Lecture 15: Properties of the Fourier Transform

$$F(i\omega) = \mathcal{F}\left[f(t)\right] = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt$$

$$f(t) = \mathcal{F}^{-1}\left[F(i\omega)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(i\omega)e^{i\omega t}\,d\omega$$

1) Linearity. $\mathcal{F}\left[\alpha f(t) + \beta g(t)\right] = \alpha\mathcal{F}\left[f(t)\right] + \beta\mathcal{F}\left[g(t)\right]$

Follows directly from linearity of the integral.

2) $\mathcal{F}\left[f(-t)\right] = F(-i\omega)$.

$$\text{Proof}: \int_{-\infty}^{\infty} f(-t)e^{-i\omega t}\,dt \underbrace{=}_{\sigma=-t} \int_{-\infty}^{\infty} f(\sigma)e^{i\omega\sigma}\,d\sigma = F(-i\omega)$$

3) If $f(t)$ is real-valued, then $F(-i\omega) = \overline{F(i\omega)}$

$$\text{Proof}: \int_{-\infty}^{\infty} f(t)e^{i\omega t}\,dt = \int_{-\infty}^{\infty} \overline{f(t)e^{-i\omega t}}\,dt = \overline{\int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt}$$

In particular, for real-valued $f(t)$ we have:

$\left|F(-i\omega)\right| = \left|F(i\omega)\right|$ (the magnitude is an even function of ω).

$\theta_F(-\omega) = -\theta_F(\omega)$ (the phase is an odd function of ω)

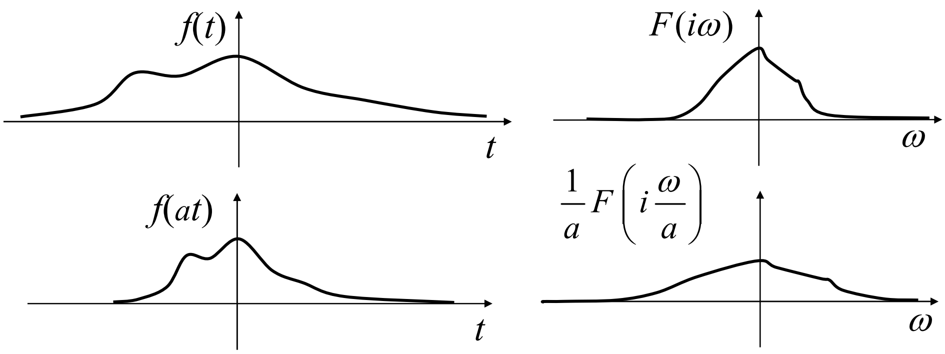
4a) $f(t)$ real and even $\left(f(t) = f(-t)\right) \Rightarrow F(i\omega)$ real (and even).

$$\text{Proof: } F(i\omega) = \mathcal{F}\left[f(t)\right] = \mathcal{F}\left[f(-t)\right] \underbrace{=}_{2)} F(-i\omega) \underbrace{=}_{3)} \overline{F(i\omega)}$$

4b) $f(t)$ real and odd $\left(f(t) = -f(-t)\right) \Rightarrow F(i\omega)$ purely imaginary (and odd). Proof: $F(i\omega) \underbrace{=}_{1)} -\mathcal{F}\left[f(-t)\right] \underbrace{=}_{2)} -F(-i\omega) \underbrace{=}_{3)} -\overline{F(i\omega)}$

5) $\mathcal{F}[f(at)] = \frac{1}{|a|} F\left(i\frac{\omega}{a}\right).$

Proof $(a>0)$: $\displaystyle\int_{-\infty}^{\infty} f(at) e^{-i\omega t}\,dt \underbrace{=}_{\sigma = at} \int_{-\infty}^{\infty} f(\sigma) e^{-i\frac{\omega}{a}\sigma} \frac{d\sigma}{a} = \frac{1}{a} F\left(i\frac{\omega}{a}\right).$



Compress time $\Longleftrightarrow$ Stretch frequency, and scale down.

6) Delay property: $\mathcal{F}[f(t-t_0)] = e^{-i\omega t_0} F(i\omega)$

Proof : $\displaystyle\int_{-\infty}^{\infty} f(t-t_0) e^{-i\omega t}\,dt \underbrace{=}_{\sigma = t - t_0} \int_{-\infty}^{\infty} f(\sigma) e^{-i\omega(\sigma + t_0)} d\sigma$

Remark: no change in magnitude $\left|e^{-i\omega t_0} F(i\omega)\right| = |F(i\omega)|$, but we add a phase that is linear in ω.

