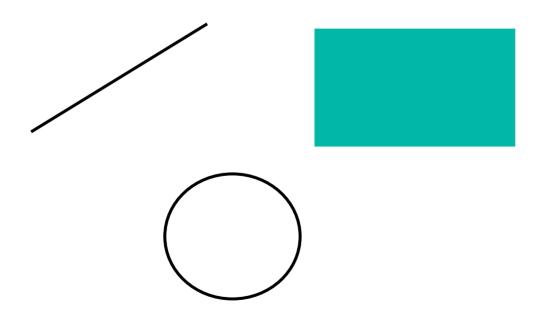
Primitives

Representations for Lines and Curves



Representations for lines and Curves

Line (in 2D)

- Explicit
- Implicit

Parametric

$$y = \frac{dy}{dx}(x - x_0) + y_0$$

$$F(x,y) = (x - x_0)dy - (y - y_0)dx$$

if
$$F(x,y) = 0$$
 then (x,y) is on line
 $F(x,y) > 0$ (x,y) is below line
 $F(x,y) < 0$ (x,y) is above line

$$x(t) = x_0 + t(x_1 - x_0)$$

$$y(t) = y_0 + t(y_1 - y_0)$$

$$t \in [0, 1]$$

$$P(t) = P_0 + t(P_1 - P_0)$$
, or $P(t) = (1 - t)P_0 + tP_1$

Circle

Explicit

$$y = \pm \sqrt{r^2 - x^2}, \quad |x| \le r$$

Implicit

$$x^{2} + y^{2} = r^{2}$$

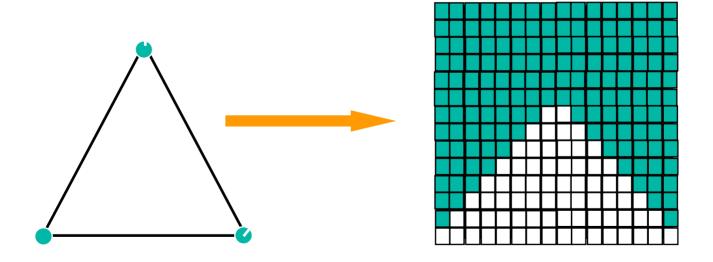
 $F(x, y) = x^{2} + y^{2} - r^{2}$

$$\begin{array}{ll} \text{if} & F(x,y) = 0 & \text{then} & (x,y) \text{ is on circle} \\ F(x,y) > 0 & (x,y) \text{ is outside} \\ F(x,y) < 0 & (x,y) \text{ is inside} \end{array}$$

Parametric

$$\begin{aligned} x(\theta) &= r \cos(\theta) \\ y(\theta) &= r \sin(\theta) \\ \theta &\in [0, 2\pi] \end{aligned}$$

Rasterization



Line rasterization

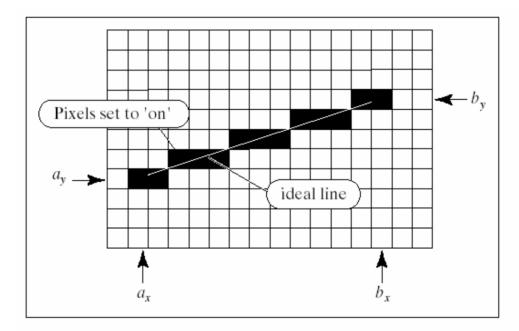


FIGURE 10.23 Drawing a straight-line-segment.

Line rasterization

Desired properties

- Straight
- Pass through end points
- Smooth
- Independent of end point order
- Uniform brightess
- Brightness independent of slope
- Efficient

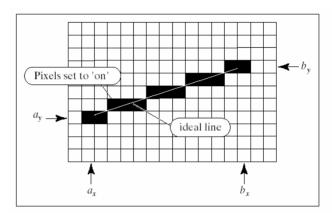


FIGURE 10.23 Drawing a straight-line-segment.

from Computer Graphics Using OpenGL, 2e, by F. S. Hill © 2001 by Prentice Hall / Prentice-Hall, Inc., Upper Sacklle River, New Jersey 07458

Straightforward Implementation

Line between two points

Better Implementation

How can we improve this algorithm?

```
DrawLine(int x1,int y1,int x2,int y2)
      float y;
       int x;
      for (x=x1; x \le x2; x++) {
               y = y1 + (x-x1)*(y2-y1)/(x2-x1)
               SetPixel(x, Round(y));
```

Better Implementation

```
DrawLine(int x1,int y1,int x2,int y2)
      float y,m;
      int x;
      dx = x2-x1;
      dy = y2-y1;
      m = dy/(float) dx;
      for (x=x1; x \le x2; x++) {
              y = y1 + m*(x-x1);
              SetPixel(x, Round(y));
```

Even Better Implementation

```
DrawLine(int x1,int y1,int x2,int y2)
      float y,m;
      int x;
      dx = x2-x1;
      dy = y2-y1;
      m = dy/(float) dx;
      y = y1 + 0.5;
      for (x=x1; x \le x2; x++) {
              SetPixel(x, Floor(y));
              y = y + m;
```

