SOME CLOSED RANGE INTEGRAL OPERATORS ON SPACES OF ANALYTIC FUNCTIONS

Austin Anderson
Department of Mathematics
University of Hawaii
Honolulu, Hawaii 96822
austina@hawaii.edu

Abstract: Our main result is a characterization of g for which the operator $S_g(f)(z) = \int_0^z f'(w)g(w)\,dw$ is bounded below on the Bloch space. We point out analogous results for the Hardy space H^2 and the Bergman spaces A^p for $1 \leq p < \infty$. We also show the companion operator $T_g(f)(z) = \int_0^z f(w)g'(w)\,dw$ is never bounded below on H^2, Bloch, nor $BMOA$, but may be bounded below on A^p.

Keywords: Volterra operator, Cesaro operator, integral operator, bounded below, closed range, Bloch, Hardy, Bergman, BMOA, multiplication operator

1. Introduction

We examine operators on Banach spaces of analytic functions on the unit disk in the complex plane. The operator T_g, with symbol $g(z)$ an analytic function on the disk, is defined by

$$T_g f(z) = \int_0^z f(w)g'(w)\,dw.$$

T_g is a generalization of the standard integral operator, which is T_g when $g(z) = z$. Letting $g(z) = \log(1/(1-z))$ gives the Cesáro operator. Discussion of the operator T_g first arose in connection with semigroups of composition operators. (see [11] for background) Characterizing the boundedness and compactness of T_g on certain spaces of analytic functions is of recent interest, as seen in [1], [2], [5] and [11], and open problems remain. T_g and its companion operator $S_g f(z) = \int_0^z f'(w)g(w)\,dw$ are related to the multiplication operator $M_g f(z) = g(z)f(z)$, since integration by parts gives

$$M_g f = f(0)g(0) + T_g f + S_g f.$$

If any two of M_g, S_g, and T_g are bounded, then so is the third. But in some situations one operator is bounded while two are unbounded. Boundedness of T_g on the Hardy and Bergman spaces and $BMOA$ is characterized in [1], [2] and [11]. The pointwise multipliers of these and many other spaces are well known. See [12] for $BMOA$.

In this paper we examine the property of being bounded below for T_g and S_g on spaces of analytic functions. We examine aspects of the problems on Hardy and Bergman spaces, the Bloch space, and $BMOA$. In doing so we must assume the

Date: March 9, 2010.

The author was supported by NSF-DGE-0841223.

operators are bounded, and we study characterizations of the symbols for which the operators are bounded. Consideration of M_g is useful as well.

2. Preliminaries

The notation $f \lesssim g$ will mean there exists a universal constant C such that $f \leq Cg$. $f \approx g$ will mean $f \lesssim g \lesssim f$.

Let D be the unit disk in the complex plane. Let $H(D)$ denote the set of analytic functions on D. For $1 \leq p < \infty$, the Hardy space H^p on D is

$$\{f \in H(D) : \|f\|_p = \sup_{0<r<1} \int_0^{2\pi} |f(re^{it})|^p \, dt < \infty\}.$$

The space of bounded analytic functions on D is

$$H^\infty = \{f \in H(D) : \|f\|_\infty = \sup_{z \in D} |f(z)| < \infty\}.$$

We define weighted Bergman spaces, for $\alpha > -1$,

$$A^p_\alpha = \{f \in H(D) : \|f\|_{A^p_\alpha} = \int_D |f(z)|^p (1-|z|^2)^\alpha \, dA(z) < \infty\},$$

where $dA(z)$ refers to Lebesgue area measure on D.

The Bloch space is

$$\mathcal{B} = \{f \in H(D) : \|f\|_{\mathcal{B}} = \sup_{z \in \mathbb{D}} |f'(z)|(1-|z|^2) < \infty\}.$$

Note that $\| \|_{\mathcal{B}}$ is a semi-norm. The true norm accounts for functions differing by an additive constant.

A complex measure μ on D is called a (Hardy space) Carleson measure if there exists $C > 0$ such that $\mu(S(I)) \leq C|I|$ for all arcs $I \subseteq \partial D$, where $S(I) = \{re^{i\theta} : 1-|I| < r < 1, e^{i\theta} \in I\}$ is the Carleson rectangle associated with I, and $|I|$ is the length of I. The smallest such C is called the Carleson constant for the measure μ. Define, for $f \in H(D)$, $d\mu_f(z) = |f'(z)|^2(1-|z|^2) \, dA(z)$. The space of analytic functions of bounded mean oscillation, $BMOA$, is the set of f for which μ_f is Carleson. The $BMOA$ norm $\|f\|_*$ is comparable to the square root of the Carleson constant for μ_f. The space of analytic functions of vanishing mean oscillation, $VMOA$, is the set of f for which

$$\lim_{|I| \to 0} \frac{\mu_f(S(I))}{|I|} = 0.$$

Zhu [14] is a good reference for background on all these spaces.

