Econometric Supplement for “Testing functional inequalities conditional on estimated functions”

Andres Aradillas-Lopez[†]

Abstract

This document details the step-by-step derivation of the main econometric results in the paper, along with the details of the examples of estimators that satisfy Assumption 1 in the paper. Every section in this document has the format **SX.X** and every equation has the format **(SX.X.X)**. Any section or equation that we reference here which does not have this format refers to a section or an equation in the main paper or in its online Appendix.

S1 Introduction

Sections S2-S4 are describe the step-by-step results leading to Propositions 1A and 1B in the paper, along with the details of the influence function estimators discussed in Section 3.5.1 of the paper and in Appendix A2. Section S5 describes the details of four types of estimators that satisfy Assumption 1 in the paper.

S2 A useful maximal inequality result

Definition: Euclidean classes of functions

Let $\mathcal{T}$ be a space and d be a pseudometric defined on $\mathcal{T}$. For each $\varepsilon > 0$, define the *packing number* $D(\varepsilon, d, \mathcal{T})$ to be the largest number D for which there exist points $m_1, \ldots, m_D$ in $\mathcal{T}$ such that $d(m_i, m_j) > \varepsilon$ for each $i \neq j$. Packing numbers are a measure of how big $\mathcal{T}$ is with respect to d and they appear in the upper bounds derived in Sherman (1994). The latter result in straightforward expressions for classes of functions $\mathscr{G}$ with a particular type of upper bound for packing numbers. These are Euclidean classes and we present their definition next. First, we say that G is an *envelope* for $\mathscr{G}$ is $\sup_{\mathscr{G}} |g(\cdot)| \leq G(\cdot)$. Let μ be a measure on $\mathcal{S}_Z^k$ and denote $\mu h \equiv \int h(z_1, \ldots, z_k) d\mu(z_1, \ldots, z_k)$. We say that the class of functions $\mathscr{G}$ is Euclidean (A, V) for the envelope G if, for any measure μ such that $\mu G^2 < \infty$, we have

$$D(x, d_\mu, \mathscr{G}) \leq A x^{-V}, \quad 0 < x \leq 1,$$

[†]Department of Economics, Pennsylvania State University, University Park, PA 16802, United States. Email: aaradill@psu.edu

where, for $g_1, g_2 \in \mathscr{G}$,

$$d_\mu(g_1, g_2) = \left(\mu|g_1 - g_2|^2/\mu G^2\right)^{1/2}.$$

The constants A and V must not depend on μ. Immediate-to-verify criteria for determining the Euclidean property have long been established, for example, in Nolan and Pollard (1987) and Pakes and Pollard (1989).

S2.1 A maximal inequality for degenerate U-processes

The following result is taken from Sherman (1994), who obtained maximal inequalities for degenerate U-Processes. Let $Z_1, \ldots, Z_n$ be i.i.d observations from a distribution F on a set $\mathcal{S}_Z$. Let k be a positive integer and $\mathscr{G}$ a class of real-valued functions on $\mathcal{S}_Z^k = \mathcal{S}_Z \otimes \cdots \otimes \mathcal{S}_Z$ (k factors). For each $g \in \mathscr{G}$, define

$$U_n^k g = (n)_k^{-1} \sum_{\boldsymbol{i}_k} g(Z_{i_1}, \ldots, Z_{i_k}),$$

where $(n)_k = n(n-1)\cdots(n-k+1)$ and $\sum_{\boldsymbol{i}_k}$ denotes the sum over the $(n)_k$ distinct integers $\{i_1, \ldots, i_k\}$ from the set $\{1, \ldots, n\}$. $U_n^k g$ is a U-statistic of order k and the collection $\{U_n^k g \colon g \in \mathscr{G}\}$ is called a U-process of order k, indexed by $\mathscr{G}$. If every $g \in \mathscr{G}$ is such that

$$\underbrace{E_F\left[g(s_1, \ldots, s_{i-1}, Z, s_{i+1}, \ldots, s_k)\right] \equiv 0}_{E_F[g(Z_1,\ldots,Z_k)|Z_1=s_1,\ldots,Z_{i-1}=s_{i-1},Z_{i+1}=s_{i+1},\ldots,Z_k=s_k]\equiv 0} \quad , \quad i = 1, \ldots, k,$$

then $\mathscr{G}$ is called an $F-$*degenerate* class of functions on $\mathcal{S}_Z^k$ and $\{U_n^k g \colon g \in \mathscr{G}\}$ is a *degenerate U-process* of order k.

Result S1 ***(Sherman (1994, Corollary 4A))*** *Let $\mathscr{G}$ be a class of $F-$degenerate functions on $\mathcal{S}_Z^k$, $k \geq 1$. Suppose $\mathscr{G}$ is Euclidean (A, V) for an envelope G such that $E_F\left[G(Z_1, \ldots, Z_k)^{4p}\right] < \infty$ for a positive integer p. Then,*

$$E_F\left[\left(\sup_{\mathscr{G}} \left|n^{k/2} U_n^k g\right|\right)^p\right] \leq \Upsilon \cdot \left(E_F\left[G(Z_1, \ldots, Z_k)^{4p}\right]\right)^{1/2} \equiv \overline{M},$$

where Υ is a constant that depends only on p, A, V and $E_F\left[G(Z_1, \ldots, Z_k)^2\right]$. By a Chebyshev inequality, this implies that for each $\varepsilon > 0$,

$$P_F\left(\sup_{\mathscr{G}} \left|n^{k/2} U_n^k g\right| > \varepsilon\right) \leq \frac{\overline{M}}{\varepsilon^p} \quad \textit{and therefore} \quad P_F\left(\sup_{\mathscr{G}} \left|U_n^k g\right| > \varepsilon\right) \leq \frac{\overline{M}}{\left(n^{k/2} \cdot \varepsilon\right)^p}.$$

S3 Propositions 1A and 1B

S3.1 Asymptotic properties of $\widehat{f_g}(x,\widehat{\theta})$

Using Chebyshev's inequality, the conditions in Assumption 1 yield

$$\sup_{F\in\mathcal{F}} P_F\left(\left\|\frac{1}{n}\sum_{i=1}^{n}\psi_F^{\theta}(Z_i)\right\|\geq c\right)=O\left(\frac{1}{\left(n^{1/2}\cdot c\right)^q}\right)\quad\forall\ c>0,$$

and combined with the assumed properties of ε_n^{θ}, this yields,

$$\begin{aligned}\sup_{F\in\mathcal{F}} P_F\left(\left\|\widehat{\theta}-\theta_F^*\right\|\geq c\right)&\leq\sup_{F\in\mathcal{F}} P_F\left(\left\|\frac{1}{n}\sum_{i=1}^{n}\psi_F^{\theta}(Z_i)\right\|\geq\frac{c}{2}\right)+\sup_{F\in\mathcal{F}} P_F\left(\left\|\varepsilon_n^{\theta}\right\|\geq\frac{c}{2}\right)\\&=O\left(\frac{1}{\left(n^{1/2}\cdot c\right)^q}\right)+O\left(\frac{1}{(r_n\cdot c)^q}\right)\quad\forall\ c>0\end{aligned}\tag{S3.1.1}$$

Therefore, for each $0<\delta<\overline{\delta}$,

$$\sup_{F\in\mathcal{F}} P_F\left(\left\|\widehat{\theta}-\theta_F^*\right\|\geq c_n\right)=o\left(\frac{1}{n^{1/2+\delta}}\right)\quad\forall\ c_n:\ \left(n^{1/2}\wedge r_n\right)\cdot n^{-\left(\frac{1+2\delta}{2q}\right)}\cdot c_n\longrightarrow\infty.\tag{S3.1.2}$$

The conditions in Assumption 1 also imply that

$$\left\|\widehat{\theta}-\theta_F^*\right\|=O_p\left(\frac{1}{n^{1/2}}\right)\quad\text{uniformly over }\mathcal{F},\tag{S3.1.3}$$

meaning that, for every $\delta>0$, there exists a finite $M_\delta>0$ and N_δ such that

$$\sup_{F\in\mathcal{F}} P_F\left(n^{1/2}\cdot\left\|\widehat{\theta}-\theta_F^*\right\|\geq M_\delta\right)<\delta\quad\forall\ n\geq N_\delta.$$

For $\{s,d\}$ in $1,\ldots,D$, and $\psi\equiv(\psi_1,\ldots,\psi_D)'$, let

$$\nabla_d K(\psi)\equiv\kappa^{(1)}(\psi_d)\cdot\prod_{\ell\neq d}\kappa(\psi_\ell),\quad\nabla_{ds}K(\psi)\equiv\begin{cases}\kappa^{(1)}(\psi_d)\cdot\kappa^{(1)}(\psi_s)\cdot\prod_{\ell\neq d,s}\kappa(\psi_\ell)&\text{if } d\neq s\\\kappa^{(2)}(\psi_d)\cdot\prod_{\ell\neq d}\kappa(\psi_\ell)&\text{if } d=s\end{cases}\tag{S3.1.4}$$

Also, as we defined previously, for any $x_1\in\mathcal{S}_X$, $x_2\in\mathcal{S}_X$ and $\theta\in\Theta$, let

$$\Delta g_d(x_1,x_2,\theta)\equiv g_d(x_1,\theta)-g_d(x_2,\theta),\quad\text{and}\quad\Delta g(x_1,x_2,\theta)\equiv(\Delta g_1(x_1,x_2,\theta),\ldots,\Delta g_D(x_1,x_2,\theta))'.$$

For a given $x \in \mathcal{S}_X$, $\theta \in \Theta$ and $h > 0$, define

$$\begin{aligned}
\Upsilon^{\ell,m}_{f_g}(X_i,x,\theta,h) &\equiv \sum_{d=1}^{D}\sum_{s=1}^{D} \nabla_{ds}K\left(\frac{\Delta g(X_i,x,\theta)}{h}\right)\frac{\partial \Delta g_d(X_i,x,\theta)}{\partial \theta_\ell}\frac{\partial \Delta g_s(X_i,x,\theta)}{\partial \theta_m},\\
\Phi^{\ell,m}_{f_g}(X_i,x,\theta,h) &\equiv \sum_{d=1}^{D} \nabla_{d}K\left(\frac{\Delta g(X_i,x,\theta)}{h}\right)\frac{\partial^2 \Delta g_d(X_i,x,\theta)}{\partial \theta_m \partial \theta_\ell}
\end{aligned} \tag{S3.1.5}$$

By the conditions described in Assumptions 3 and 4, there exist $\overline{\mu}_\Upsilon < \infty$ and $\overline{\mu}_\Phi < \infty$ such that

$$\sup_{\substack{(s,\theta)\in\mathcal{S}_X\times\Theta\\ h>0}} \left|E_F\left[\Upsilon^{\ell,m}_{f_g}(X,s,\theta,h)\right]\right| \le \overline{\mu}_\Upsilon \quad \text{and} \quad \sup_{\substack{(s,\theta)\in\mathcal{S}_X\times\Theta\\ h>0}} \left|E_F\left[\Phi^{\ell,m}_{f_g}(X,s,\theta,h)\right]\right| \le \overline{\mu}_\Phi \quad \forall\, F \in \mathcal{F}. \tag{S3.1.6}$$

