The Method of Characteristics

2D Isentropic Compressible Flow

IRROT.

$$\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=0$$

MASS+MOMENTUM

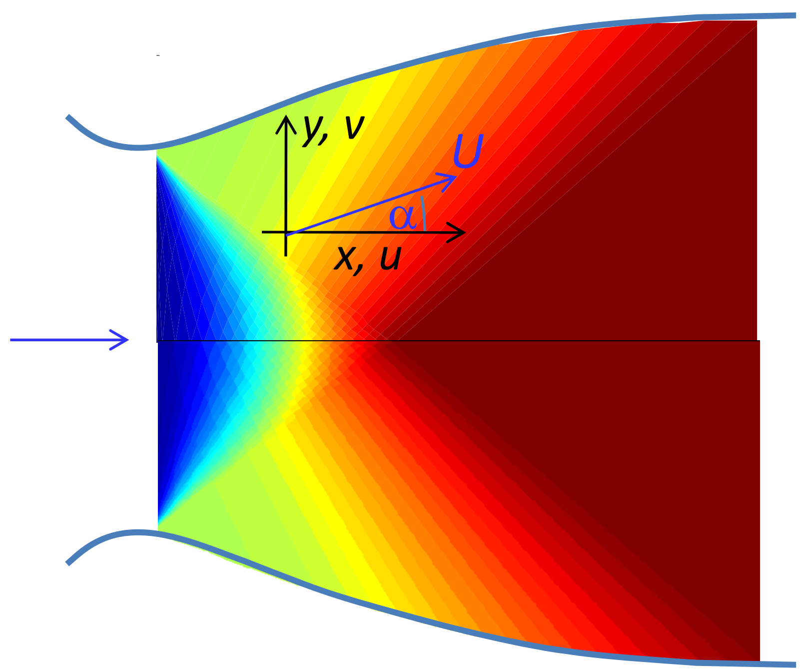
$$(1-\frac{u^2}{a^2})\frac{\partial u}{\partial x}+(1-\frac{v^2}{a^2})\frac{\partial v}{\partial y}-\frac{2uv}{a^2}\frac{\partial v}{\partial x}=0$$

ORGANIZE SO AS TO SOLVE FOR THE VELOCITY DERIVATIVES…

Partial Differential Eqns.?

TO MATRIX SOLVE FOR VELOCITY DERIVATIVES…

$$
\begin{aligned}
&(1-\frac{u^2}{a^2})\frac{\partial u}{\partial x} &&+(1-\frac{v^2}{a^2})\frac{\partial v}{\partial y} &&-\frac{2uv}{a^2}\frac{\partial v}{\partial x} &&=0\\
& &&dy\frac{\partial v}{\partial y} &&+dx\frac{\partial v}{\partial x} &&=dv\\
&dx\frac{\partial u}{\partial x} && &&+dy\frac{\partial v}{\partial x} &&=du
\end{aligned}
$$



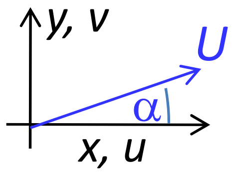
Mach 2 Nozzle Flow

Matrix Solution (Cramer's Rule)

$$\begin{pmatrix} (1-u^2/a^2) & (1-v^2/a^2) & -2uv/a^2 \\ 0 & dy & dx \\ dx & 0 & dy \end{pmatrix} \begin{pmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \partial v/\partial x \end{pmatrix} = \begin{matrix} 0 \\ dv \\ du \end{matrix}$$

SOLVING FOR $\partial v/\partial x$...

$$\partial v/\partial x = \frac{\begin{vmatrix} (1-u^2/a^2) & (1-v^2/a^2) & 0 \\ 0 & dy & dv \\ dx & 0 & du \end{vmatrix}}{\begin{vmatrix} (1-u^2/a^2) & (1-v^2/a^2) & -2uv/a^2 \\ 0 & dy & dx \\ dx & 0 & dy \end{vmatrix}} = \frac{N}{D}$$



Denominator

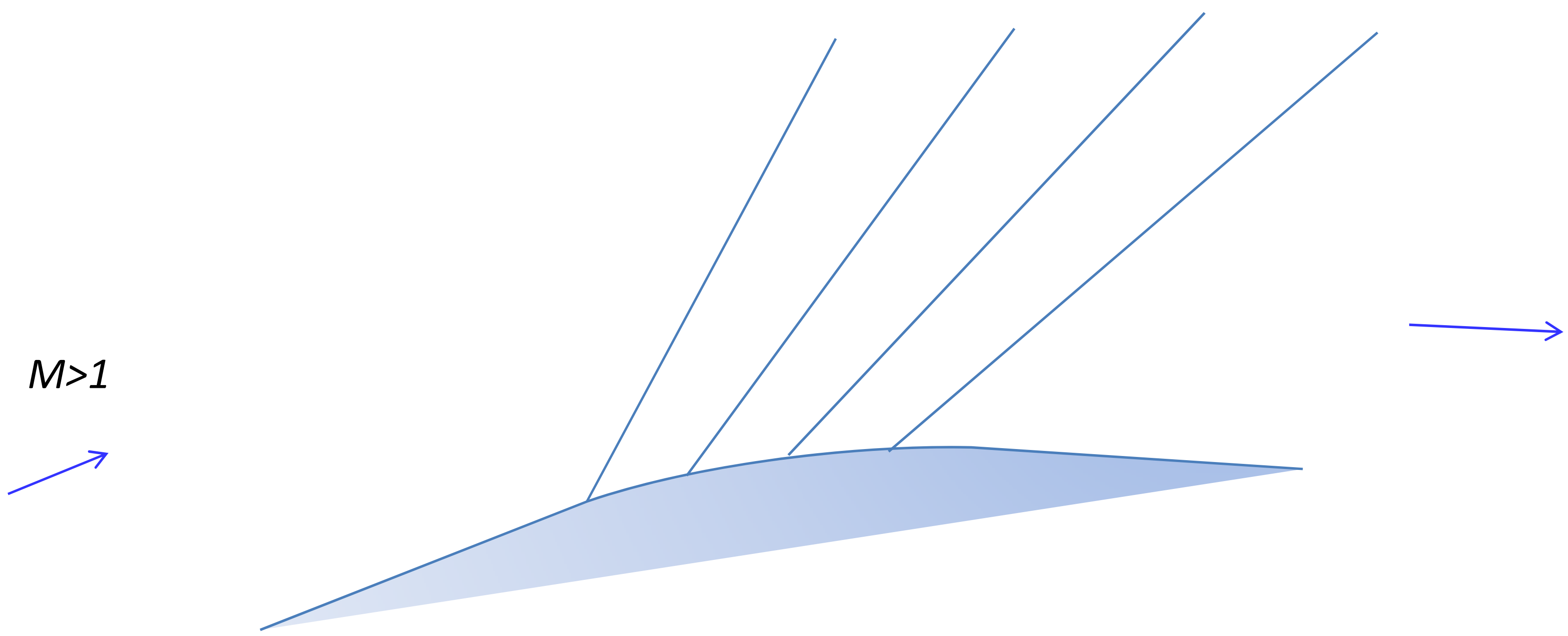
CHARACTERISTIC EQUATIONS

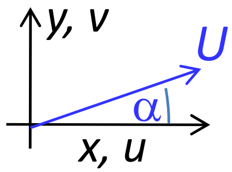
$$\begin{vmatrix} (1-u^2/a^2) & (1-v^2/a^2) & -2uv/a^2 \\ 0 & dy & dx \\ dx & 0 & dy \end{vmatrix} = D$$

