REPORT

A New Variable Structure PID-Controller Design for Robot Manipulators

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Abstract:

In this journal paper, the tracking stabilization of robot motion is designed using a new variable structure proportional-integral-derivative (PID) controller. For tracking control of a robotic manipulator, the work involves exploiting the advantages of a certain PID sliding mode controller with a PID sliding surface. In contrast to the general approach, this controller does not employ the conventional equivalent control term since it would require the use of exact full robot dynamics knowledge and matching conditions, which involves unavailability of parameter certainties. The existence of a sliding mode and gain selection guideline are clearly investigated, considering that the sliding surface also includes the integral error term, complicating the robot tracking control problem. Additionally, in contrast to uniformly ultimately boundedness, the proposed controller for the robot system's global asymptotic stability is examined. The full quadratic form of Lyapunov and the upper and lower matrix norm inequalities are used to define the sliding and global stability conditions. Additionally, a reduced design for the controller is also discussed. Through simulations, the suggested control algorithm is implemented on a two-link direct drive robot arm. The control law uses a pure signum function in place of a saturation function to address the chattering phenomena. The saturation function shall result in a smooth transient performance.

I. INTRODUCTION:

The robot manipulator is a complex network of interconnected links with extremely nonlinear dynamics. The amount of interaction between linkages grows as manipulator speed does, further complicating the control issue. The current control solutions are constrained to the conventional approach which depend on low speed-operation, where dynamic interaction is negligible, and the model can be successfully linearized and decoupled. Future applications, though, will operate at high speeds, making this strategy useless. For this reason, more advanced control techniques must be taken into consideration to enhance the manipulator's functionality and operational range. Sliding mode control is one such approach.

In control systems, sliding mode control (SMC) is a nonlinear control method in control systems that affects the dynamics of a nonlinear system by applying a discontinuous control signal (or, more precisely, a set-valued control signal) that compels the system to "slide" over a cross-section of the system's normal behaviour. The state-feedback control law is not a continuous function of time. Instead, depending on where it is in the state space, it can change from one continuous structure to another. Consequently, sliding mode control is a variable structure control method [1] – [4]. Since multiple control structures are designed such that trajectories always advance toward an adjacent region with a different control structure, the final trajectory won't entirely exist within one control structure. Instead, it will slide along the boundaries of control structures. The system's sliding motion along these limits is referred to as a sliding mode, and the geometric locus made up of these boundaries is known as the sliding surface. Any variable structure system, such as a system operating under SMC, may be seen as a special case of a hybrid dynamical system in the context of modern control theory because it transitions between different discrete control modes as well as a continuous state space.

In general, sliding mode control entails a non-linear feedback control law that switches discontinuously on a predetermined surface which is embedded in the state space of a dynamical system. This works in such a way that a force is applied to bring the state of the system back to the sliding surface when the open-loop system deviates from this surface naturally. In this way, a trajectory starting on this surface can only move or slide along it. This trajectory must undoubtedly satisfy the algebraic relation that characterizes the surface. On the other hand, by limiting the dynamics to a suitable surface, a desired algebraic relationship can be imposed on the state space of the open-loop system. For instance, the sliding surface can be represented as a linear algebraic equation to incorporate the position error and velocity. This relationship then defines a new equation that governs the dynamics on the surface. The position error and its velocity must tend to zero if this differential equation's homogeneous solution is asymptotically stable. Note that, as long as the trajectory stays on the surface, this relationship between the states is unaffected by any disturbances or modelling errors. In conclusion, selecting a surface and a control that points the vector field in the direction of the surface is required for the application of sliding mode control.

As the description above illustrates, sliding mode control is very well suited for the manipulator control problem for the following reasons. Firstly, if there is no unmodeled structural uncertainty, sliding mode control can be applied even without having a precise knowledge of system dynamics. This property is desired since the exact calculation of the dynamics is difficult, if not impossible, due to the complexity of the manipulator dynamics. Secondly, the performance of the system can be made insensitive to bounded disturbances when a sliding mode control is used. The rejection of effects brought on by Coulomb and viscous friction depends on this feature. It is also crucial that when the manipulator is transporting payloads, the force that a payload applies to the manipulator's gripper can be translated into forces or disturbances at each of the joints. As a result, the use of sliding mode control produces performance that is resistant to disturbances and modelling errors, while providing accurate tracking.

Variable structure control is a powerful control technology. It is often used to handle the worst-case control environment. This includes friction and complexity, nonlinearities, external disturbances, parametric perturbations with lower and upper bounds, etc. It is not necessary to use precise dynamic models and implementing the control methods is simple. In [5] and [6], the set-point regulation control problem is discussed. For robot manipulators with parameter perturbations, a relay type sliding mode controller with an equivalent control strategy is examined in [7]. In [8] and [9], a continuous sliding mode control law is used. In [10], a sliding mode controller is developed that considers the simultaneous force and position control of constraint robot manipulators. [11] presents an integral type of variable structure control technique for a guided missile system. The control law for sliding mode controls is typically composed of two components. The conventional equivalent control is one, while the switching part is another. Similar structure is also well-documented for the switching part in [12]. Furthermore, [12] also considers an adaptive variable structure control for robot manipulators.

[13] considers a design for an enhanced sliding surface for robot manipulators. Issues with both sliding and stability are considered. Chern and Wu have created integral variable structure controllers for robot manipulators [14] and electrohydraulic velocity servo systems [15]. A preliminary analysis of robust control has been published in [16]. An overview of six different robust control systems, including the current robot state coordinates for robot manipulators, is presented in another survey paper [17]. Robot control systems use a discontinuous min-max control term in conjunction with a linear control term. However, this brief does not take sliding and stability circumstances into account. [18] makes a new proposal for a combined variable structure controller with PID sliding surfaces for robot manipulators. The linear PID control and the discontinuous unit vector term with PID sliding surface make up the two components of this controller. Here, both regular and adaptive versions of the controller are presented. However, a nonlinear vector norm is a part of the control law. Investigations are conducted for the robot system's uniformly ultimately boundedness and exponentially stable sliding conditions. According to the simulation results, the PID sliding surface offers a quicker reaction than a traditional PD-manifold controller. [19] is a more current work that addresses the chattering issue. Passivity-based adaptive and nonadaptive chattering-free sliding mode controllers are presented. A desired transient response along with global exponential convergence of tracking errors is achieved.

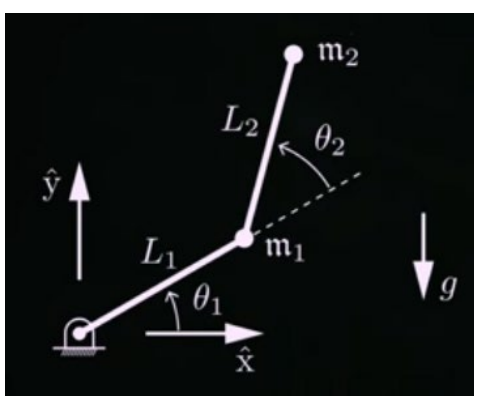
In this brief, the tracking stabilization of robot motion is designed using a new variable structure proportional-integral-derivative (PID) controller. For tracking control of a robotic manipulator, the work involves exploiting the advantages of a certain PID sliding mode controller with a PID sliding surface. In contrast to the general approach, this controller does not employ the conventional equivalent control term since it would require the use of exact full robot dynamics knowledge

and matching conditions, which involves unavailability of parameter certainties. The existence of a sliding mode and gain selection guideline are clearly investigated, considering that the sliding surface also includes the integral error term, complicating the robot tracking control problem. Additionally, in contrast to uniformly ultimately boundedness, the proposed controller for the robot system's global asymptotic stability is examined. The full quadratic form of Lyapunov and the upper and lower matrix norm inequalities are used to define the sliding and global stability conditions. Additionally, a reduced design for the controller is also discussed. Through simulations, the suggested control algorithm is implemented on a two-link direct drive robot arm. The simulation results show that the robot system's control performance is satisfactory. The control law uses a pure signum function in place of a saturation function to address the chattering phenomena. The saturation function results in a smooth transient performance. In terms of benefits and control performances, the proposed approach is compared with the existing alternative sliding mode controllers for robot manipulators. After a comparative examination using numerous simulation data, it is confirmed that the performance of the developed variable structure PID controller outperforms both the classical PID controller and an existing variable structure controller with a PID-sliding surface.

II. VARIABLE STRUCTURE PID-CONTROLLER DESIGN WITH INEXACT ROBOT PARAMETERS

A new robot tracking variable structure PID controller with a PID sliding surface design approach is described that differs from what is currently in use. In this controller, the typical equivalent control term is not employed since we believe that we do not fully understand the dynamics and characteristics of the robot arm system. By measuring the corresponding sensors, we can only determine the current angular position and angular velocity values. In general, it is not completely known and understood how the robot's mass, inertia, Coriolis and centrifugal force effects, friction, and gravity effects, etc., work. These observed robot parameters are extremely challenging nonlinear functions of joint position and joint velocity, and they significantly fluctuate at various intervals depending on the variation of joint position and joint velocity. However, since the robot parameters involve a nominal part and only some variation, the general form of the inertia matrix, Coriolis, centrifugal, friction, and gravity effects can be modelled. As a result, the upper and lower matrix norm of these parameters can be obtained and used in the choice of the design parameters. This is a benefit of this design process. Additionally, a parameter uncertainty is not explicitly provided in robot dynamics. By examining the sliding and global asymptotical stability criteria, the design parameters of the variable structure controller are parametrically determined. These requirements are expressed in terms of various min-max matrices norm inequalities and the entire quadratic form of Lyapunov.

A. Dynamics of the Robot Manipulator



The dynamics of robotic manipulators [18] can be represented by:

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau + d$$

Where,

$q \in R^n$ is the vector of joint angles, $\tau \in R^n$ is the vector of joint torque,

$M(q) \in R^{n\times n}$ is the inertia matrix,

$C(q,\dot{q})\dot{q} \in R^n$ is the vector of Coriolis and centripetal torque,

$g(q) \in R^n$ is the vector of gravitational torque,

$d \in R^n$ includes disturbances and unmodelled dynamics.

