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Note

Progress in the No-Three-in-Line-Problem

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New solutions to the no-three-in-line problem for $n = 11, 12, \ldots, 32, 34, 35, 36, 38, 40, 42$ 44, 46.

We consider a square $n \times n$ grid in the euclidean plane. The task is to mark as many of the intersection points as possible under the condition that no three of the marked points lie in a straight line. One can obviously mark at most $2n$ points. The problem of finding for which n this value is reached is known as the no-three-in-line problem [6]. One can formulate a related question: How many points can maximally be marked under this restriction? P. Erdös [15] has shown that $(1-\varepsilon)n$ points can be placed in a given grid. R. R. Hall, T. H. Jackson, A. Sudbery, and K. Wild [10] made the substantial improvement to $(\frac{3}{2}-\varepsilon)n$. This is the best known lower bound. But R. Guy and P. Kelly [9] used a probabilistic argument to support their conjecture that for large grids the limit does not reach $2n$ and will tend to $(2\pi^2/3)^{1/3}\, n \approx 1.8738n$.

From their approximation formulae one can compute that the expected number of valid grid solutions is smaller than 1 for $n > 500$. Because the upper bound $2n$ results from conditions on the rows and columns, it is more precise to choose as a basic set, the set of all configurations which satisfy these condition, as opposed to the set of all $\binom{n^2}{2n}$ unrestricted configurations. The formula for the number of row- and column-restricted configurations is given in [3], called the number of "bipermutations." The heuristic considerations of Guy and Kelly applied to this restricted set imply that there are no solutions for $n > 600$.

The largest n, for which there are solutions known, has been improved by various authors by explicit construction. Kelly obtained the results of the table for $n \leqslant 9$ [12] and for $n = 10$ [2]. Craggs and Hughes-Jones [5] gave examples for $n = 11$ and 12, including all with symmetry rot $\pi/2$.

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Gardner [7] published solutions by various people for $13 \leqslant n \leqslant 16$. Kløve gave rot $\pi/2$ solutions for $n = 14, 16$ [13] and 18, 20 [14]. Anderson [4] gave the numbers 1, 1, 3, 4, 7, 4, 13, 13, 7, 16 of solutions with rot $\pi/2$ for $n = 2, 4, \ldots, 20$ (these numbers evidently include those with full symmetry). He also found solutions for $n = 22$, 24, and 26. Kelly [12] searched up to $n = 32$ without finding any solution with the full symmetry of the square apart from the previously known $n = 2$, 4, and 10, and used this evidence as the basis of his conjecture 2 (by examining the table included in this paper, one can generalize this conjecture to: There exist exactly five solutions having orthogonal reflection symmetry). Harborth, Oertel, and Prellberg [11] gave examples for $n = 17$ and 19. Recently Harborth has reported the discovery by Gerken of two new solutions for $n = 21$ and 23.

I have been able to increase the value of n, for which such a configuration is known to exist, up to $n = 42$ using a back-track algorithm and massive computing power. For example, all solutions for $n = 20$ with rot $\pi/2$ symmetry were computed in 5 min on a HP320 computer, whereas all solutions for $n = 30$ with rot $\pi/2$ symmetry were found after no less than 25 h of cpu-time on a HP850 computer. The run to find just one rot π symmetry solution for $n = 27$ took about 960 h on a HP320.

In my investigations, I partitioned the problem into the following eight symmetry classes: Configurations having full symmetry (abbreviation: full), having only rotational symmetry (half rotation: rot π; quarter rotation: rot $\pi/2$), having only diagonal reflection symmetry (in one main-diagonal: dia1; in both main-diagonals: dia2), those having only orthogonal reflection symmetry (in one mid-perpendicular: ort1; in both mid-perpendiculars: ort2) and those having no symmetry (abbreviation: iden).

In Table I, my results are presented divided into symmetry classes. A numerical entry in the following table indicates that all solutions for that symmetry and the value of n have been found: The dots indicate that solutions are known. If n is odd then there are not any solutions in classes ort1, ort2, full and rot $\pi/2$ by reason of symmetry. The heading "total" in the second column means the sum over all symmetries for fixed n.

