A New Variable Structure PID-Controller Design for Robot Manipulators

E. M. Jafarov, M. N. A. Parlakçı, and Y. Istefanopulos

Abstract—**In this brief, a new variable structure proportional-integral-derivative (PID) controller design approach is considered for the tracking stabilization of robot motion. The work corroborates the utility of a certain PID sliding mode controller with PID sliding surface for tracking control of a robotic manipulator. Different from the general approach, the conventional equivalent control term is not used in this controller because that needs to use the matching conditions and exact full robot dynamics knowledge, which involves unavailable parameter uncertainties. Though the sliding surface includes also the integral error term, which makes the robot tracking control problem complicated, the existence of a sliding mode and gain selection guideline are clearly investigated. Moreover, different from uniformly ultimately boundedness, the global asymptotic stability of the robot system with proposed controller is analyzed. The sliding and global stability conditions are formulated in terms of Lyapunov full quadratic form and upper and lower matrix norm inequalities. Reduced design is also discussed. The proposed control algorithm is applied to a two-link direct drive robot arm through simulations. The simulation results indicate that the control performance of the robot system is satisfactory. The chattering phenomenon is handled by the use of a saturation function replaced with a pure signum function in the control law. The saturation function results in a smooth transient performance. The proposed approach is compared with the existing alternative sliding mode controllers for robot manipulators in terms of advantages and control performances. A comparative analysis with a plenty of simulation results soundly confirmed that the performance of developed variable structure PID controller is better under than those of both classical PID controller and an existing variable structure controller with PID-sliding surface.**

Index Terms—**Lyapunov full quadratic form, robot tracking control, sliding mode, variable structure PID controller.**

I. Introduction

A WELL KNOWN approach to the control of uncertain system by nonlinear feedback laws is the variable structure control [1]–[4], etc. In recent years, the variable structure principles are widely used for the stabilization of robot motion. Variable structure control is a powerful control technology. It is often used to handle the worst-case control environment: parametric perturbations with lower and upper bounds, nonlinearities, external disturbances, friction and complexity, etc. Precise dynamic models are not required and the control algorithms can be easily implemented. Set-point regulation control problem is considered in [5] and [6]. A relay type of sliding mode controller with equivalent control approach for robot manipulators with parameter perturbations is investigated in [7]. A continuous sliding mode control law is used in [8] and [9]. A sliding mode controller is proposed in [10], where simultaneous position and force control of constraint robot manipulators are taken into account. An integral type of variable structure control approach is presented in [11] for a guided missile system. The general approach in the sliding mode control is that the control law consists of two parts. One is the conventional equivalent control and another is the switching part. For the switching part, similar structure is also well documented in [12]. An adaptive variable structure control for robot manipulators is considered in [12].

An augmented sliding surface design for robot manipulators is considered in [13]. Both sliding and stability issues are taken into account. Integral variable structure controllers for robot manipulators [14] and electrohydraulic velocity servo systems [15] are designed by Chern and Wu. A first survey of robust control has been presented in [16]. Another survey paper [17] presents an overview of six different robust control schemes including current robot state coordinates for robot manipulators. Discontinuous min–max control term combined with the linear control term is used in robot control systems. However, sliding and stability conditions are not considered in this brief. A new combined variable structure controller with PID sliding surfaces for robot manipulators is proposed in [18]. This controller consists of two parts: 1) linear PID control and 2) discontinuous unit vector term with PID sliding surface. Both regular and adaptive versions of the controller are presented. However, the control law involves a nonlinear vector norm. Exponentially stable sliding conditions and uniformly ultimately boundedness of robot system are investigated. The simulation results have demonstrated that the PID sliding surface provides faster response than that of traditional PD-manifold controller. A more recent paper that deals with chattering issue can be seen in [19]. Passivity-based adaptive and nonadaptive chattering-free sliding mode controllers are proposed. A desired transient response with global exponential convergence of tracking errors is obtained.

In this brief, a new variable structure PID controller design approach is considered for the tracking stabilization of robot motion. The work corroborates the utility of a certain PID sliding mode controller with PID sliding surface for tracking control of a robotic manipulator. Different from the general approach, the conventional equivalent control term is not used in this controller because that needs to use the matching conditions and exact full robot dynamics knowledge, which involves unavailable parameter uncertainties. Though the sliding surface includes also the integral error term, which makes the robot tracking control problem complicated, the existence of a sliding mode and gain selection guideline are clearly investigated. Moreover, different from uniformly ultimately boundedness, the global asymptotic stability of the robot system with proposed controller is analyzed. The sliding and global stability conditions are formulated in terms of Lyapunov full quadratic

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E. M. Jafarov is with the Department of Aearonautical and Astronautical Engineering, İstanbul Technical University, Maslak, 80626 İstanbul, Turkey (e-mail: cafer@itu.edu.tr).

M. N. A. Parlakçı is with the Department of Computer Science, İstanbul Bilgi University, Dolapdere, 34440 İstanbul, Turkey (e-mail: aparlakci@bilgi.edu.tr).

Y. Istefanopulos is with the Electrical and Electronics Engineering Department, Boğaziçi University, Bebek, 80815 İstanbul, Turkey (e-mail: istef@boun.edu.tr).

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form and upper and lower matrix norm inequalities. Reduced design is also discussed. The proposed control algorithm is applied to a two-link direct drive robot arm through simulations. The simulation results indicate that the control performance of the robot system is satisfactory. The chattering phenomenon is handled by the use of a saturation function replaced with a pure signum function in the control law. The saturation function results in a smooth transient performance. The proposed approach is compared with the existing alternative sliding mode controllers for robot manipulators in terms of advantages and control performances. A comparative analysis with a plenty of simulation results soundly confirmed that the performance of developed variable structure PID controller is better under than those of both classical PID controller and an existing variable structure controller with PID-sliding surface.

II. Variable Structure PID-Controller Design With Inexact Robot Parameters

Different from what exists currently, a new robot tracking variable structure PID controller with a PID sliding surface design method is presented. The conventional equivalent control term is not used in this controller, because we assume that we do not know exact full knowledge about the dynamics and parameters of the robot arm system. We only know the current value of the angular position and the angular velocity by measuring the corresponding sensors. In general, the robot parameters mass, inertia, Coriolis and centrifugal force effect, friction, and gravity effects, etc. are not perfectly known. These measured robot parameters are very difficult nonlinear functions of joint position and joint velocity, and they intensively change in some intervals variously dependent on the variation of joint position and joint velocity. However, the general form of the inertia matrix, Coriolis, centrifugal, friction, and gravity effects can be modeled because robot parameters involve a nominal part and some variation, Therefore, the upper and lower matrix norm of these parameters can be obtained and used in the selection of the design parameters. This is an advantage of our design methods. Moreover, a parameter uncertainty is not presented in evident form in robot dynamics. The design parameters of the variable structure controller are parametrically obtained by analyzing the sliding and global asymptotical stability conditions. These conditions are formulated in terms of Lyapunov full quadratic form and some min–max matrix norm inequalities.

A. Dynamics of the Robot Manipulator

Introducing $\psi(t) = \int \theta(t)\,dt$, one can consider an n-link manipulator [7], [20], whose augmented dynamics are given by

$$\begin{aligned}&\dot{\psi} = \theta, \quad \dot{\theta} = \omega\\ &\dot{\omega} = M(\theta)^{-1}[-B(\theta,\dot{\theta})\dot{\theta} - f(\dot{\theta}) - g(\theta) + u(t)]\end{aligned} \tag{1}$$

where ψ is an $n \times 1$ vector of angular joint position integral, θ is an $n \times 1$ vector of angular joint position, ω is an $n \times 1$ vector of angular joint velocity, $M(\theta)$ is an $n \times n$ symmetric positive–definite inertia matrix, $B(\theta,\dot{\theta})$ is an $n \times n$ coriolis and centripetal force matrix, $f(\dot{\theta})$ is an $n \times 1$ vector of joint friction, $g(\theta)$ is an $n \times 1$ vector of gravity, and $u(t)$ is an $n \times 1$ vector of control input. The state vector of the dynamics is given as $x = [\psi\ \theta\ \omega]$. For the proceeding analysis, we need to construct the error dynamics of the robot arm system. We denote $\theta_d(t)$ as a time-varying reference position vector, the integral of the reference position obtained as $\psi_d(t) = \int_0^t \theta_d(t')\,dt'$ and the reference velocity given by $\omega_d(t) = \dot{\theta}_d(t)$. Then, the deviation of the actual position integral, position, and velocity from the reference counterparts are denoted by $\tilde{\psi}(t) = \psi(t) - \psi_d(t), \tilde{\theta}(t) = \theta(t) - \theta_d(t), \tilde{\omega}(t) = \omega(t) - \omega_d(t)$, respectively. Then, the error dynamics can be written as follows:

$$\begin{aligned}\dot{\tilde{\psi}} = \tilde{\theta}, \quad \dot{\tilde{\theta}} = \tilde{\omega}, \quad \dot{\tilde{\omega}} = M(\theta)^{-1}[-B(\theta,\dot{\theta})\tilde{\omega} - f(\dot{\theta})\\ - g(\theta) + u(t)] - M(\theta)^{-1}B(\theta,\dot{\theta})\dot{\theta}_d(t) - \ddot{\theta}_d(t).\end{aligned} \tag{2}$$

The aim of the proposed controller is to design a general purpose PID sliding mode tracking controller for the robot manipulator system such that the system (1) will be globally asymptotically stable.

