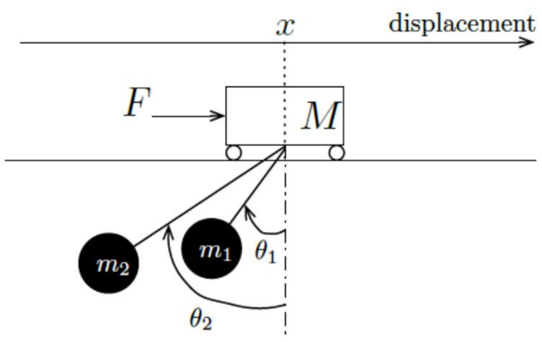
University of Maryland

College Park

PROJECT 2

ENPM667 Control of Robotic Systems

Controller Design and Simulation of a Crane System with Two Suspended Loads



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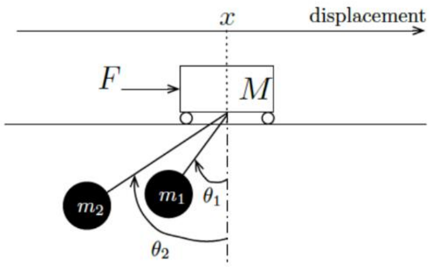
INTRODUCTION

1. PROBLEM DESCRIPTION

A crane moves along a one-dimensional track and behaves as a frictionless cart with mass M. It is actuated by an external force F that serves as an input to the system. Two loads of masses m_1 and m_2 are suspended from this crane with the help of rigid cables of different lengths, l_1 and l_2, respectively. The following diagram depicts the entire system with the associated variables.



2. APPROACH

The project begins with finding the equations of motion using Lagrangian method and the associated non-linear state space representation of the system. After obtaining the dynamic equations and the state space representation, the proceeding step involves linearizing the system around the equilibrium point specified by $x = 0$, $\theta_1 = \theta_2 = 0$ and representing it as a state-space equation.

Following this, the conditions for controllability of the system are obtained based on M, m_1, m_2, l_1 and l_2. Assigning values for these variables of the system, the controllability of the system is checked. In this case, since the system is controllable, an LQR controller is designed for the given system. Simulation responses are then recorded for the original non-linear system and linearized system. After adjusting the parameters of LQR for achieving the satisfactory responses, the closed-loop system is then checked for stability using Lyapunov's indirect method.

After designing the LQR controller, the observability of the linearized system is then checked using the parameters that were used above and selecting the appropriate output vectors.

The "best" Luenberger observer is then computed for each one of the output vectors for which the system is observable. The response to initial conditions and unit step input is then simulated when the observer is applied to both the original non-linear system and the linearized system.

Finally, the last step of the project involves designing an output feedback controller for the "smallest" output vector. Using the LQG method, the resulting output feedback controller is then applied to the original non-linear system and simulated in MATLAB.

EQUATIONS OF MOTION AND NON-LINEAR STATE SPACE REPRESENTATION

- Position of mass m_1 can be represented as a function of θ_1:

$$x_1 = (x - l_1 sin(\theta_1))\hat{\imath} + (-l_1 cos(\theta_1))\hat{\jmath}$$

- Differentiating the above equation with respect to time results in velocity equation:

$$v_1 = \left(\dot{x} - l_1 \cos(\theta_1)\,\dot{\theta}_1\right)\hat{\imath} + \left(l_1 \sin(\theta_1)\,\dot{\theta}_1\right)\hat{\jmath}$$

- Similarly, the position of mass m_2 can be represented as a function of θ_2:

$$x_2 = (x - l_2 \sin(\theta_2))\hat{\imath} + (-l_2 \cos(\theta_2))\hat{\jmath}$$

- Velocity equation for mass m_2 can be given as:

$$v_2 = \left(\dot{x} - l_2 \cos(\theta_2)\,\dot{\theta}_2\right)\hat{\imath} + \left(l_2 \sin(\theta_2)\,\dot{\theta}_2\right)\hat{\jmath}$$

- Using the velocity equations derived above, the kinetic energy can then be given as:

$$\mathrm{K} = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m_1\left(\dot{x} - l_1\dot{\theta}_1 \cos(\theta_1)\right)^2 + \frac{1}{2}m_1\left(l_1\dot{\theta}_1(\sin(\theta_1))\right)^2 + \frac{1}{2}m_2(\dot{x} -$$
$$\left.\dot{\theta}_2 l_2 \cos(\theta_2)\right)^2 + \frac{1}{2}m_2\left(l_2\dot{\theta}_2(\sin(\theta_2))\right)^2$$

- The potential energy of the system can be written as:

$$\begin{aligned} P &= -m_1 g l_1 \cos(\theta_1) - m_2 g l_2 \cos(\theta_2) \\ &= -g[m_1 l_1 \cos(\theta_1) + m_2 g l_2 \cos(\theta_2) \end{aligned}$$

