

A Gray Code Mediated Data-Oblivious (0,1)-Matrix-Vector Product Algorithm

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(0,1)-Matrix

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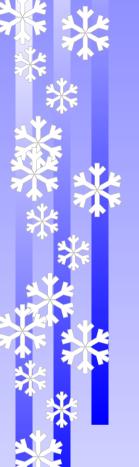
$$A\left(\in \{0,1\}^{3\times 4}\right) = \begin{pmatrix} 1 & 0 & 1 & 1\\ 0 & 0 & 1 & 0\\ 1 & 1 & 1 & 1 \end{pmatrix}$$



(0,1)-Matrices arise from problems in a variety of application areas including:

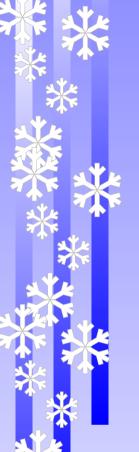


- (0,1)-Matrices arise from problems in a variety of application areas including:
 - adjacency matrices for simple graphs, representing the connectivity relationships between vertices;



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✓ term-by-document matrices generated by binary vector space model based computational information retrieval (BVIR) algorithms and applications such as search engines; an element w_{ij} in a weight matrix $W \in \{0,1\}^{n \times m}$ is set to 1 if a term t_j appears in document D_i , with 0 otherwise;



- (0,1)-Matrices arise from problems in a variety of application areas including:
 - ✓ matrix calculus applications in statistics and econometrics which generate special (0,1)-matrices such as selection, permutation, commutation, elimination, duplication, and shifting matrices;



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$$A = \sum_{i=1}^{m} \sum_{j=1}^{n} \alpha_{ij} E_{ij}, \{ E_{ij} = \mathbf{e}_i \mathbf{e}_j^T, 1 \le i \le m; 1 \le j \le m \}$$



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E.g.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$



A matrix-vector product operation may be represented by the equation

$$Ax = y$$

where the components are conformal: $A \in \{0, 1\}^{m \times n}$, $x \in \mathbb{R}^n$, and $y \in \mathbb{R}^m$.

The vectors may actually be over any algebraic ring for which addition and multiplication is closed and well defined.



E.g.

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} e \\ f \\ g \end{pmatrix}$$

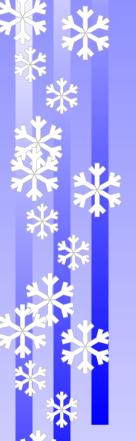


The general matrix-vector product

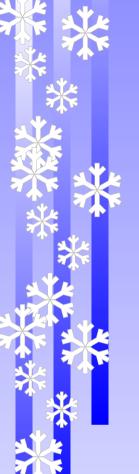
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Hamming Distance

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E.g., the two strings,

have a Hamming distance of two.



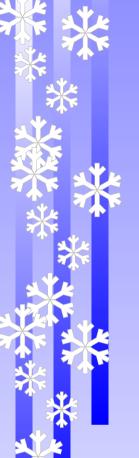
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- √ All adjacent Gray code strings have a Hamming distance of unity.
- ✓ One can label the vertices of a *hypercube* with bitstrings such that all *adjacent* neighbors have a bitstring differing in exactly one bit.
- √ Each Gray code corresponds to a *Hamiltonian* cycle about the vertices of a hypercube of a dimension corresponding to the string length.



 $Ax = y, A \in \mathbb{R}^{m \times n}, x \in \mathbb{R}^n$, and $y \in \mathbb{R}^m$, is computed via

$$y_i = \sum_{j=1}^n \alpha_{ij} x_j, 1 \le i \le m$$

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We'll select that ordering which performs an inner product $(v^T u = \alpha)$



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- \checkmark either the product $\alpha_{ij}x_j$ contributes to the sum as x_j when $\alpha_{ij}=1$
- \checkmark or it is skipped, when $\alpha_{ij} = 0$.
- \checkmark \Rightarrow All multiplications are obviated leaving only $\leq mn$ additions. (Note: For each row, one addition may be replaced by a load.)



Consider computing the difference between two elements of y, y_i and y_k :

$$y_k - y_i = \sum_{j=1}^n \alpha_{kj} x_j - \sum_{j=1}^n \alpha_{ij} x_j$$
$$= \sum_{j=1}^n (\alpha_{kj} x_j - \alpha_{ij} x_j)$$
$$= \sum_{j=1}^n x_j (\alpha_{kj} - \alpha_{ij})$$



However, if y_i has already been computed,

$$y_k = y_i + \sum_{j=1}^{n} x_j (\alpha_{kj} - \alpha_{ij})$$

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 \Rightarrow each subsequent element of y can be computed as the sum of a previously computed element with the inner product of x and the difference vector of the two rows of A, d.



