Lecture 11

Fourier series:

- Definition
- Finding the Fourier coefficients
- Convergence of Fourier series
- Representation: amplitude and phase spectra.

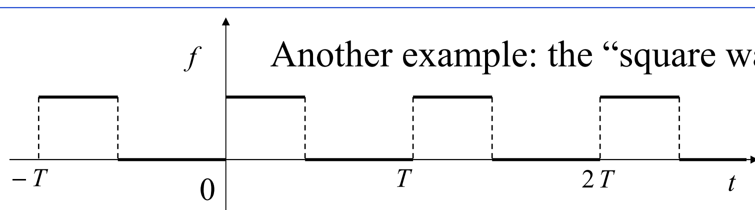
Recall: Periodic functions

A function $f(t)$ defined on $t \in (-\infty, +\infty)$ is called periodic if there exists $T > 0$ such that $f(t+T) = f(t) \quad \forall\ t \in (-\infty, +\infty)$

The smallest T satisfying the above is called the period of $f(t)$

Example: sinusoids $\sin(\omega_0 t)$, $\cos(\omega_0 t)$, $\cos(\omega_0 t + \varphi)$, $e^{i\omega_0 t}$ are periodic, with period $T = \dfrac{2\pi}{\omega_0}$.

Another example: the "square wave"



If $f_1(t)$, $f_2(t)$ are periodic, period T_1, T_2, then $f_1(t) + f_2(t)$ is periodic provided T_1, T_2 have a common multiple.

An infinite sum of sinusoids

Let ω_0 be a given frequency, and define $T = \dfrac{2\pi}{\omega_0}$. Consider

$f(t) = \sum_{n=-\infty}^{+\infty} F_n e^{in\omega_0 t}$, where the F_n are complex numbers.

Each sinusoid $F_n e^{in\omega_0 t}$ has period $\dfrac{2\pi}{n\omega_0} = \dfrac{T}{n}$.

Since these have a common multiple T, the series, if convergent, defines a periodic function of time.

Which periodic functions can we get in this way?
It turns out that all sufficiently regular periodic functions admit the above representation, called a Fourier Series.

Fourier Series

Definition: a periodic function $f(t)$ admits a Fourier series expansion if it can be expressed as

$$f(t) = \sum_{n=-\infty}^{+\infty} F_n e^{in\omega_0 t}$$

Here:

- ω_0 is called the fundamental frequency.
  The frequencies appearing in the terms $e^{in\omega_0 t}$ of the Fourier expansion are integer multiples of ω_0.
- The period of $f(t)$ is $T = \dfrac{2\pi}{\omega_0}$.
- The numbers F_n are called the Fourier coefficients of $f(t)$.
- The series is interpreted as $f(t) = \lim_{N\to+\infty} \sum_{n=-N}^{N} F_n e^{in\omega_0 t}$

Finding the Fourier coefficients

Given $f(t)$ of period T, fundamental frequency $\omega_0 = \frac{2\pi}{T}$.

Assume it has a Fourier expansion $f(t) = \sum_{n=-\infty}^{+\infty} F_n e^{in\omega_0 t}$

How do we find the coefficients F_n ?

Let's try integrating $f(t)$ and the expansion over the period.

$$\int_0^T f(t)\,dt = \int_0^T \sum_{n=-\infty}^{+\infty} F_n e^{in\omega_0 t} dt = \sum_{n=-\infty}^{+\infty} F_n \int_0^T e^{in\omega_0 t} dt = F_0 T$$

Two cases:

- For $n \neq 0$, $\int_0^T e^{in\omega_0 t} dt = \left. \frac{e^{in\omega_0 t}}{in\omega_0} \right|_0^T \underbrace{=}_{\omega_0 T = 2\pi} \frac{1}{in\omega_0}\left(e^{in2\pi} - e^0\right) = 0.$
- For $n = 0$, $\int_0^T e^{in\omega_0 t} dt = \int_0^T dt = T$

$$\boxed{F_0 = \frac{1}{T}\int_0^T f(t)\,dt}$$

Finding the Fourier expansion $f(t) = \sum_{n=-\infty}^{+\infty} F_n e^{in\omega_0 t}$

Extend the idea to find the coefficient F_n, $n \neq 0$.

