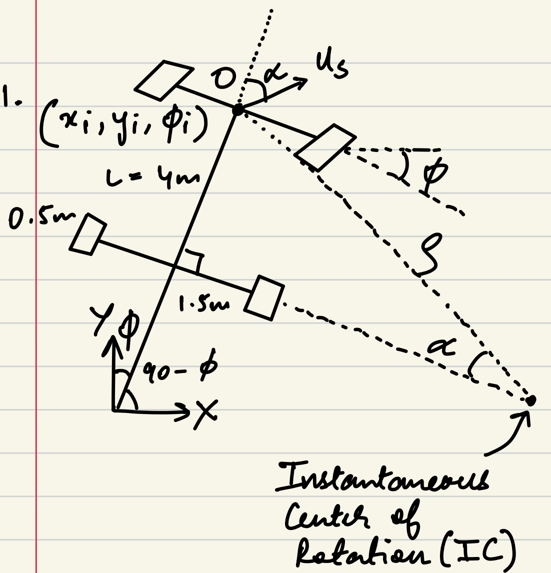
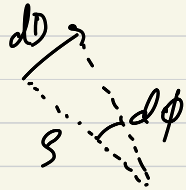
MODELING - HW1

1.



Initial pose : (x_i, y_i, ϕ_i)
Drive speed : ω
Steering angle : α
Duration : T



$\therefore dD = \rho d\phi$

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Substituting ρ from ①

$$dD = \frac{L}{\sin\alpha} d\phi$$

$$\dot{\phi} = \frac{\sin\alpha}{L} dD$$

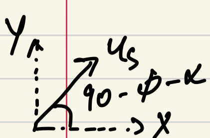
$$\dot{\phi} = \frac{\sin\alpha}{L} \dot{D}$$

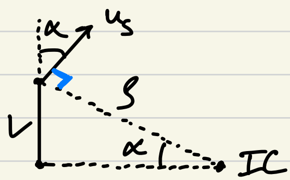
Also, $\dot{D} = u_s$

$$\therefore \dot{\phi} = \frac{u_s}{L} \sin\alpha \quad — ④$$

$$\sin\alpha = \frac{L}{\rho}$$

$$\rho = \frac{L}{\sin\alpha} \quad — ①$$





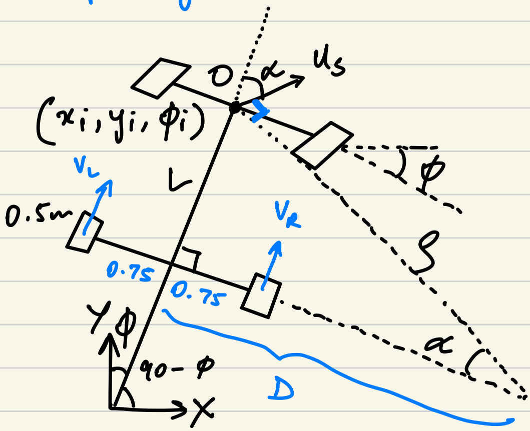
$$V_x = \dot{x} = u_s \cos(90 - \phi - \alpha)$$

$$\therefore \dot{x} = u_s \sin(\phi + \alpha) \quad — ②$$

$$V_y = \dot{y} = u_s \sin(90 - \phi - \alpha)$$

$$\therefore \dot{y} = u_s \cos(\phi + \alpha) \quad — ③$$

For finding u_s:



$$\tan\alpha = \frac{L}{D}$$

$$\Rightarrow D = \frac{L}{\tan\alpha} \quad — (6)$$

$V = r\omega$

$$\omega_{left} + \omega_{right} = 2\omega \quad — (5)$$

$$\dot{\phi} = \frac{V_L}{D+0.75} = \frac{V_R}{D-0.75} = \frac{u_s \sin\alpha}{L}$$

$$V_L(D-0.75) = V_R(D+0.75)$$

$$\cancel{r}\,\omega_L(D-0.75) = \cancel{r}\,\omega_R(D+0.75)$$

$$\omega_L = \frac{\omega_R(D+0.75)}{D-0.75}$$

$$\frac{\omega_R(D+0.75)}{D-0.75} + \omega_R = 2\omega; \text{ Substituting } \omega_L \text{ in } (5)$$