- Following this, the Lagrangian of this system is calculated as follows:

$$L = K - P$$

- Substituting the values:

$$L = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_1 l_1^2 \dot{\theta_1}^2 \cos^2(\theta_1) - m_1 l_1 \dot{\theta_1}\dot{x}\cos(\theta_1) + \frac{1}{2}m_1 l_1^2 \dot{\theta_1}^2 \sin^2(\theta_1)$$
$$+\frac{1}{2}m_2\dot{x}^2 + \frac{1}{2}m_2 l_2^2 \dot{\theta}_2^2 \cos^2(\theta_2) - m_2 l_2 \dot{\theta}_2 \dot{x}\cos(\theta_2) + \frac{1}{2}m_2 l_2^2 \dot{\theta}_2^2 \sin^2(\theta_1) + g[m_1 l_1 \cos(\theta_1) +$$
$$m_2 l_2 \cos(\theta_2)]$$

- Simplifying:

$$L = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}(m_1 + m_2)\dot{x}^2 + \frac{1}{2}m_1 l_1^2 \dot{\theta_1}^2 + \frac{1}{2}m_2 l_2^2 \dot{\theta_2}^2 - \dot{x}\left(m_1 l_1 \dot{\theta_1}\cos(\theta_1) + m_2 l_2 \dot{\theta}_2 \cos(\theta_2)\right)$$
$$+g[m_1 l_1 \cos(\theta_1) + m_2 l_2 \cos(\theta_2)]$$

- The Lagrange's equation is then given as:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \left(\frac{\partial L}{\partial x}\right) = F$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta_1}}\right) - \left(\frac{\partial L}{\partial \theta_1}\right) = 0$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta_2}}\right) - \left(\frac{\partial L}{\partial \theta_2}\right) = 0$$

- Solving the first equation of Langrange's equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \left(\frac{\partial L}{\partial x}\right) = F$$

$$\frac{\partial L}{\partial \dot{x}} = M\dot{x} + (m_1 + m_2)\dot{x} - m_1 l_1 \dot{\theta_1}\cos(\theta_1) - m_2 l_2 \dot{\theta_2}\cos(\theta_2)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = M\ddot{x} + (m_1 + m_2)\ddot{x} - [m_1 l_1 \ddot{\theta}_1 \cos(\theta_1)) - m_1 l_1 \dot{\theta}_1^2 \sin(\theta_1)]$$
$$- \left[m_2 l_2 \ddot{\theta}_2 \cos(\theta_2) - m_2 l_2 \dot{\theta}_2^2 \sin(\theta_2)\right]$$

- Since $\frac{\partial L}{\partial x} = 0$, the first equation can be written as:

$$[M + m_1 + m_2]\ddot{x} - m_1 l_1 \ddot{\theta}_1 \cos(\theta_1) + m_1 l_1 \dot{\theta}_1^{\ 2} \sin(\theta_1) - m_2 l_2 \ddot{\theta}_2 \cos(\theta_2) + m_2 l_2 \dot{\theta}_2^{\ 2} \sin(\theta_2) = F$$

- Solving the second equation of Langrange's equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \left(\frac{\partial L}{\partial \theta_1}\right) = 0$$
$$\frac{\partial L}{\partial \dot{\theta}_1} = m_1 l_1^2 \dot{\theta}_1 - m_1 \dot{x} l_1 \cos(\theta_1)$$

- Differentiating with respect to time:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = m_1 l_1^2 \ddot{\theta}_1 - \left[m_1 l_1 \ddot{x} \cos(\theta_1) - m_1 \dot{x} l_1 \dot{\theta}_1 \sin(\theta_1)\right]$$

And,

$$\frac{\partial L}{\partial \theta_1} = m_1 l_1 \dot{\theta}_1 \dot{x} \sin(\theta_1) - m_1 l_1 g \sin(\theta_1)$$

- Formulating the second equation by substituting the above equations:

$$m_1 l_1^2 \ddot{\theta}_1 - m_1 \ddot{x} l_1 \cos(\theta_1) + m_1 \dot{\theta}_1 \dot{x} l_1 \sin(\theta_1) - m_1 \dot{\theta}_1 \dot{x} l_1 \sin(\theta_1) + m_1 l_1 g \sin(\theta_1) = 0$$

- By simplifying and eliminating the equivalent terms, the second equation can be written as:

$$m_1 l_1^2 \ddot{\theta}_1 - m_1 \ddot{x} l_1 \cos(\theta_1) + m_1 l_1 g \sin(\theta_1) = 0$$

- Now, solving the third Langrange's equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right)-\left(\frac{\partial L}{\partial \theta_2}\right)=0$$

$$\frac{\partial L}{\partial \dot{\theta}_2}=m_2 l_2^2 \dot{\theta}_2 - m_2 \dot{x} l_2 \cos(\theta_2)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right)=m_2 l_2^2 \ddot{\theta}_2 - \left[m_2 \ddot{x} l_2 \cos(\theta_2) - m_2 \dot{\theta}_2 \dot{x} l_2 \sin(\theta_2)\right]$$

$$\left(\frac{\partial L}{\partial \theta_2}\right)=m_2 \dot{x} l_2 \dot{\theta}_2 \sin(\theta_2) - m_2 l_2 g \sin(\theta_2)$$

