MULTIPLICATION AND INTEGRAL OPERATORS ON SPACES OF ANALYTIC FUNCTIONS

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Abstract

We investigate operators on Banach spaces of analytic functions on the unit disk D in the complex plane. The operator T_g, with symbol $g(z)$ an analytic function on the disk, is defined by $T_g f(z) = \int_0^z f(w)g'(w)\, dw$. The operator T_g and its companion $S_g f(z) = \int_0^z f'(w)g(w)\, dw$ are related to the multiplication operator $M_g f(z) = g(z)f(z)$, since integration by parts gives $M_g f = f(0)g(0) + T_g f + S_g f$.

Characterizing boundedness of T_g and S_g on the Dirichlet space, Bloch space, and $BMOA$ illuminates well known results on the multipliers (i.e., symbols g for which M_g is bounded) of these spaces. The multipliers must satisfy two conditions, which depend on the space. The operators T_g and S_g split the two conditions on the multipliers. M_g is bounded only if both S_g and T_g are bounded, yet one of S_g or T_g may be bounded when M_g is unbounded. We note a similar phenomenon on the Hardy spaces $H^p, 1 \leq p < \infty$, and Bergman spaces A^p. An open problem is to distinguish the g for which S_g and T_g are bounded or compact on H^∞, the space of bounded analytic functions. We give partial results toward solving this problem, including an example of a function $g \in H^\infty$ such that T_g is not bounded on H^∞.

Finally, we study the symbols for which T_g and S_g have closed range on H^2, A^p, Bloch, and $BMOA$. Our main result regarding closed range operators is a characterization of g for which S_g has closed range on the Bloch space. We point out analogous

results for H^2 and the Bergman spaces. We also show T_g is never bounded below on H^2, Bloch, nor $BMOA$, but may be bounded below on A^p.

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Chapter 1

Introduction

We investigate operators on Banach spaces of analytic functions on the unit disk D in the complex plane. The operator T_g, with symbol $g(z)$ an analytic function on the disk, is defined by

$$T_g f(z) = \int_0^z f(w)g'(w)\,dw \qquad (z \in D).$$

T_g is a generalization of the standard integral operator, which is T_g when $g(z) = z$. Letting $g(z) = \log(1/(1-z))$ gives the Cesáro operator [1]. Discussion of the operator T_g first arose in connection with semigroups of composition operators (see [17] for background). Characterizing the boundedness and compactness of T_g on certain spaces of analytic functions is of recent interest, as seen in [2], [4], [8] and [17], and open problems remain. T_g and its companion operator $S_g f(z) = \int_0^z f'(w)g(w)\,dw$ are related to the multiplication operator $M_g f(z) = g(z)f(z)$, since integration by parts gives

$$M_g f = f(0)g(0) + T_g f + S_g f. \tag{1.1}$$

If any two of M_g, S_g, and T_g are bounded on a space in which point evaluation is bounded, then all three operators are bounded. But on many spaces, there exist functions g for which one operator is bounded and two are unbounded. The pointwise multipliers of the Hardy, Bergman and Bloch spaces are well known, as well as David Stegenga's results on multipliers of the Dirichlet space and $BMOA$. Theorem 2.7 below states these results. We examine boundedness and compactness of T_g and S_g on the Hardy, Bergman, Dirichlet, and Bloch spaces, as well as $BMOA$. According to [2], boundedness of the operator T_g on H^2 was first characterized by Christian Pommerenke. Boundedness and compactness of T_g was characterized on the Hardy spaces H^p for $p < \infty$ by Alexandru Aleman and Joseph Cima in [2], and on the Bergman spaces by Aleman and Aristomenis Siskakis in [4]. In [17], Siskakis and Ruhan Zhao proved T_g is bounded (and compact) on $BMOA$ if and only if $g \in LMOA$. As seen in sections 3.1, 3.2, and 3.3, boundedness of S_g is equivalent to g being bounded, while the conditions for T_g are more complicated.

An interesting interplay of the three operators M_g, T_g, and S_g occurs. In characterizing the multipliers of the Dirichlet and Bloch spaces and $BMOA$, two conditions on g are required. It turns out that the operators T_g and S_g split the conditions on the multipliers. One condition characterizes boundedness of T_g, and the other condition characterizes when S_g is bounded. In the case of the Hardy and Bergman spaces, the condition for T_g to be bounded subsumes that for S_g and M_g. Action on the space H^∞ provides an example in which M_g is bounded while T_g and S_g are not. This phenomenon is unique among the other spaces studied here, and a complete characterization of the symbols that make T_g and S_g bounded on H^∞ is unknown.

We also examine conditions on the symbol g that cause T_g and S_g to have closed range on certain spaces. We examine aspects of the problems on Hardy, weighted

Bergman, and Bloch spaces, and $BMOA$. On the spaces studied, T_g and S_g have closed range if and only if they are bounded below (Theorem 2.11). In Theorem 4.7, we characterize the symbols g for which S_g is bounded below on the Bloch space. We also point out analogous results for the Hardy space H^2 and the weighted Bergman spaces A^p_α for $1 \leq p < \infty, \alpha > -1$. In Theorem 4.1 we show the companion operator T_g is never bounded below on H^2, Bloch, nor $BMOA$. We subsequently mention an example from [16] demonstrating T_g may be bounded below on A^p.

Chapter 2

Background

2.1 General Preliminaries

For two nonnegative quantities f and g, the notation $f \lesssim g$ will mean there exists a universal constant C such that $f \leq Cg$. $f \sim g$ will mean $f \lesssim g \lesssim f$.

Let $D = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk in the complex plane and $H(D)$ the set of analytic functions on D.

Theorem 2.2 is a generalization of a result on multipliers of Banach spaces in which point evaluation is a bounded linear functional. We state the result for multipliers first.

Theorem 2.1. *Let X be a Banach space of analytic functions on which point evaluation is bounded for each point $z \in D$. Suppose M_g is bounded on X for some $g \in X$. Then*

$$|g(z)| \leq \|M_g\|.$$

The proof is similar to Theorem 2.2, so we omit it here. (See, e.g., [10, Lemma 11].)

Theorem 2.2. *Let X and Y be Banach spaces of analytic functions, $z \in D$, and let λ_z and λ'_z be linear functionals defined by $\lambda_z f = f(z)$ and $\lambda'_z f = f'(z)$ for $f \in X \cup Y$. Suppose λ_z and λ'_z are bounded on X and Y.*

(i) *If S_g maps X boundedly into Y, then*

$$|g(z)| \le \|S_g\| \frac{\|\lambda'_z\|_Y}{\|\lambda'_z\|_X}.$$

(ii) *If T_g maps X boundedly into Y, then*

$$|g'(z)| \le \|T_g\| \frac{\|\lambda'_z\|_Y}{\|\lambda_z\|_X}.$$

Proof. Note that, for $f \in X$,

$$|f'(z)||g(z)| = |\lambda'_z S_g(f)| \le \|\lambda'_z\|_Y \|S_g\| \|f\|_X. \tag{2.1}$$

Since

$$\sup_{\|f\|_X = 1} |f'(z)| = \|\lambda'_z\|_X,$$

taking the supremum of both sides of (2.1) over all f in X with norm 1 gives us

$$\|\lambda'_z\|_X |g(z)| \le \|S_g\| \|\lambda'_z\|_Y.$$

Hence (i) holds. Similarly,

$$|f(z)||g'(z)| = |\lambda'_z T_g(f)| \le \|\lambda'_z\|_Y \|T_g\| \|f\|_X.$$

Taking the supremum over $\{f \in X : \|f\|_X = 1\}$, we get

$$\|\lambda_z\|_X |g'(z)| \leq \|T_g\| \|\lambda'_z\|_Y .$$

This completes the proof. $\square$

When $Y = X$, we obtain the following corollary.

Corollary 2.3. *If X is a Banach space of analytic functions on which point evaluation of the derivative is a bounded linear functional, and S_g is bounded on X, then g is bounded.*

Corollary 2.3 will be used frequently below, because λ'_z is bounded for each $z \in D$ on the spaces in which we are interested.

2.2 Spaces of Analytic Functions

We define several Banach spaces of analytic functions on which we will compare various properties of S_g, T_g, and M_g. Zhu [21] is a good reference for background on the spaces defined in this section.

For $1 \leq p < \infty$, the Hardy space H^p on D is

$$\{f \in H(D) : \|f\|_p^p = \sup_{0<r<1} \int_0^{2\pi} |f(re^{it})|^p \, dt < \infty\}.$$

The space of bounded analytic functions on D is

$$H^\infty = \{f \in H(D) : \|f\|_\infty = \sup_{z \in D} |f(z)| < \infty\}.$$

We define weighted Bergman spaces, for $\alpha > -1$, $1 \le p < \infty$,

$$A^p_\alpha = \{f \in H(D) : \|f\|^p_{A^p_\alpha} = \int_D |f(z)|^p (1-|z|^2)^\alpha \, dA(z) < \infty\},$$

where $dA(z)$ refers to Lebesgue area measure on D. We denote the unweighted Bergman space $A^p = A^p_0$.

The Bloch space is

$$\mathcal{B} = \{f \in H(D) : \|f\|_{\mathcal{B}} = \sup_{z\in D} |f'(z)|(1-|z|^2) < \infty\}.$$

Note that $\| \, \|_{\mathcal{B}}$ is a semi-norm. The true norm is $|f(0)|+\|f\|_{\mathcal{B}}$, accounting for functions differing by an additive constant. It is well known that H^∞ a subspace of $\mathcal{B}$, and $\|f\|_{\mathcal{B}} \le \|f\|_\infty$ for all $f \in H^\infty$ [21, Proposition 5.1]. For $\alpha > 0$, the α-Bloch space $\mathcal{B}_\alpha$ and logarithmic Bloch space $\mathcal{B}_{\alpha,\ell}$ are the following sets of analytic functions defined on the disk:

$$\mathcal{B}_\alpha = \{f \in H(D) : \ \|f\|_{\mathcal{B}_\alpha} = \sup_{z\in D} |f'(z)|(1-|z|^2)^\alpha < \infty\},$$

$$\mathcal{B}_{\alpha,\ell} = \{f \in H(D) : \ \|f\|_{\mathcal{B}_{\alpha,\ell}} = \sup_{z\in D} |f'(z)|(1-|z|^2)^\alpha \log \frac{1}{1-|z|} < \infty\}.$$

Since $\mathcal{B} = \mathcal{B}_1$, we define $\mathcal{B}_\ell := \mathcal{B}_{1,\ell}$. For $0 < \alpha < 1$, $\mathcal{B}_\alpha$ are (analytic) Lipschitz class spaces (see [9, Theorem 5.1]).