B. Variable Structure PID Controller With PID Sliding Surface

In order to achieve this particular goal, a sliding mode control law is formed as

$$\begin{aligned}u(t) = -[K_r + K_p\|\tilde{\theta}(t)\| + K_i\|\tilde{\psi}(t)\|\\ + K_d\|\tilde{\omega}(t)\|]\text{sign}(s(t))\end{aligned} \tag{3}$$

where K_r is a positive scalar relay gain constant, K_p is a positive scalar feedback proportional gain constant, K_i is a positive scalar feedback integral gain constant, and K_d is a positive scalar feedback derivative gain constant parameters to be selected, $\|(\cdot)\| = \sqrt{(\cdot)^T(\cdot)}$ is the Euclidean norm, T is the transpose of vector or matrix, $\text{sign}(s(t)) = [\text{sign}(s_1(t)),\ldots,\text{sign}(s_n(t))]^T$ is the signum function vector. The conventionally used sliding surface definition involving the position error and the velocity error of the form $s(t) = \dot{e} + \Lambda e$ can be extended to the one also including the integral error of the form $s(t) = \dot{e} + \Lambda_1 e + \Lambda_2 \int e\,dt$, where Λ_1 and Λ_2 are constant positive–definite matrices and e denotes the state error term. Therefore, the augmented sliding surface function is appropriately introduced as

$$s(t) = C_1\tilde{\psi}(t) + C_2\tilde{\theta}(t) + \tilde{\omega}(t) \tag{4}$$

where C_1 and C_2 are constant design matrices. Concerning the investigation of how these design matrices can be selected to establish a stable sliding surface, one can set $s(t) = 0$ along with its state–space representation as

$$\begin{bmatrix}\dot{\tilde{\psi}}(t)\\ \dot{\tilde{\theta}}(t)\\ \dot{\tilde{\omega}}(t)\end{bmatrix} = \begin{bmatrix}0 & \mathrm{I} & 0\\ 0 & 0 & \mathrm{I}\\ -C_1 & -C_2 & 0\end{bmatrix}\begin{bmatrix}\tilde{\psi}(t)\\ \tilde{\theta}(t)\\ \tilde{\omega}(t)\end{bmatrix}. \tag{5}$$

Choosing C_1 and C_2 as positive–definite design matrices yields a stable reduced error system given in (5). Having selected the sliding mode control law (3) with sliding surface function (4), the next step is to choose the design parameters such that the sufficient conditions for the existence of a sliding mode are fulfilled and then the closed-loop system will be globally asymptotically stable. For illustration of robustness versus parameter variations, we shall use only one of robust techniques so- called second method of Lyapunov similar to [21]. The potentialities of variable structure controller providing the better control performances and robustness are fully exploited.

C. Sliding Conditions

In this section, we introduce a Lemma in order to derive the sufficient conditions for the existence of the sliding mode in the robot-manipulator system. Once the system is in the sliding mode, it becomes robust to the parameter uncertainties.

Lemma 1: The stable sliding mode on $s(t) = 0$ (4) always exists in a dynamic system (2) driven by controller (3), (4), if the following conditions hold:

$$
\begin{aligned}
K_r \geq\ & \max_{\dot\theta} \|f(\dot\theta)\| + \max_{\theta} \|g(\theta)\| \\
& + \max_{\theta,\dot\theta} \|B(\theta,\dot\theta)\|_F \max_t \|\dot\theta_d(t)\| \\
& + \max_{\theta} \|M(\theta)\|_F \max_t \|\ddot\theta_d(t)\| \\
K_d \geq\ & \eta + \max_{\theta} \|M(\theta)\|_F \|C_2\|_F \\
K_p \geq\ & \|C_2\|_F K_d - \max_{\theta} \|M(\theta)\|_F \|C_2\|_F^2 \\
& + \max_{\theta} \|M(\theta)\|_F \|C_1\|_F \\
& + \max_{\theta,\dot\theta} \|B(\theta,\dot\theta)\|_F \|C_2\|_F \\
K_i \geq\ & \|C_1\|_F K_d - \max_{\theta} \|M(\theta)\|_F \|C_1\|_F \|C_2\|_F \\
& + \max_{\theta,\dot\theta} \|B(\theta,\dot\theta)\|_F \|C_1\|_F
\end{aligned} \tag{6}
$$

where $\|(\cdot)\|_F = \sqrt{\mathrm{trace}((\cdot)^T(\cdot))}$ is the Frobenius norm, η is a constantly selected positive scalar.

Proof: Let us choose a Lyapunov function candidate as in [6]

$$V(s(t)) = s(t)^T M(\theta) s(t)/2. \tag{7}$$

The time derivative of V along the state error trajectory of system (2)–(4) is given by

$$
\begin{aligned}
\dot V &= s(t)^T M(\theta)\dot s(t) + s(t)^T \dot M(\theta) s(t)/2 \\
&= s(t)^T M(\theta)[C_1\dot{\tilde\psi} + C_2\dot{\tilde\theta} + \dot{\tilde\omega}] + s(t)^T \dot M(\theta) s(t)/2 \\
&= s(t)^T M(\theta)\{C_1\tilde\theta + C_2\tilde\omega - M(\theta)^{-1} \\
&\quad \cdot [B(\theta,\dot\theta)\tilde\omega + f(\dot\theta) + g(\theta) \\
&\quad + (K_r + K_p\|\tilde\theta\| + K_i\|\tilde\psi\| + K_d\|\tilde\omega\|) \\
&\quad \cdot \mathrm{sign}(s(t))] - M(\theta)^{-1} B(\theta,\dot\theta)\dot\theta_d(t) - \ddot\theta_d(t)\} \\
&\quad + s(t)^T \dot M(\theta) s(t)/2.
\end{aligned} \tag{8}
$$

Let us use an identity given as $B(\theta,\dot\theta) = [\dot M(\theta) - J]/2$ [7] where J is a skew symmetric matrice. Then, we can rewrite $\dot V$

$$
\begin{aligned}
\dot V &= s(t)^T \lfloor M(\theta) C_1\tilde\theta + M(\theta) C_2\tilde\omega - B(\theta,\dot\theta)\tilde\omega - f(\dot\theta) \\
&\quad - g(\theta) - (K_r + K_p\|\tilde\theta\| + K_i\|\tilde\psi\| + K_d\|\tilde\omega\|)\mathrm{sign}(s(t)) \\
&\quad - B(\theta,\dot\theta)\dot\theta_d(t) - M(\theta)\ddot\theta_d(t)] + s(t)^T B(\theta,\dot\theta)[C_1\tilde\psi \\
&\quad + C_2\tilde\theta + \tilde\omega] + s(t)^T J s(t)/2 \\
&= s(t)^T\{B(\theta,\dot\theta) C_1\tilde\psi + [M(\theta) C_1 \\
&\quad + B(\theta,\dot\theta) C_2]\tilde\theta + M(\theta) C_2\tilde\omega \\
&\quad - f(\dot\theta) - g(\theta)\} - (K_r + K_p\|\tilde\theta\| + K_i\|\tilde\psi\| + K_d\|\tilde\omega\|) \\
&\quad \cdot s(t)^T \mathrm{sign}(s(t)) - s(t)^T B(\theta,\dot\theta)\dot\theta_d(t) - s(t)^T M(\theta)\ddot\theta_d(t).
\end{aligned} \tag{9}
$$

Since $s(t)^T \mathrm{sign}(s(t)) \geq \|s(t)\|$, and $s(t)^T J s(t)/2 = 0$ [7] and taking the norm of the remaining terms, we obtain

$$
\begin{aligned}
\dot V \leq\ & \|s(t)\| [\max_{\theta,\dot\theta} \|B(\theta,\dot\theta)\|_F \|C_1\|_F \|\tilde\psi\| \\
& + (\max_{\theta} \|M(\theta)\|_F \|C_1\|_F \\
& + \max_{\theta,\dot\theta} \|B(\theta,\dot\theta)\|_F \|C_2\|_F)\|\tilde\theta\| \\
& + \max_{\theta} \|M(\theta)\|_F \|C_2\|_F \|\tilde\omega\| \\
& + \max_{\dot\theta} \|f(\dot\theta)\| + \max_{\theta} \|g(\theta)\|] - (K_r + K_p\|\tilde\theta\| + K_i\|\tilde\psi\| \\
& + K_d\|\tilde\omega\|) \cdot \|s(t)\| + \|s(t)\| \\
& \times \max_{\theta,\dot\theta} \|B(\theta,\dot\theta)\|_F \max_t \|\dot\theta_d(t)\| \\
& + \|s(t)\| \max_{\theta} \|M(\theta)\|_F \max_t \|\ddot\theta_d(t)\|.
\end{aligned} \tag{10}
$$