Characteristic Lines

$$\frac{dy}{dx} = \tan(\alpha + \mu)$$

$$\frac{dy}{dx} = \tan(\alpha - \mu)$$



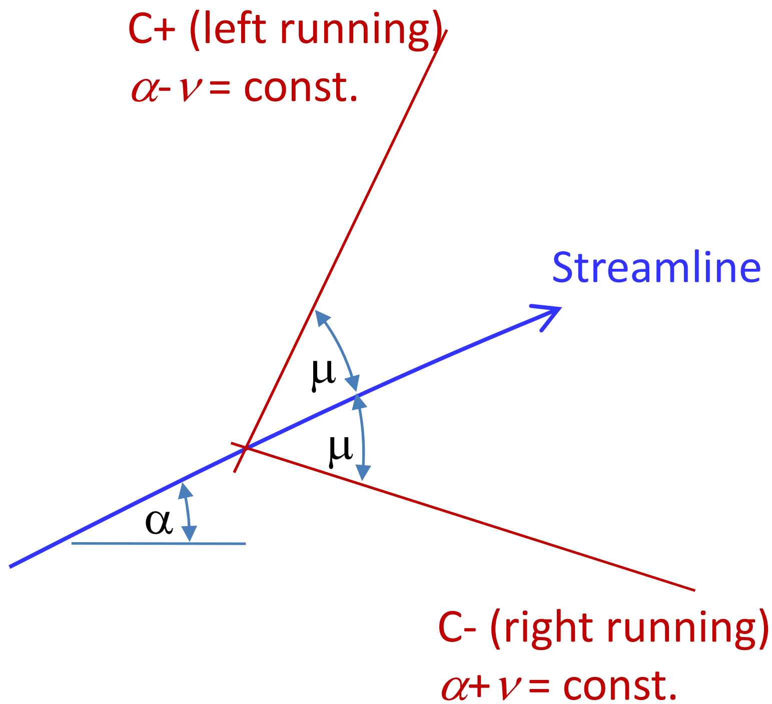


Numerator

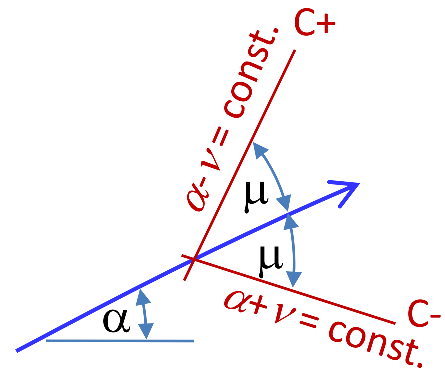
COMPATABILITY RELATIONS

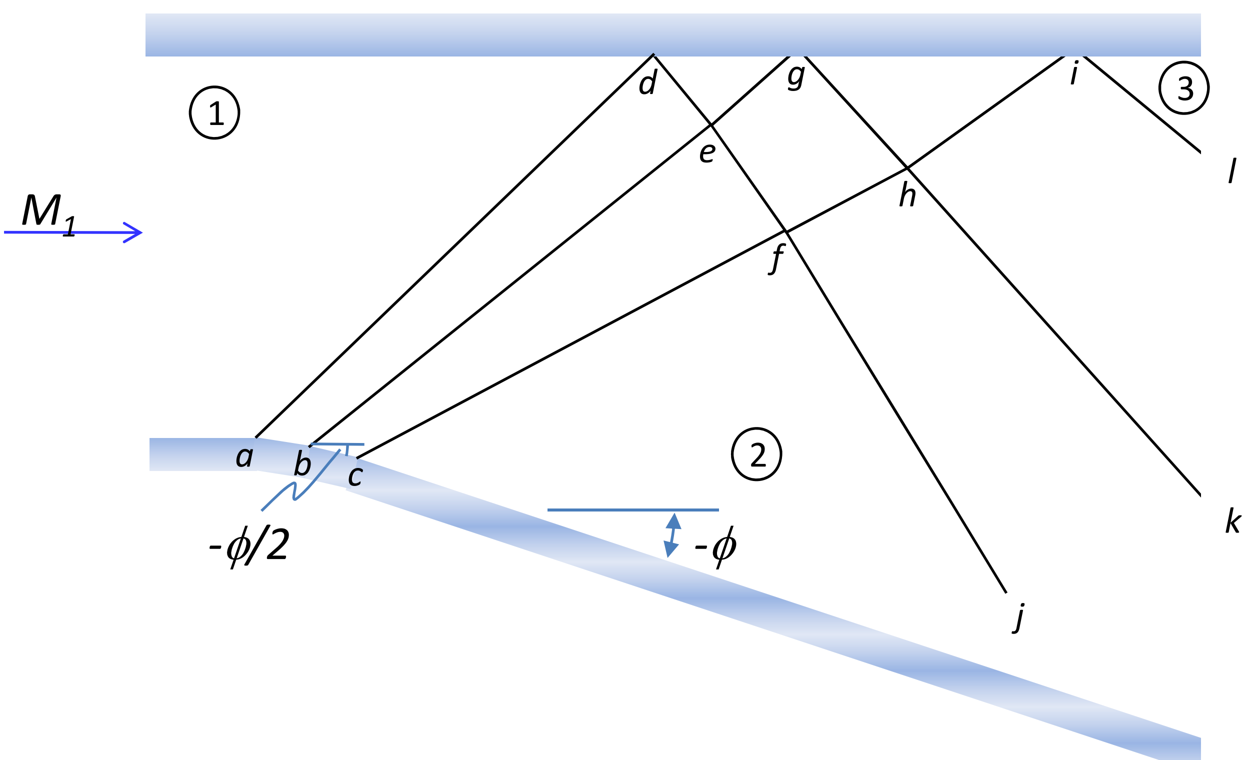
$$\frac{\begin{vmatrix} (1-u^2/a^2) & (1-v^2/a^2) & 0 \\ 0 & dy & dv \\ dx & 0 & du \end{vmatrix}}{\begin{vmatrix} (1-u^2/a^2) & (1-v^2/a^2) & -2uv/a^2 \\ 0 & dy & dx \\ dx & 0 & dy \end{vmatrix}} = \frac{N}{D}$$

