David Morin - Inverted Pendulum

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Exercise 6.5

A pendulum consists of a mass m at the end of a massless stick of length ℓ. The other end of the stick is made to oscillate vertically with a position given by $y(t) = A\cos\omega t$, where $A << \ell$. See Fig. 6.12. It turns out that if ω is large enough, and if the pendulum is initially nearly upside-down, then surprisingly it will *not* fall over as time goes by. Instead, it will (sort of) oscillate back and forth around the vertical position. Find the equation of motion for the angle of the pendulum (measured relative to its upside-down position). Explain why the pendulum doesn't fall over, and find the frequency of the back and forth motion.

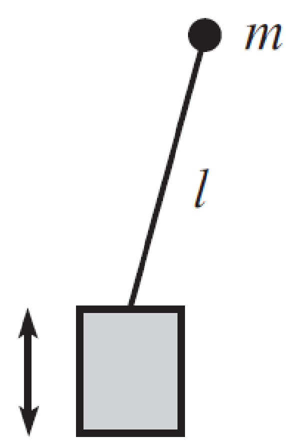

Fig. 6.12

Solution (provided by Morin):

The position of the mass if given by

$$(x, y) = (\ell \sin\theta, \ell\cos\theta + y(t))$$

, where $y(t) = A\cos\omega t$

The Lagrangian is therefore

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgh \\ &= \frac{1}{2}m(\ell^2\dot{\theta}^2 + \dot{y}^2 - 2\ell\dot{y}\dot{\theta}\sin\theta) - mg(y + \ell\cos\theta)\end{aligned}$$

And by the Euler-Lagrange equation, we get the equation of motion of the angle,

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) = \frac{\partial\mathcal{L}}{\partial\theta} &\Rightarrow \ell\ddot{\theta} - \ddot{y}\sin\theta = g\sin\theta \\ &\Rightarrow \ell\ddot{\theta} + \sin\theta\left(A\omega^2\cos\omega t - g\right) = 0\end{aligned}$$

For analysing the pendulum's behaviour, we will make use of the small angle approximation, $\sin\theta \approx \theta$.

And so,

$$\ddot{\theta} + \theta\left(a\omega^2\cos\omega t - \omega_0^2\right) = 0$$
$$\text{where, } \omega_0 = \sqrt{g/\ell} \text{ and } a = A/\ell$$

We also make the assumptions that $a \ll 1$ and $a\omega^2 \gg \omega_0^2$, for reasons explained later.

Note that when the bottom of the pendulum oscillates up, θ increases and vise-versa. On average, the position of the pendulum doesn't change much over one of these intervals (since $A \ll \ell$), and so we can approximate a solution to the equation of motion of the form

$$\theta(t) \approx C + b\cos(\omega t), b \ll C$$

where C will change over time, but on the scale of $1/\omega$ it is essentially constant if $a = A/\ell$ is small enough.

Plugging it into the equation of motion with $a \ll 1$ and $a\omega^2 \gg \omega_0^2$ yields

$$-b\omega^2\cos\omega t + Ca\omega^2\cos\omega t = 0$$

Thus, $b = aC$. And our solution for θ is

$$\theta \approx C(1 + a\cos\omega t)$$

From our equation of motion, the average acceleration of θ over a period of $2\pi/\omega$ is

$$\begin{aligned}
\overline{\ddot{\theta}} &= \overline{-\theta\left(a\omega^2\cos\omega t - \omega_0^2\right)} \\
&\approx \overline{-C\left(1 + a\cos\omega t\right)\left(a\omega^2\cos\omega t - \omega_0^2\right)} \\
&= -C\left(a^2\omega^2\overline{\cos^2\omega t} - \omega_0^2\right) \\
&= -C\left(\frac{a^2\omega^2}{2} - \omega_0^2\right) \\
&= -C\Omega^2, \text{ where } \Omega = \sqrt{\frac{a^2\omega^2}{2} - \frac{g}{\ell}}
\end{aligned}$$

Taking two derivatives of θ yields $\overline{\ddot{\theta}} = \ddot{C}$, thus

$$\ddot{C}(t) + \Omega^2 C(t) \approx 0 \text{ (simple harmonic motion)}$$

Therefore, C oscillates with frequency Ω, which is the motion keeping the pendulum upright. Note for this frequency to exist, we must have $a\omega > \sqrt{2}\omega_0$. This is achieved by our assumption $a << 1$, where $a^2\omega^2 > 2\omega_0^2$ implies $a\omega^2 >> \omega_0^2$, which is consistent.

If $a\omega^2 \gg \omega_0^2$, then the $a\omega^2\cos\omega t$ term dominates the ω_0^2. The one exception is when $\cos\omega t \approx 0$, but this occurs for negligible time if $a\omega^2 \gg \omega_0^2$. And, if $a \ll 1$, then we can ignore the $\ddot{C}$ term when substituting the approximate solution for θ into the approximate equation of motion. This is because our assumptions lead to $\ddot{C}$ being roughly proportional to $Ca^2\omega^2$. Since the other terms are proportional to $Ca\omega^2$, we need $a \ll 1$ in order for the $\ddot{C}$ term to be negligible. In short, $a << 1$ is the condition under which C varies slowly on the time scale of $1/\omega$. Thus, C will change over time, but on the scale of $1/\omega$ it is essentially constant if $a = A/\ell$ is small enough.