Chapter 10

Gasdynamics of nozzle flow

A nozzle is an extremely efficient device for converting thermal energy to kinetic energy. Nozzles come up in a vast range of applications. Obvious ones are the thrust nozzles of rocket and jet engines. Converging-diverging ducts also come up in aircraft engine inlets, wind tunnels and in all sorts of piping systems designed to control gas flow. The flows associated with volcanic and geyser eruptions are influenced by converging-diverging nozzle geometries that arise naturally in geological formations.

10.1 Area-Mach number function

In Chapter 8 we developed the area-averaged equations of motion.

$$
\begin{aligned}
&d(\rho U) = \frac{\delta \dot{m}}{A} - \rho U \frac{dA}{A} \\
&d(P - \tau_{xx}) + \rho U dU = -\frac{1}{2}\rho U^2 \left(\frac{4C_f dx}{D}\right) + \frac{(U_{xm} - U)\,\delta \dot{m}}{A} - \frac{\delta F_x}{A} \\
&d\left(h_t - \frac{\tau_{xx}}{\rho} + \frac{Q_x}{\rho U}\right) = \delta q_w - \delta w + \left(h_{tm} - \left(h_t - \frac{\tau_{xx}}{\rho} + \frac{Q_x}{\rho U}\right)\right)\frac{\delta \dot{m}}{\rho U A}
\end{aligned}
\tag{10.1}
$$

Assume the only effect on the flow is streamwise area change so that

$$\delta \dot{m} = C_f = \delta F_x = \delta q = \delta w = 0. \tag{10.2}$$

Also assume that streamwise normal stresses and heat fluxes τ_{xx}, Q_x are small enough to be neglected. With these assumptions the governing equations (10.1) together with the perfect gas law reduce to

$$
\begin{aligned}
& d\left(\rho U A\right) = 0 \\
& dP + \rho U dU = 0 \\
& C_p dT + U dU = 0 \\
& P = \rho R T.
\end{aligned}
\tag{10.3}
$$

Introduce the Mach number

$$
U^2 = \gamma R T M^2. \tag{10.4}
$$

Each of the equations in (10.3) can be expressed in fractional differential form.

$$
\begin{aligned}
& \frac{d\rho}{\rho} + \frac{dU^2}{2U^2} + \frac{dA}{A} = 0 \\
& \frac{dP}{P} + \frac{\gamma M^2}{2}\frac{dU^2}{U^2} = 0 \\
& \frac{dT}{T} + \frac{\left(\gamma - 1\right) M^2}{2}\frac{dU^2}{U^2} = 0 \\
& \frac{dP}{P} = \frac{d\rho}{\rho} + \frac{dT}{T}
\end{aligned}
\tag{10.5}
$$

Equation (10.4) can also be written in fractional differential form.

$$
\frac{dU^2}{U^2} = \frac{dT}{T} + \frac{dM^2}{M^2} \tag{10.6}
$$

Use the equations for mass, momentum and energy to replace the terms in the equation of state.

$$
-\frac{\gamma M^2}{2}\frac{dU^2}{U^2} = -\frac{dU^2}{2U^2} - \frac{dA}{A} - \frac{\left(\gamma - 1\right) M^2}{2}\frac{dU^2}{U^2} \tag{10.7}
$$

Solve for dU^2/U^2.

$$\frac{dU^2}{U^2} = \left(\frac{2}{M^2-1}\right)\frac{dA}{A} \tag{10.8}$$

Equation (10.8) shows the effect of streamwise area change on the speed of the flow. If the Mach number is less than one then increasing area leads to a decrease in the velocity. But if the Mach number is greater than one then increasing area leads to an increase in flow speed. Use (10.8) to replace dU^2/U^2 in each of the relations in (10.5).

$$\begin{aligned}\frac{d\rho}{\rho} &= -\left(\frac{M^2}{M^2-1}\right)\frac{dA}{A}\\ \frac{dP}{P} &= -\left(\frac{\gamma M^2}{M^2-1}\right)\frac{dA}{A}\\ \frac{dT}{T} &= -\left(\frac{(\gamma-1)M^2}{M^2-1}\right)\frac{dA}{A}\end{aligned} \tag{10.9}$$

Equations (10.9) describe the effects of area change on the thermodynamic state of the flow. Now use (10.8) and the temperature equation in (10.6).

$$\left(\frac{2}{M^2-1}\right)\frac{dA}{A} = -\frac{(\gamma-1)M^2}{M^2-1}\frac{dA}{A} + \frac{dM^2}{M^2} \tag{10.10}$$

Rearrange (10.10). The effect of area change on the Mach number is

$$\frac{dA}{A} = \frac{M^2-1}{2\left(1+\left(\frac{\gamma-1}{2}\right)M^2\right)}\frac{dM^2}{M^2}. \tag{10.11}$$

Equation (10.11) is different from (10.8) and (10.9) in that it can be integrated from an initial to a final state. Integrate (10.11) from an initial Mach number M to one.

$$\int_{M^2}^{1}\frac{M^2-1}{2\left(1+\left(\frac{\gamma-1}{2}\right)M^2\right)}\frac{dM^2}{M^2} = \int_{A}^{A^*}\frac{dA}{A} \tag{10.12}$$

The result is

$$ln\left(\frac{A^*}{A}\right) = \left\{-ln\left(M\right) + \ln\left(2\left(1 + \left(\frac{\gamma-1}{2}\right)M^2\right)^{\frac{\gamma+1}{2(\gamma-1)}}\right)\right\}\Bigg|_{M^2}^{1}. \tag{10.13}$$

Evaluate (10.13) at the limits

$$ln\left(\frac{A^*}{A}\right) = \ln\left(\left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{2(\gamma-1)}}\right) - \left\{-ln\left(M\right) + \ln\left(\left(1 + \left(\frac{\gamma-1}{2}\right)M^2\right)^{\frac{\gamma+1}{2(\gamma-1)}}\right)\right\} \tag{10.14}$$

which becomes

$$ln\left(\frac{A^*}{A}\right) = \ln\left(\left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{2(\gamma-1)}}\frac{M}{\left(1 + \left(\frac{\gamma-1}{2}\right)M^2\right)^{\frac{\gamma+1}{2(\gamma-1)}}}\right). \tag{10.15}$$

Exponentiate both sides of (10.15). The result is the all-important area-Mach number equation.

$$f\left(M\right) = \frac{A^*}{A} = \left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{2(\gamma-1)}}\frac{M}{\left(1 + \left(\frac{\gamma-1}{2}\right)M^2\right)^{\frac{\gamma+1}{2(\gamma-1)}}} \tag{10.16}$$

In (10.16) we referenced the integration process to $M = 1$. The area A^* is a reference area at some point in the channel where $M = 1$ although such a point need not actually be present in a given problem. The area-Mach-number function is plotted below for three values of γ.