H^∞ is a subspace of $BMOA$, which in turn is a subspace of H^2. H^∞ is also a subspace of $\mathcal{B}$. The next lemma will be useful later when studying T_g.

Lemma 2.1. *Let $f_n(z) = z^n$, $n = 1, 2, \ldots$. $\|f_n\|_X \approx 1$ for all n and $X = H^2, \mathcal{B}, BMOA$.*

Proof: It is well-known that $\|f_n\|_{H^2} = 1$ for all n. Checking the Bloch norm with a calculation, we get $\|f_n\|_{\mathcal{B}} \approx \sup_{0<r<1} nr^{n-1}(1-r) = (1-\frac{1}{n})^{n-1} \to 1/e$ as $n \to \infty$. Finally, $1 = \|f_n\|_\infty \lesssim \|f_n\|_{BMOA} \lesssim \|f_n\|_{H^2}$. (see [14]) $\square$

When studying T_g and S_g, it is useful to be able to compare the norm of a function to the norm of its derivative. For $p \geq 1$, $\alpha > -1$, the differentiation operator and its inverse, the indefinite integral, are isometries between $A^p_\alpha/\mathbb{C}$ and $A^p_{\alpha+p}$, i.e.,

$$\|f\|_{A^p_\alpha} \approx |f(0)| + \|f'\|_{A^p_{\alpha+p}}. \tag{2.1}$$

(see [14, 4.28]) Making the natural definition $A^2_{-1} = H^2$, the identity holds for $p = 2$, $\alpha = -1$ as well. This is the well-known Littlewood-Paley identity, $\|f\|_{H^2} \approx |f(0)| + \int_D |f'(z)|^2(1-|z|^2)\, dA(z)$.

On all the spaces mentioned, point evaluation is a bounded linear functional. The norm of point evaluation at z in A^p_α is comparable to $1/(1-|z|)^{(2+\alpha)/p}$. (see [14, Theorem 4.14]) In $\mathcal{B}$ and $BMOA$, the norm of point evaluation is comparable to $\log(2/(1-|z|))$. The following theorem is a generalization of a result on multipliers of Banach spaces in which point evaluation is a bounded. See, for example, [6, Lemma 11].

Theorem 2.2. *Let X, Y be Banach spaces of analytic functions, and let λ^0_z and λ^1_z be linear functionals on X and Y defined by $\lambda^0_z f = f(z)$ and $\lambda^1_z f = f'(z)$. Suppose λ^0_z and λ^1_z are bounded.*

a) *Suppose S_g maps X boundedly into Y. Then*

$$|g(z)| \leq \|S_g\| \frac{\|\lambda^1_z\|_Y}{\|\lambda^1_z\|_X}$$

b) *Suppose T_g maps X boundedly into Y. Then*

$$|g'(z)| \leq \|T_g\| \frac{\|\lambda^1_z\|_Y}{\|\lambda^0_z\|_X}$$

Proof: Note that, for $f \in X$,

$$|f'(z)||g(z)| = |\lambda^1_z S_g(f)| \leq \|\lambda^1_z\|_Y \|S_g\| \|f\|_X$$

Since $\sup_{\|f\|_X=1} |f'(z)| = \|\lambda^1_z\|_X$, taking the sup over $\|f\|_X = 1$ of both sides gives us

$$\|\lambda^1_z\|_X |g(z)| \leq \|S_g\| \|\lambda^1_z\|_Y.$$

Hence a). Similarly,

$$|f(z)||g'(z)| = |\lambda^1_z T_g(f)| \leq \|\lambda^1_z\|_Y \|T_g\| \|f\|_X.$$

Taking the sup over f with norm 1, we get

$$\|\lambda^0_z\|_X |g'(z)| \leq \|T_g\| \|\lambda^1_z\|_Y.$$

This completes the proof. □

Corollary 2.3. *If X is a Banach space of analytic functions on which point evaluation of the derivative is a bounded linear functional, and S_g is bounded on X, then g is bounded.*

In [6], we see a similar result for M_g, i.e., boundedness of M_g on a Banach space in which point evaluation is bounded implies g is bounded. On the Hardy and Bergman spaces on the disk, both M_g and S_g are bounded if and only if g is bounded. That this is necessary for S_g is Corollary 2.3. That it is sufficient follows from integration by parts since T_g and M_g are bounded if g is bounded. (see [1] and [2] concerning T_g) A similar situation holds for $\mathcal{B}$.

Proposition 2.4. *S_g is bounded on $\mathcal{B}$ if and only if $g \in H^\infty$.*

Proof: It is clear $g \in H^\infty$ implies S_g is bounded, since

$$\|S_g f\|_{\mathcal{B}} = \sup_{z\in\mathbb{D}}(|f'(z)||g(z)|(1-|z|^2)) \leq \|g\|_{H^\infty}\|f\|_{\mathcal{B}}$$

The converse follows from Corollary 2.3. □

3. The property of being bounded below

An operator T is said to be bounded below if there exists $C > 0$ such that $\|Tf\| \geq C\|f\|$ for all f.

