David Morin - Cycloidal Pendulum

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Exercise 6.29

The standard pendulum frequency of $\sqrt{g/\ell}$ holds only for small oscillations. The frequency becomes smaller as the amplitude grows. It turns out that if you want to build a pendulum whose frequency is independent of the amplitude, you should hang it from the cusp of a cycloid of a certain size, as shown in Fig. 6.30. As the string wraps partially around the cycloid, the effect is to decrease the length of string in the air, which in turn increases the frequency back up to a constant value. In more detail: A cycloid is the path taken by a point on the rim of a rolling wheel. The upside-down cycloid in Fig. 6.30 can be parameterized by $(x, y) = R(\theta - \sin\theta, -1 + \cos\theta)$, where $\theta = 0$ corresponds to the cusp. Consider a pendulum of length $4R$ hanging from the cusp, and let α be the angle the string makes with the vertical, as shown.

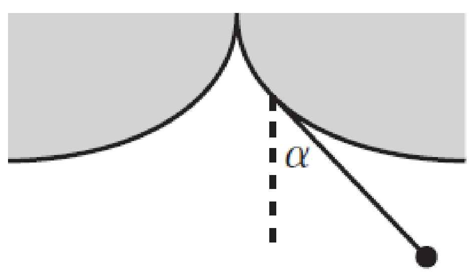

Fig. 6.30

(a)

In terms of α, find the value of the parameter θ associated with the point where the string leaves the cycloid.

Solution:

From the question, $(x, y) = R(\theta - \sin\theta, -1 + \cos\theta)$ and therefore, $(dx, dy) = Rd\theta(1 - \cos\theta, -\sin\theta)$.

Note that the part of the string leaving the cycloid is tangent to the cycloid at the point where it leaves. So, by basic trigonometry and our previous observation, $\tan\alpha = |\frac{dx}{dy}| = \frac{1-\cos\theta}{\sin\theta} = \tan\frac{\theta}{2}$.

And, finally, $\theta = 2\alpha$.

(b)

In terms of α, find the length of the string touching the cycloid.

Solution:

Consider the differential arclength ds touching the cycloid: $ds^2 = dx^2 + dy^2 = R^2 d\theta^2((1 - \cos\theta)^2 + (-\sin\theta)^2) = 2R^2 d\theta^2 (1 - \cos\theta) = 4R^2 d\theta^2 (\frac{1-\cos\theta}{2}) = 4R^2 d\theta^2 \sin^2\frac{\theta}{2} \Rightarrow ds = 2Rd\theta \sin\frac{\theta}{2} = 4Rd\alpha \sin\alpha$.

By arc length formula, the length of the string is $\int \sqrt{dx^2 + dy^2} = \int ds = \int_0^\alpha 4R \sin\alpha d\alpha = -4R\cos\alpha + 4R\cos 0 = 4R(1 - \cos\alpha)$.

(c)

In terms of α, find the Lagrangian.

Solution:

In general, the Lagrangian is:

$$\mathcal{L} = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2\right) - mgy$$

Note that the length of string in the air for given α is $4R - 4R(1 - \cos\alpha) = 4R\cos\alpha$.

Since, $\theta = 2\alpha$ at the contact point, the position of the contact point is given by $R(2\alpha - \sin 2\alpha, -1 + \cos 2\alpha)$.

And so, the position of the mass relative to the contact point is given by, $4R\cos\alpha(\sin\alpha, -\cos\alpha) = R(2\sin 2\alpha, -2 - 2\cos 2\alpha)$.

Thus, the position of the mass is $(x, y) = R(2\alpha - \sin 2\alpha, -1 + \cos 2\alpha) + R(2\sin 2\alpha, -2 - 2\cos 2\alpha) = R(2\alpha + \sin 2\alpha, -3 - \cos 2\alpha)$.

Then, $(\dot{x}, \dot{y}) = 2R\dot{\alpha}(1 + \cos 2\alpha, \sin 2\alpha)$ and $\dot{x}^2 + \dot{y}^2 = 8R^2\dot{\alpha}^2(1 + \cos 2\alpha)$.

So, we have

$$\mathcal{L} = 4mR^2\dot{\alpha}^2(1 + \cos 2\alpha) + mgR(3 + \cos 2\alpha)$$

(d)

Show that the quantity $\sin\alpha$ undergoes simple harmonic motion with frequency $\sqrt{g/4R}$, independent of the amplitude.

Solution:

By the Euler-Lagrange Equation,

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{d\dot{\alpha}}\right) = \frac{\partial\mathcal{L}}{\partial\alpha}$$

And so,

$$\begin{aligned}
\frac{d}{dt}\big(8mR^2\dot{\alpha}(1 + \cos 2\alpha)\big) &= -8mR^2\dot{\alpha}^2\sin 2\alpha - 2mgR\sin 2\alpha \\
4R\ddot{\alpha}(1 + \cos 2\alpha) - 8R\dot{\alpha}^2\sin 2\alpha &= -4R\dot{\alpha}^2\sin 2\alpha - g\sin 2\alpha \\
\ddot{\alpha}(1 + \cos 2\alpha) - \dot{\alpha}^2\sin 2\alpha &= -(g/4R)\sin 2\alpha \\
\ddot{\alpha}(2\cos^2\alpha) - \dot{\alpha}^2(2\sin\alpha\cos\alpha) &= -(g/4R)2\sin\alpha\cos\alpha \\
\ddot{\alpha}\cos\alpha - \dot{\alpha}^2\sin\alpha &= -(g/4R)\sin\alpha \\
\frac{d^2}{dt^2}(\sin\alpha) &= -(g/4R)\sin\alpha \\
\sin\alpha &= A\cos\left(\sqrt{\frac{g}{4R}}t + \phi\right)
\end{aligned}$$

Thus, $\sin\alpha$ undergoes SHM with frequency $\sqrt{\frac{g}{4R}}$, independent of the amplitude A.

(e)

In place of parts (c) and (d), solve the problem again by using $F = ma$. This actually gives a much quicker solution.

Solution:

We have tangential speed about the contact point: $v = (4R\cos\alpha)\dot{\alpha}$.

So,

$$F = ma \Rightarrow -mg\sin\alpha = m\frac{d}{dt}(4R\cos\alpha\dot{\alpha}) \Rightarrow -\frac{g}{4R}\sin\alpha = \frac{d^2}{dt^2}(\sin\alpha)$$

as desired.