$$\omega_R(D+0.75) + \omega_R(D-0.75) = 2\omega(D-0.75)$$

$$\cancel{2}\,\omega_R D = \cancel{2}\,\omega(D-0.75)$$

$$\omega_R D = \omega(D-0.75)$$

$$\omega_R = \frac{\omega}{D}(D-0.75)$$

$$\dot{\phi} = \frac{V_\ell}{D-0.75} = \frac{u_s \sin\alpha}{L}$$

$$\frac{r\omega_R}{D-0.75} = \frac{u_s \sin\alpha}{L}$$

$$\frac{r}{\cancel{D-0.75}} \times \frac{\omega}{D}\left(\cancel{D-0.75}\right) = \frac{u_s \sin\alpha}{L}$$

$$u_s = L\frac{r\omega}{D} \cdot \sin\alpha$$

$$u_s = \cancel{L}\frac{r\omega}{\cancel{L}} \cdot \tan\alpha \quad \sin\alpha \ ; \text{ Substituting } D \text{ from } 6$$

$$\boxed{u_s = \frac{r\omega}{\cos\alpha}}$$

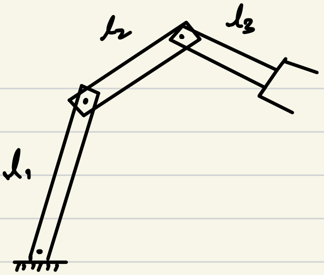
From 2, 3 & 4, The state space model is given by:

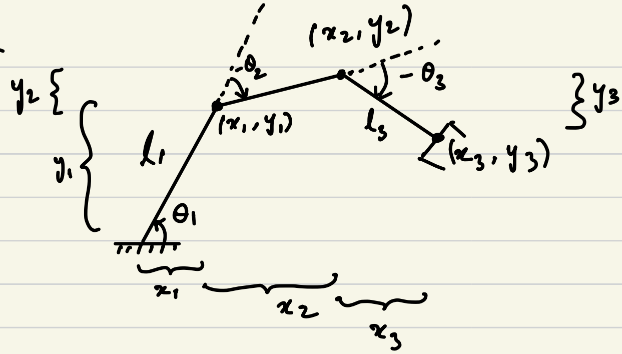
$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} u_s \sin(\phi+\alpha) \\ u_s \cos(\phi+\alpha) \\ u_s/_L \sin\alpha \end{bmatrix}$$

Substituting u_s

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} \frac{r\omega}{\cos\alpha} \sin(\phi+\alpha) \\ \frac{r\omega}{\cos\alpha} \cos(\phi+\alpha) \\ \frac{r\omega}{L\cos\alpha} \cdot \sin\alpha \end{bmatrix} = \begin{bmatrix} \frac{r\omega}{\cos\alpha} \cdot \sin(\phi+\alpha) \\ \frac{r\omega}{\cos\alpha} \cos(\phi+\alpha) \\ \frac{r\omega}{L} \tan\alpha \end{bmatrix}$$

2.

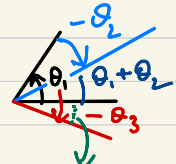




$$x_1 = l_1 \cos\theta_1$$
$$x_2 = l_2 \cos(\theta_1 + \theta_2)$$
$$x_3 = l_3 \cos\left(-\theta_3 - (\theta_1 + \theta_2)\right) = l_3 \cos\left[-(\theta_1 + \theta_2 + \theta_3)\right]$$
$$= l_3 \cos(\theta_1 + \theta_2 + \theta_3)$$

