SOME THOUGHTS ON THE NO-THREE-IN-LINE PROBLEM

Michael A. Adena, Derek A. Holton,

University of Melbourne,

and

Patrick A. Kelly,

Mount Royal College, Canada.

ABSTRACT

Given an $n \times n$ grid of n^2 points we must select as many as possible so that no three are in a straight line. This paper reviews results concerning the problem and provides a few minor proofs, additions and generalisations.

1. Introduction

Given an $n \times n$ grid of n^2 points, the problem is to select as many as possible, say $D(n)$, so that no three are in any straight line. The problem was stated for $n=8$ by Dudeney [3] and Rouse Ball [2].

2. $D(n)=2n$

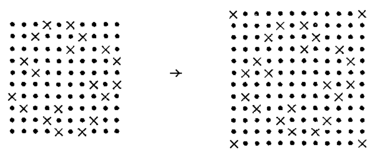
By the pigeon-hole principle $D(n) \leq 2n$, since more would give three points in some row. A lower bound is $D(n) \geq \frac{1}{2}n$ (see later).

For $n \leq 9$ some solutions for $D(n)=2n$ have been found by hand [1], [5]. Kelly found all solutions for $n \leq 9$ by computer and these are given in his Master's thesis [9]. They have been independently found and confirmed by Adena and Holton using a different algorithm. Kelly has subsequently found all solutions for $n=10$ (see Appendix A). A solution for $n=12$ has been found by hand (Figure 1). The method also converts one $n=8$ solution to an $n=10$ solution but is unlikely to generalise for larger n.

The number of solutions for which $D(n)=2n$ is shown in Table 1. Table 1 also shows the number of solutions having various symmetries. (K - about one diagonal; L - about both diagonals; M - about one bisector of opposite sides; N - none; O - about both bisectors of pairs of opposite sides; P - a half turn; Q - that of a square; R - a quarter turn.) S and T give totals; T considers isomorphic solutions as distinct.

TABLE 1

n	K	L	M	N	O	P	Q	R	S	T
2							1		1	1
3		1							1	2
4		1	1			1	1		4	11
5	2			3					5	32
6		2		4		2		3	11	50
7	1			11		10			22	132
8	5		1	40		7		4	57	380
9	3			41		7			51	368
10	3	1		132		13	1	6	156	1135



n=10 n=12

FIGURE 1

In [6], [7] and [9] Kelly and Guy have shown some minor results concerning symmetries, e.g. "for n odd there are no solutions with symmetry O". (Proof is by considering a column with a point in the $\frac{1}{2}(n+1)$th row; the column must have another distinct point in it and so, by the symmetry about the $\frac{1}{2}(n+1)$th row, there is a third point, providing a contradiction.) They provide some conjectures, e.g. "for $n > 10$, there is no solution with symmetry Q". This is supported by an unsuccessful computer search to $n=32$.

The computer algorithms used independently by the authors (Kelly, Adena and Holton) to find solutions to $D(n)=2n$ differ in their philosophies. Both use a trial and error approach. Adena and Holton have explicitly used the no-three-in-a-line property by directly manipulating the number of points in each line. This necessitated much storage to contain details of how full each line was and the subsequent development of sophisticated storage and addressing techniques in an assembly language. Kelly worked directly with the points by crossing out those points already on lines. Although the former program realises a significant saving in time (and ironically storage) for small n (at n=9, a saving of 2 to 3 adjusting for machine speed differences), the second algorithm would be faster for large n. However that question is largely academic because the time taken (increasing by a factor of at least 10 from n to n+1) renders both algorithms impractical for $n > 10$. Any algorithm employing trial and error will be too time consuming.

3. Properties of solutions of D(n)=2n

Symmetry has been used to divide the solutions into classes (see Table 1). Similarly, after identifying points with 1's and spaces with 0's, the resulting 0-1 matrices were examined for special properties which might enable solutions for higher n to be found. Indices of primitivity (the lowest power for which there are no zero elements) were calculated for all solutions with $2 \leq n \leq 10$ (Table 2). No significant patterns were observed; research is continuing along these lines. The directed graphs of the five n=10 solutions with infinite primitivity are shown in Appendix B.

