Lecture 8

- Inverse Laplace transform:
  - Other method for complex roots.
  - Non-strictly proper case.
- Using Laplace to solve differential equations
- Initial and Final value theorems.

Complex roots, another method.

Since $F(s)$ has typically real coefficients, the complex roots come in conjugate pairs. Treating these two roots at once, we can invert the transform without complex operations.

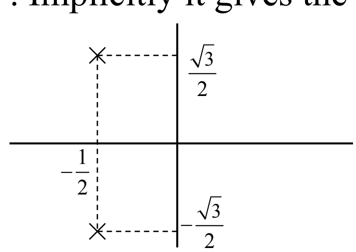
For complex roots $\alpha \pm i\beta$:

$$(s-\alpha-i\beta)(s-\alpha+i\beta)=(s-\alpha)^2+\beta^2$$

Taking a second order polynomial to the form on the right is called "completing the square". Implicitly it gives the roots. Example:

$$s^2+s+1=\left(s+\frac{1}{2}\right)^2+\frac{3}{4}$$

$$\rightarrow \quad \alpha=-\frac{1}{2},\ \beta=\frac{\sqrt{3}}{2}$$



If we combine the terms $\dfrac{A}{s-\alpha-i\beta}$ and $\dfrac{B}{s-\alpha+i\beta}$ from the expansion, we get something of the form $\dfrac{Ms+N}{(s-\alpha)^2+\beta^2}$.

Here M, N are constants, typically real.

Example: $F(s)=\dfrac{s-1}{(s+1)(s^2+s+1)} = \dfrac{A}{s+1}+\dfrac{Ms+N}{(s^2+s+1)}$

By the usual method, $A=\dfrac{\overbrace{s}^{-1}-1}{(\underbrace{s}_{-1}{}^{2}+\underbrace{s}_{-1}+1)} = -2$

Evaluating $F(s)$ at $s=0$: $\quad -1=A+N \qquad \rightarrow N=1.$

Limit of $sF(s)$ as $s\rightarrow\infty$: $\quad 0=A+M \qquad \rightarrow M=2$

Finding

$\mathcal{L}^{-1}\left[\dfrac{Ms+N}{(s-\alpha)^2+\beta^2}\right]$

from the table

$f(t)$	$F(s)$
$\cos(\beta t)u(t)$	$\dfrac{s}{s^2+\beta^2}$
$e^{\alpha t}\cos(\beta t)u(t)$	$\dfrac{s-\alpha}{(s-\alpha)^2+\beta^2}$
$\sin(\beta t)u(t)$	$\dfrac{\beta}{s^2+\beta^2}$
$e^{\alpha t}\sin(\beta t)u(t)$	$\dfrac{\beta}{(s-\alpha)^2+\beta^2}$
$e^{\alpha t}\left(C_1\cos(\beta t)+C_2\sin(\beta t)\right)u(t)$	$\dfrac{C_1(s-\alpha)+C_2\beta}{(s-\alpha)^2+\beta^2}$

Equating numerators, $\quad Ms+N=C_1 s+(C_2\beta-C_1\alpha)$

Now solve for C_1, C_2.

$$Ms + N = C_1 s + (C_2\beta - C_1\alpha) \rightarrow \begin{Bmatrix} C_1 = M \\ N = C_2\beta - M\alpha \end{Bmatrix} \rightarrow \begin{cases} C_1 = M \\ C_2 = \dfrac{N + M\alpha}{\beta} \end{cases}$$

Recapitulating:

$$\mathcal{L}^{-1}\left[\frac{Ms+N}{(s-\alpha)^2+\beta^2}\right] = e^{\alpha t}\left(M\cos(\beta t) + \frac{N+\alpha M}{\beta}\sin(\beta t)\right)u(t)$$

Back to example:

$$F(s) = \frac{s-1}{(s+1)(s^2+s+1)} = \frac{-2}{s+1} + \frac{2s+1}{\left(s+\frac{1}{2}\right)^2 + \frac{3}{4}}$$

Second fraction has $M = 2,\ N = 1,\ \alpha = -\frac{1}{2},\ \beta = \frac{\sqrt{3}}{2}$.

$$f(t) = -2e^{-t}u(t) + e^{-\frac{1}{2}t}\,2\cos\left(\frac{\sqrt{3}}{2}t\right)u(t)$$

II. Proper (but not strictly proper) case

This means that $m = \deg(P(s)) = \deg(Q(s)) = n$

$$F(s) = \frac{p_n s^n + \cdots + p_1 s + p_0}{s^n + \cdots + q_1 s + q_0} = \frac{P(s)}{Q(s)} = p_n + \frac{R(s)}{Q(s)},$$

where we define $R(s) := P(s) - p_n Q(s)$.