Midpoint algorithm (Bresenham)

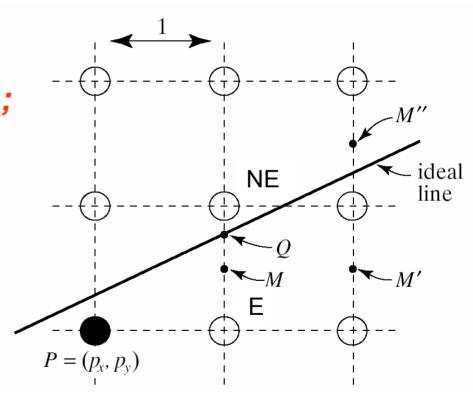
Line in the first quadrant (0<slope < 45 deg)

Implicit function:

$$F(x,y) = xdy - ydx + c,$$

 $dx,dy > 0$ and $dy/dx <= 1.0$;

- Current choice P = (x,y).
- How do we chose next of P,P'= (x+1,y') ?



Midpoint algorithm (Bresenham)

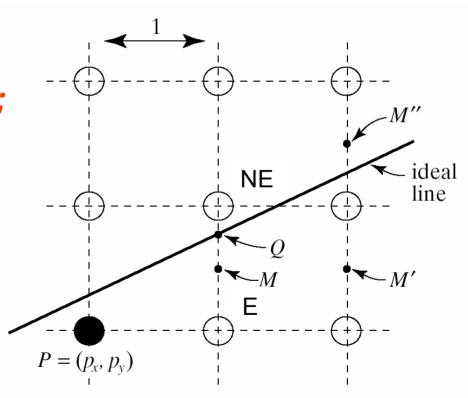
Line in the first quadrant (0<slope < 45 deg)

Implicit function:

$$F(x,y) = xdy - ydx + c,$$

 $dx,dy > 0$ and $dy/dx <= 1.0$;

- Current choice P = (x,y).
- How do we chose next of P,
 P'= (x+1,y') ?
 If(F(M) = F(x+1,y+0.5) < 0)
 M above line so E
 else
 M below line so NE

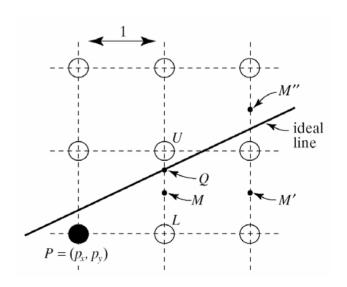


Midpoint algorithm (Bresenham)

```
DrawLine(int x1, int y1, int x2, int y2, int color)
        int x,y,dx,dy;
        y = Round(y1);
        for (x=x1; x \le x2; x++) {
                 SetPixel(x, y);
                                                                               ideal
                                                              NE
                 if (F(x+1,y+0.5)>0) {
                                                                               line
                          y = y + 1;
                                                               Ε
                                           P=(p_x,p_y)
```

 We are at pixel (x,y) we evaluate F at M = (x+1,y+0.5) and E=(x+1,y) or NE=(x+1,y+1) accordingly.

(Reminder: F(x,y) = xdy - ydx + c)



 We are at pixel (x,y) we evaluate F at M = (x+1,y+0.5) and E=(x+1,y) or NE=(x+1,y+1) accordingly.

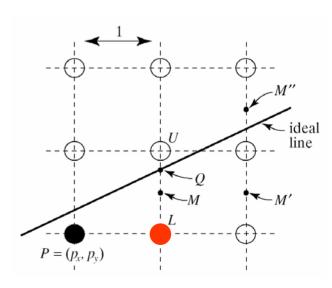
(Reminder:
$$F(x,y) = xdy - ydx + c$$
)

If we chose E for x+1 the next criteria will be at M':

$$F(x+2,y+0.5) = (x+1)dy + dy - (y+0.5)*dx +c \rightarrow$$

$$F(x+2,y+0.5) = F(x+1,y+0.5) + dy \rightarrow$$

$$F_E = F + dy = F + dF_E$$



We are at pixel (x,y) we evaluate F at M = (x+1,y+0.5) and E=(x+1,y) or NE=(x+1,y+1) accordingly.
 (Reminder: F(x,y) = xdy - ydx + c)

If we chose E for x+1 the next criteria will be at M':