- First, multiply by $e^{-in\omega_0 t}$ (changing "dummy" index to k)

$$f(t)e^{-in\omega_0 t} = \left(\sum_{k=-\infty}^{+\infty} F_k e^{ik\omega_0 t}\right) e^{-in\omega_0 t} = \sum_{k=-\infty}^{+\infty} F_k e^{i(k-n)\omega_0 t}$$

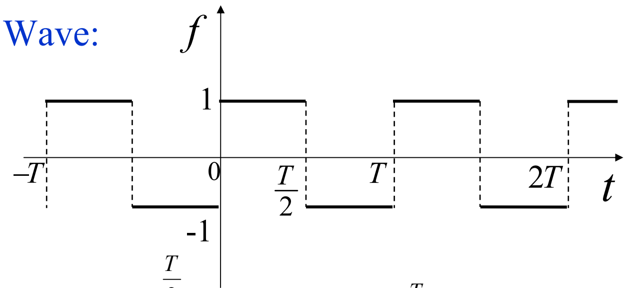
- Next, integrate over the period [0,T]

$$\int_0^T f(t) e^{-in\omega_0 t} dt = \sum_{k=-\infty}^{+\infty} F_k \int_0^T e^{i(k-n)\omega_0 t} dt \overbrace{=}^{\text{ONLY NONZERO TERM IS } k=n} F_n T$$

$$\boxed{F_n = \frac{1}{T}\int_0^T f(t)\, e^{-in\omega_0 t} dt}$$

Remark: we could instead integrate over $\left[-T/2, T/2\right]$, or another interval of length T

Example: Square Wave:



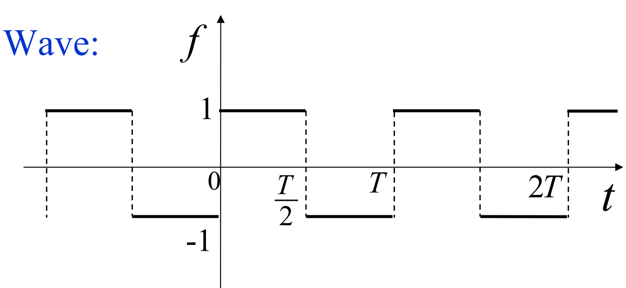
$$F_0 = \frac{1}{T}\int_0^T f(t)\,dt = 0$$

$$F_n = \frac{1}{T}\int_0^T f(t)\,e^{-in\omega_0 t}dt = \frac{1}{T}\int_0^{\frac{T}{2}} e^{-in\omega_0 t}\,dt - \frac{1}{T}\int_{\frac{T}{2}}^{T} e^{-in\omega_0 t}\,dt$$

$$= \frac{1}{T}\left.\frac{e^{-in\omega_0 t}}{-in\omega_0}\right|_0^{\frac{T}{2}} - \frac{1}{T}\left.\frac{e^{-in\omega_0 t}}{-in\omega_0}\right|_{\frac{T}{2}}^{T} = \frac{e^{-in\omega_0\frac{T}{2}} - 1}{-in\omega_0 T} - \frac{e^{-in\omega_0 T} - e^{-in\omega_0\frac{T}{2}}}{-in\omega_0 T}$$

$$\underbrace{=}_{\omega_0 T = 2\pi} \frac{1}{in2\pi}\left(-e^{-in\pi} + 1 + 1 - e^{-in\pi}\right) \underbrace{=}_{e^{-i\pi} = -1} \frac{1}{in\pi}\left(1 - (-1)^n\right)$$

Example: Square Wave:



$$F_0 = \frac{1}{T}\int_0^T f(t)\,dt = 0$$

$$F_n = \frac{1}{in\pi}\left(1 - (-1)^n\right) = \begin{cases} 0 & \text{if } n \text{ is even.} \\ \dfrac{2}{in\pi} & \text{if } n \text{ is odd.} \end{cases}$$

