On the motivic Adams' conjecture

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Topology:

$\mathcal{E}/X$ - vector bundle $\rightsquigarrow$
$$\begin{array}{ccc} \mathbb{P}(\mathcal{E}) & \to & \mathbb{P}(\mathcal{E}\oplus\mathbb{1}) \\ \downarrow & & \downarrow \\ X & \to & Th(\mathcal{E}) \end{array}$$
Thom space of $\mathcal{E}$, fiberwise 1-pt compactif. of $\mathcal{E}$ $\leftarrow$ sphere bundle $/X$

[Q] For which $\mathcal{E}_1, \mathcal{E}_2 / X$ one has $Th(\mathcal{E}_1) \simeq Th(\mathcal{E}_2)$ - fiberwise homotopy equivalent?

Stably: $Th(\mathcal{E}_1 \oplus \mathcal{E}_2) \simeq Th(\mathcal{E}_1) \wedge_X Th(\mathcal{E}_2) \rightsquigarrow Th: K_0(X) \longrightarrow \underset{\substack{\uparrow \\ \text{group of } \wedge\text{-invertible spectra}/X}}{Pic}(SH(X))$

Adams' conjecture ('63): X - finite CW-complex, $\mathcal{E}/X$, $k \in \mathbb{Z} \Rightarrow$
$\Rightarrow \exists N$ s.t. $Th(\mathcal{E}^{\oplus k^N}) \simeq Th((\psi^k\mathcal{E})^{\oplus k^N})$,
i.e. $Th(\mathcal{E} - \psi^k(\mathcal{E}))$ is k-power torsion in $Pic(SH(X))$.

Application: $X = S^n \rightsquigarrow K_0(S^n) \to Pic(SH(S^n)) \to \pi_{n-1}(\mathbb{S})$ - some knowledge on stable stems.

Proofs: Quillen '71, Sullivan '74, Friedlander '73, Becker-Gottlieb '75, Brown '73, unpublished

Motivic homotopy:

S - scheme $\rightsquigarrow SH(S) = \underset{\substack{\uparrow \\ \text{presheaves of } S^1\text{-spectra on } Sm_S}}{Spt(S)}[\text{Nis. local equiv}^{-1}, \mathbb{A}^1_X \simeq X][(\wedge\mathbb{P}^1)^{-1}]$ - motivic stable htpy cat.

$\mathcal{E}/S \rightsquigarrow Th(\mathcal{E}) = \Sigma^\infty \mathbb{P}(\mathcal{E}\oplus\mathbb{1})/\mathbb{P}(\mathcal{E})$

$\rightsquigarrow Th: K_0(S) \longrightarrow Pic(SH(S))$

$[\mathcal{E}_1] - [\mathcal{E}_2] \mapsto Th(\mathcal{E}_1) \wedge Th(\mathcal{E}_2)^\vee$, where $Th(\mathcal{E}_2)^\vee = \mathcal{H}om(Th(\mathcal{E}_2), \mathbb{1}_S)$ - inverse to $Th(\mathcal{E}_2)$ under $\wedge$-product

Th (A., Elmanto, Röndigs, Yakerson)

S - regular over field F, $\mathcal{E}/S$ - v.b., $k \in \mathbb{Z} \Rightarrow$
$\Rightarrow \exists N \in \mathbb{N}_0$: $Th(\underset{\mathcal{E}^{\oplus k^N}}{\underbrace{k^N \otimes \mathcal{E}}}) \simeq Th(k^N \otimes \psi^k \mathcal{E})$ in $SH(S)[1/e]$, e = exp. char. F

Ex: $\mathcal{L}/X$ - line bundle $\Rightarrow$
$$\begin{array}{ccc} Th(\mathcal{L}) & \simeq & Cone(\mathcal{L} - X \to X) \\ \wr\!| & \Longleftarrow & \wr\!| \quad\quad \| \\ Th(\mathcal{L}^\vee) & \simeq & Cone(\mathcal{L}^\vee - X \to X) \end{array}$$
, similarly for $\mathcal{E}$ & $\mathcal{E}^\vee$

$\rightsquigarrow$ Adams' conjecture for ψ^{-1}.

