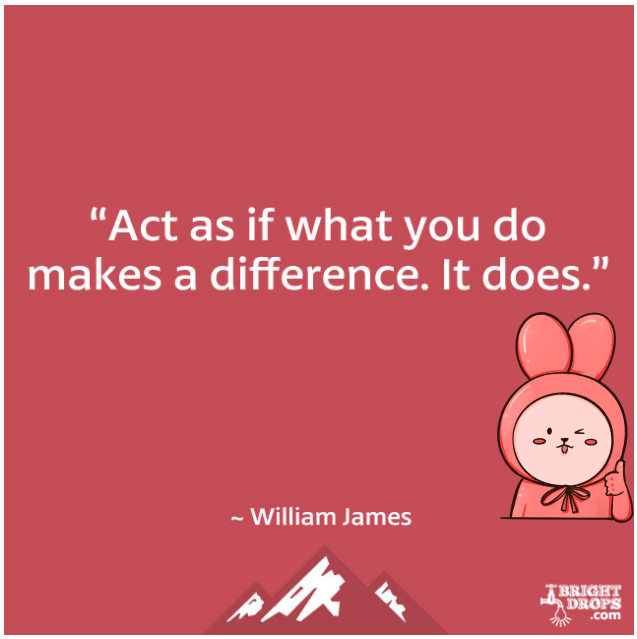
NOTES: COMBINATORICS

Good Evening



Today's content

→ Addition & Multiplication rule

→ Permutation basics

→ Combination basics & properties

→ Pascal Triangle

→ ${}^{n}C_{r}$ % p

→ Find n^{th} column title

01. Given 3 True/False questions, every question has to be answered. In how many ways can you answer all the questions.

$$\underline{\quad 2 \quad} * \underline{\quad 2 \quad} * \underline{\quad 2 \quad} = 8 \text{ ways}$$

T F T F T F

F F F
F F T
F T F
F T T
T F F
T F T
T T F
T T T

8 ways

Q2 Given 10 girls & 7 boys. How many different pairs can we form?

Note :- pair → 1 Girl + 1 Boy

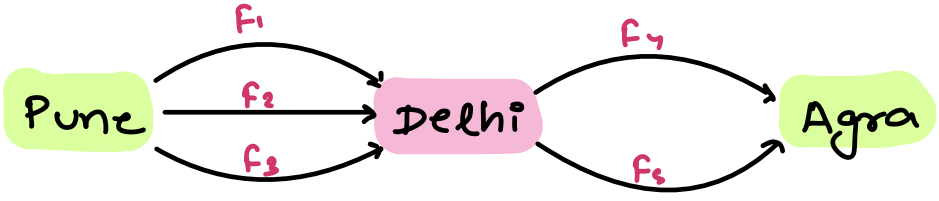
Boys	Girls
B_1	G_1
B_2	G_2
B_3	G_3
B_4	G_4
B_5	⋮
B_6	
B_7	G_{10}