Example: let $\mathcal{F}[f(t)] = F(i\omega) = \frac{1}{1+i\omega}$. Find $\mathcal{F}[f(2t-4)]$

Set $g(t) = f(2t)$. Then $G(i\omega) = \frac{1}{2} F\left(i\frac{\omega}{2}\right) = \frac{1}{2} \frac{1}{1 + i\frac{\omega}{2}} = \frac{1}{2+i\omega}$.

Now $f(2t-4) = f(2(t-2)) = g(t-2)$.

$$\Rightarrow \mathcal{F}[f(2t-4)] = e^{-i\omega 2} G(i\omega) = \frac{e^{-i\omega 2}}{2+i\omega}.$$

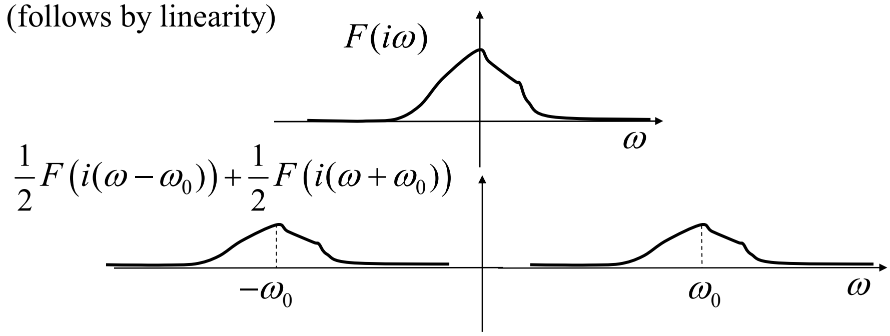
7) Modulation property: $\mathcal{F}\left[e^{i\omega_0 t} f(t)\right] = F\left(i(\omega-\omega_0)\right)$

Proof : $\int_{-\infty}^{\infty} e^{i\omega_0 t} f(t) e^{-i\omega t}\, dt = \int_{-\infty}^{\infty} f(t) e^{-i(\omega-\omega_0)t}\, dt$

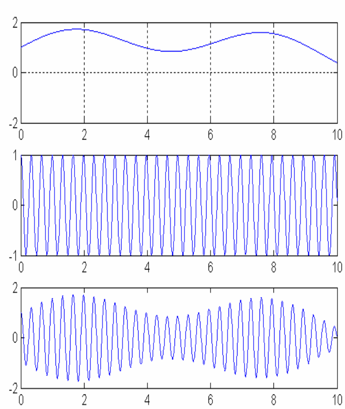
Note: dual of the delay property.

Another version: $\mathcal{F}[\cos(\omega_0 t) f(t)] = \frac{1}{2} F\left(i(\omega-\omega_0)\right) + \frac{1}{2} F\left(i(\omega+\omega_0)\right)$

(follows by linearity)



In the time domain, this is amplitude modulation (AM)



$f(t)$ "Baseband" signal (e.g. audio).

$\cos(\omega_0 t)$ "Carrier"; e.g., $\omega_0 = 2\pi\, 790 KHz$

$f(t)\cos(\omega_0 t)$ Modulated signal (what is broadcast by the antenna)

In the time domain, modulation is complicated. However in the frequency domain, it amounts to nothing more than shifting in frequency.

8) Derivative property: $\mathcal{F}\left[\frac{df}{dt}\right] = (i\omega)F(i\omega)$

Note: similar to Laplace case, but no initial condition here.

Proof : differentiating $f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(i\omega)e^{i\omega t}\, d\omega$

under the sign, with respect to t gives

$$\frac{df}{dt} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{d}{dt}\left(F(i\omega)e^{i\omega t}\right) d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} (i\omega)F(i\omega)e^{i\omega t} d\omega$$

9) Dual: $\mathcal{F}\left[(-it)f(t)\right] = \frac{d}{d\omega}F(i\omega)$

Analogous proof.

10) Parseval relation

$$\int_{-\infty}^{\infty} |f(t)|^2\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(i\omega)|^2\, d\omega$$

Integral square is preserved between time and frequency.