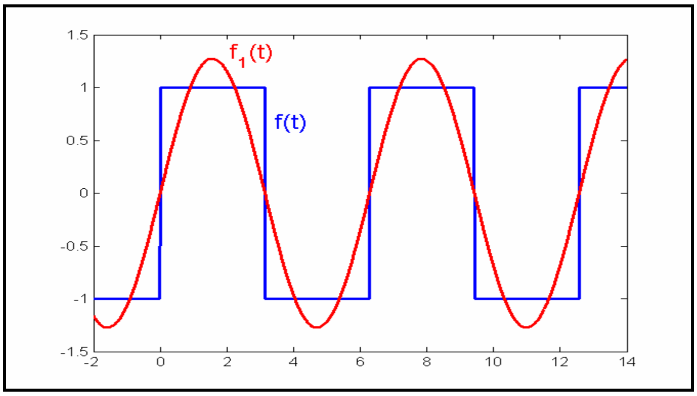
$$f(t) = \sum_{\substack{n=-\infty \\ n \text{ odd}}}^{+\infty} \frac{2}{in\pi} e^{in\omega_0 t}$$

Defining $f_N(t) = \sum_{\substack{n=-N \\ n \text{ odd}}}^{N} \frac{2}{in\pi} e^{in\omega_0 t}$,

we should have $f(t) = \lim_{N\to+\infty} f_N(t)$

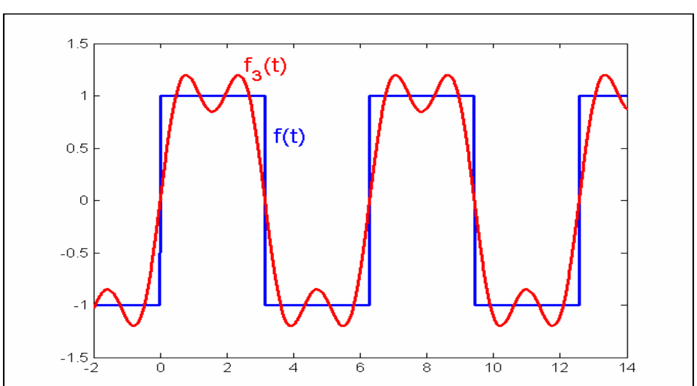
Approximations to the square wave: $f_N(t) = \sum_{\substack{n=-N \\ n \text{ odd}}}^{N} \frac{2}{in\pi} e^{in\omega_0 t}$

$$f_1(t) = \frac{2}{i\pi} e^{i\omega_0 t} - \frac{2}{i\pi} e^{-i\omega_0 t} = \frac{4}{\pi} \frac{e^{i\omega_0 t} - e^{-i\omega_0 t}}{2i} = \frac{4}{\pi} \sin(\omega_0 t)$$



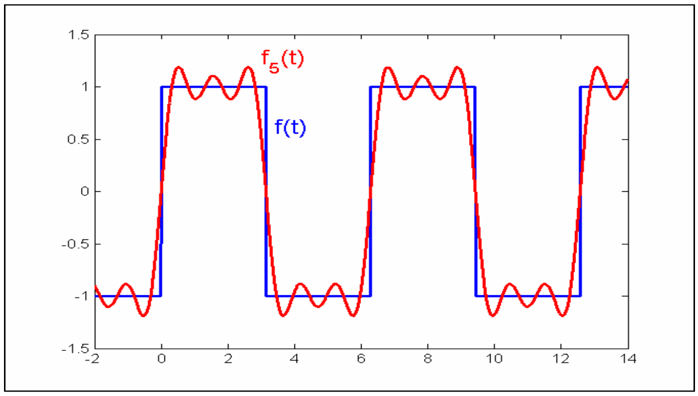
Approximations to the square wave: $f_N(t) = \sum_{\substack{n=-N \\ n \text{ odd}}}^{N} \frac{2}{in\pi} e^{in\omega_0 t}$

$$f_3(t) = f_1(t) + \frac{2}{i3\pi} e^{i3\omega_0 t} - \frac{2}{i3\pi} e^{-i3\omega_0 t} = \frac{4}{\pi} \sin(\omega_0 t) + \frac{4}{3\pi} \sin(3\omega_0 t)$$



