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DOMAIN DECOMPOSITION METHODS FOR PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT

Domain decomposition methods are iterative methods for the solution of linear or nonlinear systems that use explicit information about the geometry, discretization, and/or partial differential equation (PDE) that gave rise to the (non)linear system. A large amount of research in domain decomposition methods for partial differential equations has been carried out in the past dozen years. Recently, these techniques have begun to be applied to "real-world" engineering problems. This summary introduces the basic ideas in domain decomposition methods for PDEs. Though no particular applications are discussed, references to several recent uses of domain decomposition are given.

1. Introduction

Domain decomposition methods are parallel, potentially fast, robust algorithms for the solution of the linear (or nonlinear) equations that arise from discretizations of partial differential equations (PDEs). Some of the motivations for the use of these methods include (1) potential for efficient parallelization through the use of data locality, (2) ability to deal with PDEs on complicated physical geometries, (3) ability to deal with PDEs that exhibit different behavior on different parts of the domain (heterogeneous operators), and (4) superior convergence properties of the iterative method even on sequential machines.

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The linear (or nonlinear) systems that arise from the discretization of PDEs inherit many algebraic properties from properties of the underlying PDE. By understanding and using these relationships it is possible to derive fast linear (and nonlinear) solvers.

For linear problems it is customary to view domain decomposition methods as preconditioners for Krylov subspace methods such as the conjugate gradient method or GMRES; see, for instance, the chapter in this volume, *Linear System Solvers: Sparse Iterative Methods*, by Chan and Van der Vorst. Rather than discuss the theory of preconditioners we simply state that for the solution of the linear system

$$Au = f,$$

the application of a preconditioner B should approximate the action of A^{-1} well and should be inexpensive (and parallel) to apply.

The most comprehensive reference to domain decomposition methods is *Domain Decomposition: Parallel Multilevel Methods for Elliptic Partial Differential Equations*, by Smith, Bjørstad, and Gropp, (1995). A recent eighty-page survey of domain decomposition methods may be found in Chan and Mathew (1994). Two other limited surveys are on implementation issues by Gropp and Smith (1995) and non-self-adjoint problems by Cai (1994). The proceedings of the first seven international conferences on domain decomposition methods contain reports on a wide variety of applications and techniques (Chan, Glowinski, et al., 1989, 1990; Keyes, Chan, et al., 1992; Glowinski, Golub, et al., 1988; Glowinski, Kuznetsov, et al., 1991; Quarteroni, Periaux, et al., 1994, Keyes and Xu, 1995).

2. Background and Model Problems

Domain decomposition algorithms may be applied to a variety of partial differential equations. To simplify the presentation, however, we restrict attention to linear, second-order, elliptic PDEs,

$$\begin{aligned} L\boldsymbol{u} &= \boldsymbol{f} && \text{in } \Omega, \\ B\boldsymbol{u} &= \boldsymbol{g} && \text{on } \partial\Omega. \end{aligned}$$

A wide variety of discretizations may be applied. Again, to simplify notations, we assume that the problem is to be discretized with conforming finite elements; finite differences, spectral methods, or finite

volume methods may also be used. Domain decomposition methods may be applied equally well on structured or unstructured grids.

The variational form of the PDE may be written as: find $\boldsymbol{u} \in V$ such that

$$a(\boldsymbol{u}, \boldsymbol{v}) = F(\boldsymbol{v}) \qquad \forall \boldsymbol{v} \in V. \tag{1}$$

Here, $a(\boldsymbol{u}, \boldsymbol{v})$ is a bilinear form, while $F(\boldsymbol{v})$ is linear in $\boldsymbol{v}$. The space V is a suitable function space.

For instance, for the homogeneous Dirichlet boundary value problem

$$-\triangle \boldsymbol{u} = \boldsymbol{f},$$

$V = H_0^1(\Omega)$, while $a(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \nabla \boldsymbol{u} \nabla \boldsymbol{v}$ and $F(\boldsymbol{v}) = \int_\Omega \boldsymbol{f}\boldsymbol{v}$.