Note that $\deg(R(s)) \le n-1$, so $\dfrac{R(s)}{Q(s)}$ is strictly proper.

$$\mathcal{L}^{-1}\left[\frac{P(s)}{Q(s)}\right] = \mathcal{L}^{-1}[p_n] + \mathcal{L}^{-1}\left[\frac{R(s)}{Q(s)}\right] = p_n\delta(t) + \mathcal{L}^{-1}\left[\frac{R(s)}{Q(s)}\right]$$

The second inverse can be found using the previous case.

III. Improper case: $m = \deg(P(s)) > \deg(Q(s)) = n$

Rarely occurs in practice. Can be solved via polynomial division

$$P(s) = C(s)Q(s) + R(s),$$

with quotient polynomial $C(s)$, $\deg(C(s)) = m - n$.
and remainder polynomial $R(s)$, $\deg(R(s)) < \deg(Q(s))$

$$\rightarrow \quad \frac{P(s)}{Q(s)} = C(s) + \frac{R(s)}{Q(s)} = c_0 + c_1 s + \cdots + c_{m-n} s^{m-n} + \frac{R(s)}{Q(s)}$$

$$\mathcal{L}^{-1}\left[\frac{P(s)}{Q(s)}\right] = c_0\delta(t) + c_1\mathcal{L}^{-1}[s] + \cdots c_{m-n}\mathcal{L}^{-1}\left[s^{m-n}\right] + \mathcal{L}^{-1}\left[\frac{R(s)}{Q(s)}\right]$$

strictly proper

$$\mathcal{L}^{-1}[s] = \frac{d\delta}{dt}, \text{ since } \mathcal{L}\left[\frac{d\delta}{dt}\right] = s\mathcal{L}[\delta] - \delta(0-) = s.$$

Similarly, $\mathcal{L}^{-1}\left[s^k\right] = \dfrac{d^k\delta}{dt^k}$. $\rightarrow$ rarely arises.

Using Laplace to solve differential equations.

Linear, non-homogeneous, order n, constant coefficients.

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_1\frac{dy}{dt} + a_0 y(t) = f(t),$$

with given initial conditions $y(0-), \dfrac{dy}{dt}(0-)\cdots, \dfrac{d^{n-1}y}{dt^{n-1}}(0-)$

Find transforms of the derivatives: $\mathcal{L}\left[\dfrac{dy}{dt}\right] = sY(s) - y(0-),$

$$\mathcal{L}\left[\frac{d^2y}{dt^2}\right] = s\mathcal{L}\left[\frac{dy}{dt}\right] - \frac{dy}{dt}(0-) = s^2Y(s) - y(0-)s - \frac{dy}{dt}(0-)$$

$$\vdots$$

$$\mathcal{L}\left[\frac{d^n y}{dt^n}\right] = s^nY(s) - y(0-)s^{n-1} - \frac{dy}{dt}(0-)s^{n-2} - \cdots - \frac{d^{n-1}y}{dt^{n-1}}(0-)$$

$$\mathcal{L}\left[\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_1\frac{dy}{dt} + a_0 y(t)\right] = \mathcal{L}\left[f(t)\right],$$

$$\left(s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0\right)Y(s) - \begin{pmatrix}\text{Terms involving}\\ \text{initial conditions}\end{pmatrix} = F(s)$$

Example: $\dfrac{d^2y}{dt^2} + 3\dfrac{dy}{dt} + 2y(t) = t, \quad y(0-) = 0,\ \dfrac{dy}{dt}(0-) = 1.$

$$\mathcal{L}\left[\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y\right] = \left(s^2Y(s) - \underbrace{y(0-)}_{0}s - \underbrace{\frac{dy}{dt}(0-)}_{1}\right) +$$

$$3\left(sY(s) - \underbrace{y(0-)}_{0}\right) + 2Y(s) = \left(s^2 + 3s + 2\right)Y(s) - 1 = \mathcal{L}\left[t\right] = \frac{1}{s^2}$$

Solve for $Y(s) = \dfrac{s^2+1}{s^2(s^2+3s+2)}$, and use $\mathcal{L}^{-1}$ to get $y(t)$

General solution, homogeneous case.

