Institute of Mathematics
Faculty of Mathematics and Computer Science
Diploma Thesis

The Null Gluing Problem for Modified Wave Equations

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1 Introduction

General relativity is the theory of space, time and gravitation formulated by Einstein in 1915. It has been a groundbreaking discovery towards the grand goal of describing reality. The idea is that, in opposition to other forces which are represented by fields defined on spacetime, "gravity" is a manifestation of the curvature of spacetime itself. Using differential geometry to link gravity to the presence of matter and energy, Einstein managed to develop a theory that is of such beauty that it is widely regarded as the most beautiful theory in all of physics. The whole theory of general relativity is centered around the famous Einstein field equation,

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi T_{\mu\nu}, \tag{1.1}$$

where we have the Ricci tensor $R_{\mu\nu}$, the Ricci scalar R, our spacetime metric $g_{\mu\nu}$ and the stress energy tensor $T_{\mu\nu}$. The equation prescribes how the curvature of the spacetime on the left reacts to the presence of energy-momentum on the right. The claim is that the spacetime we live in obeys the laws imposed in the Einstein field equations. Although the beauty of the equation and its implications to our universe, the real challenge is to actually solve it and find a fitting spacetime $(\mathcal{M}, g)$. Hidden behind the elegant notation using tensors is a set of ten coupled nonlinear partial differential equations, which, by the Bianchi identities, can be reduced to six independent nonlinear partial differential equations. This leaves four degrees of freedom for choosing a coordinate system.
In mathematics, if a problem is too hard, try solving a simpler one. In particular, in the setting of the Einstein equations, one can set the energy stress tensor $T_{\mu\nu} = 0$ to reduce 1.1 to the vacuum Einstein equation

$$R_{\mu\nu} = 0. \tag{1.2}$$

Although this equation may seem quite simple, solving it explicitly is a difficult task, as it still consists of a set of highly nonlinear partial differential equations. There are only a few explicit vacuum solutions aside from the flat Minkowski spacetime. Surprisingly, only a few months after Einstein proposed his vacuum field equations, Karl Schwarzschild found a solution to the problem, which is now known as the Schwarzschild metric.

Due to the complexity in solving the Einstein field equations, it is always helpful to consider a coordinate system, that eases the explicit calculations. A clever choice of a coordinate system is the so-called double null coordinate system. This is based on spacetime paths on which light travels. In this work, we will always consider the setting of a flat Minkowski spacetime together with the mentioned double null coordinates.

The goal of this thesis is to investigate the obstructions of the gluing problem for a variety of wave equations in Minkowski spacetime. This is connected to perturbations of spacetime as one obtains wave equations by linearizing the Einstein vacuum field equations. The obstructions for the linear wave equation $\Box\psi = 0$ is more generally covered by S. Aretakis. In his paper [1] he showed that in the case of the linear homogeneous wave equation $\Box\psi = 0$ on null hypersurfaces on general four-dimensional Lorentzian manifolds, obstructions of the gluing property are related to the existence of certain conservation laws along $\mathcal{H}$ and their associated charges. In this diploma thesis, we take a more direct approach and restrict the observations to flat Minkowski spacetime. This will make it possible to provide explicit gluing constructions using mainly the tools of spherical harmonic expansion, along with specially constructed bump and cutoff functions. First, we define some fundamentals, such as the metric tensor, spacetime manifolds, and Minkowski spacetime together with the double null coordinates. After investigating how the wave operator appears in the new coordinate system, we introduce the null gluing problem. The first gluing problem we investigate is the linear wave equation in chapter 4. In this chapter, we already see some tricks and correlations which will also appear later on. After that, we move to a more generalized, but still linear wave equation in Chapter 5. The main result of this chapter will include the linear homogeneous wave equation as a special case. After a glance at higher-order gluing for this type of linear equations, we will dive into the nonlinear section. There we will also investigate the gluing problem and determine where our established tools and tricks reach their limits.

2 Double Null Coordinates

To study differential equations on Lorentzian manifolds, it is always helpful to find a suitable foliation. In the following, we will introduce the so-called double null coordinates on Minkowski spacetime, in which the level sets of coordinates u, v will be null cones. The union of these null cones forms the double null foliation of Minkowski spacetime.

2.1 Lorentzian Manifolds and Minkowski Spacetime

We first introduce the spaces and objects with which we are working. The following introduction is based on Semi-Riemannian Geometry by Barrett O'Neill [2] and Lecture Notes on General Relativity by S. Aretakis [3].

Definition 2.1 *A Lorentzian manifold* $(\mathcal{M}, g)$ *is a differentiable manifold of dimension* $n+1$*, together with a Lorentzian metric* g*, which is a differentiable assignment of a nondegenerate symmetric bilinear form* g_p *with signature* $(-,+,\dots,+)$ *in* T_pM *at each* $p \in \mathcal{M}$*.*

Definition 2.2 *A spacetime manifold* $(\mathcal{M}, g)$ *is an orientable four-dimensional Lorentzian manifold.*

Definition 2.3 *The Minkowski 4-space* $\mathbb{R}^{3+1}$ *is the real vector space* $\mathbb{R}^4$ *together with the scalar product*

$$\langle v_p, w_p \rangle = -v^0 w^0 + v^1 w^1 + v^2 w^2 + v^3 w^3. \tag{2.1}$$

Definition 2.4 *A spacetime* $(\mathcal{M}, g)$ *is called "locally Minkowski" if for every open subset* $U \subseteq \mathcal{M}$ *there exists an open subset* $V \subseteq \mathbb{R}^{3+1}$ *and an isometry* $\Psi : (U, g) \to (V, \langle ., . \rangle)$*. Then there is a local basis* $\{e_0, e_1, e_2, e_3\}$ *such that*

$$g(e_\mu, e_\nu) = g_{\mu\nu}, \tag{2.2}$$

where the matrix $g_{\mu\nu}$ *is given by*

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.3}$$

One can then construct a coordinate system (x^0, x^1, x^2, x^3) *of* $\mathbb{R}^4$ *such that* $e_\mu = \partial_\mu$ *for* $\mu = 0, 1, 2, 3$*. With respect to the above coordinate system the metric can be expressed as the* $(0,2)$ *tensor*

$$g = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2. \tag{2.4}$$

2.2 Causality

Definition 2.5 *A vector $X \in \mathbb{R}^{3+1}$ is called*

1. ***spacelike***, *if $g(X,X) > 0$,*
2. ***timelike***, *if $g(X,X) < 0$,*
3. ***null***, *if $g(X,X) = 0$.*

Definition 2.6 (Hypersurfaces) *A hypersurface $\mathcal{H}$ is called*

1. ***spacelike***, *if $\forall x \in \mathcal{H}$ the normal N_x is timelike. In this case, $g\mid_{T_x\mathcal{H}}$ is positive definite.*
2. ***timelike***, *if $\forall x \in \mathcal{H}$ the normal N_x is spacelike. In this case, $g\mid_{T_x\mathcal{H}}$ is degenerate.*
3. ***null***, *if $\forall x \in \mathcal{H}$ the normal N_x is null. In this case, $g\mid_{T_x\mathcal{H}}$ has the signature $(-,+,+)$.*

Examples of null hypersurfaces

- Consider a null vector $n = (n_0, n_1, n_2, n_3)$. Then the plane
$$P_n = \{(x^0, x^1, x^2, x^3) : n_0 x^0 = n_1 x^1 + n_2 x^2 + n_3 x^3\} \tag{2.5}$$
is a null hypersurface, since its normal is n everywhere.
- Consider the cones
$$C = \left\{(x^0, x^1, x^2, x^3) : t = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}\right\}, \tag{2.6}$$
and
$$\underline{C} = \left\{(x^0, x^1, x^2, x^3) : t = -\sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}\right\}. \tag{2.7}$$
At each end point of a null vector n on the cone, the tangent plane is P_n. Hence, the normal is the null vector n and the cones are null hypersurfaces.

2.3 Double Null Coordinates in Minkowski Spacetime

The setting of this work will always be flat Minkowski spacetime. We introduce the following change of coordinates, where we first transform our Cartesian $\mathbb{R}^3$ part into spherical coordinates and then substitute t, r for u, v by using $u = \frac{1}{2}(t - r), v = \frac{1}{2}(t + r)$.

$$\left(t, x^1, x^2, x^3\right) \xrightarrow[\text{coord.}]{\text{sph.}} \left(t, r, \theta^1, \theta^2\right) \xrightarrow[v=\frac{1}{2}(t+r)]{u=\frac{1}{2}(t-r)} \left(u, v, \theta^1, \theta^2\right). \tag{2.8}$$

Proposition 2.1 *The Minkowski metric g in double null coordinates $\left(u, v, \theta^1, \theta^2\right)$ is the following:*
$$g = -4dudv + (v-u)^2 g_{\mathbb{S}^2}. \tag{2.9}$$

Proof. First, we calculate the metric in spherical coordinates $(t, r, \theta^1, \theta^2)$. The coordinate transformation is

$$\begin{cases} t &= t \\ x^1 &= r \sin\theta^1 \cos\theta^2 \\ x^2 &= r \sin\theta^1 \sin\theta^2 \\ x^3 &= r \cos\theta^1. \end{cases} \tag{2.10}$$

To calculate the metric components of the new metric, we use the metric transformation formula

$$g_{\mu\nu} = \frac{\partial x^\alpha}{\partial y^\mu} \frac{\partial x^\beta}{\partial y^\nu} g_{\alpha\beta}. \tag{2.11}$$

The metric components of Minkowski metric in spherical coordinates are the following

$$\begin{aligned}
g_{tt} &= -1 && (2.12)\\
g_{rr} &= \frac{\partial t}{\partial r}\frac{\partial t}{\partial r} g_{tt} + \frac{\partial x^1}{\partial r}\frac{\partial x^1}{\partial r} g_{x^1x^1} + \frac{\partial x^2}{\partial r}\frac{\partial x^2}{\partial r} g_{x^2x^2} + \frac{\partial x^3}{\partial r}\frac{\partial x^3}{\partial r} g_{x^3x^3} \\
&= 0 + \sin^2\theta^1 \cos^2\theta^2 \cdot 1 + \sin^2\theta^1 \sin^2\theta^2 \cdot 1 + \cos^2\theta^1 \cdot 1 = 1 && (2.13)\\
g_{\theta^1\theta^1} &= 0 + r^2\cos^2\theta^1\cos^2\theta^2 + r^2\cos^2\theta^1\sin^2\theta^2 + r^2\sin^2\theta^1 = r^2 && (2.14)\\
g_{\theta^2\theta^2} &= 0 + r^2\sin^2\theta^1\sin^2\theta^2 + r^2\sin^2\theta^1\cos^2\theta^2 = r^2\sin^2\theta^1 && (2.15)\\
g_{tr} &= g_{rt} = 0 && (2.16)\\
g_{t\theta^1} &= g_{t\theta^2} = g_{\theta^1 t} = g_{\theta^2 t} = 0 && (2.17)\\
g_{\theta^1\theta^2} &= g_{\theta^2\theta^1} = -r^2\cos\theta^1\sin\theta^1\cos\theta^2\sin\theta^2 + r^2\cos\theta^1\sin\theta^1\cos\theta^2\sin\theta^2 = 0. && (2.18)
\end{aligned}$$

Hence the matrix representation of the Minkowski metric in spherical coordinates reads

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta^1 \end{pmatrix}. \tag{2.19}$$

Using $u = \frac{1}{2}(t - r), v = \frac{1}{2}(t + r)$, we get the double null coordinate system $(u, v, \theta^1, \theta^2)$. Note that $t = v + u, r = v - u$. The metric components of the Minkowski metric in the double null coordinate system are

$$\begin{aligned}
g_{uu} &= \frac{\partial t}{\partial u}\frac{\partial t}{\partial u} g_{tt} + \frac{\partial r}{\partial u}\frac{\partial r}{\partial u} g_{rr} = -1 + 1 = 0 && (2.20)\\
g_{vv} &= -1 + 1 = 0 && (2.21)\\
g_{uv} &= g_{vu} = -1 + (-1) = -2 && (2.22)\\
g_{\theta^1\theta^1} &= r^2 && (2.23)\\
g_{\theta^2\theta^2} &= r^2\sin\theta^1. && (2.24)
\end{aligned}$$

All the other components are 0. The matrix representation is

$$g_{\mu\nu} = \begin{pmatrix} 0 & -2 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta^1 \end{pmatrix}. \tag{2.25}$$

and the Minkowski metric g in double null coordinates reads

$$g = -4dudv + r^2 g_{\mathbb{S}^2}, \tag{2.26}$$

where $g_{\mathbb{S}^2} = (d\theta^1)^2 + \sin^2\theta^1 (d\theta^2)^2$ is the metric on the unit sphere. $\square$

The null cone in double null. In this work we will deal with hypersurfaces of constant u: $\mathcal{H} = \{u = c\}$. This is indeed a null hypersurface by the following observation. Note that $\mathcal{H}$ is a level set of the function u. Hence the gradient $\nabla u|_p$ is normal at each $p \in \mathcal{H}$. We also have the relation

$$(\nabla u)^\mu = g^{\mu\nu}\partial_\nu u \tag{2.27}$$

and therefore

$$g_p(\nabla u, \nabla u) = g_{\alpha\beta} g^{\alpha\mu}\partial_\mu u g^{\beta\nu}\partial_\nu u = \delta^\mu_\beta \partial_\mu u g^{\beta\nu}\partial_\nu u = g^{\mu\nu}\partial_\mu u \partial_\nu u = g^{uu} = 0, \tag{2.28}$$

where $g^{\mu\nu}$ is the inverse matrix of $g_{\mu\nu}$, see (2.32).

