PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 83, Number 3, November 1981

ON COMPLETE BIORTHOGONAL SYSTEMS

ROBERT M. YOUNG

ABSTRACT. Fundamental to the study of bases in a separable Hilbert space H is the notion of a *biorthogonal system*. Two sequences $\{f_n\}$ and $\{g_n\}$ of elements from H are said to be biorthogonal if $(f_n, g_m) = \delta_{nm}$. A complete sequence that possesses a biorthogonal sequence is called *exact*. Despite the symmetry of the definition of biorthogonality, simple examples show that $\{f_n\}$ may be exact while $\{g_n\}$ fails to be exact. For sequences of complex exponentials in $L^2(-\pi, \pi)$, the situation is dramatically different–if the sequence $\{e^{i\lambda_n t}\}$ is exact, then its biorthogonal sequence is also exact.

1. Introduction. Fundamental to the study of bases in a separable Hilbert space H is the notion of a biorthogonal system (see, e.g., [**2**] and [**3**] and the references therein).

DEFINITION. Two sequences $\{f_n\}$ and $\{g_n\}$ of elements from H are said to be *biorthogonal* if $(f_n, g_m) = \delta_{nm}$. A sequence that admits a biorthogonal sequence will be called *minimal*.

It is easy to show that a sequence $\{f_n\}$ is minimal if and only if none of its elements can be approximated by linear combinations of the others. If this is the case, then a biorthogonal sequence will be uniquely determined if and only if $\{f_n\}$ is *complete*. (This means that the set of all linear combinations $c_1 f_1 + \cdots + c_n f_n$ is dense in H, or, equivalently, that the zero vector alone is perpendicular to every f_n.) A sequence that is both minimal and complete will be called *exact*.

Despite the symmetry of the definition, it is apparent that for a minimal sequence the property of being complete is not inherited by its biorthogonal sequence. (Cf. if $\{f_n\}$ is a basis, then $\{g_n\}$ is also a basis [**4**, p. 29].) Indeed, if $\{e_n\}$ is an orthonormal basis for H, then $\{e_n + e_1\}_{n=2}^{\infty}$ is complete and has $\{e_n\}_{n=2}^{\infty}$ as its biorthogonal sequence. The following generalization is perhaps even more striking.

EXAMPLE. For each infinite-dimensional closed subspace K of H, there exists a biorthogonal system $\{f_n\}$, $\{g_n\}$ such that $\{f_n\}$ is complete and the closed subspace spanned by $\{g_n\}$ is K. Indeed, let $\{e_n\}$ be an orthonormal basis for K and B an orthonormal basis for $K^{\perp}$. We consider the sequence $\{e_n + y_n\}$ where the y_n are chosen from B in such a way that each element of B appears as a y_n infinitely many times. It is readily shown that $\{e_n + y_n\}$ and $\{e_n\}$ are biorthogonal and that the former sequence is complete.

Received by the editors February 2, 1981.
1980 *Mathematics Subject Classification*. Primary 42C30; Secondary 30D99.
Key words and phrases. Biorthogonal system, exact sequence, Paley-Wiener space.

© 1981 American Mathematical Society
0002-9939/81/0000-0521/$02.00

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use

The purpose of this note is to show that for minimal sequences of *complex exponentials* $\{e^{i\lambda_n t}\}$ in $L^2(-\pi, \pi)$ the situation is dramatically different–the completeness of such a sequence always ensures the completeness of its biorthogonal sequence.

THEOREM. *If the sequence of complex exponentials* $\{e^{i\lambda_n t}\}$ *is exact in* $L^2(-\pi, \pi)$, *then its biorthogonal sequence is also exact.*

2. The Paley-Wiener space. We shall make use of the Paley-Wiener space P consisting of all entire functions of exponential type at most π that are square-integrable on the real axis. It is clear that P is a vector space under pointwise addition and scalar multiplication; it is also an inner product space with respect to the inner product

$$(f, g) = \int_{-\infty}^{\infty} f(x)\overline{g(x)}\, dx.$$

If $f \in P$, then the Paley-Wiener theorem shows that

$$f(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi(t)e^{izt}\, dt,$$

with $\phi \in L^2(-\pi, \pi)$. Since the Fourier transform is an isometry, P is a separable Hilbert space, isometrically isomorphic to $L^2(-\pi, \pi)$.