The entries in columns ort2 (resp. full) continue to be zero for $n \leqslant 40$ (resp. 60). Those in column rot $\pi/2$ for $n = 24, 26, \ldots, 36$ are 23, 36, 58, 62, 99, 172, 281 and are positive for even $n \leqslant 46$. The numbers in dia2 for $23 \leqslant n \leqslant 41$ are 1, 2, 2, 0, 0, 0, 1, 1, 2, 1, 0, 0, 1, 1, 0, 1, 0, 0, 0. Otherwise in column rot π a single solution is known for $n = 27$.

Finally some new solutions are listed in Table II for values of n for which, previously, no solutions having a particular symmetry were known. Each line in the listing presents a solution. Each marked point of the grid is identified by its column coordinate. They are ordered from left to right in each row, and the rows from top to bottom. The rows are separated by semicolons. In symmetry classes dia2 and rot $\pi/2$ (rot π) only the points of a generating quarter (half) of the grid are given.

TABLE I

Number of Configurations

n	total	iden	dia1	ort1	rotπ	dia2	ort2	rotπ/2	full
1	0	0	0		0	0			
2	1	0	0	0	0	0	0	0	1
3	1	0	0		0	1			
4	4	0	0	1	1	1	0	0	1
5	5	3	2		0	0			
6	11	4	0	0	2	2	0	3	0
7	22	11	1		10	0			
8	57	40	5	1	7	0	0	4	0
9	51	41	3		7	0			
10	156	132	3	0	13	1	0	6	1
11	158	122	6		30	0			
12	566	524	3	0	33	2	0	4	0
13	499	407	9		82	1			
14	1366	1284	5	0	61	3	0	13	0
15	..		13		283	1			
16	..		14	0	189	1	0	13	0
17	..		12		282	0			
18	..		14	0	328	2	0	7	0
19	..		16		594	0			
20	..		17	0	693	2	0	16	0
21	..		13		2413	0			
22	..		18	0		1	0	8	0

TABLE II

n = 21, sym = rotπ: (15,16;5,10;8,13;8,13;5,20;3,18;15,18;6,21;20,21;3,11;10)
n = 22, sym = dia2: (9,16;12,13;5,16;8,9;17;12;;8;;13)
n = 23, sym = dia2: (16,17;7,10;13,16;10,18;5,6;;;;11,12)
n = 24, sym = dia2: (11,12;15,17;11,20;18,20;;9,15;13;8;16)
n = 25, sym = dia2: (12,15;9,18;20,23;8,10;17,20;;10,12;;;;13)
n = 27, sym = rotπ: (8,9;17,18;16,21;8,9;11,15;3,4;4,23;22,26;7,16;18,23;1,6;26,27;14,25;13)
n = 28, sym = rotπ/2: (11;7;13;7,9;12;2,5;;10,12;1;;3;;10;14)
n = 29, sym = dia2: (10,16;7,13;19,24;14,22;8,18;23;;;11,15;18;;;17)
n = 30, sym = dia2: (18,19;10,14;6,27;24;5,20;15;17;9,15;20;12;;;18)
n = 30, sym = rotπ/2: (4,10;;;;8,13;4,10;;3;3,11;;;7,11;2;2,7;15)
n = 31, sym = dia2: (5,24;11,26;4,9;15;15;21;14,19;22;14;12;;20;16)
n = 32, sym = dia2: (24,25;15,16;11,13;8,19;6,16;26;23;;11;19;;12,15;20)
n = 32, sym = rotπ/2: (12;;9,14;;14,15;2;13;7;2;6;4,8;4;1;;10;16)
n = 34, sym = rotπ/2: (7,10;15;;12;9;13;2;5,10;13;;3;11;;3;6;4,14;17)
n = 35, sym = dia2: (11,14;17,22;6,25;12,18;9,28;12;23,28;;20;15,17;;;13;;16)
n = 36, sym = dia2: (8,20;15,24;24,26;10,16;10,29;26,31;9,15;;19;;;21,23;;17;;;;18)
n = 36, sym = rotπ/2: (14,15;10;13;9,15;6;11;5;12,14;;7;;3;;;;2,9;11,13;18)
n = 38, sym = rotπ/2: (11;;18;6,14;12;14;3,9;12,15;17;;10;;2,10;;;1,2;5;15;19)
n = 40, sym = rotπ/2: (15;11,17;5;;7;12;12;17;4,13;18;4;;10;3,15;;8;;6;1,16;20)
n = 42, sym = rotπ/2: (12,19;;12,17;9;11;15;9,17;6;;;4;;10,14;10;2;;;5,8;16;2,16;21)
n = 44, sym = rotπ/2: (18;10;8;18;6;21;5,13;;14;3;21;11;19;4;15;19,12;1,8;;22;2,22)
n = 46, sym = rotπ/2: (19;;11,20;9;13,16;4,18;13;19,23;;9;;12;;2;10;7;2,14;11;;23;1;15,21)