2.4 The Wave Operator in Double Null Coordinates

Using the metric tensor we can calculate the wave operator in double null coordinates, which is the Laplace-Beltrami-Operator with our metric:

$$\square = \frac{1}{\sqrt{|g|}}\partial_\mu \left(\sqrt{|g|} g^{\mu\nu}\partial_\nu\right), \tag{2.29}$$

where

$$\sqrt{|g|} = \sqrt{|\det(g_{\mu\nu})|} = \sqrt{4r^4 \sin^2\theta^1} = 2r^2 \sin\theta^1. \tag{2.30}$$

Proposition 2.2 *The wave operator of a C^2 (or H^2) function $\psi : \mathbb{R}^2 \times [0, \pi] \times [0, 2\pi) \to \mathbb{R}$ in Minkowski spacetime using double null coordinates is the following*

$$\square\psi = -\frac{1}{r}\partial_u\partial_v(r\psi) + \frac{1}{r^2}\cancel{\Delta}_{\mathbb{S}^2}\psi. \tag{2.31}$$

Proof. The matrix representation $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$ are given by

$$g_{\mu\nu} = \begin{pmatrix} 0 & -2 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta^1 \end{pmatrix}, \quad g^{\mu\nu} = \begin{pmatrix} 0 & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & r^{-2} & 0 \\ 0 & 0 & 0 & r^{-2}\sin^{-2}\theta^1 \end{pmatrix}. \tag{2.32}$$

Therefore the wave operator looks like

$$
\begin{aligned}
\sqrt{|g|}\,\Box\psi =& \partial_u(2r^2\sin\theta^1 g^{uv}\partial_v\psi) + \partial_v(2r^2\sin\theta^1 g^{vu}\partial_u\psi) \\
&+ \partial_{\theta^1}(2r^2\sin\theta^1 g^{\theta^1\theta^1}\partial_{\theta^1}\psi) + \partial_{\theta^2}(2r^2\sin\theta^1 g^{\theta^2\theta^2}\partial_{\theta^2}\psi) \\
=& \cancel{2}\sin\theta^1\left(-\frac{1}{\cancel{2}}\right)\partial_u(r^2\partial_v\psi) + \cancel{2}\sin\theta^1\left(-\frac{1}{\cancel{2}}\right)\partial_v(r^2\partial_u\psi) \\
&+ 2\cancel{r^2}\cancel{r^{-2}}\partial_{\theta^1}(\sin\theta^1\partial_{\theta^1}\psi) + 2\cancel{r^2}\cancel{r^{-2}}\cancel{\sin\theta^1}\sin^{\cancel{-2}-1}\theta^1\partial_{\theta^2}\partial_{\theta^2}\psi \\
=& \sin\theta^1(2r\partial_v\psi - r^2\partial_u\partial_v\psi) + \sin\theta^1(-2r\partial_u\psi - r^2\partial_u\partial_v\psi) \\
&+ 2(\cos\theta^1\partial_{\theta^1}\psi + \sin\theta^1\partial_{\theta^1}\partial_{\theta^1}\psi) + 2\sin^{-1}\theta^1\partial_{\theta^2}\partial_{\theta^2}\psi \\
=& 2r\sin\theta^1(\partial_v\psi - \partial_u\psi - r\partial_u\partial_v\psi) + 2(\cos\theta^1\partial_{\theta^1}\psi + \sin\theta^1\partial_{\theta^1}\partial_{\theta^1}\psi) \\
&+ 2\sin^{-1}\theta^1\partial_{\theta^2}\partial_{\theta^2}\psi \\
=& -2r\sin\theta^1\partial_u\partial_v(r\psi) + 2(\cos\theta^1\partial_{\theta^1}\psi + \sin\theta^1\partial_{\theta^1}\partial_{\theta^1}\psi) + 2\sin^{-1}\theta^1\partial_{\theta^2}\partial_{\theta^2}\psi
\end{aligned}
\tag{2.33}
$$

where we used $\partial_u r^2 = \partial_u(v-u)^2 = -2(v-u) = -2r = -\partial_v(v-u)^2 = -\partial_v r^2$. Dividing both sides by $\sqrt{|g|} = 2r^2\sin\theta^1$ yields

$$
\Box\psi = -\frac{1}{r}\partial_u\partial_v(r\psi) + \frac{1}{r^2}\left(\frac{\cos\theta^1}{\sin\theta^1}\partial_{\theta^1}\psi + \partial_{\theta^1}\partial_{\theta^1}\psi + \sin^{-2}\theta^1\partial_{\theta^2}\partial_{\theta^2}\psi\right), \tag{2.34}
$$

which is exactly

$$
\Box\psi = -\frac{1}{r}\partial_u\partial_v(r\psi) + \frac{1}{r^2}\slashed{\Delta}_{\mathbb{S}^2}\psi. \tag{2.35}
$$

$\Box$

3 The Characteristic Gluing Problem

Consider the Minkowski space with double null coordinates $(u, v, \theta^1, \theta^2)$. We will study wave equations of the form

$$\Box\psi + F(\psi, \partial_u\psi, \partial_v\psi) = 0. \tag{3.1}$$

It follows from [4] by A. D. Rendall that the characteristic initial value problem is locally well posed when characteristic initial data is given on two transversely-intersecting null hypersurfaces. In my work I will use the following definition of first order gluing by S. Aretakis in [1]. Consider the null hypersurface $\mathcal{H} = \{u = 0\}$ and the two-dimensional spheres S_0, S_1 on $\mathcal{H}$, as well as the characteristic initial data $(\psi|_{S_0}, \partial_u\psi|_{S_0})$ and $(\psi|_{S_1}, \partial_u\psi|_{S_1})$ at the spheres.

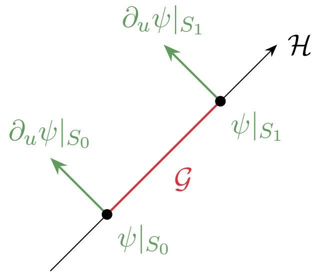

Figure 3.1: characteristic initial data

Definition 3.1 *We shall say that "we can perform first order gluing along $\mathcal{H}$" of the characteristic data $(\psi|_{S_0}, \partial_u\psi|_{S_0})$, $(\psi|_{S_1}, \partial_u\psi|_{S_1})$ if we can smoothly extend the data in $\mathcal{G}$ such that the arising solutions*

- *ψ_0 with data given on $S_0 \cup \mathcal{G}$, and*
- *ψ_1 with data given in S_1*

agree on S_1 up to first order tangential to $\mathcal{H}$ and up to first order in directions transversal to $\mathcal{H}$, that is, $\psi_0 = \psi_1$, $\partial_v\psi_0 = \partial_v\psi_1$ and $\partial_u\psi_0 = \partial_u\psi_1$ at S_1.

We can think of the data at S_0, S_1 as follows

$$\text{Data}(S_0) = \{\psi|_{S_0}, \partial_u\psi|_{S_0}, \partial_v\psi|_{S_0}\} \tag{3.2}$$

and

$$\text{Data}(S_1) = \{\psi|_{S_1}, \partial_u\psi|_{S_1}, \partial_v\psi|_{S_1}\}. \tag{3.3}$$

Then our problem is to smoothly extend ψ on $\mathcal{H}$ between S_0 and S_1 such that $\partial_u\psi$ is continuous on $\mathcal{H} \cap \{0 \leq v \leq 1\}$.

Analogously, we define the k-th order gluing problem, asking the following. Given

$$\text{Data}(S_0) = \{\psi|_{S_0}, \partial_v\psi|_{S_0}, \partial_u^l\psi|_{S_0}, k \geq l \geq 1\} \tag{3.4}$$

and

$$\text{Data}(S_1) = \{\psi|_{S_0}, \partial_v\psi|_{S_1}, \partial_u^l\psi|_{S_1}, k \geq l \geq 1\}, \tag{3.5}$$

can we then smoothly extend $\psi, \partial_u\psi, \ldots, \partial^{k-1}\psi$ on $\mathcal{H}$ between S_0 and S_1 such that $\partial_u^l\psi$ is continuous on $\mathcal{H} \cap \{0 \leq v \leq 1\}$ up to k-th order?

4 The Linear Wave Equation

The linear wave equation in double null coordinates in Minkowski Spacetime reads

$$\partial_v \partial_u (r\psi) = \frac{1}{r} \Delta\!\!\!\!/_{\mathbb{S}^2} \psi. \tag{4.1}$$

If we integrate this formula

$$\int_{v_0}^{v_1} dv \partial_v \partial_u (r\psi) = \int_{v_0}^{v_1} \frac{1}{r} \Delta\!\!\!\!/_{\mathbb{S}^2} \psi dv, \tag{4.2}$$

we get

$$[\partial_u (r\psi)]_{v_0}^{v_1} = \int_{v_0}^{v_1} \frac{1}{r} \Delta\!\!\!\!/_{\mathbb{S}^2} \psi dv. \tag{4.3}$$

This equation determines the transport of $\partial_u \psi$ along v. For the right side, we define

$$\rho(v_0, v_1, \theta^1, \theta^2) := [\partial_u (r\psi)]_{v_0}^{v_1}, \tag{4.4}$$

which is a function on the sphere and depends on the prescribed boundary data $\psi, \partial_u \psi$ at v_0, v_1. Since the Laplacian does not depend on r, we can write (4.3) as

$$\rho(v_0, v_1, \theta^1, \theta^2) = \Delta\!\!\!\!/_{\mathbb{S}^2} \left(\int_{v_0}^{v_1} \frac{1}{r} \psi dv \right) \tag{4.5}$$

for this to hold, ρ must be contained in the image $\mathrm{Im}(\Delta\!\!\!\!/_{\mathbb{S}^2})$. Since the Laplace operator on the sphere is self-adjoint, we have

$$\mathrm{Im}(\Delta\!\!\!\!/_{\mathbb{S}^2}) = (\mathrm{Ker}(\Delta\!\!\!\!/_{\mathbb{S}^2}))^{\perp}. \tag{4.6}$$

Let $\psi \in H^2(\mathbb{S}^2)$ and $\psi \in \mathrm{Ker}(\Delta\!\!\!\!/_{\mathbb{S}^2})$. Then

$$0 = \langle \Delta\!\!\!\!/_{\mathbb{S}^2} \psi, \psi \rangle = \int_{\mathbb{S}^2} \Delta\!\!\!\!/_{\mathbb{S}^2} \psi \, \psi \, d\mu_{\mathbb{S}^2} = - \int_{\mathbb{S}^2} (\nabla \psi)^2 d\mu_{\mathbb{S}^2} + \underbrace{\int_{\partial \mathbb{S}^2 = \emptyset} \frac{\partial \psi}{\partial \nu} \psi \, dS}_{=0} = -||\nabla \psi||^2_{L^2(\mathbb{S}^2)}. \tag{4.7}$$

Thus ψ must be constant and $\mathrm{Ker}(\Delta\!\!\!\!/_{\mathbb{S}^2}) \subseteq \langle 1 \rangle$. On the other hand, the Laplace operator of a constant is clearly zero and therefore $\mathrm{Ker}(\Delta\!\!\!\!/_{\mathbb{S}^2}) = \langle 1 \rangle$.
We conclude that for ψ to satisfy the wave equation (4.1) on $\mathcal{H}$ the data at v_0, v_1 must be chosen for ρ to be orthogonal to 1. Indeed, that is the only obstruction.