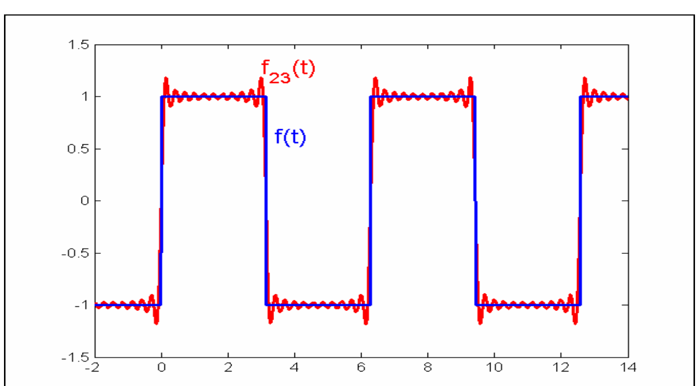
Approximations to the square wave: $f_N(t) = \sum_{\substack{n=-N \\ n \text{ odd}}}^{N} \frac{2}{in\pi} e^{in\omega_0 t}$

$$f_5(t) = \frac{4}{\pi}\sin(\omega_0 t) + \frac{4}{3\pi}\sin(3\omega_0 t) + \frac{4}{5\pi}\sin(5\omega_0 t)$$



Approximations to the square wave: $f_N(t) = \sum_{\substack{n=-N \\ n \text{ odd}}}^{N} \frac{2}{in\pi} e^{in\omega_0 t}$

$$f_{23}(t)$$



In what sense does $f_N(t)$ converge to $f(t)$?

Sequence of functions, many notions are possible:

- Pointwise convergence: for fixed time t, $\lim_{N\to+\infty} f_N(t) = f(t)$
- Convergence in mean square: $\lim_{N\to+\infty} \int_0^T \left|f_N(t) - f(t)\right|^2 dt = 0$ (more on this later).
- Weak convergence: $\lim_{N\to+\infty} \int_0^T \left(f_N(t) - f(t)\right) g(t)\, dt = 0$ for for every function $g(t)$. Used when f contains δ functions.

A theorem on pointwise convergence

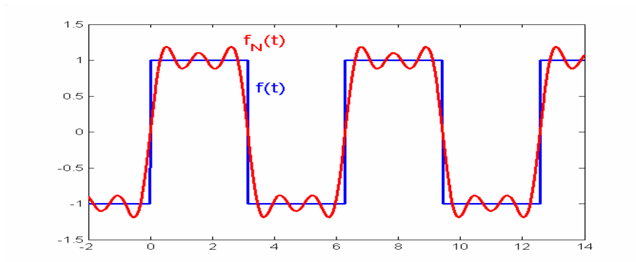
Assume $f(t)$ satisfies the "Dirichlet conditions":

- $\int_0^T \left|f(t)\right| dt$ is finite.
- $f(t)$ has a finite number of maxima and minima in the interval $[0,T]$.
- $f(t)$ has a finite number of discontinuities in $[0,T]$, which are finite jumps: i.e., at these points the lateral limits $f(t+)$ and $f(t-)$ are well defined and finite.

Then $\lim_{N\to+\infty} f_N(t) = \dfrac{f(t+)+f(t-)}{2}$. In particular at points where $f(t)$ is continuous, the series $\sum_{n=-\infty}^{+\infty} F_n e^{in\omega_0 t}$ converges pointwise to $f(t)$. Proof is beyond our scope.

The Dirichlet conditions are general enough to cover most functions one expects to encounter in physical applications. In particular, the square wave.

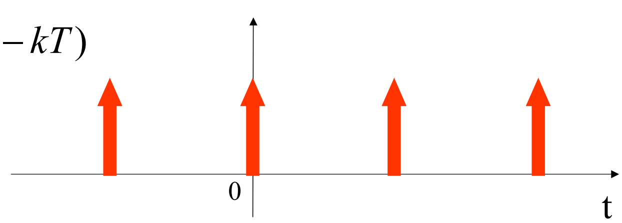
- At all times except for jumps: $\lim_{N \to +\infty} f_N(t) = f(t)$
- At the jumps (e.g. $t = 0$): $f_N(0) = 0 = \dfrac{f(0+) + f(0-)}{2}$