Taking all the terms into the parantheses of $-\|s(t)\|$ gives

$$
\begin{aligned}
\dot V \leq -\|s(t)\| [& K_r - \max_{\dot\theta} \|f(\dot\theta)\| \\
& - \max_{\theta,\dot\theta} \|B(\theta,\dot\theta)\|_F \max_t \|\dot\theta_d(t)\| \\
& - \max_{\theta} \|M(\theta)\|_F \max_t \|\ddot\theta_d(t)\| \\
& + (K_i - \max_{\theta,\dot\theta} \|B(\theta,\dot\theta)\|_F \|C_1\|_F)\|\tilde\psi\| \\
& + (K_p - \max_{\theta} \|M(\theta)\|_F \|C_1\|_F \\
& - \max_{\theta,\dot\theta} \|B(\theta,\dot\theta)\|_F \|C_2\|_F)\|\tilde\theta\| \\
& + (K_d - \max_{\theta} \|M(\theta)\|_F \|C_2\|_F)\|\tilde\omega\|].
\end{aligned} \tag{11}
$$

If K_r is chosen as in (6), and K_i, K_p, K_d are chosen as $K_i > \max_{\theta,\dot\theta} \|B(\theta,\dot\theta)\|_F \|C_1\|_F$, and $K_p > \max_{\theta} \|M(\theta)\|_F \|C_1\|_F + \max_{\theta,\dot\theta} \|B(\theta,\dot\theta)\|_F \|C_2\|_F$, and $K_d > \max_{\theta} \|M(\theta)\|_F \|C_2\|_F$ then $\dot V < 0$ is ensured and a sliding mode is always generated in the robot arm system. However, we wish to utilize the η-reaching condition [2], [20] which is given for our multivariable case by

$$
\begin{aligned}
\dot V &= s(t)^T M(\theta)\dot s(t) + s(t)^T \dot M(\theta) s(t)/2 \\
&\leq -\|s(t)\| \cdot (K_d - \max_{\theta} \|M(\theta)\|_F \|C_2\|_F)[\|C_1\|_F \|\tilde\psi\| \\
&\quad + \|C_2\|_F \|\tilde\theta\| \\
&\quad + \|\tilde\omega\| + [(K_i - \max_{\theta,\dot\theta} \|B(\theta,\dot\theta)\|_F \|C_1\|_F)/(K_d \\
&\quad - \max_{\theta} \|M(\theta)\|_F \|C_2\|_F) - \|C_1\|_F]\|\tilde\psi\| \\
&\quad + (K_p - \max_{\theta} \|M(\theta)\|_F \|C_1\|_F - \max_{\theta,\dot\theta} \|B(\theta,\dot\theta)\|_F \\
&\quad \times \|C_2\|_F)/(K_d - \max_{\theta} \|M(\theta)\|_F \|C_2\|_F) \\
&\quad - \|C_2\|_F)\|\tilde\theta\|] \leq -\eta\|s(t)\|^2.
\end{aligned} \tag{12}
$$

Let us choose K_i and K_p as in (6) and assuming that $K_d > \max_{\theta} \|M(\theta)\|_F \|C_2\|_F$, we can rewrite (12) as follows:

$$
\begin{aligned}
\dot V \leq -\|s(t)\| (K_d - \max_{\theta} \|M(\theta)\|_F \|C_2\|_F)[& \|C_1\|_F \|\tilde\psi\| \\
& + \|C_2\|_F \|\tilde\theta\| + \|\tilde\omega\|] \leq -\eta\|s(t)\|^2.
\end{aligned} \tag{13}
$$

We can utilize the Cauchy–Schwarz triangle inequality

$$\|C_1\|_F\|\tilde{\psi}(t)\| + \|C_2\|_F\|\tilde{\theta}(t)\| + \|\tilde{\omega}(t)\| \geq \|C_1\tilde{\psi}(t) + C_2\tilde{\theta}(t) + \tilde{\omega}(t)\| = \|s(t)\|.$$

Then (13) is reduced into

$$\dot{V} \leq -(K_d - \max_{\theta}\|M(\theta)\|_F\|C_2\|_F)\|s(t)\|^2 \leq -\eta\|s(t)\|^2. \tag{14}$$

We notice that (15) implies

$$-(K_d - \max_{\theta}\|M(\theta)\|_F\|C_2\|_F - \eta)\|s(t)\|^2 \leq 0. \tag{15}$$

Hence, if conditions in (6) are fully satisfied, then sliding inequality (15) reduces to $\dot{V} < 0$ for all $s(t) \neq 0$ and we conclude that a stable sliding motion is always generated on the switching surface $s(t) = 0$ (4). However, the designed system must be also globally asymptotically stable with respect to the state coordinates in large. This problem is considered in the next section. □

D. Global Asymptotical Stability

The following theorem summarizes the global asymptotic stability conditions for the closed-loop robot system.

Theorem 1: Suppose that conditions (6) of Lemma 1 hold, then the dynamic system (2) driven by controller (3), (4) is globally asymptotically stable, if the following conditions hold:

$$\begin{aligned} &R_3(\theta,\dot{\theta}) > 0; \quad \lambda_{\min}(R_1) > \max_{\theta,\dot{\theta}}\|P_3(\theta,\dot{\theta})\|_F^2 \\ &\qquad /\min_{\theta,\dot{\theta}}\lambda[R_3(\theta,\dot{\theta})] \\ &K_d > \|P_1\|_F^2\left\{\lambda_{\min}\left(C_1^TC_1\right)[\lambda_{\min}(R_1)\right. \\ &\qquad \left.-\max_{\theta,\dot{\theta}}\|P_3(\theta,\dot{\theta})\|_F^2\Big/\min_{\theta,\dot{\theta}}\lambda[R_3(\theta,\dot{\theta})]]\right\}^{-1} \\ &K_i \geq \|C_1\|_F K_d; \quad K_p \geq \|C_2\|_F K_d \end{aligned} \tag{16}$$

where

$$\begin{aligned} P_1 &= A/2 \\ R_1 &= \left[C_2^T\left(C_1^T\right)^{-1}A + A^TC_1^{-1}C_2 - W\right]/2 \\ P_2(\theta,\dot{\theta}) &= W/2 + C_1^T\dot{M}(\theta)/2 - C_1^TB(\theta,\dot{\theta})/2 \quad \text{and} \\ R_2(\theta,\dot{\theta}) &= \left(C_1^T\right)^{-1}P_2(\theta,\dot{\theta}) + P_2(\theta,\dot{\theta})^TC_1^{-1} \\ &\quad - \left[C_2^TM(\theta) + M(\theta)C_2\right]/2 \end{aligned}$$

assuming P_1 is invertible, then

$$\begin{aligned} &P_3(\theta,\dot{\theta}) \\ &\quad = C_1^TM(\theta)/2 + D/2 + C_2^T\dot{M}(\theta)/2 - C_2^TB(\theta,\dot{\theta})/2 \\ &\qquad - C_2^T\left(C_1^T\right)^{-1}P_2(\theta,\dot{\theta}) - P_1^TC_1^{-1} + R_1P_1^{-1}P_2(\theta,\dot{\theta}) \\ &R_3(\theta,\dot{\theta}) \\ &\quad = R_2(\theta,\dot{\theta}) - P_2(\theta,\dot{\theta})^T\left(P_1^T\right)^{-1}R_1P_1^{-1}P_2(\theta,\dot{\theta}) \\ &\qquad + P_2(\theta,\dot{\theta})^T\left(P_1^T\right)^{-1}P_3(\theta,\dot{\theta}) + P_3(\theta,\dot{\theta})^TP_1^{-1}P_2(\theta,\dot{\theta}). \end{aligned}$$