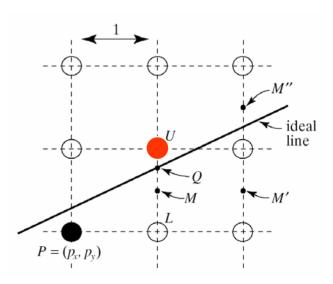
$$F(x+2,y+0.5) = (x+1)dy + dy - (y+0.5)*dx +c \rightarrow$$

$$F(x+2,y+0.5) = F(x+1,y+0.5) + dy \rightarrow F_E = F + dy$$

 If we chose NE then the next criteria will be at M":

$$F(x+2,y+1+0.5) =$$

 $F(x+1,y+0.5) + dy - dx \rightarrow$
 $F_{NE} = F + dy - dx$



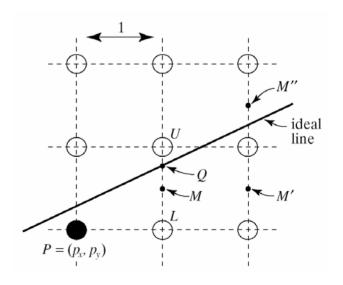
 We are at pixel (x,y) we evaluate F at M = (x+1,y+0.5) and E=(x+1,y) or NE=(x+1,y+1) accordingly.

(Reminder:
$$F(x,y) = xdy - ydx + c$$
)

If we chose E for x+1 the next criteria will be at M':

$$F_E = F + dy$$

 If we chose NE then the next criteria will be at M":



$$F_{NE} = F + dy - dx$$

Criterion update

Update

$$F_E = F + dy = F + dF_E$$

 $F_{NE} = F + dy - dx = F + dF_{NE}$

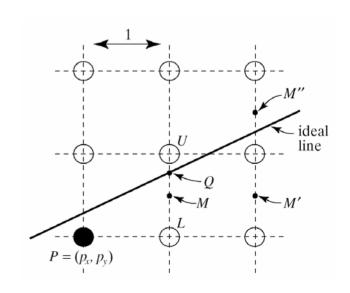
Starting value?

Assume line starts at pixel (x0,y0) $F_{\text{start}} = F(x0+1,y0+0.5) \rightarrow$

$$F_{start} = (x0+1)dy - (y0+0.5)dx + c$$

 $c = y0dx-x0dy$

$$F_{\text{start}} = dy-0.5dx$$



Criterion update (Integer version)

Update

$$F_{start} = dy -0.5dx$$

$$F_{E} = F + dy = F + dF_{E}$$

$$F_{NE} = F + dy - dx = F + dF_{NE}$$

Everything is integer except F_{start.}

Multiply by 2
$$\rightarrow$$
 $F_{start} = 2dy - dx$
 $dF_E = 2dy$
 $dF_{NE} = 2(dy-dx)$

Midpoint algorithm

```
DrawLine(int x1, int y1, int x2, int y2, int color)
        int x,y,dx,dy,dE, dNE;
        dx = x2-x1:
        dy = y2-y1;
        d = 2*dy-dx; // initialize d
        dE = 2*dy;
        dNE = 2*(dy-dx);
        y = y1;
        for (x=x1; x \le x2; x++)
                   SetPixel(x, y, color);
                   if (d>0) { // chose NE
                             d = d + dNE;
                             y = y + 1;
                   } else { // chose E
                             d = d + dE;
```

Incremental algorithms for polynomials

$$F(x) = a_n x^n + a_{n-1} x^{n-1} \dots + a_1 x + a_0, a_n \neq 0$$

$$F(x+d) = a_n (x+d)^n + a_{n-1} (x+d)^{n-1} \dots + a_1 (x+d) + a_0 =$$

$$= a_n (x+d)^n + P^{n-1} (x)$$

$$= a_n \sum_{k=0}^n \binom{n}{k} x^{n-k} d^k + P^{n-1} (x)$$

$$= a_n \sum_k \left(\frac{n}{k!(n-k)!} \right) x^{n-k} d^k + P^{n-1} (x)$$

$$= a_n x^n + \sum_{k=1}^n \left(\frac{n}{k!(n-k)!} \right) x^{n-k} d^k + P^{n-1} (x)$$

$$= a_n x^n + R^{n-1} (x) + P^{n-1} (x)$$

N-order differences

$$F(x) = a_n x^n + a_{n-1} x^{n-1} \dots + a_1 x + a_0, a_n \neq 0$$

$$F(x+d) = a_n x^n + R^{n-1}(x) + P^{n-1}(x)$$

First order

$$\Delta F = F(x+d) - F(x) = R^{n-1}(x) + P^{n-1}(x) = G_1^{n-1}(x)$$

N-order

$$\Delta^{2}F(x) = \Delta F(x+d) - \Delta F(x) = G_{2}^{n-2}(x)$$

$$\vdots$$

$$\Delta^{n}F(x) = \Delta^{n-1}F(x+d) - \Delta^{n-1}F(x) = G_{n}^{0} = c$$

N-order difference update

$$F(x) = a_n x^n + a_{n-1} x^{n-1} \dots + a_1 x + a_0, a_n \neq 0$$

$$F(x+d) = a_n x^n + R^{n-1}(x) + P^{n-1}(x)$$

$$F(x+d) = F(x) + \Delta F(x)$$

$$\Delta F(x+d) = \Delta F(x) + \Delta^2 F(x)$$

$$\vdots$$

$$\Delta^{n-1} F(x+d) = \Delta^{n-1} F(x) + \Delta^n F(x)$$

$$\Delta^n F(x+d) = c$$

We need n initial conditions to initialize the differences.