1 Boy 1 Girl

↓

7 * 10 = 70 ways

Eg 3



No. of ways to reach Agra from Pune via Delhi

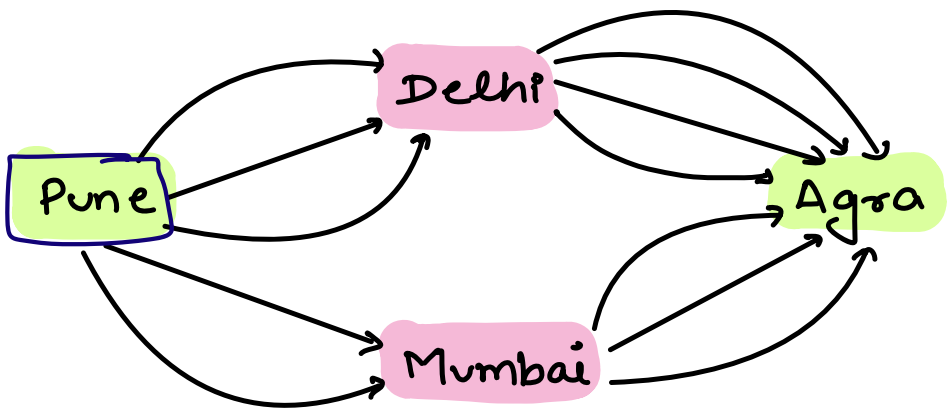
Pune ⟶ Delhi AND Delhi ⟶ Agra

3 * 2 = 6 ways

$F_1 F_4$ $F_2 F_4$ $F_3 F_4$

$F_1 F_5$ $F_2 F_5$ $F_3 F_5$

Eg 4



No. of ways to reach Agra from pune ?

via Delhi OR via Mumbai

Pune → Delhi & Delhi → Agra

P → M & M → A

3 * 4 + 2 * 3

⇒ 12 + 6 = 18 ways

Eg S Say we need to buy a valentine's gift

{pen & book} {flower & chocolate} {ring}

↓ ↓ ↓ ↓ ↓

3 5 7 3 3

No. of ways in which you can select a gift?

Ans = $3 * 5 + 7 * 3 + 3 = 15 + 21 + 3 = 39$

AND → *

OR → +

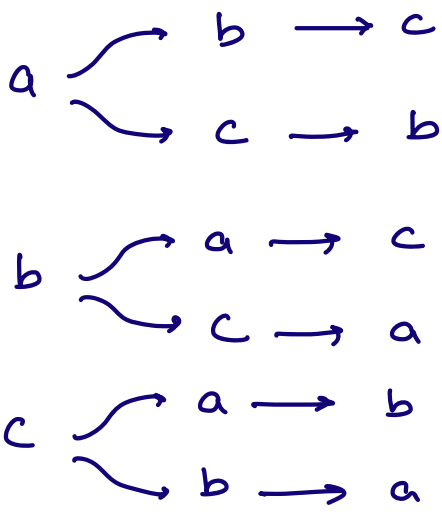
* Permutation → Arrangements of object

$(i, j) \neq (j, i)$ → order matter

Q6. Given 3 distinct characters. In how many ways, we can arrange them?

s = "abc"

$\underline{3} * \underline{2} * \underline{1} \Rightarrow 3!$



For 4 distinct characters = $4!$

Q - In how many ways n distinct characters can be arranged ?

Ans = $n * (n-1) * (n-2) * (n-3) * \ldots * 1 = n!$

Given 5 distinct characters, in how many can you arrange 2 characters

{a, b, c, d, e} $\underline{5} * \underline{4}$ = 20 ways

Q → Given n distinct characters, in how many ways can you arrange 3 characters

$$\underline{n} * \underline{(n-1)} * \underline{(n-2)} = n(n-1)(n-2) \text{ ways}$$

Q → Given n distinct characters, in how many ways can you arrange r characters

Ans = $n * (n-1) * (n-2) * \ldots * (n-(r-1))$

$$= \frac{n * (n-1) * (n-2) * \ldots (n-r+1) * (n-r) * (n-r-1) * (n-r-2) * \ldots * 1}{(n-r) * (n-r-1) * (n-r-2) * \ldots * 1}$$