As we stated in the paragraph following Assumption 3, the conditions there, combined with Lemma 2.13 in Pakes and Pollard (1989) imply that, for each $d = 1,\ldots,D$ and $\ell, m \in 1,\ldots,k$, the classes of functions

$$\begin{aligned}
\mathscr{H}^{\ell}_{1d} &= \left\{ m : \mathcal{S}_X \longrightarrow \mathbb{R}\colon\ m(x) = \frac{\partial \Delta g_d(x,s,\theta)}{\partial \theta_\ell} \ \text{ for some } s \in \mathcal{S}_X,\ \theta\in\Theta \right\},\\
\mathscr{H}^{\ell,m}_{2d} &= \left\{ m : \mathcal{S}_X \longrightarrow \mathbb{R}\colon\ m(x) = \frac{\partial^2 \Delta g_d(x,s,\theta)}{\partial \theta_\ell \partial\theta_m} \ \text{ for some } s \in \mathcal{S}_X,\ \theta\in\Theta \right\}
\end{aligned}$$

are Euclidean for an envelope $G_1(x)$ that satisfies $E_F\left[G_1(X)^{4q}\right] \le \overline{\mu}_{G_1} < \infty$ for all $F \in \mathcal{F}$, with q being the integer described in Assumption 1. Now consider the following two classes of functions

$$\begin{aligned}
\mathscr{G}^{\ell,m}_{1} &= \left\{ r : \mathcal{S}_X \longrightarrow \mathbb{R}\colon\ r(x) = \Upsilon^{\ell,m}_{f_g}(x,u,\theta,h) \ \text{ for some } u \in \mathcal{S}_X,\ \theta\in\Theta,\ h>0 \right\},\\
\mathscr{G}^{\ell,m}_{2} &= \left\{ r : \mathcal{S}_X \longrightarrow \mathbb{R}\colon\ r(x) = \Phi^{\ell,m}_{f_g}(x,u,\theta,h) \ \text{ for some } u \in \mathcal{S}_X,\ \theta\in\Theta,\ h>0 \right\},
\end{aligned}$$

By the Euclidean properties of the classes $\mathscr{H}^{\ell}_{1d}$ and $\mathscr{H}^{\ell,m}_{2d}$, and by the conditions described in Assumptions 3 and 4, the Euclidean-preserving properties described, e.g, in Lemma 2.14 in Pakes and Pollard (1989), yield that both $\mathscr{G}^{\ell,m}_{1}$ and $\mathscr{G}^{\ell,m}_{2}$ are Euclidean classes for an envelope $\overline{G}(\cdot)$ for which there exists $\overline{\mu}_{\overline{G}}$ such that $E_F\left[\overline{G}(X)^{4q}\right] \le \overline{\mu}_{\overline{G}}$ for all $F \in \mathcal{F}$, with q being the integer described in Assumption 1. Next, define

$$\begin{aligned}
\nu^{\ell,m}_{\Upsilon_f,n}(x,\theta,h) &= \frac{1}{n}\sum_{i=1}^{n}\left(\Upsilon^{\ell,m}_{f_g}(X_i,x,\theta,h) - E_F\left[\Upsilon^{\ell,m}_{f_g}(X_i,x,\theta,h)\right]\right),\\
\nu^{\ell,m}_{\Phi_f,n}(x,\theta,h) &= \frac{1}{n}\sum_{i=1}^{n}\left(\Phi^{\ell,m}_{f_g}(X_i,x,\theta,h) - E_F\left[\Phi^{\ell,m}_{f_g}(X_i,x,\theta,h)\right]\right),
\end{aligned} \tag{S3.1.7}$$

By the Euclidean property of the classes of functions $\mathscr{G}_1^{\ell,m}$ and $\mathscr{G}_2^{\ell,m}$ and the integrability condition of the corresponding envelope, applying Result S1 we obtain that there exists $\overline{M}_1 < \infty$ such that, for any $b > 0$,

$$\begin{aligned}
\sup_{F\in\mathcal{F}} P_F\left(\sup_{\substack{(x,\theta)\in\mathcal{S}_X\times\Theta\\ h>0}} \left|\nu^{\ell,m}_{\Upsilon_f,n}(x,\theta,h)\right| \geq b\right) &\leq \frac{\overline{M}_1}{\left(n^{1/2}\cdot b\right)^q} = O\left(\frac{1}{\left(n^{1/2}\cdot b\right)^q}\right)\\
\sup_{F\in\mathcal{F}} P_F\left(\sup_{\substack{(x,\theta)\in\mathcal{S}_X\times\Theta\\ h>0}} \left|\nu^{\ell,m}_{\Phi_f,n}(x,\theta,h)\right| \geq b\right) &\leq \frac{\overline{M}_1}{\left(n^{1/2}\cdot b\right)^q} = O\left(\frac{1}{\left(n^{1/2}\cdot b\right)^q}\right)
\end{aligned} \tag{S3.1.8}$$

for each $\{\ell, m\} \in 1,\ldots,k$. Note that (S3.1.8) implies, in particular, that

$$\begin{aligned}
\sup_{\substack{(x,\theta)\in\mathcal{S}_X\times\Theta\\ h>0}} \left|\nu^{\ell,m}_{\Upsilon_f,n}(x,\theta,h)\right| &= O_p\left(\frac{1}{n^{1/2}}\right) \quad \text{uniformly over } \mathcal{F}\\
\sup_{\substack{(x,\theta)\in\mathcal{S}_X\times\Theta\\ h>0}} \left|\nu^{\ell,m}_{\Phi_f,n}(x,\theta,h)\right| &= O_p\left(\frac{1}{n^{1/2}}\right) \quad \text{uniformly over } \mathcal{F}
\end{aligned} \tag{S3.1.9}$$

Equipped with (S3.1.6), (S3.1.8) and (S3.1.9), along with the conditions described in Assumption 1, we can analyze $\widehat{f_g}(g(x,\widehat{\theta})) - \widehat{f_g}(g(x,\theta^*_F))$ over $\mathcal{F}$. Denote

$$\underbrace{\frac{\partial\widehat{f_g}(g(x,\theta))}{\partial\theta}}_{1\times k} = \left(\frac{\partial\widehat{f_g}(g(x,\theta))}{\partial\theta_1},\ldots,\frac{\partial\widehat{f_g}(g(x,\theta))}{\partial\theta_k}\right), \quad \underbrace{\frac{\partial^2\widehat{f_g}(g(x,\theta))}{\partial\theta\partial\theta'}}_{k\times k} = \begin{pmatrix} \frac{\partial^2\widehat{f_g}(g(x,\theta))}{\partial\theta_1^2} & \cdots & \frac{\partial^2\widehat{f_g}(g(x,\theta))}{\partial\theta_1\partial\theta_k}\\ \vdots & \ddots & \vdots\\ \frac{\partial^2\widehat{f_g}(g(x,\theta))}{\partial\theta_k\partial\theta_1} & \cdots & \frac{\partial^2\widehat{f_g}(g(x,\theta))}{\partial\theta_k^2}\end{pmatrix}.$$

For a given $(x,\theta) \in \mathcal{S}_X \times \Theta$ and $F \in \mathcal{F}$, using second-order approximation we have

$$\widehat{f_g}(g(x,\widehat{\theta})) = \widehat{f_g}(g(x,\theta^*_F)) + \frac{\partial\widehat{f_g}(g(x,\theta^*_F))}{\partial\theta}\left(\widehat{\theta}-\theta^*_F\right) + \frac{1}{2}\left(\widehat{\theta}-\theta^*_F\right)'\frac{\partial^2\widehat{f_g}(g(x,\overline{\theta}_x))}{\partial\theta\partial\theta'}\left(\widehat{\theta}-\theta^*_F\right) \tag{S3.1.10}$$

where $\overline{\theta}_x$ belongs in the line segment connecting $\widehat{\theta}$ and θ^*_F. Thus, since Θ is taken to be convex, we have $\overline{\theta}_x \in \Theta$. Let us begin by analyzing the quadratic term; to this end, for a given $(x,\theta) \in \mathcal{S}_X \times \Theta$ and $F \in \mathcal{F}$, let

$$\xi^{f_g}_{b,n}(x,\theta) = \left(\widehat{\theta}-\theta^*_F\right)'\frac{\partial^2\widehat{f_g}(g(x,\theta))}{\partial\theta\partial\theta'}\left(\widehat{\theta}-\theta^*_F\right). \tag{S3.1.11}$$

Going back to the definitions in (S3.1.5)-(S3.1.7), we have

$$
\begin{aligned}
\frac{\partial^2 \widehat{f_g}(g(x,\theta))}{\partial\theta_\ell \partial\theta_m} &= \frac{1}{n\cdot h_n^{D+2}} \sum_{i=1}^{n} \Upsilon_{f_g}^{\ell,m}(X_i,x,\theta,h_n) + \frac{1}{n\cdot h_n^{D+1}} \sum_{i=1}^{n} \Phi_{f_g}^{\ell,m}(X_i,x,\theta,h_n) \\
&= \frac{1}{h_n^{D+2}} \cdot E_F\left[\Upsilon_{f_g}^{\ell,m}(X,x,\theta,h_n)\right] + \frac{1}{h_n^{D+2}} \cdot \nu_{\Upsilon_f,n}^{\ell,m}(x,\theta,h_n) \\
&+ \frac{1}{h_n^{D+1}} \cdot E_F\left[\Phi_{f_g}^{\ell,m}(X,x,\theta,h_n)\right] + \frac{1}{h_n^{D+1}} \cdot \nu_{\Phi_f,n}^{\ell,m}(x,\theta,h_n)
\end{aligned} \tag{S3.1.12}
$$

From here, we have

$$
\begin{aligned}
\xi_{b,n}^{f_g}(x,\theta) &= \sum_{\ell=1}^{k}\sum_{m=1}^{k} \frac{\partial^2 \widehat{f_g}(g(x,\theta))}{\partial\theta_\ell \theta_m} \cdot \left(\widehat{\theta}_m - \theta_{m,F}^*\right) \cdot \left(\widehat{\theta}_\ell - \theta_{\ell,F}^*\right) \\
&= \sum_{\ell=1}^{k}\sum_{m=1}^{k} \frac{1}{h_n^{D+2}} \cdot E_F\left[\Upsilon_{f_g}^{\ell,m}(X,x,\theta,h_n)\right] \cdot \left(\widehat{\theta}_m - \theta_{m,F}^*\right) \cdot \left(\widehat{\theta}_\ell - \theta_{\ell,F}^*\right) \\
&+ \sum_{\ell=1}^{k}\sum_{m=1}^{k} \frac{1}{h_n^{D+2}} \cdot \nu_{\Upsilon_f,n}^{\ell,m}(x,\theta,h_n) \cdot \left(\widehat{\theta}_m - \theta_{m,F}^*\right) \cdot \left(\widehat{\theta}_\ell - \theta_{\ell,F}^*\right) \\
&+ \sum_{\ell=1}^{k}\sum_{m=1}^{k} \frac{1}{h_n^{D+1}} \cdot E_F\left[\Phi_{f_g}^{\ell,m}(X,x,\theta,h_n)\right] \cdot \left(\widehat{\theta}_m - \theta_{m,F}^*\right) \cdot \left(\widehat{\theta}_\ell - \theta_{\ell,F}^*\right) \\
&+ \sum_{\ell=1}^{k}\sum_{m=1}^{k} \frac{1}{h_n^{D+1}} \cdot \nu_{\Phi_f,n}^{\ell,m}(x,\theta,h_n) \cdot \left(\widehat{\theta}_m - \theta_{m,F}^*\right) \cdot \left(\widehat{\theta}_\ell - \theta_{\ell,F}^*\right)
\end{aligned} \tag{S3.1.13}
$$