Then we have saved $||A_k||_1 - ||d||_1 - 1$ operations in computing y_k , where $||d||_1$ is the *Hamming distance* between two rows of A.



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For a duplicated row in A, load its previously computed value at the expense of no additions.



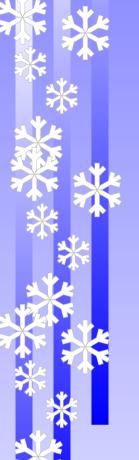
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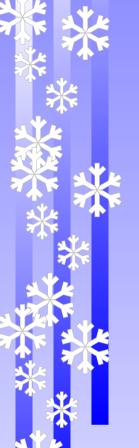
If the remaining rows are unique, what's the lowest number of additions that we need? There must be a Hamming distance of at least one between each row of A and each other row. The optimum in this case would be for there to be, for every row of A, some other row of A of unit Hamming distance. We have already defined a sequence of Boolean vectors which satisfies this condition,



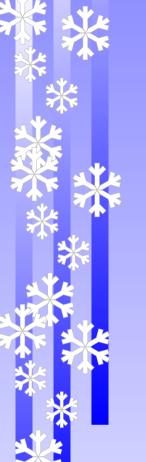
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- ✓ Using the differencing method above, a $r \times q$ Gray code matrix-vector product can be computed in a maximum of r additions.

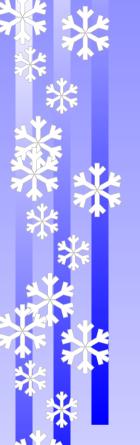


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- ✓ Using the differencing method above, a $r \times q$ Gray code matrix-vector product can be computed in a maximum of r additions.
- ✓ If one substitutes a load for an addition when there are at most 1 nonzero bits in the row, the maximum becomes r q 1 required additions.



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If we have a general (0, 1)-matrix matrix having q columns and any number of rows,

- we can first precompute a scratch y using the Gray code Matrix vector product with x.
- \checkmark Then, for each row of the general matrix A, we load into y the entry in the scratch y corresponding to the bitpattern for that row.
- The only additions are from the precomputation of the scratch y from the Gray code matrix. If m > r, we have a greater reduction in the maximum number of additions for the Gray code matrix itself.



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- $\sqrt{n \mod q} \neq 0$ in which case we will use $\lfloor n/q \rfloor$ Gray code matrix stripes of width q, and one additional Gray code matrix stripe of width $n \mod q$ with a subsequent sum across the stripes for all m.



If we define w(q, m, n) as the number of additions required for a general (0, 1)-matrix. Then for an arbitrary m and n, ignoring the q + 1 savings from above,

$$w(q, m, n) = \begin{cases} 2^{n} - n - 1 & \text{if } n < q \\ 2^{q} - q - 1 & \text{if } n = q \\ \frac{n}{q} (2^{q} - q - 1 + m) - m & \text{if } n > q \text{ and } 0 = n \text{ mod } q \\ \lfloor \frac{n}{q} \rfloor (2^{q} - q - 1 + m) & \text{if } n > q \text{ and } 0 \neq n \text{ mod } q \\ +2^{n \text{ mod } q} - n \text{ mod } q - 1 \end{cases}$$

The q which minimizes w may be computed by searching for the smallest w for a reasonably small set of q's bounded by $\lceil \lg m \rceil$.

(**Note:** *w* is discontinuous – when searching, beware of local minima.)



Matrix Dimensions											
ROWS	COLUMNS										
	5^1	5^2	5^3	5^4	5^5	5^6	5^7	5^8	5^9	5^{10}	
5^1	3	3	3	3	3	3	3	3	3	3	
$\int 5^2$	5	4	4	4	4	4	4	4	4	4	
$\int 5^3$	5	5	5	5	5	5	5	5	5	5	
$\int 5^4$	5	7	7	7	7	7	7	7	7	7	
5^5	5	9	9	9	9	9	9	9	9	9	
$\int 5^6$	5	13	11	11	11	11	11	11	11	11	
$\int 5^7$	5	13	14	13	13	13	13	13	13	13	
$\int 5^8$	5	13	16	15	15	15	15	15	15	15	
$\int 5^9$	5	13	18	17	17	17	17	17	17	17	
5^{10}	5	13	18	19	20	20	20	20	20	20	