Example: $y = x^2$

$$y(x+d) = x^{2} + 2xd + d^{2} = y(x) + 2xd + d^{2}$$

$$\rightarrow y(x+d) = y(x) + \Delta y(x)$$

$$where \ \Delta y(x) = 2xd + d^{2}$$

$$\Delta y(x+d) = 2(x+d)d + d^{2} = \Delta y(x) + 2d^{2}$$

$$\rightarrow \Delta y(x+d) = \Delta y(x) + \Delta^{2}y(x)$$

$$where \ \Delta^{2}y(x) = 2d^{2}$$

The incremental algorithm to compute y = x^2

```
computePar(int d)
     float y = 0;
     int x = 0;
     DY = d^2 ; // at x = 0
     DDY = 2*d^2 :
     for( x = 0 ; x < X MAX ; x++ ) {
            printf("d, %f\", x,y);
            y = y + DY;
            DY = DY + DDY;
```

Polygon

Collection of points connected with lines

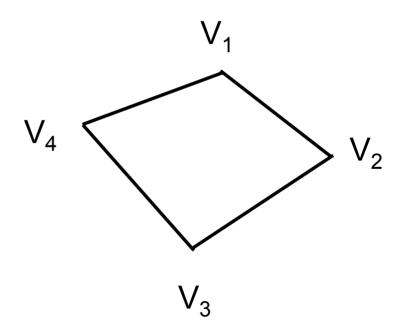
- Vertices: v1,v2,v3,v4
- Edges:

$$e_1 = v_1 v_2$$

$$e_2 = v_2 v_3$$

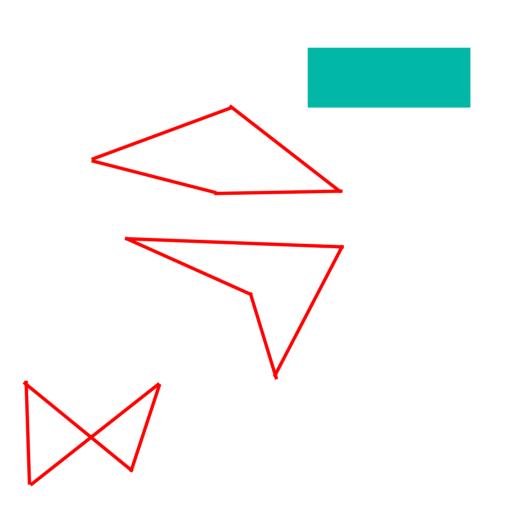
$$e_3 = v_3 v_4$$

$$e_4 = v_4 v_1$$



Polygons

- Open / closed
- Planar / non-planar
- Filled / wireframe
- Convex / concave
- Simple / non-simple



Triangles

The most common primitive

- Convex
- Planar
- Simple



Background

Plane equations

Implicit

$$F(x, y, z) = Ax + By + Cz + D = \mathbf{N} \bullet P + D$$

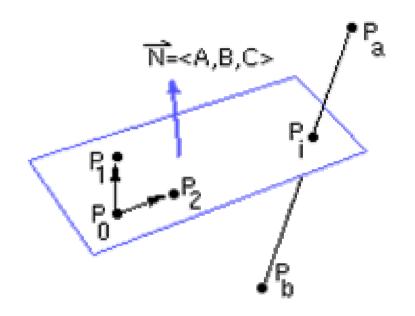
Points on Plane $F(x, y, z) = 0$

Parametric

$$\begin{aligned} Plane(s,t) &= P_0 + s(P_1 - P_0) + t(P_2 - P_0) \\ P_0, P_1, P_2 & \text{not colinear} \\ \text{or} \\ Plane(s,t) &= (1 - s - t)P_0 + sP_1 + tP_2 \\ Plane(s,t) &= P_0 + sV_1 + tV_2 \text{ where } V_1, V_2 \text{ basis vectors} \end{aligned}$$

Explicit

$$z = -(A/C)x - (B/C)y - D/C, C \neq 0$$

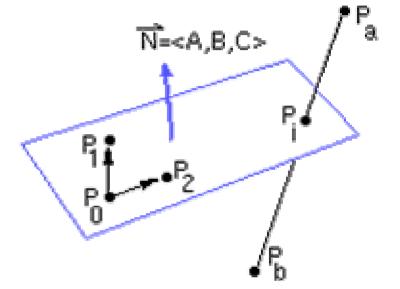


Point normal form

Plane equation

$$F(x, y, z) = Ax + By + Cz + D = \mathbb{N} \bullet P + D$$

Points on Plane $F(x, y, z) = 0$



Observation: Let's take an arbitrary vector u that lies on the plane which can be defined by two points e.g. P1, P2 on the plane.