If we use $\{\phi_i\}_{i=1}^n$ to denote the finite element basis functions and let $V^h = \text{span}\{\phi_i\}$, then the finite-dimensional variational (finite element) problem may be given as: find $\boldsymbol{u}^h = \sum_{i=1}^n u_i \phi_i \in V^h$ such that

$$a(\boldsymbol{u}^h, \boldsymbol{v}^h) = F(\boldsymbol{v}^h) \qquad \forall \boldsymbol{v}^h \in V^h.$$

This is equivalent to the linear system

$$Au = f,$$

where $A_{ji} = a(\phi_i, \phi_j)$,

$$u = \begin{pmatrix} u_1 \\ \cdots \\ u_n \end{pmatrix} \qquad \text{and} \qquad f = \begin{pmatrix} F(\phi_1) \\ \cdots \\ F(\phi_n) \end{pmatrix}.$$

To motivate the design of domain decomposition methods, we consider the important special case when the operator L is self-adjoint and uniformly elliptic. In this case, the matrix A is symmetric, positive definite and hence defines an inner product $(u, v)_A = u^T A v$ and a corresponding norm $||u||_A^2 = u^T A u$.

3. Overlapping Methods

Domain decomposition methods can be broadly classified as either overlapping or nonoverlapping methods. In this section we introduce the key ideas behind overlapping methods.

3.1. *Local corrections*

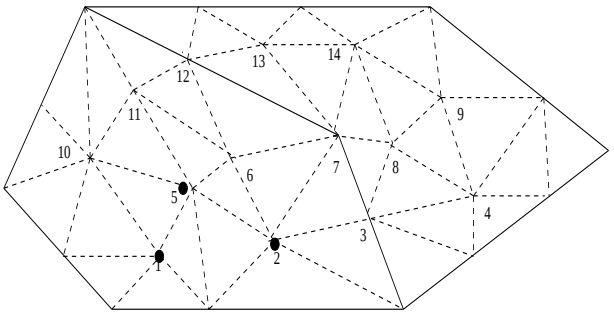

Figure 1: Determining a local correction

Consider the domain as depicted in Figure 1, and assume that a second-order, self-adjoint, uniformly elliptic PDE has been discretized by using piecewise linear finite elements on the given grid. If an approximate solution u is known, how may one improve the given solution by adjusting the values on the indicated nodes?

To quantify this question, we need to introduce some notation. Let R denote the matrix that when applied to the vector u returns only those values associated with the indicated nodes. For instance, for the nodes 1, 2, and 5, the matrix R is given by

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & 0 & \cdots \end{pmatrix}.$$

The transpose of R simply inserts the given values into the larger array:

$$\begin{pmatrix} w_1 \\ w_2 \\ 0 \\ 0 \\ w_3 \\ 0 \\ \cdots \end{pmatrix} = R^T \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}.$$

The matrix R is often referred to as the *restriction* operator, while R^T is the *interpolation* matrix.

Our "best" local correction is then defined by

$$\min_w ||u^* - (u^n + R^T w)||_A^2$$

or, equivalently,

$$\min_w (u^* - (u^n + R^T w))^T A (u^* - (u^n + R^T w)).$$

Here u^* is the exact solution to the linear system. If we take the derivative with respect to the unknowns w,

$$-RA(u^* - u^n - R^T w) = 0$$

or

$$RAR^T w = R(Au^* - Au^n).$$

Thus the "local" correction is given by

$$\begin{aligned} \text{correction} &= R^T w \\ &= R^T (RAR^T)^{-1} R(f - Au^n). \end{aligned}$$

The matrix RAR^T is simply the subblock of A associated with the given nodes.

Hence, the process of extracting out a logical block from the matrix A and solving the reduced linear system with respect to these unknowns is a *projection* of the error in the A norm. It turns out that, in addition, on the PDE side the computational process may be viewed as a projection of the error onto a subspace of the finite element space V^h.

In the more general case when A is not symmetric, positive definite, the corrections are no longer orthogonal projections of the error. However, they may still have certain desirable qualities.