Strategy: S-connected regular over a field F, $\xi \in S$-generic pt, exp. char $F = e$

1. Criterion for a map $\theta: Th(\mathcal{E}_1) \to Th(\mathcal{E}_2)$ being an iso
2. ~~Reduction to the case of $(k,e)=1$~~
3. Reduction to bundles with structure group $N_{GL_r}(T)$
4. Reduction to line bundles
5. Proof for line bundles

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(1) **Lm** ~~[illegible]~~, $\xi \in S$-generic pt, $\mathcal{E}_1, \mathcal{E}_2 / S$ $rk\,\mathcal{E}_1 = rk\,\mathcal{E}_2$

$\theta: Th(\mathcal{E}_1) \to Th(\mathcal{E}_2)$ - iso $\Leftrightarrow$ $\theta|_\xi : Th(\mathcal{E}_1|_\xi) \to Th(\mathcal{E}_2|_\xi)$ - iso in $SH(F(\xi))$

Rem. Trivialising $\mathcal{E}_1|_\xi$, $\mathcal{E}_2|_\xi$ $\rightsquigarrow$ $\theta|_\xi \in End_{SH(F(\xi))}(\mathbb{1}_{F(\xi)}) \simeq GW(F(\xi))$

$\rightsquigarrow deg^{\mathbb{A}^1}\theta|_\xi \in GW(F(\xi))/GW(F(\xi))^{*}$; θ-iso $\Leftrightarrow deg^{\mathbb{A}^1}\theta|_\xi \in GW(F(\xi))^{\times}$

Rem. S-variety$/F$, $\theta: Th(\mathcal{E}_1) \to Th(\mathcal{E}_2) \rightsquigarrow deg^{\mathbb{A}^1}\theta|_\xi$ does not necessary belong to $GW(F)$

Pf θ-iso $\Leftarrow$ θ-iso locally, so assume S local & trivialise $\mathcal{E}_1, \mathcal{E}_2$.

$$\rightsquigarrow \quad End_{SH(S)}(\mathbb{1}_S) \longrightarrow End_{SH(F(\xi))}(\mathbb{1}) \simeq GW(F(\xi))$$

$$\alpha_\theta \longmapsto \alpha_\theta|_\xi \text{-unit} \Rightarrow (\alpha_\theta)^N = 1$$

$$\Rightarrow \alpha_\theta^N = 1 + \nu,\ \nu|_\xi = 0 \Rightarrow \nu\text{-nilpotent} \Rightarrow \square$$

Thm ~~[illegible]~~ $A \in SH(F)$ - commut. ring spectrum, $\alpha \in A^p(S) := Hom_{SH(S)}(\mathbb{1}_S, A[p])$,

$\alpha|_\xi = 0 \Rightarrow \exists n: \alpha^n = 0$

Pf: α is supported on codim $\geq 1 \Rightarrow \alpha^n$ supported on codim $\geq n$ ~~[illegible]~~

uses Morel's connectivity theorem.

(2) char $F = p$.

$$\begin{tikzcd} \mathcal{E} \arrow[dr, "Frob_{/S}"] \arrow[drr, bend left, "Frob"] \arrow[ddr, bend right] & & \\ & \mathcal{E}^{[p]} \arrow[r] \arrow[d] \arrow[dr, phantom, "*"] & \mathcal{E} \arrow[d] \\ & S \arrow[r, "Frob"] & S \end{tikzcd}$$

, $\mathcal{E}^{[p]} \simeq \psi^p \mathcal{E}$

Lm $Frob/_S : Th(\mathcal{E}) \to Th(\mathcal{E}^{[p]}) \simeq Th(\psi^p \mathcal{E})$ - iso in $SH(S)[1/p]$

Pf: $(Frob/_S)|_\xi$: $\mathbb{A}^r/\mathbb{A}^r - \{0\} \to \mathbb{A}^r/\mathbb{A}^r - \{0\}$