Characteristics - Summary



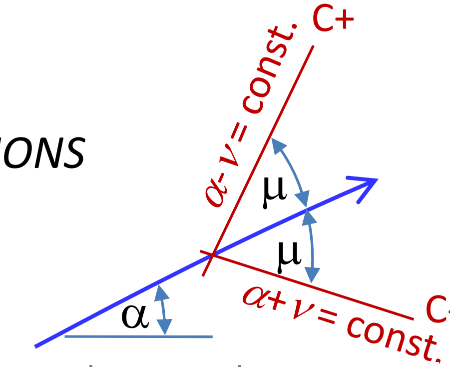
Reflected Expansion Fan



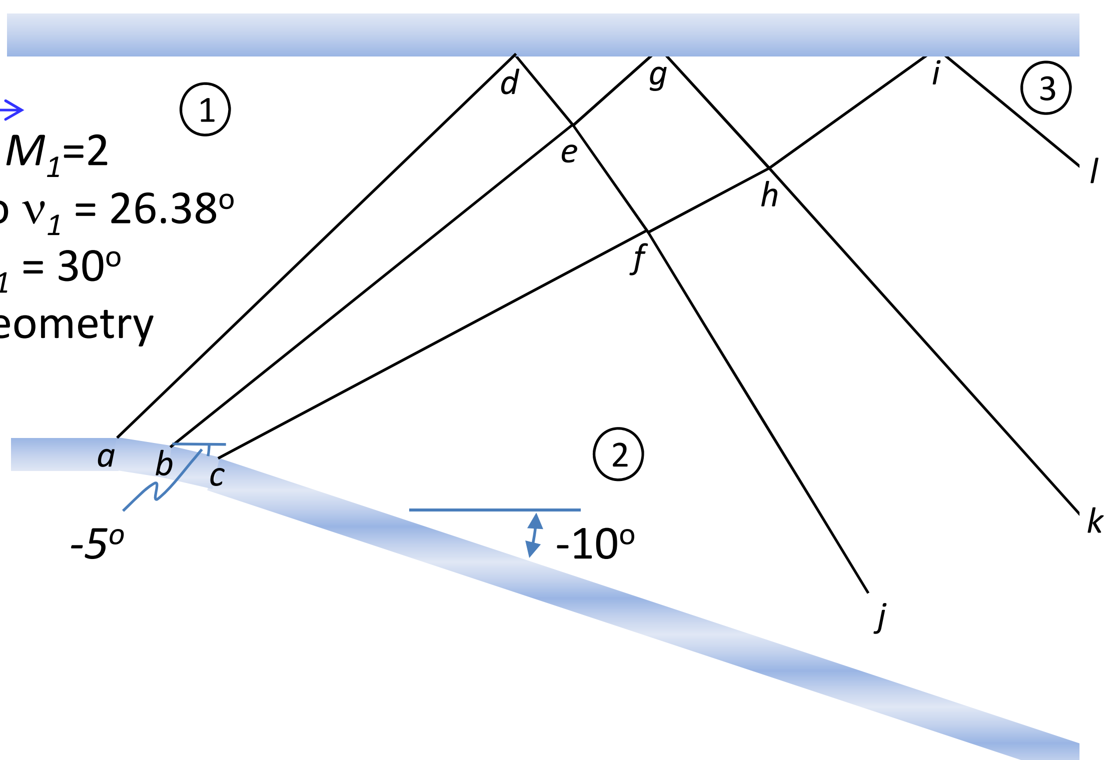


Reflected Expansion Fan

SOLVING FOR FLOW PROPERTIES IN THE UNIFORM AND SIMPLE REGIONS



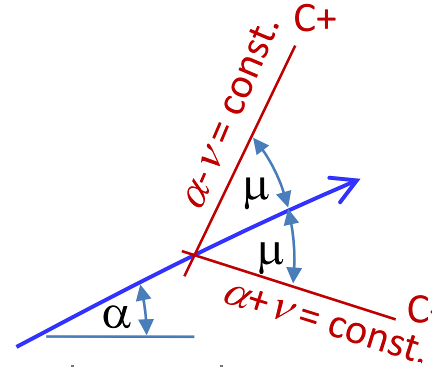
Given M_1=2
and so ν_1 = 26.38°
and μ_1 = 30°
and geometry



	α	ν
1		
2		
3		
a		
b		
c		
d		
e		
f		
g		
h		
i		

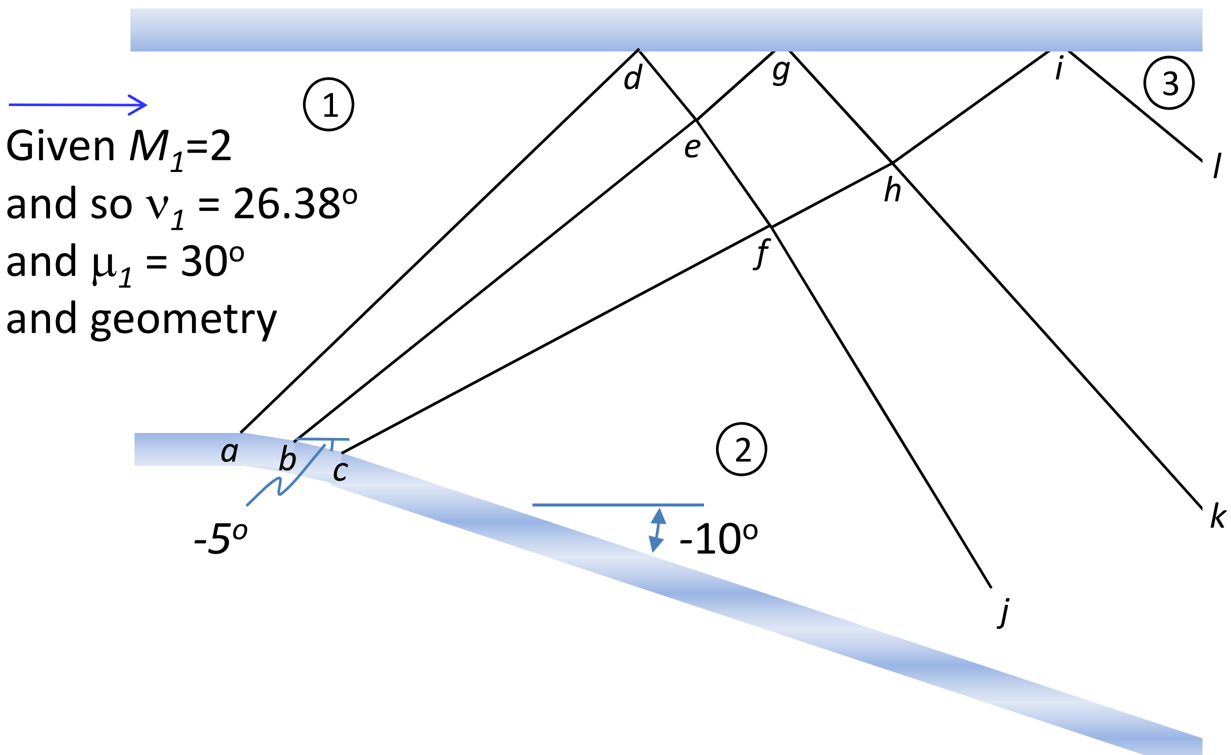
Reflected Expansion Fan

SOLVING FOR FLOW PROPERTIES IN THE COMPLEX REGION



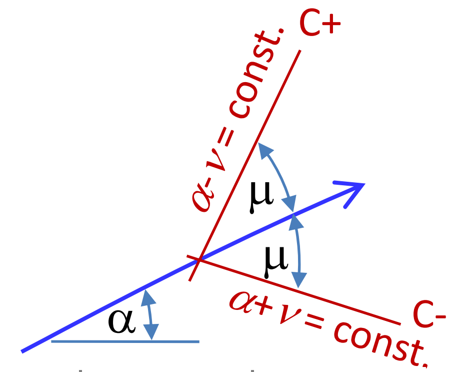
	α	ν
1		
2		
3		
a		
b		
c		
d		
e		
f		
g		
h		
i		