Introducing angular joint position integral as $\psi(t) = \int \theta_d(t)dt$, one can formulate the dynamics of an n-link manipulator [7], [20] using Lagrangian equations. The enhanced dynamics are then given by:

$$\dot{\psi} = \theta, \quad \dot{\theta} = \omega$$
$$\dot{\omega} = M(\theta)^{-1}\left[-B\left(\theta,\dot{\theta}\right)\dot{\theta} - f\left(\dot{\theta}\right) - g(\theta) + u(t)\right] \tag{1}$$

Where,

θ is an $n \times 1$ vector of angular joint position ($q\ from\ above$),

ψ is an $n \times 1$ vector of angular joint position integral,

ω is an $n \times 1$ vector of angular joint velocity,

$M(\theta)$ is an $n \times n$ symmetric positive–definite inertia matrix,

$B\left(\theta,\dot{\theta}\right)$ is an $n \times n$ Coriolis and centripetal force matrix,

$f\left(\dot{\theta}\right)$ is an $n \times 1$ vector of joint friction,

$g(\theta)$ is an $n \times 1$ vector of gravity,

$u(t)$ is an $n \times 1$ vector of control input.

The state vector of the dynamics is given as $x = [\psi\ \theta\ \omega]$. Proceeding further, we need to construct the error dynamics of the robot arm system. We denote $\theta_d(t)$ as a time-varying reference position vector, $\psi_d(t) = \int_0^t \theta_d(t')d\,t'$ as the integral of reference joint position and the reference joint velocity as $\omega_d(t) = \dot{\theta_d}(t)$. The difference between the actual position integral, position, and velocity and their reference counterparts are then denoted by $\tilde{\psi}(t) = \psi(t) - \psi_d(t)$, $\tilde{\theta}(t) = \theta(t) - \theta_d(t)$, $\tilde{\omega}(t) = \omega(t) - \omega_d(t)$, respectively. The error dynamics can then be written as follows:

$$\dot{\tilde{\psi}} = \tilde{\theta}, \qquad \dot{\tilde{\theta}} = \tilde{\omega},$$

$$\dot{\tilde{\omega}} = M(\theta)^{-1}\left[-B(\theta,\dot{\theta})\tilde{\omega} - f(\dot{\theta}) - g(\theta) + u(t)\right] - M(\theta)^{-1}B(\theta,\dot{\theta})\dot{\theta}_d(t) - \ddot{\theta}_d(t) \tag{2}$$

The proposed controller's goal is to create a general purpose PID sliding mode tracking controller for the robot manipulator system (1) that is globally asymptotically stable.

B. *Variable Structure PID Controller with PID Sliding Surface*

To reach global asymptotical stability, the following sliding mode control law is formed as:

$$u(t) = -\left[K_r + K_p\|\tilde{\theta}(t)\| + K_i\|\tilde{\psi}(t)\| + K_d\|\tilde{\omega}(t)\|\right]sign(s(t)) \tag{3}$$

Where,

K_r is a positive scalar relay gain constant,

K_p is a positive scalar feedback proportional gain constant,

K_i is a positive scalar feedback integral gain constant,

K_d is a positive scalar feedback derivative gain constant,

$\|(\cdot)\| = \sqrt{(\cdot)^{\top}(\cdot)}$ is the Euclidean norm,

$\top$ is the transpose of vector or matrix,

$sign(s(t)) = [sign(s_1(t)),\ldots, sign(s_n(t))]^{\top}$ is the signum function vector.

The commonly used sliding surface definition, which includes the position error and velocity error of the form $s(t) = \dot{e} + \Lambda e$ can be extended to include the integral error as $s(t) = \dot{e} + \Lambda_1 e + \Lambda_2 \int e\, dt$, where Λ_1 and Λ_2 denotes constant positive-definite matrices and e symbolizes the state error term. Consequently, the augmented sliding surface function is introduced as:

$$s(t) = C_1\tilde{\psi}(t) + C_2\tilde{\theta}(t) + \tilde{\omega}(t) \tag{4}$$

Where C_1 and C_2 are constant design matrices. To investigate how these design matrices can be chosen to establish a stable sliding surface, one can set $s(t) = 0$ along with its state-space representation as shown below:

$$\begin{bmatrix} \dot{\tilde{\psi}}(t) \\ \dot{\tilde{\theta}}(t) \\ \dot{\tilde{\omega}}(t) \end{bmatrix} = \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \\ -C_1 & C_2 & 0 \end{bmatrix} \begin{bmatrix} \tilde{\psi}(t) \\ \tilde{\theta}(t) \\ \tilde{\omega}(t) \end{bmatrix} \tag{5}$$

Using C_1 and C_2 as positive-definite design matrices result in a stable reduced error system as shown in (5). Following the selection of the sliding mode control law (3) with sliding surface function (4), the next step is to pick the design parameters such that the necessary requirements for the existence of a sliding mode are met and the closed-loop system is globally asymptotically stable. To demonstrate robustness vs parameter variations, only one of the robust techniques is used (known as the second Lyapunov method), which is identical to [21]. The variable structure controller's capabilities for improved control performance and robustness are utilized completely.

C. *Sliding Conditions*

This section introduces a Lemma in order to derive the sufficient conditions for the presence of the sliding mode in the robot-manipulator system. Once it reaches the sliding mode, the system becomes resistant to parameter uncertainty.

Lemma 1: The stable sliding mode on $s(t) = 0$ (4) always exists in a dynamic system (2) driven by controller (3), (4), if the following conditions hold:

$$\begin{aligned} K_r \geqslant\ & \max_{\dot{\theta}} \left\| f(\dot{\theta}) \right\| + \max_{\theta} \| g(\theta) \| \\ & + \max_{\theta,\dot{\theta}} \left\| B(\theta,\dot{\theta}) \right\|_F \max_t \left\| \dot{\theta}_d(t) \right\| \\ & + \max_{\theta} \| M(\theta) \|_F \max_t \left\| \ddot{\theta}_d(t) \right\| \end{aligned}$$

$$K_d \geqslant \eta + \max_{\theta} \|M(\theta)\|_F \, \|C_2\|_F$$

$$\begin{aligned} K_p \geqslant\ & \|C_2\|_F K_d - \max_{\theta} \|M(\theta)\|_F \, \|C_2\|_F^2 \\ & + \max_{\theta} \|M(\theta)\|_F \, \|C_1\|_F \\ & + \max_{\theta,\dot{\theta}} \left\|B\left(\theta, \dot{\theta}\right)\right\|_F \, \|C_2\|_F \end{aligned}$$

$$\begin{aligned} K_i \geqslant\ & \|C_1\|_F K_d - \max_{\theta} \|M(\theta)\|_F \, \|C_1\|_F \|C_2\|_F \\ & + \max_{\theta,\dot{\theta}} \left\|B\left(\theta, \dot{\theta}\right)\right\|_F \, \|C_1\|_F \end{aligned} \qquad (6)$$

Where $\|(\cdot)\|_F = \sqrt{trace\left((\cdot)^{\top}(\cdot)\right)}$ is the Frobenius norm and η is a constantly chosen positive scalar.

Proof: Choosing a Lyapunov function candidate as in [6]

$$V\left(s(t)\right) = s(t)^{\top} M(\theta) s(t)/2 \qquad (7)$$

The time derivative of V along the state error trajectory of system (2) – (4) is given by:

$$\begin{aligned} \dot{V} &= s(t)^{\top} M(\theta) \dot{s}(t) + s(t)^{\top} \dot{M}(\theta) s(t)/2 \\ &= s(t)^{\top} M(\theta) \left[C_1 \dot{\tilde{\psi}} + C_2 \dot{\tilde{\theta}} + \dot{\tilde{\omega}}\right] + s(t)^{\top} \dot{M}(\theta) s(t)/2 \\ &= s(t)^{\top} M(\theta) \{C_1 \tilde{\theta} + C_2 \tilde{\omega} - M(\theta)^{-1} \\ &\quad \cdot [B(\theta, \dot{\theta}) \tilde{\omega} + f(\dot{\theta}) + g(\theta) \\ &\quad + (K_r + K_p \|\tilde{\theta}\| + K_i \|\tilde{\psi}\| + K_d \|\tilde{\omega}\|) \\ &\quad \cdot sign(s(t))] - M(\theta)^{-1} \, B\left(\theta, \dot{\theta}\right) \dot{\theta}_d(t) - \ddot{\theta}_d(t)\} \end{aligned}$$

$$+ s(t)^\top \dot{M}(\theta) s(t)/2 \tag{8}$$

Using an identity $B(\theta, \dot{\theta}) = [\dot{M}(\theta) - J]/2$ given in [7], where J is a skew symmetric matrix. Then rewriting $\dot{V}$, we get:

$$\begin{aligned}
\dot{V} = {} & s(t)^\top \big[M(\theta) C_1 \tilde{\theta} + M(\theta) C_2 \tilde{\omega} - B(\theta, \dot{\theta}) \tilde{\omega} - f(\dot{\theta}) \\
& - g(\theta) - \big(K_r + K_p \|\tilde{\theta}\| + K_i \|\tilde{\psi}\| + K_d \|\tilde{\omega}\| \big) sign\big(s(t)\big) \\
& - B(\theta, \dot{\theta}) \dot{\theta}_d(t) - M(\theta) \ddot{\theta}_d(t) \big] + s(t)^\top B(\theta, \dot{\theta}) \big[C_1 \tilde{\psi} \\
& + C_2 \tilde{\theta} + \tilde{\omega} \big] + s(t)^\top J s(t)/2
\end{aligned}$$

$$\begin{aligned}
= {} & s(t)^\top \big\{ B(\theta, \dot{\theta}) C_1 \tilde{\psi} + [M(\theta) C_1 \\
& + B(\theta, \dot{\theta}) C_2] \tilde{\theta} + M(\theta) C_2 \tilde{\omega} \\
& - f(\dot{\theta}) - g(\theta) \} - \big(K_r + K_p \|\tilde{\theta}\| + K_i \|\tilde{\psi}\| + K_d \|\tilde{\omega}\| \big) \\
& \cdot s(t)^\top sign\big(s(t)\big) - s(t)^\top B(\theta, \dot{\theta}) \dot{\theta}_d(t) - s(t)^\top M(\theta) \ddot{\theta}_d(t)
\end{aligned} \tag{9}$$