Take any $b > 0$. From (S3.1.13), we have

$$
\begin{aligned}
P_F\left(\sup_{(x,\theta)\in\mathcal{S}_X\times\Theta}\left|\xi^{f_g}_{b,n}(x,\theta)\right| \geq b\right) \leq{}& \underbrace{\sum_{\ell=1}^{k}\sum_{m=1}^{k} P_F\left(\frac{\overline{\mu}_{\Upsilon}}{h_n^{D+2}}\cdot\left|\widehat{\theta}_m-\theta^*_{m,F}\right|\cdot\left|\widehat{\theta}_\ell-\theta^*_{\ell,F}\right| \geq \frac{b}{4k^2}\right)}_{(A)} \\
&+ \underbrace{\sum_{\ell=1}^{k}\sum_{m=1}^{k} P_F\left(\frac{1}{h_n^{D+2}}\cdot\sup_{\substack{(x,\theta)\in\mathcal{S}_X\times\Theta\\ h>0}}\left|\nu^{\ell,m}_{\Upsilon_f,n}(x,\theta,h)\right|\cdot\left|\widehat{\theta}_m-\theta^*_{m,F}\right|\cdot\left|\widehat{\theta}_\ell-\theta^*_{\ell,F}\right| \geq \frac{b}{4k^2}\right)}_{(B)} \\
&+ \underbrace{\sum_{\ell=1}^{k}\sum_{m=1}^{k} P_F\left(\frac{\overline{\mu}_{\Phi}}{h_n^{D+1}}\cdot\left|\widehat{\theta}_m-\theta^*_{m,F}\right|\cdot\left|\widehat{\theta}_\ell-\theta^*_{\ell,F}\right| \geq \frac{b}{4k^2}\right)}_{(C)} \\
&+ \underbrace{\sum_{\ell=1}^{k}\sum_{m=1}^{k} P_F\left(\frac{1}{h_n^{D+1}}\cdot\sup_{\substack{(x,\theta)\in\mathcal{S}_X\times\Theta\\ h>0}}\left|\nu^{\ell,m}_{\Phi_f,n}(x,\theta,h)\right|\cdot\left|\widehat{\theta}_m-\theta^*_{m,F}\right|\cdot\left|\widehat{\theta}_\ell-\theta^*_{\ell,F}\right| \geq \frac{b}{4k^2}\right)}_{(D)}
\end{aligned}
\tag{S3.1.14}
$$

We will analyze the bounds that result for the terms on the right-hand side of (S3.1.14) from the conditions in Assumptions 1-4. First, note that for any $c > 0$ and any $\{\ell, m\} \in 1,\ldots,k$, we have

$$
\begin{aligned}
P_F\left(\left|\widehat{\theta}_m-\theta^*_{m,F}\right|\cdot\left|\widehat{\theta}_\ell-\theta^*_{\ell,F}\right| \geq c\right) &\leq P_F\left(\left\|\widehat{\theta}-\theta^*_F\right\|^2_\infty \geq c\right) = P_F\left(\left\|\widehat{\theta}-\theta^*_F\right\|_\infty \geq c^{1/2}\right) \\
&\leq P_F\left(\left\|\widehat{\theta}-\theta^*_F\right\| \geq m_k\cdot c^{1/2}\right),
\end{aligned}
$$

where the last inequality follows from the equivalence of norms in Euclidean space and m_k is a constant that depends only on k (the dimension of θ). From here and the conditions in Assumption 1 and the resulting bound in equation (S3.1.2), we can immediately analyze the terms (A) and

(C) in (S3.1.14). Using the result in equation (S3.1.1), we have

$$\sup_{F\in\mathcal{F}} \underbrace{\sum_{\ell=1}^{k}\sum_{m=1}^{k} P_F\left(\frac{\overline{\mu}_\Upsilon}{h_n^{D+2}}\cdot\left|\widehat{\theta}_m-\theta^*_{m,F}\right|\cdot\left|\widehat{\theta}_\ell-\theta^*_{\ell,F}\right|\geq\frac{b}{4k^2}\right)}_{(A)} \leq k^2\cdot\sup_{F\in\mathcal{F}}\ P_F\left(\left\|\widehat{\theta}-\theta^*_F\right\|\geq m_k\cdot\left(\frac{h_n^{D+2}\cdot b}{4\overline{\mu}_\Upsilon k^2}\right)^{1/2}\right)$$

$$= O\left(\frac{1}{\left(n^{1/2}\cdot h_n^{\frac{D+2}{2}}\cdot b^{1/2}\right)^q}\right)+O\left(\frac{1}{\left(r_n\cdot h_n^{\frac{D+2}{2}}\cdot b^{1/2}\right)^q}\right),$$

$$\sup_{F\in\mathcal{F}} \underbrace{\sum_{\ell=1}^{k}\sum_{m=1}^{k} P_F\left(\frac{\overline{\mu}_\Phi}{h_n^{D+1}}\cdot\left|\widehat{\theta}_m-\theta^*_{m,F}\right|\cdot\left|\widehat{\theta}_\ell-\theta^*_{\ell,F}\right|\geq\frac{b}{4k^2}\right)}_{(C)} \leq k^2\cdot\sup_{F\in\mathcal{F}}\ P_F\left(\left\|\widehat{\theta}-\theta^*_F\right\|\geq\left(\frac{h_n^{D+1}\cdot b}{4\overline{\mu}_\Phi k^2}\right)^{1/2}\right)$$

$$= O\left(\frac{1}{\left(n^{1/2}\cdot h_n^{\frac{D+1}{2}}\cdot b^{1/2}\right)^q}\right)+O\left(\frac{1}{\left(r_n\cdot h_n^{\frac{D+1}{2}}\cdot b^{1/2}\right)^q}\right)$$

Next we analyze the terms (B) and (D) in (S3.1.14). Note first that, for any $c>0$ and using the results in equations (S3.1.2) and (S3.1.8), we have

$$\begin{aligned}
&\sup_{F\in\mathcal{F}}\ P_F\left(\sup_{\substack{(x,\theta)\in\mathcal{S}_X\times\Theta\\ h>0}}\left|\nu^{\ell,m}_{\Upsilon_f,n}(x,\theta,h)\right|\cdot\left|\widehat{\theta}_m-\theta^*_{m,F}\right|\cdot\left|\widehat{\theta}_\ell-\theta^*_{\ell,F}\right|\geq c\right)\\
\leq&\sup_{F\in\mathcal{F}}\ P_F\left(\sup_{\substack{(x,\theta)\in\mathcal{S}_X\times\Theta\\ h>0}}\left|\nu^{\ell,m}_{\Upsilon_f,n}(x,\theta,h)\right|\geq c^{1/2}\right)+\sup_{F\in\mathcal{F}}\ P_F\left(\left|\widehat{\theta}_m-\theta^*_{m,F}\right|\cdot\left|\widehat{\theta}_\ell-\theta^*_{\ell,F}\right|\geq c^{1/2}\right)\\
\leq&\sup_{F\in\mathcal{F}}\ P_F\left(\sup_{\substack{(x,\theta)\in\mathcal{S}_X\times\Theta\\ h>0}}\left|\nu^{\ell,m}_{\Upsilon_f,n}(x,\theta,h)\right|\geq c^{1/2}\right)+\sup_{F\in\mathcal{F}}\ P_F\left(\left\|\widehat{\theta}-\theta^*_F\right\|\geq m_k\cdot c^{1/4}\right)\\
=&O\left(\frac{1}{\left(n^{1/2}\cdot c^{1/2}\right)^q}\right)+O\left(\frac{1}{\left(n^{1/2}\cdot c^{1/4}\right)^q}\right)+O\left(\frac{1}{\left(r_n\cdot c^{1/4}\right)^q}\right)
\end{aligned}$$

From here we have that, for any $b > 0$,

$$\underbrace{\sup_{F\in\mathcal{F}} \sum_{\ell=1}^{k}\sum_{m=1}^{k} P_F\left(\frac{1}{h_n^{D+2}} \cdot \sup_{\substack{(x,\theta)\in\mathcal{S}_X\times\Theta \\ h>0}} \left|\nu_{\Upsilon_f,n}^{\ell,m}(x,\theta,h)\right| \cdot \left|\widehat{\theta}_m - \theta^*_{m,F}\right| \cdot \left|\widehat{\theta}_\ell - \theta^*_{\ell,F}\right| \geq \frac{b}{4k^2}\right)}_{(B)}$$

$$= O\left(\frac{1}{\left(n^{1/2}\cdot h_n^{\frac{D+2}{2}}\cdot b^{1/2}\right)^q}\right) + O\left(\frac{1}{\left(n^{1/2}\cdot h_n^{\frac{D+2}{4}}\cdot b^{1/4}\right)^q}\right) + O\left(\frac{1}{\left(r_n\cdot h_n^{\frac{D+2}{4}}\cdot b^{1/4}\right)^q}\right)$$

Similarly, the results in equations (S3.1.2) and (S3.1.8) yield for any $b > 0$,

$$\underbrace{\sup_{F\in\mathcal{F}} \sum_{\ell=1}^{k}\sum_{m=1}^{k} P_F\left(\frac{1}{h_n^{D+1}} \cdot \sup_{\substack{(x,\theta)\in\mathcal{S}_X\times\Theta \\ h>0}} \left|\nu_{\Phi_f,n}^{\ell,m}(x,\theta,h)\right| \cdot \left|\widehat{\theta}_m - \theta^*_{m,F}\right| \cdot \left|\widehat{\theta}_\ell - \theta^*_{\ell,F}\right| \geq \frac{b}{4k^2}\right)}_{(D)}$$

$$= O\left(\frac{1}{\left(n^{1/2}\cdot h_n^{\frac{D+1}{2}}\cdot b^{1/2}\right)^q}\right) + O\left(\frac{1}{\left(n^{1/2}\cdot h_n^{\frac{D+1}{4}}\cdot b^{1/4}\right)^q}\right) + O\left(\frac{1}{\left(r_n\cdot h_n^{\frac{D+1}{4}}\cdot b^{1/4}\right)^q}\right)$$

Next note that, since $h_n \longrightarrow 0$, we have

$$\frac{1}{h_n^{\frac{D+1}{2}}} = \frac{h_n^{1/2}}{h_n^{\frac{D+2}{2}}} = o\left(\frac{1}{h_n^{\frac{D+2}{2}}}\right).$$

Also, for n large enough we have $h_n^{1/2} < h_n^{1/4}$. Therefore, from the results above and (S3.1.14) we obtain that, for any $b > 0$,

$$\sup_{F\in\mathcal{F}} P_F\left(\sup_{(x,\theta)\in\mathcal{S}_X\times\Theta} \left|\xi_{b,n}^{f_g}(x,\theta)\right| \geq b\right) = O\left(\frac{1}{\left(h_n^{\frac{D+2}{2}}\cdot(n^{1/2}\wedge r_n)\cdot(b^{1/2}\wedge b^{1/4})\right)^q}\right) \tag{S3.1.15}$$

Next, going back to (S3.1.12), and using the results in (S3.1.6) and (S3.1.9), we have

$$\begin{aligned}
\sup_{(x,\theta)\in\mathcal{S}_X\times\Theta}\left|\frac{\partial^2\widehat{f_g}(g(x,\theta))}{\partial\theta_\ell\partial\theta_m}\right| &\leq \frac{1}{h_n^{D+2}}\cdot\sup_{(x,\theta)\in\mathcal{S}_X\times\Theta}\left|E_F\left[\Upsilon^{\ell,m}_{f_g}(X,x,\theta,h_n)\right]\right| + \frac{1}{h_n^{D+2}}\cdot\sup_{(x,\theta)\in\mathcal{S}_X\times\Theta}\left|\nu^{\ell,m}_{\Upsilon_f,n}(x,\theta,h_n)\right| \\
&+ \frac{1}{h_n^{D+1}}\cdot\sup_{(x,\theta)\in\mathcal{S}_X\times\Theta}\left|E_F\left[\Phi^{\ell,m}_{f_g}(X,x,\theta,h_n)\right]\right| + \frac{1}{h_n^{D+1}}\cdot\sup_{(x,\theta)\in\mathcal{S}_X\times\Theta}\left|\nu^{\ell,m}_{\Phi_f,n}(x,\theta,h_n)\right| \\
&\leq \frac{1}{h_n^{D+2}}\cdot\overline{\mu}_\Upsilon + \frac{1}{h_n^{D+2}}\cdot O_p\left(\frac{1}{\sqrt{n}}\right) + \frac{1}{h_n^{D+1}}\cdot\overline{\mu}_\Phi + \frac{1}{h_n^{D+1}}\cdot O_p\left(\frac{1}{\sqrt{n}}\right) \\
&\quad \text{uniformly over } \mathcal{F}.
\end{aligned}$$