$(x_1, \dots x_r) \mapsto (x_1^p, \dots, x_r^p)$ $\rightsquigarrow deg^{\mathbb{A}^1} Frob/_S = p_\varepsilon^r$, where $p_\varepsilon = \langle 1\rangle + \langle -1\rangle + \dots$

-invertible in $GW(F)[1/p]$

- $p_\varepsilon^r = 2^r$, $p=2$
- $p_\varepsilon^r \cdot \bar{p}_\varepsilon^r = p^r$, $p \neq 2$, $\bar{p}_\varepsilon = \langle 1\rangle - \langle -1\rangle + \langle 1\rangle + \dots$

③ Brown's trick: reduction to $N_{GL_r}(T)$

$\mathcal{E}/S \rightsquigarrow X/S$ - associated GL_r-torsor (= choices of bases in fibers)

$\rightsquigarrow Y := X/N_{GL_r}(T)$, $f: Y \to S$ - Zariski loc. trivial GL_r/N fiber bundle

- $f^*\mathcal{E}$ admits canonical reduction of the structure group
- $Tr_f : \mathbb{1}_S \to f_\# f^* \mathbb{1}_S$ - Becker-Gottlieb transfer

$k^r \otimes \mathcal{E}$ $k^r \otimes \psi^k \mathcal{E}$

Lm $\Theta: f^* Th(\mathcal{E}_1) \to f^* Th(\mathcal{E}_2)$ - iso $\Rightarrow$

$$\Rightarrow Th(\mathcal{E}_1) \wedge \mathbb{1}_S \xrightarrow{id \wedge Tr_f} Th(\mathcal{E}_1) \wedge f_\# f^* \mathbb{1}_S \xrightarrow{\sim} f_\# f^* Th(\mathcal{E}_1) \xrightarrow{\hat{\Theta}} Th(\mathcal{E}_2)$$

$\hat{\Theta}$ ← adj. to Θ

- iso.

Pf pullback to a point, may assume $Th(\mathcal{E}_1) = Th(\mathcal{E}_2) = \mathbb{1}_S$

$\hat{\Theta}$ ← adjoint to iso $f^*\mathbb{1}_F \simeq f^*\mathbb{1}_F$

$$\rightsquigarrow \mathbb{1}_F \xrightarrow{Tr_f} f_\# f^* \mathbb{1}_F \xrightarrow{\hat{\Theta}} \mathbb{1}_F$$

$$\Sigma^\infty pt_+ \to \Sigma^\infty (GL_r/N)_+ \to \Sigma^\infty pt_+$$

Rem: if Tr_f = inclusion of a rational ~~pt~~ $\mathbb{Q}/N$ $\Rightarrow$ composition $= \Theta|_x$ - iso. (adjunction)

(?) Tr_f = inclusion? (in topology - yes, since $GL_r(\mathbb{R})/N$ - connected + Euler char $= 1$)

(A) No in general: $Hom_{SH(F)}(\mathbb{1}_F, \Sigma^\infty(GL_r/N)_+) =: \pi_0^{st}(GL_r/N) \neq GW(F)$,

i.e. GL_r/N is not stably $\mathbb{A}^1$-connected. GL_r/N = variety of max. tori, non-conj. tori give inequiv. pts.

We take ℓ-adic (easy) & real (involved, topology $(GL_r/N)(\mathbb{R})$) realisations $\mathbb{1}_F \xrightarrow{Tr_f} f_\# f^* \mathbb{1}_F \to \mathbb{1}_F$, they are isos $\Rightarrow$ iso

④ Reduction to line bundles.

Y/S - $N_{GL_r}(T)$ - torsor, associated to $\mathcal{E}$.