Proposition 4.1 *Consider the wave equation $\Box \psi = 0$ on $\mathcal{H}$ and let ρ be as defined in (4.4). Then one can perform first order gluing along $\mathcal{H}$ if and only if $\rho \in \langle 1 \rangle^{\perp}$.*

Proof. We already showed that for $\rho \not\perp 1$ gluing is impossible. Let now $\rho \in \langle 1 \rangle^{\perp}$. Without loss of generality, we can assume $\psi|_{\mathcal{H}} \equiv 0$ in neighborhoods around v_0 and v_1, by choosing a smaller interval $[v_0', v_1']$, cutting off ψ smoothly and transporting $\partial_u \psi$ along $\mathcal{H}$. Using spherical harmonics $Y^{lm} : \mathbb{S}^2 \to \mathbb{R}$, we can write ψ as

$$\psi(u, v, \theta^1, \theta^2) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \psi^{lm}(u, v) \, Y^{lm}(\theta^1, \theta^2). \tag{4.8}$$

The right hand side of (4.3) becomes

$$\begin{aligned}\int_{v_0}^{v_1} \frac{1}{r} \cancel{\Delta}_{\mathbb{S}^2} \psi(v, \theta^1, \theta^2) dv &= \int_{v_0}^{v_1} \frac{1}{r} \cancel{\Delta}_{\mathbb{S}^2} \left(\sum_{l=0}^{\infty} \sum_{m=-l}^{l} \psi^{lm}(v)\, Y^{lm}(\theta^1, \theta^2) \right) dv \\ &= \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \int_{v_0}^{v_1} \frac{1}{r} \psi^{lm}(v) \underbrace{\cancel{\Delta}_{\mathbb{S}^2} Y^{lm}(\theta^1, \theta^2)}_{=-l(l+1)Y^{lm}(\theta^1,\theta^2)} dv \\ &= \sum_{l=1}^{\infty} \sum_{m=-l}^{l} -l(l+1) Y^{lm}(\theta^1, \theta^2) \int_{v_0}^{v_1} \frac{1}{r} \psi^{lm}(v)\, dv. \end{aligned} \tag{4.9}$$

Similarly for the left hand side we get

$$\rho(v_0, v_1, \theta^1, \theta^2) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \rho^{lm}(v_0, v_1)\, Y^{lm}(\theta^1, \theta^2). \tag{4.10}$$

Since Y^{lm} is an orthonormal basis of $L^2(\mathbb{S}^2)$, (4.3) is equivalent to: for each $l \geq 0$, $m = -l, \dots, l$ we have

$$\rho^{lm}(v_0, v_1) = -l(l+1) \int_{v_0}^{v_1} \frac{1}{r} \psi^{lm}(v)\, dv. \tag{4.11}$$

The fact that the constant part ρ^{00} is zero is equivalent to ρ being orthogonal to the constant functions. This correlation is further seen in Chapter 5. Our goal is to model initial data of ψ on $\mathcal{H} \cap [v_0, v_1]$ which coincides with the data prescribed at v_0, v_1 and satisfies (4.3). To do so we take a bump function $\chi : [v_0, v_1] \to [0, 1]$ with the following properties:

a) χ is smooth,

b) $\int_{v_0}^{v_1} \frac{1}{r} \chi dv = 1$ and

c) $\mathrm{supp}(\chi) \subset\subset [v_0, v_1] \setminus \{0\}$.

The existence of such a function is covered in Section A.1. Now we choose our functions ψ^{lm} to be the following: $\psi^{00} = 0$ and

$$\psi^{lm}(v) = d^{lm} \chi(v), \tag{4.12}$$

where each $d^{lm} \in \mathbb{R}$ is a constant. Then

$$\rho^{lm}(v_0, v_1) = -l(l+1) \int_{v_0}^{v_1} \frac{1}{r} d^{lm} \chi(v)\, dv = -l(l+1) d^{lm}. \tag{4.13}$$

We can isolate d^{lm}:

$$d^{lm} = \frac{-\rho^{lm}(v_0, v_1)}{l(l+1)} \tag{4.14}$$

and therefore

$$\psi(v, \theta^1, \theta^2) = \chi(v) \sum_{l=1}^{\infty} \sum_{m=-l}^{l} d^{lm} Y^{lm}(\theta^1, \theta^2). \tag{4.15}$$

By construction, this satisfies the desired transport of $\partial_u \psi$ along $\mathcal{H} \cap [v_0, v_1]$ stated in (4.3) and therefore ψ agrees up to first order at v_0 and v_1 transversal to $\mathcal{H}$. Also by choosing ψ to be constant zero in neighborhoods of v_0 and v_1, they match smoothly with the initial data along $\mathcal{H}$. Therefore the data ψ is sufficient for solving the gluing problem. $\square$

Using this explicit construction, we can show not only that the spherical regularity of ψ depends on the regularity of ρ, but that ψ is even more regular than ρ.

Proposition 4.2 *Let $\rho|_{v_0,v_1} \in L^2(\mathbb{S}^2)$. Then $\psi \in H^2(\mathbb{S}^2)$.*

Proof. Note that for $\rho \in L^2(\mathbb{S}^2)$ it is equivalent to

$$\begin{aligned} ||\rho||^2_{L^2(\mathbb{S}^2)} &= \int_{\mathbb{S}^2} \rho^2 d\mu_{\mathbb{S}^2} \\ &= \int_{\mathbb{S}^2} \left(\sum_{l,m} \rho^{lm} Y^{lm}(\theta^1, \theta^2) \right)^2 \mu_{\mathbb{S}^2} \\ &\overset{(1)}{=} \int_{\mathbb{S}^2} \sum_{l,m} \left(\rho^{lm} Y^{lm}(\theta^1, \theta^2) \right)^2 \mu_{\mathbb{S}^2} \\ &\overset{(2)}{=} \sum_{l,m} \left(\rho^{lm} \right)^2 \underbrace{\int_{\mathbb{S}^2} \left(Y^{lm}(\theta^1, \theta^2) \right)^2 \mu_{\mathbb{S}^2}}_{=\left\langle Y^{lm}, Y^{lm} \right\rangle = \delta_{ll}\delta mm = 1} \\ &= \sum_{l,m} \left(\rho^{lm} \right)^2 < \infty, \end{aligned} \tag{4.16}$$

where in (1) we used that of Y^{lm} are orthonormal and in (2) monotone convergence of the partial sum. Now for ψ we get

$$||\psi||^2_{L^2(\mathbb{S}^2)} = \chi^2 \sum_{l,m} \left(d^{lm} \right)^2 = \chi^2 \sum_{l,m} \frac{1}{l^2(l+1)^2} \left(\rho^{lm} \right)^2 \le \chi^2 \sum_{l,m} \left(\rho^{lm} \right)^2 = \underbrace{\chi^2}_{<\infty} ||\rho||^2_{L^2(\mathbb{S}^2)} < \infty. \tag{4.17}$$

$$\begin{aligned} ||\nabla\psi||^2_{L^2(\mathbb{S}^2)} &= \int_{\mathbb{S}^2} \nabla\psi\nabla\psi \, d\mu_{\mathbb{S}^2} \\ &= - \int_{\mathbb{S}^2} \psi \not{\Delta}_{\mathbb{S}^2} \psi \, d\mu_{\mathbb{S}^2} \\ &= - \int_{\mathbb{S}^2} \chi(v) \sum_{l,m} d^{lm} Y^{lm} \not{\Delta}_{\mathbb{S}^2} \left(\chi(v) \sum_{l',m'} d^{l'm'} Y^{l'm'} \right) d\mu_{\mathbb{S}^2} \\ &= - \int_{\mathbb{S}^2} \chi^2(v) \sum_{l,m} d^{lm} Y^{lm} \sum_{l',m'} d^{l'm'} \underbrace{\not{\Delta}_{\mathbb{S}^2} Y^{l'm'}}_{=-l'(l'+1)Y^{l'm'}} d\mu_{\mathbb{S}^2} \\ &= \chi^2(v) \sum_{l,m,l',m'} l'(l'+1) d^{lm} d^{l'm'} \underbrace{\int_{\mathbb{S}^2} Y^{lm} Y^{l'm'} d\mu_{\mathbb{S}^2}}_{=\delta_{l,l'}\delta_{m,m'}} \\ &= \chi^2(v) \sum_{l,m} l(l+1) \left(d^{lm} \right)^2 \\ &= \chi^2(v) \sum_{l,m} \frac{\cancel{l(l+1)}}{l^{\cancel{2}}(l+1)^{\cancel{2}}} \left(\rho^{lm} \right)^2 \le \chi^2(v) \, ||\rho||^2_{L^2} < \infty \end{aligned} \tag{4.18}$$

$$
\begin{aligned}
||\Delta\!\!\!/_{\mathbb{S}^2}\psi||^2_{L^2(\mathbb{S}^2)} &= \int_{\mathbb{S}^2} (\Delta\!\!\!/_{\mathbb{S}^2}\psi)^2 \, d\mu_{\mathbb{S}^2} \\
&= \chi^2(v) \sum_{l,m} \left(d^{lm}\right)^2 l^2(l+1)^2 \int_{\mathbb{S}^2} \left(Y^{lm}\right)^2 d\mu_{\mathbb{S}^2} \\
&= \chi^2(v) \sum_{l,m} \left(\rho^{lm}\right)^2 = \chi^2(v)||\rho||^2_{L^2(\mathbb{S}^2)} < \infty. \qquad (4.19)
\end{aligned}
$$

This implies $\psi \in H^2(\mathbb{S}^2)$. □

To conclude this section, we showed that for data with $\rho \notin \langle 1 \rangle^\perp$, gluing is impossible and for $\rho \in \langle 1 \rangle^\perp$, we have a formula (4.15) depending only on ρ. The fact that ρ must be orthogonal to the constant functions corresponds to the fact that Y^{00} is constant in v. Hence it is insightful to write $\rho \in \langle 1 \rangle^\perp = \langle Y^{00} \rangle^\perp$.

Conservation Laws. In his work [1] Aretakis showed the connection of conservation laws and the characteristic gluing problem for the wave equation on null hypersurfaces. A simple special case is the wave equation in Minkowski spacetime, for which the quantity

$$\int_{\mathbb{S}^2} \partial_u(r\psi)\, d\mu_{\mathbb{S}^2} \qquad (4.20)$$

is conserved along the null hypersurfaces $\{u = c\}$. This conservation is equivalent to

$$\langle \rho, 1 \rangle = \int_{\mathbb{S}^2} \left[\partial_u(r\psi)\right]^{v_1}_{v_0} \, d\mu_{\mathbb{S}^2} = 0. \qquad (4.21)$$

5 A Generalized Linear Wave Equation

Let us now introduce a generalized version of the wave equation. Let $M \in \mathbb{R}$. Then the equation

$$\Box\psi + \frac{M}{r^2}\psi = 0 \tag{5.1}$$

in double null coordinates reads

$$\partial_u\partial_v(r\psi) = \left(\not{\Delta}_{\mathbb{S}^2} + M\right)\frac{\psi}{r}. \tag{5.2}$$

Integrating along $\mathcal{H} = \{u = c\}$ from v_0 to v_1 gives

$$\left[\partial_u(r\psi)\right]_{v_0}^{v_1} = \left(\not{\Delta}_{\mathbb{S}^2} + M\right)\left(\int_{v_0}^{v_1}\frac{\psi}{r}dv\right). \tag{5.3}$$

Note that this holds only if $\left[\partial_u(r\psi)\right]_{v_0}^{v_1} \in \operatorname{Im}\left(\not{\Delta}^M\right)$, where $\not{\Delta}^M := \not{\Delta}_{\mathbb{S}^2} + M$. This operator is still self-adjoint and therefore

$$\operatorname{Im}\left(\not{\Delta}^M\right) = \left(\operatorname{Ker}\left(\not{\Delta}^M\right)\right)^{\perp}. \tag{5.4}$$

But this time the kernel looks different. A function ψ is in the kernel if and only if

$$\not{\Delta}_{\mathbb{S}^2}\psi = -M\psi. \tag{5.5}$$

This holds only for $M = L(L+1)$ for some $L \in \mathbb{N}$ and corresponding Y^{Lm}.

Remark 5.1 *If M does not have this form, the kernel is trivial and the image of $\not{\Delta}^M$ is the entire function space.*

If $M = L(L+1)$, we have

$$\operatorname{Ker}\left(\not{\Delta}^{L(L+1)}\right) = \left\{Y^{Lm} \text{ spherical harmonic function} : m = -L, \dots, L\right\} \tag{5.6}$$

and therefore ρ must be orthogonal to each of those spherical harmonics. Note that the case $M = L(L+1) = 0$ is the original linear wave equation in Chapter 4.