$$\mathbf{u} = P2 - P1$$

$$\begin{vmatrix}
\mathbf{N} \bullet P1 + D = 0 \\
\mathbf{N} \bullet P2 + D = 0
\end{vmatrix} \Rightarrow \mathbf{N} \bullet (P2 - P1) = 0 \Rightarrow \mathbf{N} \bullet \mathbf{u} = 0 \Rightarrow \mathbf{N} \perp \mathbf{u}$$

Computing point normal form from 3 Points

$$F(x, y, z) = Ax + By + Cz + D = \mathbf{N} \bullet P + D$$

Points on Plane $F(x, y, z) = 0$

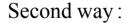
First way:

$$\mathbf{N} \bullet P0 + D = 0$$

$$\mathbf{N} \bullet P1 + D = 0$$

$$\mathbf{N} \bullet P2 + D = 0$$

 $| \mathbf{N} | = 1$ (arbitrary choice)



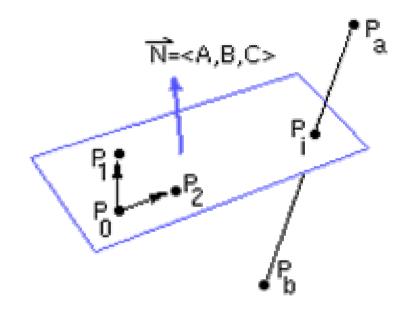
N is normal to F

Let's find a normal vector:

$$\mathbf{N} = (P1 - P0) \times (P2 - P0)$$

Compute *D*:

$$D = -\mathbf{N} \bullet P0$$



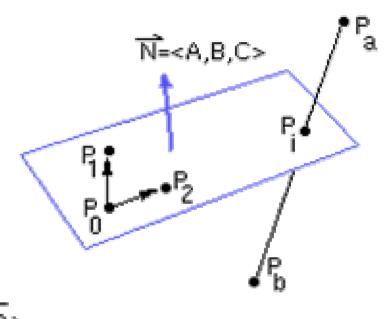
Intersection of lines and planes

Plane: PI(P) = N*P + D = 0

Line: Pa+t(Pb-Pa), t in R

$$t = \frac{-D - \overrightarrow{N} \cdot P_a}{\overrightarrow{N} \cdot P_b - \overrightarrow{N} \cdot P_a} = \frac{-F(P_a)}{F(P_b) - F(P_a)}$$

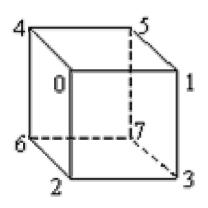
Substitute t in L(t)



Polygonal models/ data structures

[Hill: p. 287-291. Foley & van Dam: p. 471-477]

Indexed face set



fa	ces	ve	rtex list
#	vertex list	#	x,y,z
0 1 2 3 4 5	0,2,3,1 1,3,7,5 5,7,6,4 4,6,2,0 4,0,1,5 2,6,7,3	0 1 2 3 4 5 6 7	0,1,1 1,1,1 0,0,1 1,0,1 0,1,0 1,1,0 0,0,0

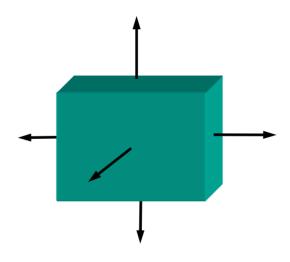
Polygon attributes

Per vertex or per face

- Normal
- Color

Per vertex

Texture coordinates



Computing the normal of a polygon

One way:

$$N = (V_{n-1} - V_0) \times (V_1 - V_0)$$

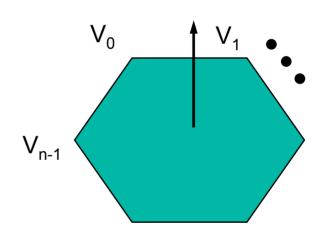
Newell's method (page 292)

$$N_{x} = \sum_{i=0}^{n-1} (y_{i} - y_{next(i)})(z_{i} + z_{next(i)})$$

$$N_{y} = \sum_{i=0}^{n-1} (z_{i} - z_{next(i)})(x_{i} + x_{next(i)})$$

$$N_{z} = \sum_{i=0}^{n-1} (x_{i} - x_{next(i)})(y_{i} + y_{next(i)})$$

$$where \quad next(j) = (j+1) \mod n$$



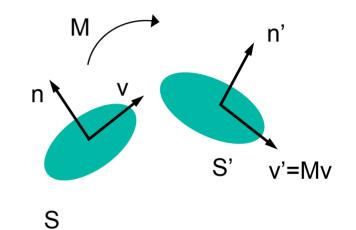
Transforming Normals

Normal vectors are transformed along with vertices and polygons.

- How do you transform a normal?
- What about unit magnitude?