3.2. *Classical block methods*

Using this interpretation, we can see that the classical block Jacobi method calculates a correction by simultaneously (and hence potentially in parallel) projecting the error onto a sequence of subspaces. For instance, if we use three blocks, the block Jacobi preconditioner is simply

$$\begin{aligned} B &= \begin{pmatrix} A_1^{-1} & 0 & 0 \\ 0 & A_2^{-1} & 0 \\ 0 & 0 & A_3^{-1} \end{pmatrix} \\ &= R_1^T A_1^{-1} R_1 + R_2^T A_2^{-1} R_2 + R_3^T A_3^{-1} R_3, \end{aligned}$$

where R_i denotes the restriction operator with respect to the ith block of unknowns.

In the classical block Gauss-Seidel algorithm one performs the projections sequentially, using the latest information for each update,

$$B = \begin{pmatrix} A_1 & 0 & 0 \\ A_{21} & A_2 & 0 \\ A_{31} & A_{32} & A_3 \end{pmatrix}^{-1}.$$

Using the R_i notation introduced above, B may be written in the slightly cumbersome way

$$B = (I - (I - R_3^T A_3^{-1} R_3 A)(I - R_2^T A_2^{-1} R_2 A)(I - R_1^T A_1^{-1} R_1 A))A^{-1}.$$

Though this explicit formula for B makes it appear that B is not easily computed, it turns out that B may be applied to a vector r using the algorithm

$$\begin{aligned} v &\leftarrow R_1^T A_1^{-1} R_1 r \\ v &\leftarrow v + R_2^T A_2^{-1} R_2 (r - Av) \\ v &\leftarrow v + R_3^T A_3^{-1} R_3 (r - Av). \end{aligned}$$

3.3. *One-level overlapping Schwarz methods*

The domain decomposition overlapping Schwarz methods may be viewed as generalizations of block Jacobi and Gauss-Seidel preconditioners where the blocking is determined from the geometric relationship between the unknowns rather than merely from the order that they happen to stored in the vectors, as is often the case for classical block methods.

In the additive Schwarz methods the domain is first partitioned into overlapping regions, each of which contains roughly the same number of unknowns (see Figure 2). The additive Schwarz preconditioner is given by

$$B = \sum_{i=1}^{p} R_i^T A_i^{-1} R_i.$$

The restriction operator R_i simply returns the coefficients associated with overlap region number i. Typically, we have one overlapping region per processor.

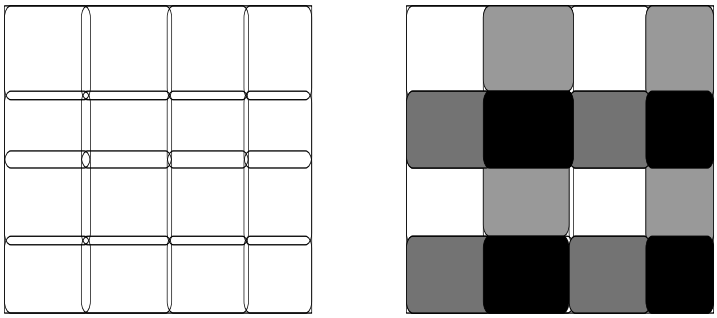

Figure 2: Overlap regions for Schwarz

The multiplicative Schwarz preconditioner first requires that the subdomains be colored so that no two overlapping regions have the same color. In our example in Figure 2 we require four colors. Corrections are then calculated simultaneously on different processors for each subdomain of a particular color. The application of preconditioner B to a vector r may be calculated by

$$\begin{aligned}
v &\leftarrow \sum_{i \in Color_1} R_i^T A_i^{-1} R_i r \\
v &\leftarrow v + \sum_{i \in Color_2} R_i^T A_i^{-1} R_i (r - Av) \\
v &\leftarrow v + \sum_{i \in Color_3} R_i^T A_i^{-1} R_i (r - Av) \\
v &\leftarrow v + \sum_{i \in Color_4} R_i^T A_i^{-1} R_i (r - Av).
\end{aligned}$$

The number of sequential steps is the number of colors. The number of parallel processors that may be used is given by the number of subdomains of a given color.

In the limit of no overlap between subdomains, these methods revert back to classical block Jacob and Gauss-Seidel methods. In the limit when each domain contains a single node, these are the point relaxation methods.

