- Classical XY-model $\|\vec\sigma\| = 1$ $\vec\sigma = (\cos\varphi, \sin\varphi)$

$$H = -J \sum_{\langle ij\rangle} \vec\sigma_i \cdot \vec\sigma_j = -J \sum_{\langle ij\rangle} \cos(\varphi_i - \varphi_j)$$

in general:

$$H = -\sum_{i\neq j} J_{ij}\, \vec\sigma_i \cdot \vec\sigma_j - \sum_j \vec h_j \cdot \vec\sigma_j$$

$$\rightarrow H = -\sum_{\langle ij\rangle} J_{ij} \cos(\varphi_i - \varphi_j) - \sum_j h_j \cos\varphi_j$$

- continuum limit $\mathcal{F}[\varphi(\vec r)] = \int d^d r\ \frac{1}{2} g (\nabla\varphi)^2$ $g = J$

Gaussian fluctuations $(a=0)$: $\langle \varphi(\vec q)\varphi(\vec q\,')\rangle = \delta_{\vec q + \vec q\,'} \frac{k_B T V}{g |\vec q|^2}$

fluctuations $\langle \varphi(\vec r)^2\rangle = \frac{1}{V^2} \sum_{\vec q \vec q\,'} \langle \varphi(\vec q)\varphi(\vec q\,')\rangle = \frac{1}{V} \sum_q \frac{k_B T}{g |\vec q|^2}$

$$\simeq \frac{1}{(2\pi)^d} \int d^d q \frac{k_B T}{g |\vec q|^2} \sim \frac{k_B T}{g} \int_{1/r}^{\Lambda} dq\, q^{d-3}$$

- $d=1$: $\langle\varphi^2\rangle \sim \frac{k_B T}{g} (r - \Lambda^{-1}) \sim r$

no ordered phase

$d>2$: $\langle\varphi^2\rangle \sim \frac{k_B T}{g} (\Lambda^{d-2} - r^{2-d}) \sim \frac{k_B T}{g}$ bound $\rightarrow$ order for T sma

$d=2$: $\langle\varphi^2\rangle = \frac{1}{2\pi^2} \cdot 2\pi \frac{k_B T}{g} \int_{1/r}^{\Lambda} \frac{dq}{q} = \frac{k_B T}{\pi g} \ln(\Lambda r) \xrightarrow{r\to\infty} \infty$

no long ranged order

- no second order phase transition in 2D (Mermin-Wagner-theorem)

$$\Rightarrow G(r) = e^{-\frac{1}{2}\langle(\varphi(r)-\varphi(0))^2\rangle} \sim r^{-\frac{k_B T}{2\pi g}}$$

$\sigma^x + i\sigma^y = e^{i\varphi}$

$\langle \vec\sigma \cdot \vec\sigma'\rangle = \langle e^{i(\varphi-\varphi')}\rangle$

$= e^{-\frac{1}{2}\langle(\varphi-\varphi')^2\rangle}$

$\Leftarrow$

Single vortex

$$E_v = \int d^2r\, \tfrac{1}{2} J |\nabla\varphi|^2 \qquad \nabla\varphi = \tfrac{1}{r} e_\varphi$$

$$= \tfrac{1}{2} J\, 2\pi \int_{\Lambda^{-1}}^{R} dr\, \tfrac{1}{r} = \pi J \ln(R\Lambda)$$

$\oint d\vec{r}\,\nabla\varphi = 2\pi \left(= \int d\varphi = \varphi_e - \varphi_i\right)$

$\Downarrow$

$\nabla\varphi \sim \frac{1}{r}$

positional Entropy $\quad \Omega \sim (R\Lambda)^2 \qquad S_v = k_B \ln\Omega = 2 k_B \ln R\Lambda$

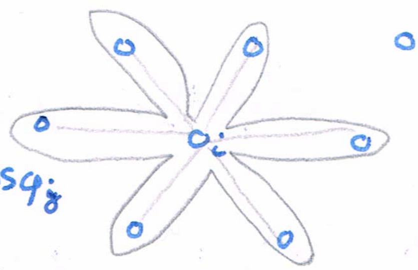
$$F_v = E_v - TS_v = (\pi J - 2k_B T) \ln R\Lambda$$

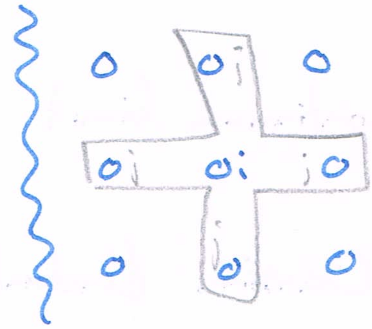
Kosterlitz - Thouless transition at $T_c = \dfrac{\pi J}{2 k_B}$

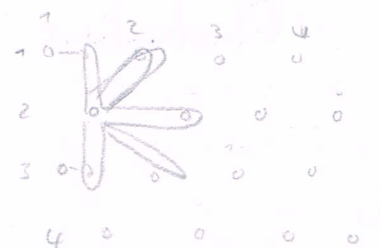
- $\{\vec{\sigma}_i\} = \{\varphi_i\}$
- $H_j = -J \sum_{\langle ij\rangle} \cos(\varphi_i - \varphi_j) - h \cos\varphi_j$
- $j = \text{rand}(\{i\})$; $\tilde{\varphi}_j = \text{rand}() \cdot 2\pi$
- if $\text{rand}() < \min\left(1, \exp\left(\dfrac{H_j - \tilde{H}_j}{k_B T}\right)\right)$

  $\varphi_j = \tilde{\varphi}_j$
- repeat

Metropolis-step







i-1, j-1	i j-1	i+1, j-1
i-1, j	i j	i+1, j
i-1, j+1	i, j+1	i+1, j+1

- $\{\varphi_i\}_{init}$, T, h
- thermalize MetroStep$(H, \{\varphi_i\}, T, \text{repeat} = 1000)$

- measurement + vary params T, h
- thermalize (100)

$M = \sum_i \vec{\sigma}_i$

$\chi = \dfrac{\partial M}{\partial h}$

$\langle \vec{\sigma}_i \vec{\sigma}_j \rangle = \langle \cos(\varphi_i - \varphi_j)$

---

visualize:

$$\text{gitter} = \begin{pmatrix} \vec{0} & \vec{g}_1 & \cdots & L\vec{g}_1 \\ \vec{g}_2 & \vec{g}_2 + \vec{g}_1 & \cdots & \vec{g}_2 + L\vec{g}_1 \\ \vdots & & & \\ L\vec{g}_2 & & \cdots & L\vec{g}_2 + L\vec{g}_1 \end{pmatrix} = \{\vec{x}_i\}$$

arrowmap$(\{\vec{x}_i\}, \{\varphi_i\})$ $\qquad$ add heatmap$(\{H_i\})$