- Formulating the third equation:

$$m_2 l_2^2 \ddot{\theta}_2 - m_2 l_2 \ddot{x}\cos(\theta_2) + m_2 \dot{\theta}_2 \dot{x} l_2 \sin(\theta_2) - m_2 \dot{\theta}_2 \dot{x} l_2 \sin(\theta_2) + m_2 g l_2 \sin(\theta_2)$$

- Simplifying and eliminating the equivalent terms:

$$m_2 l_2^2 \ddot{\theta}_2 - m_2 l_2 \ddot{x}\cos(\theta_2) + m_2 g l_2 \sin(\theta_2) = 0$$

- Finally, the double differentiation elements are derived from the equations above for obtaining the state variables:

$$\ddot{x}=\frac{1}{M+m_1+m_2}\left[m_1 l_1 \ddot{\theta}_1 \cos\theta_1 + m_2 l_2 \ddot{\theta}_2 \cos\theta_2 - m_1 l_1 \dot{\theta}_1^2 \sin\theta_1 - m_2 l_2 \dot{\theta}_2^2 \sin\theta_2 + F\right]$$

$$\ddot{\theta}_1=\frac{\ddot{x}\cos\theta_1}{l_1}-\frac{g\sin\theta_1}{l_1}$$

$$\ddot{\theta}_2=\frac{\ddot{x}\cos\theta_2}{l_2}-\frac{g\sin\theta_2}{l_2}$$

- Taking into account the state variables of the system, the state space representation of the non-linear system is then given as:

$$
\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta}_1 \\ \ddot{\theta}_1 \\ \dot{\theta}_2 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \dfrac{-m_1 g \sin\theta_1 \cos\theta_2 - m_2 g \sin\theta_2 \cos\theta_2 - m_1 l_1 \dot{\theta}_1^2 \sin\theta_1 - m_2 l_2 \dot{\theta}_2^2 \sin\theta_2 + F}{M + m_1 + m_2 - m_1 \cos^2\theta_1 - m_2 \cos^2\theta_2} \\ \dot{\theta}_1 \\ \dfrac{-m_1 g \sin\theta_1 \cos\theta_2 - m_2 g \sin\theta_2 \cos\theta_2 - m_1 l_1 \dot{\theta}_1^2 \sin\theta_1 - m_2 l_2 \dot{\theta}_2^2 \sin\theta_2 + F}{(M + m_1 + m_2 - m_1 \cos^2\theta_1 - m_2 \cos^2\theta_2) l_1} - \dfrac{g \sin\theta_1}{l_1} \\ \dot{\theta}_2 \\ \dfrac{-m_1 g \sin\theta_1 \cos\theta_2 - m_2 g \sin\theta_2 \cos\theta_2 - m_1 l_1 \dot{\theta}_1^2 \sin\theta_1 - m_2 l_2 \dot{\theta}_2^2 \sin\theta_2 + F}{(M + m_1 + m_2 - m_1 \cos^2\theta_1 - m_2 \cos^2\theta_2) l_2} - \dfrac{g \sin\theta_2}{l_2} \end{bmatrix}
$$

LINEARIZATION AROUND EQUILIBRIUM POINT

Linearization is a mathematical technique that is used to approximate the behaviour of a nonlinear system near an equilibrium point. An equilibrium point is a state at which the system is in balance and the variables that describe the system do not change over time.

To perform linearization, the equilibrium point of the system first needs to be identified. Then, the system's variables can be expressed as deviations from their equilibrium values. These deviations are called perturbations.

Next, the Taylor series expansion is used to approximate the nonlinear functions in the system with linear functions. This allows one to represent the system with a set of linear equations, which are much easier to solve than non-linear equations.

Linearization is useful because it helps to study the behaviour of a nonlinear system using the tools of linear systems theory. It can be used to predict the stability of an equilibrium point, find the response of the system to small perturbations, and design control systems to stabilize the system around the equilibrium point.

However, linearization is only valid in the vicinity of the equilibrium point, and the accuracy of the linearized model decreases as the system moves further away from the equilibrium point. Therefore, it is important to validate the linearized model with simulations or experiments to ensure that it accurately reflects the behaviour of the system.

As the derived state space equation has sine and cosine terms, the equation is a non-linear representation of the system. For linearization according to conditions given in the problem, the equilibrium point is selected as $x = 0$, $\theta_1 = 0$ and $\theta_2 = 0$. Assuming the limiting conditions at equilibrium:

$$\sin\theta_1 \approx \theta_1$$

$$\sin\theta_2 \approx \theta_2$$

$$\cos\theta_1 = \cos\theta_2 \approx 1$$

$$\dot{\theta}_1^2 = \dot{\theta}_2^2 \approx 0$$

Taking the above approximations leads to the following results for double differentiation elements of the state space equation:

$$\ddot{x} = \frac{1}{M + m_1 + m_2}\left[m_1 l_1 \ddot{\theta}_1 \cos\theta_1 + m_2 l_2 \ddot{\theta}_2 \cos\theta_2 - m_1 l_1 \dot{\theta}_1^2 \sin\theta_1 - m_2 l_2 \dot{\theta}_2^2 \sin\theta_2 + F\right]$$

$$\ddot{\theta}_1 = \frac{\ddot{x}\cos\theta_1}{l_1} - \frac{g\sin\theta_1}{l_1}$$

$$\ddot{\theta}_2 = \frac{\ddot{x}\cos\theta_2}{l_2} - \frac{g\sin\theta_2}{l_2}$$