$$\Rightarrow \frac{n!}{(n-r)!} = {}^{n}P_{r}$$

No. of ways to arrange r places from n distinct characters

* Combinations → {selection} = No. of ways to select something

$\{i, j\} = \{j, i\}$ → order of selection doesn't matter

Q Given 4 players, count no. of ways of selecting 3 players

$\{P_1 \quad P_2 \quad P_3 \quad P_4\}$

$P_1 \quad P_2 \quad P_3$

$P_1 \quad P_2 \quad P_4$

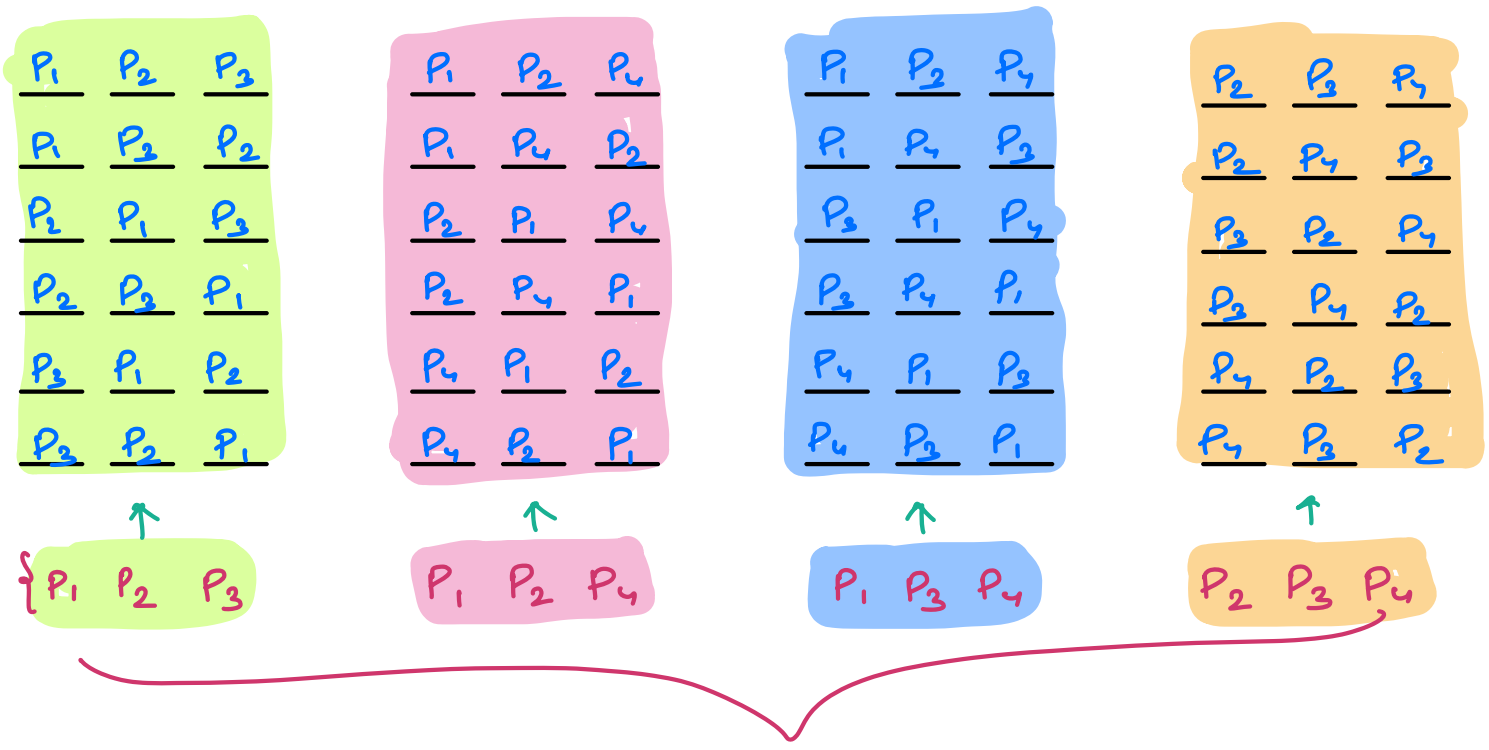
$P_1 \quad P_3 \quad P_4$

$P_2 \quad P_3 \quad P_4$

} 4 ways

Q Given 4 players, write the no. of ways to arrange players in 3 slots

{ P_1 P_2 P_3 P_4 }



Selections

For every selection = 6 arrangements

No. of selection * No. of arrangement for each selection = Total no. of arrangements