Theorem 5.1 *Consider the generalized linear wave equation $\Box\psi = -\frac{M}{r^2}\psi$ on $\mathcal{H}$ and let ρ be as defined above. If $M \neq L(L+1)$ for all $L \in \mathbb{N}$, then first order gluing has no obstructions and is always possible. If $M = L(L+1)$ for some $L \in \mathbb{N}$ we can perform first order gluing along $\mathcal{H}$ if and only if*

$$\rho \in \left\langle Y^{Lm} \textit{ spherical harmonic function} : m = -L, \dots, L\right\rangle^{\perp}. \tag{5.7}$$

Proof. Let $M \in \mathbb{R}$ and $\rho \in \left(\mathrm{Ker}\left(\slashed{\Delta}^M\right)\right)^\perp$. Without loss of generality, we can assume $\psi \equiv 0$ in neighborhoods around v_0, v_1.

$$\begin{aligned}\int_{v_0}^{v_1} \left(\slashed{\Delta}_{\mathbb{S}^2} + M\right) \frac{\psi}{r} dv &= \int_{v_0}^{v_1} \frac{1}{r} \left(\sum_{l=0}^{\infty} \sum_{m=-l}^{l} \psi^{lm}(v) \left(\slashed{\Delta}_{\mathbb{S}^2} + M\right) Y^{lm}(\theta^1, \theta^2) \right) dv \\ &= \int_{v_0}^{v_1} \frac{1}{r} \left(\sum_{l=0}^{\infty} \sum_{m=-l}^{l} \psi^{lm}(v) \left(-l(l+1) + M\right) Y^{lm}(\theta^1, \theta^2) \right) dv \\ &= \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left(-l(l+1) + M\right) Y^{lm}(\theta^1, \theta^2) \int_{v_0}^{v_1} \frac{1}{r} \psi^{lm}(v) dv. \end{aligned} \tag{5.8}$$

For clarity, we set $\sigma_M^l := -l(l+1) + M$. Let again $\psi^{lm}(v) = \chi(v) d^{lm}$, where χ is the same bump function used before at (4.12). Then

$$\sum_{l=0}^{\infty} \sum_{m=-l}^{l} \sigma_M^l \, Y^{lm}(\theta^1, \theta^2) \int_{v_0}^{v_1} \frac{1}{r} \psi^{lm}(v) dv = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \sigma_M^l \, Y^{lm}(\theta^1, \theta^2) d^{lm} \underbrace{\int_{v_0}^{v_1} \frac{1}{r} \chi(v) dv}_{=1}. \tag{5.9}$$

Using the spherical harmonic decomposition of ρ, we get that for each $l \in \mathbb{N}$, $m = -l, \ldots, l$ it must hold that

$$\rho^{lm} = \sigma_M^l d^{lm}. \tag{5.10}$$

Note that if $M = L(L+1)$ for some $L \in \mathbb{N}$ then $\rho^{Lm} = 0$ for all $m = -L, \ldots, L$.

Remark 5.2 *This is equivalent to* $\rho \perp \ker(\slashed{\Delta}^M)$ *which is, written out using spherical harmonics,*

$$\langle \rho, \phi \rangle = \int_{\mathbb{S}^2} \sum_{l,m} \rho^{lm} Y^{lm} \sum_{l,m} \phi^{lm} Y^{lm} d\mu_{\mathbb{S}^2} = \sum_{l,l',m,m'} \rho^{lm} \phi^{l'm'} \int_{\mathbb{S}^2} Y^{lm} Y^{l'm'} d\mu_{\mathbb{S}^2}, \tag{5.11}$$

where $\phi \in \ker(\slashed{\Delta}^M)$ *and therefore* $\phi^{lm} = 0$ *for* $l \neq L$. *Then the sum reduces to*

$$\sum_{m=-L}^{L} \rho^{Lm} \phi^{Lm} \int_{\mathbb{S}^2} Y^{Lm} Y^{Lm} d\mu_{\mathbb{S}^2} = \sum_{m=-L}^{L} \rho^{Lm} \phi^{Lm}. \tag{5.12}$$

For this to be zero for all $\phi \in \ker(\slashed{\Delta}^M)$, ρ^{Lm} *must be zero for all* $m = -L, \ldots, L$.

Now, if $M \neq L(L+1)$ for all $L \in \mathbb{N}$ we have no problem calculating each d^{lm} and writing out the formula. Hence in this case we can glue arbitrary data. Let now $L \in \mathbb{N}$ such that $M = L(L+1)$. Then for all $l \neq L$ we set

$$d^{lm} = \frac{\rho^{lm}}{-l(l+1) + L(L+1)}. \tag{5.13}$$

Then the gluing data is given by

$$\psi(v, \theta^1, \theta^2) = \chi(v) \sum_{\substack{l \geq 0 \\ l \neq L \\ m=-l,\ldots,l}} d^{lm} Y^{lm}(\theta^1, \theta^2). \tag{5.14}$$

This data satisfies the desired transport of $\partial_u \psi$ along $\mathcal{H} \cap [v_0, v_1]$ and therefore ψ agrees up to first order at v_0 and v_1 transversal to $\mathcal{H}$. Also, by choice of ψ being constant zero in neighborhoods of v_0 and v_1, they match smoothly with the initial data along $\mathcal{H}$. Therefore, the data ψ is sufficient for solving the gluing problem. □

Proposition 5.1 *Let $\rho|_{v_0,v_1} \in L^2(\mathbb{S}^2)$. Then $\psi \in H^2(\mathbb{S}^2)$.*

Proof. Let $\rho|_{v_0,v_1} \in L^2(\mathbb{S}^2)$. This is equivalent to $\sum\limits_{l,m} \left(\rho^{lm}\right)^2 < \infty$. Now

$$||\psi||^2_{L^2(\mathbb{S}^2)} = \chi^2 \sum_{l,m} \left(d^{lm}\right)^2 = \chi^2 \sum_{l,m} \frac{1}{(-l(l+1)+L(L+1))^2} \left(\rho^{lm}\right)^2 < \infty \tag{5.15}$$

and

$$\begin{aligned}
||\nabla\psi||^2_{L^2(\mathbb{S}^2)} &= \int_{\mathbb{S}^2} \nabla\psi\nabla\psi \, d\mu_{\mathbb{S}^2} \\
&= -\int_{\mathbb{S}^2} \psi \not\Delta_{\mathbb{S}^2} \psi \, d\mu_{\mathbb{S}^2} \\
&= \chi^2(v) \sum_{l,m,l',m'} l'(l'+1) d^{lm} d^{l'm'} \underbrace{\int_{\mathbb{S}^2} Y^{lm} Y^{l'm'} d\mu_{\mathbb{S}^2}}_{=\delta_{l,l'}\delta_{m,m'}} \\
&= \chi^2(v) \sum_{l,m} l(l+1) \left(d^{lm}\right)^2 \\
&= \chi^2(v) \sum_{l,m} \frac{l(l+1)}{(-l(l+1)+L(L+1))^2} \left(\rho^{lm}\right)^2 .
\end{aligned} \tag{5.16}$$

We can split this series into a finite sum of finite values and an infinite sum starting at some l where the quotient is smaller than one. Therefore, $||\nabla\psi||^2_{L^2(\mathbb{S}^2)} < \infty$ and $\psi \in H^2(\mathbb{S}^2)$. □

5.1 Higher Order Gluing

For second order gluing we need to prescribe data such that $\partial_u^2\psi$ is transported along $\mathcal{H}$ accordingly to the values at v_0, v_1. To find the necessary and sufficient property to do so, we look at the ∂_u-derivative of the linear wave equation (5.2).

$$\partial_u^2 \partial_v (r\psi) = (\not\Delta_{\mathbb{S}^2} + M) \left(\partial_u \left(\frac{1}{r}\psi\right)\right). \tag{5.17}$$

Integration along $\mathcal{H}$ from v_0 to v_1 gives

$$\left[\partial_u^2 (r\psi)\right]_{v_0}^{v_1} = (\not\Delta_{\mathbb{S}^2} + M) \left(\frac{1}{r}\partial_u\psi + \frac{1}{r^2}\psi\right). \tag{5.18}$$

Once again, the left-hand side should lie in the image of $\not\Delta_{\mathbb{S}^2} + M$ for gluing to be possible. Hence the following generalization of Proposition 5.1:

Proposition 5.2 *Consider the modified linear wave equation $\Box\psi = -\frac{M}{r^2}\psi$ on $\mathcal{H} = \{u = c\}$ and let ρ be as defined in the first order gluing case. Let $\rho^{(q)} = [\partial_u^q(r\psi)]_{v_0}^{v_1}$. If $M \neq L(L+1)$ for all $L \in \mathbb{N}$, then gluing up to any order has no obstructions. If $M = L(L+1)$ for some $L \in \mathbb{N}$ we can perform k-th order gluing along $\mathcal{H}$ if and only if*

$$\rho^{(q)} \in \left\langle Y^{Lm} \text{ spherical harmonic function} : m = -L, \dots, L \right\rangle^{\perp} \tag{5.19}$$

for all $q \leq k$.

Proof. Let $k \geq 0$ be fixed. The idea is once again to decompose ψ using spherical harmonics, and then using cutoff and bump functions to calculate the coefficients. We start by formulating the integral conditions to calculate the coefficients. For each $0 \leq q \leq k$, the ∂_u^{q-1}-derivative of (5.2) is

$$\partial_u^q \partial_v (r\psi) = (\not{\Delta}_{\mathbb{S}^2} + M)\left(\partial_u^{q-1}\left(\frac{1}{r}\psi\right)\right). \tag{5.20}$$

For the right side we have the formula

$$\partial_u^{q-1}\left(\frac{1}{r}\psi\right) = \sum_{j=1}^{q} \frac{(q-1)!}{(q-j)!} \frac{\partial_u^{(q-j)}\psi}{(v-u)^j}. \tag{5.21}$$

By spherical harmonic expansion and the linearity of this equation we can treat each coefficient separately. Then (5.20) turns into

$$\partial_u^q \partial_v (r\psi^{lm}) = \sum_{j=1}^{q} \frac{(-l(l+1)+M)(q-1)!}{(q-j)!} \frac{\partial_u^{(q-j)}\psi^{lm}}{r^j}. \tag{5.22}$$

For clarity, we write $\sigma_M^l = -l(l+1) + M$. Integrating along $\mathcal{H}$ gives

$$\left[\partial_u^q (r\psi^{lm})\right]_{v_0}^{v_1} = \sigma_M^l \sum_{j=1}^{q} \frac{(q-1)!}{(q-j)!} \int_{v_0}^{v_1} \frac{\partial_u^{(q-j)}\psi^{lm}}{r^j} dv. \tag{5.23}$$

If $M = L(L+1)$ for some positive integer L, then for any all $q \leq k$ it must hold that $\left[\partial_u^q(r\psi^{Lm})\right]_{v_0}^{v_1} = 0$. This is the stated orthogonality condition of ρ. The claim is that we can glue all other terms. To do so, let $\chi_0, \chi_1, \bar{\chi} : [v_0, v_1] \to \mathbb{R}^+$ be functions with the following properties:

a) $\chi_0, \chi_1, \bar{\chi}$ are smooth,

b) $\chi_0(v_0) = \chi_1(v_1) = 1$ and $\partial_v^p \chi_0(v_0) = \partial_v^p \chi_1(v_1) = 0$ for any $p \geq 0$,

c) χ_0 and χ_1 are cutoff functions from v_0 and v_1 respectively,

d) $\bar{\chi}$ is a bump function and its support is disjoint from χ_0 and χ_1.

All of them only depend on $\mathcal{H}$ and the interval $[v_0, v_1]$. Now we model $\partial_u^q \psi^{lm}$ to be

$$\partial_u^q \psi^{lm}(v) = \partial_u^q \psi^{lm}|_{v_0} \chi_0(v) + d^{lmq}\bar{\chi}(v) + \partial_u^q \psi^{lm}|_{v_0} \chi_1(v). \tag{5.24}$$

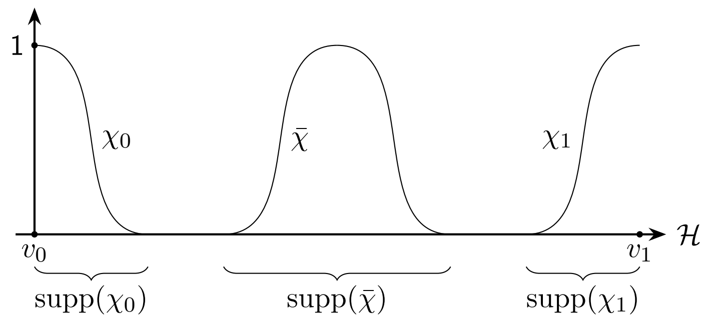

Figure 5.1: Sketch of cutoff functions χ_0, χ_1 and a bump function $\bar{\chi}$

Since every number but d^{lmq} is fixed and prescribed, we can control the integral condition along $\mathcal{H}$ by choosing d^{lmq}. Now the claim is that we can build up solutions recursively by applying the integral condition and isolating each d^{lmq}. If we apply (5.23) to $q = 0$, we get

$$\begin{aligned}\Big[\partial_u(r\psi^{lm})\Big]_{v_0}^{v_1} &= \sigma_M^l \int_{v_0}^{v_1} \frac{\psi^{lm}}{r} dv \\ &= \sigma_M^l \left(\psi^{lm}|_{v_0} \int_{v_0}^{v_1} \frac{\chi_0(v)}{r} dv + d^{lm0} \int_{v_0}^{v_1} \frac{\bar{\chi}(v)}{r} dv + \psi^{lm}|_{v_0} \int_{v_0}^{v_1} \frac{\chi_1(v)}{r} dv \right). \end{aligned} \tag{5.25}$$

Hence

$$d^{lm0} = \frac{\frac{\left[\partial_u(r\psi^{lm})\right]_{v_0}^{v_1}}{\sigma_M^l} - \psi^{lm}|_{v_0} \int_{v_0}^{v_1} \frac{\chi_0(v)}{r} dv - \psi^{lm}|_{v_0} \int_{v_0}^{v_1} \frac{\chi_1(v)}{r} dv}{\int_{v_0}^{v_1} \frac{\bar{\chi}(v)}{r} dv}. \tag{5.26}$$

Now for each $q \geq 1$ one can use (5.23) and (5.24) to recursively calculate all the d^{lmq}, eventually getting to the base case (5.26). Given d^{lmq} one can finally construct sufficient initial data along $\mathcal{H}$, which by construction is smooth along $\mathcal{H}$ and satisfies the characteristic gluing problem. $\square$

6 Nonlinear Wave Equations

In this chapter, we will take a look at the null gluing of some nonlinear wave equations. We will use a similar strategy to achieve the gluing data. Due to nonlinearity, we will encounter some difficulties that need to be overcome. The equations we will look at are

6.1 $\Box\psi = \psi^2$ and

6.2 $\psi r^2 \Box\psi = \partial_u(r\psi)\partial_v(r\psi)$.