The simplified state space form is then given as:

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta_1} \\ \ddot{\theta_1} \\ \dot{\theta_2} \\ \ddot{\theta_2} \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \frac{1}{M}\left[F - m_1 g\theta_1 - m_1 l_1 \dot{\theta}_1^2 \theta_1 - m_2 g\theta_2 - m_2 l_2 \dot{\theta}_2^2 \theta_2\right] \\ \dot{\theta_1} \\ \frac{1}{Ml_1}\left(F \ + Mg\theta_1 - m_1 g\theta_1 - m_1 l_1 \dot{\theta}_1^2 \theta_1 - m_2 g\theta_2 - m_2 l_2 \dot{\theta}_2^2 \theta_2\right) \\ \dot{\theta_2} \\ \frac{1}{Ml_2}\left(F \ + Mg\theta_2 - m_1 g\theta_1 - m_1 l_1 \dot{\theta}_1^2 \theta_1 - m_2 g\theta_2 - m_2 l_2 \dot{\theta}_2^2 \theta_2\right) \end{bmatrix}$$

$$y = CX + DU$$

Where C is a 6 X 6 matrix and D is 0 in this case.

This is similar to the general form of state space representation where F is given as the controlled input "u" on the cart:

$$\dot{X} = AX + Bu$$

Taking X, θ_1 and θ_2 as 0 at the equilibrium condition, the following matrices are obtained.

A matrix:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{g\, m_1}{M} & 0 & -\frac{g\, m_2}{M} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{g\ (M + m_1)}{M\ l_1} & 0 & -\frac{g\, m_2}{M\ l_1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{g\, m_1}{M\ l_2} & 0 & -\frac{g\ (M + m_2)}{M\ l_2} & 0 \end{pmatrix}$$

B matrix:

$$\begin{pmatrix} 0 \\ \frac{1}{M} \\ 0 \\ \frac{1}{M\ l_1} \\ 0 \\ \frac{1}{M\ l_2} \end{pmatrix}$$

CONDITIONS FOR CONTROLLABILITY

In control systems engineering, the controllability of a system is the measure of how well the system can be controlled by an external input. A system is considered controllable if it is possible to drive the system from any initial state to any final state in a finite amount of time using a suitable input signal.

For a linear time-varying system, the system is controllable if and only if the controllability matrix $C(A, B)$ has full rank. If the controllability matrix is full rank, then the system is controllable. If the controllability Gramian is rank deficient, then the system is not controllable. The matrix C is then given by:

$$C = [B \quad AB \quad A^2B \quad A^3B \quad A^4B \quad A^5B]$$

While checking for controllability, for $l_1 = l_2$, the determinant becomes zero. Also, when $l_1 = 0$ or $l_2 = 0$, the controllability matrix becomes undefined. Therefore, the conditions obtained for this system are:

$$l_1 \neq 0$$

$$l_2 \neq 0$$

$$l_1 \neq l_2$$

With these conditions for controllability, the rank is satisfied, and the system is stable.

DESIGN AND SIMULATION OF LQR CONTROLLER

The Linear-Quadratic Regulator (LQR) is a classical control design method that is used to design a controller for a linear control system. The goal of LQR design is to find a feedback control law that minimizes a quadratic cost function, which measures the deviation of the system's states and inputs from their desired values.

The cost function of a linear-quadratic regulator (LQR) is a measure of the performance of the system being controlled. It is a quadratic function that is minimized by the LQR controller to stabilize the system.

The LQR cost function is defined as the sum of the squared differences between the desired state of the system and the actual state of the system, multiplied by weighed matrices. These weighted matrices allow the designer to specify the relative importance of different states or inputs in the cost function.

The cost function is defined as follows:

$$J\left(k, \vec{X}(0)\right) = \int_0^\infty \vec{X}^T(t) Q \, \vec{X}(t) + \vec{U}_k^T(t) R \vec{U}_k(t) dt$$

where Q and R are the weighting matrices for the state and input terms, and the sum is taken over all time steps. Here, Q matrix penalizes the bad performance and R matrix penalizes the actuator effort, i.e. the energy consumed by the system.

In summary, the LQR controller minimizes this cost function by computing the optimal input sequence that drives the system to the desired state while minimizing the cost.

For the purpose of optimal feedback control, different values of Q and R are taken into consideration. This is verified by simulating in MATLAB and obtaining the output graphs.

Using the MATLAB function given below, the controllability is given as:

$$CTR\_rank = rank(ctrb(A, B))$$

Since this system is controllable, the next step would be to design the LQR controller for this controllable system.

To design the LQR controller, the C matrix is defined as an 6X6 identity matrix and D is assumed to be 0.