The value of f at these jumps has no physical relevance. So for practical purposes, the Fourier series converges to f

Example (not covered by Dirichlet): periodic impulse train.

$$f(t) = \sum_{k=-\infty}^{+\infty} \delta(t - kT)$$

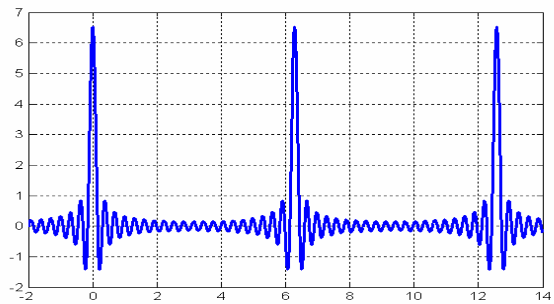


$$F_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\, e^{-in\omega_0 t} dt = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \delta(t)\, e^{-in\omega_0 t} dt = \frac{1}{T}$$

Note: we must integrate in a period, including only one of the Delta's. It could be from 0- to T-, but the above is clearer.

Example: periodic impulse train $f(t) = \sum_{k=-\infty}^{+\infty} \delta(t - kT)$

$$f_N(t) = \frac{1}{T} \sum_{n=-N}^{N} e^{in\omega_0 t} = \frac{1}{T} + \frac{2}{T}\cos(\omega_0 t) + \cdots + \frac{2}{T}\cos(N\omega_0 t)$$



$f_N(t)$ converges to $f(t)$ in the same sense pulse functions converge to δ; a rigorous treatment is beyond our scope here.

Fourier Series - Recap

Given a function $f(t)$, of period T, fundamental frequency $\omega_0 = \frac{2\pi}{T}$. We find the Fourier expansion by computing

$$F_n = \frac{1}{T} \int_0^T f(t)\, e^{-in\omega_0 t}\, dt$$

(alternatively, integrate over any interval of length T).

Then except for pathological cases, and under suitable interpretation for the series convergence, we have

$$f(t) = \sum_{n=-\infty}^{+\infty} F_n\, e^{in\omega_0 t}$$

In short: all "decent" periodic functions are superpositions of sinusoids, and the above formula gives the way to find the right superposition for a given function.

Representation of the Fourier coefficients.

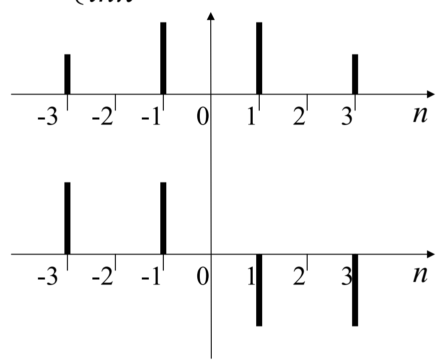
F_n is a complex number, $F_n = |F_n| e^{i\theta_n}$

The amplitude and phase (line) spectra are plots of $|F_n|$ and θ_n as a function of n.

For the square wave example: $F_n = \begin{cases} 0 \text{ if } n \text{ is even.} \\ \dfrac{2}{in\pi} \text{ if } n \text{ is odd.} \end{cases}$

$|F_n| = \begin{cases} 0 \text{ if } n \text{ is even.} \\ \dfrac{2}{|n|\pi} \text{ if } n \text{ is odd.} \end{cases}$,

$\theta_n = \begin{cases} \text{undefined if } n \text{ is even.} \\ -\dfrac{\pi}{2}\text{sign}(n) \text{ if } n \text{ is odd.} \end{cases}$



Symmetries in line spectra.

Property: if $f(t)$ is real-valued, then $F_{-n} = \overline{F_n}$. Proof

$$F_{-n} = \frac{1}{T}\int_0^T f(t)\, e^{in\omega_0 t} dt = \frac{1}{T}\int_0^T \overline{f(t) e^{-in\omega_0 t}} dt = \overline{\frac{1}{T}\int_0^T f(t)\, e^{-in\omega_0 t} dt} = \overline{F_n}$$

Consequence:

$|F_n| = |F_{-n}|$: the amplitude is an even function of n.

$\theta_n = -\theta_{-n}$: the phase is an odd function of n.