Lecture 12

- Fourier series using trigonometric functions.
- Interpretation of Fourier series as an expansion on an orthonormal basis.
- RMS value of a periodic function
- Parseval's relation
- Application to power conversion.

Fourier series with trigonometric functions.

Suppose the periodic function $f(t)$ is real-valued.

$$F_n = \frac{1}{T}\int_0^T f(t)\, e^{-in\omega_0 t} dt = \frac{1}{T}\int_0^T f(t)[\cos(n\omega_0 t) - i\sin(n\omega_0 t)]dt$$

$$= \underbrace{\frac{1}{T}\int_0^T f(t)\cos(n\omega_0 t)dt}_{a_n} \; - i \underbrace{\frac{1}{T}\int_0^T f(t)\sin(n\omega_0 t)dt}_{b_n} = a_n - ib_n$$

In particular, for $n=0$, we have $F_0 = a_0$.

$$\text{Also,} \quad f(t) = \sum_{n=-\infty}^{+\infty} F_n\, e^{in\omega_0 t} = F_0 + \sum_{n=1}^{+\infty}\left(F_n\, e^{in\omega_0 t} + F_{-n}\, e^{-in\omega_0 t}\right)$$

$$\underbrace{=}_{F_{-n}=\overline{F_n}} a_0 + \sum_{n=1}^{+\infty}\left(F_n\, e^{in\omega_0 t} + \overline{F_n\, e^{in\omega_0 t}}\right) = a_0 + 2\sum_{n=1}^{+\infty}\mathrm{Re}\left[F_n\, e^{in\omega_0 t}\right]$$

Now,

$$\mathrm{Re}\left[F_n e^{in\omega_0 t}\right] = \mathrm{Re}\left[(a_n - ib_n)[\cos(n\omega_0 t) + i\sin(n\omega_0 t)]\right] =$$
$$= a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)$$

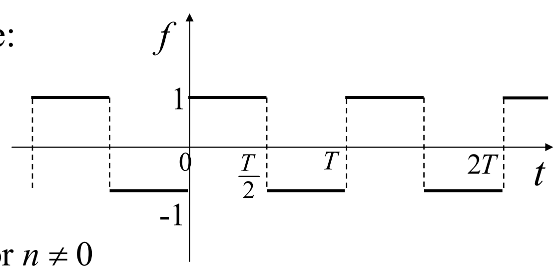
Therefore $f(t) = a_0 + 2\sum_{n=1}^{+\infty}\left(a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)\right)$,

where $a_n = \frac{1}{T}\int_0^T f(t)\cos(n\omega_0 t)dt$, $b_n = \frac{1}{T}\int_0^T f(t)\sin(n\omega_0 t)dt$.

Fourier series with trigonometric functions, for real *f(t)*

The term $a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)$ in the sum is called the "n-th harmonic" of the function $f(t)$.

Example: Square Wave:



Recall:

$F_0 = 0,$

$F_n = \frac{1}{in\pi}\left(1-(-1)^n\right)$ for $n \neq 0$

Setting $F_n = a_n - ib_n$ we get

$a_n = \mathrm{Re}\left[F_n\right] = 0$ for all n, $\quad b_n = \begin{cases} 0 & \text{if } n \text{ is even.} \\ \frac{2}{n\pi} & \text{if } n \text{ is odd.} \end{cases}$

Therefore $f(t) = 2\sum_{n=1}^{+\infty} b_n \sin(n\omega_0 t) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{+\infty} \frac{4}{n\pi}\sin(n\omega_0 t)$

Fourier Series - Recap

Given a function $f(t)$, of period T, fundamental frequency $\omega_0 = \dfrac{2\pi}{T}$. The complex Fourier series expansion is

$$f(t) = \sum_{n=-\infty}^{+\infty} F_n e^{in\omega_0 t}, \quad \text{where } F_n = \frac{1}{T}\int_0^T f(t)\, e^{-in\omega_0 t} dt$$