Therefore, under Assmptions 3 and 4,

$$\sup_{(x,\theta)\in\mathcal{S}_X\times\Theta}\left|\frac{\partial^2\widehat{f_g}(g(x,\theta))}{\partial\theta_\ell\partial\theta_m}\right| = O_p\left(\frac{1}{h_n^{D+2}}\right) \quad \text{uniformly over } \mathcal{F}. \tag{S3.1.16}$$

for all $\{\ell,m\}\in 1,\ldots,k$. Let us go back to the definition of $\xi^{f_g}_{b,n}(x,\theta)$ in (S3.1.11). Combining (S3.1.16) with the result in equation (S3.1.3) (which results from Assumption 1), we have that Assumptions 1-4 yield,

$$\begin{aligned}
\sup_{(x,\theta)\in\mathcal{S}_X\times\Theta}\left|\xi^{f_g}_{b,n}(x,\theta)\right| &\leq \left\|\widehat{\theta}-\theta^*_F\right\|^2 \times \sup_{(x,\theta)\in\mathcal{S}_X\times\Theta}\left\|\frac{\partial^2\widehat{f_g}(g(x,\theta))}{\partial\theta\partial\theta'}\right\| \\
&= O_p\left(\frac{1}{n\cdot h_n^{D+2}}\right) \quad \text{uniformly over } \mathcal{F}.
\end{aligned} \tag{S3.1.17}$$

Let us go back to the second-order approximation in (S3.1.10). We have,

$$\widehat{f_g}(g(x,\widehat{\theta})) = \widehat{f_g}(g(x,\theta^*_F)) + \frac{\partial\widehat{f_g}(g(x,\theta^*_F))}{\partial\theta}\left(\widehat{\theta}-\theta^*_F\right) + \underbrace{\frac{1}{2}\left(\widehat{\theta}-\theta^*_F\right)'\frac{\partial^2\widehat{f_g}(g(x,\overline{\theta}_x))}{\partial\theta\partial\theta'}\left(\widehat{\theta}-\theta^*_F\right)}_{=\xi^{f_g}_{b,n}(x,\overline{\theta}_x)}. \tag{S3.1.18}$$

From (S3.1.15) and (S3.1.17),

$$\begin{aligned}
\sup_{F\in\mathcal{F}} P_F\left(\sup_{x\in\mathcal{S}_{\mathcal{X}}}\left|\xi^{f_g}_{b,n}(x,\overline{\theta}_x)\right|\geq b\right) &\leq \sup_{F\in\mathcal{F}} P_F\left(\sup_{(x,\theta)\in\mathcal{S}_X\times\Theta}\left|\xi^{f_g}_{b,n}(x,\theta)\right|\geq b\right)\\
&= O\left(\frac{1}{\left(h_n^{\frac{D+2}{2}}\cdot(n^{1/2}\wedge r_n)\cdot(b^{1/2}\wedge b^{1/4})\right)^q}\right) \quad \forall\ b>0, \quad \text{and}\\
\sup_{x\in\mathcal{S}_X}\left|\xi^{f_g}_{b,n}(x,\overline{\theta}_x)\right| &\leq \sup_{(x,\theta)\in\mathcal{S}_X\times\Theta}\left|\xi^{f_g}_{b,n}(x,\theta)\right|\\
&= O_p\left(\frac{1}{n\cdot h_n^{D+2}}\right) \quad \text{uniformly over } \mathcal{F}.
\end{aligned} \tag{S3.1.19}$$

Let us now turn our attention to the linear term $\frac{\partial\widehat{f_g}(g(x,\theta^*_F))}{\partial\theta}\left(\widehat{\theta}-\theta^*_F\right)$ in the approximation (S3.1.17). Let $\nabla_d K(\cdot)$ be as defined in (S3.1.4) and for each $\ell=1,\ldots,k$, and a given $x\in\mathcal{S}_X$, $\theta\in\Theta$ and $h>0$, let

$$\begin{aligned}
\Lambda^{\ell}_{f_g}(X_i,x,\theta,h) &\equiv \sum_{d=1}^{D}\nabla_d K\left(\frac{\Delta g(X_i,x,\theta)}{h}\right)\frac{\partial\Delta g_d(X_i,x,\theta)}{\partial\theta_\ell},\\
\underbrace{\Lambda_{f_g}(X_i,x,\theta,h)}_{1\times k} &\equiv \left(\Lambda^{1}_{f_g}(X_i,x,\theta,h),\ldots,\Lambda^{k}_{f_g}(X_i,x,\theta,h)\right)
\end{aligned} \tag{S3.1.20}$$

By the conditions described in Assumptions 3 and 4, there exists $\overline{\mu}_\Lambda<\infty$ such that

$$\sup_{\substack{(x,\theta)\in\mathcal{S}_X\times\Theta\\ h>0}}\left\|E_F\left[\Lambda_{f_g}(X,x,\theta,h)\right]\right\|\leq\overline{\mu}_\Lambda \quad \forall\ F\in\mathcal{F}. \tag{S3.1.21}$$

Using the linear representation property of $\widehat{\theta}-\theta^*_F$ described in Assumption 1, we have

$$\begin{aligned}
\frac{\partial\widehat{f_g}(g(x,\theta^*_F))}{\partial\theta}\left(\widehat{\theta}-\theta^*_F\right) &= \left(\frac{1}{n\cdot h_n^{D+1}}\sum_{i=1}^{n}\Lambda_{f_g}(X_i,x,\theta^*_F,h_n)\right)\left(\widehat{\theta}-\theta^*_F\right)\\
&= \frac{1}{n^2\cdot h_n^{D+1}}\sum_{i=1}^{n}\sum_{j=1}^{n}\Lambda_{f_g}(X_i,x,\theta^*_F,h_n)\psi^{\theta}_F(Z_j)+\left(\frac{1}{n\cdot h_n^{D+1}}\sum_{i=1}^{n}\Lambda_{f_g}(X_i,x,\theta^*_F,h_n)\right)\cdot\varepsilon^{\theta}_n
\end{aligned}$$

Next, define

$$\begin{aligned}
\nu^{\ell}_{\Lambda_{f_g},n}(x,\theta,h) &= \frac{1}{n}\sum_{i=1}^{n}\left(\Lambda^{\ell}_{f_g}(X_i,x,\theta,h)-E_F\left[\Lambda^{\ell}_{f_g}(X_i,x,\theta,h)\right]\right),\\
\underbrace{\nu_{\Lambda_{f_g},n}(x,\theta,h)}_{1\times k} &= \left(\nu^{1}_{\Lambda_{f_g},n}(x,\theta,h),\ldots,\nu^{k}_{\Lambda_{f_g},n}(x,\theta,h)\right)
\end{aligned}$$

and consider the following class of functions on $\mathcal{S}_X$,

$$\mathscr{G}_3^\ell = \left\{ m : \mathcal{S}_X \longrightarrow \mathbb{R}\colon\ m(x) = \Lambda^\ell_{f_g}(x,s,\theta,h) \ \text{ for some } s \in \mathcal{S}_X,\ \theta \in \Theta,\ h > 0 \right\}$$

By the conditions described in Assumptions 3 and 4, and by the Euclidean-preserving properties in Lemma 2.14 in Pakes and Pollard (1989), for all $\ell \in 1,\ldots,k$, $\mathscr{G}_3^\ell$ is a Euclidean class for an envelope $\overline{G}_3(\cdot)$ for which (given the conditions in Assumptions 3 and 4) there exists $\overline{\mu}_{\overline{G}_3}$ such that $E_F\left[\overline{G}_3(X)^{4q}\right] \leq \overline{\mu}_{\overline{G}_3}$ for all $F \in \mathcal{F}$, with q being the integer described in Assumption 1.

By the Euclidean property of the class of functions $\mathscr{G}_3^\ell$ and the integrability property of its envelope, applying Result S1 we obtain that there exists $\overline{M}_3 < \infty$ such that, for any $b > 0$,

$$\sup_{F\in\mathcal{F}} P_F\left(\sup_{\substack{(x,\theta)\in\mathcal{S}_X\times\Theta \\ h>0}} \left\| \nu_{\Lambda_{f_g},n}(x,\theta,h) \right\| \geq b \right) \leq \frac{\overline{M}_3}{\left(n^{1/2}\cdot b\right)^q} = O\left(\frac{1}{\left(n^{1/2}\cdot b\right)^q} \right) \tag{S3.1.22}$$

Note that (S3.1.22) implies, in particular, that

$$\sup_{\substack{(x,\theta)\in\mathcal{S}_X\times\Theta \\ h>0}} \left\| \nu_{\Lambda_{f_g},n}(x,\theta,h) \right\| = O_p\left(\frac{1}{n^{1/2}}\right) \quad \text{uniformly over } \mathcal{F} \tag{S3.1.23}$$

Now, consider the following class of functions on $\mathcal{S}_V^2$,

$$\mathscr{G}_{4,F} = \left\{ m : \mathcal{S}_V^2 \longrightarrow \mathbb{R}\colon\ m(v_1,v_2) = \Lambda_{f_g}(x_1,s,\theta,h)\psi_F^\theta(z_2) \ \text{ for some } s \in \mathcal{S}_X,\ \theta \in \Theta,\ h > 0 \right\}$$

By the conditions described in Assumptions 1, 3 and 4, and by the Euclidean-preserving properties in Lemma 2.14 in Pakes and Pollard (1989), $\mathscr{G}_4$ is a Euclidean class for an envelope $\overline{G}_4(\cdot)$ for which (given the conditions in Assumptions 3 and 4) there exists $\overline{\mu}_{\overline{G}_4}$ such that $E_F\left[\overline{G}_4(X_1,Z_2)^{4q}\right] \leq \overline{\mu}_{\overline{G}_4}$ for all $F \in \mathcal{F}$ (with $(X_1,Z_2) \sim F\otimes F$), with q being the integer described in Assumption 1. Define the following U-statistic

$$U_{f_g,n}(x,\theta,h) = \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j\neq i} \Lambda_{f_g}(X_i,x,\theta,h)\psi_F^\theta(Z_j) \tag{S3.1.24}$$

Note that, since $E_F\left[\psi_F^\theta(Z)\right] = 0$, using iterated expectations we have $E_F\left[\Lambda_{f_g}(X_i,x,h)\psi_F^\theta(Z_j)\right] = 0$ and therefore,

$$E_F\left[U_{f_g,n}(x,\theta,h)\right] = 0 \ \forall\ x \in \mathcal{S}_X,\ \theta \in \Theta,\ h > 0.$$