6.1 The Case: $\Box\psi = \psi^2$

Consider the nonlinear wave equation

$$\Box\psi = \psi^2. \tag{6.1}$$

In Minkowski, using double null coordinates, this reads

$$-\frac{1}{r}\partial_u\partial_v(r\psi) + \frac{1}{r^2}\not{\Delta}_{\mathbb{S}^2}\psi = \psi^2. \tag{6.2}$$

The approach will again be to decompose ψ using spherical harmonics. But this time the set of equations is not linear anymore. So, it is generally not possible to treat each l, m linear independently as before. To solve this issue, we assume the data $\psi \mid_{\mathcal{H}}$ to only consist in a linear combination of only the first four spherical harmonics, where $l = 0, 1$

$$\psi(v, \theta^1, \theta^2) := \sum_{l=0,1} \sum_{m=-l,0,l} \psi^{lm}(v) Y^{lm}(\theta^1, \theta^2). \tag{6.3}$$

Although this choice simplifies the problem by a lot, the gluing result will be sufficient to cover the important quantities such as mass, linear momentum, angular momentum and center of mass. Furthermore, in this section, we assume $v_0 = 1$ and $v_1 = 2$ and $\mathcal{H} = \{u = 0\}$. Although one could generalize the following results for an arbitrary finite interval $[v_0, v_1]$ away from zero, this assumption will make explicit calculations significantly easier. Before stating the main statement of this section, we make some general observations about the PDE. Note that the right side of (6.2) is always nonnegative. Therefore a solution ψ to (6.2) must satisfy

$$\partial_u\partial_v(r\psi) \leq \frac{1}{r}\not{\Delta}_{\mathbb{S}^2}\psi. \tag{6.4}$$

As before, we introduce the notation

$$\rho^{lm} = [\partial_u(r\psi^{lm})]_1^2. \tag{6.5}$$

If we now apply (6.3) and and integrate over $\mathbb{S}^2$, we get

$$\partial_u\partial_v(r\psi^{00}) \leq 0 \tag{6.6}$$

and hence

$$\rho^{00} \leq 0. \tag{6.7}$$

This will be our first obstruction in the following theorem.

Theorem 6.1 *Consider the nonlinear wave equation (6.2) and let* $\psi(v,\theta^1,\theta^2)=\sum_{l=0}^{1}\sum_{m=-l}^{l}\psi^{lm}(v)Y^{lm}(\theta^1,\theta^2)$*. Let* ρ^{lm} *be as defined above. Then if* $\rho^{00}>0$*, first order gluing is not possible. If* $\rho^{00}\geq 0$ *and if*

$$-\frac{\rho^{00}}{Y^{00}}\geq\sum_{M=-1}^{1}\frac{(\alpha_{00}^{1M}+\rho^{1M})^2}{(\beta^{1M})^2}+\sum_{l=0}^{1}\sum_{m=-l}^{l}\alpha^{lm} \tag{6.8}$$

where α^{lm}*,* α_{00}^{1M} *and* β^{1M} *are defined in (6.22), (6.23) and (6.24) and only depend on initial data of* ψ *and* $\partial_v\psi$ *at* $v=1,2$*, then one can perform first-order gluing along* $\mathcal{H}$*. In addition, if for some* $\varepsilon>0$ *we have*

$$||\psi||^2_{\partial\mathcal{H}_{[1,2]}}:=\sum_{v\in\{1,2\}}||\psi||^2_{L^2(S_v)}+||\partial_v\psi||^2_{L^2(S_v)}+||\partial_u\psi||^2_{L^2(S_v)}<\varepsilon, \tag{6.9}$$

and (6.8) still holds, then there is a constant $K<\infty$*, such that the gluing data satisfies the following estimate*

$$||\psi||^2_{\mathcal{H}_{[1,2]}}<K\varepsilon \tag{6.10}$$

where $||\psi||^2_{\mathcal{H}_{[1,2]}}:=||\psi||^2_{L^2(\mathcal{H}_{[1,2]})}+||\partial_v\psi||^2_{L^2(\mathcal{H}_{[1,2]})}+||\nabla_{\mathbb{S}^2}\psi||^2_{L^2(\mathcal{H}_{[1,2]})}$ *is a norm on* $\mathcal{H}\cap[1,2]$*.*

Remark 6.1 *Essentially, we stated an explicit condition on initial data that allows us to construct sufficient gluing data. In addition, if this boundary data can be estimated using the defined norm, the gluing data can be estimated as well.*

Proof. Note that the statement in (6.9) is equivalent to

$$\sum_{l=0}^{1}\sum_{m=-l}^{l}\sum_{v\in\{1,2\}}|\psi^{lm}(v)|^2+|\partial_v\psi^{lm}(v)|^2+|\partial_u\psi^{lm}(v)|^2<\varepsilon. \tag{6.11}$$

To see that, calculate for example

$$||\psi||^2_{L^2(S_1)}=\int_{S_1}\left(\sum_{l=0}^{1}\sum_{m=-l}^{l}\psi^{lm}(1)Y^{lm}\right)^2d\mu_{\mathbb{S}^2}=\sum_{l=0}^{1}\sum_{m=-l}^{l}(\psi^{lm}(1))^2. \tag{6.12}$$

We first integrate the (6.2) against Y^{LM}. We get for each $L=0,1,M=-L,\dots,L$

$$\begin{aligned}-\frac{1}{r}\partial_u\partial_v\left(r\sum_{l=0}^{1}\sum_{m=-l}^{l}\psi^{lm}\int_{S^2}Y^{LM}Y^{lm}d\mu_{\mathbb{S}^2}\right)+\frac{1}{r^2}\sum_{l=0}^{1}\sum_{m=-l}^{l}\psi^{lm}\int_{S^2}Y^{LM}\underbrace{\Delta\!\!\!\!/_{\mathbb{S}^2}Y^{lm}}_{=-l(l+1)Y^{lm}}d\mu_{\mathbb{S}^2}\\=\sum_{l,l'=0,1}\sum_{\substack{m=-l,\dots,l\\m'=-l',\dots,l}}\psi^{lm}\psi^{l'm'}\int_{S^2}Y^{LM}Y^{lm}Y^{l'm'}d\mu_{\mathbb{S}^2},\end{aligned} \tag{6.13}$$

which simplifies to

$$-\partial_u\partial_v\left(r\psi^{LM}\right)-\frac{1}{r}\psi^{LM}L(L+1)=r\sum_{l,l'=0,1}\sum_{\substack{m=-l,\dots,l\\m'=-l',\dots,l}}\psi^{lm}\psi^{l'm'}\int_{S^2}Y^{LM}Y^{lm}Y^{l'm'}d\mu_{\mathbb{S}^2}. \tag{6.14}$$

The first of four integral conditions arises from plugging in $Y^{LM} = Y^{00}$ into (6.14)

$$-\partial_u\partial_v\left(r\psi^{00}\right) = \sum_{l,l'=0,1}\sum_{\substack{m=-l,\dots,l\\ m'=-l',\dots,l}} r\psi^{lm}\psi^{l'm'}\underbrace{\int_{S^2} Y^{00}Y^{lm}Y^{l'm'}d\mu_{\mathbb{S}^2}}_{Y^{00}\delta_l^{l'}\delta_m^{m'}} = rY^{00}\sum_{l=0}^{1}\sum_{m=-l}^{l}\left(\psi^{lm}\right)^2. \tag{6.15}$$

On the other hand, plugging in $Y^{LM} = Y^{1M}$ yields

$$-\partial_u\partial_v\left(r\psi^{1M}\right) - \frac{2}{r}\psi^{1M} = \sum_{l,l'=0,1}\sum_{\substack{m=-l,\dots,l\\ m'=-l',\dots,l}} r\psi^{lm}\psi^{l'm'}\int_{S^2} Y^{1M}Y^{lm}Y^{l'm'}d\mu_{\mathbb{S}^2}. \tag{6.16}$$

By calculations further explained in A.3 it follows that only the terms where $l \neq l'$ and $m = M$ do not vanish and are equal to $\psi^{00}\psi^{1M}Y^{00}$. Since there are exactly two of these terms we get for each $M = -1, 0, 1$ another integral condition

$$-\partial_u\partial_v\left(r\psi^{1M}\right) = 2Y^{00}r\psi^{00}\psi^{1M} + \frac{2}{r}\psi^{1M}. \tag{6.17}$$

For the following computation, we split our domain $[1,2]$ into four sub intervals $I_m = [\frac{m+5}{6}, \frac{m+6}{6}]$, where $m = 1, \dots, 6$. Now we can explicitly choose four bump functions $\bar{\chi}^{lm}$ and cutoff functions χ_1, χ_2 as described in A. We stipulate the following properties

a) $\bar{\chi}^{lm}, \chi_1$ and χ_2 are nonnegative and smooth,

b) $\mathrm{supp}(\bar{\chi}^{00}) = I_2$, $\mathrm{supp}(\bar{\chi}^{1-1}) = I_3$, $\mathrm{supp}(\bar{\chi}^{10}) = I_4$, $\mathrm{supp}(\bar{\chi}^{11}) = I_5$,

c) $\int_1^2 \left(\bar{\chi}^{lm}\right)^2 r\, dv = 1$,

d) $\chi_1(1) = \chi_2(2) = 1$ and

e) $\mathrm{supp}(\chi_1) = I_1$, $\mathrm{supp}(\chi_2) = I_6$.

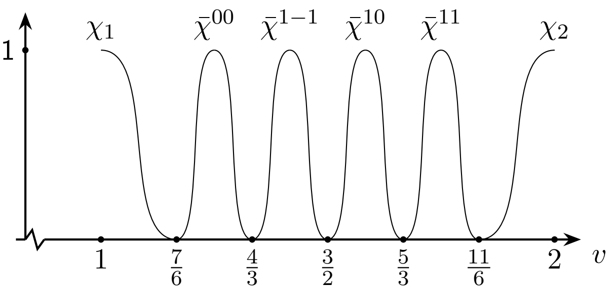

Figure 6.1: Sketch of cutoff functions χ_1, χ_2 and bump functions $\bar{\chi}^{lm}$. Note that by condition c), the height of each $\bar{\chi}$ varies. However, this is just a technical detail.