TABLE 2

Index of Primitivity

n	1	2	3	4	5	6	7	8	9	10	∞	Total
2	1											1
3		1										1
4		2	1								1	4
5			2	3								5
6			1	3	7							11
7			3	7	8	1				1	2	22
8			1	5	21	18	2				10	57
9				3	26	10	2	2			8	51
10				3	61	66	14	1	6		5	156

An unsuccessful approach was to identify points with .5's and spaces with 0's. No additional properties to those commonly known for doubly stochastic matrices were revealed.

Similarly a study of eigenvalues has so far yielded nothing.

4. A determinant condition

A determinant condition that three points in the $n \times n$ grid, (x_1,y_1), (x_2,y_2) and (x_3,y_3) $(0 \leq x_i, y_i < n)$ are not collinear is

$$\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} \neq 0.$$

This determinant condition has not yielded a more tractable computer algorithm. It can be used (trivially) to show that solutions, when transformed by the symmetries of Table 1 remain admissible. It may provide a powerful method for proving some conjectures.

5. A lower bound for D(n)

It is not known even if $D(n) \geq n$. However if n is prime, n points can be placed so that no three are in line. This construction was originally given by Erdos [10]; this proof is found in Kelly [9].

Consider points of the form $(x, x^2 \bmod n)$.

Now $$\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2-pn & y^2-qn & z^2-rn \end{vmatrix}$$

$$= (y-x)(z-x)(z-y) + n[(p-r)(y-x)+(q-p)(z-x)].$$

The first term is non-zero because x, y and z are distinct and this term is not divisible by n because all its factors are less than n which is prime. Therefore the whole expression is non-zero, and so any three distinct points cannot be collinear. Note that since by Bertrand's Theorem there is a prime number between any two numbers n and 2n, $D(n) \geq \frac{1}{2}n$.

The above construction fails for n composite. Observe that for

(a) $n=s(rs+1)$ $(r>0, s>1)$ the points $(0, 0^2)$, $(1, 1^2)$ and $(rs+1, (rs+1)^2)$ lie on the leading diagonal, $y \equiv x \bmod n$.

(b) $n=r^2s$ $(r>2, s>0)$ the points $(0, 0^2)$, $(rs, (rs)^2)$ and $(2rs, (2rs)^2)$ lie on the line $y \equiv 0 \bmod n$.

(c) $n=r(r+2s)$ $(r>1,\ s>1)$ the points $(s,\ s^2)$, $(r+s,\ (r+s)^2)$ and $(n-s,\ (n-s)^2)$ lie on the line $y \equiv s^2 \bmod n$.

Now if n is composite n can be expressed as $(2m+1)^2$ or $(2m+1)(2m+1+2p)$ if odd and $2(2m+1)$, $(2m)^2$ or $4m(m+p)=2m(2m+2p)$ if even (n and p are integers). All these cases have three points collinear by considering (b), (c) and (a), (b), (c) respectively. $n=4$ slips through the boundary conditions of the above constructions and the construction is valid. A more obvious "proof" using $n=r(r+2s+1)$ or $r(r+2s)$ fails because although for $n=r(r+2s+1)$ the points $(s+1,\ (s+1)^2)$, $(r+s+1,\ (r+s+1)^2)$ and $(n-s,\ (n-s)^2)$ lie on the straight line $y \equiv x+s(s+1) \bmod n$, that line is in two parts, see the "line" in Figure 2, which falsifies the proof.

r=s=2 n=14

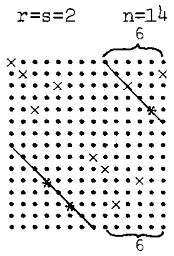


FIGURE 2

$y \equiv x+6 \bmod 14$

(In this case the leading diagonal causes the construction to fail because $14=2(2.3+1)$ cf (a).)