Since $s(t)^\top sign\big(s(t)\big) \geqslant \|s(t)\|$, and $s(t)^\top J s(t)/2 = 0$ as in [7], taking the norm of the remaining terms, we obtain:

$$\begin{aligned}
\dot{V} \leq \|s(t)\| \Big[& \max_{\theta, \dot{\theta}} \big\| B(\theta, \dot{\theta}) \big\|_F \|C_1\|_F \, \|\tilde{\psi}\| \\
& + \Big(\max_{\theta} \|M(\theta)\|_F \, \|C_1\|_F \\
& + \max_{\theta, \dot{\theta}} \big\| B(\theta, \dot{\theta}) \big\|_F \, \|C_2\|_F \Big) \|\tilde{\theta}\| \\
& + \max_{\theta} \|M(\theta)\|_F \, \|C_2\|_F \|\tilde{\omega}\| \\
& + \max_{\dot{\theta}} \big\| f(\dot{\theta}) \big\| \; + \; \max_{\theta} \|g(\theta)\| \Big] \\
& \qquad - \big(Kr + K_p \|\tilde{\theta}\| + K_i \|\tilde{\psi}\|
\end{aligned}$$

$$+K_d\|\tilde{\omega}\|)\cdot\|s(t)\|+\|s(t)\|$$
$$\times\max_{\theta,\dot{\theta}}\left\|B\left(\theta,\dot{\theta}\right)\right\|_F\max_t\left\|\dot{\theta}_d(t)\right\|$$
$$+\|s(t)\|\max_{\theta}\|M(\theta)\|_F\max_t\left\|\ddot{\theta}_d(t)\right\| \qquad (10)$$

Taking all the terms into the parentheses of $-\|s(t)\|$ gives:

$$\dot{V}\leq-\|s(t)\|\left[K_r-\max_{\dot{\theta}}\left\|f(\dot{\theta})\right\|\right.$$
$$-\max_{\theta,\dot{\theta}}\left\|B\left(\theta,\dot{\theta}\right)\right\|_F\max_t\left\|\dot{\theta}_d(t)\right\|$$
$$-\max_{\theta}\|M(\theta)\|_F\max_t\left\|\ddot{\theta}_d(t)\right\|$$
$$+\left(K_i-\max_{\theta,\dot{\theta}}\left\|B\left(\theta,\dot{\theta}\right)\right\|_F\|C_1\|_F\right)\|\tilde{\psi}\|$$
$$+\left(K_P-\max_{\theta}\|M(\theta)\|_F\|C_1\|_F\right.$$
$$\left.-\max_{\theta,\dot{\theta}}\left\|B\left(\theta,\dot{\theta}\right)\right\|_F\|C_2\|_F\right)\|\tilde{\theta}\|$$
$$\left.+\left(K_d-\max_{\theta}\|M(\theta)\|_F\|C_2\|_F\right)\|\tilde{\omega}\|\right] \qquad (11)$$

If K_r is chosen as in (6), and K_i, K_P, K_d are chosen such that –

$$K_i>\max_{\theta,\dot{\theta}}\left\|B\left(\theta,\dot{\theta}\right)\right\|_F\|C_1\|_F$$
$$K_P>\max_{\theta}\|M(\theta)\|_F\|C_1\|_F+\max_{\theta,\dot{\theta}}\left\|B\left(\theta,\dot{\theta}\right)\right\|_F\|C_2\|_F$$
$$K_d>\max_{\theta}\|M(\theta)\|_F\|C_2\|_F$$

Then $\dot{V}<0$ will always be ensured and a sliding mode will be generated in the robot arm system. However, since the η-reaching condition is required [2], [20] it is given for our multivariable case by:

$$\dot{V}=s(t)^{\top}M(\theta)\dot{s}(t)+s(t)^{\top}\dot{M}(\theta)s(t)/2$$

$$
\begin{aligned}
&\leq -\|s(t)\| \cdot \left(K_d - \max_{\theta}\|M(\theta)\|_F \|C_2\|_F\right) \left[\|C_1\|_F \|\tilde{\psi}\|\right. \\
&\quad +\|C_2\|_F \|\tilde{\theta}\| \\
&\quad +\|\widetilde{\omega}\| + \left[\left(K_i - \max_{\theta,\dot{\theta}}\left\|B(\theta,\dot{\theta})\right\|_F \|C_1\|_F\right)/K_d\right. \\
&\quad \left. - \max_{\theta}\|M(\theta)\|_F \|C_2\|_F\right) - \|C_1\|_F\Big] \|\tilde{\psi}\| \\
&\quad + \left(K_P - \max_{\theta}\|M(\theta)\|_F \|C_1\|_F - \max_{\theta,\dot{\theta}}\left\|B(\theta,\dot{\theta})\right\|_F\right. \\
&\quad \times \|C_2\|_F)/\left(K_d - \max_{\theta}\|M(\theta)\|_F \|C_2\|_F\right) \\
&\quad -\|C_2\|_F)\|\tilde{\theta}\| \leq -\eta\|s(t)\|^2
\end{aligned} \tag{12}
$$

Choosing K_i and K_P as in (6)) and assuming that $K_d > \max_{\theta}\|M(\theta)\|_F \|C_2\|_F$, (12) can be rewritten as:

$$
\begin{aligned}
\dot{V} \leq -\|s(t)\| \left(K_d - \max_{\theta}\|M(\theta)\|_F \|C_2\|_F\right) &\left[\|C_1\|_F \|\tilde{\psi}\|\right. \\
&\left. +\|C_2\|_F \|\tilde{\theta}\| + \|\widetilde{\omega}\|\right] \leq -\eta\|s(t)\|^2
\end{aligned} \tag{13}
$$

Utilizing Cauchy–Schwarz triangle inequality:

$$
\begin{aligned}
&\|C_1\|_F \|\tilde{\psi}(t)\| + \|C_2\|_F \|\tilde{\theta}(t)\| + \|\tilde{\omega}(t)\| \\
&\qquad\qquad \geq \left\|C_1\tilde{\psi}(t) + C_2\tilde{\theta}(t) + \tilde{\omega}(t)\right\| = \|s(t)\|
\end{aligned}
$$

Then (13) can be reduced to:

$$
\dot{V} \leq -\left(K_d - \max_{\theta}\|M(\theta)\|_F \|C_2\|_F\right) \|s(t)\|^2 \leq -\eta\|s(t)\|^2 \tag{14}
$$

We notice that (14) implies that:

$$
-\left(K_d - \max_{\theta}\|M(\theta)\|_F \|C_2\|_F - \eta\right) \|s(t)\|^2 \leq 0 \tag{15}
$$

Therefore, the sliding inequality in (15) is reduced to $\dot{V} < 0$ for all $s(t) \neq 0$ if the conditions in (6) are fully satisfied. We can conclude that a stable sliding motion will always be generated on the switching surface $s(t) = 0$ (4). The designed system, however, must also be globally asymptotically stable with respect to the state coordinates in large. This problem is further described in the next section.

D. Global Asymptotical Stability

The global asymptotic stability conditions for the closed-loop robot system are summarized in the following theorem.

Theorem 1: If conditions (6) of Lemma 1 are met, then the dynamic system (2) driven by controller (3), (4) is globally asymptotically stable if the following conditions are met:

$$R_3(\theta,\dot{\theta}) > 0; \quad \lambda_{\min}(R_1) > \max_{\theta,\dot{\theta}} \|P_3(\theta,\dot{\theta})\|_F^2 / \min_{\theta,\dot{\theta}} \lambda[R_3(\theta,\dot{\theta})]$$

$$K_d > \|P\|_F^2 \{\lambda_{\min}(C_1^\top C_1)\, [\lambda_{\min}(R_1) \\ -\max_{\theta,\dot{\theta}} \|P_3(\theta,\dot{\theta})\|_F^2 \Big/ \min_{\theta,\dot{\theta}} \lambda\,[R_3(\theta,\dot{\theta})]\Big]^{-1}\Big\}$$

$$K_i \geq \|C_1\|_F K_d; \quad K_p \geq \|C_2\|_F K_d \qquad (16)$$

Where,

$$P_1 = A/2$$

$$R_1 = [C_2^\top (C_1^\top)^{-1} A + A^\top C_1^{-1} C_2 - W]/2$$

$$P_2(\theta,\dot{\theta}) = W/2 + C_1^\top \dot{M}(\theta)/2 - C_1^\top B(\theta,\dot{\theta})/2$$

$$R_2(\theta,\dot{\theta}) = (C_1^\top)^{-1} P_2(\theta,\dot{\theta}) + P_2(\theta,\dot{\theta})^\top C_1^{-1} \\ -[C_2^T M(\theta) + M(\theta) C_2]/2$$

Assuming P_1 is invertible:

$$P_3(\theta,\dot{\theta}) = C_1^\top M(\theta)/2 + D/2 + C_2^\top \dot{M}(\theta)/2 - C_2^\top B(\theta,\dot{\theta})/2$$
$$-C_2^\top (C_1^\top)^{-1} P_2(\theta,\dot{\theta}) - P_1^\top C_1^{-1} + R_1 P_1^{-1} P_2(\theta,\dot{\theta})$$
$$R_3(\theta,\dot{\theta}) = R_2(\theta,\dot{\theta}) - P_2(\theta,\dot{\theta})^\top (P_1^\top)^{-1} R_1 P_1^{-1} P_2(\theta,\dot{\theta})$$
$$+P_2(\theta,\dot{\theta})^\top (P_1^\top)^{-1} P_3(\theta,\dot{\theta}) + P_3(\theta,\dot{\theta})^\top P_1^{-1} P_2(\theta,\dot{\theta})$$

Proof: Introducing a positive-definite full quadratic form of Lyapunov function candidate:

$$V(\tilde{\psi},\tilde{\theta},\tilde{\omega})$$
$$= \frac{1}{2}\begin{bmatrix}\tilde{\psi}\\ \tilde{\theta}\\ \tilde{\omega}\end{bmatrix}^\top \begin{bmatrix} A & W & C_1^\top M(\theta)\\ W^\top & D & C_2^\top M(\theta)\\ M(\theta)C_1 & M(\theta)C_2 & M(\theta)\end{bmatrix}\begin{bmatrix}\tilde{\psi}\\ \tilde{\theta}\\ \tilde{\omega}\end{bmatrix}$$
$$= \tilde{\psi}^\top A\tilde{\psi}/2 + \tilde{\psi}^\top W\tilde{\theta} + \tilde{\psi}^\top C_1^\top M(\theta)\tilde{\omega}$$
$$+\tilde{\theta}^\top D\tilde{\theta}/2 + \tilde{\theta}^\top C_2^\top M(\theta)\tilde{\omega} + \tilde{\omega}^\top M(\theta)\tilde{\omega}/2 \qquad (17)$$