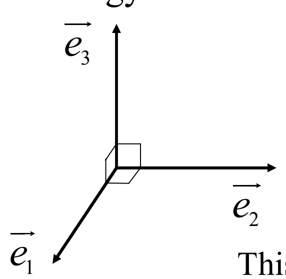
For a real-valued $f(t)$, the sine-cosine Fourier series is

$$f(t) = a_0 + 2\sum_{n=1}^{+\infty} a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t), \quad \text{where}$$

$$a_n = \frac{1}{T}\int_0^T f(t)\cos(n\omega_0 t)dt, \qquad b_n = \frac{1}{T}\int_0^T f(t)\sin(n\omega_0 t)dt.$$

Interpretation of Fourier series: expansion in an orthonormal basis.

Analogy: orthonormal basis in 3-dimensional space:



Vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3$ are orthonormal when they:

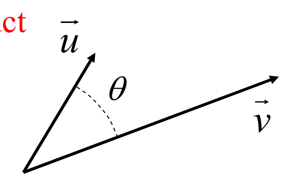
- are mutually orthogonal: $\vec{e}_m \perp \vec{e}_n$ for $m \neq n$
- have unit length: $\left|\vec{e}_n\right| = 1$ for $n = 1, 2, 3$.

This can be expressed as $\left\langle \vec{e}_m, \vec{e}_n \right\rangle = \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n \end{cases},$

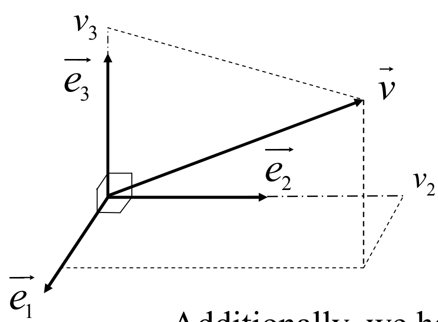
where $\langle \vec{u}, \vec{v} \rangle$ denotes the inner product

(or dot product) of vectors $\vec{u}, \vec{v}$.

Recall, $\langle \vec{u}, \vec{v} \rangle = |\vec{u}||\vec{v}|\cos(\theta)$



Expanding a vector in an orthonormal basis.



$\vec{v} = v_1\overrightarrow{e_1} + v_2\overrightarrow{e_2} + v_3\overrightarrow{e_3}$,
where the scalar coefficients
v_1, v_2, v_3 can be expressed as
$v_n = \langle \vec{v}, \overrightarrow{e_n} \rangle$ for $n = 1, 2, 3$.

Additionally, we have the relationship

$$\langle \vec{u}, \vec{v} \rangle = u_1 v_1 + u_2 v_2 + u_3 v_3,$$

and in particular the vector length satisfies

$$|\vec{v}|^2 = \langle \vec{v}, \vec{v} \rangle = v_1{}^2 + v_2{}^2 + v_3{}^2.$$

Analogy between vector and Fourier expansions

Vectors:

$$\vec{v} = \sum_{n=1}^{3} v_n \overrightarrow{e_n},$$

$$v_n = \langle \vec{v}, \overrightarrow{e_n} \rangle$$

Fourier Series:

$$f(t) = \sum_{n=-\infty}^{+\infty} F_n e^{in\omega_0 t},$$

$$F_n = \frac{1}{T}\int_0^T f(t)\, e^{-in\omega_0 t}\, dt$$

The analogy can be made precise by defining an inner product between $T-$periodic functions: $\langle f(t), g(t) \rangle = \frac{1}{T}\int_0^T f(t)\overline{g(t)}\, dt$

Then, the basis functions $\phi_n(t) = e^{in\omega_0 t}$ play the role of the $\overrightarrow{e_n}$,

and we have $f(t) = \sum_{n=-\infty}^{+\infty} F_n \phi_n(t), \quad F_n = \langle f(t), \phi_n(t) \rangle$

For a complete analogy, we check the orthonormality of the basis functions $\phi_n(t) = e^{in\omega_0 t}$.