Accordingly, problems involving complex exponentials in $L^2(-\pi, \pi)$ can be examined via their transform image in P. Thus, for example, the sequence $\{e^{i\lambda_n t}\}$ is complete in $L^2(-\pi, \pi)$ if and only if $\{\lambda_n\}$ is a *set of uniqueness* for P, i.e., if the relations

$$f \in P \quad \text{and} \quad f(\lambda_n) = 0 \qquad (n = 1, 2, 3, \dots)$$

imply that $f \equiv 0$. (A detailed account of the Paley-Wiener space and its role in the study of completeness and expansion properties of sets of complex exponentials can be found in [**4**].)

3. Proof of the theorem. It is convenient to describe the biorthogonal sequence in terms of an entire generating function which vanishes at the λ_n. Let S be the class of entire functions $f(z)$ of exponential type at most π such that $f(x)/(1 + x^2)^{1/2} \in L^2(-\infty, \infty)$. The existence of a function $f \in S$, equal to zero at every λ_n, but not identically zero, is necessary and sufficient for $\{e^{i\lambda_n t}\}$ to be minimal [**1**, p. 419]. Notice that $f(z)$ can have no zeros different from the λ_n. Otherwise, additional exponentials could be added to the sequence $\{e^{i\lambda_n t}\}$ without destroying its minimality, and this contradicts its completeness. It follows in the same way that all the zeros of $f(z)$ must be simple, and so $f'(\lambda_n)$ is never zero. Furthermore, if $g(z)$ is any other function in S, equal to zero at every λ_n, then $g(z) = Af(z)$. For if λ is any one of the λ_n, then the function $f(z)/f'(\lambda)(z - \lambda) - g(z)/g'(\lambda)(z - \lambda)$ belongs to P (provided $g \not\equiv 0$), equals zero at every λ_n, and so must be identically zero.

By virtue of the Paley-Wiener theorem, we can write

$$\frac{f(z)}{f'(\lambda_n)(z - \lambda_n)} = \frac{1}{2\pi}\int_{-\pi}^{\pi} g_n(t)e^{izt}\, dt, \tag{1}$$

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use

with $g_n \in L^2(-\pi, \pi)$. Thus the sequence $\{\overline{g_n(t)}\}$ is biorthogonal to $\{e^{i\lambda_n t}\}$. To prove it is complete, suppose that for some function $h \in L^2(-\pi, \pi)$,

$$\int_{-\pi}^{\pi} h(t)g_n(t)\,dt = 0 \qquad (n = 1, 2, 3, \ldots).$$

It is to be shown that $h(t) = 0$ almost everywhere. Replacing $h(t)$ by its Fourier series $h(t) = \sum_{-\infty}^{\infty} a_k e^{ikt}$ ($\sum_{-\infty}^{\infty}|a_k|^2 < \infty$), and then integrating term-by-term, we find

$$\text{(2)} \qquad \sum_{-\infty}^{\infty} a_k \int_{-\pi}^{\pi} g_n(t)e^{ikt}\,dt = 0 \qquad (n = 1, 2, 3, \ldots).$$

Let us suppose first that no λ_n is an integer. Then (1) and (2) show that

$$\text{(3)} \qquad \sum_{-\infty}^{\infty} a_k \frac{f(k)}{k - \lambda_n} = 0 \qquad (n = 1, 2, 3, \ldots).$$