6. D(n) for large n

Probability arguments of Kelly and Guy [6], [7] and [9] support the conjecture that $D(n) \approx \sqrt[3]{2\pi^2/3}\ n \approx 1.87n$. First they showed that the number of sets of three collinear points that can be chosen from the $n \times n$ grid is $(3/\pi^2)\ n^4 \log n + O(n^4)$. They did this by considering the number of ways three points can be selected from the points of each line in the grid. From this they found the probability that three random points are not collinear. Assuming independence, the expected number of solutions with kn points in the grid was equated to 1 yielding the approximate upper bound for D(n).

7. Related problems

A number of related problems suggest themselves. Define d(n) as the smallest number of points in the $n \times n$ grid such that the addition of one more point would cause three points to become collinear. Clearly $d(n) \leqslant D(n)$.

Table 4 gives upper bounds for d(n), calculated by hand.

TABLE 4

n	3	4	5	6	7	8	9	10
d(n)	4	4	6	6	8	8	12	12

A generalisation of the problem is to consider an $m \times n$ grid ($m<n$). $D(m,n) \leq 2m$. Probability arguments similar to those described above give the number of sets of three collinear points as $(3/2\pi^2)\ m^2\ n^2 \log mn + O(m^2\ n^2)$. For $D(m,n)=2m$, this gives a conjectured upper bound of $m/n \simeq .9$.

Another generalisation currently under investigation is no-m-in-line in an $n \times n$ grid. With $m=4$ let the maximum number of points that can be placed in the $n \times n$ grid be $E(n)$. $E(n)=3n$, at least for $3 \leq n \leq 7$. Table 5 gives some values for $U(n)$, the number of $E(n)=3n$ solutions.

TABLE 5

n	3	4	5	6	7
u(n)	1	2	35	≃1500	?

N.B. (1) U(6) is an approximation. A computer search has produced 5939 admissable solutions, but many of these are isomorphic. We suspect the number of isomorphic arrays is of the order of 1500.

(2) Hand calculations have produced an admissible solution for n=7.

8. References

[1] Acland-Hood Bull. Malayan Math. Soc., 0(1952-3), E11-12.

[2] W.W. Rouse Ball 'Mathematical Recreations and Essays' (Macmillan 1939 p 189, 1962 pp165-170)

[3] H.E. Dudeney 'Amusements in Mathematics' (Nelson 1917 pp94, 222)

[4] F. Gruenberg and G. Jaffray 'Problems for Computer Solution' (Wiley 1965 p98)

[5] Richard K. Guy Bull. Malayan Math. Soc., 0(1952-3), E22.

[6] Richard K. Guy and Patrick A. Kelly The No-three-in-line Problem (Research Paper #33 Univ of Calgary Preprint 1968)

[7] Richard K. Guy and Patrick A. Kelly The No-three-in-line Problem, Canad. Math. Bull. 11 527-531 1968

[8] Richard K. Guy Unsolved Combinatorial Problems in 'Combinatorial Mathematics and its Applications' ed Welsh (Academic 1971 p124)

[9] Patrick A. Kelly 'The Use of the Computer in Game Theory' (Master's Thesis, Univ. of Calgary 1967 pp44-55)

[10] K.F. Roth On a problem of Heilbronn J. Lond. Math. Soc. 26(1951) pp198-204 esp 204.

[11] S.K. Stein 'Mathematics - The Man-made Universe' (Freeman 1969 p291)