Appendix

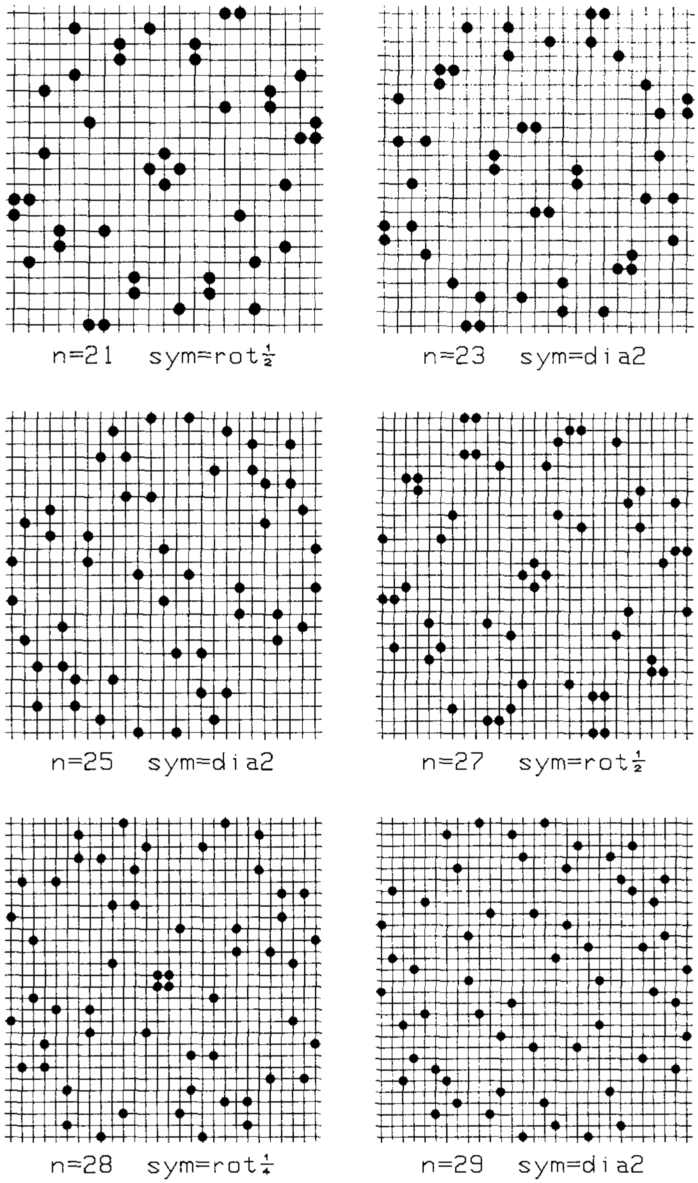

FIG. 1. Pictorial versions of a few of the examples.