$$\langle \phi_m(t), \phi_n(t) \rangle = \frac{1}{T}\int_0^T e^{im\omega_0 t}\, \overline{e^{in\omega_0 t}}\, dt$$

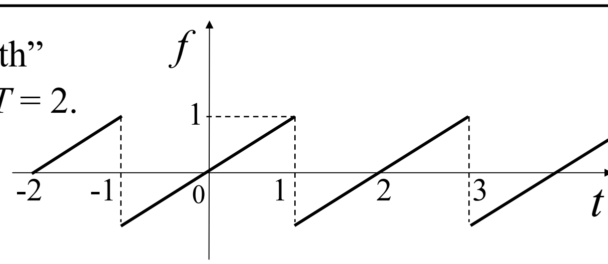
$$= \frac{1}{T}\int_0^T e^{i(m-n)\omega_0 t}\, dt \overset{\overbrace{\text{PREVIOUS LECTURE}}}{=} \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n \end{cases}$$

So the functions $\phi_n(t)$ are mutually orthogonal in this abstract sense, and of unit "size". What is this notion of size?

$$\|f\| = \sqrt{\langle f(t), f(t) \rangle} = \sqrt{\frac{1}{T}\int_0^T f(t)\overline{f(t)}\, dt} = \sqrt{\frac{1}{T}\int_0^T |f(t)|^2\, dt}$$

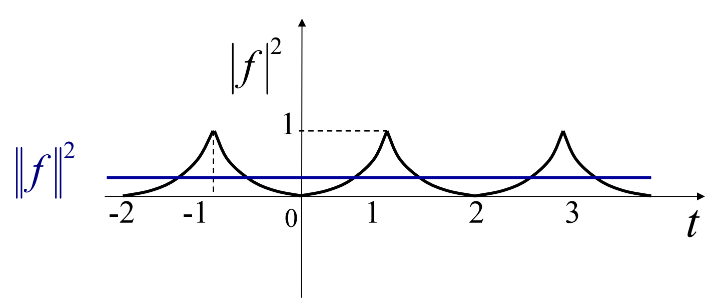
This is called the root-mean-square (RMS) value of the periodic function $f(t)$.

Example: "sawtooth" function, of period $T = 2$.



RMS value of $f(t)$ is

$$\|f\| = \sqrt{\frac{1}{T}\int_0^T |f(t)|^2\, dt} = \sqrt{\frac{1}{2}\int_{-1}^1 t^2 dt} = \sqrt{\left.\frac{t^3}{6}\right|_{-1}^1} = \frac{1}{\sqrt{3}}$$



Interpretation for the RMS value $\|f\| = \sqrt{\frac{1}{T}\int_0^T |f(t)|^2\, dt}$

It is one of many possible "norms"
(i.e. notions of size) of a function. Why this choice?

- Mathematically, it has nice properties, like vector length.
- Physical interpretation based on power:

Example: *f(t)= I(t)*, current going through a unit resistor R=1.
Instantaneously, the dissipated power is $I(t)^2$. If $I(t)$ is periodic,
the mean power dissipated over one period is $\frac{1}{T}\int_0^T |I(t)|^2\, dt = \|I\|^2_{\text{RMS}}$
Equals the power burned by the constant (DC) current $I(t) = \|I\|_{\text{RMS}}$

Special case (familiar from AC circuits): $I(t) = I_0 \cos(\omega t)$. Here

$$\|I\|_{\text{RMS}} = I_0\sqrt{\frac{1}{T}\int_0^T \cos^2(\omega t)dt} = I_0\sqrt{\frac{1}{T}\left(\frac{t}{2} + \frac{\sin(2\omega t)}{4\omega}\right)\Bigg|_0^T} = \frac{I_0}{\sqrt{2}}$$