Deriving transformation of normal

$$\mathbf{n} = (n_x, n_y, n_z, 0)^T$$
 normal to S
 $\mathbf{v} = (v_x, v_y, v_z, 0)^T$ tangent to S
 $S' = MS$, what is \mathbf{n}' ?



$$\mathbf{n} \cdot \mathbf{v} = \mathbf{n}^T \mathbf{v} = 0$$

$$\mathbf{n}^T \mathbf{v} = 0 \to \mathbf{n}^T I \mathbf{v} = 0 \to \mathbf{n} (M^{-1} M) \mathbf{v} = 0$$

$$\to (\mathbf{n}^T M^{-1})(M \mathbf{v}) = 0 \to (M^{-T} \mathbf{n})^T (M \mathbf{v}) = 0$$

$$\to (M^{-T} \mathbf{n}) \cdot (M \mathbf{v}) = 0$$

Normalization

glEnable(GL_NORMALIZE) ;

- Transformation includes scale or shear
- We provide non unit normals

Polygons in OpenGL

```
glPolygonMode(GL FRONT,GL FILL);
glPolygonMode(GL BACK,GL LINE);
glBegin(GL POLYGON)
glNormal3f(v1,v2,v3);
gIVertex3f(x1,y1,z1);
glNormal3f(v1n,v2n,v3n);
glVertex3f(xn,yn,zn);
glEnd();
```

Polygon Rasterization

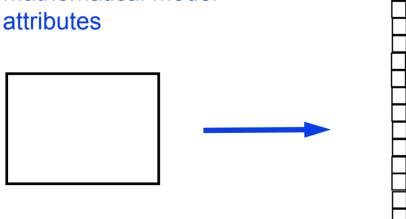
[Hill: 570-576. Foley & vanDam: p. 92-99]

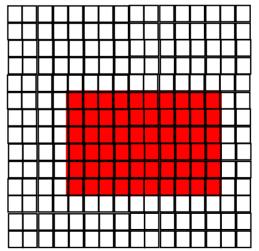
Scan conversion

shade pixels lying within a closed polygon efficiently.

Mathematical model +

Screen space





Polygon Rasterization

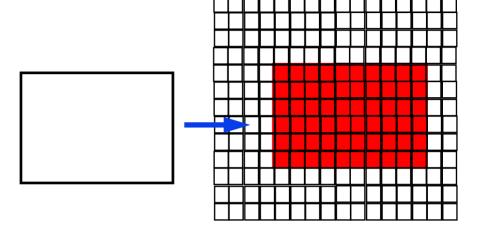
[Hill: 570-576. Foley & vanDam: p. 92-99]

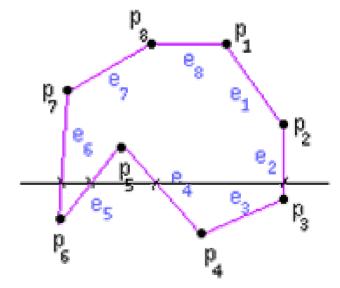
Scan conversion

shade pixels lying within a closed polygon efficiently.

Algorithm

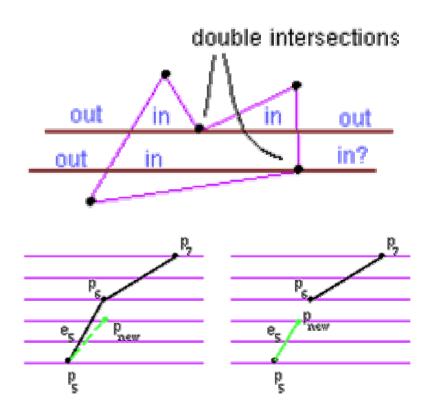
- intersect each scanline with all edges
- sort intersections in x
- calculate parity of intersections to determine in/out
- fill the 'in' pixels





Special cases

- Horizontal edges can be excluded
- Vertices lying on scanlines
 - Change in sign of y_i-y_{i+1}: count twice
 - No change: shorten edge by one scanline



Efficiency?

Many intersection tests can be eliminated by taking advantage of coherence between adjacent scanlines.

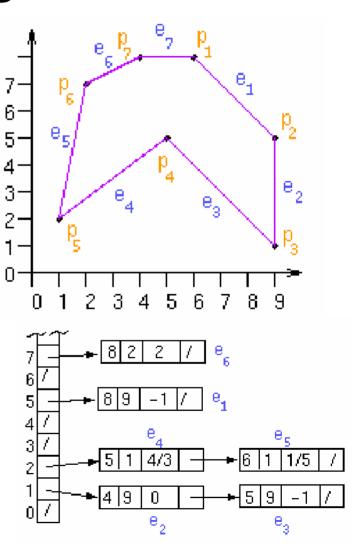
- Edges that intersect scanline y are likely to intersect y+1
- x changes predictably from scanline y to y+1

$$y = mx + a \rightarrow x = 1/m(y + a) \rightarrow x(y + 1) = x(y) + 1/m$$

Data structure 1: Edge table

Building edge table

- Traverse edges
- Eliminate horizontal edges
- If not local extremum, shorten upper vertex
- Add edge to linked-list for the scanline corresponding to the lower vertex, storing:
 - y_upper: last scanline to consider
 - x_lower: starting x coordinate for edge
 - 1/m: for incrementing x; compute before shortening



Data structure 2: Active Edge List (AEL)

- The AEL is a linked list of active edges on the current scanline, y.
- Each active edge has the following information:
 - y_upper: last scanline to consider
 - x: edge's intersection with current y)
 - 1/m: for incrementing x

The active edges are kept sorted by x.

