Modern Robotics: Mechanics, Planning, and Control

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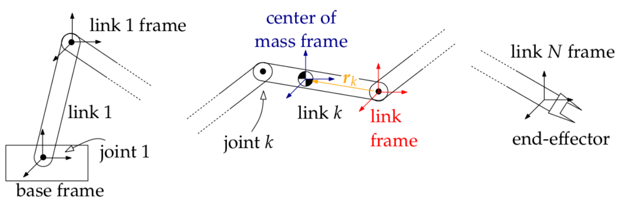
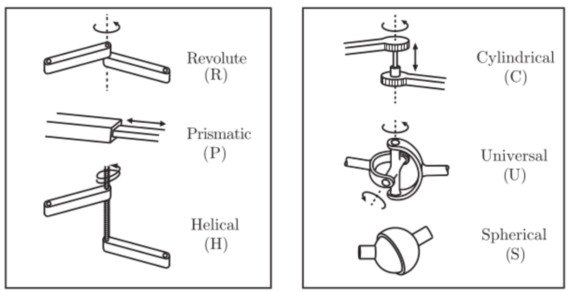
1 Foundations of Robot Motion

1.1 Degrees of Freedom of a Rigid Body

Configuration: a specification of the positions of all the points of the robot.
Robots are constructed of rigid bodies called links. These links are connected together by joints.
C-space: is the space of all configurations of the robot.
The number of degrees of freedom (DOF) is the dimension of the C-space, or the minimum number of real numbers you need to represent the configuration



Rigid body in space has six total degrees of freedom, three of which are linear, or x-y-z, and three of which are angles, sometimes called roll, pitch, and yaw.
dof $= \sum$ (freedoms of bodies) $-$ numbers of independent constraints.
A rigid body in n-dimensional space has m total degrees of freedom. How many of these m degrees of freedom are angular (not linear)? $m-n$=$n(n-1)/2$



Typical robot joints.

Joint type	dof f	Constraints c between two planar rigid bodies	Constraints c between two spatial rigid bodies
Revolute (R)	1	2	5
Prismatic (P)	1	2	5
Helical (H)	1	N/A	5
Cylindrical (C)	2	N/A	4
Universal (U)	2	N/A	4
Spherical (S)	3	N/A	3

The number of degrees of freedom f and constraints c provided by common joints.

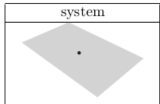
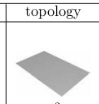
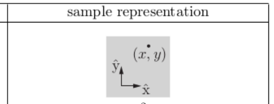
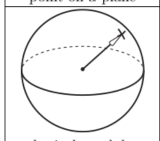
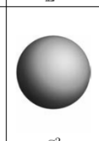
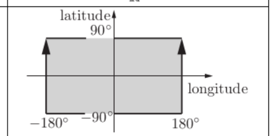
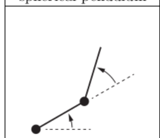
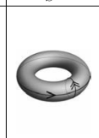
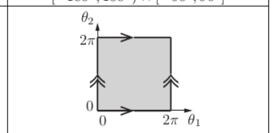
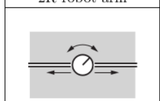
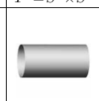
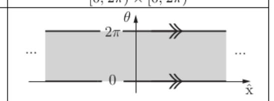
$N = \#$ of bodies, including ground.
$J = \#$ of joints, $m = 6$ for spatial bodies, 3 for planar

$$\text{dof} = \underbrace{m(N-1)}_{\text{rigid body freedoms}} - \underbrace{\sum_{i=1}^{J} c_i}_{\text{joint constraints}} = m(N-1) - \sum_{i=1}^{J}(m - f_i)$$

Grübler's formula (all constraints independent)

$$\text{dof} = m(N-1-J) + \sum_{i=1}^{J} f_i$$

1.2 Configuration Space Topology

system	topology	sample representation
 point on a plane	 $\mathbb{E}^2$	 $\mathbb{R}^2$
 spherical pendulum	 S^2	 $[-180°, 180°) \times [-90°, 90°]$
 2R robot arm	 $T^2 = S^1 \times S^1$	 $[0, 2\pi) \times [0, 2\pi)$
 rotating sliding knob	 $\mathbb{E}^1 \times S^1$	 $\mathbb{R}^1 \times [0, 2\pi)$

We say two spaces have the same "shape", or more formally that they are TOPOLOGICALLY EQUIVALENT, if one can be smoothly deformed into the other, without cutting or gluing. The topology of the space is independent of the representation of the space.