One more ingredient in the analogy

Vectors:

$$\vec{v} = \sum_{n=1}^{3} v_n \overrightarrow{e_n},$$

$$v_n = \left\langle \vec{v}, \overrightarrow{e_n} \right\rangle$$

$$\left|\vec{v}\right|^2 = v_1{}^2 + v_2{}^2 + v_3{}^2$$

Fourier Series:

$$f(t) = \sum_{n=-\infty}^{+\infty} F_n\, e^{in\omega_0 t},$$

$$F_n = \frac{1}{T}\int_0^T f(t)\, e^{-in\omega_0 t} dt = \left\langle f(t), e^{in\omega_0 t} \right\rangle$$

$$\boxed{\|f\|^2_{\text{RMS}} = \frac{1}{T}\int_0^T |f(t)|^2\, dt = \sum_{n=-\infty}^{+\infty} |F_n|^2}$$

PARSEVAL'S RELATION

Proof:

$$\frac{1}{T}\int_0^T |f(t)|^2\, dt = \frac{1}{T}\int_0^T f(t)\overline{f(t)}\, dt = \frac{1}{T}\int_0^T f(t)\, \overline{\sum_{n=-\infty}^{+\infty} F_n\, e^{in\omega_0 t}}\, dt$$

$$= \sum_{n=-\infty}^{+\infty} \overline{F_n}\; \underbrace{\frac{1}{T}\int_0^T f(t) e^{-in\omega_0 t} dt}_{F_n} = \sum_{n=-\infty}^{+\infty} |F_n|^2$$

PARSEVAL'S RELATION: $\frac{1}{T}\int_0^T |f(t)|^2\,dt = \sum_{n=-\infty}^{+\infty} |F_n|^2$

Note that if we isolate from the expansion $f(t) = \sum_{n=-\infty}^{+\infty} F_n\, e^{in\omega_0 t}$,

the harmonic term $F_n\, e^{in\omega_0 t}$, its mean square value is equal to

$$\frac{1}{T}\int_0^T \left|F_n\, e^{in\omega_0 t}\right|^2 dt = \frac{1}{T}\int_0^T |F_n|^2\, dt = |F_n|^2$$

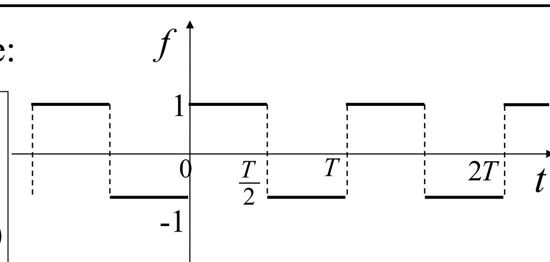
So we get the following interpretation for Parseval:
the mean square value (i.e. the mean power) of $f(t)$ is equal to the sum of the mean powers of its harmonics.

Also, for a real-valued $f(t) = a_0 + 2\sum_{n=1}^{+\infty} a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)$,

we have the Parseval relation $\frac{1}{T}\int_0^T f(t)^2\, dt = a_0{}^2 + 2\sum_{n=1}^{+\infty}\left(a_n^2 + b_n^2\right)$

Example: Square Wave:

$$f(t) = 2\sum_{n=1}^{+\infty} b_n \sin(n\omega_0 t) = \sum_{\substack{n=1 \\ n\text{ odd}}}^{+\infty} \frac{4}{n\pi}\sin(n\omega_0 t)$$



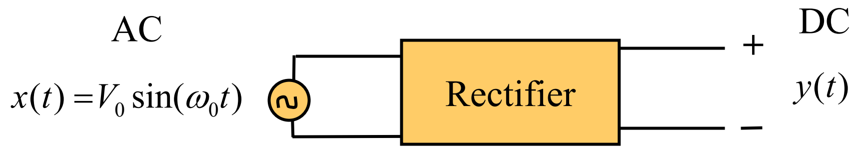
$$\frac{1}{T}\int_0^T f(t)^2\, dt = 2\sum_{n=1}^{+\infty} b_n^2 = 2\sum_{\substack{n=1 \\ n\text{ odd}}}^{+\infty}\left(\frac{2}{n\pi}\right)^2 = \frac{8}{\pi^2}\sum_{\substack{n=1 \\ n\text{ odd}}}^{+\infty}\frac{1}{n^2}$$

Since $f(t)^2 \equiv 1$, the left-hand side is easily found be to be 1.