Proof: Let us introduce a positive–definite full quadratic form of Lyapunov function candidate as

$$\begin{aligned} &V(\tilde{\psi},\tilde{\theta},\tilde{\omega}) \\ &\quad = \frac{1}{2}\begin{bmatrix}\tilde{\psi}\\ \tilde{\theta}\\ \tilde{\omega}\end{bmatrix}^T\begin{bmatrix}A & W & C_1^TM(\theta)\\ W^T & D & C_2^TM(\theta)\\ M(\theta)C_1 & M(\theta)C_2 & M(\theta)\end{bmatrix}\begin{bmatrix}\tilde{\psi}\\ \tilde{\theta}\\ \tilde{\omega}\end{bmatrix} \\ &\quad = \tilde{\psi}^TA\tilde{\psi}/2 + \tilde{\psi}^TW\tilde{\theta} + \tilde{\psi}^TC_1^TM(\theta)\tilde{\omega} \\ &\qquad + \tilde{\theta}^TD\tilde{\theta}/2 + \tilde{\theta}^TC_2^TM(\theta)\tilde{\omega} + \tilde{\omega}^TM(\theta)\tilde{\omega}/2 \end{aligned} \tag{17}$$

where C_1 and C_2 are sliding surface slope gain matrices, A and D are symmetric gain matrices, and W is another gain matrix all with appropriate dimensions such that

$$\begin{aligned} &A > 0; \quad D - W^TA^{-1}W > 0 \\ &M(\theta) - M(\theta)C_1A^{-1}C_1^TM(\theta) \\ &\quad - [M(\theta)C_2 - M(\theta)C_1A^{-1}W] \\ &\quad \cdot [D - W^TA^{-1}W]^{-1}\left[C_2^TM(\theta) - W^TAC_1^TM(\theta)\right] > 0. \end{aligned}$$

We proceed with time derivative of $V(\tilde{\psi},\tilde{\theta},\tilde{\omega},t)$ along the system dynamics and the state trajectory of system (2)– (4)

$$\begin{aligned} &\dot{V}(\tilde{\psi},\tilde{\theta},\tilde{\omega},t) \\ &\quad = \tilde{\psi}^TA\dot{\tilde{\psi}} + \dot{\tilde{\psi}}^TW\tilde{\theta} + \tilde{\psi}^TW\dot{\tilde{\theta}} \\ &\qquad + \dot{\tilde{\psi}}^TC_1^TM(\theta)\tilde{\omega} + \tilde{\psi}^TC_1^TM(\theta)\dot{\tilde{\omega}} \\ &\qquad + \tilde{\psi}^TC_1^T\dot{M}(\theta)\tilde{\omega} + \tilde{\theta}^TD\dot{\tilde{\theta}} + \dot{\tilde{\theta}}^TC_2^TM(\theta)\tilde{\omega} \\ &\qquad + \tilde{\theta}^TC_2^TM(\theta)\dot{\tilde{\omega}} + \tilde{\theta}^TC_2^T\dot{M}(\theta)\tilde{\omega} + \tilde{\omega}^TM(\theta)\dot{\tilde{\omega}} \\ &\qquad + \tilde{\omega}^T\dot{M}(\theta)\tilde{\omega}/2. \end{aligned} \tag{18}$$

Substituting system dynamics (2) into $\dot{V}(\tilde{\psi},\tilde{\theta},\tilde{\omega},t)$, we get

$$\begin{aligned} &\dot{V}(\tilde{\psi},\tilde{\theta},\tilde{\omega},t) \\ &\quad = \tilde{\psi}^TA\tilde{\theta} + \tilde{\theta}^TW\tilde{\theta} + \tilde{\psi}^TW\tilde{\omega} + \tilde{\theta}^TC_1^TM(\theta)\tilde{\omega} \\ &\qquad + \tilde{\psi}^TC_1^TM(\theta)\dot{\tilde{\omega}} + \tilde{\psi}^TC_1^T\dot{M}(\theta)\tilde{\omega} + \tilde{\theta}^TD\tilde{\omega} \\ &\qquad + \tilde{\omega}^TC_2^TM(\theta)\tilde{\omega} + \tilde{\theta}^TC_2^TM(\theta)\dot{\tilde{\omega}} + \tilde{\theta}^TC_2^T\dot{M}(\theta)\tilde{\omega} \\ &\qquad + \tilde{\omega}^TM(\theta)\dot{\tilde{\omega}} + \tilde{\omega}^T\dot{M}(\theta)\tilde{\omega}/2 \\ &\quad = \tilde{\psi}^TA\tilde{\theta} + \tilde{\psi}^T\left[W + C_1^T\dot{M}(\theta)\right]\tilde{\omega} + \tilde{\theta}^TW\tilde{\theta} \\ &\qquad + \tilde{\theta}^T[C_1^TM(\theta) + D + C_2^T\dot{M}(\theta)]\tilde{\omega} + \tilde{\omega}^T\left[C_2^TM(\theta)\right. \\ &\qquad \left.+ \dot{M}(\theta)/2\right]\tilde{\omega} + \left[\tilde{\psi}^TC_1^T + \tilde{\theta}^TC_2^T + \tilde{\omega}^T\right]M(\theta)\dot{\tilde{\omega}}. \end{aligned} \tag{19}$$

The last summing term of (19) can be rewritten as

$$\begin{aligned} (*) &= \left[\tilde{\psi}^TC_1^T + \tilde{\theta}^TC_2^T + \tilde{\omega}^T\right]M(\theta)\dot{\tilde{\omega}} \\ &= \left[\tilde{\psi}^TC_1^T + \tilde{\theta}^TC_2^T + \tilde{\omega}^T\right] \\ &\quad \cdot M(\theta)\{M(\theta)^{-1}[-B(\theta,\dot{\theta})\tilde{\omega} \\ &\quad - f(\dot{\theta}) - g(\theta) + u] - M(\theta)^{-1} \\ &\quad \cdot B(\theta,\dot{\theta})\dot{\theta}_d(t) - \ddot{\theta}_d(t)\} \\ &= -\tilde{\psi}^TC_1^TB(\theta,\dot{\theta})\tilde{\omega} - \tilde{\theta}^TC_2^TB(\theta,\dot{\theta})\tilde{\omega} \\ &\quad - \tilde{\omega}^TB(\theta,\dot{\theta})\tilde{\omega} - s(t)^T[f(\dot{\theta}) \\ &\quad + g(\theta) + B(\theta,\dot{\theta})\dot{\theta}_d(t) + M(\theta)\ddot{\theta}_d(t)] \\ &\quad - K_rs(t)^T\text{sign}(s(t)) - K_d[\|C_1\|_F\|\tilde{\psi}\| + \|C_2\|_F\|\tilde{\theta}\| \\ &\quad + \|\tilde{\omega}\| + (K_i/K_d - \|C_1\|_F)\|\tilde{\psi}\| \\ &\quad + (K_p/K_d - \|C_2\|_F)\|\tilde{\theta}\|]s(t)^T\text{sign}(s(t)). \end{aligned}$$

If we choose K_p and K_i in accordance with (16), and utilize $s(t)^T \mathrm{sign}(s(t)) \geq \|s(t)\|$, then taking the norm gives

$$\begin{aligned}(*) \leq & -\tilde{\psi}^T C_1^T B(\theta,\dot{\theta})\tilde{\omega} - \tilde{\theta}^T C_2^T B(\theta,\dot{\theta})\tilde{\omega} - \tilde{\omega}^T B(\theta,\dot{\theta})\tilde{\omega} \\ & - [K_r - \max_{\dot{\theta}} \|f(\dot{\theta})\| - \max_{\theta} \|g(\theta)\| \\ & - \max_{\theta,\dot{\theta}} \|B(\theta,\dot{\theta})\|_F \max_t \|\dot{\theta}_d(t)\| \\ & - \max_{\theta} \|M(\theta)\| \|\ddot{\theta}_d(t)\|] \|s(t)\| \\ & - K_d [\|C_1\|_F \|\tilde{\psi}\| + \|C_2\|_F \|\tilde{\theta}\| + \|\tilde{\omega}\|] \|s(t)\|.\end{aligned}$$

Choosing K_r in accordance with (16) and using Cauchy–Schwartz triangle inequality, we get

$$\begin{aligned}(*) \leq & -\tilde{\psi}^T C_1^T B(\theta,\dot{\theta})\tilde{\omega} - \tilde{\theta}^T C_2^T B(\theta,\dot{\theta})\tilde{\omega} \\ & - \tilde{\omega}^T B(\theta,\dot{\theta})\tilde{\omega} - K_d \|s(t)\|^2 \\ \leq & -\tilde{\psi}^T C_1^T B(\theta,\dot{\theta})\tilde{\omega} - \tilde{\theta}^T C_2^T B(\theta,\dot{\theta})\tilde{\omega} - \tilde{\omega}^T B(\theta,\dot{\theta})\tilde{\omega} \\ & - [C_1\tilde{\psi} + C_2\tilde{\theta} + \tilde{\omega}]^T K_d [C_1\tilde{\psi} \\ & + C_2\tilde{\theta} + \tilde{\omega}] \leq -\tilde{\psi}^T C_1^T K_d C_1 \tilde{\psi} \\ & - 2\tilde{\psi}^T C_1^T K_d C_2 \tilde{\theta} - \tilde{\psi}^T \left(2C_1^T K_d + C_1^T B(\theta,\dot{\theta})\right)\tilde{\omega} \\ & - \tilde{\theta}^T C_2^T K_d C_2 \tilde{\theta} \\ & - \tilde{\theta}^T \left(2C_2^T K_d + C_2^T B(\theta,\dot{\theta})\right)\tilde{\omega} \\ & - \tilde{\omega}^T (K_d \mathrm{I} + B(\theta,\dot{\theta}))\tilde{\omega}.\end{aligned}$$