It typically is the case for one-to-one operators on Banach spaces that boundedness below is equivalent to having closed range. The analogue of Theorem 3.2 for composition operators is found in Cowen and MacCluer [4]. We include the proof for T_g and S_g, essentially the same, for easy reference.

Lemma 3.1. *T_g is one-to-one for nonconstant g.*

Proof: If $T_g f_1 = T_g f_2$, taking derivatives gives $f_1(z)g'(z) = f_2(z)g'(z)$. Thus $f_1(z) = f_2(z)$ except possibly at the (isolated) points where g vanishes. Since f_1 and f_2 are analytic, $f_1 = f_2$. □

When considering the property of being bounded below for S_g, we note that S_g maps any constant function to the 0 function. Thus, it is only useful to consider spaces of analytic functions modulo the constants.

Theorem 3.2. *Let Y be a Banach space of analytic functions on the disk. For nonconstant g, T_g is bounded below on Y if and only if it has closed range. S_g is bounded below on $Y/\mathbb{C}$ if and only if it has closed range on $Y/\mathbb{C}$.*

Proof: Assume T_g is bounded below, i.e., there exists $\varepsilon > 0$ such that $\|T_g f\| \geq \varepsilon\|f\|$ for all f. Suppose $\{T_g f_n\}$ is a Cauchy sequence in the range of T_g. Since $\|f_n - f_m\| \lesssim \|T_g f_n - T_g f_m\|$, $\{f_n\}$ is also a Cauchy sequence. Letting $f = \lim f_n$, we have $T_g f_n \to T_g f$, showing $T_g f_n$ converges in the range of T_g. Hence the range is closed.

Conversely, assume $T_g : Y \to Y$ is closed range. Let $\{f_n\}$ be a sequence in Y such that $\|T_g f_n\| \to 0$. T_g is one-to-one by Lemma 3.1. Let the closed range of T_g be X. X is a Banach space, and we can define the inverse $T_g^{-1} : X \to Y$. Suppose $\{x_n\}$ converges to $x = T_g h$ in X, and $T_g^{-1}x_n$ converges to y in Y. Applying T_g to $\{T_g^{-1}x_n\}$, this means x_n converges to $T_g y$. Hence $T_g y = T_g h$. Since T_g is one-to-one, $y = h$, and $x = T_g^{-1}y$. By the Closed Graph Theorem, T_g^{-1} is continuous. Thus, $\|f_n\| = \|T_g^{-1}(T_g f_n)\| \to 0$, implying T_g is bounded below.

The same argument holds for S_g as well, but only on spaces modulo constants, since S_g is not one-to-one otherwise. □

We will show that T_g is never bounded below on H^2, $\mathcal{B}$, nor $BMOA$. The sequence $\{z^n\}$ demonstrates the result in each space, since the functions z^n have norm comparable to 1, independent of n. (Lemma 2.1)

Theorem 3.3. *T_g is never bounded below on H^2, $\mathcal{B}$, nor $BMOA$.*

Proof: Let $f_n(z) = z^n$. For H^2,

$$\lim_{n\to\infty}\|T_g f_n\|^2 \approx \lim_{n\to\infty}\int_D |z^n|^2|g'(z)|^2(1-|z|^2)\,dA(z)$$

We assume T_g is bounded, so $g \in BMOA$ by a result of Aleman and Siskakis. [2] Thus μ_g is a Carleson measure, allowing us to bring the limit inside the integral by the Dominated Convergence Theorem.

$$\lim_{n\to\infty} \|T_g f_n\|^2 \approx \int_D \lim_{n\to\infty} |z^n|^2 |g'(z)|^2 (1-|z|^2)\, dA(z) = 0.$$

Since $\|f_n\|_2 = 1$ for all n, T_g is not bounded below.

If T_g is bounded on $\mathcal{B}$, then, by Theorem 2.2, $|g'(z)|(1-|z|) = O(1/\log(1/(1-|z|)))$ as $|z| \to 1$.

$$\|T_g f_n\|_{\mathcal{B}} = \sup_{z\in D} |z^n||g'(z)|(1-|z|) \lesssim \sup_{0\le r<1} r^n \frac{1}{\log(2/1-r)}.$$

Given $\varepsilon > 0$, there exists $\delta < 1$ such that $1/\log(2/(1-r)) < \varepsilon$ for $\delta < r < 1$. For large n, $r^n < \varepsilon$ for $0 < r < \delta$. Thus, $\lim_{n\to\infty} \|T_g f_n\|_{\mathcal{B}} = 0$, and Lemma 2.1 implies T_g is not bounded below on $\mathcal{B}$.