Note that for smaller values of γ it takes an extremely large area ratio to generate high Mach number flow. A value of $\gamma = 1.2$ would be typical of the very high temperature mixture of gases in a rocket exhaust. Conversely, if we want to produce a high Mach number flow in a reasonable size nozzle, say for a wind tunnel study, an effective method is to select a monatomic gas such as Helium which has $\gamma = 1.66$. A particularly interesting feature of (10.16) is the insensitivity of $f\left(M\right)$ to γ for subsonic flow.

10.1.1 Mass conservation

The result (10.16) can also be derived simply by equating mass flows at any two points in the channel and using the mass flow relation.

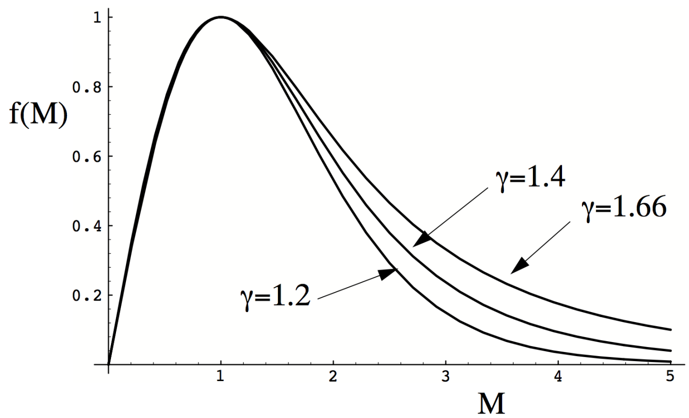

Figure 10.1: *Area-Mach number function.*

$$\dot{m} = \rho U A \tag{10.17}$$

This can be expressed as

$$\dot{m} = \rho U A = \frac{P}{RT}(\gamma R T)^{1/2} M A. \tag{10.18}$$

Insert

$$\begin{aligned} \frac{T_t}{T} &= 1 + \frac{\gamma - 1}{2} M^2 \\ \frac{P_t}{P} &= \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\frac{\gamma}{\gamma - 1}} \end{aligned} \tag{10.19}$$

into (10.18) to produce

$$\dot{m} = \rho U A = \frac{\gamma}{\left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{2(\gamma-1)}}} \left(\frac{P_t A}{\sqrt{\gamma R T_t}}\right) f(M). \tag{10.20}$$

If we equate the mass flows at any two points in a channel (10.20) gives

$$
\begin{aligned}
&\dot{m}_1 = \dot{m}_2 \\
&\frac{P_{t1} A_1}{\sqrt{T_{t1}}} f\left(M_1\right) = \frac{P_{t2} A_2}{\sqrt{T_{t2}}} f\left(M_2\right).
\end{aligned}
\tag{10.21}
$$

In the case of an adiabatic ($T_t = constant$), isentropic ($P_t = constant$) flow in a channel (10.16) provides a direct relation between the local area and Mach number.

$$
A_1 f\left(M_1\right) = A_2 f\left(M_2\right) \tag{10.22}
$$

10.2 A simple convergent nozzle

Figure 10.2 shows a large adiabatic reservoir containing an ideal gas at pressure P_t. The gas exhausts through a simple convergent nozzle with throat area A_e to the ambient atmosphere at pressure $P_{ambient}$. Gas is continuously supplied to the reservoir so that the reservoir pressure is effectively constant. Assume the gas is calorically perfect, ($P = \rho RT$, C_p and C_v are constant) and assume that wall friction is negligible.

Let's make this last statement a little more precise. Note that we do not assume that the gas is inviscid since we want to accommodate the possibility of shock formation somewhere in the flow. Rather, we make use of the fact that, if the nozzle is large enough, the boundary layer thickness will be small compared to the diameter of the nozzle enabling most of the flow to be treated as irrotational and isentropic.

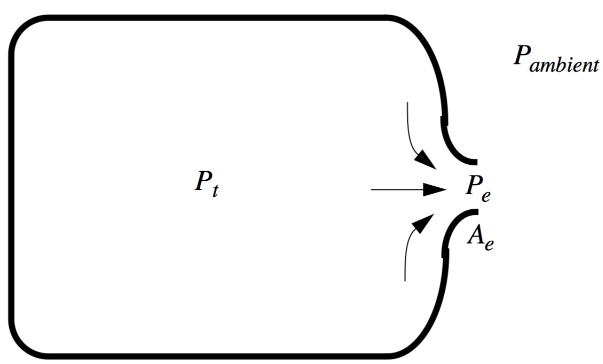

Figure 10.2: *Reservoir with a convergent nozzle.*

The isentropic assumption works quite well for nozzles that are encountered in most applications. But if the plenum falls below a few centimeters in size with a nozzle diameter less than a few millimeters then a fully viscous, non-isentropic treatment of the flow is required. Accurate nozzle design, regardless of size, virtually always requires that the boundary layer on the wall of the plenum and nozzle is taken into account.