Therefore, we conclude that $\frac{\pi^2}{8} = \sum_{\substack{n=1 \\ n\text{ odd}}}^{+\infty}\frac{1}{n^2} = 1 + \frac{1}{9} + \frac{1}{25} + \cdots$

Not an obvious sum!

Application of Fourier series: power conversion.



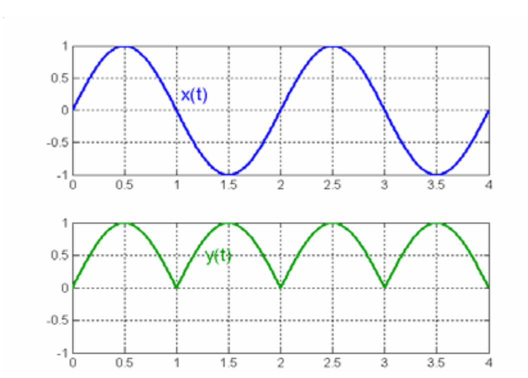
The signal $x(t)$ is a pure sinusoid (AC power) as in the voltage you find in the wall outlet. Its Fourier expansion is immediate:

$$x(t) = V_0 \frac{e^{i\omega_0 t} - e^{-i\omega_0 t}}{2i}, \quad \text{so} \quad X_{\pm 1} = \pm \frac{V_0}{2i}, \text{ and } X_n = 0 \text{ for } n \neq \pm 1$$

- A rectifier is a circuit that converts the power to DC (used by electronic equipment such as computers, audio, ...). Ideally, *y(t)* should be a perfectly flat, constant DC voltage.
- In practice, one gets an approximation to DC, with some remaining oscillations ("AC component").

A simple rectifier: $y(t) = |x(t)|$

Nonlinear, time invariant and memoryless system.
Can be approximately implemented by a diode circuit.



$x(t) = V_0 \sin(\omega_0 t)$.
Period is T.

$y(t) = V_0 |\sin(\omega_0 t)|$

Period is $T_y = \dfrac{T}{2}$;

$\omega_{0y} = 2\omega_0$

Not quite flat, but let's see how much of y is DC.

Represent $y(t)$ by a Fourier Series $y(t) = \sum_{n=-\infty}^{+\infty} Y_n\, e^{in\omega_{0y}t}$

$$Y_n = \frac{1}{T_y}\int_0^{T_y} y(t)\, e^{-in\omega_{0y}t}dt = \frac{V_0}{T_y}\int_0^{T_y} \left|\sin(\omega_0 t)\right| e^{-in\omega_{0y}t}dt$$

$$T_y = \frac{T}{2} \;\Rightarrow\; \text{for } 0 \le t \le T_y \text{ we have } 0 \le \omega_0 t \le \pi \;\Rightarrow\; \sin(\omega_0 t) \ge 0$$

$$\text{Therefore } Y_n = \frac{2V_0}{T}\int_0^{\frac{T}{2}} \sin(\omega_0 t) e^{-in2\omega_0 t}dt$$

$$= \frac{2V_0}{T}\int_0^{\frac{T}{2}} \frac{e^{i\omega_0 t} - e^{-i\omega_0 t}}{2i} e^{-in2\omega_0 t}dt$$

$$= \frac{V_0}{Ti}\int_0^{\frac{T}{2}} \left(e^{i\omega_0(1-2n)t} - e^{-i\omega_0(1+2n)t}\right)dt$$

$$Y_n = \frac{V_0}{Ti}\int_0^{\frac{T}{2}} \left(e^{i\omega_0(1-2n)t} - e^{-i\omega_0(1+2n)t}\right)dt$$