Explicit Representation:
It uses n parameters to represent a n-dimensional space. But it has sigularities, for example the North/South pole situation.
There are two solutions for the sigularities:
-Use multiple coordinate charts (atlas).
-Use implicit representation.

Implicit Representation:
It uses more parameters than the space's dof. It views the n-dimensional space as embedded in a Euclidean space of more than n dimensions. It uses the coordinates of the higher-dimensional space, but subjects these coordinates to constraints that reduce the number of degrees of freedom. It has advantage in representing closed-chain mechanism.

1.3 Holonomic constraints

Constraints that reduce the dimension of C-space. The C-space can be viewed as a surface of dimension $n-k$ embedded in $\mathbb{R}^n$, where n indicates the number of parameters that are used to represent the space, and k indicates the number of constraints holonomic. (In the example of the planar, closed-chain, four-bar linkage, $n=4$ and $k=3$)

$$\theta = [\theta_1 \cdots_n]^T \quad g(\theta) = \begin{bmatrix} g_1(\theta_1, \ldots, \theta_n) \\ g_k(\theta_1, \ldots, \theta_n) \end{bmatrix} = 0$$

$$\begin{bmatrix} \frac{\partial g_1}{\partial \theta_1}(\theta)\dot{\theta}_1 + \cdots + \frac{\partial g_1}{\partial \theta_n}(\theta)\dot{\theta}_n \\ \vdots \\ \frac{\partial g_k}{\partial \theta_1}(\theta)\dot{\theta}_1 + \cdots + \frac{\partial g_k}{\partial \theta_n}(\theta)\dot{\theta}_n \end{bmatrix} = 0$$

$$\begin{bmatrix} \frac{\partial g_1}{\partial \theta_1}(\theta) & \cdots & \frac{\partial g_1}{\partial \theta_n}(\theta) \\ \vdots & \ddots & \vdots \\ \frac{\partial g_k}{\partial \theta_1}(\theta) & \cdots & \frac{\partial g_k}{\partial \theta_n}(\theta) \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \vdots \\ \dot{\theta}_n \end{bmatrix} = A(\theta)\dot{\theta} = 0$$

where the A matrix has k rows and n columns. Velocity constraints like this are called **Pfaffian constraints**.
Holonomic Constraints: Constraints on configuration. Constraints that reduce the dimension of C-space.
Nonholonomic Constraints: Constraints on velocity. Pfaffian constraints that are non-integrable. (example: A coin rolling on a plane without slipping)

1.4 Task Space and Workspace

Task Space: The space in which the robot's task is naturally described.

Work Space: A specification of the reachable configurations of the end-effector.

Both spaces are distinct from the robot's C-space.

1.5 Rotation Matrices

A rotation matrix R is in $SO(3)$(special orthogonal group). The special orthogonal group $SO(3)$ is the set of all 3×3 real matrices R satisfying $R^{\mathrm{T}}R = I$ and $\det R = 1$

Properties of rotation matrices

inverse: $R^{-1} = R^{\mathrm{T}} \in SO(3)$
closure: $R_1 R_2 \in SO(3)$
associative: $(R_1 R_2) R_3 = R_1 (R_2 R_3)$
not commutative:$R_1 R_2 \neq R_2 R_1$
$x \in \mathbb{R}^3, \|Rx\| = \|x\|$

Usages of Rotation Matrices:

- Representing an orientation
Here the rotation matrix R_{sb} can be considered as the representation of the three axes in b at s.

- Changing the reference frame
Here the rotation matrix R_{sb} can be considered as an operation to change the reference frame of a matrix from s to b. A subscription cancellation rule can be exploited.