Where,

C_1 and C_2 are sliding surface slope gain matrices,

A and D are symmetric gain matrices,

W is another gain matrix with appropriate dimensions such that:

$$A > 0; \quad D - W^\top A^{-1} W > 0$$
$$M(\theta) - M(\theta)C_1 A^{-1} C_1^\top M(\theta) - [M(\theta)C_2 - M(\theta)C_1 A^{-1} W]$$
$$\cdot [D - W^\top A^{-1} W]^{-1} [C_2^\top M(\theta) - W^\top A C_1^\top M(\theta)] > 0$$

Taking the time derivative of $\dot{V}(\tilde{\psi},\tilde{\theta},\tilde{\omega},t)$ along the system dynamics and the state trajectory of the system (2) – (4):

$$\dot{V}(\tilde{\psi},\tilde{\theta},\tilde{\omega},t)$$

$$
\begin{aligned}
&= \tilde{\psi}^\top A\dot{\tilde{\psi}} + \dot{\tilde{\psi}}^\top W\tilde{\theta} + \tilde{\psi}^\top W\dot{\tilde{\theta}} \\
&\quad + \dot{\tilde{\psi}}^\top C_1^\top M(\theta)\tilde{\omega} + \tilde{\psi}^\top C_1^\top M(\theta)\dot{\tilde{\omega}} \\
&\quad + \tilde{\psi}^\top C_1^\top \dot{M}(\theta)\tilde{\omega} + \tilde{\theta}^\top D\dot{\tilde{\theta}} + \dot{\tilde{\theta}}^\top C_2^\top M(\theta)\tilde{\omega} \\
&\quad + \tilde{\theta}^\top C_2^\top M(\theta)\dot{\tilde{\omega}} + \tilde{\theta}^\top C_2^\top \dot{M}(\theta)\tilde{\omega} + \tilde{\omega}^\top M(\theta)\dot{\tilde{\omega}} \\
&\quad + \tilde{\omega}^\top \dot{M}(\theta)\tilde{\omega}/2
\end{aligned} \tag{18}
$$

Substituting system dynamics (2) into $\dot{V}\left(\tilde{\psi}, \tilde{\theta}, \tilde{\omega}, t\right)$:

$$
\begin{aligned}
&\dot{V}\left(\tilde{\psi}, \tilde{\theta}, \tilde{\omega}, t\right) \\
&= \tilde{\psi}^\top A\tilde{\theta} + \tilde{\theta}^\top W\tilde{\theta} + \tilde{\psi}^\top W\tilde{\omega} + \tilde{\theta}^\top C_1^\top M(\theta)\tilde{\omega} \\
&\quad + \tilde{\psi}^\top C_1^\top M(\theta)\dot{\tilde{\omega}} + \tilde{\psi}^\top C_1^\top \dot{M}(\theta)\tilde{\omega} + \tilde{\theta}^\top D\tilde{\omega} \\
&\quad + \tilde{\omega}^\top C_2^\top M(\theta)\tilde{\omega} + \tilde{\theta}^\top C_2^\top M(\theta)\dot{\tilde{\omega}} + \tilde{\theta}^\top C_2^\top \dot{M}(\theta)\tilde{\omega} \\
&\quad + \tilde{\omega}^\top M(\theta)\dot{\tilde{\omega}} + \tilde{\omega}^\top \dot{M}(\theta)\tilde{\omega}/2 \\
&= \tilde{\psi}^\top A\tilde{\theta} + \tilde{\psi}^\top\left[W + C_1^\top \dot{M}(\theta)\right]\tilde{\omega} + \tilde{\theta}^\top W\tilde{\theta} \\
&\quad + \tilde{\theta}^\top\left[C_1^\top M(\theta) + D + C_2^\top \dot{M}(\theta)\right]\tilde{\omega} + \tilde{\omega}^\top\left[C_2^\top M(\theta)\right. \\
&\quad \left. + \dot{M}(\theta)/2\right]\tilde{\omega} + \left[\tilde{\psi}^\top C_1^\top + \tilde{\theta}^\top C_2^\top + \tilde{\omega}^\top\right]M(\theta)\dot{\tilde{\omega}}
\end{aligned} \tag{19}
$$

The last term $\left[\tilde{\psi}^\top C_1^\top + \tilde{\theta}^\top C_2^\top + \tilde{\omega}^\top\right]M(\theta)\dot{\tilde{\omega}}$ of (19) can be rewritten as:

$$
\begin{aligned}
(*) &= \left[\tilde{\psi}^\top C_1^\top + \tilde{\theta}^\top C_2^\top + \tilde{\omega}^\top\right]M(\theta)\dot{\tilde{\omega}} \\
&= \left[\tilde{\psi}^\top C_1^\top + \tilde{\theta}^\top C_2^\top + \tilde{\omega}^\top\right] \\
&\quad \cdot M(\theta)\{M(\theta)^{-1}\left[-B(\theta, \dot{\theta})\,\tilde{\omega}\right. \\
&\quad \left. - f(\dot{\theta}) - g(\theta) + u\right] - M(\theta)^{-1} \\
&\quad \cdot B\left(\theta, \dot{\theta}\right)\dot{\theta}_d(t) - \ddot{\theta}_d(t)\}
\end{aligned}
$$

$$
\begin{aligned}
&= -\tilde{\psi}^\top C_1^\top B(\theta,\dot{\theta})\tilde{\omega} - \tilde{\theta}^\top C_2^\top B(\theta,\dot{\theta})\,\tilde{\omega} \\
&\quad -\tilde{\omega}^\top B(\theta,\dot{\theta})\,\tilde{\omega} - s(t)^\top[f(\dot{\theta}) \\
&\quad +g(\theta) + B(\theta,\dot{\theta})\dot{\theta}_d(t) + M(\theta)\ddot{\theta}_d(t)] \\
&\quad -K_r s(t)^\top sign(s(t)) - K_d\big[\|C_1\|_F\|\tilde{\psi}\| + \|C_2\|_F\|\tilde{\theta}\| \\
&\quad +\|\tilde{\omega}\| + (K_i/K_d - \|C_1\|_F)\|\tilde{\psi}\| \\
&\quad +(K_p/K_d - \|C_2\|_F)\|\tilde{\theta}\|\big]s(t)^\top sign(s(t))
\end{aligned}
$$

Choosing K_p and K_i from (16) and utilizing $s(t)^\top sign(s(t)) \geq \|s(t)\|$, then taking the norm gives:

$$
\begin{aligned}
(*) &\leq -\tilde{\psi}^\top C_1^\top B(\theta,\dot{\theta})\tilde{\omega} - \tilde{\theta}^\top C_2^\top B(\theta,\dot{\theta})\tilde{\omega} - \tilde{\omega}^\top B(\theta,\dot{\theta})\tilde{\omega} \\
&\quad -\Big[K_r - \max_{\dot{\theta}}\|f(\dot{\theta})\| - \max_{\theta}\|g(\theta)\| \\
&\quad - \max_{\theta,\dot{\theta}}\|B(\theta,\dot{\theta})\|_F \max_t\|\dot{\theta}_d(t)\| \\
&\quad - \max_{\theta}\|M(\theta)\|_F\,\|\ddot{\theta}_d(t)\|\Big]\,\|s(t)\| \\
&\quad -K_d\big[\|C_1\|_F\|\tilde{\psi}\| + \|C_2\|_F\|\tilde{\theta}\| + \|\tilde{\omega}\|\big]\|s(t)\|
\end{aligned}
$$

Choosing K_r from (16) and using Cauchy–Schwartz triangle inequality, we get:

$$
\begin{aligned}
(*) &\leq -\tilde{\psi}^\top C_1^\top B(\theta,\dot{\theta})\tilde{\omega} - \tilde{\theta}^\top C_2^\top B(\theta,\dot{\theta})\tilde{\omega} \\
&\quad -\tilde{\omega}^\top B(\theta,\dot{\theta})\tilde{\omega} - K_d\|s(t)\|^2 \\
&\leq -\tilde{\psi}^\top C_1^\top B(\theta,\dot{\theta})\tilde{\omega} - \tilde{\theta}^\top C_2^\top B(\theta,\dot{\theta})\tilde{\omega} - \tilde{\omega}^\top B(\theta,\dot{\theta})\tilde{\omega} \\
&\quad -[C_1\tilde{\psi} + C_2\tilde{\theta} + \tilde{\omega}]^\top K_d[C_1\tilde{\psi} \\
&\quad +C_2\tilde{\theta} + \tilde{\omega}] \leq -\tilde{\psi}^\top C_1^\top K_d C_1\tilde{\psi} \\
&\quad -2\tilde{\psi}^\top C_1^\top K_d C_2\tilde{\theta} - \tilde{\psi}^\top\left(2C_1^\top K_d + C_1^\top B(\theta,\dot{\theta})\right)\tilde{\omega}
\end{aligned}
$$

$$-\tilde{\theta}^\top C_2^T K_d C_2 \tilde{\theta}$$

$$-\tilde{\theta}^\top \left(2C_2^\top K_d + C_2^\top B(\theta, \dot{\theta})\right) \tilde{\omega}$$

$$-\tilde{\omega}^\top \left(K_d I + B(\theta, \dot{\theta})\right) \tilde{\omega}$$

Substituting $(*)$ back in (19) and simplifying, we get:

$$\dot{V}(\tilde{\psi}, \tilde{\theta}, \tilde{\omega}, t)$$

$$\leq -\tilde{\psi}^\top C_1^\top K_d C_1 \tilde{\psi} - 2\tilde{\psi}^\top (C_1^\top K_d C_2 - A/2)\tilde{\theta} - 2\tilde{\psi}^\top$$

$$\cdot \left[C_1^\top K_d + C_1^\top B(\theta, \dot{\theta})/2 - W/2 - C_1^\top \dot{M}(\theta)/2\right]\tilde{\omega}$$

$$-\tilde{\theta}^\top (C_2^T K_d C_2 - W/2)\tilde{\theta} - 2\tilde{\theta}^\top$$

$$\times \left[C_2^T K_d + C_2^\top B(\theta, \dot{\theta})/2 - C_1^\top M(\theta)/2 - D/2\right.$$

$$\left. - C_2^\top \dot{M}(\theta)/2\right]$$

$$\times \tilde{\omega} - \tilde{\omega}^\top \left(K_d I + B(\theta, \dot{\theta}) - C_2^\top M(\theta) - \dot{M}(\theta)/2\right)\tilde{\omega} \qquad (20)$$