Scan conversion algorithm

```
for each scanline

add edges in edge table to AEL

if AEL <> NIL

sort AEL by x

fill pixels between edge pairs

delete finished edges

update each edge's x in AEL
```

Example

for each scanline
add edges in edge table to AEL
if AEL <> NIL
sort AEL by x
fill pixels between edge pairs
delete finished edges
update each edge's x in AEL

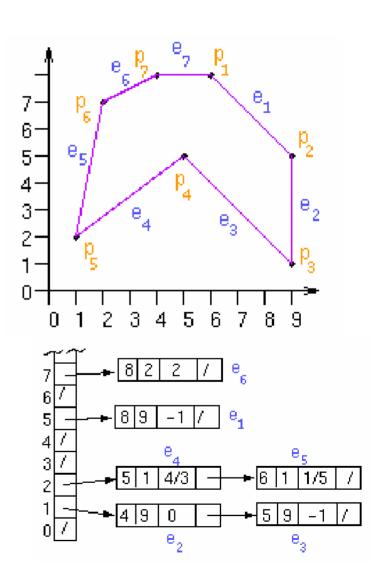
Reminder:

Edge table

y_upper x_lower 1/m

AEL:

y_upper |x_current | 1/m



Special cases

Triangles – Convex Polygons

Maximum two edges per scanline

Overlaping polygons

priorities

Color, patterns

Z for visibility

Interpolating information (incrementally)

Color, Normal, Texture coordinates

Right edge (1,2):

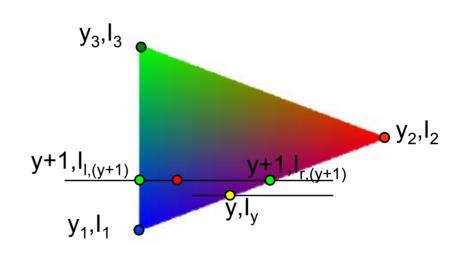
$$\frac{I_{r,(y+1)} - I_{r,y}}{(y+1) - y} = \frac{I_1 - I_2}{y_1 - y_2} \Rightarrow I_{r,(y+1)} = I_{r,y} + \frac{I_1 - I_2}{y_1 - y_2}$$

Left Edge (1,3):

$$\frac{I_{l,(y+1)} - I_{l,y}}{(y+1) - y} = \frac{I_1 - I_3}{y_1 - y_3} \Rightarrow I_{l,(y+1)} = I_{l,y} + \frac{I_1 - I_3}{y_1 - y_3}$$

Along scanline:

$$\frac{I_{(x+1)} - I_x}{(x+1) - x} = \frac{I_r - I_l}{x_r - x_l} \Longrightarrow I_{r,(y+1)} = I_{r,y} + \frac{I_r - I_l}{x_r - x_l}$$



Interpolating information (incrementally)

Color, Normal, Texture coordinates

Right edge (1,2):

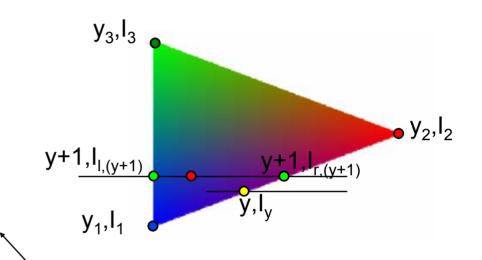
$$\frac{I_{r,(y+1)} - I_{r,y}}{(y+1) - y} = \frac{I_1 - I_2}{y_1 - y_2} \Rightarrow I_{r,(y+1)} = I_{r,y} + \frac{I_1 - I_2}{y_1 - y_2}$$

Left Edge (1,3):

$$\frac{I_{l,(y+1)} - I_{l,y}}{(y+1) - y} = \frac{I_1 - I_3}{y_1 - y_3} \Rightarrow I_{l,(y+1)} = I_{l,y} + \frac{I_1 - I_3}{y_1 - y_3}$$

Along scanline:

$$\frac{I_{(x+1)} - I_x}{(x+1) - x} = \frac{I_r - I_l}{x_r - x_l} \Rightarrow I_{r,(y+1)} = I_{r,y} + \frac{I_r - I_l}{x_r - x_l}$$