- Rotating a vector or a frame
Here the rotation matrix R_{sb} can be considered as an operation that rotates s about an axis $\hat{\omega}$ for θ to b. We have $R = Rot(\hat{\omega}, \theta)$.

1.6 Angular Velocities

Angular velocity is defined as $w = \hat{w}\dot{\theta}$. It's a 3-vector, where $\hat{w}$ indicates the axis of rotation and $\dot{\theta}$ indicates the rate of rotation.

For a given body frame, whose orientation is represented by a rotation matrix $R = \begin{bmatrix} r_1 & r_2 & r_3 \end{bmatrix}$. As it rotate with the fixed angular velocity ω_s, which is described in s, R is a function of time t. The rate of change of R can then be represented as:

$$\dot{R} = \begin{bmatrix} \omega_s \times r_1 & \omega_s \times r_2 & \omega_s \times r_3 \end{bmatrix} = \omega_s \times R$$

Notation: skew-symmetric matrices

$$x \times y = [x]y$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3, \quad [x] = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \quad [x] = -[x]^{\mathrm{T}}$$

The set of all 3×3 skew-symmetric real matrices is called so(3).

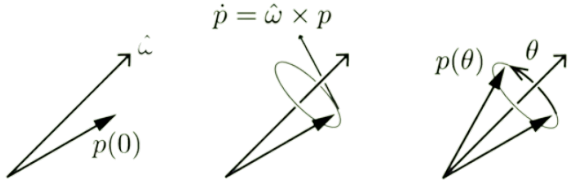
$$\dot{R}_{sb} = \begin{bmatrix} \dot{\hat{x}}_b & \dot{\hat{y}}_b & \dot{\hat{z}}_b \end{bmatrix} = [\omega_s] R_{sb}$$
$$\omega_b = R_{bs}\omega_s = R_{sb}^{-1}\omega_s = R_{sb}^{\mathrm{T}}\omega_s$$

then

$$[\omega_b] = R^{-1}\dot{R} = R^{\mathrm{T}}\dot{R}$$
$$[\omega_s] = \dot{R}R^{-1} = \dot{R}R^{\mathrm{T}}$$

1.7 Exponential Coordinates of Rotation

Any orientation can be achieved from an initial orientation aligned with the space frame by rotating about some unit axis by a particular angle. We call the unit axis $\hat{\omega}$ and the rotation distance θ. If we multiply these two together, we get the 3-vector$\hat{\omega}\theta$. We call these 3 parameters the exponential coordinates representing the orientation of one frame relative to another. This is an alternative representation to a rotation matrix. We call these exponential coordinates because of the connection to linear differential equations.



During the rotation, we have:

$$\dot{p} = [\hat{\omega}]p$$

this equation can be considered as a differential equation of the form $\dot{x} = Ax$, the solution is:

$$p(t) = e^{[\hat{\omega}]t}p(0)$$

where $p(0)$ is the initial position of vector p. Since the rate of rotation is 1 rad/s, we can replace t by θ, so we have:

$$p(\theta) = e^{[\hat{\omega}]\theta}p(0)$$

$e^{[\hat{\omega}]\theta}$ serves as a rotation matrix that rotates vector $p(0)$ to $p(\theta)$. $e^{[\hat{\omega}]\theta}$, after expansion, can be written as:

$$e^{[\hat{\omega}]\theta} = I + \sin\theta[\hat{\omega}] + (1-\cos\theta)[\hat{\omega}]^2$$

For a matrix $A \in \mathbb{R}^{n\times n}$, e^A is the matrix exponential of A. Here the equation that uses the matrix exponential of $[\hat{\omega}]\theta$ to represent a rotation matrix is called Rodrigues' formula:

$$Rot(\hat{\omega}, \theta) = e^{[\hat{\omega}]\theta} = I + \sin\theta[\hat{\omega}] + (1-\cos\theta)[\hat{\omega}]^2 \in SO(3)$$

Recall that, when using rotation matrix as rotation operation, $\hat{\omega}$ is coordinate-dependent. Thus pre-multiplying and post-multiplying a rotation matrix can have difference results.