Using the identity given in [7] with $(\cdot)^\top J(\cdot) = 0$ for the last term of (20):

$$(**) = -\tilde{\omega}^\top \left(K_d I + B(\theta, \dot{\theta}) - C_2^T M(\theta) - \dot{M}(\theta)/2\right)\tilde{\omega}$$

$$= -\tilde{\omega}^\top \left[K_d I + \dot{M}(\theta)/2 - J/2 - C_2^\top M(\theta) - \dot{M}(\theta)/2\right)\tilde{\omega}$$

$$= -\tilde{\omega}^\top [K_d I - C_2^T M(\theta)]\tilde{\omega}$$

Substituting $(**)$ in (20) and converting $\dot{V}(\tilde{\psi}, \tilde{\theta}, \tilde{\omega}, t)$ (20) to quadratic form as:

$$\dot{V} \leq -x^\top H x$$

Where H is a symmetric $3n \times 3n$ matrix with –

$$H_{11} = C_1^\top K_d C_1$$

$$H_{12} = C_1^T K_d C_2 - P_1,$$

$$H_{13} = C_1^\top K_d - P_2(\theta, \dot{\theta}),$$

$$H_{23} = C_2^\top K_d + C_2^\top B(\theta, \dot{\theta})/2 - C_1^\top M(\theta)/2 - D/2 - C_2^\top \dot{M}(\theta)/2$$

$$H_{33} = K_d I + R_2(\theta, \dot{\theta}) - (C_1^\top)^{-1} P_2(\theta, \dot{\theta}) - P_2(\theta, \dot{\theta})^\top C_1^{-1}$$

And P_1, R_1, $P_2(\theta, \dot{\theta})$, $R_2(\theta, \dot{\theta})$, $P_3(\theta, \dot{\theta})$, $R_3(\theta, \dot{\theta})$ being defined with established Theorem 1. Hence, we need to investigate appropriate K_d's such that H always remains positive-definite making $\dot{V}(\tilde{\psi}, \tilde{\theta}, \widetilde{\omega}, t)$ always negative-definite for a global asymptotically stable robot arm system.

For the next step, we must choose K_d such that it makes H non-negative. To achieve this, Schur's complement is used since H is a partitioned matrix. Using Schur's complement's definition, the following in-equalities are established:

$$H_{11} > 0, \quad H_{22} - H_{12}^\top H_{11}^{-1} H_{12} > 0 \tag{22}$$

$$H_{33} - H_{13}^\top H_{11}^{-1} H_{13} - (H_{23} - H_{12}^\top H_{11}^{-1} H_{13})^\top \cdot (H_{22} - H_{12}^\top H_{11}^{-1} H_{12})^{-1} (H_{23} - H_{12}^\top H_{11}^{-1} H_{13}) > 0 \tag{23}$$

Considering the first inequality of (22), $C_1^\top K_d C_1$ which is equal to H_{11} will always be positive-definite since K_d is a positive scalar constant. For the second inequality of (22), we have:

$$H_{22} - H_{12}^\top H_{11}^{-1} H_{12} = R_1 - P_1^\top C_1^{-1} K_d^{-1} (C_1^\top)^{-1} P_1 > 0 \tag{24}$$

If $R_1 > 0$, and given that $K_d > 0$, we obtain:

$$K_d > \|P_1\|_F^2 (\lambda_{\min}(C_1^T C_1) \lambda_{\min}(R_1) = \|A\|_F^2 / \left[4\lambda_{\min}(C_1^\top C_1)\lambda_{\min}(R_1)\right] \tag{25}$$

Before proceeding to analyse the third condition in (23), let us make a change of variable as given below:

$$H_{22} - H_{12}^\top H_{11}^{-1} H_{12} = R_1 - P_1^\top C_1^{-1} K_d^{-1} \cdot (C_1^\top)^{-1} P_1 = K_{d1}$$

Assuming that P_1 is invertible, we compute:

$$H_{33} - H_{13}^\top H_{11}^{-1} H_{13} = R_2(\theta, \dot{\theta})$$

$$-P_2(\theta,\dot{\theta})^\top C_1^{-1} K_d^{-1} (C_1^\top)^{-1}$$

$$\cdot P_2(\theta,\dot{\theta}) = R_2(\theta,\dot{\theta}) + P_2(\theta,\dot{\theta})^\top (P_1^\top)^{-1}$$

$$\times K_{d1} P_1^{-1} P_2(\theta,\dot{\theta})$$

$$- P_2(\theta,\dot{\theta})^\top (P_1^\top)^{-1} R_1 P_1^{-1} P_2(\theta,\dot{\theta})$$

$$H_{23} - H_{12}^\top H_{11}^{-1} H_{13} = C_2^\top B(\theta,\dot{\theta})/2 - C_1^\top M(\theta)/2 - D/2$$

$$-C_2^\top \dot{M}(\theta)/2 + C_2^T (C_1^T)^{-1} P_2(\theta,\dot{\theta}) + P_1^\top C_1^{-1}$$

$$-P_1^\top C_1^{-1} K_d^{-1} (C_1^\top)^{-1} P_2(\theta,\dot{\theta})$$

$$= K_{d1} P_1^{-1} P_2(\theta,\dot{\theta}) - P_3(\theta,\dot{\theta})$$

Then the condition in (23) can be rewritten as:

$$H_{33} - H_{13}^\top H_{11}^{-1} H_{13} - (H_{23}^\top - H_{13}^\top H_{11}^{-1} H_{12})$$

$$\times (H_{22} - H_{12}^\top H_{11}^{-1} H_{12})^{-1} \cdot (H_{23} - H_{12}^\top H_{11}^{-1} H_{13})$$

$$= R_3(\theta,\dot{\theta}) - P_3(\theta,\dot{\theta})^\top K_{d1}^{-1} P_3(\theta,\dot{\theta}) > 0 \qquad (26)$$

Assuming that $R_3(\theta,\dot{\theta}) > 0$, then:

$$\lambda_{\min}(R_1) > \max_{\theta,\dot{\theta}} \left\| P_3(\theta,\dot{\theta}) \right\|_F^2 / \min_{\theta,\dot{\theta}} \lambda \left[R_3(\theta,\dot{\theta}) \right]$$

Since $K_{d1} > 0$, it follows from (26) that:

$$\lambda_{\min}(K_{d1}) > \lambda_{\min}(R_1) - \|P_1\|_F^2 / \lambda_{\min}(C_1^\top C_1) K_d$$

$$> \max_{\theta,\dot{\theta}} \left\| P_3(\theta,\dot{\theta}) \right\|_F^2 / \min_{\theta,\dot{\theta}} \lambda \left[R_3(\theta,\dot{\theta}) \right] \qquad (27)$$

Taking the in-equality (27) into account, if all the in-equalities in (16) are satisfied then (20) reduces to:

$$\dot{V} \leq -x^\top H x < 0 \qquad (28)$$

Hence, Theorem 1 is proved and finally, the design is complete.

III. REDUCED DESIGN

In this section, we shall use Theorem 1 to systematically choose various design parameters for reducing the convolution of the system. Since the existence of such a Lyapunov function is based purely on mathematical conditions provided by (16), we shall investigate certain "relaxed conditions" that are satisfied independently by using matrix norms of the system parameters.

First, if we select the parameters in (17) as:

$\mathrm{A} = \mathrm{a}_0 \mathrm{I}_\mathrm{n}, \qquad \mathrm{W} = \mathrm{w}_0 \mathrm{I}_\mathrm{n}, \qquad \mathrm{D} = \mathrm{d}_0 \mathrm{I}_\mathrm{n}, \qquad \mathrm{C}_1 = \mathrm{c}_1 \mathrm{I}_\mathrm{n}, \qquad \mathrm{C}_2 = \mathrm{I}_\mathrm{n}$

Then the stability conditions in (16) can be reduced. The Lyapunov function candidate to be positive-definite, the parameters should then satisfy the following conditions:

$$
\begin{aligned}
&i)\ a_0 > 0 \\
&ii)\ d_0 - \mathrm{w}_0^2 a_0^{-1} > 0 \\
&iii)\ (*) = M(\theta) - c_1^2 a_0^{-1} [M(\theta)]^2 \\
&\qquad -(c_2 - c_1 w_0 a_0^{-1})^2 (d_0 - w_0^2 a_0^{-1})^{-1} [M(\theta)]^2 > 0
\end{aligned}
\tag{29}
$$

Elaborately working on condition iii) in (29), we obtain:

$$
\begin{aligned}
(*) > &\min_\theta \lambda\left(M(\theta)\right) \\
&-\lfloor c_1^2 a_0^{-1} + (c_2 - c_1 w_0 a_0^{-1})^2 (d_0 - w_0^2 a_0^{-1})^{-1} \rfloor \cdot \max_\theta \|M(\theta)\|_F^2 > 0
\end{aligned}
$$

