < 1$, $\omega_O < \alpha$

- For a switch closure at t = 0

$$i(t) = A_1 e^{S_1 t} + A_2 e^{S_2 t}$$

- If current is zero at t = 0,

$$A_1 + A_2 = 0$$

- and

$$v_C(t) = v_S + A_1' e^{S_1 t} + A_2' e^{S_2 t}$$

$$v_C(t = o) = v_S + A_1' + A_2'$$

CRITICALLY DAMPED SERIES RLC CIRCUIT

- For a switch closure at t = 0

$$i(t) = (D_1 + D_2 t)e^{-\alpha t}$$

- If current is zero at t = 0,

$$i(t = 0) = 0 = D_1$$

- and

$$v_C(t) = v_S + (D_1' + D_2' t)e^{-\alpha t}$$

UNDERDAMPED SERIES RLC CIRCUIT

- For a switch closure at t = 0

$$i(t) = e^{-\alpha t}\left[B_1 \cos(\omega_d t) + B_2 \sin(\omega_d t)\right]$$

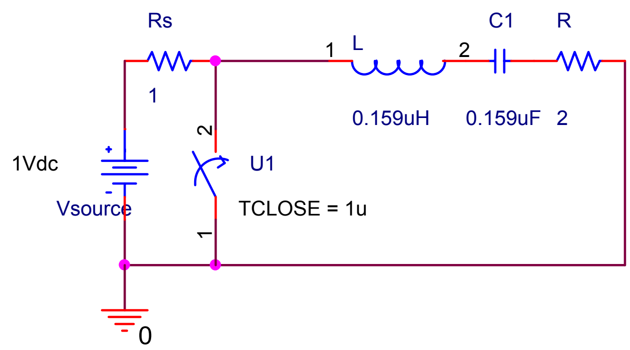
- If current is zero at t = 0,

$$i(t = 0) = 0 = B_1$$

- and

$$v_C(t) = v_S + e^{-\alpha t}\left[B_1' \cos(\omega_d t) + B_2' \sin(\omega_d t)\right]$$

SERIES RLC OVERDAMPED BEHAVIOR DEMONSTATION



- Now, in the PSpice demonstration C = 0.159 µF and L = 0.159µH. And

$$\alpha \equiv \frac{R}{2L}$$

- And also

$$\omega_O \equiv \frac{1}{\sqrt{LC}} \cong 2\pi(1MHz)$$

- With these values, we further have:

$$Q = \frac{2\omega_O L}{R} = \frac{2}{R}$$

- So, for critical damping, R = 2Ω, for underdamped conditions, R < 2Ω, and for overdamped conditions R > 2Ω.
- Here are three results for these three conditions.
- Note that at t = 0, current is zero for each system. Then, as the voltage source is suddenly removed, current first flows then decays as the capacitor discharges its initial energy.
- Also note that voltage is equal to V_S at t = 1 µs, the point of switch closure.

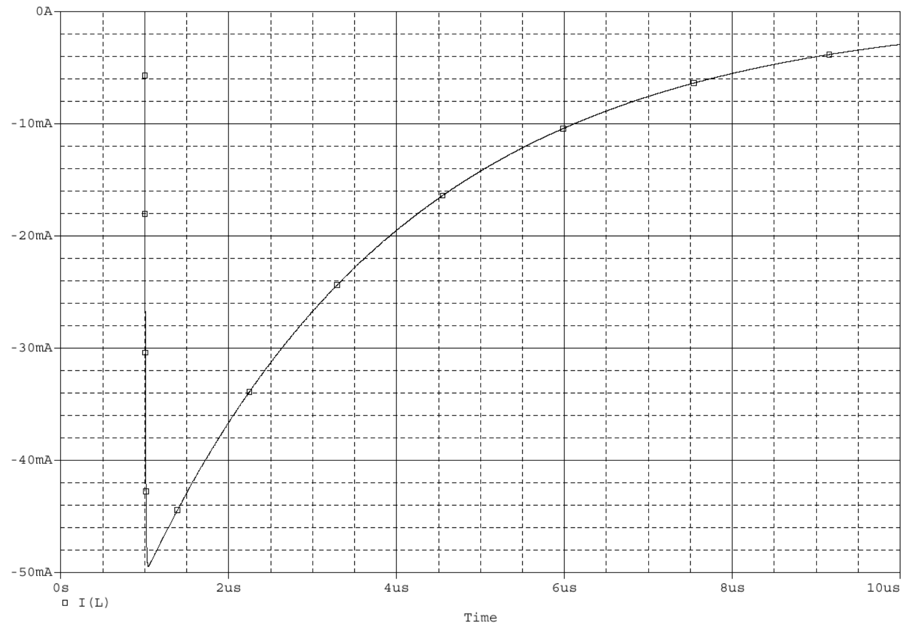


Figure 12. Inductor Current for Over Damped Circuit Conditions: R *= 20, Q = 0.1*