$$\begin{aligned} \exp: &\quad [\hat{\omega}]\theta \in so(3) \to R \in SO(3) \\ \log: &\quad R \in SO(3) \to [\hat{\omega}]\theta \in \mathrm{so}(3) \end{aligned}$$

$$\mathrm{Rot}(\hat{\omega}, \theta) = \\ \begin{bmatrix} c_\theta + \hat{\omega}_1^2(1-c_\theta) & \hat{\omega}_1\hat{\omega}_2(1-c_\theta) - \hat{\omega}_3 s_\theta & \hat{\omega}_1\hat{\omega}_3(1-c_\theta) + \hat{\omega}_2\ \mathrm{s}_\theta \\ \hat{\omega}_1\hat{\omega}_2(1-c_\theta) + \hat{\omega}_3\ \mathrm{S}\theta & c_\theta + \hat{\omega}_2^2(1-c_\theta) & \hat{\omega}_2\hat{\omega}_3(1-\mathrm{c}_\theta) - \hat{\omega}_1\ \mathrm{s}_\theta \\ \hat{\omega}_1\hat{\omega}_3(1-\mathrm{c}_\theta) - \hat{\omega}_2\ \mathrm{S}_\theta & \hat{\omega}_2\hat{\omega}_3(1-c_\theta) + \hat{\omega}_1\ \mathrm{S}\theta & \mathrm{c}_\theta + \hat{\omega}_3^2(1-c_\theta) \end{bmatrix}$$

$$\mathrm{tr}R = r_{11} + r_{22} + r_{33} = 1 + 2\cos\theta$$

1.8 Homogeneous Transformation Matrices

The special Euclidean group $SE(3)$ is the set of all 4×4 real matrices T of the form

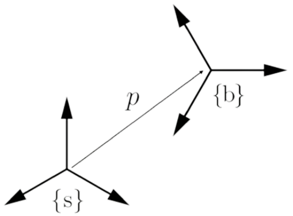
$$T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_1 \\ r_{21} & r_{22} & r_{23} & p_2 \\ r_{31} & r_{32} & r_{33} & p_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R \in SO(3), p \in \mathbb{R}^3$

$$\text{inverse: } T^{-1} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} R^{\mathrm{T}} & -R^{\mathrm{T}}p \\ 0 & 1 \end{bmatrix} \in SE(3)$$

$$\text{closure: } T_1T_2 \in SE(3)$$
$$\text{associative: } (T_1T_2)T_3 = T_1(T_2T_3)$$
$$\text{not commutative: } T_1T_2 \neq T_2T_1$$

Usages of Transformation Matrices
Analogous to rotation matrices, transformation matrices have three usages:
Representing a configuration



$$T_{sb} = \begin{bmatrix} R_{sb} & p \\ 0 & 1 \end{bmatrix}$$

$$T_{sb} = T_{bs}^{-1}$$

Changing the reference frame of a vector or a frame
The subscript cancellation rule also applies here:

$$T_{ab}T_{bc} = T_{ac}$$

$$T_{ab}v_b = v_a$$

Displacing(rotating and translating) a vector or a frame We have:
A transformation matrix can be seen as a combination of roation and translation operations.

Same with rotation matrix, transformation matrix here is coordinate-dependent. Whether we pre-multiply or post-multiply T_{sb} by $T = (R, p)$ determines whether the $\hat{\omega}$-axis and p are interpreted as in the fixed frame or the body frame:
$\hat{\omega}$ is interpreted as in the fixed frame

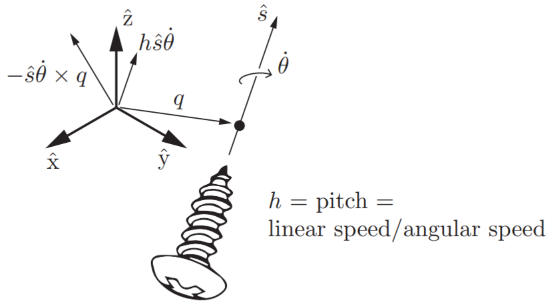
$$T_{sb'} = TT_{sb} = Trans(p) \underbrace{Rot(\hat{\omega}, \theta)T_{sb}}_{\text{rotate, then translate}}$$