Given M_1=2
and so ν_1 = 26.38°
and μ_1 = 30°
and geometry



Reflected Expansion Fan

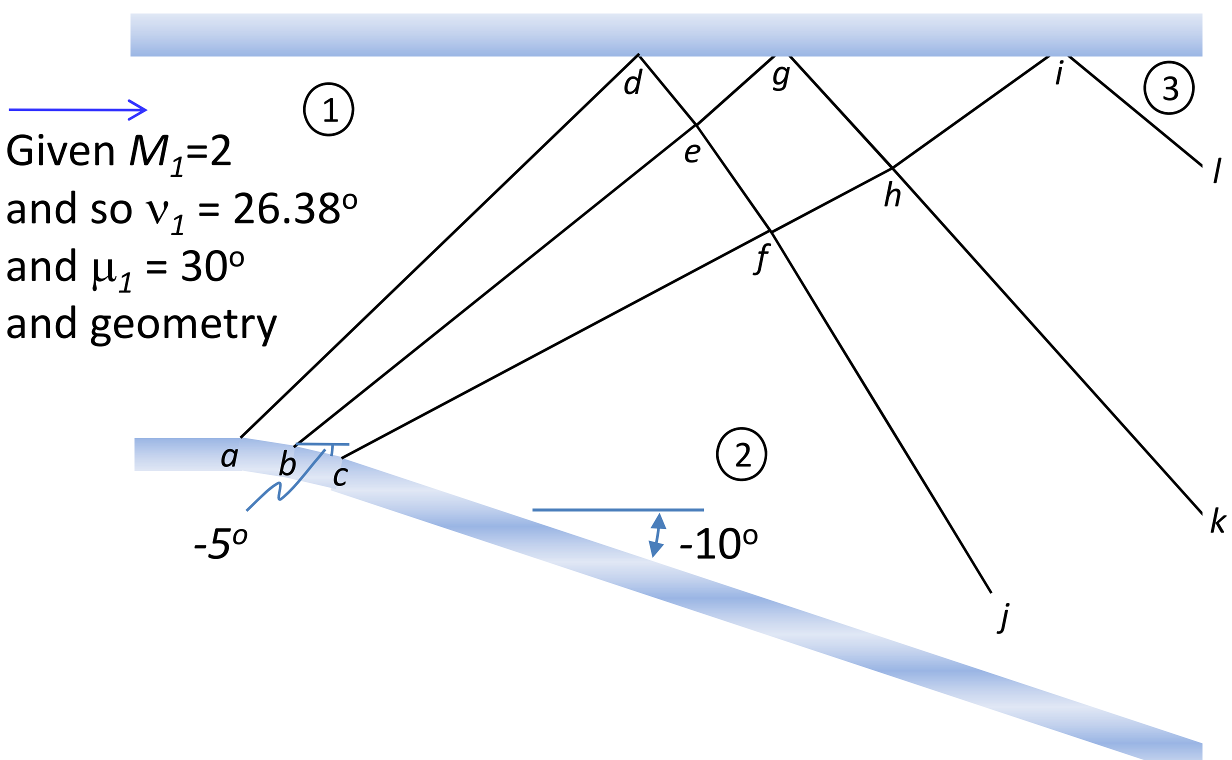
SOLVING FOR THE WAVE GEOMETRY



	α	ν	μ
1	0	26.4	30
2	-10	36.4	24.8
3	0	46.4	20.5
a	0	26.4	30
b	-5	31.4	27.2
c	-10	36.4	24.8
d	0	26.4	30
e	-5	36.4	24.8
f	-10	36.4	24.8
g	0	36.4	24.8
h	-5	41.4	22.6
i	0	46.4	20.5

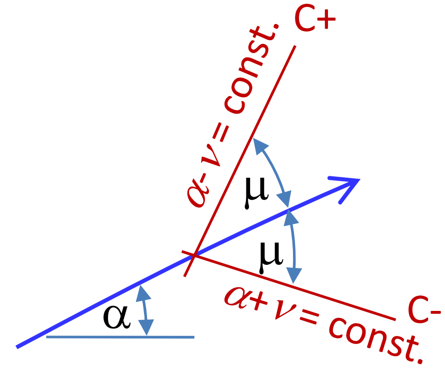
PowerPoint Solution

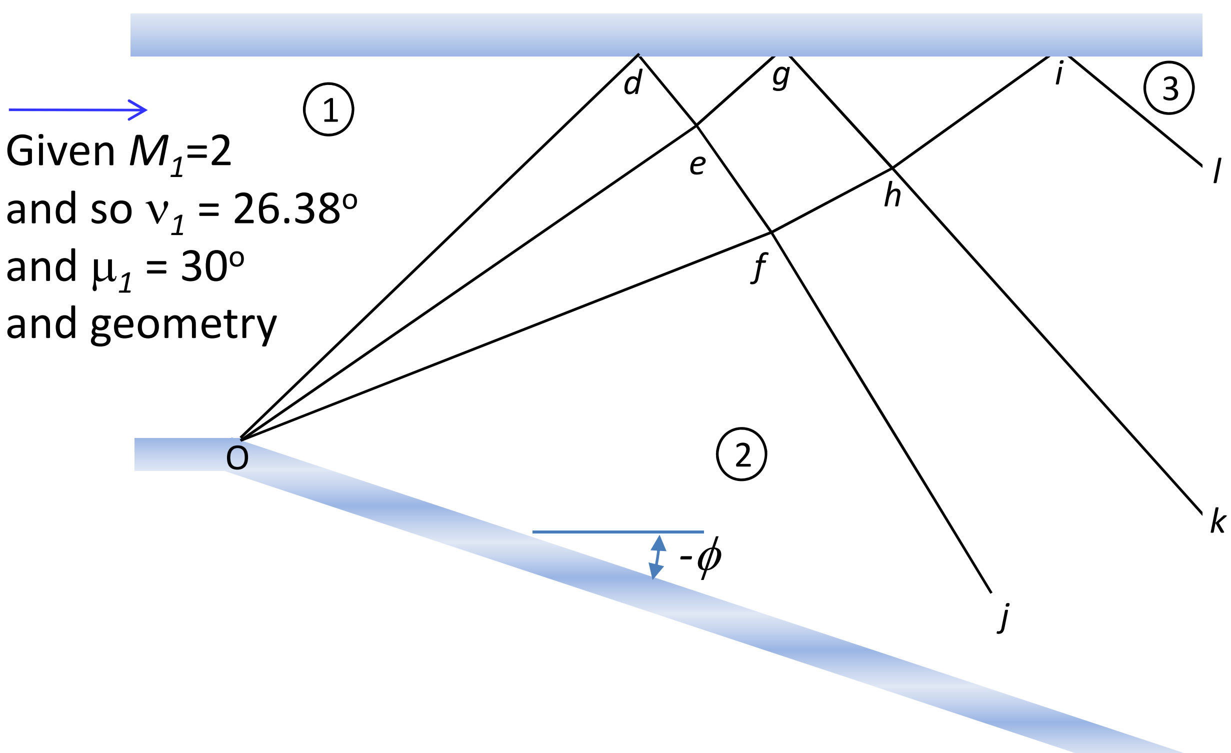
Given M_1=2
and so ν_1 = 26.38°
and μ_1 = 30°
and geometry