Similarly:

$$y_1 = l_1 \sin\theta_1$$
$$y_2 = l_2 \sin(\theta_1 + \theta_2)$$
$$y_3 = -l_3 \sin(\theta_1 + \theta_2 + \theta_3)$$



$$-\theta_3 - (\theta_1 + \theta_2)$$
$$= -\left[\theta_1 + \theta_2 + \theta_3\right]$$

$$x = x_1 + x_2 + x_3$$
$$y = y_1 + y_2 + y_3$$

$$x = l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) + l_3 \cos(\theta_1 + \theta_2 + \theta_3)$$
$$\dot{x} = -l_1 \dot{\theta}_1 \sin\theta_1 - l_2 \sin(\theta_1 + \theta_2) . (\dot{\theta}_1 + \dot{\theta}_2) - l_3 \sin(\theta_1 + \theta_2 + \theta_3) . (\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3)$$

$$\dot{x} = \dot{\theta}_1 \left[-l_1 \sin\theta_1 - l_2 \sin(\theta_1 + \theta_2) - l_3 \sin(\theta_1 + \theta_2 + \theta_3)\right]$$
$$+ \dot{\theta}_2 \left[-l_2 \sin(\theta_1 + \theta_2) - l_3 \sin(\theta_1 + \theta_2 + \theta_3)\right]$$
$$+ \dot{\theta}_3 \left[-l_3 \sin(\theta_1 + \theta_2 + \theta_3)\right]$$

$$y = l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2) - l_3 \sin(\theta_1 + \theta_2 + \theta_3)$$
$$\dot{y} = l_1 \dot{\theta}_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2)(\dot{\theta}_1 + \dot{\theta}_2) - l_3 \cos(\theta_1 + \theta_2 + \theta_3) . (\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3)$$

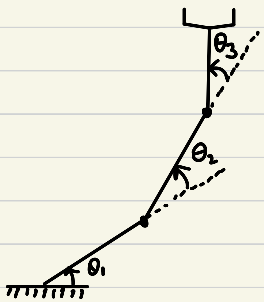
$$\dot{y} = \dot{\theta}_1 \left[l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) - l_3 \cos(\theta_1 + \theta_2 + \theta_3)\right]$$
$$+ \dot{\theta}_2 \left[l_2 \cos(\theta_1 + \theta_2) - l_3 \cos(\theta_1 + \theta_2 + \theta_3)\right]$$
$$+ \dot{\theta}_3 \left[-l_3 \cos(\theta_1 + \theta_2 + \theta_3)\right]$$

$\therefore$ Forward Kinematics is given by:

$$\begin{bmatrix} -l_1\sin\theta_1 - l_2\sin(\theta_1+\theta_2) - l_3\sin(\theta_1+\theta_2+\theta_3) & -l_2\sin(\theta_1+\theta_2) - l_3\sin(\theta_1+\theta_2+\theta_3) & -l_3\sin(\theta_1+\theta_2+\theta_3) \\ l_1\cos\theta_1 + l_2\cos(\theta_1+\theta_2) - l_3\cos(\theta_1+\theta_2+\theta_3) & l_2\cos(\theta_1+\theta_2) - l_3\cos(\theta_1+\theta_2+\theta_3) & -l_3\cos(\theta_1+\theta_2+\theta_3) \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \dot\theta_1 \\ \dot\theta_2 \\ \dot\theta_3 \end{bmatrix} = \begin{bmatrix} \dot x \\ \dot y \\ \dot\phi \end{bmatrix}$$

Inverse Kinematics:

$$\begin{bmatrix} \dot\theta_1 \\ \dot\theta_2 \\ \dot\theta_3 \end{bmatrix} = \begin{bmatrix} -l_1\sin\theta_1 - l_2\sin(\theta_1+\theta_2) - l_3\sin(\theta_1+\theta_2+\theta_3) & -l_2\sin(\theta_1+\theta_2) - l_3\sin(\theta_1+\theta_2+\theta_3) & -l_3\sin(\theta_1+\theta_2+\theta_3) \\ l_1\cos\theta_1 + l_2\cos(\theta_1+\theta_2) - l_3\cos(\theta_1+\theta_2+\theta_3) & l_2\cos(\theta_1+\theta_2) - l_3\cos(\theta_1+\theta_2+\theta_3) & -l_3\cos(\theta_1+\theta_2+\theta_3) \\ 1 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \dot x \\ \dot y \\ \dot\phi \end{bmatrix}$$



For generalised equation, we take the following orientation where θ_1, θ_2 & θ_3 are positive.

