MODELING - HW2

1.1 When rotating & translating with respect to current axis, we <u>post-multiply</u> the rotation and translation matrices.

When rotating & translating with respect to world axis, we <u>pre-multiply</u> the rotation and translation matrices.

1. Rotate by ϕ about the world x-axis.
2. Translate by y along the current y-axis.
3. Rotate by θ about the world z-axis.
4. Rotate by ψ about the current x-axis.

Step 1 : $H = R_{x,\phi}$

Step 2 : $H = R_{x,\phi} \cdot T_{y,y}$

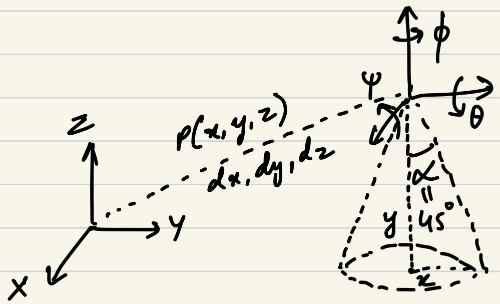
Step 3 : $H = R_{z,\theta} \cdot R_{x,\phi} \cdot T_{y,y}$

Step 4 : $H = R_{z,\theta} \, R_{x,\phi} \, T_{y,y} \, R_{x,\psi}$

$$= R_\theta R_\phi T_y R_\psi$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi & 0 \\ 0 & \sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi & 0 \\ 0 & \sin\psi & \cos\psi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

1.2



(ψ, θ, ϕ)

$$H = \left[\begin{array}{ccc|c} & R & & T \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

The rotation matrix is given by:

$$R = R_{z,\phi} R_{y,\theta} R_{x,\psi}$$

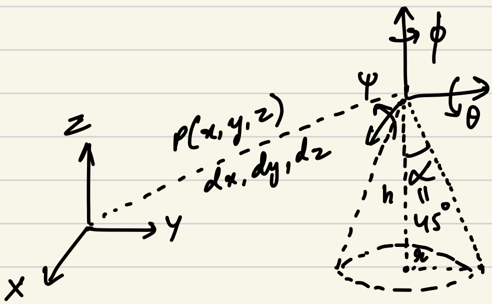
$$= \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{bmatrix}$$

$$= \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta\sin\psi & \sin\theta\cos\psi \\ 0 & \cos\psi & -\sin\psi \\ -\sin\theta & \cos\theta\sin\psi & \cos\theta\cos\psi \end{bmatrix}$$

$$= \begin{bmatrix} C_\phi C_\theta & C_\phi S_\theta S_\psi - S_\phi C_\psi & S_\phi S_\psi + C_\phi S_\theta C_\psi \\ S_\phi C_\theta & C_\phi C_\psi + S_\phi S_\theta S_\psi & S_\phi S_\theta C_\psi - C_\phi S_\psi \\ -S_\theta & C_\theta S_\psi & C_\theta C_\psi \end{bmatrix}$$

Homogeneous Transformation matrix is given by:

$$H = \begin{bmatrix} C_\phi C_\theta & C_\phi S_\theta S_\psi - S_\phi C_\psi & S_\phi S_\psi + C_\phi S_\theta C_\psi & dx \\ S_\phi C_\theta & C_\phi C_\psi + S_\phi S_\theta S_\psi & S_\phi S_\theta C_\psi - C_\phi S_\psi & dy \\ -S_\theta & C_\theta S_\psi & C_\theta C_\psi & dz \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



(ψ, θ, ϕ)

$\tan\alpha = \dfrac{r}{h}$

$\tan 45 = \dfrac{r}{h}$

$h = r$

Equation of cone :

$$x^2 + y^2 = \left(\frac{r^2}{h^2}\right) z^2$$

$$\Rightarrow x^2 + y^2 = z^2$$

$$\Rightarrow x^2 + y^2 - z^2 = 0$$

This equation can be written in matrix form as:

$$\begin{bmatrix} x & y & -z \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

OR

$$\begin{bmatrix} x & y & z & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = 0$$

To get the equation of ellipse, we set $z = 0$

$$\begin{bmatrix} x & y & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ 0 \\ 1 \end{bmatrix} = 0$$

Applying the homogeneous matrix to transform to the new coordinates:

$$\left[H.\begin{bmatrix} x & y & 0 & 1 \end{bmatrix}^T \right]^T \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} H. \begin{bmatrix} x \\ y \\ 0 \\ 1 \end{bmatrix} = 0$$

After matrix multiplication, we get equation of the form:

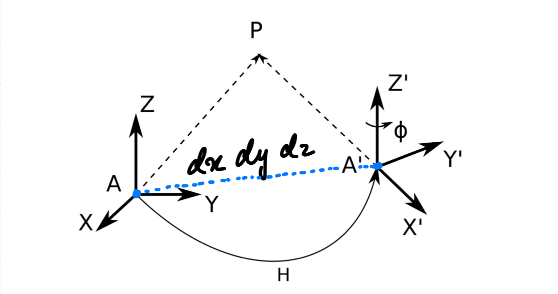
$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0$$

Extracting the coefficients, we substitute them in the equation below to find the area of ellipse

$$A = \frac{-\pi}{(ac - b^2)^{3/2}} \begin{vmatrix} a & b & d \\ b & c & e \\ d & e & f \end{vmatrix}$$

All computations are done in the python file.