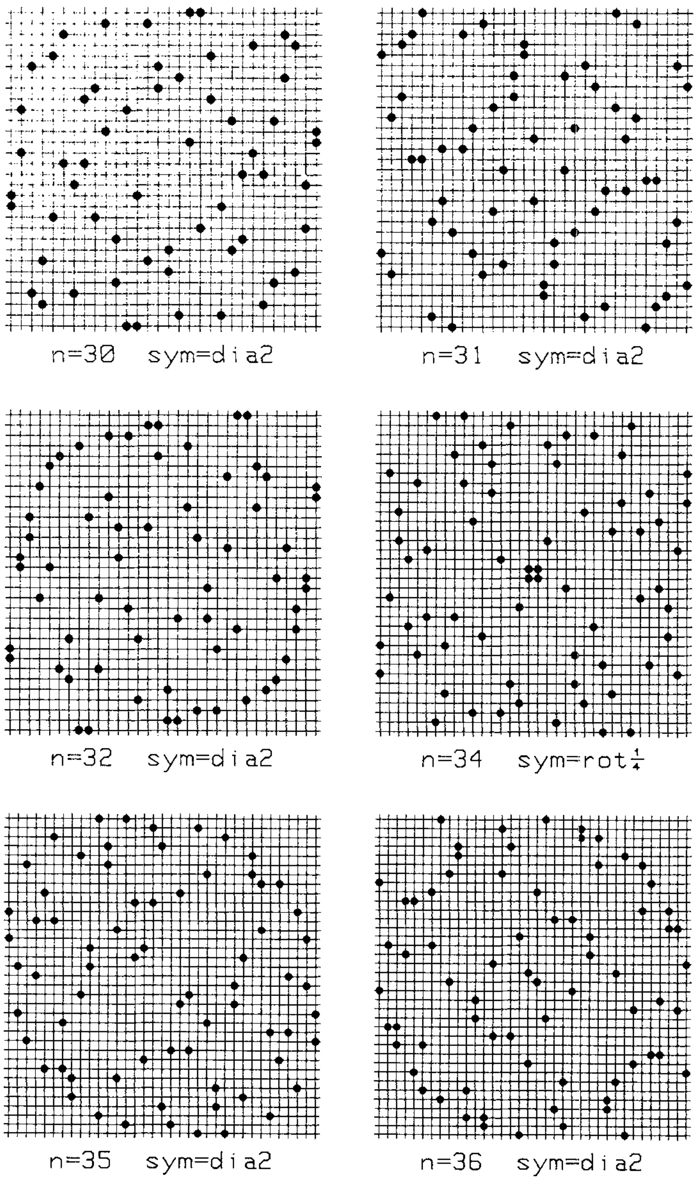

FIGURE 1—*Continued*

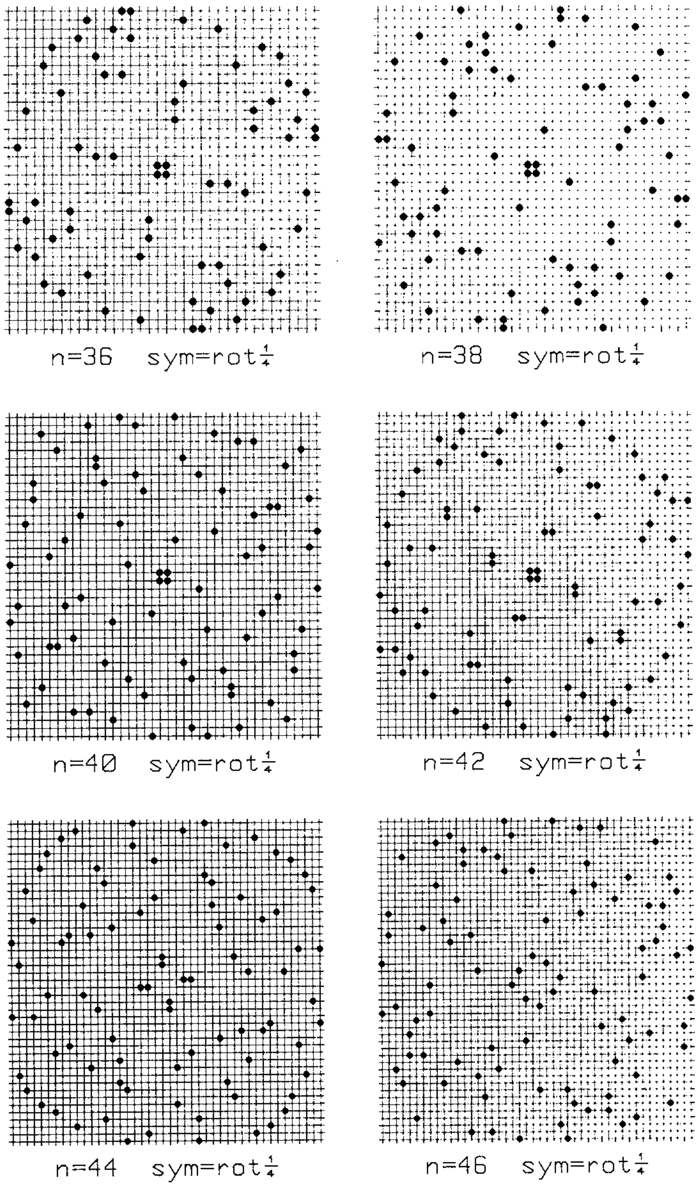

FIGURE 1—*Continued*

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