Substituting $(*)$ into $\dot{V}(\tilde{\psi},\tilde{\theta},\tilde{\omega},t)$ appropriately, we obtain

$$\begin{aligned}&\dot{V}(\tilde{\psi},\tilde{\theta},\tilde{\omega},t) \\ &\leq -\tilde{\psi}^T C_1^T K_d C_1 \tilde{\psi} - 2\tilde{\psi}^T \left(C_1^T K_d C_2 - A/2\right)\tilde{\theta} - 2\tilde{\psi}^T \\ &\cdot \left[C_1^T K_d + C_1^T B(\theta,\dot{\theta})/2 - W/2 - C_1^T \dot{M}(\theta)/2\right]\tilde{\omega} \\ &- \tilde{\theta}^T \left(C_2^T K_d C_2 - W/2\right)\tilde{\theta} - 2\tilde{\theta}^T \\ &\times \Big[C_2^T K_d + C_2^T B(\theta,\dot{\theta})/2 - C_1^T M(\theta)/2 - D/2 \\ &- C_2^T \dot{M}(\theta)/2\Big] \\ &\times \tilde{\omega} - \tilde{\omega}^T \left(K_d \mathrm{I} + B(\theta,\dot{\theta}) - C_2^T M(\theta) - \dot{M}(\theta)/2\right)\tilde{\omega}.\end{aligned} \tag{20}$$

We can use the identity given in [7] with $(\cdot)^T J(\cdot) = 0$ for the last summing term of (20) as

$$\begin{aligned}&(**) \\ &= -\tilde{\omega}^T \left(K_d \mathrm{I} + B(\theta,\dot{\theta}) - C_2^T M(\theta) - \dot{M}(\theta)/2\right)\tilde{\omega} \\ &= -\tilde{\omega}^T \left[K_d \mathrm{I} + \dot{M}(\theta)/2 - J/2 - C_2^T M(\theta) - \dot{M}(\theta)/2\right]\tilde{\omega} \\ &= -\tilde{\omega}^T \left[K_d \mathrm{I} - C_2^T M(\theta)\right]\tilde{\omega}.\end{aligned}$$

Now, substituting $(**)$ into (20), we convert $\dot{V}(\tilde{\psi},\tilde{\theta},\tilde{\omega},t)$ (20) into quadratic form as

$$\dot{V} \leq -x^T H x \tag{21}$$

where H is a symmetric $3n \times 3n$ matrix with $H_{11} = C_1^T K_d C_1$

$$\begin{aligned}H_{12} &= C_1^T K_d C_2 - P_1, \\ H_{13} &= C_1^T K_d - P_2(\theta,\dot{\theta}), \\ H_{22} &= C_2^T K_d C_2 + R_1 - C_2^T \left(C_1^T\right)^{-1} P_1 + P_1^T C_1^{-1} C_2, \\ H_{23} &= C_2^T K_d + C_2^T B(\theta,\dot{\theta})/2 \\ &\quad - C_1^T M(\theta)/2 - D/2 - C_2^T \dot{M}(\theta)/2, \\ H_{33} &= K_d \mathrm{I} + R_2(\theta,\dot{\theta}) - \left(C_1^T\right)^{-1} P_2(\theta,\dot{\theta}) - P_2(\theta,\dot{\theta})^T C_1^{-1}\end{aligned}$$

with P_1, R_1, $P_2(\theta,\dot{\theta}), R_2(\theta,\dot{\theta}), P_3(\theta,\dot{\theta})$, and $R_3(\theta,\dot{\theta})$ being defined in the statement of Theorem 1. Therefore, we need to investigate appropriate K_d's such that H remains always positive–definite allowing $\dot{V}(\tilde{\psi},\tilde{\theta},\tilde{\omega},t)$ to be always negative definite for a global asymptotically stable robot arm system. Now, the next step is to choose K_d that makes H non-negative. Since H is a partitioned matrix, Schur complement is used for this goal. In accordance with Schur's complement, the following inequalities are established as

$$H_{11} > 0, \quad H_{22} - H_{12}^T H_{11}^{-1} H_{12} > 0 \tag{22}$$

$$\begin{aligned}&H_{33} - H_{13}^T H_{11}^{-1} H_{13} - \left(H_{23} - H_{12}^T H_{11}^{-1} H_{13}\right)^T \\ &\quad \cdot \left(H_{22} - H_{12}^T H_{11}^{-1} H_{12}\right)^{-1} \left(H_{23} - H_{12}^T H_{11}^{-1} H_{13}\right) > 0.\end{aligned} \tag{23}$$

First of all, considering the first half of (22) we notice that $C_1^T K_d C_1$ is always positive–definite since K_d is a positive scalar constant. For the second half of (22), we have

$$H_{22} - H_{12}^T H_{11}^{-1} H_{12} = R_1 - P_1^T C_1^{-1} K_d^{-1} \left(C_1^T\right)^{-1} P_1 > 0. \tag{24}$$

Assuming that $R_1 > 0$ and since $K_d > 0$, we obtain

$$\begin{aligned}K_d &> \|P_1\|_F^2 / \lambda_{\min}\left(C_1^T C_1\right)\lambda_{\min}(R_1) \\ &= \|A\|_F^2 / \left[4\lambda_{\min}\left(C_1^T C_1\right)\lambda_{\min}(R_1)\right].\end{aligned} \tag{25}$$

Before analyzing the third condition in (23), let us make a change of variable as $H_{22} - H_{12}^T H_{11}^{-1} H_{12} = R_1 - P_1^T C_1^{-1} K_d^{-1} \cdot (C_1^T)^{-1} P_1 = K_{d1}$, assuming that P_1 is invertible, we compute

$$\begin{aligned}&H_{33} - H_{13}^T H_{11}^{-1} H_{13} = R_2(\theta,\dot{\theta}) \\ &\quad - P_2(\theta,\dot{\theta})^T C_1^{-1} K_d^{-1} \left(C_1^T\right)^{-1} \\ &\quad \cdot P_2(\theta,\dot{\theta}) = R_2(\theta,\dot{\theta}) + P_2(\theta,\dot{\theta})^T \left(P_1^T\right)^{-1} \\ &\quad \times K_{d1} P_1^{-1} P_2(\theta,\dot{\theta}) - P_2(\theta,\dot{\theta})^T \left(P_1^T\right)^{-1} R_1 P_1^{-1} P_2(\theta,\dot{\theta}) \\ &H_{23} - H_{12}^T H_{11}^{-1} H_{13} = C_2^T B(\theta,\dot{\theta})/2 - C_1^T M(\theta)/2 - D/2 \\ &\quad - C_2^T \dot{M}(\theta)/2 + C_2^T \left(C_1^T\right)^{-1} P_2(\theta,\dot{\theta}) + P_1^T C_1^{-1} \\ &\quad - P_1^T C_1^{-1} K_d^{-1} \left(C_1^T\right)^{-1} P_2(\theta,\dot{\theta}) \\ &\quad = K_{d1} P_1^{-1} P_2(\theta,\dot{\theta}) - P_3(\theta,\dot{\theta}).\end{aligned}$$

Then, the condition in (23) can be rewritten as follows:

$$\begin{aligned}&H_{33} - H_{13}^T H_{11}^{-1} H_{13} - \left(H_{23}^T - H_{13}^T H_{11}^{-1} H_{12}\right) \\ &\quad \times \left(H_{22} - H_{12}^T H_{11}^{-1} H_{12}\right)^{-1} \cdot \left(H_{23} - H_{12}^T H_{11}^{-1} H_{13}\right) \\ &\quad = R_3(\theta,\dot{\theta}) - P_3(\theta,\dot{\theta})^T K_{d1}^{-1} P_3(\theta,\dot{\theta}) > 0.\end{aligned} \tag{26}$$

We assume $R_3(\theta,\dot{\theta}) > 0$; then, $\lambda_{\min}(R_1) > \max_{\theta,\dot{\theta}} \|P_3(\theta,\dot{\theta})\|_F^2 / \min_{\theta,\dot{\theta}} \lambda[R_3(\theta,\dot{\theta})]$. Since $K_{d1} > 0$, it follows from (26) that

$$\begin{aligned}\lambda_{\min}(K_{d1}) &> \lambda_{\min}(R_1) - \|P_1\|_F^2 / \lambda_{\min}\left(C_1^T C_1\right) K_d \\ &> \max_{\theta,\dot{\theta}} \|P_3(\theta,\dot{\theta})\|_F^2 / \min_{\theta,\dot{\theta}} \lambda[R_3(\theta,\dot{\theta})].\end{aligned} \tag{27}$$