Determination of "good" subdomains, to ensure data locality, load balance, and low communication, can be viewed as a graph partitioning problem. See the chapter in this volume by A. Pothen and also Cai and Saad (1993). It is also possible to determine overlap by using graph information from the matrix rather than geometric

information. This approach is used in the PETSc package of Gropp and Smith (1993) and discussed in Cai and Saad (1993).

The use of overlapping Schwarz methods for nonlinear problems used in combination with Newton methods may be found in Cai, Gropp, Keyes, and Tidriri (1994). Valuable discussion may also be found in Bjørstad (1994).

3.4. *Multilevel overlapping Schwarz methods*

For elliptic PDEs there is an inherent need in the iterative method (or preconditioner) to provide for global communication of information at each iteration if a small number of iterations is desired. This communication may be achieved by calculating coarser grid approximate solutions to the PDE. The coarse grids, interpolation operators, and operators are constructed in a process similar to classical multigrid. A key difference is that in domain decomposition the ratio in mesh refinement between levels may be 10 or 100 while in multigrid it is usually 2 or 4.

If the matrix R^T represents interpolation from the coarse to the fine grid, then the *coarse grid correction* may be written as

$$R^T A_C^{-1} R r.$$

The operator A_C^{-1} represents (approximately) solving the coarse grid problem. The coarse grid problem may be solved by recursively applying the same algorithm.

Again, as with the local corrections, under certain circumstances, it can be shown that the coarse grid corrections are projections of the error onto subspaces of the solution space.

The multilevel overlapping Schwarz methods have very good convergence properties. In many circumstances the resulting iterative method has a convergence rate independent of the problem size and number of processors. The amount of work required per iteration (and hence the overall work needed in the solution process) obviously depends on how the local and coarse grid solves are performed. In many applications the use of an approximate solver on the subproblems is appropriate.

The classical V-cycle multigrid using either Gauss-Seidel or Jacobi smoothing can be interpreted as a special case of overlapping domain decomposition with small subdomains and minimal overlap between the subdomains.

For a discussion of parallel multigrid techniques see the chapter in this volume by Jones and McCormick.

Various strategies have been proposed for fast solution of problems with adaptive refinement include the FAC (Fast Adaptive Composite) and AFAC (Asynchronous FAC) grids of McCormick (1984, 1989), McCormick and Thomas (1986), Hart and McCormick (1987); see also Mandel and McCormick (1989) and the analysis of similar methods by Dryja and Widlund (1989). The approach of Bramble, Ewing, Pasciak, and Schatz (1988) and Bramble, Ewing, Parashkevov, and Pasciak (1990) is slightly different.

The original theory for overlapping Schwarz methods may be traced back to Schwarz (1890). See also Lions (1988), Dryja and Widlund (1987, 1989), Bramble, Pasciak, Wang, and Xu (1991). Multilevel versions have been studied by Dryja and Widlund (1991), Zhang (1992a, 1992b), and Griebel and Oswald (1995). Numerical studies of multilevel overlapping Schwarz methods may be found in Gropp and Smith (1994) and Bjørstad and Skogen (1992).

Recent work on multilevel overlapping Schwarz methods for unstructured grids has been performed by Cai (1995), Chan and Smith (1995) and Chan, Smith, and Zou (1994).

4. Nonoverlapping Methods

The nonoverlapping domain decomposition methods may also be viewed as combining projections of the error onto subspaces of the solution space. However, there is also a simple linear algebra interpretation, based on a reduced linear system, that we will adopt for this survey.

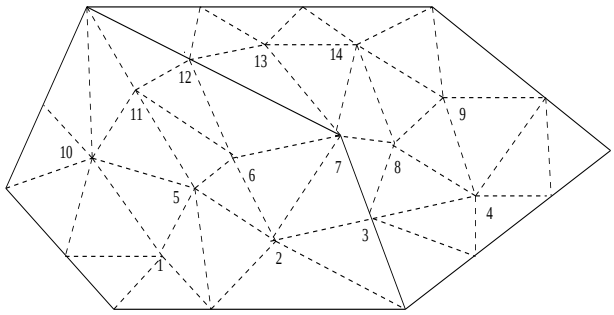

Figure 3: Two subdomains and an interface

Consider a domain divided into two nonoverlapping regions as depicted in Figure 3. We partition the unknowns into three sets: those in the first domain (denoted by $\bar{u}_1$ and containing nodes 1, 2, 5, 6, 10, 11), those in in the second domain, and those on the interface between the two domains (denoted by $\bar{u}_3$ and containing 3, 7, 12). Then the linear system may be written as