1.3



$$X' = HX$$

Transformation matrix : ${}^{A}_{A'}H = \begin{bmatrix} R_{z,\phi} & d \\ 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 & dx \\ \sin\phi & \cos\phi & 0 & dy \\ 0 & 0 & 1 & dz \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x'\cos\phi + dx - y'\sin\phi \\ x'\sin\phi + y'\cos\phi + dy \\ z' + dz \\ 1 \end{bmatrix}$$

$$x = x'\cos\phi + dx - y'\sin\phi \qquad z = z' + dz$$

$$\therefore \underline{dx = x + y'\sin\phi - x'\cos\phi} \qquad \underline{dz = z - z'}$$

$$y = x'\sin\phi + y'\cos\phi + dy$$

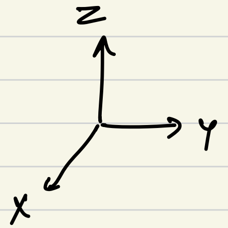
$$\therefore \underline{dy = y - x'\sin\phi - y'\cos\phi}$$

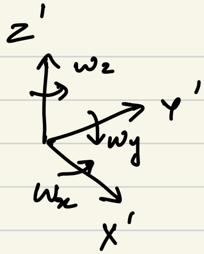
∴ The transformation matrix is given by:

$$ {}^{A}_{A'}H = \begin{bmatrix} \cos\phi & -\sin\phi & 0 & x + y'\sin\phi - x'\cos\phi \\ \sin\phi & \cos\phi & 0 & y - x'\sin\phi - y'\cos\phi \\ 0 & 0 & 1 & z - z' \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ \therefore\ {}^{A'}_{A}H = {}^{A}_{A'}H^{-1} = \begin{bmatrix} \cos\phi & \sin\phi & 0 & x' - x\cos\phi - y\sin\phi \\ -\sin\phi & \cos\phi & 0 & y' + x\sin\phi - y\cos\phi \\ 0 & 0 & 1 & z' - z \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

2.1





$\psi_g = 35°$ (X)
$\theta_g = 15°$ (Y)
$\phi_g = 20°$ (Z)

$w_{max} = 1$ deg/s

The rotation matrix with euler angles is given by:

$$R = R_{z,\phi} R_{y,\theta} R_{x,\psi}$$

$$= \begin{bmatrix} c\phi c\theta & c\phi s\theta s\psi - s\phi c\psi & s\phi s\psi + c\phi s\theta c\psi \\ s\phi c\theta & c\phi c\psi + s\phi s\theta s\psi & s\phi s\theta c\psi - c\phi s\psi \\ -s\theta & c\theta s\psi & c\theta c\psi \end{bmatrix}$$

Substituting the values of ψ, θ, ϕ in radians, we get:

$$R = \begin{bmatrix} 0.9077 & -0.1407 & 0.3954 \\ 0.3304 & 0.8205 & -0.4665 \\ -0.2588 & 0.5540 & 0.7912 \end{bmatrix}$$

For converting rotation matrix to axis-angle representation, we use the formula:

Equivalent angle :-

$$\theta = \cos^{-1}\left(\frac{r_{11} + r_{22} + r_{33} - 1}{2}\right)$$

$= 0.7079$ radians

Equivalent axis unit vector:-

$$k = \frac{1}{2\sin\theta}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$$

$$\begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix} = \begin{bmatrix} 0.7847 \\ 0.5031 \\ 0.3622 \end{bmatrix}$$

We can see that the x-component is the largest.

Since the axis has to travel the longest distance in the x-direction, we set $W_x = W_{max} = 1 \text{ deg/s}$
$= 0.0175 \text{ rad/s}$

We use the formula, $w = \dot{\theta} k$

$$w_x = \dot{\theta} k_x$$
$$0.0175 = \dot{\theta} \times 0.7847$$
$$\dot{\theta} = 0.0223 \text{ radians}$$

$$w_y = \dot{\theta} k_y$$
$$= 0.0223 \times 0.5031$$
$$= 0.0112 \text{ radians} \qquad (w_y < w_{max})$$

$$w_z = \dot{\theta} k_z$$
$$= 0.0223 \times 0.3622$$
$$= 0.008 \text{ radians} \qquad (w_z < w_{max})$$

$\therefore$ Shortest time is given by:

$$w = \frac{\theta}{T} \quad \text{where } w = \dot{\theta};\ \theta = 0.7079 \text{ rad}$$

$$T = \frac{0.7079}{0.0223} = 31.82 \text{ seconds}$$

The rotation matrix with axis-angle representation is given by:

$$R_{k,\theta} = \begin{bmatrix} k_x^2 v_\theta + c_\theta & k_x k_y v_\theta - k_z s_\theta & k_x k_z v_\theta + k_y s_\theta \\ k_x k_y v_\theta + k_z s_\theta & k_y^2 v_\theta + c_\theta & k_y k_z v_\theta - k_x s_\theta \\ k_x k_z v_\theta - k_y s_\theta & k_y k_z v_\theta + k_x s_\theta & k_z^2 v_\theta + c_\theta \end{bmatrix}$$

where $v_\theta = \text{vers}\,\theta = 1 - c_\theta$

The Euler angles can be calculated using the formulas:

$$\psi = \text{Atan2}(r_{32}, r_{33})$$

$$\theta = \text{Atan2}\left(-r_{31}, \sqrt{r_{32}^2 + r_{33}^2}\right)$$

$$\phi = \text{Atan2}(r_{21}, r_{11})$$

We can plot the euler angles with respect to time by substituting $\theta = \dot{\theta} t$ in $R_{k,\theta}$ where $\dot{\theta} = 0.0223$ rad
t = time step

Computations and plotting are done in the python file provided.