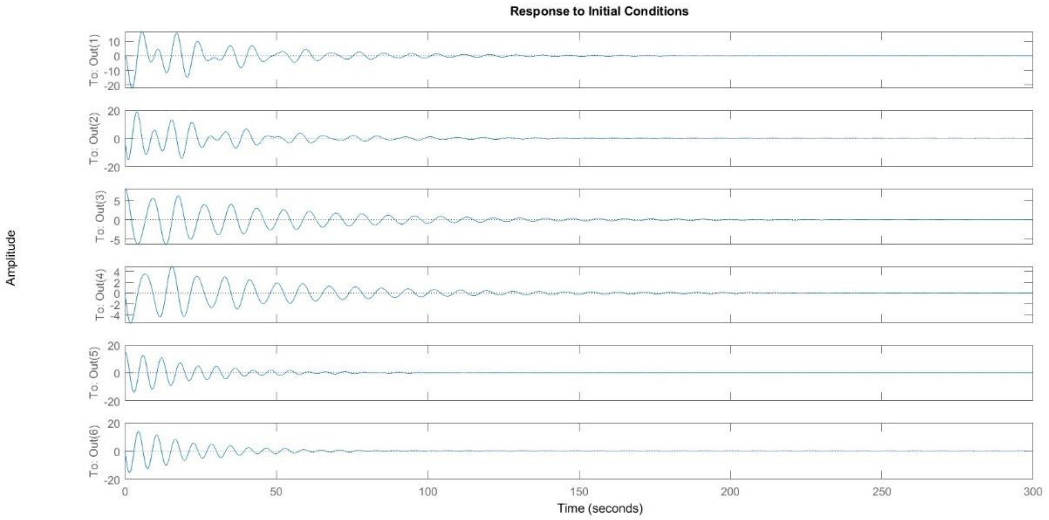
The initial state space condition is taken as:

$$X\_init = [0.1;\ 0;\ 8;\ 0;\ 15;\ 0]$$

Taking Q as:

$$Q = \begin{bmatrix} 100 & 0 & 0 & 0 & 0 & 0 \\ 0 & 80 & 0 & 0 & 0 & 0 \\ 0 & 0 & 200 & 0 & 0 & 0 \\ 0 & 0 & 0 & 80 & 0 & 0 \\ 0 & 0 & 0 & 0 & 50 & 0 \\ 0 & 0 & 0 & 0 & 0 & 300 \end{bmatrix}$$

Taking ***R*** as **0.0005**, the following response is obtained:



Lyapunov stability analysis –

Since all the eigen values have negative real part, the linearized system is locally stable around the equilibrium point.

$$eigen\ values = \begin{matrix} -0.0213 + 0.7243i \\ -0.0213 - 0.7243i \\ -0.0464 + 1.0389i \\ -0.0464 - 1.0389i \\ -0.4628 + 0.4239i \\ -0.4628 - 0.4239i \end{matrix}$$

Initial response to original non-linear system –

We could not simulate the result of initial response to non-linear system.

OBSERVABILITY OF THE SYSTEM

Observability is a property of a dynamic system that refers to the ability to determine the state of the system based on its output measurements. A system is considered observable if it is possible to uniquely determine the state of the system at any given time based on the output measurements.

For a system to be observable, it must satisfy the observability rank condition, which states that the rank of the observability matrix must be equal to the number of states in the system. If a system is not observable, it means that there is insufficient information available in the output measurements to uniquely determine the state of the system.

The observability matrix is given by:

$$O(A,C) = [\mathrm{C} \quad \mathrm{C}A \quad \mathrm{C}A^2 \quad \dots \quad \mathrm{C}A^{n\text{-}1}]^T$$

If rank of $O(A,C) =$ n, then the system is said to be observable.

For the following output vectors $x(t), (\theta_1(t), \theta_2(t)), (x(t), \theta_2(t))$ or $(x(t), \theta_1(t), \theta_2(t))$, the observability matrices are chosen as below:

$$C_1 = [1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]$$

$$C2 = \begin{matrix} [0 & 0 & 1 & 0 & 0 & 0] \\ [0 & 0 & 0 & 0 & 1 & 0] \end{matrix}$$

$$C3 = \begin{matrix} [1 & 0 & 0 & 0 & 0 & 0] \\ [0 & 0 & 0 & 0 & 1 & 0] \end{matrix}$$

$$C4 = \begin{matrix} [1 & 0 & 0 & 0 & 0 & 0] \\ [0 & 0 & 1 & 0 & 0 & 0] \\ [0 & 0 & 0 & 0 & 1 & 0] \end{matrix}$$

Based on the rank of the observability matrix defined above, the output vectors below are found to be observable.

$$x(t), \qquad (x(t), \theta_2(t)), \qquad (x(t), \theta_1(t), \theta_2(t))$$

LUENBERGER OBSERVER

A Luenberger observer is a type of mathematical control system that is used to estimate the state of a system based on a set of measured outputs. The basic idea behind a Luenberger observer is to use a mathematical model of the system to predict the state of the system based on its inputs and outputs. The observer compares the predicted state to the measured outputs and adjusts the model accordingly to minimize the error between the predicted and measured values. This allows the observer to estimate the state of the system even if it is not directly measurable.