If the ambient pressure equals the reservoir pressure there is, of course, no flow. If $P_{ambient}$ is slightly below P_t then there is a low-speed, subsonic, approximately isentropic flow from the plenum to the nozzle. If $P_t/P_{ambient}$ is less than a certain critical value then the condition that determines the speed of the flow at the exit is that the exit static pressure is very nearly equal to the ambient pressure.

$$P_e = P_{ambient} \tag{10.23}$$

The reason this condition applies is that large pressure differences cannot occur over small distances in a subsonic flow. Any such difference that might arise, say between the nozzle exit and a point slightly outside of and above the exit, will be immediately smoothed out by a readjustment of the whole flow. Some sort of shock or expansion is required to maintain a pressure discontinuity and this can only occur in supersonic flow. Slight differences in pressure are present due to the mixing zone that exists outside the nozzle but in subsonic flow these differences are very small compared to the ambient pressure. Since the flow up to the exit is approximately isentropic the stagnation pressure P_t is approximately constant from the reservoir to the nozzle exit and we can write

$$\frac{P_t}{P_e} = \left(1 + \left(\frac{\gamma - 1}{2}\right) {M_e}^2\right)^{\frac{\gamma}{\gamma-1}}. \tag{10.24}$$

Using (10.23) and (10.24) we can solve for the Mach number at the nozzle exit in terms of the applied pressure ratio.

$$M_e = \left(\frac{2}{\gamma - 1}\right)^{1/2} \left(\left(\frac{P_t}{P_{ambient}}\right)^{\frac{\gamma-1}{\gamma}} - 1\right)^{1/2} \tag{10.25}$$

Note that the nozzle area does not appear in this relationship.

10.2.1 The phenomenon of choking

The exit Mach number reaches one when

$$\frac{P_t}{P_{ambient}} = \left(\frac{\gamma + 1}{2}\right)^{\frac{\gamma}{\gamma-1}}. \tag{10.26}$$

For Air with $\gamma = 1.4$ this critical pressure ratio is $P_t/P_{ambient} = 1.893$ and the condition **??** holds for $1 \leq P_t/P_{ambient} \leq 1.893$. At $P_t/P_{ambient} = 1.893$ the area-Mach number function $f(M)$ is at its maximum value of one. At this condition the mass flow through the nozzle is as large as it can be for the given reservoir stagnation pressure and temperature and the nozzle is said to be choked.

If $P_t/P_{ambient}$ is increased above the critical value the flow from the reservoir to the nozzle throat will be unaffected; the Mach number will remain $M_e = 1$ and $P_t/P_{ambient} = 1.893$. However condition (10.23) will no longer hold because now $P_e > P_{ambient}$. The flow exiting the nozzle will tend to expand supersonically eventually adjusting to the ambient pressure through a system of expansions and shocks.

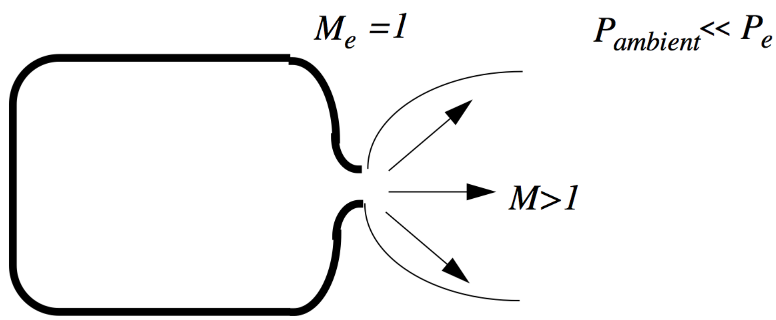

Figure 10.3: *Plenum exhausting to very low pressure.*

10.3 The converging-diverging nozzle

Now let's generalize these ideas to the situation where the nozzle consists of a converging section upstream of the throat and a diverging section downstream. Consider the nozzle geometry shown below.

The goal is to completely determine the flow in the nozzle given the pressure ratio $P_t/P_{ambient}$ and the area ratio A_e/A_{throat}. Before analyzing the flow we should first work out the critical exit Mach numbers and pressures for the selected area ratio. Solving

$$\frac{A_{throat}}{A_e} = \left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \frac{M_e}{\left(1+\left(\frac{\gamma-1}{2}\right) {M_e}^2\right)^{\frac{\gamma+1}{2(\gamma-1)}}} \tag{10.27}$$

gives two critical Mach numbers $M_{ea} < 1$ and $M_{eb} > 1$ for isentropic flow in the nozzle with $M = 1$ at the throat. The corresponding critical exit pressures are determined from

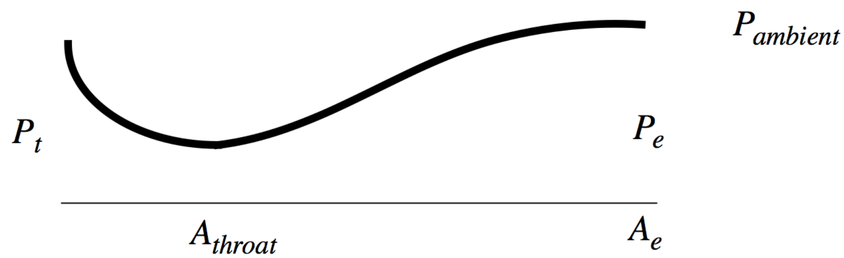

Figure 10.4: *Converging-diverging geometry*

$$
\begin{aligned}
\frac{P_t}{P_{ea}} &= \left(1 + \left(\frac{\gamma - 1}{2}\right) {M_{ea}}^2\right)^{\frac{\gamma}{\gamma-1}} \\
\frac{P_t}{P_{eb}} &= \left(1 + \left(\frac{\gamma - 1}{2}\right) {M_{eb}}^2\right)^{\frac{\gamma}{\gamma-1}}.
\end{aligned}
\tag{10.28}
$$

There are several possible cases to consider.

10.3.1 Case 1 - Isentropic subsonic flow in the nozzle

If $P_t/P_{ambient}$ is not too large then the flow throughout the nozzle will be sub- sonic and isentropic and the pressure at the exit will match the ambient pressure. In this instance the exit Mach number is determined using (10.25).

If $P_t/P_{ambient}$ is increased there is a critical value that leads to choking at the throat. This flow condition is sketched below.

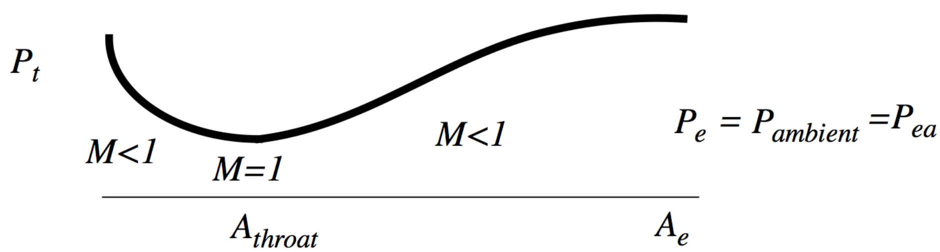

Figure 10.5: *Onset of choking.*

The exit Mach number is M_{ea} and the pressure ratio is $P_t/P_{ambient} = P_t/P_{ea}$. Note that when a diverging section is present the pressure ratio that leads to choking is less than that given by (10.26). The flow in the nozzle is all subsonic when the pressure ratio is in the range

$$1 < \frac{P_t}{P_{ambient}} < \frac{P_t}{P_{ea}}. \tag{10.29}$$

10.3.2 Case 2 - Non-isentropic flow - shock in the nozzle

If the pressure ratio is increased above P_t/P_{ea} a normal shock will form downstream of the throat, the exit Mach number remains subsonic and the exit pressure will continue to match the ambient pressure. This flow condition is shown below.