In view of (27), if (16) is satisfied, then (20) reduces to

$$\dot{V} \leq -x^T H x < 0. \tag{28}$$

Hence, Theorem 1 is proved. Thus, the design is completed. □

III. Reduced Design

In this section, we will use Theorem1 particularly in choosing various design parameters more systematically. Since the existence of such a Lyapunov function depends on purely mathematical conditions provided by (16), we will once more investigate some "relaxed conditions" which are satisfied independently from matrix norms of system parameters. First we notice that the stability conditions in (16) can be reduced, if we select the parameters in (17) as $A = a_0\mathrm{I}_n, W = w_0\mathrm{I}_n, D = d_0\mathrm{I}_n, C_1 = c_1\mathrm{I}_n, C_2 = \mathrm{I}_n$, then for the Lyapunov function candidate to be positive–definite, the parameters should satisfy

$$\begin{aligned}&\text{i) } a_0 > 0 \quad \text{ii) } d_0 - w_0^2 a_0^{-1} > 0\\ &\text{iii) } (*) = M(\theta) - c_1^2 a_0^{-1}[M(\theta)]^2\\ &\quad - \left(c_2 - c_1 w_0 a_0^{-1}\right)^2 \left(d_0 - w_0^2 a_0^{-1}\right)^{-1} [M(\theta)]^2 > 0. \quad (29)\end{aligned}$$

Elaborately working on the condition iii) in (29), we obtain

$$\begin{aligned}(*) > &\min_\theta \lambda(M(\theta))\\ &- \left[c_1^2 a_0^{-1} + \left(c_2 - c_1 w_0 a_0^{-1}\right)^2 \left(d_0 - w_0^2 a_0^{-1}\right)^{-1}\right]\end{aligned}$$

$\cdot \max_\theta \|M(\theta)\|_F^2 > 0$. Let us choose $w_0 = a_0 > 0, d_0 = 2a_0$, then i) and ii) are satisfied, and iii) takes the form

$$a_0 > \bar{c}_1 \left[c_1^2 + (c_2 - c_1)^2\right] \quad (30)$$

where $\bar{c}_1 = \max_\theta \|M(\theta)\|_F^2 (\min_\theta \lambda[M(\theta)])^{-1}$. We evaluate P_i's and R_i's, $i = 1, 2, 3$ as: $P_1 = 0.5a_0\mathrm{I}, R_1 = (c_2 c_1^{-1} - 0.5)a_0\mathrm{I}$

$$\begin{aligned}P_2(\theta,\dot\theta) &= 0.5[a_0\mathrm{I} + c_1\dot M(\theta) - c_1 B(\theta,\dot\theta)]\\ R_2(\theta,\dot\theta) &= a_0 c_1^{-1}\mathrm{I} + 0.5[\dot M(\theta) - B(\theta,\dot\theta)]^T\\ &\quad + 0.5[\dot M(\theta) - B(\theta,\dot\theta)] - c_2 M(\theta)\\ P_3(\theta,\dot\theta) &= 0.5c_1 M(\theta) + 0.5c_1^{-1}(c_1 + c_2 - 1)a_0\mathrm{I}\\ &\quad + c_1\left(c_2 c_1^{-1} - 0.5\right)\cdot[\dot M(\theta) - B(\theta,\dot\theta)]\\ R_3(\theta,\dot\theta) &= (c_1 - c_2)M(\theta) + 1.5a_0\mathrm{I}\\ &\quad + 0.5(c_1 + c_2)[\dot M(\theta) - B(\theta,\dot\theta)]^T\\ &\quad + 0.5(c_1 + c_2)[\dot M(\theta) - B(\theta,\dot\theta)]\\ &\quad + 0.5c_1^2 a_0^{-1}[\dot M(\theta) - B(\theta,\dot\theta)]^T M(\theta)\\ &\quad + 0.5c_1^2 a_0^{-1} M(\theta)[\dot M(\theta) - B(\theta,\dot\theta)]\\ &\quad + c_1^2 a_0^{-1}\left(c_2 c_1^{-1} - 0.5\right)[\dot M(\theta)\\ &\quad - B(\theta,\dot\theta)]^T[\dot M(\theta) - B(\theta,\dot\theta)].\end{aligned}$$

Now we can explicitly treat the stability conditions given in (16) as follows, assuming $c_2 c_1^{-1} > 0.5$, and noticing that $[\dot M(\theta) - B(\theta,\dot\theta)]^T[\dot M(\theta) - B(\theta,\dot\theta)] \geq 0$, we can write an inequality of the form

$$\begin{aligned}R_3(\theta,\dot\theta) > \{&c_1 \min_\theta\lambda[M(\theta)] - c_2\max_\theta\|M(\theta)\|_F + 1.5a_0\\ &- (c_1 + c_2)(\max_{\theta,\dot\theta}\|\dot M(\theta)\|_F + \max_{\theta,\dot\theta}\|B(\theta,\dot\theta)\|_F)\\ &- c_1^2 a_0^{-1}\max_\theta\|M(\theta)\|_F(\max_{\theta,\dot\theta}\|\dot M(\theta)\|_F\\ &+ \max_{\theta,\dot\theta}\|B(\theta,\dot\theta)\|_F)\mathrm{I}\} > 0 \quad (31)\end{aligned}$$

which implies to satisfy $1.5a_0 > \bar{c}_2 c_1 + \bar{c}_3 c_2 + \bar{c}_4 c_1^2 a_0^{-1}$. Thus, we should choose a_0 such that

$$a_0 > \left[\bar{c}_2 c_1 + \bar{c}_3 c_2 + \sqrt{(\bar{c}_2 c_1 + \bar{c}_3 c_2)^2 + 6\bar{c}_4 c_1^2}\right] \Big/ 3 \quad (32)$$

where

$$\bar{c}_2 = \max_{\theta,\dot\theta}\|\dot M(\theta)\|_F + \max_{\theta,\dot\theta}\|B(\theta,\dot\theta)\|_F - \min_\theta\lambda[M(\theta)]$$

and

$$\bar{c}_3 = \max_\theta\|M(\theta)\|_F + \max_{\theta,\dot\theta}\|\dot M(\theta)\|_F + \max_{\theta,\dot\theta}\|B(\theta,\dot\theta)\|_F$$

and

$$\bar{c}_4 = \max_\theta\|M(\theta)\|_F[\max_{\theta,\dot\theta}\|\dot M(\theta)\|_F + \max_{\theta,\dot\theta}\|B(\theta,\dot\theta)\|_F].$$

In order to check if

$$\lambda_{\min}(R_1) > \max_{\theta,\dot\theta}\|P_3(\theta,\dot\theta)\|_F^2 \min_{\theta,\dot\theta}\lambda[R_3(\theta,\dot\theta)]^{-1}$$

is satisfied, we first set $c_1 + c_2 = 1$ and notice the inequalities

$$\begin{aligned}\max_{\theta,\dot\theta}\|P_3(\theta,\dot\theta)\|_F < \bar{c}_5 \quad &\text{where } \bar{c}_5 = 0.5c_1\max_\theta\|M(\theta)\|_F\\ &+ c_1\left(c_2 c_1^{-1} - 0.5\right)[\max_{\theta,\dot\theta}\|\dot M(\theta)\|_F + \max_{\theta,\dot\theta}\|B(\theta,\dot\theta)\|_F].\end{aligned}$$

Moreover, we also have

$$\min_{\theta,\dot\theta}\lambda[R_3(\theta,\dot\theta)] > a_0^{-1}[1.5a_0^2 - (\bar{c}_2 c_1 + \bar{c}_3 c_2)a_0 - \bar{c}_4 c_1^2].$$

Therefore, it is possible to find a new lower bound for a_0

$$\begin{aligned}\lambda_{\min}(R_1) &= \left(c_2 c_1^{-1} - 0.5\right)a_0\\ &> a_0\bar{c}_5^2\left[1.5a_0^2 - (\bar{c}_2 c_1 + \bar{c}_3 c_2)a_0 - \bar{c}_4 c_1^2\right]^{-1}.\end{aligned}$$

Considering (29), (30), and (32) a_0 can be chosen such that $a_0 > \max(\bar{a}_1, \bar{a}_2, \bar{a}_3)$ where $\bar{a}_1 = \bar{c}_1[c_1^2 + (c_2 - c_1)^2]$, $\bar{a}_2 = [\bar{c}_2 c_1 + \bar{c}_3 c_2 + \sqrt{(\bar{c}_2 c_1 + \bar{c}_3 c_2)^2 + 6\bar{c}_4 c_1^2}]/3$, $\bar{a}_2 = [\bar{c}_2 c_1 + \bar{c}_3 c_2 + \sqrt{(\bar{c}_2 c_1 + \bar{c}_3 c_2)^2 + 6\bar{c}_4 c_1^2}]/3$, $\bar{a}_3 = [\bar{c}_2 c_1 + \bar{c}_3 c_2 + \sqrt{(\bar{c}_2 c_1 + \bar{c}_3 c_2)^2 + 6(\bar{c}_4 c_1^2 + \bar{c}_5^2(c_2 c_1^{-1} - 0.5)^{-1})}]/3$. Finally, the lower bound of the control parameter K_d is evaluated as

$$\begin{aligned}K_d > a_0\Big\{4c_1^2\Big[&c_2 c_1^{-1} - 0.5 - \bar{c}_5^2\\ &\times\left(1.5a_0^2 - (\bar{c}_2 c_1 + \bar{c}_3 c_2)a_0 - \bar{c}_4 c_1^2\right)^{-1}\Big]\Big\}^{-1}. \quad (33)\end{aligned}$$

