Lab 12: MST

1. Given a connected undirected graph $G = (V, E)$. Find the MST using Kruskal's algorithm. Implement the naive algorithm; do not use DSU.
2. Given a connected undirected graph $G = (V, E)$. Find the MST via Prim's algorithm using a) Storing cut edges in the priority queue and b) Storing nodes in the priority queue where the priority of a node is decided by the weight of the cut edges incident to that node.
3. To verify and understand the fundamental properties of the Minimum Spanning Tree (MST) using computation and reasoning. You are given a connected, weighted, undirected graph $G = (V, E)$ with possibly distinct or repeated edge weights. Perform the following experiments to verify key theoretical properties of MSTs.

   Part A — Cut Property

   Statement: For any cut $(S, V \setminus S)$ in a connected graph, the lightest edge crossing the cut belongs to at least one MST.

   Tasks:

   (a) Construct or take a connected weighted graph with 6–8 vertices.
   (b) Choose any non-trivial subset $S \subset V$.
   (c) Identify the minimum-weight edge crossing the cut $(S, V \setminus S)$.
   (d) Compute an MST of G.
   (e) Verify whether the chosen edge appears in the MST.

   Expected Observation: The lightest edge crossing any cut is always part of some MST.

   Part B — Cycle Property

   Statement: In any cycle within the graph, the heaviest edge cannot belong to any MST.

   Tasks:

   (a) Identify at least two distinct cycles in G.
   (b) For each cycle, find the heaviest edge.
   (c) Check whether these edges appear in your MST.

   Expected Observation: The heaviest edge in any cycle never appears in the MST.

Part C — Uniqueness Property

Statement: If all edge weights in the graph are distinct, then the MST is unique.

Tasks:

(a) Create two versions of the same graph:
- Version 1: All edge weights distinct.
- Version 2: Some edge weights repeated.

(b) Compute MSTs for both versions using any MST algorithm.

(c) Compare whether the resulting MSTs are identical.

Expected Observation: Distinct edge weights guarantee a unique MST. Repeated weights may lead to multiple MSTs.

Part D — Sensitivity of Edge Weights

Statement: The MST changes when certain edge weights cross specific threshold values.

Tasks:

(a) Select an MST edge e_1 and gradually increase its weight. Observe when it is no longer part of the MST.

(b) Select a non-MST edge e_2 and gradually decrease its weight. Observe when it becomes part of the MST.

(c) Record the threshold weights at which MST structure changes.

Expected Observation: Each edge has a critical threshold beyond which it enters or leaves the MST.