$x_1 = l_1 \cos\theta_1$
$x_2 = l_2 \cos(\theta_1+\theta_2)$
$x_3 = l_3 \cos(\theta_1+\theta_2+\theta_3)$

$y_1 = l_1 \sin\theta_1$
$y_2 = l_2 \sin(\theta_1+\theta_2)$
$y_3 = l_3 \sin(\theta_1+\theta_2+\theta_3)$

$$x = x_1 + x_2 + x_3$$
$$y = y_1 + y_2 + y_3$$

$$x = l_1 \cos\theta_1 + l_2 \cos(\theta_1+\theta_2) + l_3 \cos(\theta_1+\theta_2+\theta_3)$$
$$\dot{x} = -l_1 \dot{\theta}_1 \sin\theta_1 - l_2 \sin(\theta_1+\theta_2).(\dot{\theta}_1+\dot{\theta}_2) - l_3 \sin(\theta_1+\theta_2+\theta_3).(\dot{\theta}_1+\dot{\theta}_2+\dot{\theta}_3)$$
$$\dot{x} = \dot{\theta}_1\left[-l_1 \sin\theta_1 - l_2 \sin(\theta_1+\theta_2) - l_3 \sin(\theta_1+\theta_2+\theta_3)\right] + \dot{\theta}_2\left[-l_2 \sin(\theta_1+\theta_2) - l_3 \sin(\theta_1+\theta_2+\theta_3)\right] + \dot{\theta}_3\left[-l_3 \sin(\theta_1+\theta_2+\theta_3)\right]$$

$$y = l_1 \sin\theta_1 + l_2 \sin(\theta_1+\theta_2) + l_3 \sin(\theta_1+\theta_2+\theta_3)$$
$$\dot{y} = l_1 \dot{\theta}_1 \cos\theta_1 + l_2 \cos(\theta_1+\theta_2)(\dot{\theta}_1+\dot{\theta}_2) + l_3 \cos(\theta_1+\theta_2+\theta_3).(\dot{\theta}_1+\dot{\theta}_2+\dot{\theta}_3)$$
$$\dot{y} = \dot{\theta}_1\left[l_1 \cos\theta_1 + l_2 \cos(\theta_1+\theta_2) + l_3 \cos(\theta_1+\theta_2+\theta_3)\right] + \dot{\theta}_2\left[l_2 \cos(\theta_1+\theta_2) + l_3 \cos(\theta_1+\theta_2+\theta_3)\right] + \dot{\theta}_3\left[l_3 \cos(\theta_1+\theta_2+\theta_3)\right]$$

$\therefore$ Forward Kinematics is given by:

$$\begin{bmatrix} -l_1 \sin\theta_1 - l_2 \sin(\theta_1+\theta_2) - l_3 \sin(\theta_1+\theta_2+\theta_3) & -l_2 \sin(\theta_1+\theta_2) - l_3 \sin(\theta_1+\theta_2+\theta_3) & -l_3 \sin(\theta_1+\theta_2+\theta_3) \\ l_1 \cos\theta_1 + l_2 \cos(\theta_1+\theta_2) + l_3 \cos(\theta_1+\theta_2+\theta_3) & l_2 \cos(\theta_1+\theta_2) + l_3 \cos(\theta_1+\theta_2+\theta_3) & l_3 \cos(\theta_1+\theta_2+\theta_3) \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\phi} \end{bmatrix}$$

Inverse Kinematics:

$$\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} = \begin{bmatrix} -l_1 \sin\theta_1 - l_2 \sin(\theta_1+\theta_2) - l_3 \sin(\theta_1+\theta_2+\theta_3) & -l_2 \sin(\theta_1+\theta_2) - l_3 \sin(\theta_1+\theta_2+\theta_3) & -l_3 \sin(\theta_1+\theta_2+\theta_3) \\ l_1 \cos\theta_1 + l_2 \cos(\theta_1+\theta_2) + l_3 \cos(\theta_1+\theta_2+\theta_3) & l_2 \cos(\theta_1+\theta_2) + l_3 \cos(\theta_1+\theta_2+\theta_3) & l_3 \cos(\theta_1+\theta_2+\theta_3) \\ 1 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\phi} \end{bmatrix}$$