IV. Simulation Example

To illustrate the feasibility of the designed controller we consider the tracking control of a two-link SCARA type manipulator [22] shown in Fig. 1. Here $\theta = [\theta_1\ \theta_2]$ and $u = [u_1\ u_2]$, the entries of $M(\theta), B(\theta,\dot\theta), f(\dot\theta)$ and $g(\theta)$ are $M_{11} = p_1 + 2p_3\cos(\theta_2), M_{12} = p_2 + p_3\cos(\theta_2), M_{22} = p_2$

$$\begin{aligned}&B_{11} = -\dot\theta_2 p_3\sin(\theta_2), \quad B_{12} = -(\dot\theta_1 + \dot\theta_2)p_3\sin(\theta_2)\\ &B_{21} = \dot\theta_1 p_3\sin(\theta_2) \quad B_{22} = 0, \quad f_{11} = K_1\dot\theta_1 + K_2\mathrm{sgn}(\dot\theta_1)\\ &f_{21} = K_1\dot\theta_2 + K_2\mathrm{sgn}(\dot\theta_2)\\ &g_{11} = (m_1 + m_2)l_1\cos(\theta_1) + m_2 g l_2\cos(\theta_1 + \theta_2)\\ &g_{21} = m_2 g l_2\cos(\theta_1 + \theta_2)\end{aligned}$$

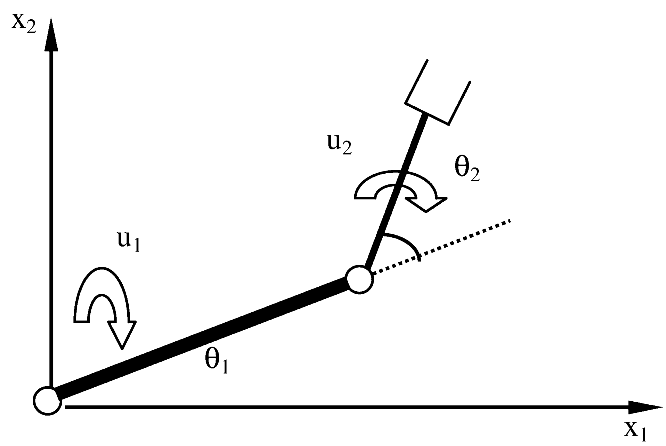

Fig. 1. Joint positions of the manipulator.

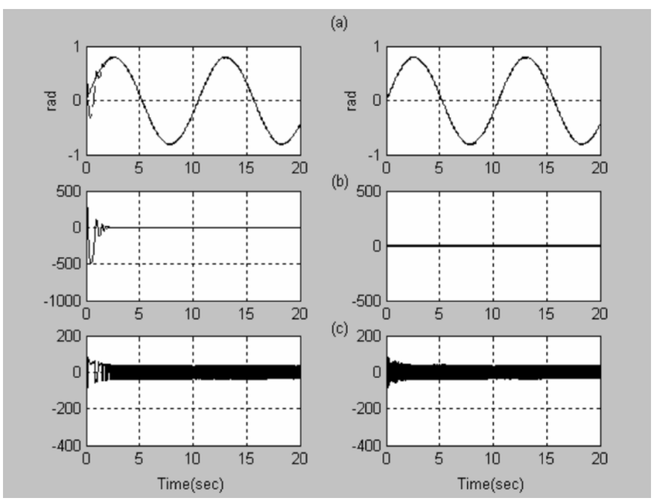

Fig. 2. This note's control. (a) Desired (dark) and actual (dash) position errors. (b) Switching functions. (c) Control inputs applied.

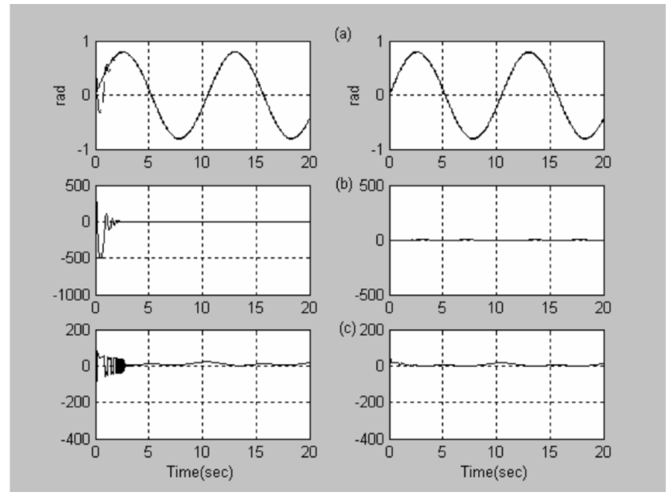

Fig. 3. This note's saturation control. (a) Desired (dark) and actual (dash) position errors. (b) Switching functions. (c) Control inputs applied.

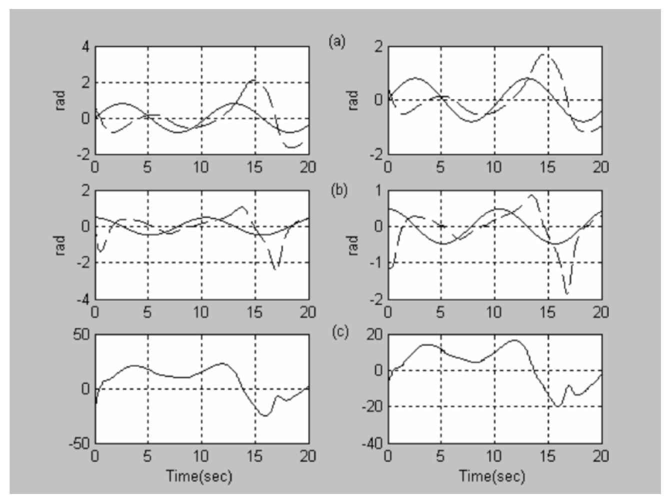

Fig. 4. Classical PID control. (a) Desired (dark) and actual (dash) position errors. (b) Desired (dark) and actual (dash) velocity errors. (c) Control inputs applied.

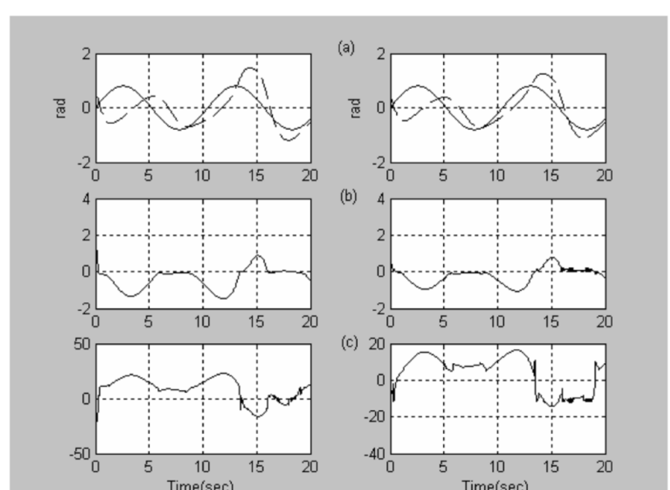

Fig. 5. Stepanenko control. (a) Desired (dark) and actual (dash) position errors. (b) Switching functions. (c) Control inputs applied.

where system parameters are given as $p_1 = 3.16, p_2 = 0.106, p_3 = 0.173$, $m_1 = 10.6$ kg, $m_1 = 4.85$ kg, $l_1 = 0.36, l_2 = 0.24, K_1 = 1, K_2 = 10, c = 1s^{-1}$. The reference trajectory for the position control problem has been chosen equally for both links as, $\theta_{d1}(t) = 0.8\sin(0.6t)$ rad. Thus, the reference trajectory for position integral and velocity are computed as $\psi_{d1}(t) = -4\cos(0.6t)/3, \omega_{d1}(t) = 0.48\cos(0.6t)$. The sampling time is used as 0.0025 in the whole simulation study. The control parameters are chosen such that $K_p = K_i = K_d = K_r = 10 \cdot \mathrm{I}, C_1 = 10 \cdot \mathrm{I}, C_2 = 1000 \cdot \mathrm{I}$ in the case of simulations by the method of this note and classical PID controller. The control parameters are determined as $K_p = K_d = \mathrm{I}, K = 10 \cdot \mathrm{I}$ and $\alpha\|Y\| = 10$ in the case of the simulation work with the method given in [18]. The simulation results are given in Figs. 2–5.