Define the conelike region with aperture $\alpha \in (0,1)$ at $e^{i\theta}$ to be

$$\Gamma_\alpha(e^{i\theta}) = \left\{ z \in D : \frac{|e^{i\theta} - z|}{1-|z|} < \alpha \right\}.$$

For a function f on D, define the *nontangential limit* of f at $e^{i\theta}$ to be

$$f^*(e^{i\theta}) = \lim_{z \to e^{i\theta}} f(z) \qquad (z \in \Gamma_\alpha(e^{i\theta})),$$

provided the limit exists. If $f \in H^1$, then the nontangential limit of f exists for almost all $e^{i\theta} \in \partial D$ (see [11, Theorem I.5.2]). In particular, we have the radial limit $\lim_{r\to 1^-} f(re^{i\theta}) = f^*(e^{i\theta})$ almost everywhere. When f is in H^1, we define $f(e^{i\theta}) = f^*(e^{i\theta})$, so that f is defined on $\overline{D}$ except for a set of measure 0 in ∂D.

For a measurable complex-valued function φ defined on ∂D, and an arc $I \subseteq \partial D$, define

$$\varphi_I = \frac{1}{|I|} \int_I \varphi(e^{it})\, dt,$$

where $|I|$ is the length of I, normalized so that $|I| \leq 1$. The function φ has *bounded mean oscillation* if

$$\sup_I \frac{1}{|I|} \int_I |\varphi(e^{it}) - \varphi_I| dt < \infty,$$

as I ranges over all arcs in ∂D.

The space $BMOA$ is the set of functions in H^1 whose radial limit functions have bounded mean oscillation. We define the semi-norm

$$\|f\|_* = \sup_I \frac{1}{|I|} \int_I |f(e^{it}) - f_I| dt,$$

so $BMOA = \{f \in H^1 : \|f\|_* < \infty\}$. We assume $f \in H^1$, although in fact, $BMOA$ is contained in H^2. One way to see this is via the duality relations $(H^1)^* \cong BMOA$ (see [21, Theorem 9.20]) and $(H^p)^* \cong H^q$, where $1/p + 1/q = 1$ (see [9, Theorem 7.3]). Since $H^2 \subset H^1$, we have $BMOA \cong (H^1)^* \subset (H^2)^* \cong H^2$. In fact, there exists $C > 0$

such that, for all $f \in BMOA$,

$$\|f\|_2 \leq C\|f\|_*. \tag{2.2}$$

Note that H^∞ is a subspace of $BMOA$, since for $f \in H^\infty$, the following calculation shows

$$\|f\|_* \leq \|f\|_\infty : \tag{2.3}$$

$$\begin{aligned}
\|f\|_* &= \sup_I \frac{1}{|I|}\int_I |f - f_I| \\
&\leq \sup_I \bigg(\frac{1}{|I|}\int_I |f - f_I|^2\bigg)^{1/2} \\
&= \sup_I \bigg(\frac{1}{|I|}\int_I (f - f_I)(\overline{f} - \overline{f_I})\bigg)^{1/2} \\
&= \sup_I \bigg(\frac{1}{|I|}\bigg(\int_I |f|^2 - \int_I f\overline{f_I} - \int_I \overline{f} f_I\bigg) + |f_I|^2\bigg)^{1/2} \\
&= \sup_I ((|f|^2)_I - |f_I|^2)^{1/2} \\
&\leq \sup_I ((|f|^2)_I)^{1/2} \leq \|f\|_\infty.
\end{aligned}$$

The function $f \in BMOA$ has *vanishing mean oscillation* if, for all $\varepsilon > 0$, there exists $\delta > 0$ such that $|I| < \delta$ implies

$$\frac{1}{|I|}\int_I |f - f_I| < \varepsilon.$$

The subspace of $BMOA$ consisting of the functions with vanishing mean oscillation is denoted $VMOA$. Another way we write the condition defining $VMOA$ is

$$VMOA = \{f \in BMOA : \lim_{|I|\to 0} \frac{1}{|I|}\int_I |f - f_I| = 0\}.$$

A noteworthy subspace of $VMOA$ is those functions whose mean oscillation vanishes as least as quickly as $1/\log(1/|I|)$. We define

$$LMOA = \{f \in VMOA : \lim_{|I|\to 0} \frac{\log(1/|I|)}{|I|} \int_I |f - f_I| < \infty\}.$$

A useful characterization of $BMOA$ for our purposes involves Carleson measures. For $1 \leq p < \infty$, a positive measure μ on D is a *Carleson measure for* H^p if there exists $C > 0$ such that

$$\int_D |f|^p \, d\mu \leq C\|f\|_p^p \quad \text{for all } f \in H^p.$$

For an arc $I \subseteq \partial D$, define the *Carleson rectangle* associated with I to be

$$S(I) = \{re^{i\theta} : 1 - |I| < r < 1, e^{i\theta} \in I\}.$$

For $1 \leq p < \infty$, the measure μ is Carleson for H^p if and only if there exists $C > 0$ such that $\mu(S(I)) \leq C|I|$ for all arcs $I \subseteq \partial D$ (a well-known result of Lennart Carleson, see [11, Theorem II.3.9]). The smallest such C is called the Carleson constant for the measure μ. Note that this characterization of Carleson measures is independent of p, and it shows the Carleson measures for H^p are the same for all p $(1 \leq p < \infty)$.

Define, for $f \in H(D)$, $d\mu_f(z) = |f'(z)|^2(1 - |z|^2) \, dA(z)$. $BMOA$ is the set of $f \in H^2$ for which μ_f is Carleson for H^2, and the $BMOA$ semi-norm $\|f\|_*$ is comparable to the square root of the Carleson constant for μ_f (see [11, Theorem VI.3.4]). The space $VMOA$ is the set of f for which

$$\lim_{|I|\to 0} \frac{\mu_f(S(I))}{|I|} = 0.$$

Also,

$$LMOA = \{f \in VMOA : \lim_{|I|\to 0} \frac{\mu_f(S(I))}{|I|} \log(1/|I|) < \infty\}.$$

The next lemma will be useful in Chapter 4 when showing T_g is not bounded below on $H^2, BMOA$, or the Bloch space.

Lemma 2.4. *If n is a positive integer, then $1 = \|z^n\|_2 \sim \|z^n\|_* \sim \|z^n\|_{\mathcal{B}}$, and the constants of comparison are independent of n.*

Proof. A straightforward calculation shows $\|z^n\|_2 = 1$ for all n. Checking the Bloch norm, we get $\|z^n\|_{\mathcal{B}} \sim \sup_{0<r<1} nr^{n-1}(1-r) = (1-\frac{1}{n})^{n-1} \to 1/e$ as $n \to \infty$. Finally, $1 = \|z^n\|_2 \lesssim \|z^n\|_* \lesssim \|z^n\|_\infty = 1$, by (2.2) and (2.3). $\square$

On all the spaces mentioned, point evaluation is a bounded linear functional. For $f \in H^p (1 \le p < \infty), |f(z)| \le 2^{1/p}\|f\|_p(1-|z|)^{-1/p}$ (see [9, p. 36]). The norm of point evaluation at z in A^p_α is comparable to $(1-|z|)^{-(2+\alpha)/p}$ [21, Theorem 4.14]. In $\mathcal{B}$, the norm of point evaluation at z is comparable to $\log(2/(1-|z|))$, which we demonstrate in the next Proposition.

Proposition 2.5. *For $f \in \mathcal{B}$, let $\lambda_z f = f(z)$ denote point evaluation at z. Then*

$$\|\lambda_z\| \sim \log(2/(1-|z|)).$$

Proof. If $f \in \mathcal{B}$, then $|f'(z)| \le \|f\|_{\mathcal{B}}/(1-|z|)$ by the definition of $\mathcal{B}$. Integrating along a ray from the origin to $z = re^{i\theta} \in D$, we have

$$\begin{aligned} |f(z) - f(0)| &= |\int_0^z f'(w)\, dw| \\ &\le \int_0^r |f'(te^{i\theta})|\, dt \\ &\le \|f\|_{\mathcal{B}} \int_0^r 1/(1-t)\, dt \\ &= \|f\|_{\mathcal{B}} \log(1/(1-r)). \end{aligned}$$

Thus, $|f(z)| \le |f(0)| + \|f\|_{\mathcal{B}} \log(1/(1-|z|))$, and $\|\lambda_z\| \lesssim \log(2/(1-|z|))$.

For $a \in D$, define the test function

$$f_a(z) = \log(1/(1-\overline{a}z)).$$

Then $|f_a'(z)| = |a|/|1-\overline{a}z|$, and $f_a \in \mathcal{B}$ with $\|f_a\|_{\mathcal{B}} \le 1$ for all $a \in D$. Also,

$$\|\lambda_a\| \ge |f_a(a)| = \log(1/(1-|a|^2)).$$

This shows $\log(2/(1-|z|)) \lesssim \|\lambda_z\|$, hence the proposition is true. $\square$

Remark. This result generalizes to the α-Bloch spaces, and for $\alpha > 1$ we have that $\|\lambda_z\|_{\mathcal{B}_\alpha} \sim (1-|z|^2)^{\alpha-1}$. The proof is similar but the test functions must be adjusted.

An application of bounded point evaluation is the following well-known result.