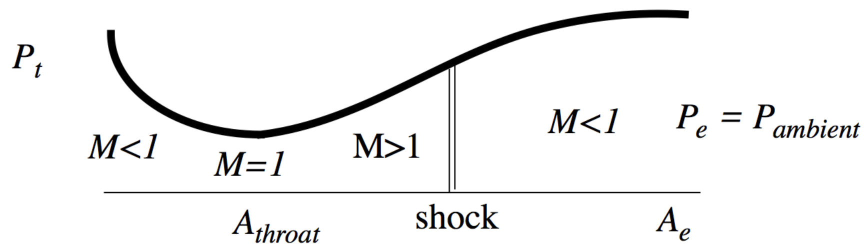

Figure 10.6: *Shock in nozzle.*

The entropy is constant up to the shock wave, increases across the wave and remains constant to the exit. To work out the flow properties, first equate mass flows at the throat and nozzle exit

$$\dot{m}_{throat} = \dot{m}_{exit} \tag{10.30}$$

or

$$P_t A_{throat} = P_{te} A_e f\left(M_e\right). \tag{10.31}$$

The key piece of information that enables us to solve for the flow is that the exit pressure still matches the ambient pressure and so we can write

$$P_{te} = P_e \left(1 + \left(\frac{\gamma-1}{2}\right) {M_e}^2\right)^{\frac{\gamma}{\gamma-1}} = P_{ambient} \left(1 + \left(\frac{\gamma-1}{2}\right) {M_e}^2\right)^{\frac{\gamma}{\gamma-1}}. \tag{10.32}$$

When (10.32) is incorporated into (10.31) the result is

$$\left(\frac{P_t}{P_{ambient}}\right)\left(\frac{A_{throat}}{A_e}\right) = \left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{2(\gamma-1)}} M_e\left(1+\left(\frac{\gamma-1}{2}\right)M_e{}^2\right)^{\frac{1}{2}}. \tag{10.33}$$

The items on the left side of (10.33) are known quantities and so one solves (10.33) implicitly for $M_e < 1$. With the exit Mach number known, (10.31) is used to determine the stagnation pressure ratio across the nozzle.

$$\left(\frac{P_{te}}{P_t}\right) = \left(\frac{A_{throat}}{A_e}\right)\frac{1}{f\left(M_e\right)} < 1 \tag{10.34}$$

Since the only mechanism for stagnation pressure loss is the normal shock, the value of P_{te}/P_t determined from (10.34) can be used to infer the shock Mach number from

$$\left(\frac{P_{te}}{P_t}\right) = \left(\frac{\left(\frac{\gamma+1}{2}\right)M^2{}_{shock}}{1+\left(\frac{\gamma-1}{2}\right)M^2{}_{shock}}\right)^{\frac{\gamma}{\gamma-1}}\left(\frac{\frac{\gamma+1}{2}}{\gamma M^2{}_{shock}-\left(\frac{\gamma-1}{2}\right)}\right)^{\frac{1}{\gamma-1}}. \tag{10.35}$$

Thus all of the important properties of the flow in the nozzle are known given the plenum to ambient pressure ratio and the nozzle area ratio.

As the nozzle pressure ratio is increased, the shock moves more and more downstream until it is situated at the nozzle exit. This flow condition is shown below.

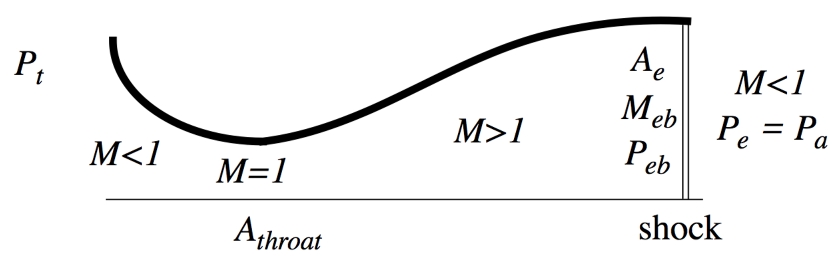

Figure 10.7: *Normal shock at the nozzle exit.*

In this case the Mach number just ahead of the shock is the supersonic critical value M_{eb} and the Mach number just behind is the corresponding subsonic value derived from normal shock relations.

$$M^2{}_{e(\text{behindshock})} = \frac{1+\left(\frac{\gamma-1}{2}\right)M^2{}_{eb}}{\gamma M^2{}_{eb}-\left(\frac{\gamma-1}{2}\right)} \tag{10.36}$$

The value of $P_t/P_{ambient}$ that produces this flow is

$$\left.\frac{P_t}{P_{ambient}}\right|_{\text{exit}\perp\text{shock}} = \left(\frac{A_e}{A_{throat}}\right)\left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{2(\gamma-1)}} M_{e(\text{behindshock})}\left(1+\left(\frac{\gamma-1}{2}\right)M^2_{e(\text{behindshock})}\right)^{\frac{1}{2}}. \tag{10.37}$$

The shock-in-the-nozzle case occurs over the range

$$\frac{P_t}{P_{ea}} < \frac{P_t}{P_{ambient}} < \left.\frac{P_t}{P_{ambient}}\right|_{\text{exit}\perp\text{shock}}. \tag{10.38}$$

10.3.3 Case 3 - Isentropic supersonic flow in the nozzle

If the nozzle pressure ratio exceeds the value given in (10.37) then no further changes occur in the flow within the nozzle. Three different cases are distinguished.

i) Over expanded flow - This corresponds to the range

$$\left.\frac{P_t}{P_{ambient}}\right|_{\text{exit}\perp\text{shock}} < \frac{P_t}{P_{ambient}} < \frac{P_t}{P_{eb}}. \tag{10.39}$$

In this case the flow passes through an oblique shock as it exhausts.