APPENDIX A

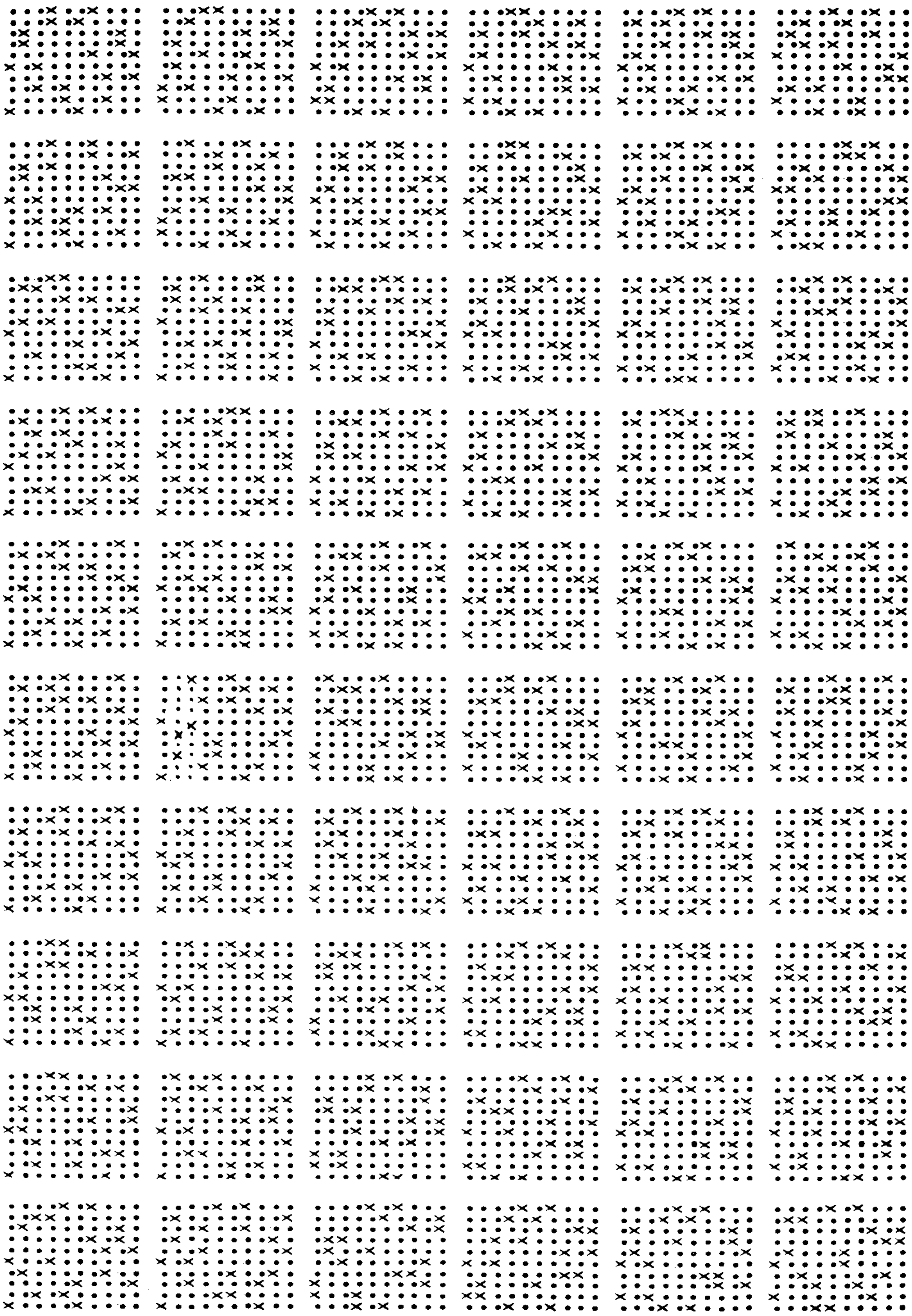
D(10)=20 solutions

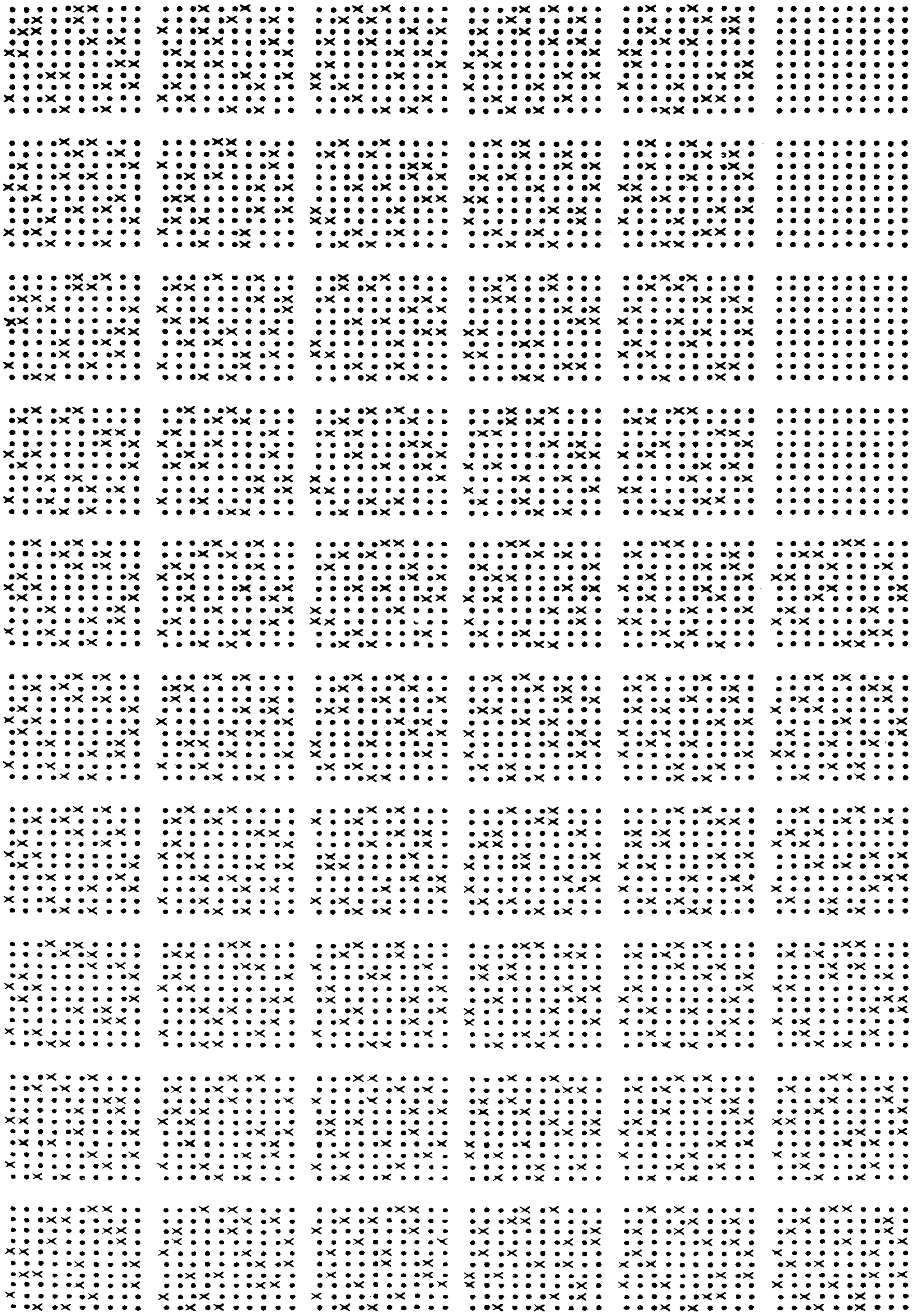
```
....XX....  ..XX......  X........X  ..X...X...  ...XX.....  ...XX.....  ...X.X....  ...X.X....  XX........  ..X.X.....
..X....X..  .....XX...  ...X.X....  ..X...X...  ...X...X..  ......XX..  ..X...X...  ......XX..  X....X....  ......X.X.
.X......X.  X.X.......  ...XX.....  ........XX  .X..X.....  .X...X....  .....X..X.  .X...X....  ......X.X.  .X...X....
...X..X...  X.......X.  .......XX.  XX........  ........XX  .X.......X  .X.......X  .X.......X  ....X...X.  X.......X.
X........X  .....X..X.  .X.....X..  ....XX....  .......X.X  ..X......X  X.X.......  X.X.......  ...X.....X  ....X..X..
X........X  .X..X.....  ..X.....X.  ....XX....  X.X.......  X......X..  .......X.X  .......X.X  .X.....X..  .....X...X
...X..X...  .X.......X  .XX.......  ........XX  XX........  X.......X.  X.......X.  X.......X.  ..X......X  X.X.......
.X......X.  .......X.X  .....XX...  XX........  .....X..X.  ....X...X.  .X..X.....  ....X...X.  .....X.X..  ...X.....X
..X....X..  ...XX.....  ....X.X...  ...X...X..  ..X...X...  ..XX......  ...X...X..  ..XX......  ..XX......  .X.....X..
....XX....  ......XX..  X........X  ...X...X..  .....XX...  .....XX...  ....X.X...  ....X.X...  ....X.X...  ...X..X...

..X..X....  X...X.....  ..XX......  ..XX......  ..XX......  ..X.X.....  ..X.X.....  ..X.X.....  ..X..X....  ..X...X...
....X..X..  ....X.X...  ..X..X....  .....X.X..  ......XX..  .X.....X..  ....X.X...  ......XX..  .X.....X..  .XX.......
.X......X.  ...X...X..  ...X.....X  ...X.....X  XX........  ....X.X...  .XX.......  .X..X.....  ....X.X...  ....X.X...