Proposition 2.6. *The following are equivalent:*

(i) *M_g is bounded on H^p for $1 \le p \le \infty$.*

(ii) *M_g is bounded on A^p_α for $1 \le p < \infty, \alpha > -1$.*

(iii) *$g \in H^\infty$.*

Proof. If $g \in H^\infty, 1 \le p < \infty$, then

$$\begin{aligned}\|M_g f\|_p^p = \|fg\|_p^p &= \sup_{0<r<1} \int_0^{2\pi} |f(re^{it})g(re^{it})|^p \, dt \\ &\le \|g\|_\infty^p \sup_{0<r<1} \int_0^{2\pi} |f(re^{it})|^p \, dt \\ &= \|g\|_\infty^p \|f\|_p^p.\end{aligned}$$

If $p = \infty$, then $\|M_g f\|_\infty = \sup_{z \in D} |f(z)g(z)| \le \|f\|_\infty \|g\|_\infty$. Conversely, if M_g is bounded on H^p $(1 \le p \le \infty)$, then $g \in H^\infty$ by Theorem 2.1. We have proved (i) and (iii) are equivalent. The proof that (ii) and (iii) are equivalent is similar. □

2.3 Differentiation Isomorphisms

When studying T_g and S_g, it is useful to be able to compare the norm of a function to the norm of its derivative. For $p \ge 1$, $\alpha > -1$, the differentiation operator and its inverse, the indefinite integral, are isomorphisms between $A^p_\alpha/\mathbb{C}$ and $A^p_{\alpha+p}$ by the relation

$$\|f\|_{A^p_\alpha} \sim |f(0)| + \|f'\|_{A^p_{\alpha+p}} \tag{2.4}$$

(see [21, Theorem 4.28]). Note that applying the Open Mapping Theorem to the bijection $f + \mathbb{C} \leftrightarrow f - f(0)$, we get that $A^p_\alpha/\mathbb{C}$ and $\{f \in A^p_\alpha : f(0) = 0\}$ are isomorphic. Making the natural definition $A^2_{-1} = H^2$, (2.4) holds for $p = 2$, $\alpha = -1$ as well. For $f \in H^2$ with $f(0) = 0$, this is the well-known Littlewood-Paley identity,

$$\frac{1}{2\pi}\|f\|_2^2 = \frac{1}{\pi}\int_D 2|f'(z)|^2 \log\frac{1}{|z|}\, dA(z)$$

(see [11, Lemma VI.3.1]). The relation (2.4) demonstrates a key connection between S_g and M_g via the differentiation operator, since $(S_g f)' = M_g f'$, and thus the following diagram is commutative.

$$\begin{array}{ccc} A^p_\alpha/\mathbb{C} & \xrightarrow{S_g} & A^p_\alpha/\mathbb{C} \\ {\scriptstyle f\mapsto f'}\big\downarrow & & \big\downarrow{\scriptstyle f\mapsto f'} \\ A^p_{\alpha+p} & \xrightarrow{M_g} & A^p_{\alpha+p} \end{array}$$

2.4 Boundedness of M_g

The multiplication operator M_g has been thoroughly studied, and conditions characterizing boundedness of M_g on the spaces mentioned are well known. Theorem 2.7 lists these results.

For a Banach space $X \subset H(D)$, let

$$M[X] = \{g \in H(D) : M_g \text{ is bounded on } X\}.$$

Suppose point evaluations λ_z on X are uniformly bounded for z in compact subsets of D. That is, given a compact set $K \subset D$, there exists $C > 0$ such that $\|\lambda_z\| < C$ for all $z \in K$. Therefore, the unit ball of X is uniformly bounded on compact subsets of D, hence the unit ball is a normal family. Using the Closed Graph Theorem, it follows that $M[X] = \{g \in H(D) : M_g X \subset X\}$.

Theorem 2.7. (i) $M[H^p] = H^\infty, 1 \leq p \leq \infty.$

(ii) $M[A^p_\alpha] = H^\infty, 1 \leq p < \infty, \alpha > -1.$

(iii) $M[\mathcal{B}] = H^\infty \cap \mathcal{B}_\ell.$

(iv) $M[BMOA] = H^\infty \cap LMOA.$

(v) *M_g is bounded on $\mathcal{D}$ if and only if $g \in H^\infty$ and the measure μ_g given by*

$d\mu_g(z) = |g'(z)|^2 dA(z)$ *is a Carleson measure for* $\mathcal{D}$.

For (i) and (ii) see Proposition 2.6. The result for the Bloch space, (iii), is due originally to Jonathan Arazy [5]. The results (iv) and (v) for $BMOA$ and the Dirichlet space $\mathcal{D}$ are due to David Stegenga (see [18] and [19]). In [19] Stegenga also characterized the Carleson measures for $\mathcal{D}$ by a condition involving capacity.

2.5 Operators with Closed Range

A bounded operator T on a space X is said to be *bounded below* if there exists $C > 0$ such that $\|Tf\| \geq C\|f\|$ for all $f \in X$. A one-to-one operator on a Banach space has closed range if and only if it is bounded below. The analogue of Theorem 2.11 for composition operators is found in Cowen and MacCluer [7]. The proof, included here for easy reference, depends on the general propositions 2.9 and 2.10. When considering when S_g is bounded below, we note that S_g maps any constant function to the 0 function. Thus, it is only useful to consider spaces of analytic functions modulo the constants.

Lemma 2.8. *T_g is one-to-one for nonconstant g, and S_g is one-to-one on $H(D)/\mathbb{C}$.*

Proof. Let $f_1, f_2 \in H(D)$. If $T_g f_1 = T_g f_2$, taking derivatives gives $f_1(z)g'(z) = f_2(z)g'(z)$. Thus $f_1(z) = f_2(z)$ except possibly at the (isolated) points where g' vanishes. Since f_1 and f_2 are analytic, $f_1 = f_2$. The proof is similar for S_g on $H(D)/\mathbb{C}$. $\square$

Let Y be a Banach space and let T be a bounded linear operator on Y.

Proposition 2.9. *If T is bounded below then T has closed range.*

Proof. Assume T is bounded below, i.e., there exists $\varepsilon > 0$ such that $\|Tf\| \geq \varepsilon \|f\|$ for all $f \in Y$. Suppose $\{Tf_n\}$ is a Cauchy sequence in the range of T. Since $\|f_n - f_m\| \lesssim \|Tf_n - Tf_m\|$, $\{f_n\}$ is also a Cauchy sequence. Letting $f = \lim f_n$, we have $Tf_n \to Tf$, showing Tf_n converges in the range of T. Hence the range of T is closed. $\square$

Proposition 2.10. *If T is one-to-one and has closed range, then T is bounded below.*

Proof. Let $\{f_n\}$ be a sequence in Y such that $\|Tf_n\| \to 0$. Let X denote the closed range of T. With the norm inherited from Y, X is a Banach space. Since T is one-to-one we can define the inverse $T^{-1} : X \to Y$. Suppose $\{x_n\}$ converges to $x = Th$ in X, and $T^{-1}x_n$ converges to y in Y. Applying T to $\{T^{-1}x_n\}$, this means x_n converges to Ty. Hence $Ty = Th$. Since T is one-to-one, $y = h = T^{-1}x$. By the Closed Graph Theorem, T^{-1} is continuous. Thus, $\|f_n\| = \|T^{-1}(Tf_n)\| \to 0$, implying T is bounded below. $\square$

Theorem 2.11. *Let Y be a Banach space of analytic functions on the disk, and let T_g and S_g be bounded on Y. For nonconstant g, T_g is bounded below on Y if and only if it has closed range. S_g is bounded below on $Y/\mathbb{C}$ if and only if it has closed range on $Y/\mathbb{C}$.*

Theorem 2.11 follows from Proposition 2.9 and Proposition 2.10 with Lemma 2.8.

The next proposition compares compact operators to operators that are bounded below. It includes an elementary argument that the closed unit ball of an infinite dimensional Banach space is not compact.

Proposition 2.12. *Let X be an infinite dimensional Banach space and $T : X \to X$ a bounded linear operator. If T is bounded below, then it is not compact.*

Proof. The special case when X is a Hilbert space is easy. Find an orthonormal sequence $\{u_n\} \subset X$. Assume T is bounded below, so there exists $\delta > 0$ such that $\|Tx\| \geq \delta\|x\|$ for all $x \in X$. Then, for $m \neq n$,

$$\|Tu_n - Tu_m\| = \|T(u_n - u_m)\| \geq \delta\|u_n - u_m\| = \delta\sqrt{2}.$$

Thus we have a uniformly separated sequence of points in the image of the closed unit ball of X under T. The separated sequence can have no convergent subsequence, showing T is not compact.

For a general Banach space X, we construct an analogous sequence. We show there is an infinite, uniformly separated sequence in the closed unit ball B of X. If this fails to be true, then for any $\varepsilon > 0$ there exists a finite set that is an ε cover of B. So suppose $\{u_1, u_2, ..., u_N\} \subset B$, and for all $u \in B$, there exists j, $1 \leq j \leq N$, such that $\|u - u_j\| < 1/2$. Let M be the span of $\{u_1, u_2, ..., u_N\}$. We will show that $M = X$, contradicting the assumption that X has infinite dimension. Let $y \in X$. Since M is finite dimensional, it is closed. If $y \notin M$, then $d = \inf_{m\in M} \|y - m\| > 0$. Let $m_0 \in M$ such that $d \leq \|y - m_0\| \leq 3d/2$, and $y_0 = (y - m_0)/\|y - m_0\| \in B$. Then

$$\begin{aligned}\inf_{m\in M} \|y_0 - m\| &= \inf_{m\in M} \|\frac{y - m_0}{\|y - m_0\|} - m\| = \inf_{m\in M} \|\frac{y - m_0 - \|y - m_0\|m}{\|y - m_0\|}\| \\ &= \frac{1}{\|y - m_0\|} \inf_{m\in M} \|y - m\| \geq \frac{d}{3d/2} = 2/3.\end{aligned}$$

This violates the fact that the basis of M is a $1/2$ cover of B. We conclude that there exists an infinite sequence $\{u_n\} \subset B$ such that $m \neq n$ implies $\|u_n - u_m\| \geq 1/2$. As we saw in the case when X is a Hilbert space, if T is bounded below then it is not compact. $\square$

Chapter 3

Boundedness of T_g and S_g

3.1 Results of Aleman, Siskakis, Cima, and Zhao

Alexandru Aleman and Aristomenis Siskakis characterized boundedness and compactness of T_g on H^p for $1 \leq p < \infty$ in [3] (Theorem 3.1 below). The dual of H^1 is $BMOA$ ([21, Theorem 9.20]), and the dual of the Bergman space A^1 is the Bloch space ([21, Theorem 5.3]). Hence, in light of duality, Theorem 3.2 is an analogue for the Bergman spaces of Theorem 3.1. Theorem 3.2 was proved by Aleman and Siskakis in [4]. Theorem 3.3 was established by Siskakis and Ruhan Zhao in [17].

Theorem 3.1. (Aleman and Siskakis [3]) *For $1 \leq p < \infty$, T_g is bounded [compact] on H^p if and only if $g \in BMOA\,[VMOA]$.*

Theorem 3.2. (Aleman and Siskakis [4]) *For $p \geq 1$, T_g is bounded [compact] on A^p if and only if $g \in \mathcal{B}\,[\mathcal{B}_0]$.*

Theorem 3.3. (Siskakis and Zhao [17]) *T_g is bounded on $BMOA$ if and only if $g \in LMOA$.*

We use Theorems 3.1 and 3.2 in the following result.