We will characterize the relevant properties of the U-process $\left\{U_{f_g,n}(x,\theta,h)\colon\ x \in \mathcal{S}_X,\ \theta \in \Theta,\ h > 0\right\}$ by first looking at its Hoeffding decomposition (see Serfling (1980, pages 177-178) or Sherman (1994, equations (6)-(7))). For a given $s \in \mathcal{S}_X$, $\theta \in \Theta$ and $h > 0$, let

$$
\begin{aligned}
\varphi_F^{f_g}(Z_i, x, \theta, h) &= E_F\Big[\Lambda_{f_g}(X, x, \theta, h)\Big]\psi_F^{\theta}(Z_i),\\
\vartheta_F(V_i, V_j, x, \theta, h) &= \Lambda_{f_g}(X_i, x, h)\psi_F^{\theta}(Z_j) + \Lambda_{f_g}(X_j, x, h)\psi_F^{\theta}(Z_i) - \varphi_F^{f_g}(Z_i, x, \theta, h) - \varphi_F^{f_g}(Z_j, x, \theta, h)
\end{aligned}
\tag{S3.1.25}
$$

Note that $\vartheta_F^{f_g}(V_i, V_j, x, \theta, h) = \vartheta_F^{f_g}(V_j, V_i, x, \theta, h)$ and $E_F\Big[\vartheta_F^{f_g}(V_i, V_j, x, \theta, h)|V_i\Big] = E_F\Big[\vartheta_F^{f_g}(V_i, V_j, x, \theta, h)|V_j\Big] = 0$. Define the following degenerate U-statistic of order 2,

$$
\widetilde{U}_{f_g,n}(x, \theta, h) = \frac{1}{n(n-1)}\sum_{i=1}^{n}\sum_{j\neq i}\vartheta_F^{f_g}(V_i, V_j, s, \theta, h)
$$

From the conditions described in Assumptions 1-4, Lemma 20 in Nolan and Pollard (1987) (or Lemma 5 in Sherman (1994)) combined with the Euclidean-preserving properties in Lemma 2.14 in Pakes and Pollard (1989), the class of functions

$$
\mathscr{G}_{5,F} = \Big\{m : \mathcal{S}_V^2 \longrightarrow \mathbb{R}\colon\ m(v_1, v_2) = \vartheta_F^{f_g}(v_1, v_2, x, \theta, h)\ \text{ for some } x \in \mathcal{S}_X,\ \theta \in \Theta,\ h > 0\Big\}
$$

is Euclidean for an envelope $\overline{G}_5(\cdot)$ that satisfies $E_F\Big[\overline{G}_5(V_1, V_2)^{4q}\Big] \leq \overline{\mu}_{\overline{G}_5}$ for all $F \in \mathcal{F}$ (with $(V_1, V_2) \sim F\otimes F$) and q being the integer described in Assumption 1. From here, applying Result S1 we obtain that there exists $\overline{M}_5 < \infty$ such that, for any $b > 0$,

$$
\begin{aligned}
&\sup_{F\in\mathcal{F}}\ P_F\left(\sup_{\substack{(x,\theta)\in\mathcal{S}_X\times\Theta\\ h>0}}\left|\widetilde{U}_{f_g,n}(x, \theta, h)\right| \geq b\right) \leq \frac{\overline{M}_5}{(n\cdot b)^q} = O\left(\frac{1}{(n\cdot b)^q}\right),\ \text{ and therefore,}\\
&\sup_{\substack{(x,\theta)\in\mathcal{S}_X\times\Theta\\ h>0}}\left|\widetilde{U}_{f_g,n}(x, \theta, h)\right| = O_p\left(\frac{1}{n}\right) \quad \text{uniformly over } \mathcal{F}
\end{aligned}
\tag{S3.1.26}
$$

Let $\varphi_F^{f_g}(Z_i, x, \theta, h)$ be as described in (S3.1.25) and note that $E_F\left[\varphi_F^{f_g}(Z_i, x, \theta, h)\right] = 0$ for all $x \in \mathcal{S}_X$, $\theta \in \Theta$ and $h > 0$. Define

$$
\nu_{\varphi_{f_g},n}(x, \theta, h) = \frac{1}{n}\sum_{i=1}^{n}\varphi_F^{f_g}(Z_i, x, \theta, h) \tag{S3.1.27}
$$

Note that $E_F\left[\varphi_F^{f_g}(Z_i, x, \theta, h)\right] = 0$ (and therefore, $E_F\left[\nu_{\varphi_{f_g},n}(x, \theta, h)\right] = 0$) for all $x \in \mathcal{S}_X$, $\theta \in \Theta$ and $h > 0$. Again, the conditions in Assumptions 1-4, Lemma 20 in Nolan and Pollard (1987) (or Lemma 5 in Sherman (1994)) combined with the Euclidean-preserving properties in Lemma 2.14 in Pakes and Pollard (1989) imply that the class of functions

$$
\mathscr{G}_{6,F} = \Big\{m : \mathcal{S}_Z \longrightarrow \mathbb{R}\colon\ m(z) = \varphi_F^{f_g}(z, x, \theta, h)\ \text{ for some } x \in \mathcal{S}_X,\ \theta \in \Theta,\ h > 0\Big\}
$$

is Euclidean for an envelope $\overline{G}_6(\cdot)$ that satisfies $E_F\left[\overline{G}_6(Z)^{4q}\right] \leq \overline{\mu}_{\overline{G}_6}$ for all $F \in \mathcal{F}$ and q being the integer described in Assumption 1. From here, applying Result S1 we obtain that there exists $\overline{M}_6 < \infty$ such that, for any $b > 0$,

$$\sup_{F \in \mathcal{F}} P_F\left(\sup_{\substack{(x,\theta)\in\mathcal{S}_X\times\Theta \\ h>0}} \left|\nu_{\varphi_{f_g},n}(x,\theta,h)\right| \geq b\right) \leq \frac{\overline{M}_6}{\left(n^{1/2}\cdot b\right)^q} = O\left(\frac{1}{\left(n^{1/2}\cdot b\right)^q}\right), \text{ and therefore,}$$
$$\sup_{\substack{(x,\theta)\in\mathcal{S}_X\times\Theta \\ h>0}} \left|\nu_{\varphi_{f_g},n}(x,\theta,h)\right| = O_p\left(\frac{1}{n^{1/2}}\right) \quad \text{uniformly over } \mathcal{F} \tag{S3.1.28}$$

Equipped with the results in (S3.1.26) and (S3.1.28), let us look at the Hoeffding decomposition of the U-statistic $U_{f_g,n}(x,\theta,h)$ defined in (S3.1.24). This is given by (see Serfling (1980, pages 177-178) or Sherman (1994, equations (6)-(7))),

$$U_{f_g,n}(x,\theta,h) = \nu_{\varphi_{f_g},n}(x,\theta,h) + \frac{1}{2}\cdot \widetilde{U}_{f_g,n}(x,\theta,h) \tag{S3.1.29}$$

Therefore, from (S3.1.26) and (S3.1.28), for any $b > 0$ we have

$$\begin{aligned}
&\sup_{F \in \mathcal{F}} P_F\left(\sup_{\substack{(x,\theta)\in\mathcal{S}_X\times\Theta \\ h>0}} \left|U_{f_g,n}(x,\theta,h)\right| \geq b\right) \\
\leq&\sup_{F \in \mathcal{F}} P_F\left(\sup_{\substack{(x,\theta)\in\mathcal{S}_X\times\Theta \\ h>0}} \left|\nu_{\varphi_{f_g},n}(x,\theta,h)\right| \geq \frac{b}{2}\right) + \sup_{F \in \mathcal{F}} P_F\left(\frac{1}{2}\cdot\sup_{\substack{(x,\theta)\in\mathcal{S}_X\times\Theta \\ h>0}} \left|\widetilde{U}_{f_g,n}(x,\theta,h)\right| \geq \frac{b}{2}\right) \\
=&O\left(\frac{1}{(n\cdot b)^q}\right) + O\left(\frac{1}{\left(n^{1/2}\cdot b\right)^q}\right) = O\left(\frac{1}{\left(n^{1/2}\cdot b\right)^q}\right)
\end{aligned} \tag{S3.1.30}$$

Equipped with these results, let us go back to the second-order approximation in (S3.1.18). For a given $(x,\theta) \in \mathcal{S}_X \times \Theta$ and $F \in \mathcal{F}$, let

$$\xi^{f_g}_{a,n}(x,\theta) = \frac{\partial \widehat{f}_g(g(x,\theta))}{\partial\theta}\left(\widehat{\theta} - \theta^*_F\right).$$

We have,

$$\widehat{f}_g(g(x,\widehat{\theta})) = \widehat{f}_g(g(x,\theta^*_F)) + \underbrace{\frac{\partial \widehat{f}_g(g(x,\theta^*_F))}{\partial\theta}\left(\widehat{\theta} - \theta^*_F\right)}_{=\xi^{f_g}_{a,n}(x,\theta^*_F)} + \underbrace{\frac{1}{2}\left(\widehat{\theta} - \theta^*_F\right)' \frac{\partial^2 \widehat{f}_g(g(x,\overline{\theta}_x))}{\partial\theta\partial\theta'}\left(\widehat{\theta} - \theta^*_F\right)}_{=\xi^{f_g}_{b,n}(x,\overline{\theta}_x)}. \tag{S3.1.31}$$

The properties of the quadratic term $\xi^{f_g}_{b,n}(x,\overline{\theta}_x)$ were analyzed previously and summarized in (S3.1.15) and (S3.1.17). We can now study the properties of the linear term $\xi^{f_g}_{a,n}(x,\theta^*_F)$. Before proceeding there is only one more process to consider, whose properties will follow from our previous results. Let

$$m^{f_g}_n(x,\theta,h) = \frac{1}{n}\sum_{i=1}^{n}\Big(\Lambda_{f_g}(X_i,x,\theta,h)\psi^{\theta}_F(Z_i) - E_F\Big[\Lambda_{f_g}(X_i,x,\theta,h)\psi^{\theta}_F(Z_i)\Big]\Big)$$

Note that, under our assumptions, there exists $\overline{\mu}_m < \infty$ such that

$$\sup_{\substack{(x,\theta)\in\mathcal{S}_X\times\Theta\\ h>0}}\Big|E_F\Big[\Lambda_{f_g}(X_i,x,\theta,h)\psi^{\theta}_F(Z_i)\Big]\Big| \leq \overline{\mu}_m \quad \forall\ F\in\mathcal{F}. \tag{S3.1.32}$$

Our previous arguments and Euclidean properties of the class of functions $\mathscr{G}_{4,F}$ imply, once again, through Result S1,

$$\begin{aligned}&\sup_{F\in\mathcal{F}} P_F\left(\sup_{\substack{(x,\theta)\in\mathcal{S}_X\times\Theta\\ h>0}}\Big|m^{f_g}_n(x,\theta,h)\Big|\geq b\right) = O\left(\frac{1}{\left(n^{1/2}\cdot b\right)^q}\right) \quad \forall\ b>0, \quad \text{and therefore,}\\ &\sup_{\substack{(x,\theta)\in\mathcal{S}_X\times\Theta\\ h>0}}\Big|m^{f_g}_n(x,\theta,h)\Big| = O_p\left(\frac{1}{n^{1/2}}\right) \quad \text{uniformly over } \mathcal{F}\end{aligned} \tag{S3.1.33}$$