Proof : $\int_{-\infty}^{\infty} |f(t)|^2\, dt = \int_{-\infty}^{\infty} f(t)\overline{f(t)}\, dt$

$$= \int_{-\infty}^{\infty} f(t)\overline{\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} F(i\omega)e^{i\omega t}\, d\omega\right)} dt =$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{F(i\omega)} \underbrace{\left(\int_{-\infty}^{\infty} f(t)e^{-i\omega t}\, dt\right)}_{F(i\omega)} d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(i\omega)|^2\, d\omega$$

Application of Fourier to LTI systems

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(t-\sigma)x(\sigma)d\sigma$$

Theorem: Given an LTI system of impulse response $h(t)$; let $X(i\omega) = \mathcal{F}\left[x(t)\right]$, $H(i\omega) = \mathcal{F}\left[h(t)\right]$, $Y(i\omega) = \mathcal{F}\left[y(t)\right]$ (assume they exist). Then $Y(i\omega) = H(i\omega)X(i\omega)$ for all ω.

Remarks:

- The Fourier transform takes convolutions into products.
- Similar to Laplace, but here we do not require causality, and $x(t)$ could be nonzero for all time.

Proof: $Y(i\omega) = \int_{-\infty}^{\infty} y(t)e^{-i\omega t}dt = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} h(t-\sigma)x(\sigma)d\sigma\right]e^{-i\omega t}dt$

$$\underbrace{=}_{\text{Change order of integration.}} \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} h(t-\sigma)e^{-i\omega t}dt\right]x(\sigma)d\sigma$$

$$\underbrace{=}_{\tau=t-\sigma} \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} h(\tau)e^{-i\omega(\sigma+\tau)}d\tau\right]x(\sigma)d\sigma$$

$$= \int_{-\infty}^{\infty} h(\tau)e^{-i\omega\tau}d\tau \int_{-\infty}^{\infty} e^{-i\omega\sigma}x(\sigma)d\sigma = H(i\omega)X(i\omega)$$

Definition: The Fourier transform $H(i\omega) = \mathcal{F}\left[h(t)\right]$ of the system impulse response function is called the frequency response function of the LTI system.

Interpretation: system response to a sinusoid

$x(t) = e^{i\omega_0 t}$

$t \in (-\infty, +\infty)$

$y(t) = H(i\omega_0)e^{i\omega_0 t}$

We've seen this before! Only difference: here we allow for a non-causal system, provided $\mathcal{F}\left[h(t)\right]$ exists.

Proof: $X(i\omega) = 2\pi\delta(\omega - \omega_0)$

$$\Rightarrow Y(i\omega) = H(i\omega)2\pi\delta(\omega - \omega_0) \underbrace{=}_{\text{PROPERTY OF } \delta} H(i\omega_0)2\pi\delta(\omega - \omega_0)$$

$$\Rightarrow y(t) = H(i\omega_0)\mathcal{F}^{-1}\left[2\pi\delta(\omega - \omega_0)\right] = H(i\omega_0)e^{i\omega_0 t}$$

For a general input $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(i\omega)e^{i\omega t}\, d\omega$ (superposition of sinusoids)

Interpretation of $Y(i\omega) = H(i\omega)X(i\omega)$: frequency components of the output $y(t)$ are filtered by the frequency response $H(i\omega)$.

Fourier and Convolution

Let $F(i\omega) = \mathcal{F}\left[f(t)\right]$, $G(i\omega) = \mathcal{F}\left[g(t)\right]$

We have already proved that $\mathcal{F}\left[f(t) * g(t)\right] = F(i\omega)G(i\omega)$ (Fourier turns convolutions into products).

Dual property: $\mathcal{F}\left[f(t)g(t)\right] = \frac{1}{2\pi}F(i\omega) * G(i\omega)$. (Fourier turns products into convolutions, up to a 2π factor). Proof:

$$\int_{-\infty}^{\infty} f(t)g(t)e^{-i\omega t}dt \underbrace{=}_{\substack{\text{INVERSE}\\ \text{FORMULA}}} \int_{-\infty}^{\infty}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} F(i\lambda)e^{i\lambda t}d\lambda\right]g(t)e^{-i\omega t}dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(i\lambda)\underbrace{\left[\int_{-\infty}^{\infty} g(t)e^{-i(\omega-\lambda)t}dt\right]}_{G(i(\omega-\lambda))}d\lambda = \frac{1}{2\pi}F(i\omega) * G(i\omega)$$