$x * 6 = 24$

$x = \frac{24}{6} = 4$

Q Given n distinct ele, in how many ways we can **select** r elements

// Given n distinct ele, arrange r ele

$$^{n}P_{r} = \frac{n!}{(n-r)!}$$

r → Arrange r elements $= r!$

No. of selection for r distinct ele = 1

1, 2, 3

Arrangement	selection
$r!$	1
$\frac{n!}{(n-r)!}$	x

$$r! * x = \frac{n!}{(n-r)!}$$

$$x = \frac{n!}{r!\,(n-r)!}$$

Given n distinct ele, select r ele $= {}^{n}C_{r} = \frac{n!}{r!\,(n-r)!}$

10:05pm → 10:15 pm

r = a b c

No. of ways in which we can arrange r ele $= r!$

No. of ways to select r elements $= 1$

* Properties of Combination

01. No. of ways to select 0 items from N

$$^{N}C_{0} = \frac{N!}{(N-0)! * 0!} = 1$$

02. No. of ways to select n items from n

$$^{n}C_{n} = \frac{n!}{(n-n)! * n!} = \frac{n!}{0! * n!} = 1$$

03. No. of ways to select r items from n

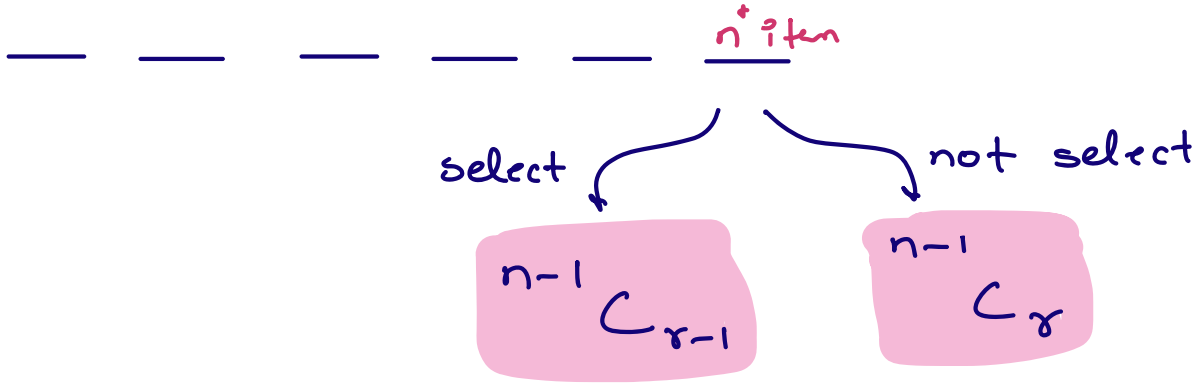
$$^{n}C_{r} = \frac{n!}{(n-r)!\, r!}$$

04. No. of ways to select (n-r) items from n

$$^{n}C_{n-r} = \frac{n!}{(n-r)! * r!}$$

$$^nC_r = {}^nC_{n-r}$$

* Given n distinct character, select r items



$$^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r$$

Microsoft

Q Pascal Triangle for n=4 by %o M

$^0C_0$

$^1C_0$ $^1C_1$

$^2C_0$ $^2C_1$ $^2C_2$

$^3C_0$ $^3C_1$ $^3C_2$ $^3C_3$

$^4C_0$ $^4C_1$ $^4C_2$ $^4C_3$ $^4C_4$

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

* Brute force Approach

↳ For each & every value, calculate the factorials & print it.