Theorem 3.4. *On A^p and H^p $(1 \leq p < \infty)$, S_g is bounded if and only if $g \in H^\infty$.*

Proof. Recall that $H^\infty \subset BMOA$ and $H^\infty \subset \mathcal{B}$. If $g \in H^\infty$, $1 \leq p < \infty$, then M_g and T_g are bounded on H^p by Theorems 2.7 and 3.1. By the product rule (1.1), S_g is also bounded. The result holds similarly for A^p using Theorem 3.2. If S_g is bounded on A^p or H^p, then $g \in H^\infty$ by Corollary 2.3. □

3.2 The α-Bloch Spaces

The natural analogue of Theorem 3.3 is that T_g is bounded on $\mathcal{B}$ precisely when $g \in \mathcal{B}_\ell$. Theorem 3.5 extends this result to the α-Bloch spaces as well [13].

Theorem 3.5. *Let $\alpha, \beta > 0$.*

(a) *The operator S_g maps $\mathcal{B}_\alpha$ boundedly into $\mathcal{B}_\beta$ if and only if*

(i) $|g(z)| = O((1-|z|^2)^{\alpha-\beta})$ *as* $|z| \to 1^-$ $(\alpha \leq \beta)$.

(ii) $g = 0$ $(\alpha > \beta)$.

(b) *The operator T_g maps $\mathcal{B}_\alpha$ boundedly into $\mathcal{B}_\beta$ if and only if*

(i) $g \in \mathcal{B}_{\beta,\ell}$ $(\alpha = 1)$.

(ii) $g \in \mathcal{B}_{1-\alpha+\beta}$ $(\alpha > 1,\ 1-\alpha+\beta \geq 0)$.

(iii) g is constant $(\alpha > 1,\ 1-\alpha+\beta < 0)$.

(iv) $g \in \mathcal{B}_\beta$ $(\alpha < 1)$.

Proof. Recall from Proposition 2.5 that if $\alpha = 1$, we have $\|\lambda_z\|_{\mathcal{B}_\alpha} \sim \log \frac{1}{1-|z|}$ as $|z| \to 1$, where $\|\lambda_z\|_{\mathcal{B}_\alpha} = \sup_{\|f\| \leq 1} |f(z)|$ is the norm in $\mathcal{B}_\alpha$ of point evaluation at $z \in D$. If $\alpha > 1$, then $\|\lambda_z\|_{\mathcal{B}_\alpha} \sim (1-|z|^2)^{\alpha-1}$ as $|z| \to 1$.

Note that $\|\cdot\|_{\mathcal{B}_\alpha}$ are seminorms, which are adequate for showing boundedness of these operators. We consider conditions such that S_g maps $\mathcal{B}_\alpha$ into $\mathcal{B}_\beta$. Define a certain growth measurement of g by

$$A_t(g) = \sup_{z\in D}((1-|z|^2)^t|g(z)|), \qquad t \geq 0.$$

If $\beta \geq \alpha$ we have

$$\begin{aligned}
\|S_g f\|_{\mathcal{B}_\beta} &= \sup_{z\in D}(|f'(z)g(z)|(1-|z|^2)^\beta) \\
&\leq \sup_{z\in D}(|f'(z)|(1-|z|^2)^\alpha)\sup_{z\in D}((1-|z|^2)^{\beta-\alpha}|g(z)|) \\
&= \|f\|_{\mathcal{B}_\beta} A_{\beta-\alpha}(g).
\end{aligned}$$

Thus, S_g maps $\mathcal{B}_\alpha$ boundedly into $\mathcal{B}_\beta$ if $A_{\beta-\alpha}(g) < \infty$.

To show this condition is necessary, suppose S_g maps $\mathcal{B}_\alpha$ boundedly into $\mathcal{B}_\beta$ for $\beta \geq \alpha$. By Theorem 2.2,

$$|g(z)| \leq \|S_g\| \frac{\|\lambda_z'\|_{\mathcal{B}_\beta}}{\|\lambda_z'\|_{\mathcal{B}_\alpha}} \sim \|S_g\| \frac{(1-|z|^2)^{-\beta}}{(1-|z|^2)^{-\alpha}}.$$

Taking the supremum over $z \in D$, we get $A_{\beta-\alpha}(g) \lesssim \|S_g\|$. Hence S_g is bounded if and only if $A_{\beta-\alpha}(g) < \infty$. In particular, S_g is bounded on B_α if and only if $g \in H^\infty$, which is evident from Corollary 2.3. If S_g maps $\mathcal{B}_\alpha$ boundedly into $\mathcal{B}_\beta$, and $\beta < \alpha$, Theorem 2.2 implies that $|g(z)| \to 0$ as $|z| \to 1^-$. Since g is analytic, this means $g = 0$. This proves (a).

Consider conditions such that T_g maps $\mathcal{B}_\alpha$ into $\mathcal{B}_\beta$. By the Closed Graph Theorem, mapping $\mathcal{B}_\alpha$ into $\mathcal{B}_\beta$ is equivalent to T_g mapping $\mathcal{B}_\alpha$ boundedly into $\mathcal{B}_\beta$. In the case

$\alpha = 1$, and $\|f\|_{\mathcal{B}} \neq 0$, we have $|f(z)| \lesssim \|f\|_{\mathcal{B}} \log(2/(1-|z|))$. Thus, if $g \in \mathcal{B}_{\beta,\ell}$, then

$$\begin{aligned}\|T_g f\|_{\mathcal{B}_\beta} &= \sup_{z\in\mathcal{D}} |f(z)||g'(z)|(1-|z|^2)^\beta \\ &\lesssim \sup_{z\in\mathcal{D}} \left(\|f\|_{\mathcal{B}} \log \frac{2}{1-|z|} |g'(z)|(1-|z|^2)^\beta \right) \\ &\leq \|f\|_{\mathcal{B}} \|g\|_{\mathcal{B}_{\beta,\ell}},\end{aligned}$$

so T_g is bounded. Conversely, T_g being bounded implies, by Theorem 2.2,

$$|g'(z)| \leq \|T_g\| \frac{\|\lambda_z'\|_{\mathcal{B}_\beta}}{\|\lambda_z\|_{\mathcal{B}}} \sim \|T_g\| \frac{(1-|z|)^{-\beta}}{\log \frac{2}{1-|z|}}.$$

Hence, T_g is bounded if and only if $g \in \mathcal{B}_{\beta,\ell}$.

In the case $\alpha > 1$, assume T_g maps $\mathcal{B}_\alpha$ into $\mathcal{B}_\beta$. Then, by Theorem 2.2,

$$|g'(z)| \leq \|T_g\| \frac{\|\lambda_z'\|_{\mathcal{B}_\beta}}{\|\lambda_z\|_{\mathcal{B}_\alpha}} \sim (1-|z|^2)^{\alpha-\beta-1}.$$

If $1-\alpha+\beta > 0$, then $g \in \mathcal{B}_{1-\alpha+\beta}$. $1-\alpha+\beta = 0$ implies g is a function whose derivative is bounded. If $1-\alpha+\beta < 0$, then g is constant.

For $\alpha > 1$, $\|f\|_{\mathcal{B}} \neq 0$, we have $|f(z)| \lesssim \|f\|_{\mathcal{B}_\alpha}(1-|z|)^{1-\alpha}$. Thus,

$$\begin{aligned}\|T_g f\|_{\mathcal{B}_\beta} &\lesssim \sup_{z\in\mathcal{D}} \big((\|f\|_{\mathcal{B}_\alpha}(1-|z|^2)^{1-\alpha})|g'(z)|(1-|z|^2)^\beta \big) \\ &\leq \|g\|_{\mathcal{B}_\beta} + \|f\|_{\mathcal{B}_\alpha} \|g\|_{\mathcal{B}_{1-\alpha+\beta}}.\end{aligned}$$

In the case $\alpha < 1$, $\mathcal{B}_\alpha$ is a Lipschitz class space (see [9]), a subspace of H^∞. Evidently T_g is bounded from $\mathcal{B}_\alpha$ to $\mathcal{B}_\beta$ if and only if $g \in \mathcal{B}_\beta$. $\square$

3.3 Splitting the Multiplier Condition

We compare the three operators M_g, T_g, and S_g. Recall $M[X] = \{g \in H(D) : M_g \text{ is bounded on } X\}$. Define $T[X]$ and $S[X]$ similarly. The Dirichlet space provides a nice example of how the condition for boundedness of M_g may split into the conditions for S_g and T_g.

A function $f \in H(D)$ is in the Dirichlet space $\mathcal{D}$ provided

$$\|f\|_{\mathcal{D}} = \int_D |f'(z)|^2 \, dA(z) < \infty,$$

where A is Lebesgue area measure on D. A complex measure μ is a *Carleson measure for* $\mathcal{D}$ if, for all $f \in \mathcal{D}$, $\int_D |f|^2 \, d\mu \leq \|f\|_{\mathcal{D}}^2$. Carleson measures for $\mathcal{D}$ were characterized by Stegenga in [19]. Stegenga uses the result to characterize the pointwise multipliers of the Dirichlet space, which depend on two conditions. One is boundedness of the symbol of the multiplier. The other condition is a capacitary condition on the measure μ in Theorem 3.6 (ii), the same condition characterizing the symbols g for which T_g is bounded. This is not surprising since the two conditions come from the two terms in the product rule, which also gives us (1.1).

Theorem 3.6. (i) *S_g is bounded on the Dirichlet space $\mathcal{D}$ if and only if $g \in H^\infty$.*

(ii) *T_g is bounded on the Dirichlet space $\mathcal{D}$ if and only if μ is a Carleson measure for $\mathcal{D}$, where $d\mu(z) = |g'(z)|^2 dA(z)$.*

Proof. S_g is bounded on $\mathcal{D}$ if and only if there exists $C > 0$ such that

$$\| S_g f \|_{\mathcal{D}}^2 = \int_D |f'(z)|^2 |g(z)|^2 \, dA(z) \leq C \, \| f \|_{\mathcal{D}}^2 \, .$$

Clearly $g \in H^\infty$ implies S_g is bounded. For the converse, note that point evaluation of the derivative in $\mathcal{D}$ is analogous to point evaluation in A^2, and $\|\lambda_z'\|_{\mathcal{D}} \sim \|\lambda_z\|_{A^2}$. Thus, Corollary 2.3 applies, proving (i).