$\hat{\omega}$ is interpreted as in the body frame

$$T_{sb''} = T_{sb}T = \underbrace{T_{sb}Trans(p)}_{\text{translate, then rotate}} Rot(\hat{\omega}, \theta)$$

1.9 Twists

Any rigid-body velocity, which consists of a linear component and an angular component, is equivalent to the instantaneous velocity about some screw axis.



screw

$$\mathcal{S} = \begin{bmatrix} \mathcal{S}_\omega \\ \mathcal{S}_v \end{bmatrix} = \begin{bmatrix} \text{angular velocity when } \dot{\theta} = 1 \\ \text{linear velocity of the origin when } \dot{\theta} = 1 \end{bmatrix}$$

$$\text{twist: } \mathcal{V} = \begin{bmatrix} \omega \\ v \end{bmatrix} = \mathcal{S}\dot{\theta}$$

- Pitch h is infinite:

$$\mathcal{S}_\omega = 0, \|\mathcal{S}_v\| = 1, \dot{\theta} = \text{ linear speed}$$

- Pitch h is finite:

$$\|\mathcal{S}_\omega\| = 1, \dot{\theta} = \text{ rotational speed}$$

If $\mathcal{S}$ is defined in $\{b\}, \mathcal{V}_b = (\omega_b, v_b) = \mathcal{S}\dot{\theta}$ is the body twist.
If $\mathcal{S}$ is defined in $\{s\}, \mathcal{V}_s = (\omega_s, v_s) = \mathcal{S}\dot{\theta}$ is the spatial twist.

Change the frame of representation of a twist
The 6×6 adjoint representation of a transformation matrix

$$T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \text{ is } [\mathrm{Ad}_T] = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix} \in \mathbb{R}^{6\times 6}$$
$$\mathcal{V}_a = [\mathrm{Ad}_{T_{ab}}]\mathcal{V}_b$$

Recall that rotation matrices have a property that:

$$\dot{R}R^{-1} = [\omega_s]$$

$$R^{-1}\dot{R} = [\omega_b]$$

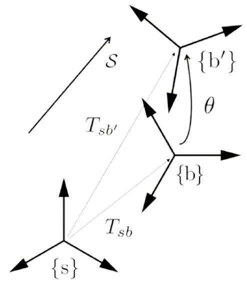
Homogeneous transformation matrices have a similar one:

$$[\mathcal{V}_b] = T^{-1}\dot{T} = \begin{bmatrix} [\omega_b] & v_b \\ 0 & 0 \end{bmatrix} \in se(3)$$

where $[\omega_b]$ is the angular velocity expressed in the b frame. $v_b = R^T\dot{p}$, which can be written as $R_{bs}p_s = p_b$, is the linear velocity of the origin of frame b expressed in b.

$$[\mathcal{V}_s] = \dot{T}T^{-1} = \begin{bmatrix} [\omega_s] & v_s \\ 0 & 0 \end{bmatrix} \in se(3)$$

$$\mathcal{S} = (\mathcal{S}_\omega, \mathcal{S}_v)$$
$$\exp: \quad se(3) \to SE(3)$$
If $\mathcal{S}_\omega = 0, \|\mathcal{S}_v\| = 1:$
$$e^{[\mathcal{S}]\theta} = \begin{bmatrix} I & \mathcal{S}_v\theta \\ 0 & 1 \end{bmatrix}$$
If $\|\mathcal{S}_\omega\| = 1:$
$$e^{[\mathcal{S}]\theta} = \begin{bmatrix} e^{[\mathcal{S}_\omega]\theta} & \left(I\theta + (1-\cos\theta)[\mathcal{S}_\omega] + (\theta - \sin\theta)[\mathcal{S}_\omega]^2\right)\mathcal{S}_v \\ 0 & 1 \end{bmatrix}$$