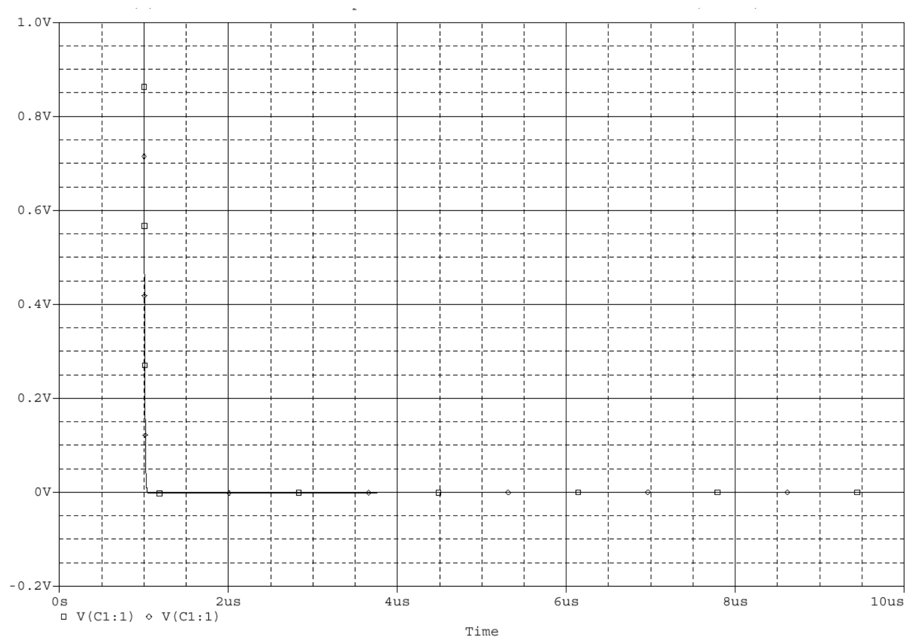


Figure 13. Capacitor Voltage for Over Damped Circuit Conditions: R *= 20, Q = 0.1*

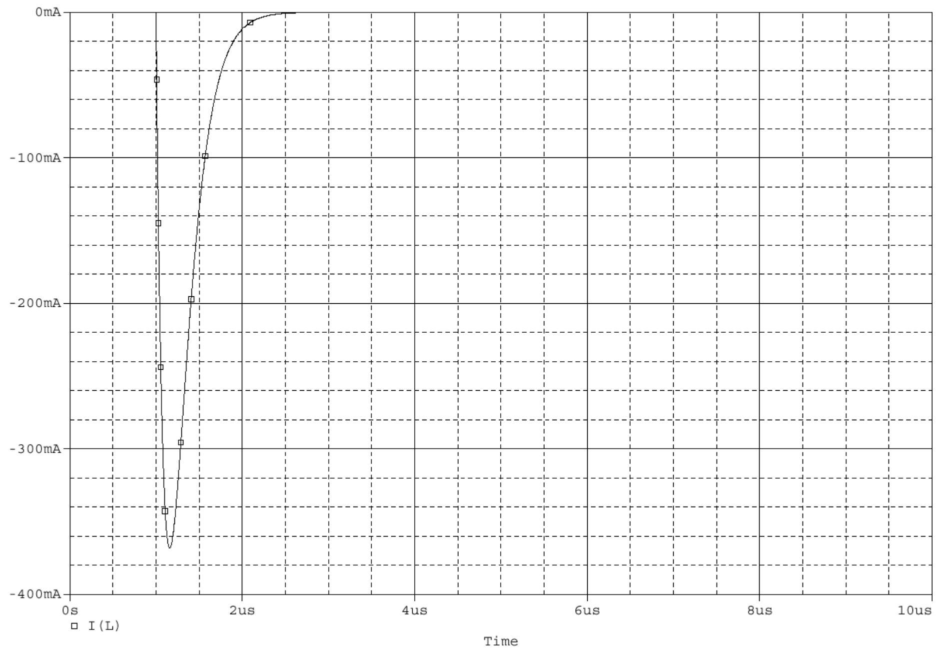


Figure 14. Inductor Current for Critically Damped Circuit Conditions: R = *2, Q = 1*

