Coordinate Geometry of Circles: A Comprehensive Guide

1. Definition and Standard Form

A circle is the locus of all points in a plane that are equidistant from a fixed point called the center. In coordinate geometry, if (h,k) is the center and r is the radius, then any point (x,y) on the circle satisfies:

$$(x-h)^2 + (y-k)^2 = r^2$$

This is known as the standard form equation of a circle.

2. General Form

The general equation of a circle is: $$x^2 + y^2 + 2gx + 2fy + c = 0$$ where:

- Center: $(-g,-f)$
- Radius: $r = \sqrt{g^2 + f^2 - c}$

3. Key Properties

3.1 Tangent and Normal

For a circle with center (h,k) and point of tangency (x_1,y_1):

- **Tangent equation**: $$(x-h)(x_1-h) + (y-k)(y_1-k) = r^2$$

- **Normal equation**: $$\frac{x-x_1}{x_1-h} = \frac{y-y_1}{y_1-k}$$

3.2 Length of Tangent

The length of tangent from an external point (x_1,y_1) to a circle $(x-h)^2 + (y-k)^2 = r^2$ is: $$T = \sqrt{(x_1-h)^2 + (y_1-k)^2 - r^2}$$

4. Power of a Point

For any point $P(x_1,y_1)$ and a circle with center O: $$\text{Power of P} = OP^2 - r^2 = (x_1-h)^2 + (y_1-k)^2 - r^2$$

Properties:

- If P is outside: Power = (Length of tangent from P)^2
- If P is on circle: Power = 0
- If P is inside: Power is negative

5. Intersection of Circles

Two circles: $$x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$$ $$x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$$

Their radical axis equation is: $$2(g_1-g_2)x + 2(f_1-f_2)y + (c_1-c_2) = 0$$

6. System of Circles

6.1 Family of Circles

The equation of family of circles passing through the intersection of two circles $S_1 = 0$ and $S_2 = 0$ is: $$S_1 + \lambda S_2 = 0$$ where λ is a parameter.

6.2 Orthogonal Circles

Two circles are orthogonal if they intersect at right angles. For circles with centers (h_1,k_1), (h_2,k_2) and radii r_1, r_2: $$(h_1-h_2)^2 + (k_1-k_2)^2 = r_1^2 + r_2^2$$

7. Special Cases and Applications

7.1 Circle through Three Points

For points (x_1,y_1), (x_2,y_2), (x_3,y_3):

$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ x_1^2+y_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0$$

7.2 Coaxial Circles

Family of circles whose centers lie on a straight line and have a common radical axis: $$x^2 + y^2 + 2gx + 2fy + c + \lambda(x^2 + y^2 + 2g'x + 2f'y + c') = 0$$

8. Important Theorems

1. **Apollonian Circles**: The locus of points whose distances from two fixed points are in a constant ratio.

2. **Angle Properties**: The angle between two circles is equal to the angle between their tangents at their point of intersection.

Practice Problems

1. Find the equation of circle passing through $(0,0)$, $(a,0)$, $(0,b)$.
2. Prove that the radical axis of two circles is perpendicular to the line joining their centers.
3. Find the condition for two circles to be orthogonal.

Note: This document covers the essential aspects of coordinate geometry of circles. For deeper understanding, practice solving problems involving these concepts.