Floating Point

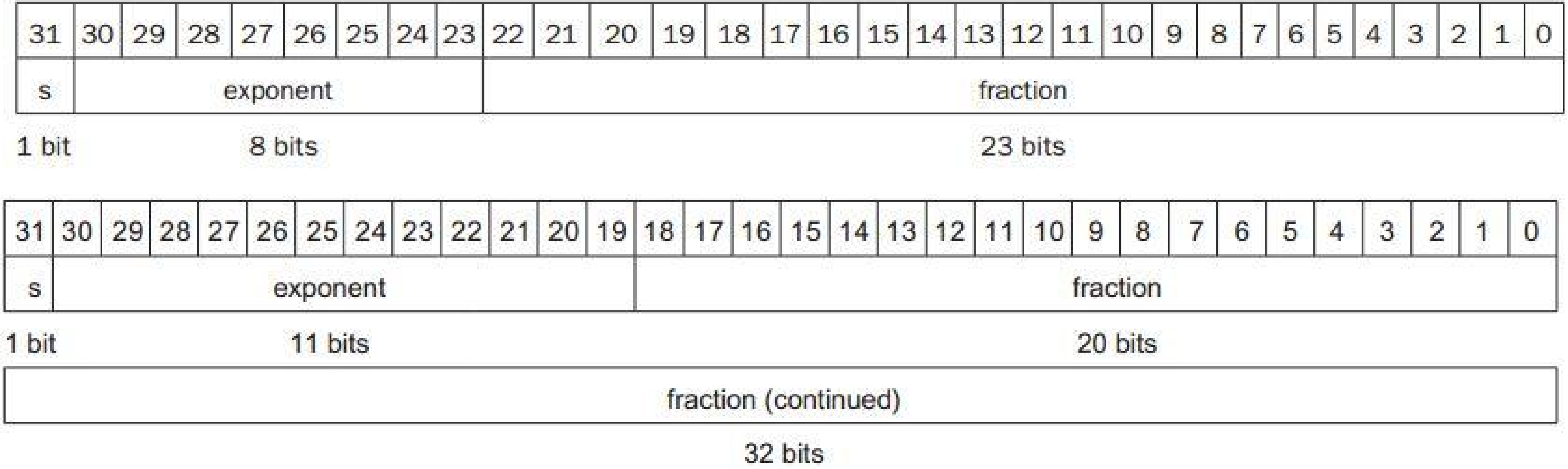
- Representation for non-integral numbers
- Including very small and very large numbers
- **Scientific notation:** A single digit to the left of the decimal point. A number in scientific notation that has no leading 0s is called a normalized number.
    - Normalized: -3.81×10^{22}
    - Not normalized: 0.006×10^{-5}, 105.75×10^{4}

- Just as in scientific notation, numbers are represented as a single nonzero digit to the left of the binary point (**floating point**). In binary, the form is:

$$\pm 1.xxxx_2 \times 2^{yyyy}$$

Floating Point Representation

- IEEE 754 Floating Point Standard
  - Single precision floating point (32-bit)
  - Double precision floating point (64-bit)



Floating Point Representation

- In general, floating-point numbers are of the form
$$(-1)^S \times (1 + Fraction) \times 2^{(Exponent - Bias)}$$
- S: sign bit (0: non-negative, 1: negative)
- Normalized significand: 1.0 ≤ |significand| < 2.0
  - Always has a leading pre-binary-point 1 bit, so no need to represent it explicitly (hidden bit)
  - Significand is Fraction with the “1.” restored
- Exponent: excess representation: actual exponent + Bias
  - Exponent is unsigned
  - Single: Bias = **127**; Double: Bias = **1023**

Floating Point Representation

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- Example: -0.75_{10}

$-0.75_{10} = -0.11 = -1.1_2 \times 2^{-1}$

S= 1

Fraction = $10000\ldots000_2$ (23 bit in single precision and 52 bit in double precision)

Exponent = –1 + Bias

Single: –1 + 127 = 126 = $0111\ 1110_2$ (8 bit)

Double: –1 + 1023 = 1022 = $011\ 1111\ 1110_2$ (11 bit)

$$(-1)^1 \times (1 + .1000\ 0000\ 0000\ 0000\ 0000\ 000_{two}) \times 2^{(126-127)}$$