Variation

1. Wave originates at a sudden turn

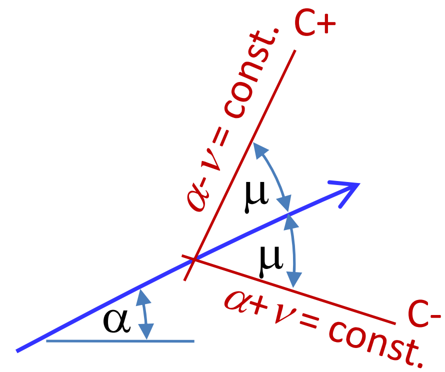


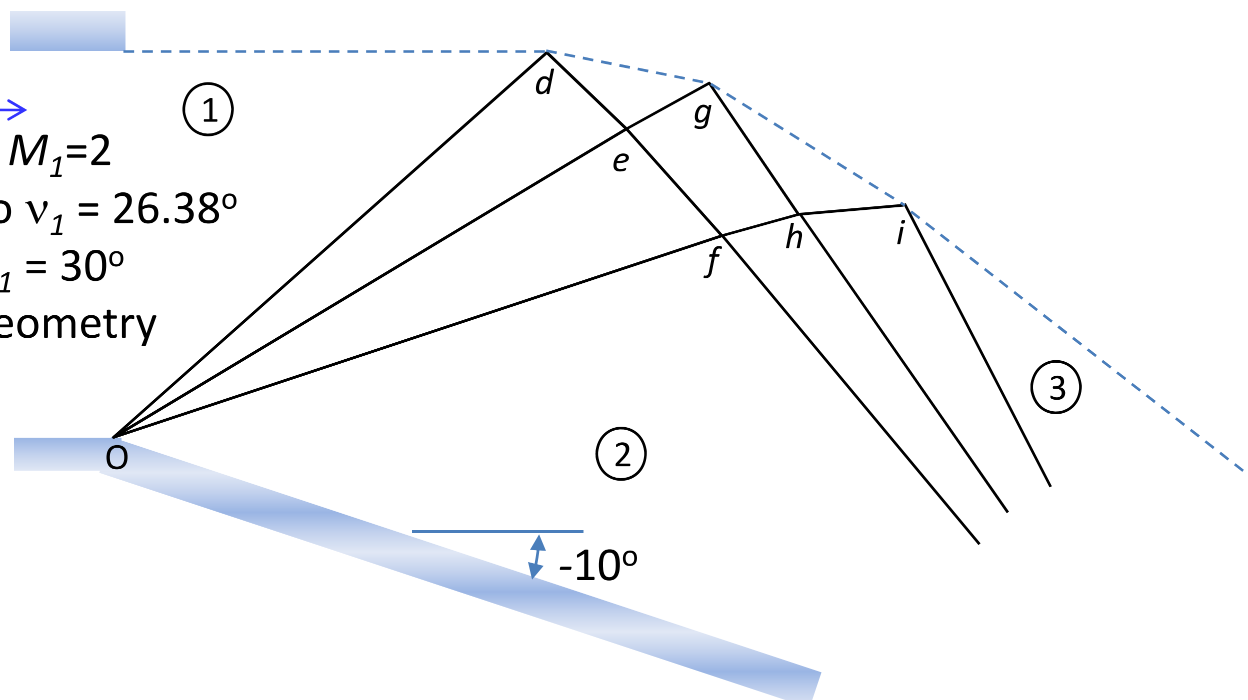


Given M_1=2
and so ν_1 = 26.38°
and μ_1 = 30°
and geometry

Variation

2. Wave reflects from a jet edge



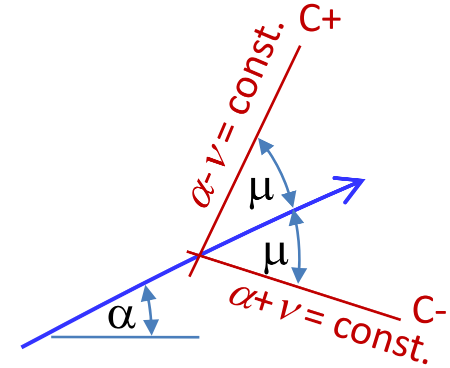


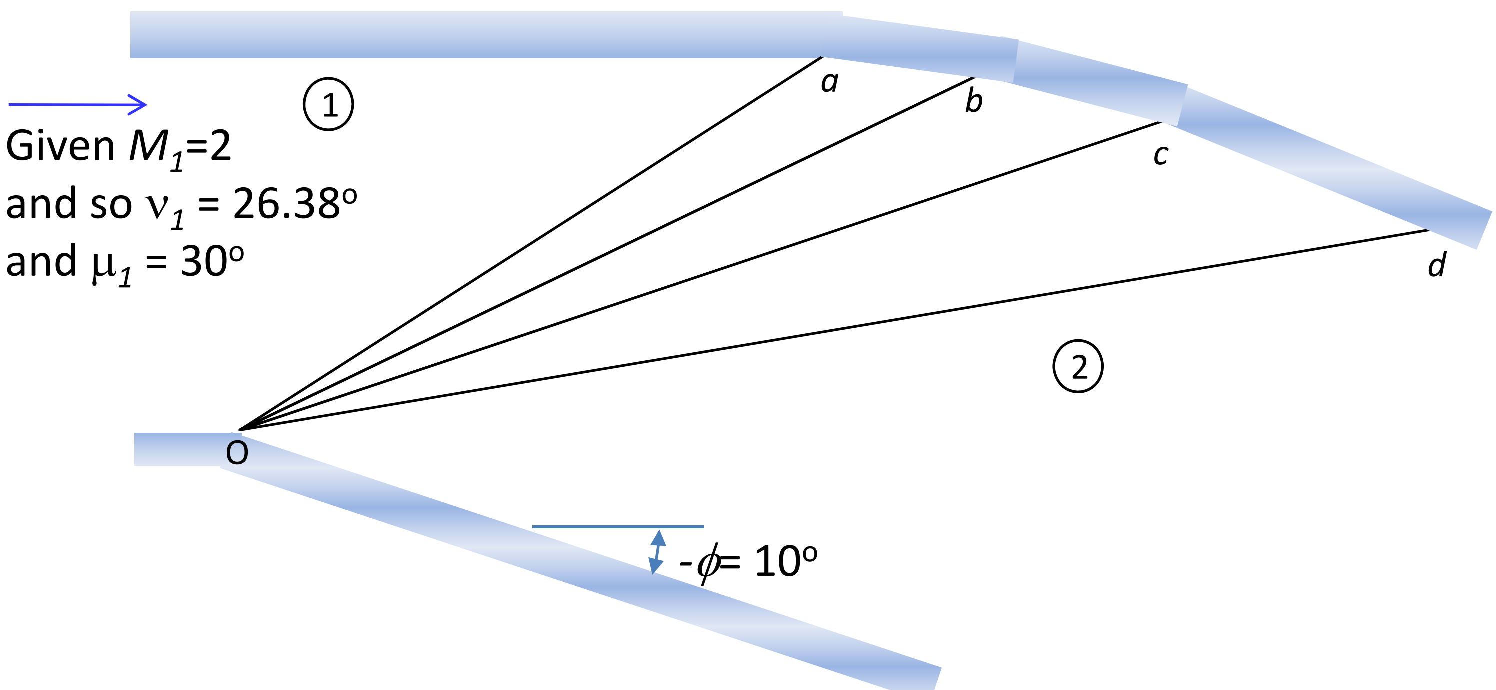
Given M_1=2
and so ν_1 = 26.38°
and μ_1 = 30°
and geometry