Remark 6.2 *The choice in c) is rather arbitrary to simplify some calculations later on. It may be chosen differently to optimize some constants of the statement.*

To set up our data on $\mathcal{H}$, we propose

$$\psi^{lm}(v) = \chi_1(v)\psi_1^{lm}(v) + \bar{\chi}^{lm}(v)d^{lm} + \chi_2(v)\psi_2^{lm}(v), \tag{6.18}$$

where $\psi_1^{lm}(v) := \psi^{lm}|_{v=1} + (v-1)\partial_v\psi^{lm}|_{v=1}$ and $\psi_2^{lm}(v) := \psi^{lm}|_{v=2} + (v-2)\partial_v\psi^{lm}|_{v=2}$. Next we apply (6.18) to the right sides of (6.15) and (6.17) and integrate each equation along $\mathcal{H}$ to get the following.

$$\begin{aligned}
-\int_1^2 dv\partial_v\partial_u(r\psi^{00}) &= Y^{00}\sum_{l=0}^{1}\sum_{m=-l}^{l}(d^{lm})^2\underbrace{\int_1^2 r(\bar{\chi}^{lm})^2dv}_{=1} + \int_1^2 r(\chi_1)^2(\psi_1^{lm})^2dv \\
&\quad + \int_1^2 r(\chi_2)^2(\psi_2^{lm})^2dv, &(6.19)\\
-\int_1^2 dv\partial_v\partial_u(r\psi^{1M}) &= d^{1M}\int_1^2 \frac{2}{r}\bar{\chi}^{1M}dv + 2Y^{00}\int_1^2 r(\chi_1)^2\psi_1^{00}\psi_1^{1M}dv \\
&\quad + 2Y^{00}\int_1^2 r(\chi_2)^2\psi_2^{00}\psi_2^{1M}dv + 2\int_1^2\frac{1}{r}\chi_1\psi_1^{1M}dv + 2\int_1^2\frac{1}{r}\chi_2\psi_2^{1M}dv. &(6.20)
\end{aligned}$$

Again, the left-hand side represents the differences of $\psi^{lm}, \partial_u\psi^{lm}$ between $v = 1, 2$. To make our life easier we introduce some notation:

$$\begin{aligned}
\rho^{lm} &= \int_1^2 dv\partial_v\partial_u(r\psi^{lm}) = [\partial_u(r\psi^{lm})]_1^2, &(6.21)\\
\alpha^{lm} &= \int_1^2 r(\chi_1)^2(\psi_1^{lm})^2dv + \int_1^2 r(\chi_2)^2(\psi_2^{lm})^2dv, &(6.22)\\
\alpha_{00}^{1M} &= 2Y^{00}\int_1^2 r(\chi_1)^2\psi_1^{00}\psi_1^{1M}dv + 2Y^{00}\int_1^2 r(\chi_2)^2\psi_2^{00}\psi_2^{1M}dv \\
&\quad + 2\int_1^2\frac{1}{r}\chi_1\psi_1^{1M}dv + 2\int_1^2\frac{1}{r}\chi_2\psi_2^{1M}dv, &(6.23)\\
\beta^{1M} &= \int_1^2\frac{2}{r}\bar{\chi}^{1M}dv \le 2\int_1^2\bar{\chi}^{1M}dv \le 2\left(\int_2^1(\bar{\chi}^{1M})^2dv\right)^{\frac{1}{2}} \le 2\left(\int_2^1(\bar{\chi}^{1M})^2rdv\right)^{\frac{1}{2}} = 2. &(6.24)
\end{aligned}$$

Using this notation, we can more neatly write

$$\begin{aligned}
-\rho^{00} &= Y^{00}\sum_{l=0}^{1}\sum_{m=-l}^{l}(d^{lm})^2 + \alpha^{lm} = Y^{00}((d^{00})^2+\alpha^{00}) + Y^{00}\sum_{M=-1}^{1}(d^{1M})^2 + \alpha^{1M}, &(6.25)\\
-\rho^{1M} &= d^{1M}\beta^{1M} + \alpha_{00}^{1M}, &(6.26)
\end{aligned}$$

from which we can calculate

$$d^{1M} = \frac{-\alpha_{00}^{1M} - \rho^{1M}}{\beta^{1M}} \tag{6.27}$$

$$d^{00} = \sqrt{-\frac{\rho^{00}}{Y^{00}} - \sum_{M=-1}^{1}(d^{1M})^2 - \sum_{l=0}^{1}\sum_{m=-l}^{l}\alpha^{lm}}, \tag{6.28}$$

and with d^{1M} plugged in,

$$d^{00} = \sqrt{-\frac{\rho^{00}}{Y^{00}} - \sum_{M=-1}^{1} \frac{(\alpha_{00}^{1M} + \rho^{1M})^2}{(\beta^{1M})^2} - \sum_{l=0}^{1} \sum_{m=-l}^{l} \alpha^{lm}}. \tag{6.29}$$

For d^{00} to be well defined, we need to make sure that

$$-\frac{\rho^{00}}{Y^{00}} \geq \sum_{M=-1}^{1} \frac{(\alpha_{00}^{1M} + \rho^{1M})^2}{(\beta^{1M})^2} + \sum_{l=0}^{1} \sum_{m=-l}^{l} \alpha^{lm}. \tag{6.30}$$

This is exactly the requirement in (6.8). Hence if it holds our stipulated gluing data works. We still need to show (6.10). We have the norm

$$||\psi||^2_{\mathcal{H}_{[1,2]}} = ||\psi||^2_{L^2(\mathcal{H}_{[1,2]})} + ||\partial_v \psi||^2_{L^2(\mathcal{H}_{[1,2]})} + ||\nabla_{\mathbb{S}^2} \psi||^2_{L^2(\mathcal{H}_{[1,2]})} \tag{6.31}$$

and need to estimate each term. First, note that

$$\begin{aligned}
||\psi||^2_{L^2(\mathcal{H}_{[1,2]})} &= \int_1^2 \int_{S_v} |\psi|^2 d\mu_{\mathbb{S}^2} dv \\
&= \int_1^2 \int_{S_v} \left(\sum_{l=0}^{1} \sum_{m=-l}^{l} \psi^{lm} Y^{lm} \right)^2 d\mu_{\mathbb{S}^2} dv \\
&= \int_1^2 \sum_{l=0}^{1} \sum_{m=-l}^{l} (\psi^{lm})^2 dv \\
&= \sum_{l=0}^{1} \sum_{m=-l}^{l} ||\psi^{lm}||^2_{L^1([1,2])}
\end{aligned} \tag{6.32}$$

and

$$\begin{aligned}
||\nabla_{\mathbb{S}^2} \psi||^2_{L^2(\mathcal{H}_{[1,2]})} &= \int_1^2 \int_{S_v} \nabla_{\mathbb{S}^2} \psi \nabla_{\mathbb{S}^2} \psi \, d\mu_{\mathbb{S}^2} dv \\
&= -\int_1^2 \int_{S_v} \psi \not{\Delta}_{\mathbb{S}^2} \psi \, d\mu_{\mathbb{S}^2} dv \\
&= -\int_1^2 \int_{S_v} \sum_{l=0}^{1} \sum_{m=-l}^{l} \psi^{lm} Y^{lm} \sum_{l'=0}^{1} \sum_{m=-l'}^{l'} \psi^{l'm'} \not{\Delta}_{\mathbb{S}^2} Y^{l'm'} d\mu_{\mathbb{S}^2} dv \\
&= 2 \int_1^2 \sum_{m=-l}^{l} (\psi^{1m})^2 dv \\
&= 2 \sum_{m=-l}^{l} ||\psi^{1m}||^2_{L^1([1,2])}.
\end{aligned} \tag{6.33}$$

Hence, if we show the estimate for each ψ^{lm} and $\partial_v \psi^{lm}$, we are done. But first we write out some more estimates. For $w=1,2$, $l=0,1$, $m=-l,\ldots,l$, $M=-1,0,1$, we have

$$\begin{aligned} |\psi^{lm}_w(v)| &= \left|\psi^{lm}|_{v=w} + (v-1)\partial_v\psi^{lm}|_{v=w}\right| \\ &\leq |\psi^{lm}(w)| + |\partial_v\psi^{lm}(w)| \\ &< 2\sqrt{\varepsilon}, \end{aligned} \tag{6.34}$$

$$|\partial_v\psi^{lm}_w(v)| = |\partial_v\psi^{lm}(w)| < \sqrt{\varepsilon}, \tag{6.35}$$

$$\begin{aligned} \alpha^{1M}_{00} &= 2Y^{00}\int_1^2 r(\chi_1)^2\psi^{00}_1\psi^{1M}_1 dv + 2Y^{00}\int_1^2 r(\chi_2)^2\psi^{00}_2\psi^{1M}_2 dv \\ &\quad + 2\int_1^2 \frac{1}{r}\chi_1\psi^{1M}_1 dv + 2\int_1^2 \frac{1}{r}\chi_2\psi^{1M}_2 dv \\ &\leq 4Y^{00}\int_1^2 \psi^{00}_1\psi^{1M}_1 dv + 4Y^{00}\int_1^2 \psi^{00}_2\psi^{1M}_2 dv + 2\int_1^2 \psi^{1M}_1 dv + 2\int_1^2 \psi^{1M}_2 dv \\ &< 8Y^{00}\varepsilon + 4\sqrt{\varepsilon}, \end{aligned} \tag{6.36}$$

and

$$\begin{aligned} \alpha^{lm} &= \int_1^2 r(\chi_1)^2(\psi^{lm}_1)^2 dv + \int_1^2 r(\chi_2)^2(\psi^{lm}_2)^2 dv, \\ &\leq 2||\psi^{lm}_1||^2_{L^2([1,2])} + 2||\psi^{lm}_2||^2_{L^2([1,2])} \\ &\leq 2|\psi^{lm}(1)|^2 + 2|\partial_v\psi^{lm}(1)|^2 + 2|\psi^{lm}(2)|^2 + 2|\partial_v\psi^{lm}(2)|^2 \\ &< 8\varepsilon. \end{aligned} \tag{6.37}$$

$$|d^{1M}| = \frac{1}{\beta^{1M}}(|\alpha^{1M}_{00}| + |\rho^{1M}|) < \frac{1}{\beta^{1M}}(8Y^{00}\varepsilon + 10\sqrt{\varepsilon}), \tag{6.38}$$

$$|d^{00}| < K\varepsilon. \tag{6.39}$$

By (6.18), we get that there exists a constant K such that

$$||\psi^{lm}||^2_{L^1([1,2])} \leq |d^{lm}|^2 + C(||\psi^{lm}_1||^2_{L^1([1,2])} + ||\psi^{lm}_2||^2_{L^1([1,2])}) < K\varepsilon, \tag{6.40}$$

and also, by the chain rule, we get

$$\begin{aligned} \partial_v\psi^{lm}(v) = \partial_v\chi_1(v)\psi^{lm}_1(v) + \chi_1(v)\partial_v\psi^{lm}_1(v) &+ \partial_v\bar{\chi}^{lm}(v)d^{lm} \\ &+ \partial_v\chi_2(v)\psi^{lm}_2(v) + \chi_2(v)\partial_v\psi^{lm}_2(v), \end{aligned} \tag{6.41}$$

$$||\partial_v\psi^{lm}||^2_{L^1([1,2])} \leq K(||\psi^{lm}_1||^2_\infty + ||\partial_v\psi^{lm}_1||^2_\infty + |d^{lm}|^2 + ||\psi^{lm}_2||^2_\infty + ||\partial_v\psi^{lm}_2||^2_\infty) < K\varepsilon. \tag{6.42}$$

Thus, we showed (6.10) and the proof is complete. □

Remark 6.3 *By using $\rho^{lm} = -[\psi^{lm}]^2_1 + [r\partial_u\psi^{lm}]^2_1$ one can get a sufficient requirement for (6.8), which separates initial data of ψ and $\partial_v\psi$ from $\partial_u\psi$:*

$$\frac{[\psi^{00}]^2_1}{Y^{00}} - \frac{[r\partial_u\psi^{00}]^2_1}{Y^{00}} \geq \sum_{M=-1}^{1}\frac{2(\alpha^{1M}_{00} - [\psi^{1M}]^2_1)^2}{(\beta^{1M})^2} + \sum_{M=-1}^{1}\frac{2([r\partial_u\psi^{1M}]^2_1)^2}{(\beta^{1M})^2} + \sum_{l=0}^{1}\sum_{m=-l}^{l}\alpha^{lm} \tag{6.43}$$

which can be written as

$$-[r\partial_u\psi^{00}]_1^2 - \sum_{M=-1}^{1} \frac{2Y^{00}([r\partial_u\psi^{1M}]_1^2)^2}{(\beta^{1M})^2} \geq \sum_{M=-1}^{1} \frac{2Y^{00}(\alpha_{00}^{1M} - [\psi^{1M}]_1^2)^2}{(\beta^{1M})^2} + \sum_{l=0}^{1}\sum_{m=-l}^{l} Y^{00}\alpha^{lm} - [\psi^{00}]_1^2. \tag{6.44}$$

We name the right-hand side

$$F(\psi^{lm}(1), \psi^{lm}(2), \partial_v\psi^{lm}(1), \partial_v\psi^{lm}(2)) := \sum_{M=-1}^{1} \frac{2Y^{00}(\alpha_{00}^{1M} - [\psi^{1M}]_1^2)^2}{(\beta^{1M})^2} + \sum_{l=0}^{1}\sum_{m=-l}^{l} Y^{00}\alpha^{lm} - [\psi^{00}]_1^2. \tag{6.45}$$

This is an explicit function which takes all the initial data from ψ and $\partial_v\psi$ and returns a number. If this number is smaller than $-[r\partial_u\psi^{00}]_1^2 - \sum_{M=-1}^{1} \frac{2Y^{00}([r\partial_u\psi^{1M}]_1^2)^2}{(\beta^{1M})^2}$, then gluing is possible.