Choosing $w_0 = a_0 > 0, d_0 = 2a_0$ then i) and ii) are satisfied, and iii) takes the form:

$$a_0 > \bar{c}_1 [c_1^2 + (c_2 - c_1)^2] \tag{30}$$

Where $\bar{c}_1 = \max_\theta \|M(\theta)\|_F^2 \left(\min_\theta \lambda\, [M(\theta)]\right)^{-1}$. We evaluate P_i's and R_i's with i=1,2,3 as $P_1 = 0.5 a_0 I, R_1 = (c_2 c_1^{-1} - 0.5) a_0 I$

$$P_2(\theta, \dot{\theta}) = 0.5\left[a_0 I + c_1 \dot{M}(\theta) - c_1 B(\theta, \dot{\theta})\right]$$

$$
\begin{aligned}
R_2(\theta, \dot{\theta}) = &\ a_0 c_1^{-1} I + 0.5\left[\dot{M}(\theta) - B(\theta, \dot{\theta})\right]^\top \\
&+ 0.5\left[\dot{M}(\theta) - B(\theta, \dot{\theta})\right] - c_2 M(\theta)
\end{aligned}
$$

$$P_3(\theta,\dot{\theta}) = 0.5c_1 M(\theta) + 0.5c_1^{-1}(c_1 + c_2 - 1)a_0 I$$
$$+c_1(c_2 c_1^{-1} - 0.5)\cdot\left[\dot{M}(\theta) - B(\theta,\dot{\theta})\right]$$
$$R_3(\theta,\dot{\theta}) = (c_1 - c_2)M(\theta) + 1.5a_0 I$$
$$+0.5(c_1 + c_2)\left[\dot{M}(\theta) - B(\theta,\dot{\theta})\right]^\top$$
$$+0.5(c_1 + c_2)\left[\dot{M}(\theta) - B(\theta,\dot{\theta})\right]$$
$$+0.5c_1^2 a_0^{-1}\left[\dot{M}(\theta) - B(\theta,\dot{\theta})\right]^\top M(\theta)$$
$$+0.5c_1^2 a_0^{-1} M(\theta)\left[\dot{M}(\theta) - B(\theta,\dot{\theta})\right]$$
$$+c_1^2 a_0^{-1}(c_2 c_1^{-1} - 0.5)\left[\dot{M}(\theta)\right.$$
$$\left.-B(\theta,\dot{\theta})\right]^\top\left[\dot{M}(\theta) - B(\theta,\dot{\theta})\right]$$

Now we can treat the stability conditions given in (16) clearly as follows, assuming that $c_2 c_1^{-1} > 0.5$, noticing $\left[\dot{M}(\theta) - B(\theta,\dot{\theta})\right]^\top\left[\dot{M}(\theta) - B(\theta,\dot{\theta})\right] \geq 0$, we can write an inequality of the form:

$$R_3(\theta,\dot{\theta}) > \Big\{c_1 \min_{\theta}\lambda[M(\theta)] - c_2 \max_{\theta}\|M(\theta)\|_F + 1.5a_0$$
$$-(c_1 + c_2)\left(\max_{\theta,\dot{\theta}}\left\|\dot{M}(\theta)\right\|_F + \max_{\theta,\dot{\theta}}\left\|B(\theta,\dot{\theta})\right\|_F\right)$$
$$-c_1^2 a_0^{-1}\max_{\theta}\|M(\theta)\|_F\left(\max_{\theta,\dot{\theta}}\left\|\dot{M}(\theta)\right\|_F\right.$$
$$\left.+\max_{\theta,\dot{\theta}}\left\|B(\theta,\dot{\theta})\right\|_F\right)I\Big\} > 0 \tag{31}$$

This implies to satisfy $1.5a_0 > \bar{c}_2 c_1 + \bar{c}_3 c_2 + \bar{c}_4 c_1^2 a_0^{-1}$. Thus, we should choose a_0 such that:

$$a_0 > \left[\bar{c}_2 c_1 + \bar{c}_3 c_2 + \sqrt{(\bar{c}_2 c_1 + \bar{c}_3 c_2)^2 + 6\bar{c}_4 c_1^2}\right]/3 \tag{32}$$

Where,

$$\bar{c}_2 = \max_{\theta,\dot{\theta}} \left\| \dot{M}(\theta) \right\|_F + \max_{\theta,\dot{\theta}} \left\| B(\theta, \dot{\theta}) \right\|_F - \min_{\theta} \lambda[M(\theta)]$$

and

$$\bar{c}_3 = \max_{\theta} \| M(\theta) \|_F + \max_{\theta,\dot{\theta}} \left\| \dot{M}(\theta) \right\|_F + \max_{\theta,\dot{\theta}} \left\| B(\theta, \dot{\theta}) \right\|_F$$

and

$$\bar{c}_4 = \max_{\theta} \| M(\theta) \|_F \left[\max_{\theta,\dot{\theta}} \left\| \dot{M}(\theta) \right\|_F + \max_{\theta,\dot{\theta}} \left\| B(\theta, \dot{\theta}) \right\|_F \right]$$

To check if

$$\lambda_{\min}(R_1) > \max_{\theta,\dot{\theta}} \left\| P_3(\theta, \dot{\theta}) \right\|_F^2 \min_{\theta,\dot{\theta}} \lambda \left[R_3(\theta, \dot{\theta}) \right]^{-1}$$

is satisfied, we first set $c_1 + c_2 = 1$ and notice the inequalities:

$$\max_{\theta,\dot{\theta}} \left\| P_3(\theta, \dot{\theta}) \right\|_F < \bar{c}_5$$

Where,

$$\bar{c}_5 = 0.5 c_1 \max_{\theta} \| M(\theta) \|_F$$
$$+ c_1 (c_2 c_1^{-1} - 0.5) \left[\max_{\theta,\dot{\theta}} \left\| \dot{M}(\theta) \right\|_F + \max_{\theta,\dot{\theta}} \left\| B(\theta, \dot{\theta}) \right\|_F \right]$$

Furthermore, we also have:

$$\min_{\theta,\dot{\theta}} \lambda \left[R_3(\theta, \dot{\theta}) \right] > a_0^{-1} [1.5 a_0^2 - (\bar{c}_2 c_1 + \bar{c}_3 c_2) a_0 - \bar{c}_4 c_1^2]$$

Therefore, it is possible to find a new lower bound for a_0:

$$\lambda_{\min}(R_1) = (c_2 c_1^{-1} - 0.5) a_0$$
$$> a_0 \bar{c}_5^2 [1.5 a_0^2 - (\bar{c}_2 c_1 + \bar{c}_3 c_2) a_0 - \bar{c}_4 c_1^2]^{-1}$$

Taking the inequalities (29), (30), and (32) into account, a_0 can be chosen as shown below:

$$a_0 > \max(\bar{a}_1, \bar{a}_2, \bar{a}_3)$$

Where,

$$\bar{a}_1 = \bar{c}_1[c_1^2 + (c_2 - c_1)^2],$$

$$\bar{a}_2 = \left[\bar{c}_2 c_1 + \bar{c}_3 c_2 + \sqrt{(\bar{c}_2 c_1 + \bar{c}_3 c_2)^2 + 6\bar{c}_4 c_1^2}\right]/3,$$

$$\bar{a}_3 = \left[\bar{c}_2 c_1 + \bar{c}_3 c_2 + \sqrt{(\bar{c}_2 c_1 + \bar{c}_3 c_2)^2 + 6(\bar{c}_4 c_1^2 + \bar{c}_5^2 (c_2 c_1^{-1} - 0.5)^{-1})}\right]/3$$

Finally, the lower bound of the control parameter K_d can be evaluated as:

$$K_d > a_0\{4c_1^2[c_2 c_1^{-1} - 0.5 - \bar{c}_5^2 \times (1.5a_0^2 - (\bar{c}_2 c_1 + \bar{c}_3 c_2)a_0 - \bar{c}_4 c_1^2)^{-1}]\}^{-1} \quad (33)$$

IV. SIMULATION

To better understand the utility of the proposed controller, the tracking control of a two-link SCARA type manipulator [22] is considered as shown in Fig 1. The variables of the dynamic equation of the robot system are then taken as:

$$\theta = [\theta_1 \quad \theta_2]$$

$$u = [u_1 \quad u_2]$$

And the matrix elements of $M(\theta), B(\theta, \dot{\theta}), f(\dot{\theta})$ and $g(\theta)$ are defined as:

$$M_{11} = p_1 + 2p_3 \cos(\theta_2)$$

$$M_{12} = M_{21} = p_2 + p_3 \cos(\theta_2)$$

$$M_{22} = p_2$$

$$B_{11} = -\dot{\theta}_2 p_3 \sin(\theta_2)$$

$$B_{12} = -(\dot{\theta}_1 + \dot{\theta}_2) p_3 \sin(\theta_2)$$

$$B_{21} = \dot{\theta}_1 p_3 \sin(\theta_2), \quad B_{22} = 0$$

$$f_{11} = K_1 \dot{\theta}_1 + K_2 sgn(\dot{\theta}_1)$$

$$f_{21} = K_1 \dot{\theta}_2 + K_2 sgn(\dot{\theta}_2)$$

$$g_{11} = (m_1 + m_2) l_1 \cos(\theta_1) + m_2 g l_2 \cos(\theta_1 + \theta_2)$$

$$g_{21} = m_2 g l_2 \cos(\theta_1 + \theta_2)$$

and the system parameters are taken as:

$$p_1 = 3.16,$$

$$p_2 = 0.106,$$

$$p_3 = 0.173,$$

$$l_1 = 0.36,$$

$$l_2 = 0.24,$$
$$K_1 = 1,$$
$$K_2 = 10,$$
$$c = 1s^{-1}$$

The reference trajectory for the position control is taken equally for the both the links as:

$$\theta_{d1}(t) = 0.8\sin(0.6t)\ rad$$

The reference trajectory for position integral and velocity are taken as:

$$\psi_{d1}(t) = -4\cos(0.6t)\,/3\,rad$$
$$\omega_{d1}(t) = 0.48\cos(0.6t)\ rad$$

The sampling time is taken as 10.0 for whole simulation and the control parameters are taken as:

$$K_p = K_i = K_d = K_r = 10.I,\ C_1 = 10.I,\ C_2 = 1000.I$$

for the new variable structure PID controller that is proposed in this note.

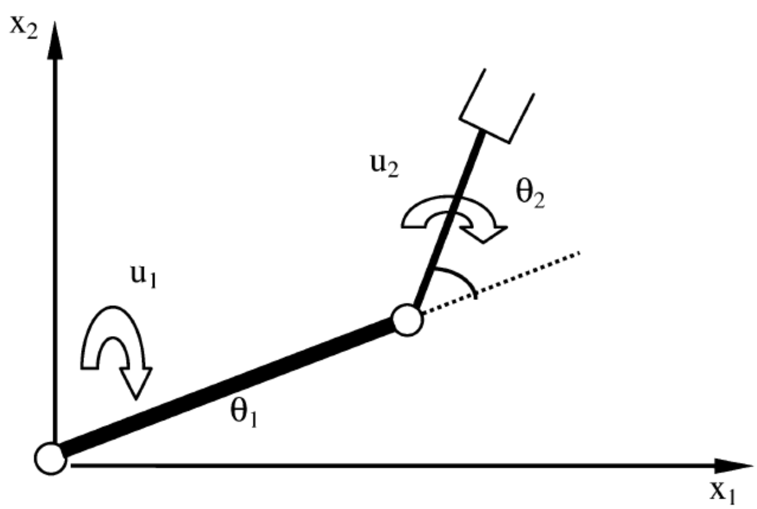


Fig 1: Two-link SCARA type manipulator

Figure 2 shows the modelling of the two-link SCARA type manipulator in Simulink.