If $\mathcal{S}$ is expressed in $\{b\}$:

$$T_{sb'} = T_{sb}e^{[\mathcal{S}_b]\theta}$$

If $\mathcal{S}$ is expressed in $\{s\}$

$$T_{sb'} = e^{[\mathcal{S}_s]\theta}T_{sb}$$

Wrenches
Given a reference frame a, a force $f_a \in \mathbb{R}^3$ creates a torque / moment $m_a \in \mathbb{R}^3$ after acting on a rigid body at point $r_a \in \mathbb{R}^3$:

$$m_a = r_a \times f_a$$

Combining moment and force together, we have the six-dimensional spatial force / wrench $\mathcal{F}_a$:

$$\mathcal{F}_a = \begin{bmatrix} m_a \\ f_a \end{bmatrix} \in \mathbb{R}^6$$

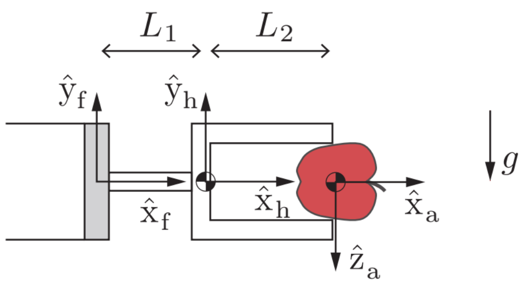
Similar with the case of twist, the reference frame of a wrench can be switched with the help of the adjoint representation of a transformation matrix. But the case is slightly different here:

$$\mathcal{F} b = Ad^T T_{ab}(\mathcal{F} a) = [AdT_{ab}]^T \mathcal{F}_a$$

The dot product of a twist and a wrench is power. Power does not depend on a coordinate frame, and therefore the power must be the same whether the wrench and twist are represented in the b frame or in the s frame.

$$\mathcal{V}_b^{\mathrm{T}} \mathcal{F}_b = \text{ power}$$

Example A robot hand holding an apple subject to gravity



$$\mathcal{F}_a = \begin{bmatrix} m_a \\ f_a \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -\mathrm{mg} \\ 0 \end{bmatrix} \quad T_{af} = \begin{bmatrix} 1 & 0 & 0 & -L \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathcal{F}_f = [\mathrm{Ad}_{T_{af}}]^{\mathrm{T}} \mathcal{F}_a$$

$$\mathcal{F}_f = \begin{bmatrix} m_f \\ f_f \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\mathrm{mg}L \\ 0 \\ -\mathrm{mg} \\ 0 \end{bmatrix}$$

Summary

Rotations	Rigid-Body Motions
$R \in SO(3) : 3 \times 3$ matrices $R^{\mathrm{T}} R = I, \det R = 1$	$T \in SE(3) : 4 \times 4$ matrices $T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$, where $R \in SO(3), p \in \mathbb{R}^3$
$R^{-1} = R^{\mathrm{T}}$	$T^{-1} = \begin{bmatrix} R^{\mathrm{T}} & -R^{\mathrm{T}} p \\ 0 & 1 \end{bmatrix}$
change of coordinate frame: $R_{ab} R_{bc} = R_{ac}, \;\; R_{ab} p_b = p_a$	change of coordinate frame: $T_{ab} T_{bc} = T_{ac}, \;\; T_{ab} p_b = p_a$
rotating a frame {b}: $R = \mathrm{Rot}(\hat{\omega}, \theta)$ $R_{sb'} = R R_{sb}$: rotate θ about $\hat{\omega}_s = \hat{\omega}$ $R_{sb''} = R_{sb} R$: rotate θ about $\hat{\omega}_b = \hat{\omega}$	displacing a frame {b}: $T = \begin{bmatrix} \mathrm{Rot}(\hat{\omega}, \theta) & p \\ 0 & 1 \end{bmatrix}$ $T_{sb'} = T T_{sb}$: rotate θ about $\hat{\omega}_s = \hat{\omega}$ (moves {b} origin), translate p in {s} $T_{sb''} = T_{sb} T$: translate p in {b}, rotate θ about $\hat{\omega}$ in new body frame
unit rotation axis is $\hat{\omega} \in \mathbb{R}^3$, where $\\|\hat{\omega}\\| = 1$	"unit" screw axis is $\mathcal{S} = \begin{bmatrix} \omega \\ v \end{bmatrix} \in \mathbb{R}^6$, where either (i) $\\|\omega\\| = 1$ or (ii) $\omega = 0$ and $\\|v\\| = 1$
	for a screw axis $\{q, \hat{s}, h\}$ with finite h, $\mathcal{S} = \begin{bmatrix} \omega \\ v \end{bmatrix} = \begin{bmatrix} \hat{s} \\ -\hat{s} \times q + h\hat{s} \end{bmatrix}$
angular velocity is $\omega = \hat{\omega}\dot{\theta}$	twist is $\mathcal{V} = \mathcal{S}\dot{\theta}$