We have

$$\begin{aligned}\xi^{f_g}_{a,n}(x,\theta) &= \left(\frac{1}{n\cdot h_n^{D+1}}\sum_{i=1}^{n}\Lambda_{f_g}(X_i,x,\theta,h_n)\right)\left(\widehat{\theta}-\theta^*_F\right)\\ &= \frac{1}{n^2\cdot h_n^{D+1}}\sum_{i=1}^{n}\sum_{j=1}^{n}\Lambda_{f_g}(X_i,x,\theta,h_n)\psi^{\theta}_F(Z_j) + \left(\frac{1}{n\cdot h_n^{D+1}}\sum_{i=1}^{n}\Lambda_{f_g}(X_i,x,\theta,h_n)\right)\varepsilon^{\theta}_n\\ &= \frac{1}{h_n^{D+1}}\cdot\left(\frac{n-1}{n}\right)\cdot U_{f_g,n}(x,\theta,h_n)\\ &+ \frac{1}{n\cdot h_n^{D+1}}\cdot E_F\Big[\Lambda_{f_g}(X,x,\theta,h_n)\psi^{\theta}_F(Z)\Big] + \frac{1}{n\cdot h_n^{D+1}}\cdot m^{f_g}_n(x,\theta,h_n)\\ &+ \frac{1}{h_n^{D+1}}\cdot E_F\Big[\Lambda_{f_g}(X,x,\theta,h_n)\Big]\varepsilon^{\theta}_n + \frac{1}{h_n^{D+1}}\cdot \nu_{\Lambda_{f_g},n}(x,\theta,h_n)\varepsilon^{\theta}_n\end{aligned} \tag{S3.1.34}$$

From (S3.1.21) and (S3.1.32), for any $F \in \mathcal{F}$ we have,

$$\left|\xi_{a,n}^{f_g}(x,\theta)\right| \leq \frac{1}{h_n^{D+1}} \cdot \left(\frac{n-1}{n}\right) \cdot \left|U_{f_g,n}(x,\theta,h_n)\right| + \frac{1}{n \cdot h_n^{D+1}} \cdot \left|m_n^{f_g}(x,\theta,h_n)\right| + \frac{1}{h_n^{D+1}} \cdot \left\|\nu_{\Lambda_{f_g},n}(x,\theta,h_n)\right\| \cdot \left\|\varepsilon_n^{\theta}\right\|$$
$$+ \frac{\overline{\mu}_{\Lambda}}{h_n^{D+1}} \cdot \left\|\varepsilon_n^{\theta}\right\| + \frac{\overline{\mu}_m}{n \cdot h_n^{D+1}}.$$

Thus, for any $b > 0$,

$$P_F\left(\left|\xi_{a,n}^{f_g}(x,\theta)\right| \geq b\right)$$
$$\leq P_F\left(\frac{1}{h_n^{D+1}} \cdot \left(\frac{n-1}{n}\right) \cdot \left|U_{f_g,n}(x,\theta,h_n)\right| + \frac{1}{n \cdot h_n^{D+1}} \cdot \left|m_n^{f_g}(x,\theta,h_n)\right| + \frac{1}{h_n^{D+1}} \cdot \left\|\nu_{\Lambda_{f_g},n}(x,\theta,h_n)\right\| \cdot \left\|\varepsilon_n^{\theta}\right\| \right.$$
$$\left. + \frac{1}{h_n^{D+1}} \cdot \overline{\mu}_{\Lambda} \cdot \left\|\varepsilon_n^{\theta}\right\| \geq b - \frac{\overline{\mu}_m}{n \cdot h_n^{D+1}}\right)$$

Therefore,

$$\begin{aligned}
\sup_{F\in\mathcal{F}} P_F\left(\sup_{(x,\theta)\in\mathcal{S}_X\times\Theta}\left|\xi_{a,n}^{f_g}(x,\theta)\right| \geq b\right) \leq & \underbrace{\sup_{F\in\mathcal{F}} P_F\left(\frac{1}{h_n^{D+1}} \cdot \left(\frac{n-1}{n}\right) \cdot \sup_{\substack{(x,\theta)\in\mathcal{S}_X\times\Theta \\ h>0}}\left|U_{f_g,n}(x,\theta,h)\right| \geq \frac{1}{4} \cdot \left(b - \frac{\overline{\mu}_m}{n \cdot h_n^{D+1}}\right)\right)}_{(A)} \\
& + \underbrace{\sup_{F\in\mathcal{F}} P_F\left(\frac{1}{n \cdot h_n^{D+1}} \cdot \sup_{\substack{(x,\theta)\in\mathcal{S}_X\times\Theta \\ h>0}}\left|m_n^{f_g}(x,\theta,h)\right| \geq \frac{1}{4} \cdot \left(b - \frac{\overline{\mu}_m}{n \cdot h_n^{D+1}}\right)\right)}_{(B)} \\
& + \underbrace{\sup_{F\in\mathcal{F}} P_F\left(\frac{1}{h_n^{D+1}} \cdot \sup_{\substack{(x,\theta)\in\mathcal{S}_X\times\Theta \\ h>0}}\left\|\nu_{\Lambda_{f_g},n}(x,\theta,h)\right\| \cdot \left\|\varepsilon_n^{\theta}\right\| \geq \frac{1}{4} \cdot \left(b - \frac{\overline{\mu}_m}{n \cdot h_n^{D+1}}\right)\right)}_{(C)} \\
& + \underbrace{\sup_{F\in\mathcal{F}} P_F\left(\frac{\overline{\mu}_{\Lambda}}{h_n^{D+1}} \cdot \left\|\varepsilon_n^{\theta}\right\| \geq \frac{1}{4} \cdot \left(b - \frac{\overline{\mu}_m}{n \cdot h_n^{D+1}}\right)\right)}_{(D)}
\end{aligned} \tag{S3.1.35}$$

Using our previous results we can analyze each of the terms on the right hand side of (S3.1.35).

If $b > 0$ is fixed[1], there exists n_0 such that $b - \frac{\overline{\mu}_m}{n \cdot h_n^{D+1}} > 0 \ \forall \ n > n_0$. From (S3.1.30), it follows that

$$\begin{aligned}
&\sup_{F\in\mathcal{F}} P_F\left(\frac{1}{h_n^{D+1}}\cdot\left(\frac{n-1}{n}\right)\cdot \sup_{\substack{(x,\theta)\in\mathcal{S}_X\times\Theta\\ h>0}}\left|U_{f_g,n}(x,\theta,h)\right| \geq \frac{1}{4}\cdot\left(b-\frac{\overline{\mu}_m}{n\cdot h_n^{D+1}}\right)\right)\\
=&\sup_{F\in\mathcal{F}} P_F\left(\sup_{\substack{(x,\theta)\in\mathcal{S}_X\times\Theta\\ h>0}}\left|U_{f_g,n}(x,\theta,h)\right| \geq \frac{1}{4}\cdot\left(\frac{n}{n-1}\right)\cdot h_n^{D+1}\cdot\left(b-\frac{\overline{\mu}_m}{n\cdot h_n^{D+1}}\right)\right) = O\left(\frac{1}{\left(n^{1/2}\cdot h_n^{D+1}\cdot\left(b-\frac{\overline{\mu}_m}{n\cdot h_n^{D+1}}\right)\right)^q}\right)
\end{aligned} \tag{S3.1.36A}$$

From (S3.1.33),

$$\begin{aligned}
&\sup_{F\in\mathcal{F}} P_F\left(\frac{1}{n\cdot h_n^{D+1}}\cdot \sup_{\substack{(x,\theta)\in\mathcal{S}_X\times\Theta\\ h>0}}\left|m_n^{f_g}(x,\theta,h)\right| \geq \frac{1}{4}\cdot\left(b-\frac{\overline{\mu}_m}{n\cdot h_n^{D+1}}\right)\right)\\
=&\sup_{F\in\mathcal{F}} P_F\left(\sup_{\substack{(x,\theta)\in\mathcal{S}_X\times\Theta\\ h>0}}\left|m_n^{f_g}(x,\theta,h)\right| \geq \frac{1}{4}\cdot n\cdot h_n^{D+1}\cdot\left(b-\frac{\overline{\mu}_m}{n\cdot h_n^{D+1}}\right)\right) = O\left(\frac{1}{\left(n^{3/2}\cdot h_n^{D+1}\cdot\left(b-\frac{\overline{\mu}_m}{n\cdot h_n^{D+1}}\right)\right)^q}\right)
\end{aligned} \tag{S3.1.36B}$$

Next, from (S3.1.22) and Assumption 1,

$$\begin{aligned}
&\sup_{F\in\mathcal{F}} P_F\left(\frac{1}{h_n^{D+1}}\cdot \sup_{\substack{(x,\theta)\in\mathcal{S}_X\times\Theta\\ h>0}}\left\|\nu_{\Lambda_{f_g},n}(x,\theta,h)\right\|\cdot\left\|\varepsilon_n^{\theta}\right\| \geq \frac{1}{4}\cdot\left(b-\frac{\overline{\mu}_m}{n\cdot h_n^{D+1}}\right)\right)\\
\leq&\sup_{F\in\mathcal{F}} P_F\left(\sup_{\substack{(x,\theta)\in\mathcal{S}_X\times\Theta\\ h>0}}\left\|\nu_{\Lambda_{f_g},n}(x,\theta,h)\right\| \geq \left(\frac{1}{4}\cdot h_n^{D+1}\cdot\left(b-\frac{\overline{\mu}_m}{n\cdot h_n^{D+1}}\right)\right)^{1/2}\right)\\
+&\sup_{F\in\mathcal{F}} P_F\left(\left\|\varepsilon_n^{\theta}\right\| \geq \left(\frac{1}{4}\cdot h_n^{D+1}\cdot\left(b-\frac{\overline{\mu}_m}{n\cdot h_n^{D+1}}\right)\right)^{1/2}\right)\\
=&O\left(\frac{1}{\left(n^{1/2}\cdot h_n^{\frac{D+1}{2}}\cdot\left(b-\frac{\overline{\mu}_m}{n\cdot h_n^{D+1}}\right)^{1/2}\right)^q}\right)+O\left(\frac{1}{\left(r_n\cdot h_n^{\frac{D+1}{2}}\cdot\left(b-\frac{\overline{\mu}_m}{n\cdot h_n^{D+1}}\right)^{1/2}\right)^q}\right)\\
=&O\left(\frac{1}{\left((n^{1/2}\wedge r_n)\cdot h_n^{\frac{D+1}{2}}\cdot\left(b-\frac{\overline{\mu}_m}{n\cdot h_n^{D+1}}\right)^{1/2}\right)^q}\right)
\end{aligned} \tag{S3.1.36C}$$

[1]This is true for any sequence $b_n > 0$ such that $b_n \cdot n \cdot h_n^{D+1} \longrightarrow \infty$.