Put $c_k = a_k f(k)/k$ ($k \neq 0$). Since $f(k)/(k - \lambda_1)$ are the Fourier coefficients of a function in $L^2(-\pi, \pi)$, $\{f(k)/k\} \in l^2$, and hence $\sum|c_k| < \infty$. This permits us to write $c = \sum c_k$. Since

$$a_k \frac{f(k)}{k - \lambda_n} = c_k \frac{k}{k - \lambda_n} = c_k\left(1 + \frac{\lambda_n}{k - \lambda_n}\right) \qquad (k \neq 0),$$

(3) becomes

$$-\frac{a_0 f(0)}{\lambda_n} + c + \lambda_n \sum_{k \neq 0} \frac{c_k}{k - \lambda_n} = 0 \qquad (n = 1, 2, 3, \ldots).$$

We define a function $g(z)$ by writing

$$g(z) = -a_0 f(0)\frac{\sin \pi z}{z} + c \sin \pi z + z \sin \pi z \sum_{k \neq 0} \frac{c_k}{k - z} = g_1 + g_2 + g_3.$$

Assertion. $g(z)$ belongs to the class S. Clearly, g_1 and g_2 belong to S. Since the set of functions $\{(\sin \pi(z - k))/\pi(z - k)\}_{-\infty}^{\infty}$ forms an orthonormal basis for P (take the Fourier transform of e^{ikt}) and $\sum|c_k|^2 < \infty$, it follows that the series

$$\sum_{k \neq 0} \pi(-1)^{k+1} c_k \frac{\sin \pi(z - k)}{\pi(z - k)} = \sin \pi z \sum_{k \neq 0} \frac{c_k}{k - z}$$

converges in the topology of P, and hence pointwise, to a function belonging to P. Thus g_3 belongs to S. This proves the assertion. Since $g(\lambda_n) = 0$ ($n = 1, 2, 3, \ldots$), we must have $g(z) = Af(z)$.

If $k \neq 0$, then

$$g(k) = k\pi(-1)^{k+1}c_k = \pi(-1)^{k+1}a_k f(k).$$

If $k = 0$, the first term is still equal to the last. Accordingly,

$$\text{(4)} \qquad A = \pi(-1)^{k+1}a_k \quad \text{for every } k,$$

since $f(k)$ is never zero (recall that no λ_n is an integer and $f(z)$ is equal to zero only at the λ_n). But $\{a_k\} \in l^2$, and therefore (4) is possible only if $a_k = 0$ for every k.

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use

Thus $h(t) = 0$ almost everywhere and the biorthogonal sequence $\{\overline{g_n(t)}\}$ is complete in $L^2(-\pi, \pi)$. This proves the theorem under the assumption that no λ_n is an integer.

In the general case, we can choose a suitable real number α such that $\lambda_n + \alpha$ is never an integer. Since multiplication by $e^{i\alpha t}$ is a unitary operator on $L^2(-\pi, \pi)$, the sequence $\{e^{i(\lambda_n + \alpha)t}\}$ is complete and hence its biorthogonal sequence $\{e^{i\alpha t}\overline{g_n(t)}\}$ is also complete. Thus $\{\overline{g_n(t)}\}$ is exact. $\square$

REMARK. The examples in §1 show that the converse of the theorem is false–there are incomplete sequences of complex exponentials that admit complete biorthogonal sequences. Whether *every* such sequence must admit a complete biorthogonal sequence appears to be an open problem.

REFERENCES

1. B. Ja. Levin, *Distribution of zeros of entire functions*, Transl. Math. Monographs, vol. 5, Amer. Math. Soc., Providence, R. I., 1964.

2. J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces*. I, Springer-Verlag, Berlin and New York, 1977.

3. I. Singer, *Bases in Banach spaces*. I, Springer-Verlag, Berlin and New York, 1970.

4. R. M. Young, *An introduction to nonharmonic Fourier series*, Pure and Appl. Math., vol. 93, Academic Press, New York, 1980.

DEPARTMENT OF MATHEMATICS, OBERLIN COLLEGE, OBERLIN, OHIO 44074

Current address: Department of Mathematics, University of California, Los Angeles, California 90024

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use