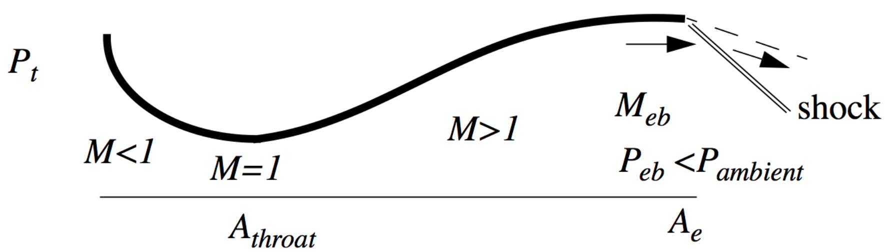

Figure 10.8: *Oblique shock at the nozzle exit.*

ii) Fully expanded flow - This is the case where

$$\frac{P_t}{P_{ambient}} = \frac{P_t}{P_{eb}}. \tag{10.40}$$

The exit pressure now matches the ambient pressure and the flow exhausts smoothly.

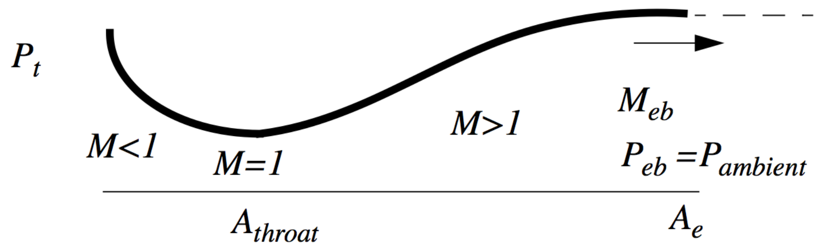

Figure 10.9: *Fully expanded flow.*

iii) Under expanded flow - This corresponds to the range

$$\frac{P_t}{P_{ambient}} > \frac{P_t}{P_{eb}}. \tag{10.41}$$

In this case the exit pressure exceeds the ambient pressure and the flow expands outward as it leaves the nozzle.

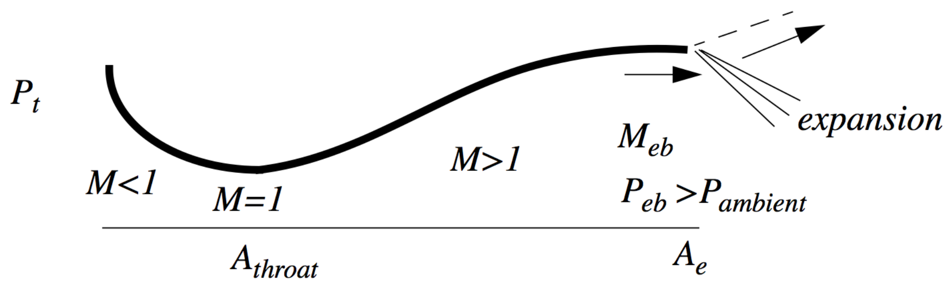

Figure 10.10: *Expansion fan at the nozzle exit.*

A good example of the occurrence of all three conditions is the Space Shuttle Main Engine which leaves the pad in an over expanded state, becomes fully expanded at high altitude and then extremely under expanded as the Shuttle approaches the vacuum of space.

10.4 Examples

10.4.1 Shock in a nozzle

A normal shock is stabilized in the diverging section of a nozzle. The area ratios are, $A_s/A_{throat} = 2$, $A_e/A_{throat} = 4$ and $A_1 = A_e$.

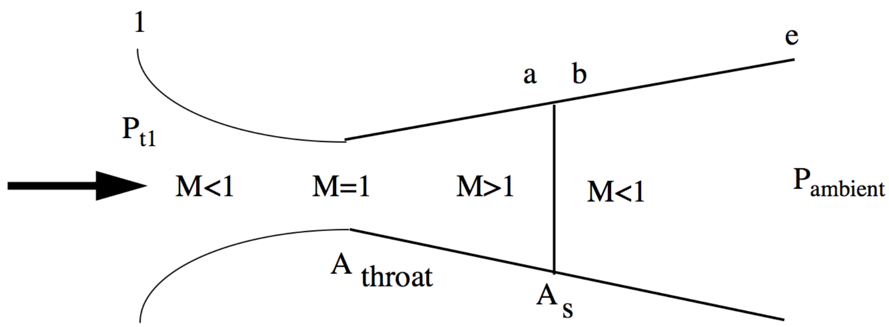

Figure 10.11: *Converging-diverging nozzle with shock.*

1) Determine M_1, M_e, the Mach number just ahead of the shock M_a, and the Mach number just behind the shock, M_b . Assume the gas is Air with $\gamma = 1.4$.

Solution

The area ratio from the throat to the shock is 2. One needs to solve

$$f\left(M_a\right) = 1/2 \tag{10.42}$$

for the supersonic root. The solution from tables or a calculator is $M_a = 2.197$. The normal shock relation for the downstream Mach number is

$$M_b^2 = \frac{1 + \left(\frac{\gamma-1}{2}\right) M_a^2}{\gamma M_a^2 - \left(\frac{\gamma-1}{2}\right)} \tag{10.43}$$

which gives $M_b = 0.547$. At station 1 the area ratio to the throat is 4 and the Mach number is subsonic. Solve for the subsonic root of

$$f\left(M_1\right) = 1/4. \tag{10.44}$$

The solution is $M_1 = 0.147$. The area ratio from behind the shock to station e is two. If we equate mass flows at both points and assume isentropic flow from station b to e we can write

$$A_b f\left(M_b\right) = A_e f\left(M_e\right). \tag{10.45}$$

Solve (10.45) for $f(M_e)$.

$$f(M_e) = \frac{A_b}{A_e} f(M_b) = (1/2)(0.794) = 0.397 \tag{10.46}$$

The exit Mach number is $M_e = 0.238$. So far the structure of the flow is as shown below.