$$\begin{pmatrix} A_1 & 0 & A_{13} \\ 0 & A_2 & A_{23} \\ A_{31} & A_{32} & A_3 \end{pmatrix} \begin{pmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \bar{u}_3 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix},$$

where

$$A_3 = A_3^{(1)} + A_3^{(2)}$$

contains contributions from both sides of the interface. The key point is that nodes in the first subdomain are completely decoupled from nodes in the second subdomain. Once the unknowns in Ω_1 and Ω_2 have been eliminated, the resulting *Schur complement* system is given by

$$\begin{aligned} S\bar{u}_3 &= (S^{(1)} + S^{(2)})\bar{u}_3 \\ &= (A_3^{(1)} - A_{31}A_1^{-1}A_{13} + A_3^{(2)} - A_{32}A_2^{-1}A_{23})\bar{u}_3 \qquad (2) \\ &= f_3 - A_{31}A_1^{-1}f_1 - A_{32}A_2^{-1}f_2 \\ &= g_3. \end{aligned}$$

This is easily generalized to any number of subdomains.

4.1. *Substructuring*

Substructuring is an efficient, parallel *direct* method for the solution of linear systems arising from discretizations of PDEs based on Schur complements. We examine this for a small problem; see Figure 4.

In this particular example the elimination is done in three levels.

- Level 1: The four groups 1, 2 and 3, 4 and 7 and 8, 9, 10 are eliminated in parallel.
- Level 2: The two groups 5, 6 and 11 are eliminated in parallel.
- Level 3: The final Schur complement involving 12, 13, 14 is eliminated.

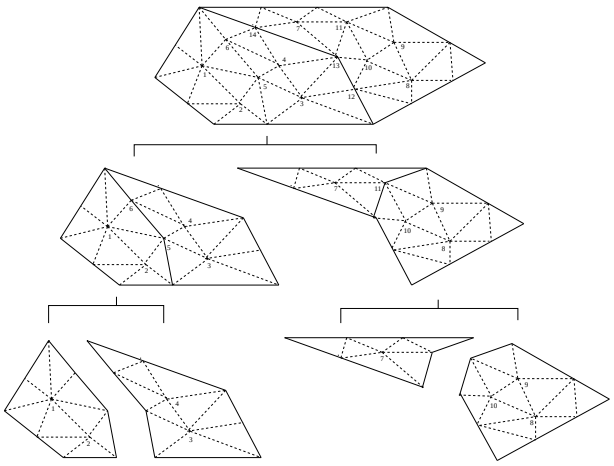

Figure 4: Example of substructuring

Once the factorization is complete, the unknowns may be calculated.

- Level 3: Solve for u_{12}, u_{13}, u_{14}.
- Level 2: Solve for u_5, u_6 and u_{11} in parallel.
- Level 1: Solve for u_1, u_2 and u_3, u_4 and u_7 and u_8, u_9, u_{10} in parallel.

In actual engineering codes much larger groups of unknowns are usually eliminated in the *static condensation* process. An introduction to substructuring from the structural engineering point of view may be found in Przemieniecki (1963, 1985). A modern discussion of the parallelization of a commercial substructuring code may be found in Hvidsten (1990). The ordering induced by this elimination process is essentially the nested dissection ordering; see George (1973) and George and Liu (1981).

4.2. *Iterative substructuring*

Iterative substructuring is a branch of domain decomposition that attempts to design preconditioners for the Schur complement linear systems such as (2) that do not require the explicit creation of the matrices S.

In this survey we discuss three such preconditioners: the Neumann-Dirichlet, the Neumann-Neumann, and lastly the balancing Neumann-

Neumann. A survey of many other iterative substructuring methods may be found in Dryja, Smith, and Widlund (1993).