6.2 The Case: $\psi r^2\Box\psi = \partial_u(r\psi)\partial_v(r\psi)$

In this section, we will look at a nonlinear wave equation that is similar to

$$\Box\psi = \frac{1}{\psi r^2}\partial_u(r\psi)\partial_v(r\psi), \tag{6.46}$$

namely

$$r\psi\partial_u\partial_v(r\psi) + \partial_u(r\psi)\partial_v(r\psi) = \psi \not\Delta_{\mathbb{S}^2}\psi. \tag{6.47}$$

If we assume $\psi|_{[v_0,v_1]} > 0$, we can rearrange them into one another. However, the problem with (6.46) is the term $\frac{1}{\psi r^2}$, which excludes $\psi = 0$ as a trivial solution. On the other hand, considering (6.47) lets us construct gluing data close to $\psi = 0$, similar to Section 6.1. Hence, in this section we will consider (6.47) and allow ψ to attain zero. Furthermore, we will consider $\mathcal{H} = \{u = 0\}$ and the interval of interest being $[1, 2]$. Our first trick is to rearrange the left side to achieve the following:

$$\partial_v(\partial_u(r\psi)r\psi) = \psi \not\Delta_{\mathbb{S}^2}\psi. \tag{6.48}$$

We will adopt the use of only four spherical harmonics as in Section 6.1.

$$\psi(v, \theta^1, \theta^2) := \sum_{l=0,1}\sum_{m=-l,0,l} \psi^{lm}(v)Y^{lm}(\theta^1, \theta^2). \tag{6.49}$$

If we plug (6.49) into (6.48) we get

$$\sum_{l,l',m,m'} Y^{lm}Y^{l'm'}\partial_v(\partial_u(r\psi^{lm})r\psi^{l'm'}) = \sum_{l,l',m,m'} -l(l+1)Y^{lm}Y^{l'm'}\psi^{lm}\psi^{l'm'}. \tag{6.50}$$

When integrated over the sphere, this turns into

$$\sum_{l=0}^{1}\sum_{m=-l}^{l} \partial_v(\partial_u(r\psi^{lm})r\psi^{lm}) = -2\sum_{m=-1}^{1}(\psi^{lm})^2. \tag{6.51}$$

Also, if we first multiply by Y^{1M} and then integrate, we get

$$\partial_v(\partial_u(r\psi^{1M})r\psi^{00}) + \partial_v(\partial_u(r\psi^{00})r\psi^{1M}) = -2\psi^{00}\psi^{1M}. \tag{6.52}$$

If we integrate (6.51) and (6.52) along $\mathcal{H}$, we get the integral conditions

$$\sum_{l=0}^{1}\sum_{m=-l}^{l}[\partial_u(r\psi^{lm})r\psi^{lm}]_1^2 = -2\sum_{m=-1}^{1}\int_1^2(\psi^{lm})^2 dv \tag{6.53}$$

$$[\partial_u(r\psi^{1M})r\psi^{00} + \partial_u(r\psi^{00})r\psi^{1M}]_1^2 = -2\int_1^2 \psi^{00}\psi^{1M}.dv \tag{6.54}$$

At this point we did not make any assumptions beside (6.49) but we already see some requirements for the initial data at 1, and 2. To state those requirements we first introduce the notation

$$\tilde{\rho}^{lm} = [\partial_u(r\psi^{lm})r\psi^{lm}]_1^2 \tag{6.55}$$

$$\tilde{\rho}^{1M}_{00} = [\partial_u(r\psi^{1M})r\psi^{00} + \partial_u(r\psi^{00})r\psi^{1M}]_1^2. \tag{6.56}$$

Note that we used $\tilde{\rho}^{lm}$ to emphasize the distinction to the ρ^{lm} used before. We observe the following using (6.53).

1. $\sum_{l=0}^{1}\sum_{m=-l}^{l}\tilde{\rho}^{lm} \leq 0$ and
2. if $\sum_{l=0}^{1}\sum_{m=-l}^{l}\tilde{\rho}^{lm} = 0$ then $\psi^{1m} = 0$ for $m = -1, 0, 1$. But since the sum of all $\tilde{\rho}^{lm}$ is zero, $\tilde{\rho}^{00}$ must also be zero. Then we achieve trivial gluing by setting $\psi \equiv \psi(1)$ constant. Thus, we impose $\sum_{l=0}^{1}\sum_{m=-l}^{l}\tilde{\rho}^{lm} < 0$ to achieve non-trivial gluing in the following theorem.

Theorem 6.2 *Consider the nonlinear partial differential equation (6.47) and let $\psi(v,\theta^1,\theta^2) = \sum_{l=0}^{1}\sum_{m=-l}^{l}\psi^{lm}(v)Y^{lm}(\theta^1,\theta^2)$. Let $\tilde{\rho}^{lm}$ and $\tilde{\rho}^{lm}_{00}$ be as defined above. Then if $\sum_{l=0}^{1}\sum_{m=-l}^{l}\tilde{\rho}^{lm} > 0$, first-order gluing is not possible. If $\sum_{l=0}^{1}\sum_{m=-l}^{l}\tilde{\rho}^{lm} < 0$, the following holds. If*

$$-\sum_{l=0}^{1}\sum_{m=-l}^{l}\tilde{\rho}^{lm} > 2\sum_{m=-1}^{1}\alpha^{1m}, \tag{6.57}$$

where α^{1m} is defined in (6.62) and only depends on values of ψ and $\partial_v\psi$ at $v = 1, 2$, then one can perform first-order gluing along $\mathcal{H}$.

Proof. To setup the gluing data, we use the same cutoff functions χ_0 and χ_1 as in Section 6.1 and three bump functions χ^{1M} with disjoint support. In this section, we impose $\int_1^2(\chi^{1M}(v))^2 dv = 1$. We propose

$$\psi^{00}(v) = \chi_0(v)\psi_1^{00}(v) + d^{00}(\tilde{\chi}^{1-1}(v) + \tilde{\chi}^{10}(v) + \tilde{\chi}^{11}(v)) + \chi_1(v)\psi_2^{00}(v) \tag{6.58}$$

and

$$\psi^{1M}(v) = \chi_0(v)\psi_1^{1M}(v) + d^{1M}\tilde{\chi}^{1M} + \chi_1(v)\psi_2^{1M}(v), \tag{6.59}$$

where $\psi_1^{lm}(v) := \psi^{lm}|_{v=1} + (v-1)\partial_v\psi^{lm}|_{v=1}$ and $\psi_2^{lm}(v) := \psi^{lm}|_{v=2} + (v-2)\partial_v\psi^{lm}|_{v=2}$. Next, we apply this scaffold function to (6.53) and (6.54) to get the following.

$$
\begin{aligned}
-\sum_{l=0}^{1}\sum_{m=-l}^{l}\tilde{\rho}^{lm} = \sum_{m=-1}^{1} 2\int_1^2 (\chi_0(v))^2(\psi_1^{1m}(v))^2 dv + 2(d^{1m})^2 \underbrace{\int_1^2 (\bar{\chi}^{1m})^2 dv}_{=1} \\
+ 2\int_1^2 (\chi_1(v))^2(\psi_2^{1m}(v))^2 dv, \qquad (6.60)\\
-\tilde{\rho}_{00}^{1M} = 2\int_1^2 (\chi_0(v))^2\psi_1^{00}(v)\psi_1^{1M}(v)dv + 2d^{00}d^{1M}\underbrace{\int_1^2 (\bar{\chi}^{1M})^2 dv}_{=1} + \\
2\int_1^2 (\chi_1(v))^2\psi_2^{00}(v)\psi_2^{1M}(v)dv. \qquad (6.61)
\end{aligned}
$$

After introducing some notation

$$\alpha^{lm} = \int_1^2 (\chi_0(v))^2(\psi_1^{1m}(v))^2 dv + \int_1^2 (\chi_1(v))^2(\psi_2^{1m}(v))^2 dv, \qquad (6.62)$$

$$\alpha_{00}^{lm} = \int_1^2 (\chi_0(v))^2\psi_1^{00}(v)\psi_1^{1M}(v)dv + \int_1^2 (\chi_1(v))^2\psi_2^{00}(v)\psi_2^{1M}(v)dv, \qquad (6.63)$$

we arrive at

$$-\sum_{l=0}^{1}\sum_{m=-l}^{l}\tilde{\rho}^{lm} = 2\sum_{m=-1}^{1}(d^{1m})^2 + 2\sum_{m=-1}^{1}\alpha^{1m}, \qquad (6.64)$$

$$-\tilde{\rho}_{00}^{1M} = 2d^{00}d^{1M} + 2\alpha_{00}^{1M}. \qquad (6.65)$$

Now we can calculate

$$d^{1M} = \frac{-\tilde{\rho}_{00}^{1M} - 2\alpha_{00}^{1M}}{2d^{00}} \qquad (6.66)$$

and

$$-\sum_{l=0}^{1}\sum_{m=-l}^{l}\tilde{\rho}^{lm} = \frac{1}{(d^{00})^2}\sum_{m=-1}^{1}\frac{1}{2}(\tilde{\rho}_{00}^{1m} + 2\alpha_{00}^{1m})^2 + 2\sum_{m=-1}^{1}\alpha^{1m}, \qquad (6.67)$$

therefore

$$d^{00} = \sqrt{\frac{-2\sum_{l=0}^{1}\sum_{m=-l}^{l}\tilde{\rho}^{lm} - 4\sum_{m=-1}^{1}\alpha^{1m}}{\sum_{m=-1}^{1}(\tilde{\rho}_{00}^{1m} + 2\alpha_{00}^{1m})^2}} \qquad (6.68)$$

and

$$d^{1M} = (-\tilde{\rho}_{00}^{1M} - 2\alpha_{00}^{1M})\sqrt{\frac{\sum_{m=-1}^{1}(\tilde{\rho}_{00}^{1m} + 2\alpha_{00}^{1m})^2}{-8\sum_{l=0}^{1}\sum_{m=-l}^{l}\tilde{\rho}^{lm} - 16\sum_{m=-1}^{1}\alpha^{1m}}}. \qquad (6.69)$$

In the calculations, we secretly assumed that certain terms do not vanish. Thus, the following statements need justification.

1. In (6.66) $d^{00} \neq 0$: The case of d^{00} being zero can only happen if $\tilde{\rho}_{00}^{1M} = -2\alpha_{00}^{1M}$ for $M = -1, 0, 1$. But then our integral conditions (6.54) are already satisfied. Then ψ^{00} is fixed automatically and we can choose d^{1M} to satisfy (6.64), independently of d^{00}.

2. In (6.68) $\sum_{m=-1}^{1}(\tilde{\rho}_{00}^{1m}+2\alpha_{00}^{1m})^2 \neq 0$: This happens only if $\tilde{\rho}_{00}^{1M} = -2\alpha_{00}^{1M}$. Thus, see above.
3. In (6.69): $\sum_{l=0}^{1}\sum_{m=-l}^{l}\tilde{\rho}^{lm} + 2\sum_{m=-1}^{1}\alpha^{1m} < 0$: This is justified in the following.

This is equivalent to

$$-\sum_{l=0}^{1}\sum_{m=-l}^{l}\tilde{\rho}^{lm} > 2\sum_{m=-1}^{1}\alpha^{1m}. \tag{6.70}$$

This is exactly the requirement in (6.57). We proved, that if this holds, our gluing data is sufficient to solve the gluing problem. $\square$

7 Conclusion

In this work, we studied the characteristic gluing problem in Minkowski spacetime for different types of wave equations. After introducing double null coordinates, we went on calculating the wave operator and stating the gluing problem. Essentially, the gluing problem raises the question whether one can stipulate initial data between two given points, where initial data is given, such that the resulting solution of a given differential equation is well defined. In most cases, one can state gluing obstructions, which are properties of the initial boundary data, that obstruct finding suitable data in between. The first type of wave equation we looked at was the linear homogeneous wave equation. We found that in this case, gluing is possible if and only if the initial boundary data satisfies certain orthogonality conditions, that is, it must be orthogonal to the constant functions in the $L^1(\mathbb{S}^2)$ sense. This result is an explicit special case of the results in [1]. Furthermore, the said orthogonality conditions are equivalent to the conservation of certain quantities, called charges. Throughout this work, we used the method of spherical harmonic decomposition to state a set of integral conditions on our gluing data. Using those conditions, we wrote out explicit formulae for the gluing data. To do so, we used predefined bump and cutoff functions.