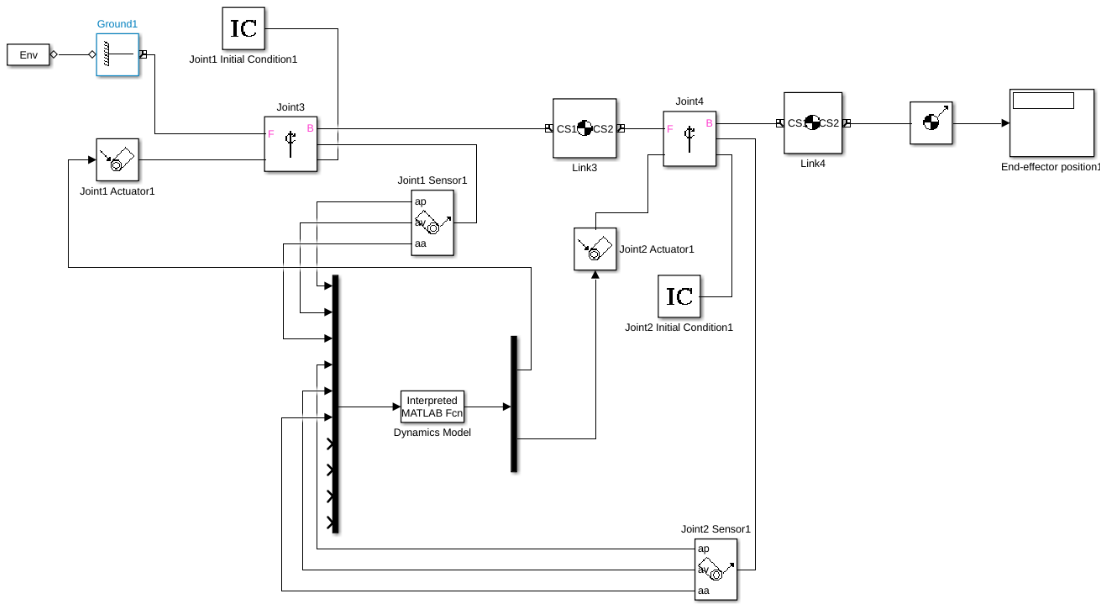


Fig 2: Dynamic model of the two-link SCARA type manipulator

Figure 3 is the design of Variable Structure PID Controller in Simulink

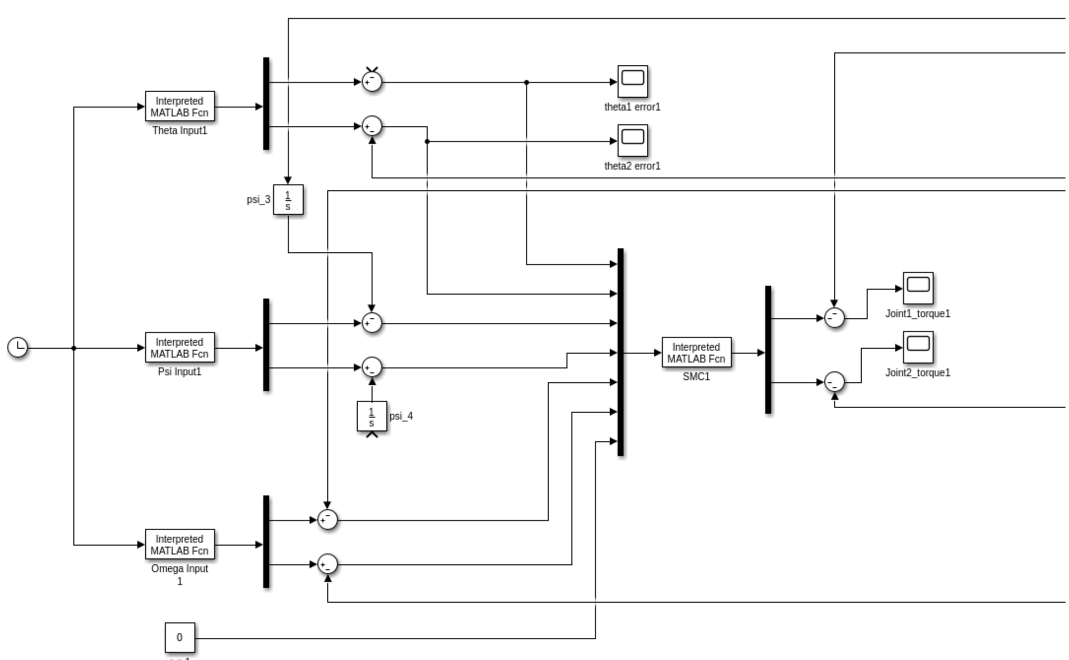


Fig 3: Variable Structure PID Controller model

Figures 4 and 5 are the computational files that calculate system dynamics and the output of the designed controller.

Editor - /home/shreejay/UMD/ENPM667/Projects/Project1/code/VariableStructurePID/dynamics.m

dynamics.m | smc1.m | +

```matlab
function output = dynamics(theta1, theta_dot1, theta_dot_dot1, theta2, theta_dot2, theta_dot_dot2, error1, error_dot1, error2, error_dot2)

    p1 = 3.16;
    p2 = 0.106;
    p3 = 0.173;
    m1 = 10.6;
    m2 = 4.85;
    l1 = 0.36;
    l2 = 0.24;
    K1 = 1;
    K2 = 10;
    g = 9.8;

    C1 = 10   * eye(2);
    C2 = 1000 * eye(2);

    M11 = p1 + 2 * p3 * cos(theta2);
    M12 = p2 + p3 * cos(theta2);
    M21 = p1 + 2 * p3 * cos(theta2);
    M22 = p2;

    B11 =  - theta_dot2 * p3 * sin(theta2);
    B12 =  -(theta_dot1 + theta_dot2) * p3 * sin(theta2);
    B21 =    theta_dot1 * p3 * sin(theta2);
    B22 =    0;

    f11 = K1 * theta_dot1 + K2 * sign(theta_dot1);
    f21 = K1 * theta_dot2 + K2 * sign(theta_dot2);

    g11 = (m1 + m2) * g * l1 * cos(theta1) + m2 * g * l2 * cos(theta1 + theta2);
    g21 = m2 * g * l2 * cos(theta1 + theta2);

    T1 = M11 * theta_dot_dot1 + M12 * theta_dot_dot2 + B11 * theta_dot1 + B12 * theta_dot2 + f11 + g11;
    T2 = M21 * theta_dot_dot1 + M22 * theta_dot_dot2 + B21 * theta_dot1 + B22 * theta_dot2 + f21 + g21;

    output(1,1) = -(C1(1,1) * error1) - (C2(1,1) * error_dot1) - T1 + theta_dot_dot1;
    output(2,1) = -(C1(2,2) * error2) - (C2(2,2) * error_dot2) - T2 + theta_dot_dot2;

end
```

Fig 4: System Dynamics script

Editor - /home/shreejay/UMD/ENPM667/Projects/Project1/code/VariableStructurePID/smc1.m

dynamics.m | smc1.m

```matlab
function output = smc1(e1, e2, e_int1, e_int2, e_dot1, e_dot2, delay)

    C1 = 10;
    C2 = 1000;
    Kr = 10;
    Kp = 10;
    Ki = 10;
    Kd = 10;

    s1 = (C1 * e_int1) + (C2 * e1) + e_dot1;
    s2 = (C1 * e_int2) + (C2 * e2) + e_dot2;

    if (abs(s1) > delay)
        sat_s1 = sign(s1);
    else
        sat_s1 = s1/delay;
    end

    if (abs(s2) > delay)
        sat_s2 = sign(s2);
    else
        sat_s2 = s2/delay;
    end

    output(1,1) = -(Kr + (Kp * e1) + (Ki * e_int1) + (Kd * e_dot1)) * sat_s1;
    output(2,1) = -(Kr + (Kp * e2) + (Ki * e_int2) + (Kd * e_dot2)) * sat_s2;
end
```

Fig 5: Variable Structure PID Controller script

V. CONCLUSION

The Variable Structure PID Controller model and two-link dynamic system are replicated in MATLAB Simulink to test the result and the robustness of the system.

However, because of lack of data available of some parameters such as input signal error of joint angles, joint angular velocities, we are not able to conclude satisfactory outcome for the tracking stabilization of robot motion using new Variable Structure PID controller we have redesigned based on original author’s conclusion.

Figure 6 - 9 indicate the outcome of this control system. The result of the error in theta values in torque input are not as per the expectations.