The general form for state space equation is given as:

$$\dot{X}(t) = AX(t) + Bu(t)$$

$$y(t) = CX(t) + Du(t)$$

By selecting the appropriate input variables for which the system is observable, a Luenberger observer is added to the system catering to the conditions mentioned in problem statement. The system with the observer can then be represented as follows:

$$\hat{X}(t) = A\hat{X}(t) + Bu(t) + L\big(y(t) - \hat{y}(t)\big)$$

$$\hat{y}(t) = C\hat{X}(t) + Du(t)$$

Here, D is assumed to be zero.

$\hat{X}(t)$ is the estimated state of the system at time t. A is the system matrix, B is the input matrix, $u(t)$ is the input to the system at time t, L is the observer gain matrix, $y(t)$ is the measured output of the system at time t and C is the output matrix

The state-space equation describes how the estimated state of the system $\hat{X}(t)$ changes over time based on the system's inputs $u(t)$ and the measured output $y(t)$.

The term $L[y(t) - C\hat{X}(t)]$ represents the correction term that is applied to the estimated state based on the difference between the measured output $y(t)$ and the predicted output $C\hat{X}(t)$. This term allows the observer to adjust its estimate of the state based on the measured output, which helps to improve the accuracy of the estimate.

The observer gain matrix L determines how aggressively the observer adjusts its estimate of the state based on the measured output. A larger value of L will result in a more aggressive correction, while a smaller value will result in a slower correction. The choice of L will depend on the characteristics of the system and the desired performance of the observer.

To obtain response to initial condition and unit step input for each C_1, C_3 and C_4, the Q and R are chosen as below:

$$Q = \begin{bmatrix} 100 & 0 & 0 & 0 & 0 & 0 \\ 0 & 80 & 0 & 0 & 0 & 0 \\ 0 & 0 & 200 & 0 & 0 & 0 \\ 0 & 0 & 0 & 80 & 0 & 0 \\ 0 & 0 & 0 & 0 & 50 & 0 \\ 0 & 0 & 0 & 0 & 0 & 300 \end{bmatrix}$$

$$R = 0.0005$$

The initial state space condition is taken as:

$$X\_init = [0.5;\ 0;\ 10;\ 0;\ 20;\ 0;\ 0;\ 0;\ 0;\ 0;\ 0]$$

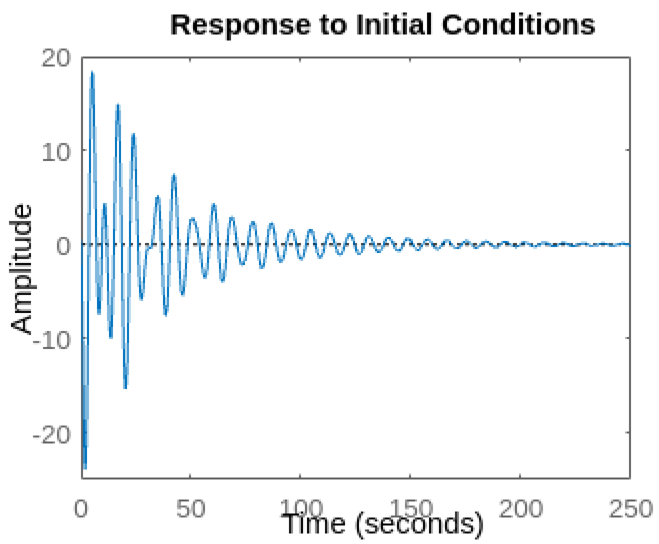
The eigen values for which the LQR controller is designed are chosen as below:

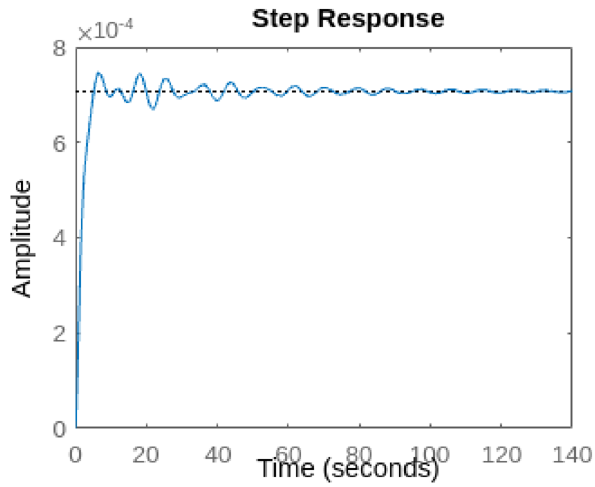
$$eigen\ values = [-2;\ -1;\ -4;\ -8;\ -5;\ -0.5]$$

For given A and $C1$, $C3$ and $C4$, three individual Luenberger observers $L1$, $L3$ and $L4$ are calculated for the above eigen values.