$$= \frac{V_0}{Ti}\left(\left.\frac{e^{i\omega_0(1-2n)t}}{i\omega_0(1-2n)}\right|_0^{\frac{T}{2}} + \left.\frac{e^{-i\omega_0(1+2n)t}}{i\omega_0(1+2n)}\right|_0^{\frac{T}{2}} \right)$$

$$= \frac{V_0}{Ti^2\omega_0}\left(\frac{\overbrace{e^{i\pi(1-2n)}}^{-1} - 1}{(1-2n)} + \frac{\overbrace{e^{-i\pi(1+2n)}}^{-1} - 1}{(1+2n)} \right)$$

$$= \frac{2V_0}{2\pi}\left(\frac{1}{(1-2n)} + \frac{1}{(1+2n)} \right) = \frac{V_0}{\pi}\left(\frac{1+2n+1-2n}{(1-4n^2)} \right)$$

$$\boxed{Y_n = \frac{2V_0}{\pi(1-4n^2)}}$$

$$y(t) = Y_0 + \sum_{n\neq 0} Y_n\, e^{in\omega_{0y}t} = \underbrace{\frac{2V_0}{\pi}}_{\substack{\text{DC}\\ \text{COMPONENT}}} + \underbrace{\sum_{n\neq 0} \frac{2V_0}{\pi(1-4n^2)} e^{in\omega_{0y}t}}_{\text{AC COMPONENTS}}$$

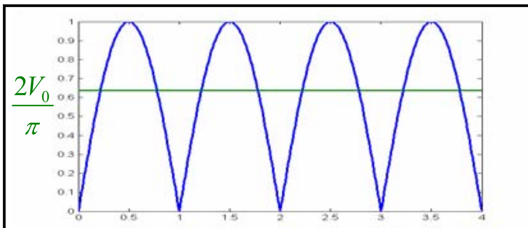
Alternatively, in sine-cosine form we get

$$a_n = \mathrm{Re}\left[Y_n\right] = Y_n,\; b_n = 0 \;\Rightarrow\; y(t) = a_0 + 2\sum_{n=1}^{+\infty} a_n \cos(n\omega_{0y}t)$$

$$= \underbrace{\frac{2V_0}{\pi}}_{\substack{\text{DC}\\ \text{COMP}}} + \underbrace{\sum_{n=1}^{+\infty} \frac{4V_0}{\pi(1-4n^2)} \cos(n\omega_{0y}t)}_{\text{AC COMPONENTS}}$$

So we obtain a desired DC component, plus harmonics of the fundamental frequency $\omega_{0y} = 2\omega_0$.

In the power supply, $\omega_0 = 2\pi \cdot 60\,Hz$. The harmonics will have frequencies $2\pi \cdot 120\,Hz,\; 2\pi \cdot 240\,Hz, \ldots$



$$y(t) = \underbrace{\frac{2V_0}{\pi}}_{\text{DC}} + \underbrace{\sum_{n\neq 0} Y_n\, e^{in\omega_{0y}t}}_{\text{AC}}$$

From the graph, the AC part looks significant.

Power analysis by Parseval: $\underbrace{\frac{1}{T}\int_0^T |y(t)|^2\, dt}_{\substack{\text{TOTAL}\\ \text{POWER}}} = \underbrace{|Y_0|^2}_{\substack{\text{DC}\\ \text{POWER}}} + \underbrace{\sum_{n\neq 0} |Y_n|^2}_{\text{AC POWER}}$

The total power is $\frac{1}{T_y}\int_0^{T_y} |y(t)|^2\, dt = \frac{V_0^2}{T_y}\int_0^{T_y} \sin^2(\omega_0 t)dt = \frac{V_0^2}{2}$

The DC power is $|Y_0|^2 = \frac{4V_0^2}{\pi^2} = 81\%$ of the total power.

The AC power is the remaining 19%.

Not too bad, not too good as a DC source.