Finally, also from Assumption 1,

$$\sup_{F\in\mathcal{F}} P_F\left(\frac{\overline{\mu}_\Lambda}{h_n^{D+1}}\cdot\left\|\varepsilon_n^\theta\right\|\geq\frac{1}{4}\cdot\left(b-\frac{\overline{\mu}_m}{n\cdot h_n^{D+1}}\right)\right)=\sup_{F\in\mathcal{F}} P_F\left(\left\|\varepsilon_n^\theta\right\|\geq\frac{h_n^{D+1}}{4\overline{\mu}_m}\cdot\left(b-\frac{\overline{\mu}_m}{n\cdot h_n^{D+1}}\right)\right)$$
$$=O\left(\frac{1}{\left(r_n\cdot h_n^{D+1}\cdot\left(b-\frac{\overline{\mu}_m}{n\cdot h_n^{D+1}}\right)\right)^q}\right) \tag{S3.1.36D}$$

For large enough n, we have[2] $b-\frac{\overline{\mu}_m}{n\cdot h_n^{D+1}}>0$ and $h_n<1$ (and therefore $h_n^{D+1}<h_n^{\frac{D+1}{2}}$). Combining (S3.1.36A)-(S3.1.36D) with (S3.1.35), we obtain,

$$\sup_{F\in\mathcal{F}} P_F\left(\sup_{(x,\theta)\in\mathcal{S}_X\times\Theta}\left|\xi_{a,n}^{f_g}(x,\theta)\right|\geq b\right)=O\left(\frac{1}{\left((n^{1/2}\wedge r_n)\cdot h_n^{D+1}\cdot\left(\left(b-\frac{\overline{\mu}_m}{n\cdot h_n^{D+1}}\right)\wedge\left(b-\frac{\overline{\mu}_m}{n\cdot h_n^{D+1}}\right)^{1/2}\right)\right)^q}\right)$$

Going back to the second-order approximation in (S3.1.31),

$$\widehat{f_g}(g(x,\widehat{\theta}))=\widehat{f_g}(g(x,\theta_F^*))+\xi_{a,n}^{f_g}(x,\theta_F^*)+\xi_{b,n}^{f_g}(x,\overline{\theta}_x), \tag{S3.1.37}$$

where, for any $b>0$

$$\begin{aligned}\sup_{F\in\mathcal{F}} P_F\left(\sup_{x\in\mathcal{S}_X}\left|\xi_{a,n}^{f_g}(x,\theta_F^*)\right|\geq b\right)&\leq\sup_{F\in\mathcal{F}} P_F\left(\sup_{(x,\theta)\in\mathcal{S}_X\times\Theta}\left|\xi_{a,n}^{f_g}(x,\theta)\right|\geq b\right)\\&=O\left(\frac{1}{\left((n^{1/2}\wedge r_n)\cdot h_n^{D+1}\cdot\left(\left(b-\frac{\overline{\mu}_m}{n\cdot h_n^{D+1}}\right)\wedge\left(b-\frac{\overline{\mu}_m}{n\cdot h_n^{D+1}}\right)^{1/2}\right)\right)^q}\right)\end{aligned} \tag{S3.1.38}$$

Note that, for any $c>0$, we have $\min\left\{c, c^{1/2}, c^{1/4}\right\}=\min\left\{c, c^{1/4}\right\}$. Using this and combining

[2] See footnote 1.

(S3.1.19) with (S3.1.38), we have that for any $b > 0$,

$$\begin{aligned}
&\sup_{F\in\mathcal{F}} P_F\left(\sup_{x\in\mathcal{S}_{\mathcal{X}}}\left|\xi^{f_g}_{a,n}(x,\theta^*_F)+\xi^{f_g}_{b,n}(x,\overline{\theta}_x)\right|\geq b\right)\\
\leq&\sup_{F\in\mathcal{F}} P_F\left(\sup_{x\in\mathcal{S}_{\mathcal{X}}}\left|\xi^{f_g}_{a,n}(x,\theta^*_F)\right|\geq \frac{b}{2}\right)+\sup_{F\in\mathcal{F}} P_F\left(\sup_{x\in\mathcal{S}_{\mathcal{X}}}\left|\xi^{f_g}_{b,n}(x,\overline{\theta}_x)\right|\geq \frac{b}{2}\right)\\
=&O\left(\frac{1}{\left((n^{1/2}\wedge r_n)\cdot h_n^{D+1}\cdot\left(\left(\frac{b}{2}-\frac{\overline{\mu}_m}{n\cdot h_n^{D+1}}\right)\wedge\left(\frac{b}{2}-\frac{\overline{\mu}_m}{n\cdot h_n^{D+1}}\right)^{1/4}\right)\right)^q}\right)
\end{aligned}\tag{S3.1.39}$$

Now let us analyze $\widehat{f_g}(g(x,\theta^*_F))$. For a given $x\in\mathcal{S}_X$, $\theta\in\Theta$ and $h>0$ let

$$\nu^{f_g}_n(x,\theta,h)=\frac{1}{n}\sum_{i=1}^n\left(K\left(\frac{\Delta g(X_i,x,\theta)}{h}\right)-E_F\left[K\left(\frac{\Delta g(X_i,x,\theta)}{h}\right)\right]\right).$$

Directly from Assumption 4, the class of functions

$$\mathscr{G}_7=\left\{m:\mathcal{S}_X\longrightarrow\mathbb{R}\colon\ m(x)=K\left(\frac{\Delta g(x,u,\theta)}{h}\right)\ \text{ for some } u\in\mathcal{S}_X,\ \theta\in\Theta,\ h>0\right\}$$

is Euclidean for the constant envelope $\overline{K}$. From here, applying Result S1 we obtain that there exists $\overline{M}_7<\infty$ such that, for any $b>0$,

$$\begin{aligned}
&\sup_{F\in\mathcal{F}} P_F\left(\sup_{\substack{(x,\theta)\in\mathcal{S}_X\times\Theta\\ h>0}}\left|\nu^{f_g}_n(x,\theta,h)\right|\geq b\right)\leq\frac{\overline{M}_7}{\left(n^{1/2}\cdot b\right)^q}=O\left(\frac{1}{\left(n^{1/2}\cdot b\right)^q}\right),\ \text{ and therefore,}\\
&\sup_{\substack{(x,\theta)\in\mathcal{S}_X\times\Theta\\ h>0}}\left|\nu^{f_g}_n(x,\theta,h)\right|=O_p\left(\frac{1}{n^{1/2}}\right)\quad\text{uniformly over }\mathcal{F}.
\end{aligned}\tag{S3.1.40}$$

For a given $x\in\mathcal{S}_X$ let

$$B^{f_g}_n(x)\equiv\underbrace{\frac{1}{h_n^D}\cdot E_F\left[K\left(\frac{\Delta g(X,x,\theta^*_F)}{h_n}\right)\right]-f_g(g(x,\theta^*_F))}_{\text{bias}}.$$

We have,

$$\widehat{f_g}(g(x,\theta^*_F))=f_g(g(x,\theta^*_F))+\frac{1}{h_n^D}\cdot\nu^{f_g}_n(x,\theta^*_F,h_n)+B^{f_g}_n(x).$$

From here, (S3.1.37) yields,

$$\widehat{f_g}(g(x,\widehat{\theta})) = f_g(g(x,\theta_F^*)) + \frac{1}{h_n^D}\cdot \nu_n^{f_g}(x,\theta_F^*,h_n) + B_n^{f_g}(x) + \xi_{a,n}^{f_g}(x,\theta_F^*) + \xi_{b,n}^{f_g}(x,\overline{\theta}_x), \tag{S3.1.41}$$

From Assumptions 4 and the smoothness conditions described in Assumption 2, an M^{th}-order approximation implies that there exists a constant $\overline{B}_{1,f} < \infty$ such that

$$\sup_{x\in\mathcal{X}} \left|B_n^{f_g}(x)\right| \leq \overline{B}_{1,f}\cdot h_n^M \quad \forall\ F\in\mathcal{F}. \tag{S3.1.42}$$

From here,[3]

$$\sup_{x\in\mathcal{X}}\left|\widehat{f_g}(g(x,\widehat{\theta})) - f_g(g(x,\theta_F^*))\right| \leq \frac{1}{h_n^D}\cdot\sup_{x\in\mathcal{X}}\left|\nu_n^{f_g}(x,\theta_F^*,h_n)\right| + \overline{B}_{1,f}\cdot h_n^M + \sup_{x\in\mathcal{X}}\left|\xi_{a,n}^{f_g}(x,\theta_F^*) + \xi_{b,n}^{f_g}(x,\overline{\theta}_x)\right|$$

Thus, for any $b>0$,

$$\begin{aligned}
&\sup_{F\in\mathcal{F}}\ P_F\left(\sup_{x\in\mathcal{X}}\left|\widehat{f_g}(g(x,\widehat{\theta})) - f_g(g(x,\theta_F^*))\right| \geq b\right)\\
\leq&\sup_{F\in\mathcal{F}}\ P_F\left(\frac{1}{h_n^D}\cdot\sup_{x\in\mathcal{X}}\left|\nu_n^{f_g}(x,\theta_F^*,h_n)\right| + \sup_{x\in\mathcal{X}}\left|\xi_{a,n}^{f_g}(x,\theta_F^*) + \xi_{b,n}^{f_g}(x,\overline{\theta}_x)\right| \geq b - \overline{B}_{1,f}\cdot h_n^M\right)\\
\leq&\sup_{F\in\mathcal{F}}\ P_F\left(\frac{1}{h_n^D}\cdot\sup_{x\in\mathcal{X}}\left|\nu_n^{f_g}(x,\theta_F^*,h_n)\right| \geq \left(\frac{b-\overline{B}_{1,f}\cdot h_n^M}{2}\right)\right)\\
+&\sup_{F\in\mathcal{F}}\ P_F\left(\sup_{x\in\mathcal{X}}\left|\xi_{a,n}^{f_g}(x,\theta_F^*) + \xi_{b,n}^{f_g}(x,\overline{\theta}_x)\right| \geq \left(\frac{b-\overline{B}_{1,f}\cdot h_n^M}{2}\right)\right)
\end{aligned} \tag{S3.1.43}$$

For a given $b>0$, there exists an n_0 such that[4]

$$\frac{b-\overline{B}_{1,f}\cdot h_n^M}{4} - \frac{\overline{\mu}_m}{n\cdot h_n^D} > 0, \quad \text{and} \quad h_n < 1 \quad \forall\ n > n_0$$

[3]Note that here we are focusing on our testing range $\mathcal{X}$.

[4]What follows is true more generally if we replace the constant b with a sequence $s_n > 0$ which may converge to zero as long as $h_n^M/s_n \longrightarrow 0$ and $s_n\cdot n\cdot h_n^{D+1}\longrightarrow 0$.

From (S3.1.39) and (S3.1.40), the inequality in (S3.1.43) yields

$$
\begin{aligned}
&\sup_{F\in\mathcal{F}} P_F\left(\sup_{x\in\mathcal{X}}\left|\widehat{f_g}(g(x,\widehat{\theta}))-f_g(g(x,\theta_F^*))\right|\geq b\right)\\
&=O\left(\frac{1}{\left(n^{1/2}\cdot h_n^D\cdot\left(\frac{b-\overline{B}_{1,f}\cdot h_n^M}{2}\right)\right)^q}\right)+O\left(\frac{1}{\left((n^{1/2}\wedge r_n)\cdot h_n^{D+1}\cdot\left(\left(\frac{b}{4}-\frac{\overline{B}_{1,f}\cdot h_n^M}{4}-\frac{\overline{\mu}_m}{n\cdot h_n^{D+1}}\right)\wedge\left(\frac{b}{4}-\frac{\overline{B}_{1,f}\cdot h_n^M}{4}-\frac{\overline{\mu}_m}{n\cdot h_n^{D+1}}\right)^{1/4}\right)\right)^q}\right)\\
&=O\left(\frac{1}{\left((n^{1/2}\wedge r_n)\cdot h_n^{D+1}\cdot\left(\left(\frac{b}{4}-\frac{\overline{B}_{1,f}\cdot h_n^M}{4}-\frac{\overline{\mu}_m}{n\cdot h_n^{D+1}}\right)\wedge\left(\frac{b}{4}-\frac{\overline{B}_{1,f}\cdot h_n^M}{4}-\frac{\overline{\mu}_m}{n\cdot h_n^{D+1}}\right)^{1/4}\right)\right)^q}\right)
\end{aligned}
\tag{S3.1.44}
$$

S3.1.1 A general result for $\sup_{F\in\mathcal{F}} P_F\left(\sup_{x\in\mathcal{X}}\left|\widehat{f_g}(g(x,\widehat{\theta}))-f_g(g(x,\theta_F^*))\right|\geq s_n\right)$