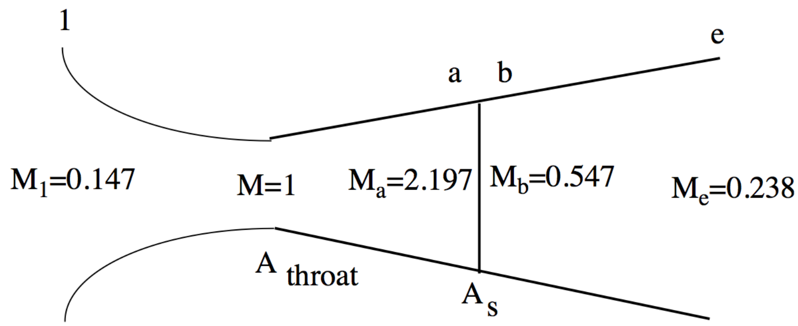

Figure 10.12: *Converging-diverging nozzle with Mach numbers labeled.*

2) Determine the pressure ratio across the nozzle, P_e/P_{t1}.

Solution

Since the exit flow is subsonic the exit pressure matches the ambient pressure. Use (10.33).

$$\begin{aligned}\frac{P_e}{P_{t1}} &= \frac{(A_{throat}/A_e)}{\left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{2(\gamma-1)}} M_e \left(1+\left(\frac{\gamma-1}{2}\right)M_e^2\right)^{1/2}} \\ &= \frac{(1/4)}{\left(\frac{1}{2}\right)^3 (0.238)\left(1+(1/5)(0.238)^2\right)^{1/2}} = 0.604\end{aligned} \tag{10.47}$$

3) What pressure ratio would be required to position the shock at station e?

Solution

The structure of the flow in this case would be as shown in 10.13. To determine the pressure ratio that produces this flow use (10.37).

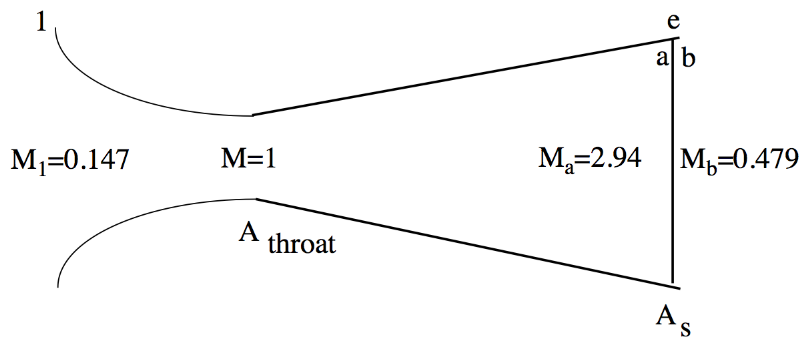

Figure 10.13: *Converging-diverging nozzle with shock at the exit.*

$$\left.\frac{P_t}{P_{ambient}}\right|_{\text{exit}\perp\text{shock}} = \left.\frac{P_{t1}}{P_e}\right|_{\text{exit}\perp\text{shock}} =$$

$$\left(\frac{A_e}{A_{throat}}\right)\left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{2(\gamma-1)}} M_{e(\text{behindshock})}\left(1+\left(\frac{\gamma-1}{2}\right)M^2_{e(\text{behindshock})}\right)^{\frac{1}{2}} = \tag{10.48}$$

$$\left.\frac{P_e}{P_{t1}}\right|_{\text{exit}\perp\text{shock}} = \frac{1}{4(1.2)^3\,(0.479)\left(1+(1/5)\,(0.479)^2\right)^{\frac{1}{2}}} = 0.295$$

This is considerably lower than the pressure ratio determined in part 2.

10.4.2 Cold gas thruster

A cold gas thruster on a spacecraft uses Helium (atomic weight 4) as the working gas. The gas exhausts through a large area ratio nozzle to the vacuum of space. Compare the energy of a parcel of gas in the fully-expanded exhaust to the energy it had when it was in the chamber.

Answer

In the chamber the energy per unit mass, neglecting kinetic energy is

$$E_{chamber} = C_v T_{chamber}. \tag{10.49}$$

Assume the expansion takes place adiabatically. Under that assumption, the stagnation enthalpy is conserved.

$$C_p T_{chamber} = C_p T + \frac{1}{2}U^2 = \text{constant} \tag{10.50}$$

Since the area ratio is large the thermal energy of the exhaust gas is small compared to the kinetic energy.

$$E_{exhaust} = C_v T_{exhaust} + \frac{1}{2}U_{exhaust}^2 \cong \frac{1}{2}U_{exhaust}^2 \cong C_p T_{chamber} \tag{10.51}$$

Divide (10.51) by ${U_1}^2$. The result is

$$\frac{E_{exhaust}}{E_{chamber}} = \frac{C_p T_{chamber}}{C_v T_{exhaust}} = \gamma = \frac{5}{3}. \tag{10.52}$$

The energy gained by the fluid element during the expansion process is due to the pressure forces that accelerate the element. In fact what is recovered is exactly the work required to create the original pressurized state.

10.4.3 Gasdynamics of a double throat, starting and unstarting supersonic flow

One of the most important applications of the gas-dynamic tools we have been developing is to a channel with multiple throats. Virtually all air-breathing propulsion systems utilize at least two throats; one to decelerate the incoming flow and a second to accelerate the exit flow. When a compressor and turbine are present several more throats may be involved. The simplest application of of two throats is to the design of a supersonic wind tunnel. Shown below is a supersonic wind tunnel that uses air as the working gas.

A very large plenum contains the gas at constant stagnation pressure and temperature, P_t , T_t. The flow exhausts to a large tank that is maintained at vacuum $P_a = 0$. The upstream nozzle area ratio is $A_2/A_1 = 6$ and the ratio of exit area to throat area is $A_e/A_1 = 2$. The test section has a constant area $A_3 = A_2$. A shock wave is stabilized in the diverging portion of the nozzle. The wall friction coefficient is very small.

1) Determine P_{te}/P_e.