On $BMOA$, Siskakis and Zhao proved T_g being bounded implies $g \in VMOA$. [11]

$$\lim_{n\to\infty} \|T_g f_n\|_*^2 \approx \lim_{n\to\infty} \sup_I \frac{1}{|I|} \int_{S(I)} |z^n|^2 |g'(z)|^2 (1-|z|^2)\, dA(z).$$

Let I be an arc in ∂D, and let $\varepsilon > 0$. Since $g \in VMOA$, there exists $\delta > 0$ such that

$$\frac{1}{|J|} \int_{S(J)} |g'(z)|^2 (1-|z|^2)\, dA(z) < \varepsilon \text{ whenever } |J| < \delta.$$

If $|I| > \delta$, divide I into K disjoint intervals of length approximately δ, so

$$I = \cup_{i=1}^K J_i, \delta/2 < |J_i| < \delta \text{ for all } i, \text{ and } \delta K \approx |I|.$$

Let $S_\delta(I) = S(I) - \cup_i S(J_i)$. For large n, $(1-\delta/2)^{2n} \le \varepsilon|I|$, and to estimate the integral over $S_\delta(I)$ we use the fact that μ_g is a Carleson measure.

$$\begin{aligned} \frac{1}{|I|} \int_{S(I)} |z^n|^2 |g'(z)|^2 (1-|z|^2)\, dA(z) &= \frac{1}{|I|} \int_{S_\delta(I)} |z^n|^2 |g'(z)|^2 (1-|z|^2)\, dA(z) \\ &+ \frac{1}{|I|} \sum_{i=1}^K \int_{S(J_i)} |z^n|^2 |g'(z)|^2 (1-|z|^2)\, dA(z) \\ &\le \frac{1}{|I|} (1-\delta/2)^{2n} C\|g\|_*^2 + \frac{1}{|I|} K\delta\varepsilon \lesssim \varepsilon \end{aligned}$$

for large n. Hence $\lim_{n\to\infty} \|T_g f_n\|_* = 0$ and T_g is not bounded below on $BMOA$. □

In contrast to Theorem 3.3, T_g can be bounded below on weighted Bergman spaces. We state the result here, but the key is Proposition 3.5, proved afterward.

Theorem 3.4. *Let $1 \le p < \infty$, $\alpha > -1$. T_g is bounded below on A^p_α if and only if there exist $c > 0$ and $\delta > 0$ such that*

$$|\{z \in D : |g'(z)|(1-|z|^2) > c\} \cap S(I)| > \delta |I|^2.$$

Proof: We must assume T_g is bounded on A^p_α. By Theorem 2.2, $g \in \mathcal{B}$. (That this is also sufficient for T_g to be bounded on A^p_0 is in [1].) T_g is bounded below on A^p_α if and only if

$$\|T_g f\|^p_{A^p_\alpha} \approx \int_D |f(z)|^p |g'(z)|^p (1-|z|^2)^{\alpha+p}\, dA(z) \gtrsim \|f\|^p_{A^p_\alpha}.$$

By Proposition 3.5, this is true if and only if there exist $c > 0$ and $\delta > 0$ such that

$$|\{z \in D : |g'(z)|^p (1-|z|^2)^p > c\} \cap S(I)| > \delta |I|^2$$

for all arcs $I \subseteq \partial D$. If this holds for some p it holds for all p. $\square$

The proof of [10, Proposition 5.4] shows this result is nonvacuous. Ramey and Ullrich construct a Bloch function g such that $|g'(z)|(1-|z|) > c_0$ if $1 - q^{-(k+1/2)} \leq |z| \leq 1 - q^{-(k+1)}$, for some $c_0 > 0$, q some large positive integer, and $k = 1, 2, \ldots$. Given a Carleson square $S(I)$, let k_I be the least positive integer such that $q^{-k_I+1/2} \leq |I|$. The annulus $E = \{z : 1 - q^{-(k_I+1/2)} \leq |z| \leq 1 - q^{-(k_I+1)}\}$ intersects $S(I)$, and

$$|E \cap S(I)| \approx |I|((1-q^{-(k_I+1)}) - (1-q^{-(k_I+1/2)})) = |I|\frac{q^{1/2}-1}{q^{k_I+1}} \geq \frac{(q^{1/2}-1)}{q^{3/2}}|I|^2.$$

Setting $c = c_0$ and $\delta \approx 1/q$ show Theorem 3.4 holds for this example of g, and T_g is bounded below on A^p_α.

We define $H^p_0 = H^p/\mathbb{C} = \{f \in H^p : f(0) = 0\}$. The operator S_g can clearly be bounded below, since $g(z) = 1$ gives the identity operator. A result due to Luecking (see [4, 3.34]) leads to a characterization of functions for which S_g is bounded below on H^2_0 and $A^p_\alpha/\mathbb{C}$. We state a reformulation useful to our purposes here.