The initial condition terms form a polynomial (say $P(s)$) of degree $n-1$. Setting $Q(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$, we have $Q(s)Y(s) - P(s) = 0 \ \rightarrow Y(s) = \dfrac{P(s)}{Q(s)}$

Therefore, if $y(t)$ is the solution of a linear, homogeneous, constant coefficient differential equation; its Laplace transform $Y(s)$ is always a strictly proper rational function

Inverting this transform by partial fractions, we find that $y(t)$ is a linear combination of terms of the forms

$$t^k e^{\alpha t},\ \ t^k e^{\alpha t}\cos(\beta t),\ \ t^k e^{\alpha t}\sin(\beta t).$$

Initial and Final Value Theorems

I.V.T.: $\lim_{t \to 0+} f(t) = f(0+) = \lim_{s \to +\infty} sF(s)$

F.V.T: $\lim_{t \to +\infty} f(t) = f(+\infty) = \lim_{s \to 0+} sF(s)$

REMARKS:

- Relates limiting values of a function and its transform. Can be used to get time-domain information from the Laplace transform.
- Opposite limits in t and in s. Here we take s to be real-valued.
- SOME RESTRICTIONS APPLY! Precise statements and proofs given below.

Final Value Theorem: Precise Statement

If the DOC of $F(s)$ includes the half-plane $\mathrm{Re}\left[s\right] > 0$,

and $f(+\infty) = \lim_{t \to +\infty} f(t)$ exists. Then $f(+\infty) = \lim_{s \to 0+} sF(s)$

Examples:

1) $f(t) = u(t) - u(t)e^{-t}$. $F(s) = \frac{1}{s} - \frac{1}{s+1}$, DOC: $\mathrm{Re}[s] > 0$

$\lim_{s \to 0+} sF(s) = \lim_{s \to 0+} 1 - \frac{s}{s+1} = 1 = f(+\infty)$

2) $f(t) = u(t)\cos(t)$, $F(s) = \frac{s}{s^2+1}$, DOC : $\mathrm{Re}[s] > 0$

$\lim_{s \to 0+} sF(s) = 0$. But $\lim_{t \to +\infty} f(t)$ does not exist!

Practical rule: use FVT if all poles of $F(s)$ are in $\mathrm{Re}\left[s\right] < 0$, or at $s = 0$.

Final Value Theorem: Proof

If the DOC of $F(s)$ includes the half-plane $\mathrm{Re}[s] > 0$,

and $f(+\infty) = \lim_{t \to +\infty} f(t)$ exists. Then $f(+\infty) = \lim_{s \to 0+} sF(s)$

Proof: from derivative property, $sF(s) = f(0-) + \int_{0-}^{\infty} e^{-st} \frac{df}{dt}\, dt$

Also, by integration we have $f(+\infty) = f(0-) + \int_{0-}^{\infty} \frac{df}{dt}\, dt$

Subtracting the above, we get $sF(s) - f(+\infty) = \int_{0-}^{\infty} \frac{df}{dt}\left(e^{-st} - 1\right) dt$

Now $\lim_{s \to 0+}\left(e^{-st} - 1\right) = 0$ for every t. Note this limit can be done within the DOC. Therefore the right-hand side goes to zero. ■

Initial Value Theorem: Precise Statement

If $f(t)$ has no impulses or higher order singularities ($\frac{d\delta}{dt}$, etc.) at $t = 0$, then $\lim_{t \to 0+} f(t) = f(0+) = \lim_{s \to +\infty} sF(s)$

Examples: 1) $f(t) = u(t)\cos(t)$, $F(s) = \frac{s}{s^2 + 1}$

$$\lim_{s \to +\infty} sF(s) = \lim_{s \to +\infty} \frac{s^2}{s^2 + 1} = 1 = f(0+)$$

2) $f(t) = \delta(t)$. $\lim_{s \to +\infty} sF(s) = \infty$. IVT does not apply

Practical rule: use IVT when $F(s)$ is a strictly proper rational function, or includes a delay term $e^{-\tau s}$.

Initial Value Theorem: Proof

If $f(t)$ has no impulses or higher order singularities

$(\frac{d\delta}{dt}$, etc.) at $t = 0$, then $\lim_{t \to 0+} f(t) = f(0+) = \lim_{s \to +\infty} sF(s)$

Proof: since there are no singularities, $F(s) = \int_{0+}^{\infty} e^{-st} f(t)\, dt$

Integration by parts gives $s\int_{0+}^{\infty} e^{-st} f(t)\, dt - f(0+) = \int_{0+}^{\infty} e^{-st} \frac{df}{dt}\, dt$

(same as in the proof of the derivative property, repacing $0-$ by $0+$)

Now $\lim_{s \to +\infty} e^{-st} = 0$ for every $t > 0$. Therefore the right-hand side goes to zero, and $\lim_{s \to +\infty} sF(s) = f(0+)$.

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