Solution

The mass balance between stations 1 and e is

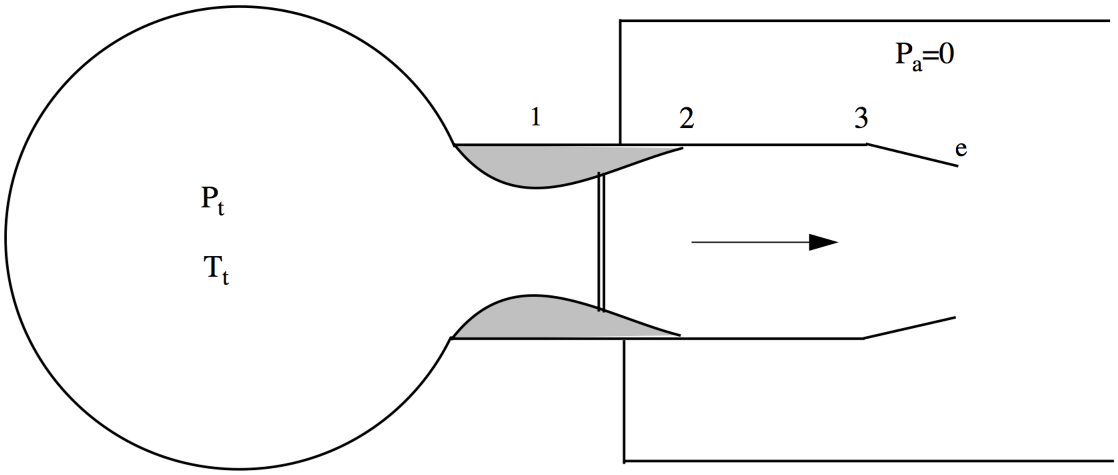

Figure 10.14: *Supersonic wind tunnel with two throats.*

$$\begin{aligned} &\dot{m}_1 = \dot{m}_e \\ &\frac{P_{t1}A_1}{\sqrt{\gamma R T_{t1}}} f(M_1) = \frac{P_{te}A_e}{\sqrt{\gamma R T_{te}}} f(M_e)\,. \end{aligned} \tag{10.53}$$

The flow exits to vacuum and so the large pressure ratio across the system essentially guarantees that both throats must be choked, $M_1 = 1$ and $M_e = 1$. Assume the flow is adiabatic and neglect wall friction. With these assumptions the mass balance (10.53). reduces to

$$\frac{P_{te}}{P_t} = \frac{A_1}{A_e} = 0.5. \tag{10.54}$$

2) Determine the shock Mach number

Solution

From the relations for shock wave flow, the shock Mach number that reduces the stagnation pressure by half for a gas with $\gamma = 1.4$ is $M_s = 2.5$.

3) Determine the Mach numbers at stations 2 and 3.

Solution

The Mach number at station 3 is determined by the area ratio from 3 to e and the fact that the exit is choked.

$$\frac{A_e}{A_3} = \frac{1}{3} \Rightarrow M_3 = 0.195 \tag{10.55}$$

Since the area of the test section is constant and friction is neglected the Mach number at station 2 is the same $M_2 = 0.195$.

4) Suppose A_e is reduced to the point where $A_e = A_1$. What happens to the shock?

Solution

Again use the mass flow equation (10.20) and equate mass flows at the two throats. In this case (10.53) is

$$\frac{P_{te}}{P_t} = \frac{A_1}{A_e} = 1.0. \tag{10.56}$$

There is no shock and therefore there is no stagnation pressure loss between the two throats. As A_e is reduced the shock moves upstream to lower Mach numbers till a point is reached when the two areas are equal. At that point the shock has essentially weakened to the point of disappearing altogether.

5) Suppose A_e is made smaller than A_1, what happens?

Solution

Since both the stagnation pressure and temperature are now constant along the channel and the exit throat is choked the mass balance (10.53) becomes

$$f\left(M_1\right) = \frac{A_e}{A_1}. \tag{10.57}$$

The Mach number at the upstream throat becomes subsonic and satisfies (10.57) as the area is further reduced.

6) Suppose A_e is increased above $A_e/A_1 = 2$. What happens to the shock?

Solution

In this case the shock moves downstream to higher Mach numbers. The highest Mach number that the shock can reach is at the end of the expansion section of the upstream nozzle where the area ratio is $A_2/A_1 = 6$. Equation (10.16) gives the Mach number of the shock at that point as $M_2 = 3.368$. The corresponding stagnation pressure ratio across the shock is $P_{te}/P_t = 0.2388$. Using the mass balance again, the throat area ratio that produces this condition is

$$\frac{A_e}{A_1} = \frac{P_t}{P_{te}} = 4.188. \tag{10.58}$$

Throughout this process the exit is at $M_e = 1$ and the flow in the test section is subsonic due to the presence of the shock. In fact the Mach number in the test section from station 2 to 3 would be the Mach number behind a Mach 3.368 shock which is 0.4566. Note that this is consistent with the area ratio $A_3/A_e = 6/4.188 = 1.433$ for which the subsonic solution of (10.16) is 0.4566.

7) Now suppose A_e/A_1 is increased just slightly above 4.188, what happens?

Solution

Again go back to the mass flow relation (10.53). Write (10.53) as

$$P_{t1}A_1 = P_{te}A_e f\left(M_e\right). \tag{10.59}$$

The upstream throat is choked and so the mass flow is fixed and the left-hand-side of (10.59) is fixed. The shock is at the highest Mach number it can reach given the area ratio of the upstream nozzle. So as A_e/A_1 is increased above 4.188 there is no way for P_{te}/P_t to decrease so as to maintain the equality (10.59) enforced by mass conservation. Instead an event occurs and that event is that the shock is swallowed by the downstream throat and supersonic flow is established in the test section. The supersonic wind tunnel is said to be started. Since there is no shock present the flow throughout the system is isentropic and the mass balance (10.59) becomes

$$f\left(M_e\right) = \frac{A_1}{A_e} = \frac{1}{4.188}. \tag{10.60}$$

The Mach number at the exit throat is now the supersonic root of (10.60), $M_e = 2.99$. If A_e/A_1 is increased further the exit Mach number increases according to Equation (10.60). If A_e/A_1 is reduced below 4.188 the exit Mach number reduces below 2.99 until it approaches one from above as $A_e/A_1 \to 1+\varepsilon$. If A_e/A_1 is reduced below one the wind tunnel unstarts and the flow between 1 and the exit is all subsonic (no shock) with $M_1 = M_e = 1$.