Proposition 3.5. (Luecking) *Let τ be a bounded, nonnegative, measurable function on D. Let $G_c = \{z \in D : \tau(z) > c\}$, $1 \leq p < \infty$, and $\alpha > -1$. There exists $C > 0$ such that the inequality*

$$\int_D |f(z)|^p \tau(z)(1-|z|)^\alpha\, dA(z) \geq C \int_D |f(z)|^p (1-|z|)^\alpha\, dA(z)$$

holds if and only if there exist $\delta > 0$ and $c > 0$ such that $|G_c \cap S(I)| \geq \delta |I|^2$ for every interval $I \subset T$.

The proof is omitted. Using the Littlewood-Paley identity we get the following:

Corollary 3.6. *S_g is bounded below on H^2_0 if and only if there exist $c > 0$ and $\delta > 0$ such that $|G_c \cap S(I)| \geq \delta |I|^2$, where $G_c = \{z \in D : |g(z)| > c\}$.*

We use Corollary 3.6 to construct a nonexample of boundedness below of S_g on H^2_0, and compare M_g on H^2 to S_g on H^2_0. If $g(z)$ is the singular inner function $\exp(\frac{z+1}{z-1})$, S_g is not bounded below on H^2_0. To see this, fix $c \in (0,1)$. G_c is the complement in D of a horodisk, a disk tangent to the unit circle, with radius $r = \frac{\log c + 1}{2(\log c - 1)}$ and center $1 - r$. Choosing a sequence of intervals $I_n \subset T$ such that 1 is the center of I_n and $|I_n| \to 0$ as $n \to \infty$, we see

$$\frac{|G_c \cap S(I_n)|}{|I_n|^2} \to 0 \text{ as } n \to \infty,$$

meaning S_g is not bounded below on H^2_0.

M_g is bounded below on H^2 if and only if the radial limit function of $g \in H^\infty$ is essentially bounded away from 0 on ∂D. ([8] has this result as a special case of weighted composition operators.) Theorem 3.8 will show this is weaker than the condition for S_g to be bounded below on H_0^2. The example above of a singular inner function then shows it is strictly weaker. To prove Theorem 3.8 we use a lemma which allows us to estimate an analytic function inside the disk by its values on the boundary. Define the conelike region with aperture $\alpha \in (0, 1)$ at $e^{i\theta}$ to be

$$\Gamma_\alpha(e^{i\theta}) = \left\{ z \in D : \frac{|e^{i\theta} - z|}{1 - |z|} < \alpha \right\}.$$

For a function $g \in H(D)$, define the nontangential limit function, for almost all e^{it},

$$|g^*(e^{it})| = \lim_{\Gamma_\alpha(e^{it}) \ni z \to e^{it}} |g(z)|.$$

For any arc $I \subseteq \partial D$ and $0 < r < 2\pi/|I|$, rI will denote the arc with the same center as I and length $r|I|$. We define the upper Carleson rectangle

$$S_\varepsilon(I) = \{re^{it} : 1 - |I| < r < (1 - \varepsilon|I|), e^{it} \in I\}, \text{ and } S^+(I) = S_{1/2}(I).$$

Lemma 3.7. *Given (1 >) $\varepsilon > 0$ and a point $e^{i\theta}$ such that $|g^*(e^{i\theta})| < \varepsilon$, there exists an arc $I \subset \partial D$ such that $|g(z)| < \varepsilon$ for $z \in S_\varepsilon(I)$.*

Proof: We can choose α close enough to 1 so that $S_\varepsilon(I) \subset \Gamma_\alpha(e^{i\theta})$ for all I centered at $e^{i\theta}$ with, say, $|I| < 1/4$. If $|g^*(e^{i\theta})| < \varepsilon$, there exists $\delta > 0$ such that

$$z \in \Gamma_\alpha(e^{i\theta}), |z - e^{i\theta}| < \delta \text{ imply } |g(z)| < \varepsilon.$$

Choosing I such that $S(I)$ is contained in a δ-neighborhood of $e^{i\theta}$ finishes the proof. □

Theorem 3.8. *If S_g is bounded below on H_0^2, then M_g is bounded below on H^2.*

Proof: Assume M_g is not bounded below on H^2. Let $\varepsilon > 0$. The radial limit function of g equals g^* almost everywhere, so there exists a point $e^{i\theta}$ such that $|g^*(e^{i\theta})| < \varepsilon$. By Lemma 3.7, there exists $S(I)$ such that $|\{z : |g(z)| \geq \varepsilon\} \cap S(I)| \leq \varepsilon|I|$. Since ε was arbitrary, this violates the condition in Proposition 3.5.□

We now characterize the symbols g which make S_g bounded below on the Bloch space. It turns out to be a common condition appearing in a few different forms in the literature. The condition appears in characterizing M_g on A_0^2 in McDonald and Sundberg [9]. Our main result is equivalence of (i)-(iii) in Theorem 3.9, and we give references with brief explanations for (iv)-(vi).