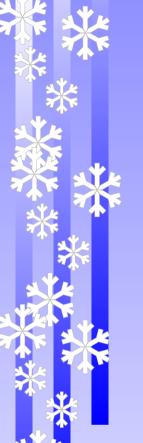


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5^1	1.1e1	8.0e1	4.1e2	2.0e3	1.0e4	5.2e4	2.6e5	1. 3e6	6.5e6	3.2e7	
5^2	2.8e1	2.2e2	1.1e3	5.7e3	2.8e4	1.4e5	7.2e5	3. 6e6	1.8e7	9.0e7	
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5^5	2.8e1	7.3e3	4.7e4	2.5e5	1.2e6	6.3e6	3.1e7	1. 5e8	7.8e8	3.9e9	
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5^7	2.8e1	9.0e4	7.6e5	4.1e6	2.0e7	1.0e8	5.1e8	2. 5e9	1.3e10	6.4e10	
5^8	2.8e1	4.0e5	3.2e6	1.7e7	8.8e7	4.4e8	2.2e9	1. 1e10	5.5e10	2.7e11	
5^9	2.8e1	1.9e6	1.3e7	7.5e7	3.8e8	1.9e9	9.5e9	4. 7e10	2.3e11	1.2e12	
5^{10}	2.8e1	9.7e6	6.0e7	3.2e8	1.6e9	8.4e9	4.2e10	2.11e11	1.06e12	5.28e12	



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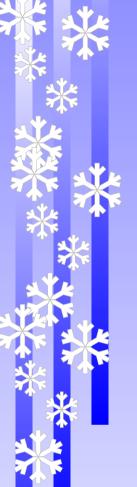


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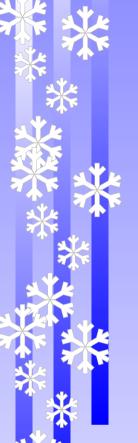


The following alternate approaches are under development:

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Then all of the E_{ij} matrices will be added together forming denser (0,1)-matrices for each set of identical α_{ij} terms.



We then get the equation

$$A = \sum_{i=1}^{k} \beta_i B_i$$

Where there are k distinct entries, β_i in A and each B_i represents a distinct (0, 1)-matrix.



For a single β , the matrix vector product can be written as:

$$Ax = y = \beta Bx = B(\beta x),$$

The scalar product has been reduced from $\Theta(mn)$ to $\Theta(n)$.



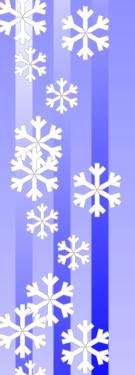
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The previous two equations may be combined:

$$Ax = y = \left(\sum_{i=1}^k \beta_i B_i\right) x = \sum_{i=1}^k B_i(\beta_i x).$$



Lets consider the case where k=2.

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- ✓ The elements of the matrix A are either of only two values, $\alpha_{ij} \in \{\beta_1, \beta_2\}$.
- ✓ The cardinality of the set that α_{ij} is drawn from is the same as that of the (0, 1)-matrix.
- Rather than use a pair of (0,1)-matrices, we can use a single one if we first affinely transform the equation with an appropriate scaling and translation. The elements of A can be scaled simply by multiplying by a scaling factor: γA getting $\gamma Ax = \gamma y$.



To translate the values of A, we need to create a conformal translation matrix, $T = \{1\}^{m \times n}$, which will serve as a basis for the uniform translation of all the entries of A.

A unit translation of A towards $+\infty$ is represented by the sum A + T.

We get
$$(A + T)x = Ax + Tx = y^{(A)} + y^{(T)}$$
,

where
$$y_j^{(T)} = \sum_{i=1}^{n} x_i$$
.



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The above equations may now be combined to represent the full affine transformation:

$$\gamma(A + \delta T)x = \gamma(Ax + \delta Tx)$$
$$= \gamma y^{(A)} + \gamma \delta y^{(T)} = \gamma(y^{(A)} + \delta y^{(T)})$$



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This can be transformed into a (0,1)-matrix based vector product by setting $\gamma=2$ and $\delta=-\frac{1}{2}$:



$$\{-1,1\}^{m \times n} x = \gamma \left(\{0,1\}^{m \times n} + \delta \{1\}^{m \times n} \right) x$$

$$= \gamma (A + \delta T) x$$

$$= \gamma (Ax + \delta Tx)$$

$$= \gamma (y^{(A)} + \delta y^{(T)})$$

$$= 2(y^{(A)} - \frac{1}{2}y^{(T)})$$

$$= 2y^{(A)} - y^{(T)}).$$



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- We were able to demonstrate a differencing method for reducing the number of arithmetic operations within a (0, 1)-matrix vector product.
- ✓ We used that method to more efficiently perform Gray code matrix vector products.



✓ Using one or more Gray code matrix vector products, we showed how to reduce the number of arithmetic operations for a general (0, 1)-matrix vector product.



- ✓ Using one or more Gray code matrix vector products, we showed how to reduce the number of arithmetic operations for a general (0, 1)-matrix vector product.
- We then showed how (0,1)-matrix vector products may be applied to the performance of certain restricted classes of more general matrix vector products.