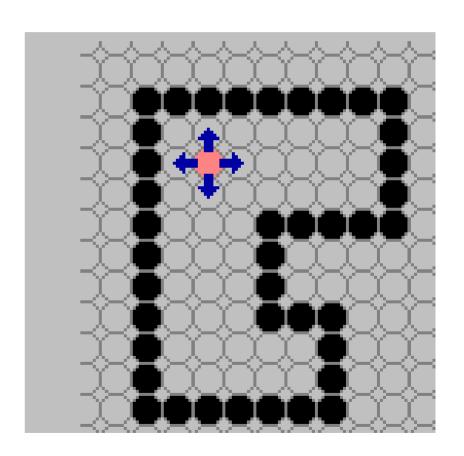


Constant along the line

Pixel Region filling algorithms[Hill 561-577]

Scan convert boundary
Fill in regions

2D paint programs



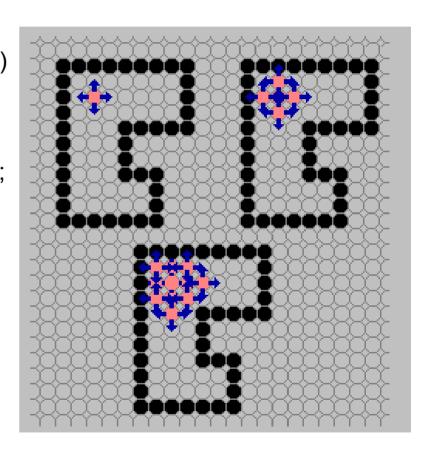
http://www.cs.unc.edu/~mcmillan/comp136/Lecture8/areaFills.html

BoundaryFill

```
boundaryFill(int x, int y, int fill, int boundary) {
    if ((x < 0) || (x >= raster.width)) return;
    if ((y < 0) || (y >= raster.height)) return;
    int current = raster.getPixel(x, y);
    if ((current != boundary) & (current != fill)) {
        raster.setPixel(fill, x, y);
        boundaryFill(x+1, y, fill,boundary);
        boundaryFill(x, y+1, fill, boundary);
        boundaryFill(x-1, y, fill, boundary);
        boundaryFill(x, y-1, fill, boundary);
    }
}
```

Flood Fill

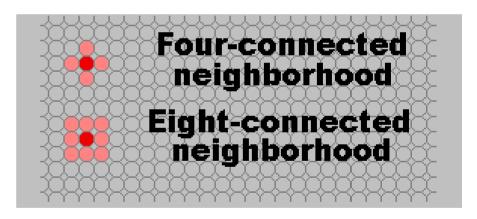
```
public void floodFill(int x, int y, int fill, int old)
     if ((x < 0) || (x >= raster.width)) return;
     if ((y < 0) || (y >= raster.height)) return;
      if (raster.getPixel(x, y) == old) {
         raster.setPixel(fill, x, y);
         floodFill(x+1, y, fill, old);
         floodFill(x, y+1, fill, old);
         floodFill(x-1, y, fill, old);
         floodFill(x, y-1, fill, old);
```



Adjacency

4-connected

8 connected



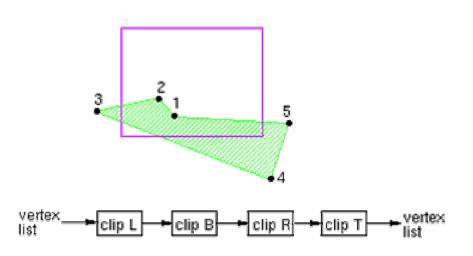
Polygon clipping (2D)[Hill 181-208]

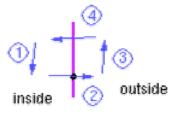
Sutherland-Hodgeman [Hill 202]

for each side of clipping window

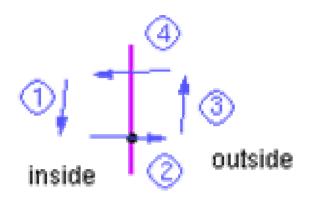
for each edge of polygon

output points based upon the following table

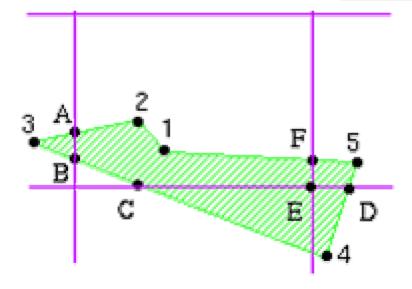




case	first	second	output
#	point	point	point(s)
1 2 3 4	inside inside outside outside	outside	second point intersection point none intersection point and second point



case	first	second	output
#	point	point	point(s)
1 2 3 4	inside inside outside outside	inside outside outside inside	



original: 1,2,3,4,5,1

clip L: 1,2,A,B,4,5,1 clip B: 1,2,A,B,C,D,5,1 clip R: 1,2,A,B,C,E,F,1 clip T: (same)

Outcodes for trivial reject/accept

[Hill 389] A vertex outcode consists of four bits: TBRL, where:

T is set if y > top,

B is set if y < bottom,

R is set if x > right, and

L is set if x < left.

Trivial accept: all vertices are inside (all outcodes are 0000, bitwise OR)

Trivial reject: all vertices are outside with respect to any given side(bitwise AND is not 0000)