T_g is bounded on $\mathcal{D}$ if and only if there exists $C > 0$ such that

$$\| T_g f \|_{\mathcal{D}}^2 = \int_D |f(z)|^2 |g'(z)|^2 \, dA(z) \leq C \| f \|_{\mathcal{D}}^2 .$$

This is precisely the statement in (ii). $\square$

The interaction of M_g, S_g, and T_g imitates the situation in the Bloch space and in $BMOA$. On $\mathcal{D}, BMOA$, and $\mathcal{B}$, M_g is bounded only if S_g and T_g are both bounded, yet it is possible for S_g or T_g to be unbounded while M_g is bounded. The two conditions given for multipliers of $\mathcal{B}$ in Theorem 2.7 are the same as the conditions for S_g and T_g to be bounded, namely that $g \in H^\infty$ and that $g \in \mathcal{B}_\ell$. On $BMOA$, S_g is bounded if and only if $g \in H^\infty$, T_g is bounded if and only if $g \in LMOA$, and M_g is bounded when $g \in H^\infty \cap LMOA$. The following table summarizes these results. ($Y = T[\mathcal{D}]$. See Theorem 3.6.)

X	$M[X]$	$S[X]$	$T[X]$
$\mathcal{D}$	$H^\infty \cap Y$	H^∞	Y
$BMOA$	$H^\infty \cap LMOA$	H^∞	$LMOA$
$\mathcal{B}$	$H^\infty \cap \mathcal{B}_\ell$	H^∞	$\mathcal{B}_\ell$
H^p	H^∞	H^∞	$BMOA$
A^p	H^∞	H^∞	$\mathcal{B}$
H^∞	H^∞	?	?

In the Hardy and Bergman spaces for finite p, the condition characterizing multipliers is simply the condition for S_g to be bounded, but the condition for boundedness of T_g is weaker, i.e., $S[X] \subset T[X]$. Thus, in all these spaces we have the phenomenon that boundedness of the multiplication operator M_g is equivalent to boundedness of both S_g and T_g. As we will see in Section 3.4 below, this phenomenon fails for the operators acting on H^∞. The question marks in the table represent unsolved problems, but some discussion and partial results will be presented.

3.4 Boundedness of T_g and S_g on H^∞

It is trivial that $M[H^\infty] = H^\infty$, i.e., the multipliers of H^∞ are precisely the functions in H^∞ themselves. Such is not the case for T_g and S_g, and characterizing boundedness of these operators on H^∞ is an open problem. The following proposition gives a necessary condition for T_g and S_g to be bounded on H^∞.

Proposition 3.7. $T[H^\infty] = S[H^\infty] \subseteq H^\infty$.

Proof. From Theorem 2.2, we see that $S[H^\infty] \subseteq H^\infty$, i.e., if S_g is bounded on H^∞, then $g \in H^\infty$. Hence M_g is bounded as well. By the product rule, (1.1), this implies T_g is also bounded. Thus, S_g is bounded implies T_g is bounded, or $S[H^\infty] \subseteq T[H^\infty]$. Letting $1 \in H^\infty$ denote the constant function, we have $T_g 1 = g$. If T_g is bounded with norm $\|T_g\|$, then

$$\|g\|_\infty = \|T_g 1\|_\infty \leq \|T_g\|.$$

Thus $T[H^\infty] \subseteq H^\infty$. If T_g is bounded then $g \in H^\infty$ and M_g is bounded, so S_g is bounded by (1.1), i.e., $T[H^\infty] \subseteq S[H^\infty]$. Hence the result holds. $\square$

We show that the inclusion in Proposition 3.7 is proper, i.e., $g \in H^\infty$ is not sufficient for T_g to be bounded. The following counterexample demonstrates this.

For $w, z \in D$, let $\rho(z,w) = \frac{|w-z|}{|1-\overline{w}z|}$ denote the pseudohyperbolic metric on D, and for $0 < r < 1$ let $D(w,r) = \{z \in D : \rho(z,w) < r\}$. We can find a sequence $\{a_n\}$

such that, for the Blaschke product B with zeros $\{a_n\}$, T_B is unbounded on H^∞. Fix a small $\varepsilon > 0$. We will choose $\{a_n\}$ such that $0 < a_n < a_{n+1} < 1$ for all n, with corresponding factors $\sigma_n(x) = \frac{a_n - x}{1 - a_n x}$, so $B = \prod \sigma_n$ is real-valued on the unit interval. For each n define $B_n = B/\sigma_n$. Also, choose the a_n to be highly separated in pseudohyperbolic distance; that is,

$$|B_n(x)| > 1 - \varepsilon \text{ for } x \in I_n,$$

where $I_n = D(a_n, 1/2) \cap \mathbb{R}$.

Let x_n and y_n be the endpoints of I_n, so

$$\rho(a_n, x_n) = \sigma_n(x_n) = 1/2$$

and $\sigma_n(y_n) = -1/2$.

Then

$$\begin{aligned}
|B(x_n) - B(y_n)| &= |B_n(x_n)\sigma_n(x_n) - B_n(x_n)\sigma_n(y_n) \\
&\qquad + B_n(x_n)\sigma_n(y_n) - B_n(y_n)\sigma_n(y_n)| \\
&\geq |B_n(x_n)\sigma_n(x_n) - B_n(x_n)\sigma_n(y_n)| \\
&\qquad - |B_n(x_n)\sigma_n(y_n) - B_n(y_n)\sigma_n(y_n)| \\
&= |B_n(x_n)||\sigma_n(x_n) - \sigma_n(y_n)| - |\sigma_n(y_n)||B_n(x_n) - B_n(y_n)| \\
&\geq (1 - \varepsilon)(1) - (1/2)\varepsilon \sim 1.
\end{aligned}$$

Hence

$$\int_{I_n} |B'(x)|\, dx \sim 1 \tag{3.1}$$

for all n.

Let J_n be the interval between I_n and I_{n+1}, so $J_n = (y_n, x_{n+1})$. We will see that B' has a zero in each J_n. If $x_n < x < y_{n+1}$, then

$$\frac{|B(x)|}{|\sigma_n(x)\sigma_{n+1}(x)|} > 1 - \varepsilon.$$

We may assume the a_n are chosen so that $\rho(a_n, x)\rho(a_{n+1}, x) = 1 - \varepsilon$ for some $x \in J_n$. Thus, there exists $x \in J_n$ such that

$$|B(x)| > |\sigma_n(x)\sigma_{n+1}(x)|(1 - \varepsilon) > (1 - \varepsilon)^2.$$

Also, $(1-\varepsilon)(1/2) < |B(y_n)| < 1/2$, and $(1-\varepsilon)(1/2) < |B(x_{n+1})| < 1/2$, while the sign of B does not change on J_n. Since ε is small, the Mean Value Theorem implies there exists $d_n \in J_n$ such that $B'(d_n) = 0$. Moreover, d_n is separated from the endpoints of J_n, i.e., there exists $\xi > 0$ such that $\rho(d_n, y_n) > \xi$ and $\rho(d_n, x_{n+1}) > \xi$ for all n.

For each n, d_n is the only zero of B' on the real line between a_n and a_{n+1}. The number of zeros in the disk is $n - 1$ for the derivative of $\prod_{j=1}^{n} \sigma_j$ by the Riemann-Hurwitz formula, and Hurwitz's theorem tells us no other zeros arise in the limit function.

Letting $-f$ denote the Blaschke product with zero sequence $\{d_n\}$, we get $f(x)B'(x) \geq 0$ for $0 < x < 1$. Note that f is an interpolating Blaschke product, and there exists $\delta > 0$ such that $|f(x)| \geq \delta$ for all n, $x \in I_n$.

Thus, using (3.1) we have

$$\begin{aligned}\lim_{r\to 1} T_B f(r) &= \lim_{r\to 1}\int_0^r B'(x)f(x)\,dx\\ &= \sum_n \int_{I_n} |B'(x)||f(x)|\,dx + \sum_n \int_{J_n} |B'(x)||f(x)|\,dx\\ &\gtrsim \sum_n \delta = \infty.\end{aligned}$$

Hence T_B is not bounded on H^∞.

3.5 Future Work

In this section we state results toward characterizing boundedness of T_g on H^∞. The *radial variation* of $g \in H(D)$ at $\theta \in \partial D$ is

$$V(g,\theta) = \int_0^1 |g'(te^{i\theta})|\,dt.$$

The weakest sufficient condition we know for characterizing $T[H^\infty]$ is a uniform bound on the radial variation of the symbol g. The strongest necessary condition we have proven is that $g \in H^\infty$, although this is not sufficient. Finally, we give the weakest sufficient condition we have for compactness of T_g in Theorem 3.13.

Theorem 3.8. *The following condition on g implies T_g is bounded on H^∞:*

$$\text{There exists } M > 0 \text{ such that for all } \theta \in \partial D, V(g,\theta) < M. \tag{3.2}$$

Proof. If (3.2) holds, then

$$\begin{aligned}\|T_g f\|_\infty &= \sup_{z\in D} \left|\int_0^z f(w)g'(w)\,dw\right| \\ &= \sup_\theta \left|\int_0^1 f(te^{i\theta})g'(te^{i\theta})\,dt\right| \\ &\le \sup_\theta \int_0^1 |f(te^{i\theta})g'(te^{i\theta})|\,dt \le \|f\|_\infty M.\end{aligned}$$

□

The following proposition is the Fejér-Riesz inequality. For a proof, see [9, Theorem 3.13].

Proposition 3.9. (Fejér-Riesz) *If $f \in H^p \quad (1 \le p < \infty)$, then the integral of $|f|^p$ along the real interval $-1 < x < 1$ converges, and*

$$\int_{-1}^1 |f(x)|^p\,dx \le \frac{1}{2}\|f\|_p^p.$$

By Theorem 3.8 and the Fejér-Riesz inequality, $g' \in H^1$ implies T_g is bounded on H^∞, with norm no greater than $\|g'\|_{H^1}/2$. However, condition 3.2 does not imply $g' \in H^1$, as we will see in Theorem 3.12. We will use a pair of theorems from univalent function theory.

For $E \subset \mathbb{C}$, let $\Lambda(E)$ denote the linear measure of E.

$$\Lambda(E) = \lim_{\varepsilon\to 0} \inf_{d_k<\varepsilon} \sum_k d_k,$$

where the infimum ranges over countable covers of E by discs D_k of diameter d_k.