31	30	29	28	27	26	25	24	23	22	21	20	19	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0
1	0	1	1	1	1	1	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

1 bit 8 bits 23 bits

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31	30	29	28	27	26	25	24	23	22	21	20	19	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0
1	0	1	1	1	1	1	1	1	1	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

1 bit 11 bits 20 bits

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

32 bits

Floating Point Representation

- In general, floating-point numbers are of the form

$$(-1)^S \times (1 + Fraction) \times 2^{(Exponent-Bias)}$$

- $(1 + Fraction)$ is known as significand
- Why is Bias here?
  - For the benefit of comparison

- Bias 127 (for single precision)
  - Lowest (0) and Highest (255) values of the exponent are reserved
  - The range of usable exponent values is [1,254]. So, the range of (exponent-127) is [-126 .. 127].

Single Precision Range

- Exponents 0000 0000 and 1111 1111 reserved
- Smallest value
  - Exponent: 0000 0001, actual exponent = 1 – 127 = –126
  - Fraction: 000…00 (23bits), significand = 1.0   (i.e., 1 +Fraction)
  - $\pm 1.0 \times 2^{-126} \approx \pm\ 1.2 \times 10^{-38}$
- Largest value
  - exponent: 1111 1110, actual exponent = 254 – 127 = +127
  - Fraction: 111…11 (23bits), significand ≈ 2.0   (i.e., 1+Fraction)
  - $\pm 2.0 \times 2^{+127} \approx \pm\ 3.4 \times 10^{+38}$

Double Precision Range

- Exponents 000 0000 0000 and 111 1111 1111 reserved
- Smallest value
  - Exponent: 000 0000 0001, actual exponent =1 – 1023 = –1022
  - Fraction: 000…00 (52bits), significand = 1.0   (i.e., 1 +Fraction)
  - $\pm 1.0 \times 2^{-1022} \approx \pm\ 2.2 \times 10^{-308}$
- Largest value
  - exponent: 111 1111 1110, actual exponent = 2046 – 1023 = +1023
  - Fraction: 111…11 (52bits), significand ≈ 2.0   (i.e., 1+Fraction)
  - $\pm 2.0 \times 2^{+1023} \approx \pm\ 1.8 \times 10^{+308}$

EEE 754 encoding of floating-point numbers

Single precision		Double precision		Object represented
Exponent	Fraction	Exponent	Fraction	
0	0	0	0	0
0	Nonzero	0	Nonzero	± denormalized number
1–254	Anything	1–2046	Anything	± floating-point number
255	0	2047	0	± infinity
255	Nonzero	2047	Nonzero	NaN (Not a Number)

Denormal Number

- Exponent = 000...0, hidden bit is 0

$$(-1)^S \times (0 + Fraction) \times 2^{-Bias}$$

- Smaller than normal numbers
  - allow for gradual underflow, with diminishing precision
  - For example, the smallest single precision denormalized number is

$$0.0000\ 0000\ 0000\ 0000\ 0000\ 001_2 \times 2^{-126} \text{ or } 1.0_2 \times 2^{-149}$$