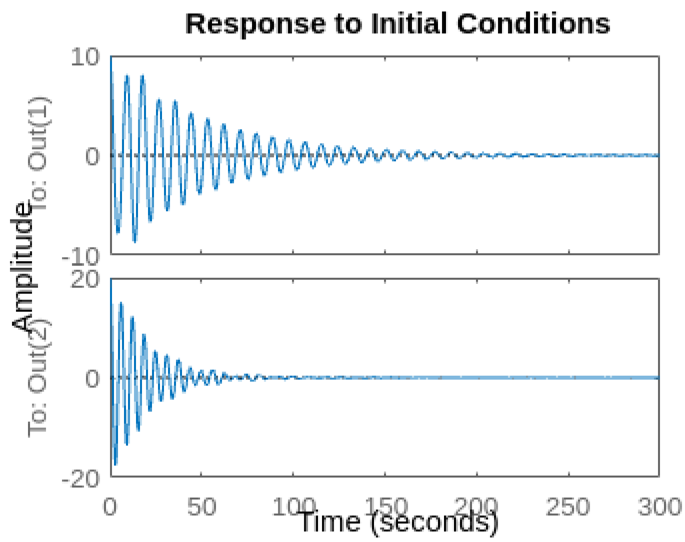
Based on LQR controller and $L1$, $L3$ and $L4$ observers, new state space representation is generated.

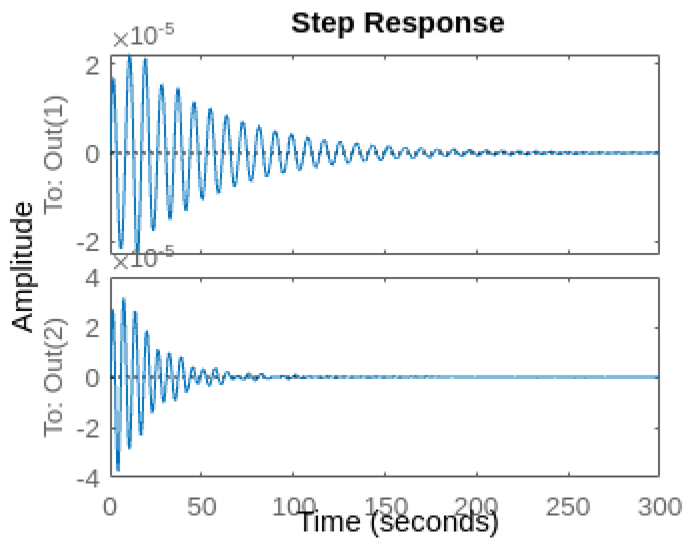
Response to initial condition and unit step input for $C1$:



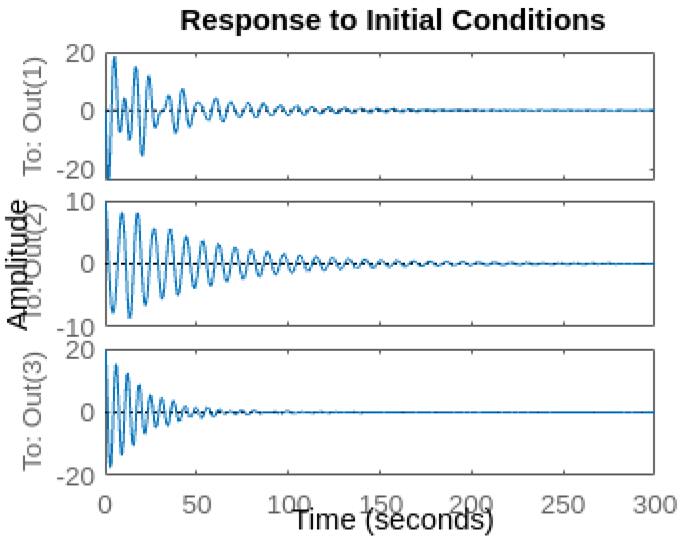


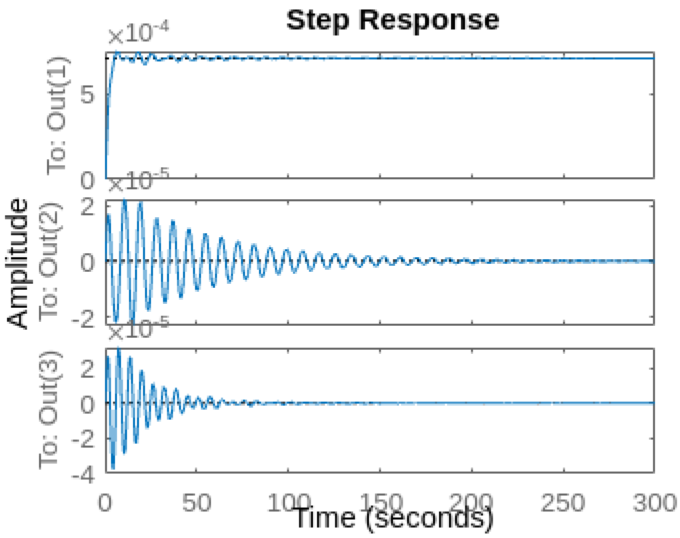
Response to initial condition and unit step input for $C3$:





Response to initial condition and unit step input for $C4$:





Initial response to original non-linear system –

We could not simulate the result of initial response to non-linear system.

DESIGN AND SIMULATION OF LQG CONTROLLER

The Linear Quadratic Gaussian (LQG) controller is a type of control system that is used to optimize the performance of a system by minimizing a cost function that is defined in terms of the system's states and inputs. It is a combination of a Linear Quadratic Regulator (LQR) controller and a Kalman filter, which is used to estimate the state of the system.