Theorem 3.10. [15, Theorem 10.11] *If $f(z)$ is analytic and univalent in D then*

$$f' \in H^1 \Leftrightarrow \Lambda(\partial f(D)) < \infty.$$

The next theorem is attributed to F. W. Gehring and W. K. Hayman.

Theorem 3.11. [15, Theorem 10.9] *Let $f(z)$ be analytic and univalent in D. If $C \subset D$ is a Jordan arc from 0 to $e^{i\theta}$ then*

$$V(f,\theta) \leq K\, l(f(C)),$$

where K is an absolute constant, and $l(f(C))$ is the arc length of $f(C)$.

Theorem 3.12. *There exists a univalent function $f \in H^\infty$ such that* (3.2) *holds but $f' \notin H^1$.*

Proof. Construct a starlike region $G \subset D$ with a boundary of infinite length by removing countably many slits from D. The Riemann map f from D to G is univalent and fails to satisfy $f' \in H^1$ by Theorem 3.10. However, the image under f of any radius from 0 to $e^{i\theta}$ is bounded in length by K from Theorem 3.11. This shows that the condition $g' \in H^1$ is strictly stronger than condition (3.2). $\square$

The next result pertains to compactness of T_g.

Theorem 3.13. *The following condition implies compactness of T_g on H^∞.*

$$\text{For all } \varepsilon > 0, \text{ there exists } r < 1 \text{ such that } \int_r^1 |g'(te^{i\theta})|\, dt < \varepsilon \text{ for all } \theta \in \partial D. \tag{3.3}$$

Proof. Let $\overline{B} = \{f \in H^\infty : \|f\|_\infty \leq 1\}$ denote the closed unit ball of H^∞. Let $\{f_n\} \subset \overline{B}$ be a sequence of functions in $\overline{B}$. Then for all n and $z \in D$, $\|f_n(z)\|_\infty \leq 1$, i.e., $\{f_n\}$ is uniformly bounded on D by 1. By Montel's Theorem $\{f_n\}$ is a normal family, and there exists $f \in H^\infty$ such that $f_{n_k} \to f$ locally uniformly in D for some subsequence $\{f_{n_k}\}$ of $\{f_n\}$. The operator T_g is compact if and only if $T_g f_n$ has a convergent subsequence in H^∞.

Replacing f_n in the above argument by $f_n - f$, we see the following criterion: T_g is compact on H^∞ if and only if $f_n \to 0$ locally uniformly in D implies $T_g f_n \to 0$ in H^∞.

Let $\varepsilon > 0$, and assume (3.3) holds. Choose N such that $n > N$ implies $|f_n(z)| < \varepsilon$ for $|z| < r$, where r satisfies (3.3). Note that (3.2) holds, since there exists M such that $|g'(z)| < M$ on $\{z : |z| \leq r\}$.

Then, for $z \in D$ and $n > N$, we have

$$\begin{aligned} \|T_g f_n\|_\infty &\leq \sup_\theta \int_0^1 |f_n(te^{i\theta})g'(te^{i\theta})|\, dt \\ &= \sup_\theta \int_0^r |f_n(te^{i\theta})g'(te^{i\theta})|\, dt \ + \int_r^1 |f_n(te^{i\theta})g'(te^{i\theta})|\, dt \\ &\leq \varepsilon M + \varepsilon. \end{aligned}$$

Hence $\|T_g f_n\|_\infty \to 0$ as $n \to \infty$. $\square$

Chapter 4

Results on Closed Range Operators

4.1 When T_g Is Bounded Below

We will show that T_g is never bounded below on H^2, $\mathcal{B}$, or $BMOA$. The sequence $\{z^n\}$ demonstrates the result in each space, since the functions z^n have norm comparable to 1, independent of n (Lemma 2.4).

Theorem 4.1. *T_g is never bounded below on H^2, $\mathcal{B}$, or $BMOA$.*

Proof. Let $f_n(z) = z^n$. For H^2, the Littlewood-Paley identity gives us

$$\lim_{n\to\infty} \|T_g f_n\|_2^2 \sim \lim_{n\to\infty} \int_D |z^n|^2 |g'(z)|^2 (1 - |z|^2)\, dA(z).$$

We assume T_g is bounded, so $g \in BMOA$ by the result of Aleman and Siskakis [4]. Thus μ_g is a Carleson measure, allowing us to bring the limit inside the integral by the Dominated Convergence Theorem in the following:

$$\lim_{n\to\infty} \|T_g f_n\|_2^2 = \int_D \lim_{n\to\infty} |z^n|^2 |g'(z)|^2 (1 - |z|^2)\, dA(z) = 0.$$

Since $\|f_n\|_2 = 1$ for all n, we have shown T_g is not bounded below.

If T_g is bounded on $\mathcal{B}$, then, by Theorem 2.2, $|g'(z)|(1-|z|) = O(1/\log(1/(1-|z|)))$ as $|z| \to 1$. Thus,

$$\|T_g f_n\|_{\mathcal{B}} = \sup_{z \in D} |z^n||g'(z)|(1-|z|) \lesssim \sup_{0 \le r < 1} r^n \frac{1}{\log(2/(1-r))}.$$

Given $\varepsilon > 0$, there exists $\delta < 1$ such that $1/\log(2/(1-r)) < \varepsilon$ for $\delta < r < 1$. For large n and $0 < r < \delta$, we have $r^n < \varepsilon$. Thus, $\lim_{n\to\infty} \|T_g f_n\|_{\mathcal{B}} = 0$, and Lemma 2.4 implies T_g is not bounded below on $\mathcal{B}$.

Siskakis and Zhao [17] proved that if T_g is bounded on $BMOA$ then $g \in LMOA$. Using the relationship between $BMOA$ and Carleson measures for H^2, we have

$$\lim_{n\to\infty} \|T_g f_n\|_*^2 \sim \lim_{n\to\infty} \sup_I \frac{1}{|I|} \int_{S(I)} |z^n|^2 |g'(z)|^2 (1-|z|^2)\, dA(z).$$

Let I be an arc in ∂D, and let $\varepsilon > 0$. Since $g \in VMOA$, there exists $\delta > 0$ such that

$$\frac{1}{|J|} \int_{S(J)} |g'(z)|^2 (1-|z|^2)\, dA(z) < \varepsilon \text{ whenever } |J| < \delta.$$

If $|I| > \delta$, divide I into K disjoint intervals of length approximately δ, so

$$I = \cup_{i=1}^K J_i, \quad \delta/2 < |J_i| < \delta \text{ for all } i, \text{ and } \delta K \sim |I|.$$

Let $S_\delta(I) = S(I) - \cup_i S(J_i)$. For large n we have $(1-\delta/2)^{2n} \le \varepsilon|I|$, and to estimate the integral over $S_\delta(I)$ we use the fact that μ_g is a Carleson measure. Thus,

$$\begin{aligned}
\frac{1}{|I|}\int_{S(I)} |z^n|^2 |g'(z)|^2 (1-|z|^2)\, dA(z) &= \frac{1}{|I|}\int_{S_\delta(I)} |z^n|^2 |g'(z)|^2 (1-|z|^2)\, dA(z) \\
&+ \frac{1}{|I|}\sum_{i=1}^{K}\int_{S(J_i)} |z^n|^2 |g'(z)|^2 (1-|z|^2)\, dA(z) \\
&\le \frac{1}{|I|}(1-\delta/2)^{2n}\, C\|g\|_*^2 + \frac{1}{|I|} K\delta\varepsilon \lesssim \varepsilon.
\end{aligned}$$

Hence $\lim_{n\to\infty} \|T_g f_n\|_* = 0$ and T_g is not bounded below on $BMOA$. $\square$

In contrast to Theorem 4.1, T_g can be bounded below on weighted Bergman spaces. We state the result here, but the key is Proposition 4.3, proved afterward. For a measurable set $G \subset D$, $|G|$ denotes the Lebesque area measure of G.

Theorem 4.2. *Let $1 \le p < \infty$, $\alpha > -1$. T_g is bounded below on A^p_α if and only if there exist $c > 0$ and $\delta > 0$ such that, for all $I \subset \partial D$,*

$$|\{z \in D : |g'(z)|(1-|z|^2) > c\} \cap S(I)| > \delta |I|^2.$$

Proof. We must assume T_g is bounded on A^p_α. By Theorem 3.2, $g \in \mathcal{B}$. T_g is bounded below on A^p_α if and only if

$$\|T_g f\|^p_{A^p_\alpha} \sim \int_D |f(z)|^p |g'(z)|^p (1-|z|^2)^{\alpha+p}\, dA(z) \gtrsim \|f\|^p_{A^p_\alpha}. \tag{4.1}$$

By Proposition 4.3, (4.1) holds if and only if there exist $c > 0$ and $\delta > 0$ such that

$$|\{z \in D : |g'(z)|^p (1-|z|^2)^p > c\} \cap S(I)| > \delta |I|^2 \tag{4.2}$$

for all arcs $I \subseteq \partial D$. If (4.2) holds for some $p \ge 1$, it holds for $1 \le p < \infty$, with an adjustment of the constant c. $\square$

The proof of [16, Proposition 5.4] shows this result is nonvacuous. Ramey and Ullrich construct a Bloch function g such that $|g'(z)|(1-|z|) > c_0$ if $1 - q^{-(k+1/2)} \leq |z| \leq 1 - q^{-(k+1)}$, for some $c_0 > 0$, q some large positive integer, and $k = 1, 2, ...$ Given a Carleson square $S(I)$, let k_I be the least positive integer such that $q^{-k_I+1/2} \leq |I|$. The annulus $E = \{z : 1 - q^{-(k_I+1/2)} \leq |z| \leq 1 - q^{-(k_I+1)}\}$ intersects $S(I)$, and

$$|E \cap S(I)| \sim |I|((1 - q^{-(k_I+1)}) - (1 - q^{-(k_I+1/2)})) = |I|\frac{q^{1/2}-1}{q^{k_I+1}} \geq \frac{(q^{1/2}-1)}{q^{3/2}}|I|^2.$$

Setting $c = c_0$ and $\delta \sim 1/q$ show Theorem 4.2 holds for this example of g, and T_g is bounded below on A^p_α.

4.2 When S_g Is Bounded Below

The operator S_g can clearly be bounded below, since $g(z) = 1$ gives the identity operator. A result due to Daniel Luecking (see [7, Theorem 3.34]) leads to a characterization of functions for which S_g is bounded below on $H^2_0 := H^2/\mathbb{C}$ and $A^p_\alpha/\mathbb{C}$. We state a reformulation useful to our purposes here.