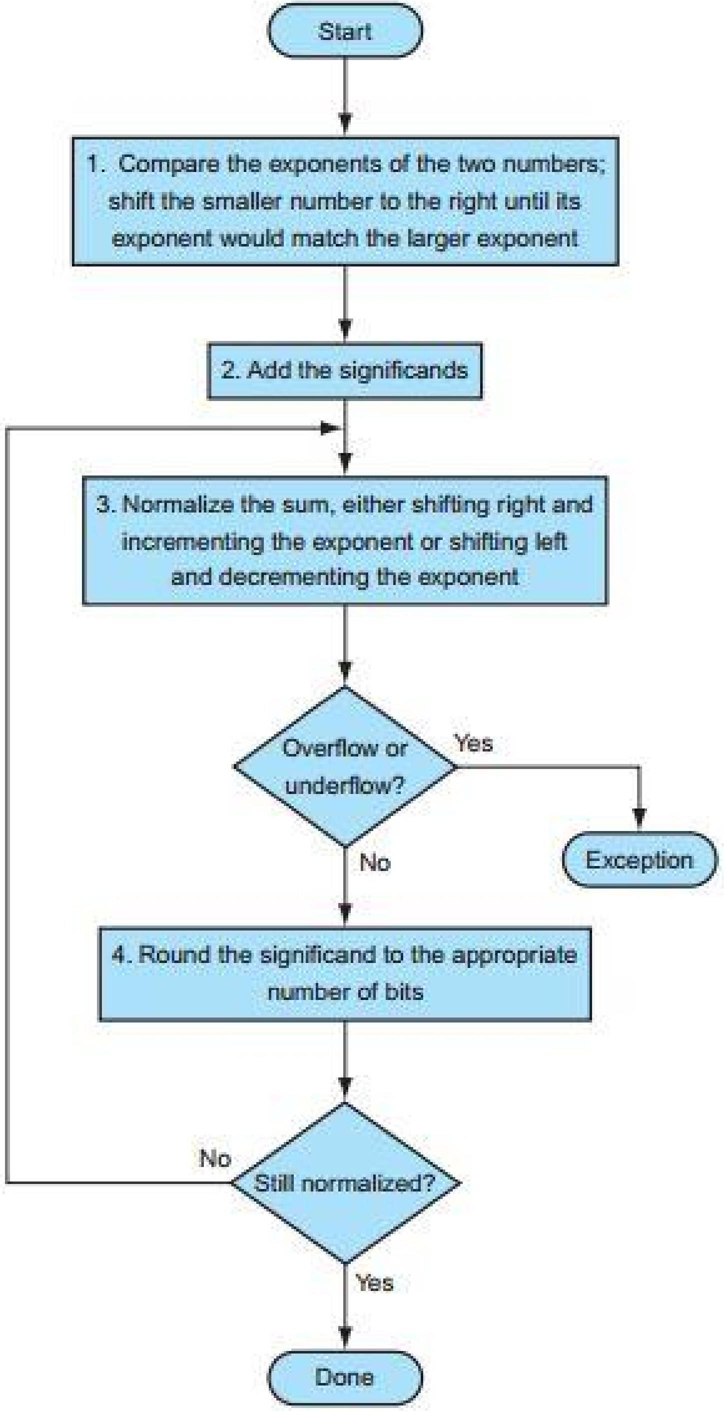
whereas the smallest positive single precision normalized number was