PowerPoint Solution

Variation

3. Wave cancelation at a wall





Design Applications

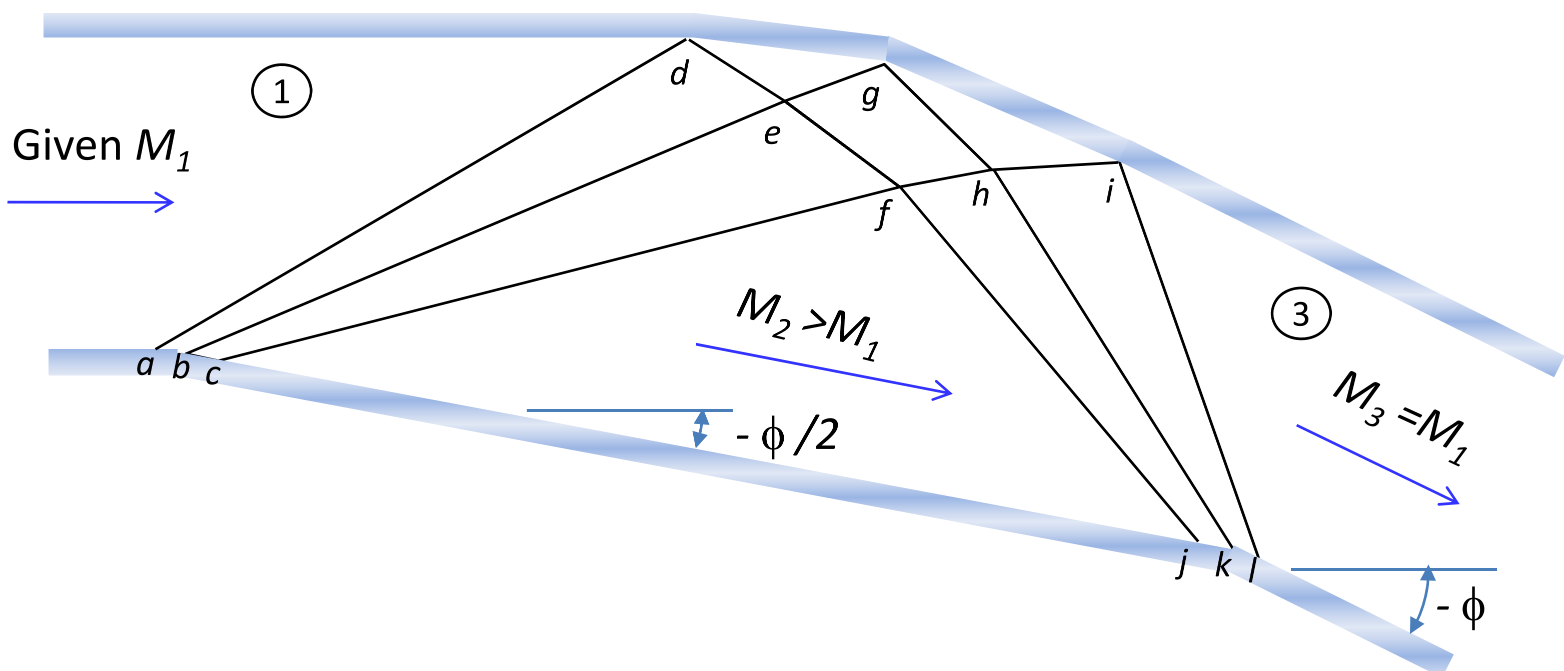
1. Constant Mach number turn
2. Wind Tunnel Type Nozzle
3. Rocket Type Nozzle

1. Constant Mach number turn

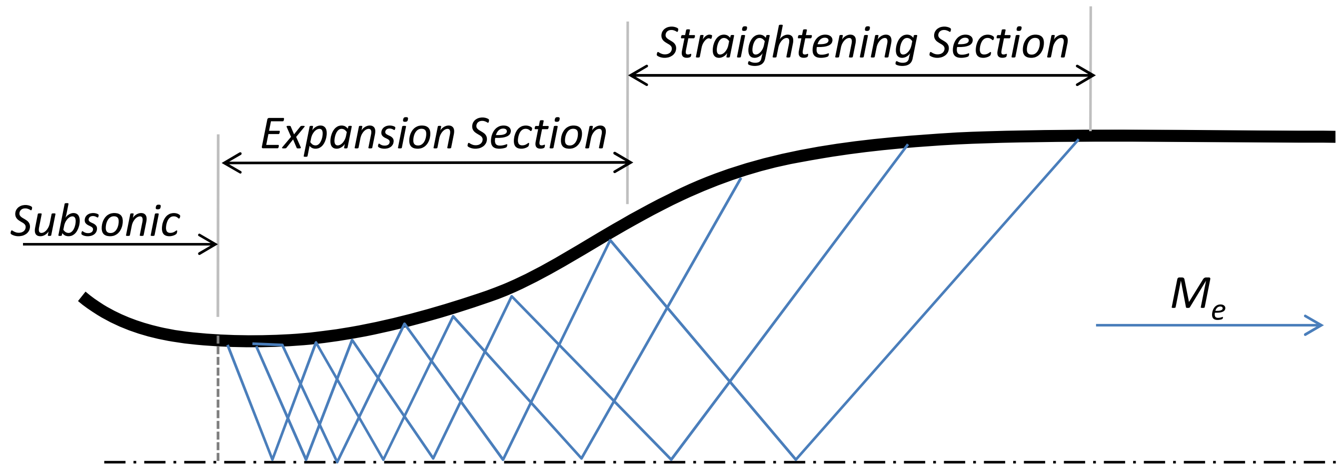
a. Choose initial turn geometry *abc* of angle $\phi/2$

b. Compute reflection of wave as though from a jet edge of constant Mach M_1. Shape of jet edge gives shape of upper wall.

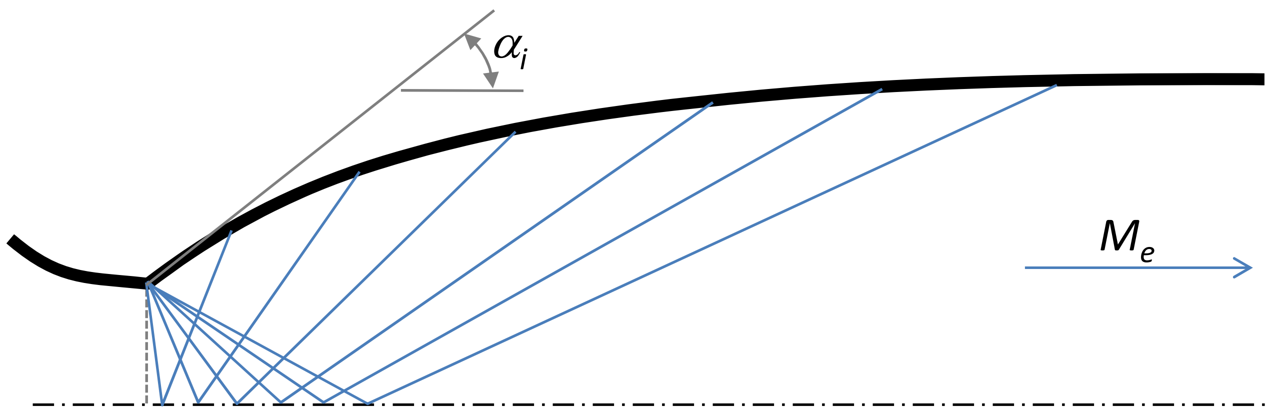
c. Choose lower wall shape to cancel reflected wave at *jkl*