Proposition 4.3. (Luecking) *Let τ be a bounded, nonnegative, measurable function on D. Let $G_c = \{z \in D : \tau(z) > c\}$, $1 \leq p < \infty$, and $\alpha > -1$. There exists $C > 0$ such that the inequality*

$$\int_D |f(z)|^p \tau(z)(1-|z|)^\alpha \, dA(z) \geq C \int_D |f(z)|^p (1-|z|)^\alpha \, dA(z)$$

holds if and only if there exist $\delta > 0$ and $c > 0$ such that $|G_c \cap S(I)| \geq \delta|I|^2$ for every interval $I \subset \partial D$.

The proof is omitted. Using the Littlewood-Paley identity we get the following:

Corollary 4.4. *S_g is bounded below on H_0^2 if and only if there exist $c > 0$ and $\delta > 0$ such that $|G_c \cap S(I)| \geq \delta |I|^2$ for all $I \subset \partial D$, where $G_c = \{z \in D : |g(z)| > c\}$.*

We use Corollary 4.4 to exhibit an example when S_g is not bounded below on H_0^2. If $g(z)$ is the singular inner function $\exp(\frac{z+1}{z-1})$, S_g is not bounded below on H_0^2. To see this, fix $c \in (0, 1)$. G_c is the complement in D of a horodisk, a disk tangent to the unit circle, with radius $r = \frac{\log c + 1}{2(\log c - 1)}$ and center $1 - r$. Choosing a sequence of intervals $I_n \subset \partial D$ such that 1 is the center of I_n and $|I_n| \to 0$ as $n \to \infty$, we see

$$\frac{|G_c \cap S(I_n)|}{|I_n|^2} \to 0 \text{ as } n \to \infty,$$

meaning S_g is not bounded below on H_0^2.

We compare M_g on H^2 to S_g on H_0^2. M_g is bounded below on H^2 if and only if the radial limit function of $g \in H^\infty$ is essentially bounded away from 0 on ∂D (a special case of weighted composition operators; see [12]). Theorem 4.6 will show this is weaker than the condition for S_g to be bounded below on H_0^2. The example above of a singular inner function then shows it is strictly weaker. To prove Theorem 4.6 we use a lemma which allows us to estimate an analytic function inside the disk by its values on the boundary.

For any arc $I \subseteq \partial D$ and $0 < r < 2\pi/|I|$, rI will denote the arc with the same center as I and length $r|I|$. We define the upper Carleson rectangle by

$$S_\varepsilon(I) = \{re^{it} : 1 - |I| < r < (1 - \varepsilon|I|), e^{it} \in I\}, \text{ and } S^+(I) = S_{1/2}(I).$$

Lemma 4.5. *Given ε, $0 < \varepsilon < 1$, and a point $e^{i\theta}$ such that $|g^*(e^{i\theta})| < \varepsilon$, there exists* *an arc $I \subset \partial D$ such that $|g(z)| < \varepsilon$ for $z \in S_\varepsilon(I)$.*

Proof. We can choose α close enough to 1 so that $S_\varepsilon(I) \subset \Gamma_\alpha(e^{i\theta})$ for all I centered at $e^{i\theta}$ with, say, $|I| < 1/4$. If $|g^*(e^{i\theta})| < \varepsilon$, there exists $\delta > 0$ such that

$$z \in \Gamma_\alpha(e^{i\theta}) \text{ and } |z - e^{i\theta}| < \delta \text{ implies } |g(z)| < \varepsilon.$$

Choosing I such that $S(I)$ is contained in a δ-neighborhood of $e^{i\theta}$ finishes the proof.
□

Theorem 4.6. *If S_g is bounded below on H_0^2, then M_g is bounded below on H^2.*

Proof. Assume M_g is not bounded below on H^2. Let $\varepsilon > 0$. The radial limit function of g equals g^* almost everywhere, so there exists a point $e^{i\theta}$ such that $|g^*(e^{i\theta})| < \varepsilon$. By Lemma 4.5, there exists $S(I)$ such that $|\{z : |g(z)| \geq \varepsilon\} \cap S(I)| \leq \varepsilon|I|$. Since ε was arbitrary, this violates the condition in Corollary 4.4. □

4.2.1 S_g on the Bloch Space

We now characterize the symbols g which make S_g bounded below on the Bloch space. It turns out to be a common condition appearing in a few different forms in the literature. The condition appears in characterizing M_g on A_0^2 in McDonald and Sundberg [14]. Our main result is equivalence of (i)-(iii) in Theorem 4.7, and we give references with brief explanations for (iv)-(vi).

An infinite *Blaschke product* is a function on D of the form

$$B(z) = z^m \prod_n \frac{-\overline{a}_n}{|a_n|} \frac{z - a_n}{1 - \overline{a}_n z},$$

where m is a nonnegative integer, $0 \neq a_n \in D$ for $n = 1, 2, 3, \ldots$, and

$$\sum_n (1 - |a_n|) < \infty.$$

A sequence $\{a_n\}$ is an *interpolating sequence* if, given $\{w_n\} \in \ell^\infty$, there exists $f \in H^\infty$ such that

$$f(a_n) = w_n \quad \text{for all } n.$$

An interpolating sequence $\{a_n\}$ must satisfy

$$\rho(a_j, a_k) > \delta \text{ for all } j \neq k \text{ and some } \delta > 0, \tag{4.3}$$

where $\rho(a_j, a_k) = |a_j - a_k|/|1 - \overline{a}_k a_j|$ denotes the pseudohyperbolic distance between a_k and a_j. A well-known result of Carleson is that the sequence $\{a_n\}$ is interpolating if and only if (4.3) holds and the measure $\sum(1 - |a_n|)\delta_{a_n}$ is a Carleson measure for H^1 (see [11, Theorem VII.1.1]).

Letting $\{a_n\}$ be the sequence of zeros of B ($a_n = 0$ is possible), the Blaschke product B is *interpolating* if $\{a_n\}$ is an interpolating sequence.

For the pseudohyperbolic disk of radius $d > 0$ and center $w \in D$, we use the notation

$$D(w, d) = \{z \in D : \rho(z, w) < d\}.$$

Theorem 4.7. *The following are equivalent for* $g \in H^\infty$*:*

(i) $g = BF$ *for a finite product* B *of interpolating Blaschke products and* F *such that* F *and* $1/F \in H^\infty$.

(ii) S_g *is bounded below on* $\mathcal{B}/\mathbb{C}$.

(iii) *There exist $r < 1$ and $\eta > 0$ such that for all $a \in D$,*

$$\sup_{z \in D(a,r)} |g(z)| > \eta.$$

(iv) *S_g is bounded below on H_0^2.*

(v) *M_g is bounded below on A_α^p for $\alpha > -1$.*

(vi) *S_g is bounded below on $A_\alpha^p/\mathbb{C}$ for $\alpha > -1$.*

Proof. (i) $\Rightarrow$ (ii): Note that for any $g_1, g_2 \in H(D)$,

$$S_{g_1 g_2} = S_{g_1} S_{g_2}. \tag{4.4}$$

If S_g is bounded on $\mathcal{B}$, then $g \in H^\infty$ by Corollary 2.3. If F and $1/F \in H^\infty$, then

$$\|S_F f\| = \sup_{z \in D} |F(z)||f'(z)|(1 - |z|^2) \geq (1/\|1/F\|_\infty)\|f\|_{\mathcal{B}}.$$

Hence S_F is bounded below.

The product of two interpolating Blaschke products is not necessarily an interpolating Blaschke product. For example, B^2 fails to satisfy (4.3) for any interpolating Blaschke product B. However, by virtue of (4.4), we may assume B is a single interpolating Blaschke product without loss of generality. Let $\{w_n\}$ be the zero sequence of B, so

$$B(z) = \prod_n \frac{\overline{w}_n}{|w_n|} \frac{w_n - z}{1 - \overline{w}_n z}.$$

Let B_j be B without its jth zero, i.e., $B_j(z) = \frac{1 - \overline{w}_j z}{w_j - z} B(z)$. Since B is interpolating, there exist $\delta > 0$ and $r > 0$ such that, for all j, $|B_j(z)| > \delta$ whenever $z \in D(w_j, r)$. In particular, the sequence $\{w_n\}$ is separated, so shrinking r if necessary, we may assume

$$\inf_{j \neq k} \rho(w_k, w_j) > 2r.$$

We compare $\|f\|$ to $\|S_B f\| = \sup_{z\in D} |B(z)||f'(z)|(1-|z|^2)$. Let $a \in D$ be a point where the supremum defining the norm of f is almost achieved, say, $|f'(a)|(1-|a|^2) > \|f\|/2$.

Consider the pseudohyperbolic disk $D(a,r)$. Inside $D(a,r)$ there may be at most one zero of B, or there may not be a zero. If the zero exists call it w_k. We examine three cases depending on the location and existence of w_k.

If $r/2 \leq \rho(w_k, a) < r$, then

$$|B(a)| = \frac{|w_k - a|}{|1 - \overline{w}_k a|} |B_k(a)| > (r/2)\delta.$$

Thus, we would have

$$\|S_B f\| \geq |B(a)||f'(a)|(1-|a|^2) > (r/2)\delta\|f\|/2,$$

and S_g would be bounded below.

On the other hand, suppose $\rho(w_k, a) < r/2$. Consider the disk $D(w_k, r/2)$, which is contained in $D(a,r)$. The expression $1-|z|^2$ is roughly constant on a pseudohyperbolic disk, i.e.,

$$\sup_{z\in D(a,r)} (1-|z|^2) > C_r(1-|a|^2) \text{ for some } C_r > 0.$$

C_r does not depend on a, and is near 1 for small r. By the maximum principle for f', there exists a point $z_a \in \partial D(w_k, r/2)$ where

$$|f'(z_a)|(1-|z_a|^2) > |f'(a)|C_r(1-|a|^2) > C_r\|f\|/2.$$

(Since $\rho(w_k, a) < r/2$ and $\rho(z_a, w_k) = r/2$, we have $\rho(z_a, a) < r$.) This shows that S_g is bounded below, for

$$\begin{aligned}
\|S_B f\| &\geq |B(z_a)||f'(z_a)|(1-|z_a|^2) \\
&> \rho(w_k, z_a)|B_k(z_a)|C_r\|f\|/2 \\
&> (r/2)\delta C_r\|f\|/2.
\end{aligned}$$

Finally, suppose no such w_k exists. Then the function $((a-z)/(1-\overline{a}z))B(z)$ is also an interpolating Blaschke product, and the previous case applies with $w_k = a$.