After the homogeneous linear case, $\Box\psi = 0$, we took those considerations a step further by adding a linear source term. It turned out that it depends on the source term whether it impacted the gluing property or not. That is, if the source term was equal to some $L(L+1)$, which is the negative of an eigenvalue of the spherical Laplace operator $\not{\Delta}_{\mathbb{S}^2}$, then the initial boundary data must be orthogonal to each spherical harmonic function of order L. On the other hand, if M is any other value, then there is no gluing obstruction for the linear case. Furthermore, we showed that solving the wave equation improved the spherical regularity of the gluing data. That is, if the initial boundary data is in $L^2(\mathbb{S}^2)$, then the constructed gluing data is $H^2(\mathbb{S}^2)$. After that, we looked at the higher order gluing problem for the linear case and stated similar orthogonality properties for the initial boundary data.

After the linear case, we looked at some nonlinear wave equations. The goal was to apply the same methods towards modeling gluing data, while also considering regularity. The nonlinearity lead to higher complexity of the calculations. To counteract this, we assumed that our functions only consist of a linear combination of four spherical harmonics Y^{lm} for $l = 0, 1$, $m = -l, \dots, l$. Although this reduction significantly eases the calculations, there is still enough information to cover basic physical quantities. In particular, we looked at the nonlinear wave equation $\Box\psi = \psi^2$. For this equation, we found an explicit requirement in the initial boundary data that makes gluing possible. Furthermore, we found that, if the said requirement is satisfied, we can not only construct a solution, but the regularity of the initial boundary data is passed onto the gluing data. The second case considered the equation $\psi r^2 \Box\psi = \partial_u(r\psi)\partial_v(r\psi)$. This time again, we found conditions under which gluing is possible, but due to the structure of the constructed gluing data, we did not find a feasible regularity condition.

A Appendix

At many points in this work, we claimed the existence of certain cutoff and bump functions. In this section, we put some rigor into those statements.

A.1 Bump Functions

Throughout this work we used the following types of bump function, which only depend on v_0, v_1 and the constant c defining the null hypersurface $\mathcal{H} = \{u = c\}$.

Lemma A.1 *Let $u = c$ and $[\tilde{v}_0, \tilde{v}_1] \subset\subset [v_0, v_1] \setminus \{0\}$ be an interval. Let also $\alpha \in \mathbb{R}$ and $\beta \geq 1$ be fixed. Suppose $\chi : [v_0, v_1] \to \mathbb{R}$ is a function with the following properties*

a) χ is smooth,

b) $\int_{v_0}^{v_1} r^\alpha \chi^\beta dv = 1$ and

c) $\mathrm{supp}(\chi) = [\tilde{v}_0, \tilde{v}_1]$.

This χ exists.

Proof. Consider

$$\eta(v) = \begin{cases} \exp\left(\frac{1}{(v - \frac{\tilde{v}_1 + \tilde{v}_0}{2})^2 - (\frac{\tilde{v}_1 - \tilde{v}_0}{2})^2}\right) & \text{if } v \in [\tilde{v}_0, \tilde{v}_1] \\ 0 & \text{else.} \end{cases} \tag{A.1}$$

Clearly, the integral of this function is finite. Let $I = \int_{v_0}^{v_1} \eta^\beta(v) dv$. Now let $\chi(v) = \frac{\eta}{(Ir^\alpha)^{\frac{1}{\beta}}}$, where still $r = v - c$. This function χ is still smooth and has the same support as η and now also satisfies b):

$$\int_{v_0}^{v_1} r^\alpha \chi^\beta(v) = \int_{v_0}^{v_1} \frac{\eta^\beta(v)}{I} dv = \frac{\int_{v_0}^{v_1} \eta^\beta(v) dv}{\int_{v_0}^{v_1} \eta^\beta(v) dv} = 1. \tag{A.2}$$

□

A.2 Cutoff Functions

Many times in this work, we used χ_0^q and χ_1^q used the following cutoff functions.

Lemma A.2 *Let $\varepsilon > 0$ and let $\chi_0^\varepsilon, \chi_1^\varepsilon : [v_0, v_1] \to \mathbb{R}^+$ be functions with the following properties:*

a) χ_0^ε and χ_1^ε are smooth,

b) $\chi_0^\varepsilon(v_0) = \chi_1^\varepsilon(v_1) = 1$ and $\partial_v^p \chi_0^\varepsilon(v_0) = \partial_v^p \chi_1^\varepsilon(v_1) = 0$ for any $p \geq 0$,

c) $\sup \chi_0^\varepsilon = [v_0, v_0 + \varepsilon]$ and $\sup \chi_1^\varepsilon = [v_1 - \varepsilon, v_1]$.

These functions χ_0^ε and χ_1^ε do exist.

Proof. We will show this only for χ_0^ε. We get χ_1^ε by mirroring and translating χ_0^ε. Consider

$$\chi_0^\varepsilon(v) = \begin{cases} \exp\left(\frac{\varepsilon^2}{(v-v_0)^2-\varepsilon^2} + 1\right) & \text{if } v \in [v_0, v_0+\varepsilon] \\ 0 & \text{if } v \geq v_0 + \varepsilon. \end{cases} \tag{A.3}$$

Concerning b), we have

$$\chi_0^\varepsilon(v_0) = \exp\left(\frac{\varepsilon^2}{-\varepsilon^2} + 1\right) = 1 \tag{A.4}$$

and

$$\lim_{v \uparrow v_0+\varepsilon} \chi_0^\varepsilon(v) = \lim_{v \uparrow v_0+\varepsilon} \exp\left(\frac{\varepsilon^2}{(v-v_0)^2-\varepsilon^2} + 1\right) = 0. \tag{A.5}$$

For c), we have

$$\partial_v \chi_0^\varepsilon(v)|_{v_0} = \chi_0^\varepsilon(v_0) \frac{2\varepsilon^2(v_0-v_0)}{((v_0-v_0)^2-\varepsilon^2)^2} = 0. \tag{A.6}$$

Since χ_0^ε also smooth it satisfies a), b), and c). $\square$

A.3 Spherical Harmonics

The theory of spherical harmonics is a classic topic and is found in many books and lecture notes. The following brief overview is from *Notes on Spherical Harmonics and Linear Representations of Lie Groups* by Jean Gallier [5].
Let $\triangle_{\mathbb{S}^2}$ be the Laplace operator on the two-dimensional sphere and let $f : \mathbb{S}^2 \to \mathbb{R}$ be a function. In spherical coordinates, the operator reads

$$\triangle_{\mathbb{S}^2} f = \frac{1}{\sin\theta^1}\partial_{\theta^1}\left(\sin\theta^1 \partial_{\theta^1} f\right) + \frac{1}{\sin^2\theta^1}\partial_{\theta^2}\partial_{\theta^2} f. \tag{A.7}$$

We are looking for eigenfunctions of the Laplace operator, namely functions that satisfy the equation

$$\triangle_{\mathbb{S}^2} f = \lambda f. \tag{A.8}$$

For that we have the following results.

Proposition A.1 *The eigenvalues of the Laplace operator on the 2-sphere are* $\{-l(l+1) \mid l \in \mathbb{N}_0\}$*. We call them* λ_l*. Each* λ_l *has multiplicity* $2l+1$*. For distinction of the eigenfunctions we introduce the index* $m = -l, \dots, l$ *and call the corresponding eigenfunctions* Y^{lm} *the real spherical harmonic functions. Furthermore*

a) Y^{lm} *are orthogonal and, by normalization, orthonormal. That is* $\int_{\mathbb{S}^2} Y^{lm} Y^{l'm'} = \delta_l^{l'} \delta_m^{m'}$.

b) For every function $f \in L^2(\mathbb{S}^2)$ *there is a unique series* $\sum_{l=0}^{\infty}\sum_{m=-l}^{l} f^{lm} Y^{lm}$ *converging to* f *in* $L^2(\mathbb{S}^2)$*, where each* f^{lm} *is given by* $f^{lm} = \int_{\mathbb{S}^2} Y^{lm} f$.

Proof. The proof is classical. See also the lecture notes [5]. □

On Triple Products of Spherical Harmonics: In Section 6.1 we claimed that

$$\sum_{l,l',m,m'} \psi^{lm}\psi^{l'm'} \int_{S^2} Y^{1M} Y^{lm} Y^{l'm'} d\mu_{\mathbb{S}^2} = 2Y^{00}\psi^{00}\psi^{1M}. \tag{A.9}$$

The explicit form of the spherical harmonic functions is not unique and depends on various choices. However, a standard choice for the $l = 0, 1$ real spherical harmonic functions is the following

$$Y^{00} = \frac{1}{2\sqrt{\pi}} \tag{A.10}$$

$$Y^{1-1} = \sqrt{\frac{3}{4\pi}} \sin\theta^1 \sin\theta^2 \tag{A.11}$$

$$Y^{10} = \sqrt{\frac{3}{4\pi}} \cos\theta^1 \tag{A.12}$$

$$Y^{11} = \sqrt{\frac{3}{4\pi}} \sin\theta^1 \cos\theta^2. \tag{A.13}$$

The integral is equal to

$$\int_{S^2} Y^{1M} Y^{lm} Y^{l'm'} d\mu_{\mathbb{S}^2} = \int_0^{2\pi}\int_0^{\pi} Y^{1M} Y^{lm} Y^{l'm'} \sin\theta^1 d\theta^1 d\theta^2. \tag{A.14}$$

At first glance we have a symmetry in interchanging (l, m) with (l', m'). Also, the trivial case of $L = l = l' = 0$ is an integral of constant functions and hence zero. By symmetry properties of cos and sin, all integrals concerning $L = l = l' = 1$ are also zero. The first mentioned symmetry, as well as the integral evaluation of the non-trivial terms and their corresponding ψ^{lm} values is seen in the following WOLFRAM MATHEMATICA computation.

```
In[1]:=  Y[0,0,theta_,phi_]  := 1/(2 Sqrt[Pi])
         Y[1,-1,theta_,phi_] := Sqrt[3/(4 Pi)]*Sin[theta]*Sin[phi]
         Y[1,0,theta_,phi_]  := Sqrt[3/(4 Pi)]*Cos[theta]
         Y[1,1,theta_,phi_]  := Sqrt[3/(4 Pi)]*Sin[theta]*Cos[phi]
         Table[Integrate[
          psi[l0,m0] Y[l0,m0,theta,phi] psi[l1,m1]
          Y[l1,m1,theta,phi] Y[1,M,theta,phi] Sin[theta],
          {theta,0,Pi}, {phi,0,2Pi}], {l0,0,1},
          {l1,0,1}, {m0,-l0,l0}, {m1,-l1,l1}, {M,-1,1}]
```

This produces the following table:

$$\begin{pmatrix} \begin{pmatrix} \{0,0,0\} \end{pmatrix} & \begin{pmatrix} \left\{\frac{\psi(0,0)\psi(1,-\mathbf{1})}{2\sqrt{\pi}},0,0\right\} & \left\{0,\frac{\psi(0,0)\psi(1,\mathbf{0})}{2\sqrt{\pi}},0\right\} & \left\{0,0,\frac{\psi(0,0)\psi(1,\mathbf{1})}{2\sqrt{\pi}}\right\} \end{pmatrix} \\ \begin{pmatrix} \left\{\frac{\psi(0,0)\psi(1,-\mathbf{1})}{2\sqrt{\pi}},0,0\right\} \\ \left\{0,\frac{\psi(0,0)\psi(1,\mathbf{0})}{2\sqrt{\pi}},0\right\} \\ \left\{0,0,\frac{\psi(0,0)\psi(1,\mathbf{1})}{2\sqrt{\pi}}\right\} \end{pmatrix} & \begin{pmatrix} \{0,0,0\} & \{0,0,0\} & \{0,0,0\} \\ \{0,0,0\} & \{0,0,0\} & \{0,0,0\} \\ \{0,0,0\} & \{0,0,0\} & \{0,0,0\} \end{pmatrix} \end{pmatrix} \tag{A.15}$$

The table consists of four sub-tables which themselves consist in one to nine sub-tables itself, depending on the indices possible. The upper left cell is the case where $l = 0, l' = 0$ is fixed and $M = -1, 0, 1$. The upper right and lower left cells are the cases where $l = 0, l' = 1$ and $l = 1, l' = 0$ are fixed respectively, and each one consists of three curly brackets corresponding to $m' = -1, 0, 1$ and $m = -1, 0, 1$, respectively. In (A.15), the **bold integers** are equal to M in each case. The bottom right table is for all cases where $L = l = l' = 1$. If we sum all those expressions for each M separately, we get the desired equation

$$\sum_{l,l',m,m'} \psi^{lm}\psi^{l'm'} \int_{S^2} Y^{1M} Y^{lm} Y^{l'm'} d\mu_{\mathbb{S}^2} = 2Y^{00}\psi^{00}\psi^{1M}. \tag{A.16}$$

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Declaration of Originality

I hereby declare that I have written this thesis independently and only with the use of the sources and aids indicated. In particular, literal or analogous quotations are marked as such. I am aware that non-compliance may lead to the subsequent withdrawal of my degree. I confirm that the electronic copy matches the printed copies.

Leipzig, August 8, 2025

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