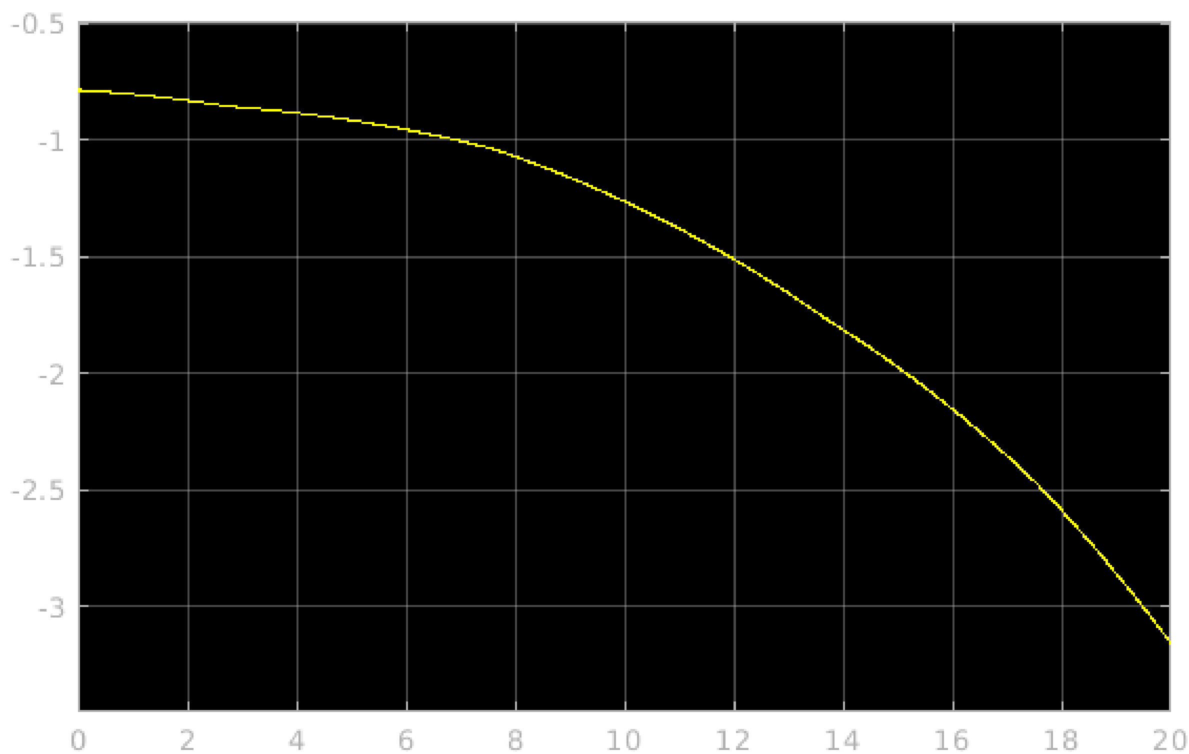


Fig 6: Error in θ_1 w.r.t time

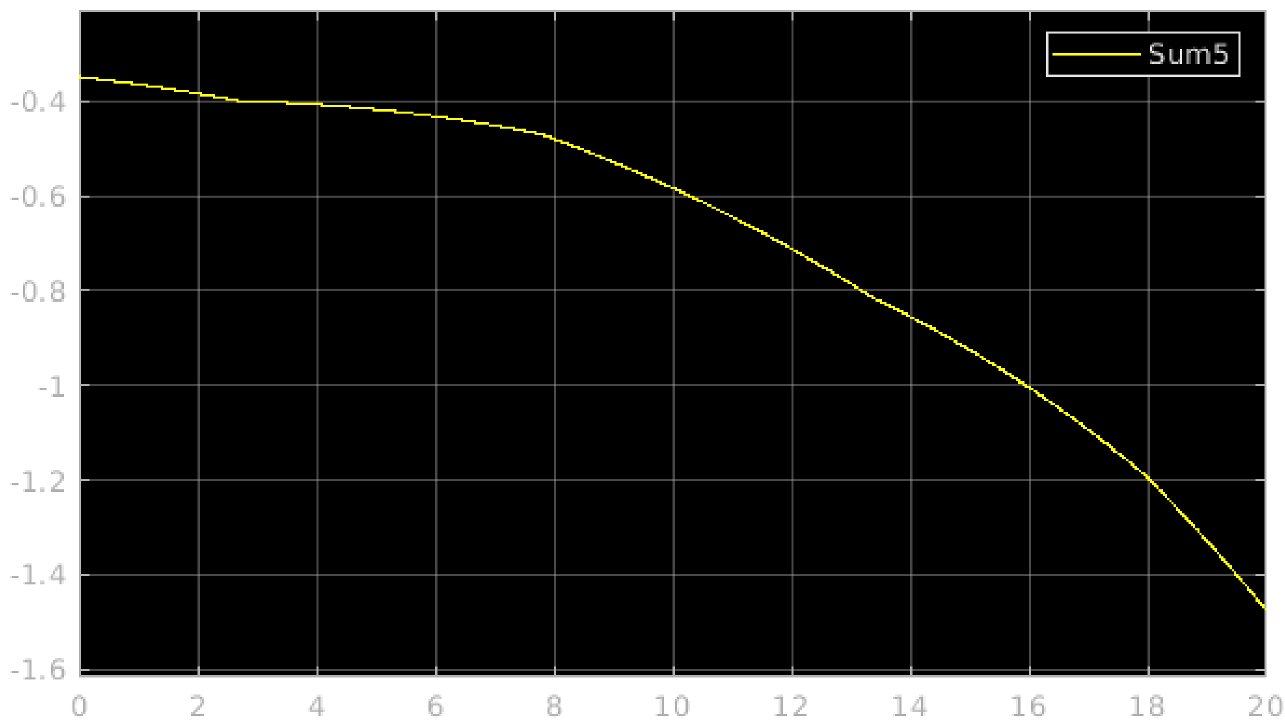


Fig 7: Error in θ_2 w.r.t time

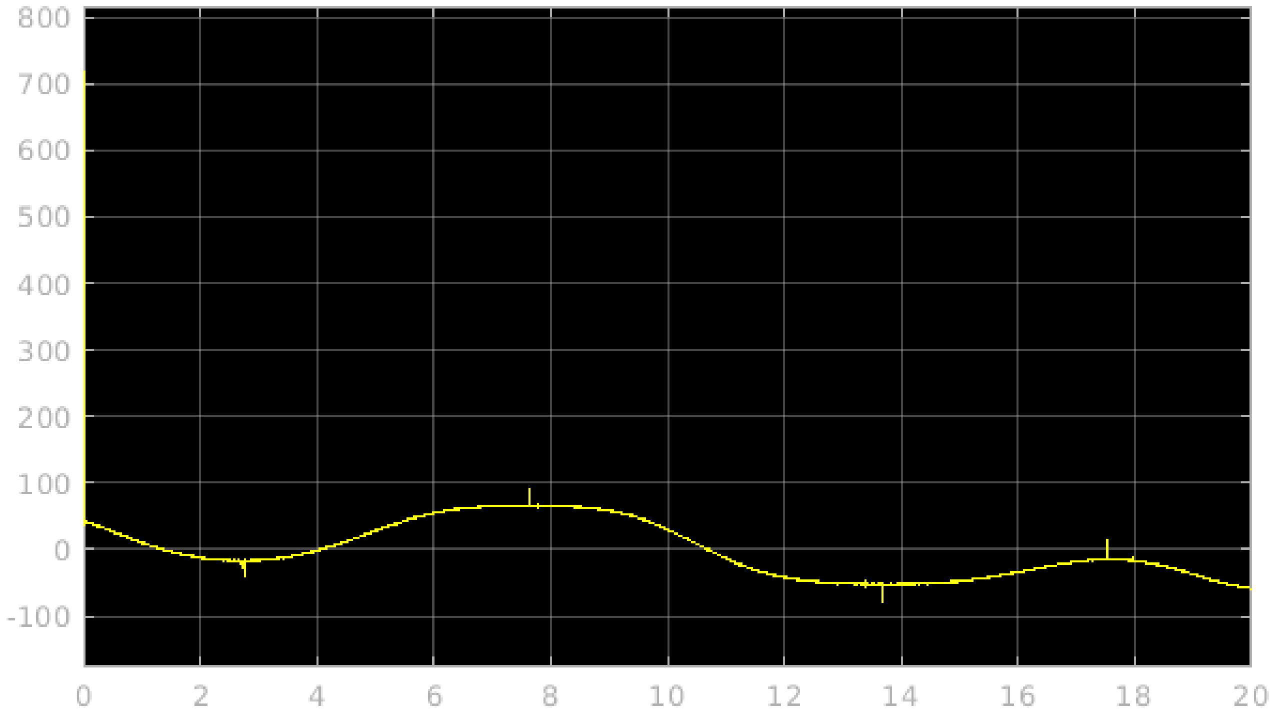


Fig 8: Torque provided to the first link over time

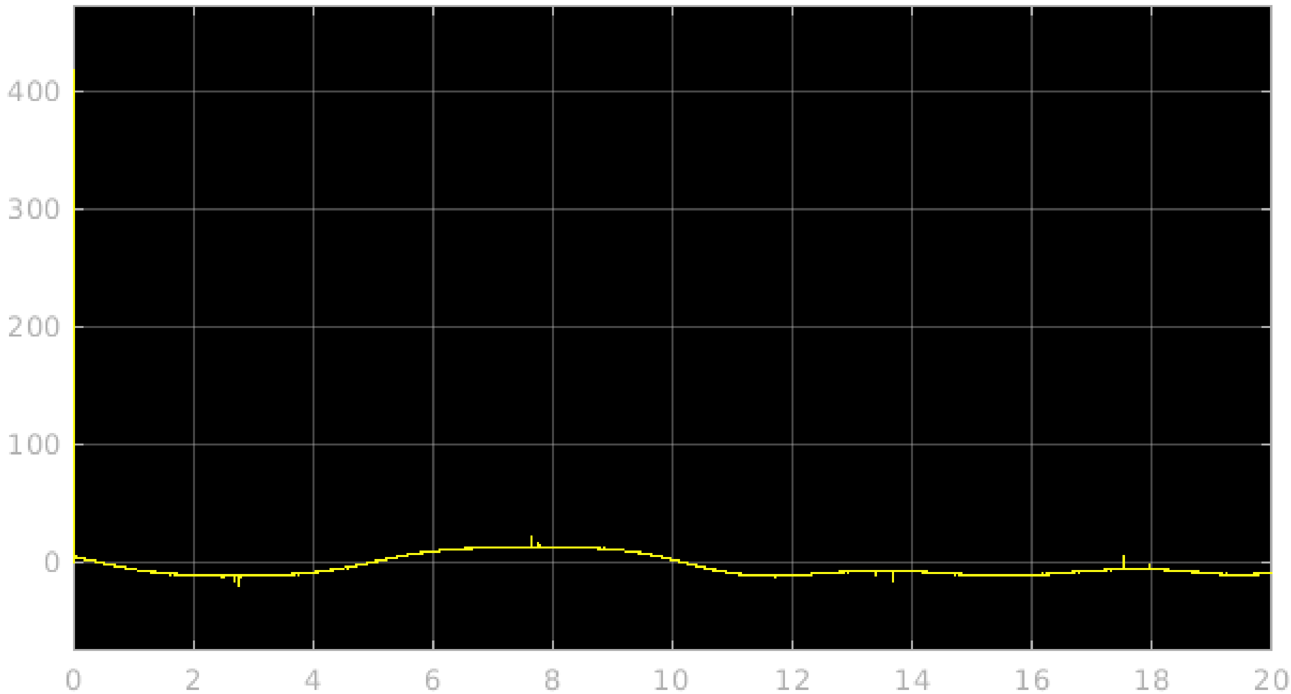


Fig 9: Torque provided to the second link over time

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