Theorem 3.9. *The following are equivalent for $g \in H^\infty$:*

(i) *$g = BF$ for a finite product B of interpolating Blaschke products and F such that $F, 1/F \in H^\infty$.*

(ii) *S_g is bounded below on $\mathcal{B}/\mathbb{C}$.*

(iii) *There exist $r < 1$ and $\eta > 0$ such that for all $a \in D$,*

$$\sup_{z \in D(a,r)} |g(z)| > \eta.$$

(iv) *S_g is bounded below on H_0^2.*

(v) *M_g is bounded below on A_α^p for $\alpha > -1$.*

(vi) *S_g is bounded below on $A_\alpha^p/\mathbb{C}$ for $\alpha > -1$.*

Proof: (i) $\Rightarrow$ (ii): Note that $S_{g_1 g_2} = S_{g_1} S_{g_2}$ for any g_1, g_2. It follows that if S_{g_1} and S_{g_2} are bounded below then $S_{g_1 g_2}$ is also bounded below. We will show that S_F and S_B are bounded below, implying the result for S_g.

It is necessary that $g \in H^\infty$ for S_g to be bounded on $\mathcal{B}$. (Corollary 2.3) If F, $1/F \in H^\infty$, then

$$\|S_F f\| = \sup_{z\in D} |F(z)||f'(z)|(1-|z|^2) \geq (1/\|1/F\|_\infty)\|f\|_{\mathcal{B}}.$$

Hence S_F is bounded below.

By virtue of the fact beginning this proof, we may assume B is a single interpolating Blaschke product without loss of generality. Let $\{w_n\}$ be the zero sequence of B, so

$$B(z) = e^{i\varphi} \prod_n \frac{w_n - z}{1 - \overline{w}_n z}.$$

Denote the pseudohyperbolic metric

$$\rho(z,w) = \frac{|w-z|}{|1-\overline{w}z|}, \text{ for any } z, w \in D.$$

For the pseudohyperbolic disk of radius $d > 0$ and center $w \in D$, we use the notation

$$D(w,d) = \{z \in D : \rho(z,w) < d\}.$$

Let B_j be B without its jth zero, i.e., $B_j(z) = \frac{1-\overline{w}_j z}{w_j - z} B(z)$. Since B is interpolating, there exist $\delta > 0$ and $r > 0$ such that, for all j, $|B_j(z)| > \delta$ whenever $z \in D(w_j, r)$. In particular, the sequence $\{w_n\}$ is separated, so shrinking r if necessary, we may assume

$$\inf_{j\neq k} \rho(w_k, w_j) > 2r.$$

We compare $\|f\|$ to $\|S_B f\| = \sup_{z\in D} |B(z)||f'(z)|(1-|z|^2)$. Let $a \in D$ be a point where the supremum defining the norm of f is almost achieved, say, $|f'(a)|(1-|a|^2) > \|f\|/2$.

Consider the pseudohyperbolic disk $D(a,r)$. Inside $D(a,r)$ there may be at most one zero of B, say w_k. We examine three cases depending on the location and existence of w_k.

If $r/2 \leq \rho(w_k, a) < r$, then

$$|B(a)| = \frac{|w_k - a|}{|1-\overline{w}_k a|} |B_k(a)| > (r/2)\delta.$$

Thus we would have

$$\|S_B f\| \geq |B(a)||f'(a)|(1-|a|^2) > (r/2)\delta\|f\|/2,$$

and S_g would be bounded below.

On the other hand, suppose $\rho(w_k, a) < r/2$. Consider the disk $D(w_k, r/2)$, which is contained in $D(a,r)$. The expression $1-|z|^2$ is roughly constant on a pseudohyperbolic disk, i.e.,

$$\sup_{z\in D(a,r)} (1-|z|^2) > C_r(1-|a|^2) \text{ for some } C_r > 0.$$

C_r does not depend on a, and is near 1 for small r. By the maximum principle for f', there exists a point $z_a \in \partial D(w_k, r/2)$ where

$$|f'(z_a)|(1-|z_a|^2) > |f'(a)|C_r(1-|a|^2) > C_r\|f\|/2.$$

(Since $\rho(w_k, a) < r/2$ and $\rho(z_a, w_k) = r/2$, we have $\rho(z_a, a) < r$.) This shows that S_g is bounded below, for

$$\begin{aligned} \|S_B f\| &\geq |B(z_a)||f'(z_a)|(1-|z_a|^2) \\ &> \rho(w_k, z_a)|B_k(z_a)|C_r\|f\|/2 \\ &> (r/2)\delta C_r\|f\|/2. \end{aligned}$$

Finally, suppose no such w_k exists. Then the function $((a-z)/(1-\overline{a}z))B(z)$ is also an interpolating Blaschke product, and the previous case applies with $w_k = a$.