X...X.....  .X.......X  .....X..X.  .....X..X.  ........XX  ......X..X  ......X..X  ......X..X  ......X..X  X...X.....
..X......X  .......XX.  ........XX  ........XX  ....XX....  .X.......X  ........XX  .X.......X  XX........  XX........
......X.X.  .XX.......  XX........  XX........  ....XX....  X.......X.  XX........  X.......X.  ........XX  ........XX
XX........  X.......X.  .X..X.....  .X..X.....  XX........  X..X......  X..X......  X..X......  X..X......  .....X...X
.....X...X  ..X...X...  X.....X...  X.....X...  ........XX  ...X.X....  .......XX.  .....X..X.  ...X.X....  ...X.X....
...X...X..  ...X.X....  ....X..X..  ..X.X.....  ..XX......  ..X.....X.  ...X.X....  ..XX......  ..X.....X.  .......XX.
...X..X...  .....X...X  ......XX..  ......XX..  ......XX..  .....X.X..  .....X.X..  .....X.X..  ....X..X..  ...X...X..

..X...X...  ..X...X...  ..X...X...  ...X.X....  XX........  XX........  X.X.......  X.X.......  X.X.......  X.X.......
.X.....X..  ..X...X...  ......XX..  ..X....X..  ....X...X.  .....X..X.  .X.....X..  ....X..X..  ....X..X..  .....XX...
....X.X...  ....X...X.  .X..X.....  ...X.X....  .....X...X  X.......X.  ......X.X.  ...X.X....  .......X.X  ..X.X.....
X...X.....  X...X.....  X...X.....  .X.......X  .....X...X  .....XX...  ....X..X..  .......X.X  ....XX....  X.......X.
XX........  .X.......X  .X.......X  X.......X.  X.X.......  ...X.....X  .X.......X  X..X......  ..X.....X.  .....X.X..
........XX  X.......X.  X.......X.  .X.......X  .X.X......  ...X.....X  ....X...X.  .X.......X  ........XX  ...X.....X