$\tilde{Y} := Y/\tilde{N}$, $\tilde{N} = N \cap \left(\begin{array}{c|c} * & 0 \\ \hline 0 & * \end{array}\right) = S_{r-1} \ltimes G_m^{\times r}$

~~[illegible]~~ $f: \tilde{Y} \to S$ - finite etale

$\Rightarrow R f_* \mathcal{L} \simeq \mathcal{E}$ for a line bundle $\mathcal{L}/\tilde{Y}$

$\xRightarrow{\text{Bachmann-Hoyois}}$ $f_* Th(\mathcal{L}) \simeq Th(\mathcal{E})$

$$f_* Th(\psi^k \mathcal{L}) \simeq Th(\psi^k \mathcal{E})$$

$$\rightsquigarrow \Theta: Th(k^r \otimes \mathcal{L}) \xrightarrow{\sim} Th(k^r \otimes \psi^k \mathcal{L}) \Rightarrow f_* \Theta: Th(k^r \otimes \mathcal{E}) \xrightarrow{\sim} Th(k^r \otimes \psi^k \mathcal{E}) \text{ - iso}$$

⑤ Line Bundles

$\mathcal{L}/S$ – line bundle, $\psi^k \mathcal{L} = \mathcal{L}^{\otimes k}$

$\varphi: Th(\mathcal{L}) \to Th(\mathcal{L}^{\otimes k})$ $\quad \deg^{\mathbb{A}^1}\varphi|_s = k_\varepsilon$

$\quad v \mapsto v\otimes\cdots\otimes v$

- k-odd ⇝ $\varphi' = k_\varepsilon\circ\varphi: Th(\mathcal{L}) \to Th(\mathcal{L}^{\otimes k})$, $\deg^{\mathbb{A}^1}\varphi' = k$
- $k = 2$ ⇝ $\varphi': Th(\mathcal{L}\oplus\mathcal{L}) \to Th(\mathcal{L}^{\otimes 2}\oplus\mathcal{L}^{\otimes 2})$, $\deg^{\mathbb{A}^1}\varphi' = 3\langle 1\rangle + \langle -1\rangle$

  $(u,v) \to (u^2-v^2, u\otimes v)$

  ⇝ $\varphi'' = (3\cdot\langle 1\rangle - \langle -1\rangle)\cdot\varphi'$, $\deg^{\mathbb{A}^1}\varphi'' = 8$
- k-even

Thm (motivic mod k Adams thm)

$(k,e)=1$, $\mathcal{E}_1,\mathcal{E}_2/S$, $\varphi: Th(\mathcal{E}_1)\to Th(\mathcal{E}_2)$, $\deg^{\mathbb{A}^1}\varphi|_s = k$

$\Rightarrow \exists N\in\mathbb{N}_0$: $Th(k^N\otimes\mathcal{E}_1) \simeq Th(k^N\otimes\mathcal{E}_2)$ in $SH(S)$.

Pf use (over clos. pt: φ/k ⇝) localisation sequences to extend (φ/k) iso from generic pt, the obstructions are nilpotent by the following.

Thm F-field, e = exp. char F, $s,w\in\mathbb{N}_{>0}$ $\Rightarrow \exists N = N(s,w)$ s.t. $(\pi_{s+w,w}\mathbb{1}_F)[1/e]$ is N-torsion.

Pf. slice spectral sequence & Bachmann-Hopkins on $\mathbb{1}[\eta^{-1}]$.

V - local, $V^\circ = V\setminus z$ (closed pt), triv. $\mathcal{E}_1,\mathcal{E}_2$, suppose have iso on V°.

⇝ $\mathbb{1}^0(V) \to \mathbb{1}^0(V^\circ) \xrightarrow{\partial} \mathbb{1}^1_z(V)$ $\quad \pi_{2d,d}(F(z))$, $d=\dim V$.

$\psi\in\mathbb{1}^0(V)$, $\varphi\in\mathbb{1}^0(V^\circ)$, θ

$\psi\mapsto 1$, $\varphi\mapsto k$, $\theta\mapsto 1$, $GW(F(S))$.

$\theta = \psi|_{V^\circ} + D$

$k\theta = \varphi|_{V^\circ}$

$= \partial(\theta^{k^m}) = 0$