Rotations (cont.)	Rigid-Body Motions (cont.)
for any 3-vector, e.g., $\omega \in \mathbb{R}^3$, $[\omega] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \in so(3)$ identities, $\omega, x \in \mathbb{R}^3, R \in SO(3)$: $[\omega] = -[\omega]^{\mathrm{T}}, [\omega]x = -[x]\omega$, $[\omega][x] = ([x][\omega])^{\mathrm{T}}, R[\omega]R^{\mathrm{T}} = [R\omega]$	for $\mathcal{V} = \begin{bmatrix} \omega \\ v \end{bmatrix} \in \mathbb{R}^6$, $[\mathcal{V}] = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} \in se(3)$ (the pair (ω, v) can be a twist $\mathcal{V}$ or a "unit" screw axis $\mathcal{S}$, depending on the context)
$\dot{R}R^{-1} = [\omega_s], \;\; R^{-1}\dot{R} = [\omega_b]$	$\dot{T}T^{-1} = [\mathcal{V}_s], \;\; T^{-1}\dot{T} = [\mathcal{V}_b]$
	$[\mathrm{Ad}_T] = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix} \in \mathbb{R}^{6\times 6}$ identities: $[\mathrm{Ad}_T]^{-1} = [\mathrm{Ad}_{T^{-1}}]$, $[\mathrm{Ad}_{T_1}][\mathrm{Ad}_{T_2}] = [\mathrm{Ad}_{T_1 T_2}]$
change of coordinate frame: $\hat{\omega}_a = R_{ab}\hat{\omega}_b, \;\; \omega_a = R_{ab}\omega_b$	change of coordinate frame: $\mathcal{S}_a = [\mathrm{Ad}_{T_{ab}}]\mathcal{S}_b, \;\; \mathcal{V}_a = [\mathrm{Ad}_{T_{ab}}]\mathcal{V}_b$
exp coords for $R \in SO(3)$: $\hat{\omega}\theta \in \mathbb{R}^3$	exp coords for $T \in SE(3)$: $\mathcal{S}\theta \in \mathbb{R}^6$
$\exp : [\hat{\omega}]\theta \in so(3) \to R \in SO(3)$ $R = \mathrm{Rot}(\hat{\omega}, \theta) = e^{[\hat{\omega}]\theta} =$ $I + \sin\theta[\hat{\omega}] + (1 - \cos\theta)[\hat{\omega}]^2$	$\exp : [\mathcal{S}]\theta \in se(3) \to T \in SE(3)$ $T = e^{[\mathcal{S}]\theta} = \begin{bmatrix} e^{[\omega]\theta} & * \\ 0 & 1 \end{bmatrix}$ where $* =$ $(I\theta + (1 - \cos\theta)[\omega] + (\theta - \sin\theta)[\omega]^2)v$
$\log : R \in SO(3) \to [\hat{\omega}]\theta \in so(3)$	$\log : T \in SE(3) \to [\mathcal{S}]\theta \in se(3)$
moment change of coord frame: $m_a = R_{ab} m_b$	wrench change of coord frame: $\mathcal{F}_a = (m_a, f_a) = [\mathrm{Ad}_{T_{ba}}]^{\mathrm{T}} \mathcal{F}_b$

References

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