.....X...X  .....X...X  .....X...X  X.......X.  .......XX.  .X.....X..  X........X  ......X.X.  XX........  .......X.X
...X.X....  .X...X....  .....X..X.  ....X.X...  ....X.X...  ....X..X..  ..X..X....  .X......X.  ...X..X...  .X.X......
..X.....X.  ...X...X..  ..XX......  ..X....X..  ..X....X..  ..X.X.....  ...X..X...  ..X..X....  .X.X......  .X....X...
...X...X..  ...X...X..  ...X...X..  ....X.X...  ...X..X...  ..X...X...  ...X.X....  ....X.X...  .....XX...  ....X...X.

X.X.......  X.X.......  X.X.......  X.X.......  X.X.......  X..X......  X..X......  X...X.....  X...X.....  X...X.....
.....XX...  .....X..X.  .....X..X.  .....X...X  .......X.X  ...X...X..  ......XX..  .X.X......  ..X.X.....  ..X..X....
.......X.X  .....X.X..  ......XX..  .....X.X..  ....X.X...  .......X.X  ....X.X...  .......X.X  ...X...X..  ..X.....X.
..X...X...  .X.X......  ..X..X....  X..X......  ..X....X..  ..X.X.....  .......XX.  ....X.X...  .X.......X  .....X...X
.X.....X..  X........X  .X.......X  .......X.X  .X.......X  ......X.X.  .X.X......  .......X.X  .......X.X  .X.....X..
...X.....X  ...X..X...  ......X.X.  ...X..X...  ....X...X.  ..X......X  XX........  .X.X......  .X.X......  XX........
X.......X.  .......X.X  X........X  .X......X.  XX........  X.....X...  ....X....X  X.X.......  .....X..X.  ........XX
.X.X......  .X..X.....  ...XX.....  .X......X.  .....X..X.  .X......X.  ..X..X....  .....X..X.  X.....X...  ...X..X...
....XX....  ..X.X.....  ....X..X..  ..X.X.....  ...X..X...  .X...X....  ........XX  ..X..X....  ..X.....X.  ......XX..
....X...X.  ......X.X.  .X.X......  ....X.X...  ...X.X....  ....XX....  ..X..X....  ......X.X.  .....XX...  ...XX.....
```