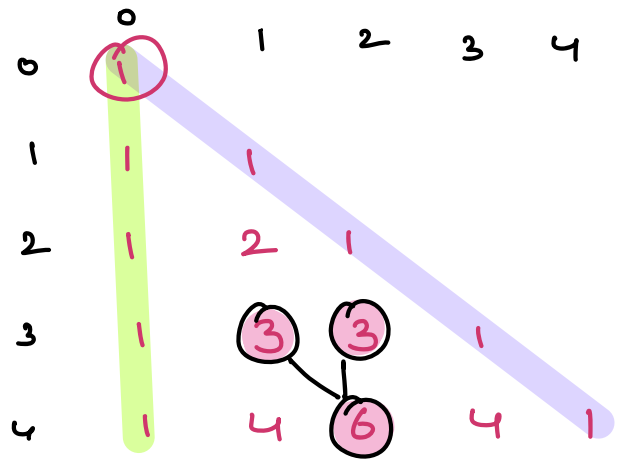
$5!_0 = 120$

$10!_0 = 3628800$

$20!_0 = 2.4 * 10^{18}$

} factorials grow very rapidly

*



* $^nC_0 = 1$

* $^nC_n = 1$

* $^nC_r = {}^{n-1}C_r + {}^{n-1}C_{r-1}$

$^4C_2 = {}^3C_2 + {}^3C_1$

$c[i][j] = c[i-1][j] + c[i-1][j-1]$

* pseudocode

```
System.out.println("1");      // 0th row

for (i=1 ; i<n; i++)
    c[i][0] = 1
    c[i][i] = 1

    System.out.print( c[i][0]); // 0th col

    for (j=1; j<i ; j++) {
        c[i][j] = (c[i-1][j] + c[i-1][j-1]) %m
        print (c[i][j]);
    }

    system.out.print (c[i][i]);  // last col

    println();
}
```

TC: $O(n^2)$

SC: $O(n^2)$

Optimise SC to $O(n)$

Q Find ${}^nC_r \% p$, $p \rightarrow$ prime no.

$n = 5 \quad r = 2 \quad p = 7$

${}^5C_2 \% 7$

$$\Rightarrow \frac{5!}{(5-2)! * 2!} \Rightarrow \frac{5 * 4 * 3!}{3! * 2} = 10 \% 7 \Rightarrow 3$$

Idea $\rightarrow$ Pascal Triangle TC: $O(n^2)$
SC: $O(n^2)$

Idea 2

$${}^nC_r \% p = \left(\frac{n!}{(n-r)! * r!} \right) \% p$$

$$= \left(n! \% p * ((n-r)!)^{-1} \% p) * (r!^{-1} * p) \right) \% p$$

Fermat theorem $\rightarrow$ $x^{-1} \% p = x^{p-2} \% p$

powermod $(x, p-2, p)$

TC: $O(\log p)$

TC to find factorial = O(n)

Now the entire formula is

$$nf = n! \% p$$

$$rf = r! \% p$$

$$nrf = (n-r)! \% p$$

$$Ans = (nf * pow(rf, p-2, p) * pow(nrf, p-2, p)) \% p$$

```
solve(n, r, p)
    long nf = n! % p
    long rf = r! % p
    long nrf = (n-r)! % p
    long rfpow = powmod(rf, p-2, p)
    long nrfpow = powmod(nrf, p-2, p)
    long ans = (((nf * rfpow) % p * nrfpow % p) % p
    return (int)ans;
```

3

Q4 Find n^{th} column title

1	2	3	...	26	27	28	...	52	53	54	...
A	B	C	...	Z	AA	AB	...	AZ	BA	BB	...

n = 3 ⟶ Ans = C

n = 30 ⟶ Ans = AD

n = 50 ⟶ Ans = AX

2	30	0
2	15	1
2	7	1
2	3	1
2	1	1
	0	

Decimal ⟶ Binary

Ans = 11110

Base for mapping A → Z = 26

*

26	26	0
26	1	1
	0	

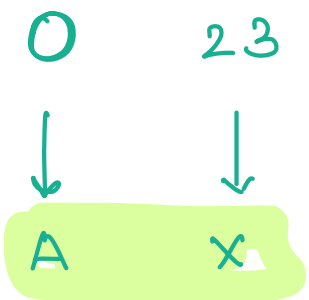
26	26 − 1 = 25	25 → Z
	0	

26	27 − 1 = 26	0 → A
26	1 − 1 = 0	0 → A
	0	

AA

50 ⟶ AX

26	50 − 1 = 49	23
26	1 − 1 = 0	0
	0	



TC: log N

SC: O(1)