4.2.1. Neumann-Dirichlet preconditioner

When subdomains Ω_1 and Ω_2 are of roughly equal size and shape, one would expect that $S^{(1)}$ and $S^{(2)}$ would have similar properties and hence one would hope that either ${S^{(1)}}^{-1}$ or ${S^{(2)}}^{-1}$ would be an effective preconditioner for $S = S^{(1)} + S^{(2)}$. This is, in fact, the case. In the *Neumann-Dirichlet* preconditioner one preconditions with ${S^{(1)}}^{-1}$ resulting in

$$(S^{(1)} + S^{(2)}){S^{(1)}}^{-1} = I + S^{(2)}{S^{(1)}}^{-1}.$$

The application of ${S^{(1)}}^{-1}$ to a vector can be achieved by noting the following formula,

$$ {S^{(1)}}^{-1} = \begin{pmatrix} 0 & I \end{pmatrix} \begin{pmatrix} A_{11} & A_{13} \\ A_{31} & A_{33}^{(1)} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ I \end{pmatrix}. $$

The center term on the left-hand side corresponds to solving a subproblem with Neumann boundary conditions on the artificial boundary. The application of $S^{(2)}$ involves the application A_{22}^{-1}, which involves the solution of a subproblem with Dirichlet boundary conditions on the artificial boundary; see (2). The Neumann-Dirichlet preconditioner is discussed, for instance, in Bjørstad and Widlund (1986).

For two subdomains the Neumann-Dirichlet preconditioner obviously offers little chance for parallelization; however, if the domain is divided into strips, the "odd" strips can be solved with Neumann boundary conditions on the artificial boundary, while for the "even" strips one may use Dirichlet values. Thus each processor can be assigned two adjacent strips.

The application of the Neumann-Dirichlet method to structural mechanics problems may be found in Bjørstad and Hvidsten (1988).

4.2.2. Neumann-Neumann method

In the Neumann-Neumann preconditioner a Dirichlet boundary value problem and a Neumann boundary value problem are each solved on each subdomain. For two subdomains this amounts to preconditioning S by ${S^{(1)}}^{-1} + {S^{(2)}}^{-1}$.

This approach may easily be extended to any number of processors; each processor would be assigned one subdomain. The drawback is that for subdomains that are completely in the interior of the domain the resulting Neumann boundary value problem that must be solved is singular. The standard way to solve this is to add a zero*th* order term to the Neumann boundary value problem to make it nonsingular.

As with the overlapping methods the standard Neumann-Neumann preconditioner converges slowly for large numbers of subdomains, since there is no global communication in the preconditioner. The Neumann-Neumann for two subdomains appeared in Bourgat et al. (1989). It was extended to several subdomains in De Roeck and Le Tallec (1991).

4.2.3. Balancing

The *balancing Neumann-Neumann method* uses a piecewise constant coarse grid correction to provide for global communication and is an almost optimal preconditioner. Balancing was introduced in Mandel (1992) and Mandel and Brezina (1992).

The convergence rate of the balancing Neumann-Neumann method is quite good. The convergence depends only mildly on the ratio of the substructure size to the discretization size. The standard implementation, however, requires that a Dirichlet boundary value problem and a Neumann boundary value problem be solved exactly for each subdomain at each iteration. This is a potentially expensive operation and is one drawback of the balancing algorithm.

The Neumann-Neumann methods have been successfully applied to structural mechanics problems by Le Tallec, De Roeck and Vidrascu (1991). Cowser, Mandel, and Wheeler (1993) have applied the balancing Neumann-Neumann method to oil reservoir simulations using mixed finite elements. A detailed mathematical analysis of related methods may be found in Dryja and Widlund (1993).

4.2.4. Coupling PDEs of different character

In certain situations it may be desirable to model a physical phenomenon by using PDEs of different basic character (for instance elliptic and hyperbolic in transonic flow) in different regions of the domain. Though this can be handled by using the overlapping Schwarz

methods (see, for instance, Cai, Gropp, Keyes, and Tidriri (1994)), it is more commonly dealt with by using nonoverlapping subdomains and special *transmission conditions* across the artificial boundaries. A simple introduction to these ideas may be found in Chan and Mathew (1994). Discussions of coupling Boltzmann and Euler equations may be found in Bourgat, Le Tallec, Perthame, and Qiu (1994). Many other papers dealing with this important topic may be found in the proceedings of the first seven international conferences on domain decomposition mentioned in the introduction.

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