Since the LQG controller combines the LQR controller and the Kalman filter, this helps in optimizing the performance of the system by minimizing the cost function J while also considering the uncertainty in the state estimate provided by the Kalman filter. This allows the LQG controller to achieve good performance even in the presence of noise and other uncertainties.

The output feedback controller is designed based on the below output vector:

$$Y\_init \ = \ [0.5;\ 0;\ 5;\ 0;\ 5;\ 0;\ 0;\ 0;\ 0;\ 0;\ 0]$$

To apply LQG method, the Q and R are chosen as below:

$$Q = \begin{bmatrix} 100 & 0 & 0 & 0 & 0 & 0 \\ 0 & 80 & 0 & 0 & 0 & 0 \\ 0 & 0 & 200 & 0 & 0 & 0 \\ 0 & 0 & 0 & 80 & 0 & 0 \\ 0 & 0 & 0 & 0 & 50 & 0 \\ 0 & 0 & 0 & 0 & 0 & 300 \end{bmatrix}$$

$$R = 0.0005$$

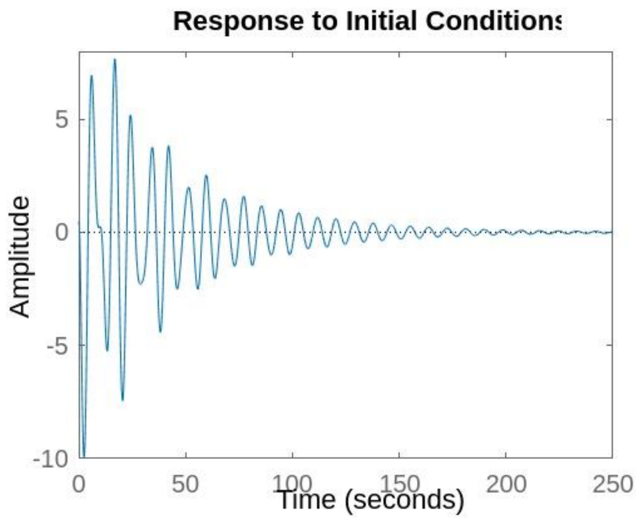
The system and measurement noises are taken as:

$$Vd \ = \ 0.1 * eye(6)$$

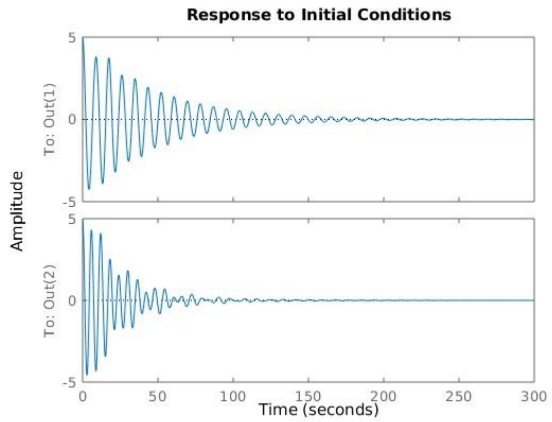
$$Vn \ = \ 1$$

The gain matrices K1, K3 and K4 are calculated using the above parameters for C1, C3 and C4. For the above chosen output vector, following responses are obtained.

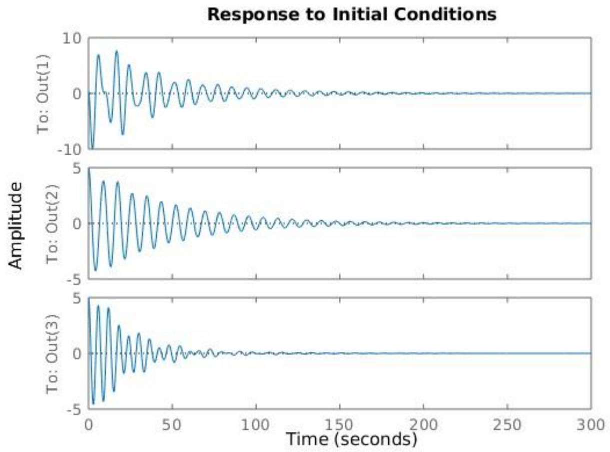
Response to initial condition and unit step input for $C1$:



Response to initial condition and unit step input for $C3$:



Response to initial condition and unit step input for $C4$:



The main aim of the most optimal Reference tracking is to minimize the following cost function for LQG controller:

$$\int_0^\infty \left(X(t) - X(d)\right)^T (t) Q \left(X(t) - X(d)\right) + (U_k - U_\infty)^T R (U_k - U_\infty) dt$$

The optimal feedback solution is given by $U(t) = K\left(X(t) - X(d)\right)$ where there exists U_∞ such that $AX_d + B_k U_\infty$. The gain matrix is given by $K = -R^{-1} B_k^T P$ where P is positive symmetric definite solution of the Ricatti equation. To decrease the ideal reference tracking cost function, the controller is modified.

With this design, the cart will be able to withstand continual force perturbations. The controller will take these disturbances into account under the supposition that they are Gaussian in nature. The cart's continual force disturbances can be rejected by the controller's design. The Gaussian process is used to account for these disturbances and reduce inaccuracy.