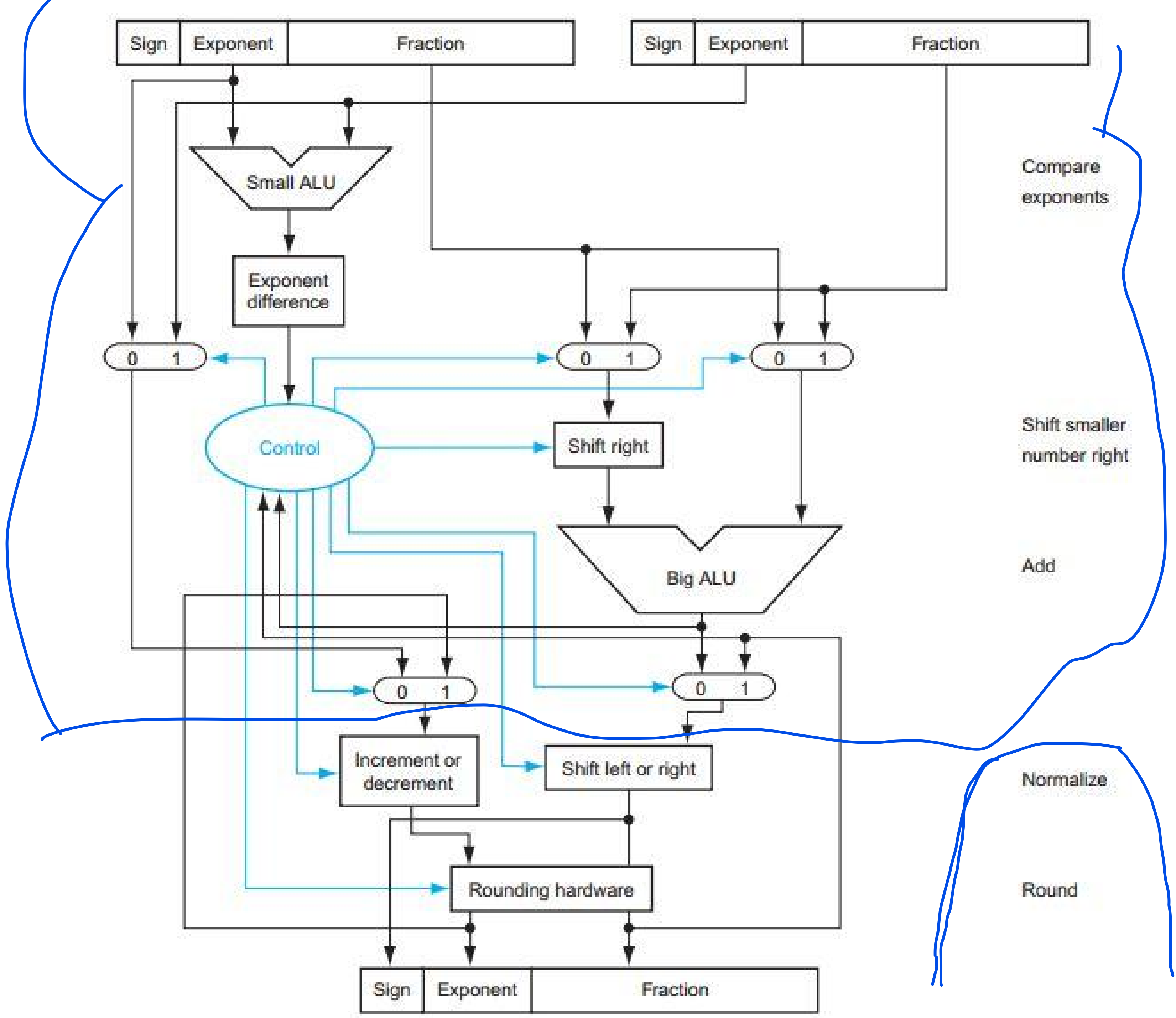
$$1.0000\ 0000\ 0000\ 0000\ 0000\ 000_2 \times 2^{-126}$$

Floating Point Addition

- Example: (Binary Floating-Point Addition) Add the number 0.5 and -04375 in binary.
- Consider a 4-digit binary example
    - $1.000_2 \times 2^{-1} + (-1.110_2 \times 2^{-2})$

1. Align binary points
    - Shift the smaller number to right until its exponent would match the larger exponent
    - $1.000_2 \times 2^{-1} + (-0.111_2 \times 2^{-1})$
2. Add significands
    - $1.000_2 \times 2^{-1} + (-0.111_2 \times 2^{-1}) = 0.001_2 \times 2^{-1}$
3. Normalize result & check for over/underflow
    - $1.000_2 \times 2^{-4}$, with no overflow / underflow
4. Round and renormalize if necessary
    - $\mathbf{1.000_2} \times \mathbf{2^{-4}}$(no change) = **0.0625**


Start
1. Compare the exponents of the two numbers; shift the smaller number to the right until its exponent would match the larger exponent
2. Add the significands
3. Normalize the sum, either shifting right and incrementing the exponent or shifting left and decrementing the exponent
Overflow or underflow?
Yes
Exception
No
4. Round the significand to the appropriate number of bits
Still normalized?
No
Yes
Done


Sign
Exponent
Fraction
Sign
Exponent
Fraction
Small ALU
Exponent difference
0
1
0
1
0
1
Control
Shift right
Big ALU
0
1
0
1
Increment or decrement
Shift left or right
Rounding hardware
Sign
Exponent
Fraction
Compare exponents
Shift smaller number right
Add
Normalize
Round

Rounding Bits

- Rounding using Guard and round digits, and sticky bit

G	R	S	Action
0	0	0	Truncate
0	0	1	Truncate
0	1	0	Truncate
0	1	1	Truncate
1	0	0	Round to Even
1	0	1	Round Up
1	1	0	Round Up
1	1	1	Round Up

$1.100GRS$

$1.10001 = 1.53125$
$1.10010 = 1.56250$
$1.10011 = 1.59375$