(ii) $\Rightarrow$ (iii): Assume (iii) fails. Given $\varepsilon > 0$, choose r near 1 so that $1 - r^2 < \varepsilon$, and choose $a \in D$ such that $|g(z)| < \varepsilon$ for all $z \in D(a, r)$. Consider the test function $f_a(z) = (a-z)/(1-\overline{a}z)$. By a well-known identity,

$$(1-|z|^2)|f_a'(z)| = 1 - (\rho(a, z))^2.$$

Thus $f_a \in \mathcal{B}$ with $\|f_a\| = 1$ for all $a \in D$. (The seminorm is 1, but the true norm is between 1 and 2 for all a.) By supposition on g,

$$\begin{aligned} \|S_g f_a\| &= \sup_{z\in D} |g(z)||f_a'(z)|(1-|z|^2) \\ &= \max\left\{ \sup_{z\in D(a,r)} |g(z)||f_a'(z)|(1-|z|^2), \sup_{z\in D\setminus D(a,r)} |g(z)||f_a'(z)|(1-|z|^2) \right\} \\ &\leq \max\left\{ \sup_{z\in D(a,r)} |g(z)|\|f_a\|, \sup_{z\in D\setminus D(a,r)} |g(z)|(1-r^2) \right\} \\ &< \max\{\varepsilon, \|g\|_\infty \varepsilon\} \leq \varepsilon(\|g\|_\infty + 1) \end{aligned}$$

Since $\|f_a\| = 1$ and ε was arbitrary, S_g is not bounded below.

(iii) $\Rightarrow$ (i): Assuming (iii) holds, we first rule out the possibility that g has a singular inner factor. We factor $g = BI_gO_g$ where B is a Blaschke product, I_g a singular inner function, and O_g an outer function. Let ν be the measure on ∂D determining I_g, so

$$I_g(z) = \exp\left(-\int \frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\nu(\theta)\right).$$

Let $\varepsilon > 0$. For any $\alpha > 1$ and for ν-almost all θ, there exists $\delta > 0$ such that

$$z \in \Gamma_\alpha(e^{i\theta}), |z - e^{i\theta}| < \delta \text{ imply } |I_g(z)| < \varepsilon. \tag{3.1}$$

This is [7, Theorem II.6.2]. δ may depend on θ and α, but for nontrivial ν there exists some θ where (3.1) holds. Given $r < 1$, choose $\alpha < 1$ such that, for every a near $e^{i\theta}$ on the ray from 0 to $e^{i\theta}$, the pseudohyperbolic disk $D(a, r)$ is contained in $\Gamma_\alpha(e^{i\theta})$. The disk $D(a, r)$ is a euclidean disk whose euclidean radius is comparable to $1 - a$. For a close enough to $e^{i\theta}$,

$$z \in D(a, r) \text{ implies } |z - e^{i\theta}| < \delta.$$

Hence $\sup_{z\in D(a,r)} |g(z)| < \varepsilon\|g\|$. This violates (iii), so ν must be trivial, and $I_g \equiv 1$.

A similar argument handles the outer function O_g. If for all $\varepsilon > 0$ there exists e^{it} such that $|O_g^*(e^{it})| < \varepsilon$, we apply Lemma 3.7. The upper Carleson square in Lemma 3.7 contains some pseudohyperbolic disk that violates (iii), so O_g^* is essentially bounded away from 0. There exists $\eta > 0$, such that $|O_g^*(e^{it})| \geq \eta$ almost everywhere. Note $1/O_g \in H^\infty$, since for all $z \in D$,

$$\log|O_g(z)| = \frac{1}{2\pi}\int_0^{2\pi} \log|O_g^*(e^{it})| \frac{1-|z|^2}{|e^{it}-z|^2}\,dt \geq \log\eta.$$

We have reduced the symbol to a function $g = BF$, where $F, 1/F \in H^\infty$ and B is a Blaschke product, say with zero sequence $\{w_n\}$. We will show that the measure $\mu_B = \sum(1 - |w_n|^2)\delta_{w_n}$ is a Carleson measure, implying B is a finite product of interpolating Blaschke products. (see, e.g., [9, Lemma 21]) Let $r < 1$ and $\eta > 0$ be as in (iii), so $\sup_{z\in D(a,r)} |B(z)| > \eta$ for all a. Given any arc $I \subseteq \partial D$, we may choose a_I and z_I such that $D(a_I, r) \subseteq S(I)$, $z_I \in D(a_I, r)$, $|B(z_I)| > \eta$, and $(1-|z_I|) \approx |I|$ as I varies. $\mu_B(S(I)) = \sum(1-|w_{n_k}|^2)$ where the subsequence $\{w_{n_k}\} = \{w_n\} \cap S(I)$. Assume without loss of generality that $|I| < 1/2$, so $|w_{n_k}| > 1/2$ for all k. This ensures $|1 - \overline{w}_{n_k} z_I| \approx |I|$. Thus we have

$$\begin{aligned}
\frac{1}{|I|}\sum_k (1-|w_{n_k}|^2) &\approx \sum_k \frac{(1-|z_I|^2)(1-|w_{n_k}|^2)}{|1-\overline{w}_{n_k} z_I|^2} \\
&= \sum_k 1 - (\rho(z_I, w_{n_k}))^2 \\
&< 2\sum_n 1 - \rho(z_I, w_n) \\
&\leq -\sum_n \log \rho(z_I, w_n) \\
&= -\log \prod_n \frac{|w_n - z_I|}{|1-\overline{w}_n z_I|} \\
&= -\log|B(z)| \leq -\log \eta.
\end{aligned}$$

This shows μ_B is a Carleson measure.