(ii) $\Rightarrow$ (iii): Assume (iii) fails. Given $\varepsilon > 0$, choose r near 1 so that $1 - r^2 < \varepsilon$, and choose $a \in D$ such that $|g(z)| < \varepsilon$ for all $z \in D(a, r)$. Consider the test function $f_a(z) = (a - z)/(1 - \overline{a}z)$. By a well-known identity (see [11, p.3]),

$$(1-|z|^2)|f_a'(z)| = 1 - (\rho(a, z))^2.$$

Thus $f_a \in \mathcal{B}$ with $\|f_a\| = 1$ for all $a \in D$. (The seminorm is 1, but the true norm is between 1 and 2 for all a.) Since (iii) fails, we have

$$\begin{aligned}
\|S_g f_a\| &= \sup_{z \in D} |g(z)||f_a'(z)|(1-|z|^2) \\
&= \max\left\{ \sup_{z \in D(a,r)} |g(z)||f_a'(z)|(1-|z|^2), \sup_{z \in D \setminus D(a,r)} |g(z)||f_a'(z)|(1-|z|^2) \right\} \\
&\leq \max\left\{ \sup_{z \in D(a,r)} |g(z)|\|f_a\|, \sup_{z \in D \setminus D(a,r)} |g(z)|(1-r^2) \right\} \\
&< \max\{\varepsilon, \|g\|_\infty \varepsilon\} \leq \varepsilon(\|g\|_\infty + 1)
\end{aligned}$$

Since $\|f_a\| = 1$ and ε was arbitrary, S_g is not bounded below.

(iii) $\Rightarrow$ (i): Assuming (iii) holds, we first rule out the possibility that g has a singular inner factor. We factor $g = BI_gO_g$ where B is a Blaschke product, I_g a singular inner function, and O_g an outer function. Let ν be the measure on ∂D determining I_g, so

$$I_g(z) = \exp\left(-\int \frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\nu(\theta)\right).$$

Let $\varepsilon > 0$. For any $\alpha > 1$ and for ν-almost all θ, there exists $\delta > 0$ such that

$$z \in \Gamma_\alpha(e^{i\theta}) \text{ and } |z - e^{i\theta}| < \delta \text{ implies } |I_g(z)| < \varepsilon \tag{4.5}$$

(see [11, Theorem II.6.2]). The constant δ may depend on θ and α, but for nontrivial ν there exists some θ for which (4.5) holds. Given $r < 1$, choose $\alpha < 1$ such that, for every a near $e^{i\theta}$ on the ray from 0 to $e^{i\theta}$, the pseudohyperbolic disk $D(a, r)$ is contained in $\Gamma_\alpha(e^{i\theta})$. The disk $D(a, r)$ is a euclidean disk whose euclidean radius is comparable to $1 - a$. For a close enough to $e^{i\theta}$,

$$z \in D(a, r) \text{ implies } |z - e^{i\theta}| < \delta.$$

Hence $\sup_{z\in D(a,r)} |g(z)| < \varepsilon\|g\|$. This violates (iii), so ν must be trivial, and $I_g \equiv 1$.

A similar argument handles the outer function O_g. If for all $\varepsilon > 0$ there exists e^{it} such that $|O_g^*(e^{it})| < \varepsilon$, we apply Lemma 4.5. The upper Carleson square in Lemma 4.5 contains some pseudohyperbolic disk that violates (iii), so O_g^* is essentially bounded away from 0. There exists $\eta > 0$, such that $|O_g^*(e^{it})| \geq \eta$ almost everywhere. Note $1/O_g \in H^\infty$, since for all $z \in D$,

$$\log|O_g(z)| = \frac{1}{2\pi}\int_0^{2\pi} \log|O_g^*(e^{it})| \frac{1-|z|^2}{|e^{it}-z|^2}\,dt \geq \log\eta.$$

We have reduced the symbol to a function $g = BF$, where $F, 1/F \in H^\infty$ and B is a Blaschke product, say with zero sequence $\{w_n\}$. We will show that the measure $\mu_B = \sum(1-|w_n|^2)\delta_{w_n}$ is a Carleson measure, implying B is a finite product of interpolating Blaschke products (see, e.g., [14, Lemma 21]). Let $r < 1$ and $\eta > 0$ be as in (iii), so $\sup_{z\in D(a,r)} |B(z)| > \eta$ for all a. Given any arc $I \subseteq \partial D$, we may choose a_I and z_I such that $z_I \in D(a_I, r) \subseteq S(I)$ and $|B(z_I)| > \eta$. We may also ensure $(1-|z_I|) \sim |I|$ as I varies. Note that $\mu_B(S(I)) = \sum(1-|w_{n_k}|^2)$, for the subsequence $\{w_{n_k}\} = \{w_n\} \cap S(I)$. Assume without loss of generality that $|I| < 1/2$, so $|w_{n_k}| > 1/2$ for all k. This ensures $|1 - \overline{w}_{n_k} z_I| \sim |I|$. Thus we have

$$
\begin{aligned}
\frac{1}{|I|}\sum_k (1-|w_{n_k}|^2) &\sim \sum_k \frac{(1-|z_I|^2)(1-|w_{n_k}|^2)}{|1-\overline{w}_{n_k} z_I|^2} \\
&= \sum_k 1 - (\rho(z_I, w_{n_k}))^2 \\
&< 2\sum_n 1 - \rho(z_I, w_n) \\
&\leq -\sum_n \log \rho(z_I, w_n) \\
&= -\log \prod_n \frac{|w_n - z_I|}{|1-\overline{w}_n z_I|} \\
&= -\log |B(z)| \leq -\log \eta.
\end{aligned}
$$

This shows μ_B is a Carleson measure, hence (iii) $\Rightarrow$ (i).

Bourdon shows in [6, Theorem 2.3, Corollary 2.5] that (i) is equivalent to the reverse Carleson condition in Corollary 4.4 above, hence (i) $\Leftrightarrow$ (iv). This reverse Carleson condition also characterizes boundedness below of M_g on weighted Bergman spaces by Proposition 4.3. Thus (iv) $\Leftrightarrow$ (v). The equivalence of (v) and (vi) is evident from the differentiation isomorphism (2.4). $\square$

4.2.2 Concluding Remarks

We suspect the results about H^2 can be extended to all H^p, $1 \leq p < \infty$, but without the Littlewood-Paley identity the proof will be more difficult. Generalizing the results on the Bloch space to the α-Bloch spaces can be done with adjusted test functions as in [20]. Finally, we have partial results concerning S_g being bounded below on $BMOA$, but have not completed proving a characterization like the one in Theorem 4.7.

Bibliography

[1] Aleman, Alexandru, *Some open problems on a class of integral operators on spaces of analytic functions.* Topics in complex analysis and operator theory, 139140, Univ. Mlaga, Mlaga, 2007.

[2] Aleman, Alexandru; Cima, Joseph A., *An integral operator on H^p and Hardy's inequality.* J. Anal. Math. 85 (2001), 157–176.

[3] Aleman, Alexandru; Siskakis, Aristomenis G. *An integral operator on H^p*. Complex Variables Theory Appl. 28 (1995), no. 2, 149–158.

[4] Aleman, Alexandru; Siskakis, Aristomenis G., *Integration operators on Bergman spaces.* Indiana Univ. Math. J. 46 (1997), no. 2, 337–356.

[5] Arazy, Jonathan, *Multipliers of Bloch Functions.* University of Haifa Publication Series 54 (1982).

[6] Bourdon, Paul S., *Similarity of parts to the whole for certain multiplication operators.* Proc. Amer. Math. Soc. 99 (1987), no. 3, 563–567.

[7] Cowen, Carl; MacCluer, Barbara, *Composition Operators on Spaces of Analytic Functions.* CRC Press, New York, 1995.

[8] Dostanić, Milutin R., *Integration operators on Bergman spaces with exponential weight.* Rev. Mat. Iberoam. 23 (2007), no. 2, 421–436.

[9] Duren, P. L., *Theory of H^p spaces.* Dover, New York, 2000.

[10] Duren, P. L.; Romberg, B. W.; Shields, A. L., *Linear functionals on H^p spaces with* $0 < p < 1$. J. Reine Angew. Math. 238 (1969), 32-60.

[11] Garnett, John B., *Bounded Analytic Functions. Revised First Edition.* Springer, New York, 2007.

[12] Kumar, Romesh; Partington, Jonathan R., *Weighted composition operators on Hardy and Bergman spaces.* Recent advances in operator theory, operator algebras, and their applications, 157–167, Oper. Theory Adv. Appl., 153, Birkhuser, Basel, 2005.

[13] Lv, Xiao-fen, *Extended Cesáro operator between Bloch-type spaces.* Chinese Quart. J. Math. 24 (2009), no. 1, 1019.

[14] McDonald, G.; Sundberg, C., *Toeplitz operators on the disc.* Indiana Univ. Math. J. 28 (1979), no. 4, 595–611.

[15] Pommerenke, Christian, *Univalent functions.* Studia Mathematica/Mathematische Lehrbücher, Band XXV. Vandenhoeck & Ruprecht, Göttingen, 1975.

[16] Ramey, Wade; Ullrich, David, *Bounded mean oscillation of Bloch pull-backs.* Math. Ann. 291 (1991), no. 4, 591–606.

[17] Siskakis, Aristomenis G.; Zhao, Ruhan, *A Volterra type operator on spaces of analytic functions.* Function spaces (Edwardsville, IL, 1998), 299–311, Contemp. Math., 232, Amer. Math. Soc., Providence, RI, 1999.

[18] Stegenga, David A., *Bounded Toeplitz operators on* H^1 *and applications of the duality between* H^1 *and the functions of bounded mean oscillation.* Amer. J. Math. 98 (1976), no. 3, 573–589.

[19] Stegenga, David A., *Multipliers of the Dirichlet space.* Illinois J. Math. 24 (1980), no. 1, 113–139.

[20] Zhang, M.; Chen, H., *Weighted composition operators of* H^∞ *into* α*-Bloch spaces on the unit ball.* Acta Math. Sin. (Engl. Ser.) 25 (2009), no. 2, 265–278.

[21] Zhu, Kehe, *Operator theory in function spaces. Second edition.* Mathematical Surveys and Monographs, 138. American Mathematical Society, Providence, RI, 2007.