2. Wind Tunnel-Type Nozzle



3. Rocket Engine-Type Nozzle



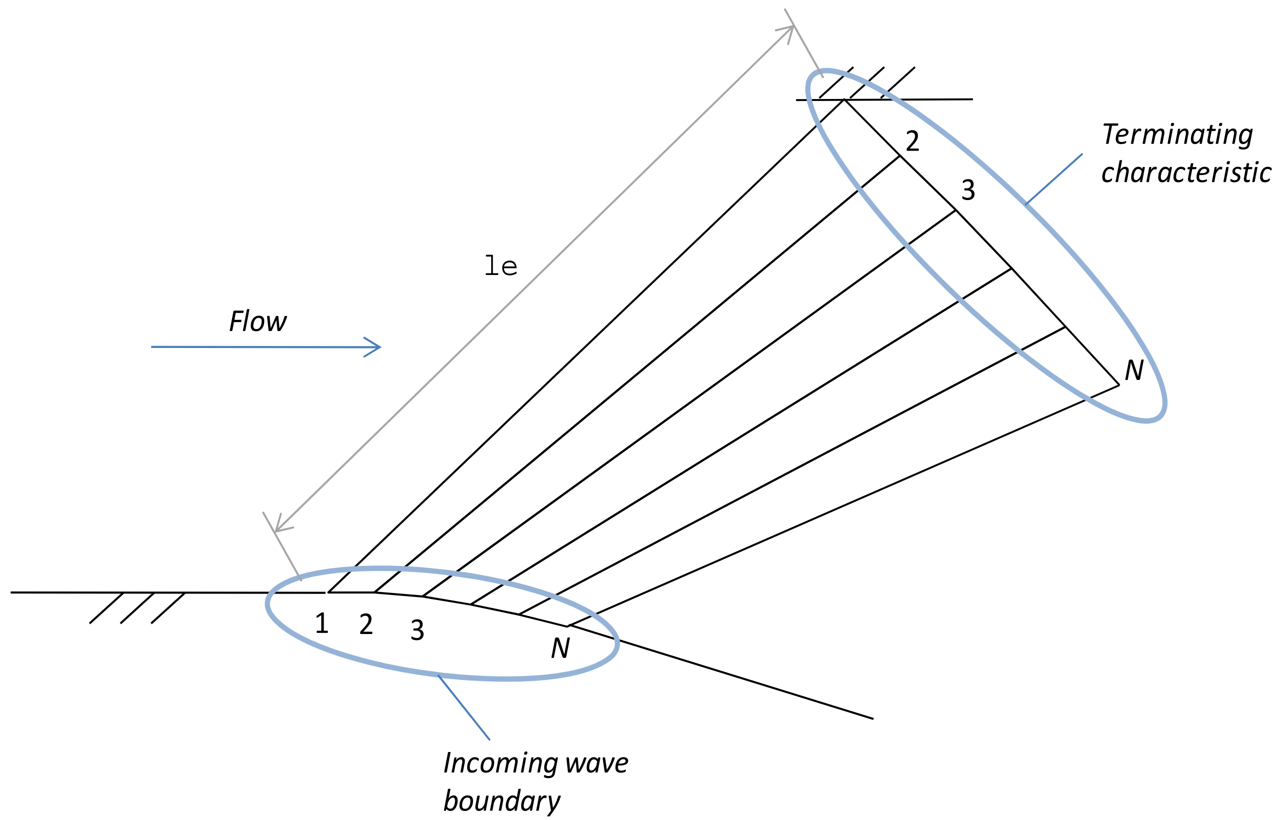
PowerPoint Example

Matlab Codes and Scripts for the Method of Characterisics

- Computing and Plotting Simple waves
- Computing and Plotting Complex Regions
- Other functions
- Application Examples

Basic Simple Wave

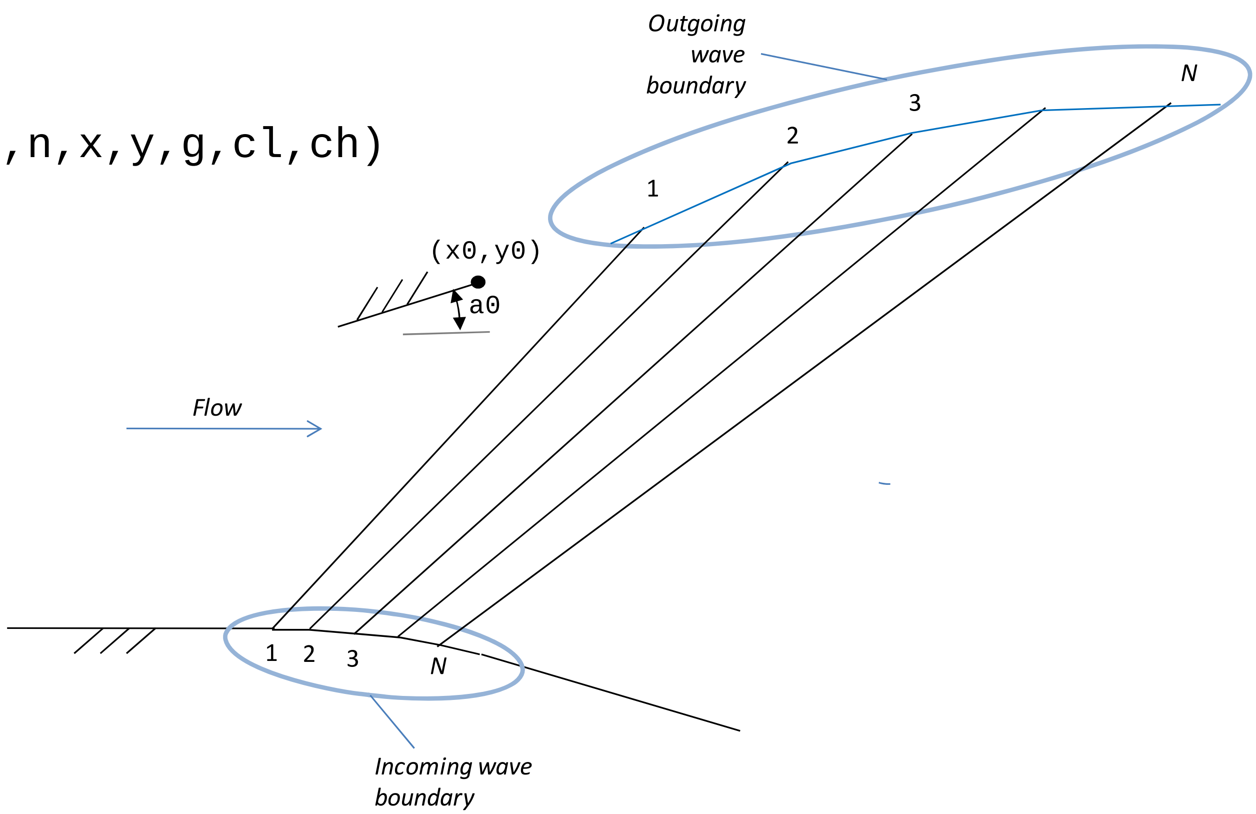
```
function [a,n,x,y]=simple(ai,ni,xi,yi,le,g)
```



Other Simple Wave Functions

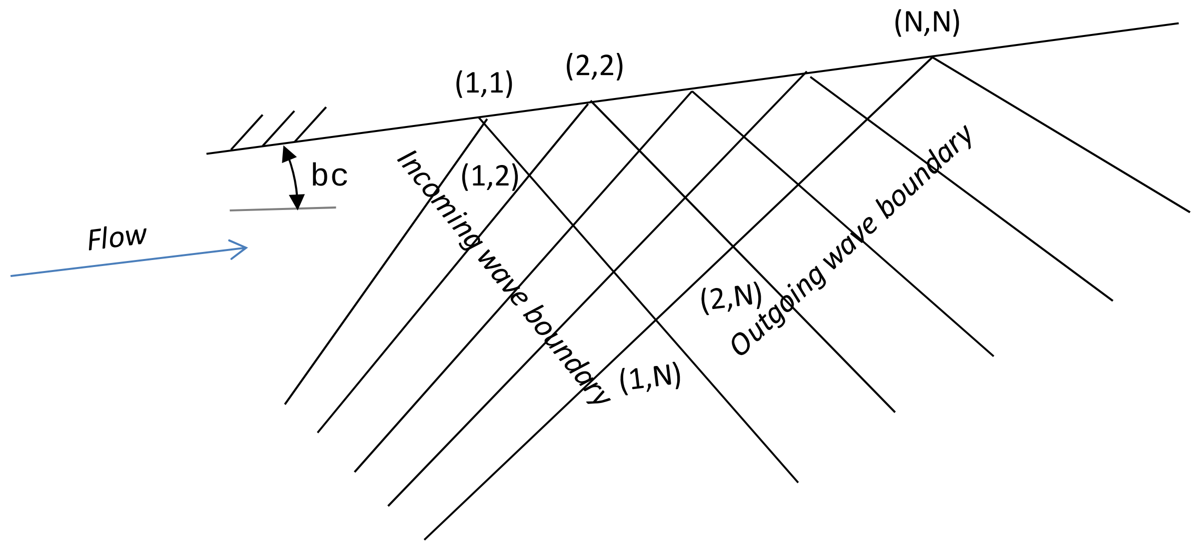
```
function [a,n,x,y]= simpleCancel(ai,ni,xi,yi,c,x0,y0,a0,g)
```

```
function simpleplot(a,n,x,y,g,cl,ch)
```