10.5 Problems

Problem 1 - Consider the expression ρU^n. The value $n = 1$ corresponds to the mass flux,

$n = 2$ corresponds to the momentum flux and $n = 3$ corresponds to the energy flux of a compressible gas. Use the momentum equation

$$dP + \rho U dU = 0 \tag{10.61}$$

to determine the Mach number (as a function of n) at which ρU^n; $n = 1$ is a maximum in steady flow.

Problem 2 - In the double-throat example above the flow exhausts into a vacuum chamber. Suppose the pressure P_a is not zero. What is the maximum pressure ratio P_a/P_t that would be required for the supersonic tunnel to start as described in the example? The exit area can be varied as required.

Problem 3 - In the double-throat example above suppose the effect of wall friction is included. How would the answers to the problem change? Would the various values calculated in the problem increase, decrease or remain the same and why?

Problem 4 - Figure 10.15 shows a supersonic wind tunnel which uses helium as a working gas. A very large plenum contains the gas at constant stagnation pressure and temperature P_t, T_t. Supersonic flow is established in the test section and the flow exhausts to a large tank at pressure P_a.

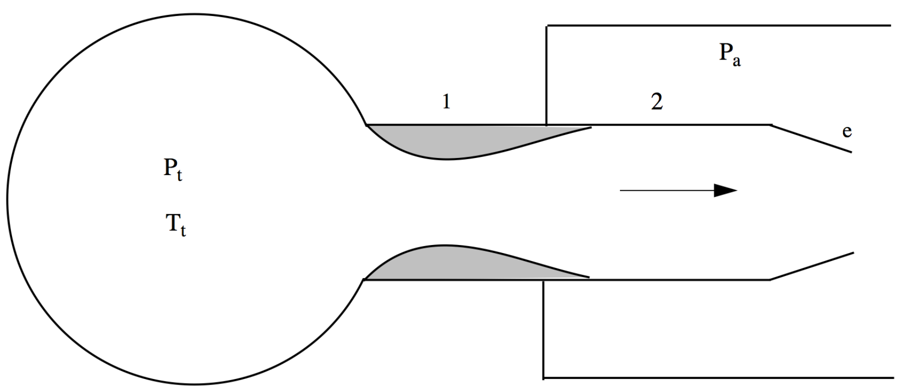

Figure 10.15: *Supersonic wind tunnel with variable area exit.*

The exit area A_e can be varied in order to change the flow conditions in the tunnel Initially $A_2/A_e = 4$, and $A_2/A_1 = 8$. The gas temperature in the plenum is $T_t = 300$K. Neglect wall friction. Let $P_t/P_a = 40$.

1) Determine the Mach numbers at A_e , A_1 and A_2.

2) Determine the velocity U_e and pressure ratio P_e/P_a.

3) Suppose A_e is reduced. Determine the value of A_e/A_2 which would cause the Mach number at A_e to approach one (from above). Suppose A_e is reduced slightly below this value - what happens to the supersonic flow in the tunnel? Determine P_{te}/P_t and the Mach numbers at A_1, A_2 and A_e for this case.

Problem 5 - Figure 10.16 shows a supersonic wind tunnel which uses air as the working gas. A very large plenum contains the gas at constant stagnation pressure and temperature, P_t, T_t. The flow exhausts to a large tank that is maintained at vacuum $P_a = 0$. The upstream nozzle area ratio is $A_2/A_1 = 3$. The downstream throat area A_e can be varied in order to change the flow conditions in the tunnel. Initially, $A_e = 0$. Neglect wall friction. Assign numerical values where appropriate.

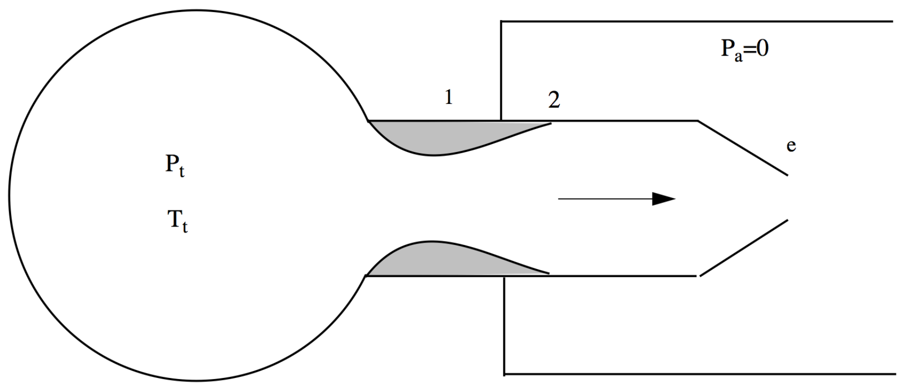

Figure 10.16: *Supersonic wind tunnel exiting to vacuum.*

1) Suppose A_e/A_1 is slowly increased from zero. Plot P_{te}/P_t as a function of A_e/A_1 for the range $0 \leq A_e/A_1 \leq 3$.

2) Now with $A_e/A_1 = 3$ initially, let A_e be decreased back to zero. Plot P_{te}/P_t as a function of A_e/A_1 for this process.

Problem 6 - In Chapter 2 we looked at the blowdown through a small nozzle of a calorically perfect gas from a large adiabatic pressure vessel at initial pressure P_i and temperature T_i to the surroundings at pressure P_a and temperature T_a. I would like you to reconsider that problem from the point of view of the conservation equations for mass and energy. Use a control volume analysis to determine the relationship between the pressure, density and temperature in the vessel as the mass is expelled. Show that the final temperature derived from a control volume analysis is the same as that predicted by integrating the Gibbs equation.

Problem 7 - Consider the inverse of Problem 6. A highly evacuated, thermally insulated flask is placed in a room with air temperature T_a. The air is allowed to enter the flask through a slightly opened stopcock until the pressure inside equals the pressure in the room. Assume the air to be calorically perfect. State any other assumptions needed to solve the problem.

(i)Use a control volume analysis to determine the relationship between the pressure, density and temperature in the vessel as mass enters the vessel.

(ii)Determine the entropy change per unit mass during the process for the gas that enters the vessel.

(iii)Determine the final temperature of the gas in the vessel.

May I suggest that you break the process into two parts. When the stopcock is first opened, the opening is choked and the flow outside the flask is steady. But after a while the opening un-chokes and the pressure at the opening increases with time. In the latter case the flow outside the flask is unsteady and one needs to think of a reasonable model of the flow in order to solve the problem.