V. Comparison Analysis of the Alternative Controllers Simulation Results

For the purpose of comparison, the simulations have been carried out by considering three types of controllers for the control of robot. The first is the one that is proposed in this work. The second is the classical PID controller and last is the one that is proposed by Stepanenko [18]. Applying our novel variable structure PID controller, it is seen in Fig. 2(a) that the actual joint positions succeed to follow the references perfectly though the first joint position makes temporary overshoots, which die out immediately. The switching functions shown in Fig. 2(b) indicate a very rapid convergence of the joint states toward the sliding manifolds. The control inputs are depicted in Fig. 2(c) in which a considerable amount of chattering effect is noticed. Since chattering is untolerable for a proper operation of actuators, there is a need to overcome this problem practically. The conventional solution in the sliding mode literature is to use a saturation function in the control law in place of the pure signum function. Under this situation, the trajectory following portrait for joint positions and the convergence of the switching functions are presented in Fig. 3(a) and (b), respectively, in which the behaviors are almost same as compared with the former case. The significant gain from this slight change in the control law can be seen in the control inputs shown in Fig. 3(c), which exhibit a chattering-free smooth and a decreasing control action. In the second stage of the simulations, the classical PID controller is used and the simulations are renewed with the same parameter settings. The actual joint positions and joint velocities versus their reference counterparts are given in Fig. 4(a) and (b). It is immediately noticed that there exist a considerable amount of deterioration in the performance of the robot manipulator. The deviations from the references of the joint positions and joint velocities are untolerable unless the control parameters are chosen high enough which is not needed in the case of our proposed controller. The control inputs depicted in Fig. 4(c) are smooth but are of nondecreasing nature. In the final stage of the simulations, the control law that is proposed by Stepanenko [18] is taken into account and the simulations are repeated. Looking at the behavior of the joint positions given in Fig. 5(a), it can be seen that there exist a certain amount of deviation from the reference trajectories which is the same situation observed in the case of classical PID controller. This outcome is also confirmed when the switching functions are checked from Fig. 5(b). There it is noticed that the joint trajectories fail to reach the sliding manifolds exactly and make fluctuations around them. However, the good thing is that the control torques shown in Fig. 5(c) are chattering free. As it is seen from Fig. 5(b) which is different from the switching functions in Figs. 2(b) and 3(b), the Stepanenko switching function case is not equal to zero. This shows that the sliding mode conditions are sometimes not satisfied. Therefore, this degrades the control performances. It also needs an attention that this performance may be improved by high-gain control action which means more energy consumption while the same goal can be achieved with less control effort by using our newly proposed variable structure saturation controller. Thus, comparison analysis of three alternative controllers with a plenty of simulation results in terms of advantages and control performances soundly confirmed that the proposed controller provided better performances than those of existing.

VI. Conclusion

In this brief, a new variable structure PID controller design approach is considered for the tracking stabilization of robot motion. The work corroborates the utility of a certain PID sliding mode controller with PID sliding surface for tracking control of a robotic manipulator. Different from the general approach, the conventional equivalent control term is not used in this controller because that needs to use the matching conditions with exact full robot dynamics knowledge and unavailable parameter information. Though the sliding surface includes also the integral error term, which makes the robot tracking control problem complicated, the existence of a sliding mode and gain selection guideline are clearly investigated. Moreover, different from uniformly ultimately boundededness, the global asymptotic stability of the robot system is analyzed with proposed controller. The sliding and global stability conditions are formulated in terms of Lyapunov full quadratic form and upper and lower matrix norm inequalities. Reduced design is also discussed. The proposed control algorithm is applied to a two-link direct drive robot arm through simulations. The results indicate that the control performance of the robot system is satisfactory. The chattering phenomenon has been overcome by the use of a saturation function in the control law in place of a pure signum function. The replacement of a saturation function with the pure signum function yields a smooth transient performance. The proposed approach is compared with the existing alternative sliding mode controllers for robot manipulators in terms of advantages and control performances. A comparative analysis with a plenty of simulation results soundly confirmed that the performances of the developed variable structure PID controller are better under than those of both classical PID controller and an existing variable structure controller with PID-sliding surface.

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References

[1] V. I. Utkin, *Sliding Modes in Control and Optimization*. New York: Springer-Verlag, 1991.

[2] J.-J. E. Slotine and W. Li, *Applied Nonlinear Control*. Englewood Cliffs, NJ: Prentice-Hall, 1991.

[3] J. Y. Hung, W. Gao, and J. C. Hung, "Variable structure control: A survey," *IEEE Trans. Ind. Electron.*, vol. 40, no. 1, pp. 2–21, Feb. 1993.

[4] A. Ferrara, L. Magnani, and R. Scattolini, "A globally stabilizing hybrid variable structure control strategy," *IEEE Trans. Automat. Contr.*, vol. 47, no. 8, pp. 1334–1337, Aug. 2002.

[5] K. D. Young, "Controller design for a manipulator using the theory of variable structure systems," *IEEE Trans. Syst., Man, Cybern.*, vol. SMC–8, no. 2, pp. 101–109, Feb. 1978.

[6] K. S. Yeung and Y. P. Chen, "A new controller design for manipulators using the theory of variable structure systems," *IEEE Trans. Automat. Cont.*, vol. 33, no. 2, pp. 200–206, Feb. 1988.

[7] E. Bailey and A. Arapostathis, "A simple sliding mode scheme applied to robot manipulators," *Int. J. Control*, vol. 45, no. 4, pp. 1197–1209, 1987.

[8] M. Zhihong and M. Palaniswami, "Robust tracking control for rigid robotic manipulators," *IEEE Trans. Automat. Contr.*, vol. 39, no. 1, pp. 154–159, Jan. 1994.

[9] K. Erbatur, M. O. Kaynak, and A. Sabanovic, "A study on robustness property of sliding-mode controllers: A novel design and experimental investigations," *IEEE Trans. Ind. Electron.*, vol. 46, no. 5, pp. 1012–1017, Oct. 1999.

[10] K.-Y. Lian and C.-R. Lin, "Sliding-mode motion/force control of constrained robots," *IEEE Trans. Automat. Contr.*, vol. 43, no. 8, pp. 1101–1103, Aug. 1998.

[11] E. M. Jafarov and R. Tasaltin, "Design of robust autopilot output integral sliding mode controllers for guided missile systems with parameter perturbations," *Int. J. Aircraft Eng. Aerospace Technol.*, vol. 33, no. 1, pp. 16–25, 2001.

[12] D. S. Yoo, H. H. Choi, and M. J. Chung, "Adaptive variable structure control for robot manipulators," in *Proc. IEEE-TENCON'92*, Melbourne, Australia, Nov., pp. 1028–1032.

[13] E. Jafarov, Y. Istefanopulos, and M. N. A. Parlakçı, "A new variable structure-PID controller for robot manipulators with parameter perturbations: An augmented sliding surface approach," in *Proc. 15th IFAC World Congr.*, Barcelona, Spain, Jul. 2002.

[14] T. L. Chern and Y. C. Wu, "Integral variable structure control approach for robot manipulators," *Proc. Inst. Electr. Eng.*, vol. 139, no. 2, pp. 161–166, 1992.

[15] ——, "Design of integral variable structure controller and application to electrohydraulic velocity servosystems," *Proc. Inst. Electr. Eng.*, vol. 138, no. 5, pp. 439–444, 1991.

[16] C. Abdallah, D. Dawson, P. Dorato, and M. Jamshidi, "Survey of robust control for rigid robots," *IEEE Control Syst. Mag.*, vol. 11, no. 2, pp. 24–30, Feb. 1991.

[17] H. G. Sage, M. F. De Mathelin, and E. Ostertag, "Robust control of robot manipulators: A survey," *Int. J. Control*, vol. 72, no. 16, pp. 1498–1522, 1999.

[18] Y. Stepanenko, Y. Cao, and C. Su, "Variable structure control of robotic manipulator with PID sliding surfaces," *Int. J. Robust Nonlinear Control*, vol. 8, pp. 79–90, 1998.

[19] V. Parra-Vega and G. Hirzinger, "Chattering-free sliding mode control for a class of nonlinear mechanical systems," *Int. J. Robust Nonlinear Control*, vol. 11, pp. 1161–1178, 2001.

[20] H. Asada and J.-J. E. Slotine, *Robot Analysis and Control*. New York: Wiley, 1986.

[21] M. Spong and M. Vidyasagar, *Robot Dynamics and Control*. New York: Wiley, 1989.

[22] *NSK Robot Manual, IMI (Integrated Motions Corporated) Direct Drive Manipulator Research and Development Package*, 1992.