(i) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v) $\Leftrightarrow$ (vi)

Bourdon shows in [3, Theorem 2.3, Corollary 2.5] that (i) is equivalent to the reverse Carleson condition in Corollary 3.6 above, hence (i) $\Leftrightarrow$ (iv). This reverse Carleson condition also characterizes boundedness below of M_g on weighted Bergman spaces by Proposition 3.5. Thus (iv) $\Leftrightarrow$ (v). A key connection is between S_g and M_g via the differentiation operator and equation (2.1), since $(S_g f)' = M_g f'$. The following diagram is commutative:

$$\begin{tikzcd}
A^p_\alpha/\mathbb{C} \arrow[r, "S_g"] \arrow[d, "f\mapsto f'"'] & A^p_\alpha/\mathbb{C} \arrow[d, "f\mapsto f'"] \\
A^p_{\alpha+p} \arrow[r, "M_g"] & A^p_{\alpha+p}
\end{tikzcd}$$

This explains (v) $\Rightarrow$ (vi). Since $A^2_{-1} = H^2$, we can combine (iv) and (vi) to say S_g is bounded below on $A^2_\alpha/\mathbb{C}$ for $\alpha \geq -1$. $\square$

Concluding Remarks

We suspect the results about H^2 can be extended to all H^p, $1 \leq p < \infty$, but without the Littlewood-Paley identity the proof is more difficult. Generalizing the results on Bloch to the α-Bloch spaces can be done with adjusted test functions as in [13]. Finally, we have partial results concerning S_g being bounded below on $BMOA$, but have not completed proving a characterization like the one in Theorem 3.9.

I would like to thank my advisor Dr. Wayne Smith for his invaluable guidance.

References

[1] Aleman, Alexandru; Cima, Joseph A., *An integral operator on H^p and Hardy's inequality.* J. Anal. Math. 85 (2001), 157–176.

[2] Aleman, Alexandru; Siskakis, Aristomenis G., *Integration operators on Bergman spaces.* Indiana Univ. Math. J. 46 (1997), no. 2, 337–356.

[3] Bourdon, Paul S., *Similarity of parts to the whole for certain multiplication operators.* Proc. Amer. Math. Soc. 99 (1987), no. 3, 563–567.

[4] Cowen, Carl; MacCluer, Barbara, *Composition Operators on Spaces of Analytic Functions.* CRC Press, New York, 1995.

[5] Dostanić, Milutin R., *Integration operators on Bergman spaces with exponential weight.* Rev. Mat. Iberoam. 23 (2007), no. 2, 421–436.

[6] Duren, P. L.; Romberg, B. W.; Shields, A. L., *Linear functionals on H^p spaces with* $0 < p < 1$. J. Reine Angew. Math. 238 (1969), 32-60.

[7] Garnett, John B., *Bounded Analytic Functions. Revised First Edition.* Springer, New York, 2007.

[8] Kumar, Romesh; Partington, Jonathan R., *Weighted composition operators on Hardy and Bergman spaces.* Recent advances in operator theory, operator algebras, and their applications, 157–167, Oper. Theory Adv. Appl., 153, Birkhuser, Basel, 2005.

[9] McDonald, G.; Sundberg, C., *Toeplitz operators on the disc.* Indiana Univ. Math. J. 28 (1979), no. 4, 595–611.

[10] Ramey, Wade; Ullrich, David, *Bounded mean oscillation of Bloch pull-backs.* Math. Ann. 291 (1991), no. 4, 591–606.

[11] Siskakis, Aristomenis G., Zhao, Ruhan, *A Volterra type operator on spaces of analytic functions.* Function spaces (Edwardsville, IL, 1998), 299–311, Contemp. Math., 232, Amer. Math. Soc., Providence, RI, 1999.

[12] Stegenga, David A., *Bounded Toeplitz operators on H^1 and applications of the duality between H^1 and the functions of bounded mean oscillation.* Amer. J. Math. 98 (1976), no. 3, 573–589.

[13] Zhang, M.; Chen, H., *Weighted composition operators of H^∞ into α-Bloch spaces on the unit ball.* Acta Math. Sin. (Engl. Ser.) 25 (2009), no. 2, 265–278.

[14] Zhu, Kehe, *Operator theory in function spaces. Second edition.* Mathematical Surveys and Monographs, 138. American Mathematical Society, Providence, RI, 2007.

Department of Mathematics, University of Hawaii, Honolulu, Hawaii 96822
E-mail address: austina@hawaii.edu