Basic Complex Wave

```
function [a,n,x,y]=complex3(ai,ni,xi,yi,bc,g)
```



Other Complex Wave Functions

```
function [a,n,x,y]=complex3curve(ai,ni,xi,yi,bc,g)
```

```
function [a,n,x,y]=complex3free(ai,ni,xi,yi,bc,g)
```

```
function [a,n,x,y]=complex4(ap,np,xp,yp,an,nn,xn,yn,g)
```

```
function complex3plot(a,n,x,y,g,cl,ch)
function complex4plot(a,n,x,y,g,cl,ch)
```

Extras

```
function uniformplot(a,n,x,y,g,cl,ch)
```

```
function n=nu(m,g)
```

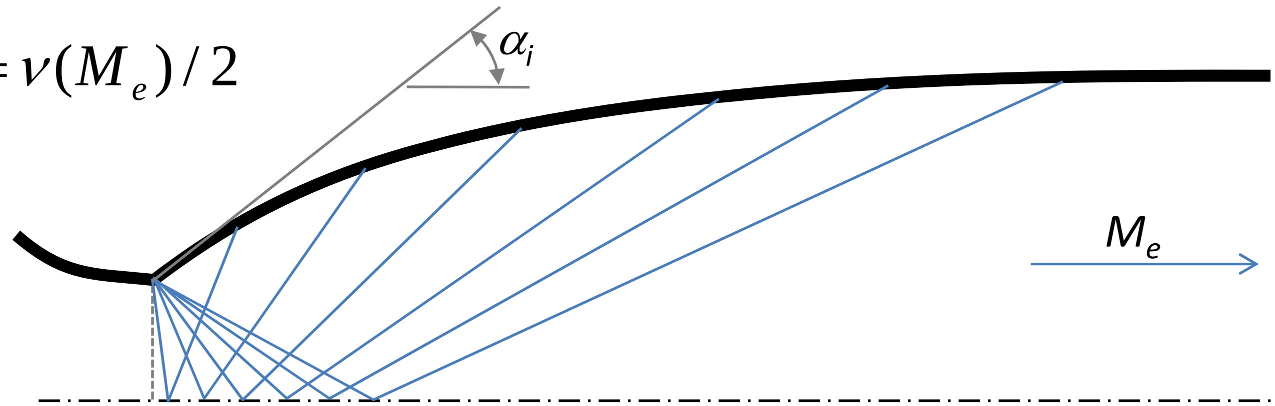
```
function m=m_nu(n,g)
```

```
function [x,y]=intercept(x1,y1,t1,x2,y2,t2)
```

```
function [x,y,a]=interceptCurve(x1,y1,t1,cf)
```

Example: Rocket Engine Nozzle

$$\alpha_i = \nu(M_e)/2$$

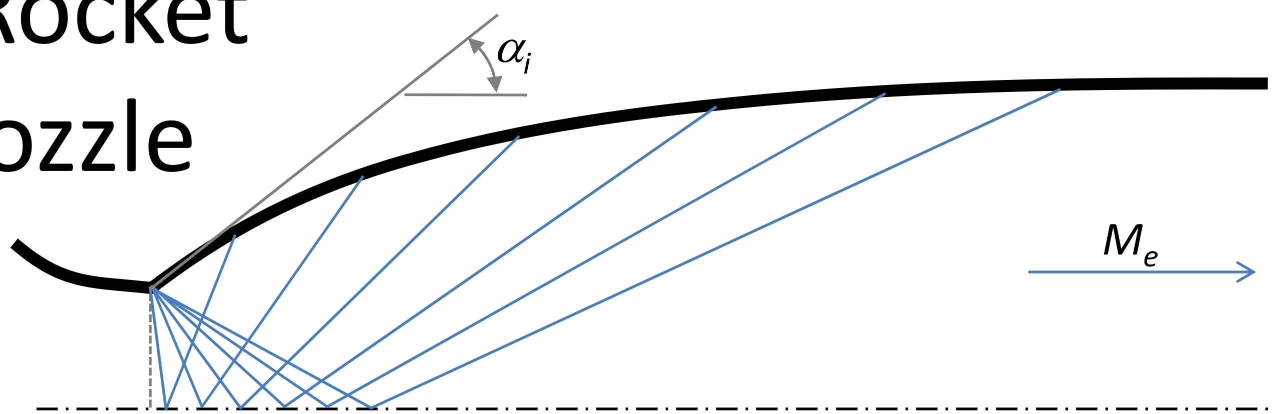



```
function [a,n,x,y]=simple(ai,ni,xi,yi,le,g)

function [a,n,x,y]=complex3(ai,ni,xi,yi,bc,g)

function [a,n,x,y]= simpleCancel(ai,ni,xi,yi,c,x0,y0,a0,g)
```

`ArocketEngineNozzle.m`

Matlab Demo

Example: Rocket Engine Nozzle




```
clear all   %Mach 2 minimum length nozzle, initial turn angle ai=0.23 radians
g=1.4;
ai=[.00001 .0001 .001 .005 .01:.01:.23]; %28 waves
xi=zeros(size(ai));yi=ones(size(ai));ni=ai;
le=-1;
figure(1);clf(1);cl=1;ch=2;

[a1,n1,x1,y1]=simple(ai,ni,xi,yi,le,g);
simpleplot(a1,n1,x1,y1,g,cl,ch);

[a2,n2,x2,y2]=complex3(a1(end,:),n1(end,:),x1(end,:),y1(end,:),0,g);  %complex wave reflection
complex3plot(a2,n2,x2,y2,g,cl,ch);

[a3,n3,x3,y3]=simpleCancel(a2(:,end),n2(:,end),x2(:,end),y2(:,end),1,xi(end),yi(end),ai(end),g);
simpleplot(a3,n3,x3,y3,g,cl,ch);

uniformplot([a1(1,1) a1(2,1) 0],[n1(1,1) n1(2,1) 0],[x1(1,1) x1(2,1) x1(1,1)],…
uniformplot([a1(1,end) a1(2,end) a3(2,1)],[n1(1,end) n1(2,end) n3(2,1)],…
uniformplot([a3(1,end) a3(2,end) a3(1,end)],[n3(1,end) n3(2,end) n3(1,end)],…

hold off;caxis([cl ch]);colorbar;axis image
```

Pre-Packaged Scripts

- Constant Mach Number Turn
- Minimum length nozzle
- Wave interaction
- Underexpanded jet
- Wind tunnel nozzle