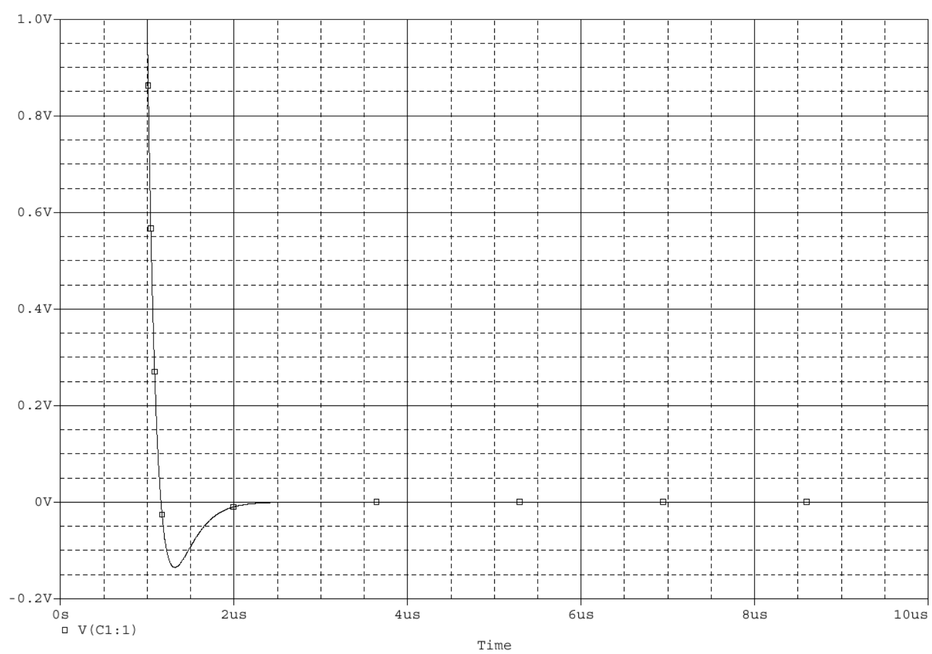


Figure 15. Capaccitor Voltage for Critically Damped Circuit Conditions: R = *2, Q = 1*

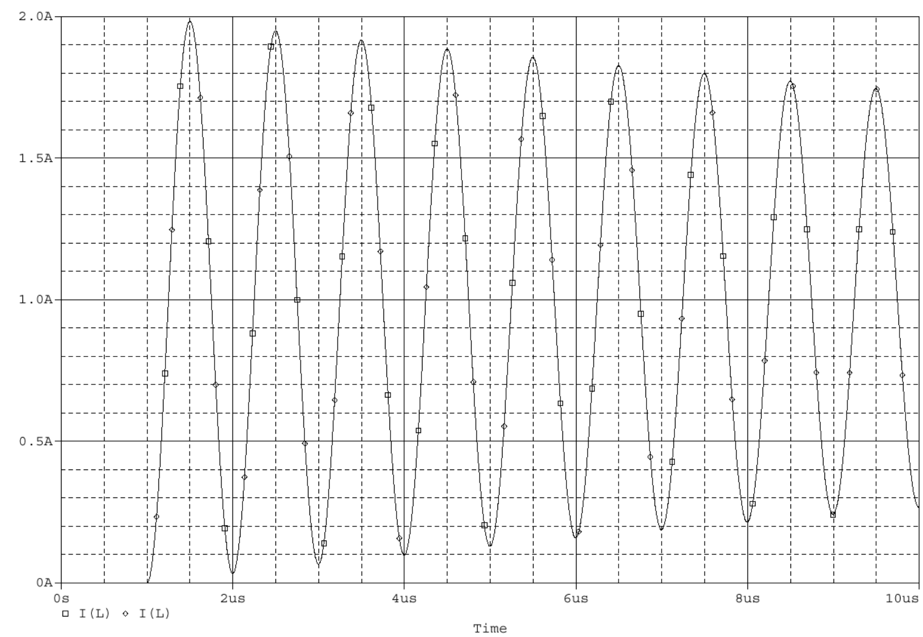


Figure 16. Inductor Current for Underdamped Circuit Conditions: R *= 0.05, Q = 40*

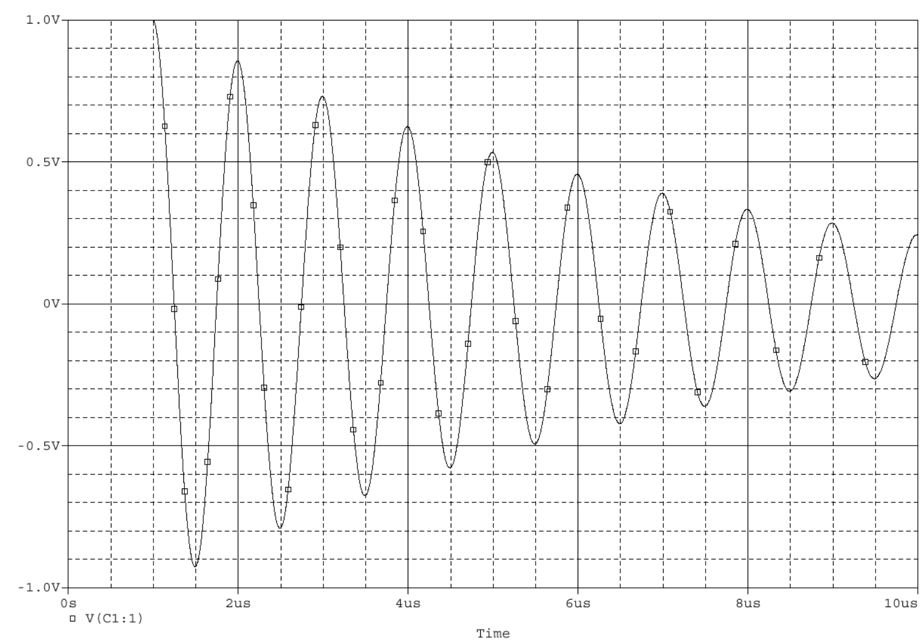


Figure 17. Capacitor Voltage for Underdamped Circuit Conditions: R *= 0.05, Q = 40*