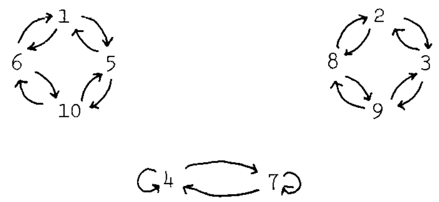


APPENDIX B

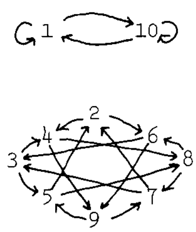
D(10)=20 solutions with infinite primitivity

Adjacency Matrix | Directed Graph

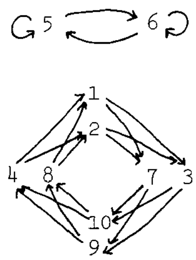
1. $\begin{bmatrix} 0000110000 \\ 0010000100 \\ 0100000010 \\ 0001001000 \\ 1000000001 \\ 1000000001 \\ 0001001000 \\ 0100000010 \\ 0010000100 \\ 0000110000 \end{bmatrix}$



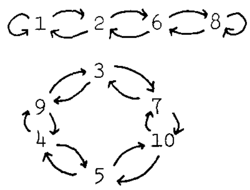
2. $\begin{bmatrix} 1000000001 \\ 0001010000 \\ 0001100000 \\ 0000000110 \\ 0100000100 \\ 0010000010 \\ 0110000000 \\ 0000011000 \\ 0000101000 \\ 1000000001 \end{bmatrix}$



3. $\begin{bmatrix} 0010001000 \\ 0010001000 \\ 0000000011 \\ 1100000000 \\ 0000110000 \\ 0000110000 \\ 0000000011 \\ 1100000000 \\ 0001000100 \\ 0001000100 \end{bmatrix}$



4. $\begin{bmatrix} 1100000000 \\ 1000010000 \\ 0000001010 \\ 0000100010 \\ 0001000001 \\ 0100000100 \\ 0010000001 \\ 0000010100 \\ 0011000000 \\ 0000101000 \end{bmatrix}$



APPENDIX B (Continued)

5. $$\begin{bmatrix} 0011000000 \\ 0000001100 \\ 1100000000 \\ 0000000011 \\ 0000110000 \\ 0000110000 \\ 1100000000 \\ 0000000011 \\ 0011000000 \\ 0000001100 \end{bmatrix}$$