Lab #2

Tasks on Inductive Proofs in Peano Arithmetic

For the task, provide the full list of rules defining the conditions (operations and functions), as well as the protocol for unification of substitution symbols. If necessary, introduce and prove new lemmas, numbering them as L1, L2, etc., for references in the unification protocol.

Definitions of the operations +, *, d with the free symbol 0 and the functional symbol s:

A: x + 0 = x
B: x + s(y) = s(x + y)
C: x * 0 = 0
D: x * s(y) = (x * y) + x
E: d(0) = 0
F: d(s(x)) = s(s(d(x)))

Task 1:

1. x * s(0) = x

Inductive Hypothesis (IH): P(x) = (x * s(0) = x)
Base case: P(0) = (0 * s(0) = 0)
Proof of the Base Case (BC):

- Left-hand side:
  - $0{*}s(0)=[D{:}x=0,y=0]=(0{*}0)+0=[A{:}x=(0{*}0)]=0{*}0=[C{:}x=0]=0$
- Right-hand side:
  - 0 = [E] = 0

Thus, P(0) holds.

P(s(x)) = s(x) * s(0) = s(x)

- Left-hand side:
  $s(x) * s(0) = [D: x = s(x), y=0] = (s(x) * 0) + s(x) = [B: x=s(x)*0, y=x] = s((s(x)*0) + x) = [C: x=s(x)] = s(x)$
- Right-hand side:
  $s(x)$

Both sides are equal, thus the inductive step holds

Task 2

1. $x + s(y) = s(x) + y$

Inductive Hypothesis (IH): $P(x) = (x + s(y) = s(x) + y)$

Base case: $P(0) = (0 + s(y) = s(0) + y)$

Proof of the Base Case (BC):

- Left-hand side:
  - $0 + s(y) = [B: x=0, y=y] = s(0 + y) = [Л2\ s(a + b) = s(a) + b: a=0, b=y]= s(0) + y$
- Right-hand side:
  - $s(0) + y$

Thus, P(0) holds.

$P(s(x)) = (s(x) + s(y) = s(s(x)) + y)$

- Left-hand side:
  $s(x) + s(y) = [B: x=s(x), y=y] = s(s(x) + y) = [Л2:s(a + b) = s(a) + b: a=s(x), b = y] = s(s(x)) + y$

Right-hand side:

$s(s(x)) + y$

Both sides are equal, thus the inductive step holds.