We can generalize the result in (S3.1.44) as follows. There exist constants $K_1>0$, $K_2>0$ and $K_3>0$ such that, for any sequence $s_n>0$ (possibly converging to zero) such that

$$\frac{h_n^M}{s_n}\longrightarrow 0\quad\text{and}\quad s_n\cdot n\cdot h_n^{D+1}\longrightarrow\infty,$$

we have

$$
\begin{aligned}
&\sup_{F\in\mathcal{F}} P_F\left(\sup_{x\in\mathcal{X}}\left|\widehat{f_g}(g(x,\widehat{\theta}))-f_g(g(x,\theta_F^*))\right|\geq s_n\right)\\
&=O\left(\frac{1}{\left((n^{1/2}\wedge r_n)\cdot h_n^{D+1}\cdot\left(\left(K_1\cdot s_n-K_2\cdot h_n^M-\frac{K_3}{n\cdot h_n^{D+1}}\right)\wedge\left(K_1\cdot s_n-K_2\cdot h_n^M-\frac{K_3}{n\cdot h_n^{D+1}}\right)^{1/4}\right)\right)^q}\right)
\end{aligned}
\tag{S3.1.45}
$$

Next, we want to obtain a linear representation for $\widehat{f_g}(g(x,\widehat{\theta}))-f_g(g(x,\theta_F^*))$ over $x\in\mathcal{X}$. To this end let us go back to (S3.1.37) and recall that

$$\widehat{f_g}(g(x,\widehat{\theta}))=\widehat{f_g}(g(x,\theta_F^*))+\xi_{a,n}^{f_g}(x,\theta_F^*)+\xi_{b,n}^{f_g}(x,\overline{\theta}_x).$$

From (S3.1.19),

$$\sup_{F\in\mathcal{F}} P_F\left(\sup_{x\in\mathcal{S}_{\mathcal{X}}}\left|\xi_{b,n}^{f_g}(x,\overline{\theta}_x)\right|\geq b\right)=O_p\left(\frac{1}{n\cdot h_n^{D+2}}\right)\quad\text{uniformly over }\mathcal{F}.$$

To derive the linear representation we will focus on more detail on the term $\xi^{f_g}_{a,n}(x,\theta^*_F)$. From (S3.1.34) and the Hoeffding decomposition of $U_{f_g,n}$ in (S3.1.29), we have

$$\xi^{f_g}_{a,n}(x,\theta^*_F) = \left(\frac{n-1}{n}\right)\cdot \underbrace{\left(\frac{1}{h_n^{D+1}}\cdot E_F\left[\Lambda_{f_g}(X,x,\theta^*_F,h_n)\right]\right)\cdot \frac{1}{n}\sum_{i=1}^{n}\psi^{\theta}_F(Z_i)}_{=\nu_{\varphi_{f_g,n}(x,\theta^*_F,h_n)}\text{ (see (S3.1.25) and (S3.1.27))}} + \frac{1}{h_n^{D+1}}\cdot E_F\left[\Lambda_{f_g}(X,x,\theta^*_F,h_n)\right]\varepsilon^{\theta}_n$$
$$+\left(\frac{n-1}{2n}\right)\cdot\frac{1}{h_n^{D+1}}\cdot\widetilde{U}_{f_g,n}(x,\theta^*_F,h_n)+\frac{1}{n\cdot h_n^{D+1}}\cdot E_F\left[\Lambda_{f_g}(X,x,\theta^*_F,h_n)\psi^{\theta}_F(Z)\right]$$
$$+\frac{1}{n\cdot h_n^{D+1}}\cdot m^{f_g}_n(x,\theta^*_F,h_n)+\frac{1}{h_n^{D+1}}\cdot\nu_{\Lambda_{f_g},n}(x,\theta^*_F,h_n)\varepsilon^{\theta}_n. \tag{S3.1.46}$$

Therefore we can express

$$\xi^{f_g}_{a,n}(x,\theta^*_F) = \left(\frac{n-1}{n}\right)\cdot\left(\frac{1}{h_n^{D+1}}\cdot E_F\left[\Lambda_{f_g}(X,x,\theta^*_F,h_n)\right]\right)\cdot\frac{1}{n}\sum_{i=1}^{n}\psi^{\theta}_F(Z_i)+\frac{1}{h_n^{D+1}}\cdot E_F\left[\Lambda_{f_g}(X,x,\theta^*_F,h_n)\right]\varepsilon^{\theta}_n+\varsigma^{f_g}_{1,n}(x) \tag{S3.1.47}$$

where

$$\varsigma^{f_g}_{1,n}(x) \equiv \left(\frac{n-1}{2n}\right)\cdot\frac{1}{h_n^{D+1}}\cdot\widetilde{U}_{f_g,n}(x,\theta^*_F,h_n)+\frac{1}{n\cdot h_n^{D+1}}\cdot E_F\left[\Lambda_{f_g}(X,x,\theta^*_F,h_n)\psi^{\theta}_F(Z)\right]$$
$$+\frac{1}{n\cdot h_n^{D+1}}\cdot m^{f_g}_n(x,\theta^*_F,h_n)+\frac{1}{h_n^{D+1}}\cdot\nu_{\Lambda_{f_g},n}(x,\theta^*_F,h_n)\varepsilon^{\theta}_n$$

Let us analyze $\varsigma^{f_g}_{1,n}(x)$. From (S3.1.26),

$$\left(\frac{n-1}{2n}\right)\cdot\frac{1}{h_n^{D+1}}\cdot\sup_{x\in\mathcal{S}_X}\left|\widetilde{U}_{f_g,n}(x,\theta^*_F,h_n)\right| = O_p\left(\frac{1}{n\cdot h_n^{D+1}}\right).$$

From (S3.1.32),

$$\frac{1}{n\cdot h_n^{D+1}}\cdot\sup_{x\in\mathcal{S}_X}\left|E_F\left[\Lambda_{f_g}(X,x,\theta^*_F,h_n)\psi^{\theta}_F(Z)\right]\right| \le \frac{1}{n\cdot h_n^{D+1}}\cdot\overline{\mu}_m \quad \forall\ F\in\mathcal{F}.$$

From (S3.1.33),

$$\frac{1}{n\cdot h_n^{D+1}}\cdot\sup_{x\in\mathcal{S}_X}\left|m^{f_g}_n(x,\theta^*_F,h_n)\right| = \frac{1}{n\cdot h_n^{D+1}}\cdot O_p\left(\frac{1}{n^{1/2}}\right) = O_p\left(\frac{1}{n^{3/2}\cdot h_n^{D+1}}\right) \quad \text{uniformly over } \mathcal{F}$$

Let $\tau > 0$ be the constant described in Assumption 1. From the conditions described there and the result in (S3.1.23),

$$\frac{1}{h_n^{D+1}} \cdot \sup_{x \in \mathcal{S}_X} \left\| \nu_{\Lambda_{f_g},n}(x, \theta_F^*, h_n) \right\| \cdot \left\| \varepsilon_n^\theta \right\| = \frac{1}{h_n^{D+1}} \cdot O_p\left(\frac{1}{n^{1/2}}\right) \cdot o_p\left(\frac{1}{n^{1/2+\tau}}\right) = o_p\left(\frac{1}{n^{1+\tau} \cdot h_n^{D+1}}\right) \quad \text{uniformly over } \mathcal{F}$$

Therefore,

$$\begin{aligned} \sup_{x \in \mathcal{S}_X} \left| \varsigma_{1,n}^{f_g}(x) \right| &= O_p\left(\frac{1}{n \cdot h_n^{D+1}}\right) + O\left(\frac{1}{n \cdot h_n^{D+1}}\right) + O_p\left(\frac{1}{n^{3/2} \cdot h_n^{D+1}}\right) + o_p\left(\frac{1}{n^{1+\tau} \cdot h_n^{D+1}}\right) \\ &= O_p\left(\frac{1}{n \cdot h_n^{D+1}}\right) \quad \text{uniformly over } \mathcal{F} \end{aligned} \tag{S3.1.48}$$

With this result at hand, going back to (S3.1.46) we see that we need to analyze the term

$$\frac{1}{h_n^{D+1}} \cdot E_F\left[\Lambda_{f_g}(X, x, \theta_F^*, h_n)\right].$$

Recall from the definitions in (S3.1.4) and (S3.1.20) that

$$\begin{aligned} \nabla_d K(\psi) &\equiv \kappa^{(1)}(\psi_d) \cdot \prod_{\ell \neq d} \kappa(\psi_\ell), \\ \Lambda_{f_g}^\ell(X_i, x, \theta_F^*, h_n) &\equiv \sum_{d=1}^{D} \nabla_d K\left(\frac{\Delta g(X_i, x, \theta_F^*)}{h_n}\right) \frac{\partial \Delta g_d(X_i, x, \theta)}{\partial \theta_\ell}, \\ \underbrace{\Lambda_{f_g}(X_i, x, \theta_F^*, h_n)}_{1 \times k} &\equiv \left(\Lambda_{f_g}^1(X_i, x, \theta_F^*, h_n), \ldots, \Lambda_{f_g}^k(X_i, x, \theta_F^*, h_n)\right) \end{aligned}$$

Therefore,

$$\frac{1}{h_n^{D+1}} \cdot E_F\left[\Lambda_{f_g}(X, x, \theta_F^*, h_n)\right] = \left(\frac{1}{h_n^{D+1}} \cdot E_F\left[\Lambda_{f_g}^1(X, x, \theta_F^*, h_n)\right], \ldots, \frac{1}{h_n^{D+1}} \cdot E_F\left[\Lambda_{f_g}^k(X, x, \theta_F^*, h_n)\right]\right).$$

For each $\ell = 1, \ldots, k$, we have

$$\frac{1}{h_n^{D+1}} \cdot E_F\left[\Lambda_{f_g}^\ell(X, x, \theta_F^*, h_n)\right] = \sum_{d=1}^{D} \frac{1}{h_n^{D+1}} \cdot E_F\left[\nabla_d K\left(\frac{\Delta g(X, x, \theta_F^*)}{h_n}\right) \frac{\partial \Delta g_d(X, x, \theta)}{\partial \theta_\ell}\right].$$

Thus, it is sufficient for us to analyze $\frac{1}{h_n^{D+1}} \cdot E_F\left[\nabla_d K\left(\frac{\Delta g(X,x,\theta_F^*)}{h_n}\right) \frac{\partial \Delta g_d(X,x,\theta)}{\partial \theta_\ell}\right]$ for any given $d = 1, \ldots, D$

and $\ell = 1,\dots,k$. We have,

$$
\begin{aligned}
&\frac{1}{h_n^{D+1}} \cdot E_F\left[\nabla_d K\left(\frac{\Delta g(X,x,\theta_F^*)}{h_n}\right)\frac{\partial \Delta g_d(X,x,\theta_F^*)}{\partial \theta_\ell}\right] \\
=&\frac{1}{h_n^{D+1}} \cdot E_F\left[\nabla_d K\left(\frac{g(X,\theta_F^*)-g(x,\theta_F^*)}{h_n}\right)\left(\frac{\partial g_d(X,\theta_F^*)}{\partial \theta_\ell} - \frac{\partial g_d(x,\theta_F^*)}{\partial \theta_\ell}\right)\right] \\
=&\frac{1}{h_n^{D+1}} \cdot E_F\left[\nabla_d K\left(\frac{g(X,\theta_F^*)-g(x,\theta_F^*)}{h_n}\right)\frac{\partial g_d(X,\theta_F^*)}{\partial \theta_\ell}\right] - \frac{1}{h_n^{D+1}} \cdot E_F\left[\nabla_d K\left(\frac{g(X,\theta_F^*)-g(x,\theta_F^*)}{h_n}\right)\right]\frac{\partial g_d(x,\theta_F^*)}{\partial \theta_\ell}
\end{aligned}
$$

As we defined in Assumption 2, for a given $g \equiv (g_1,\dots,g_D) \in \mathcal{S}_{g,F}$, and each $\ell = 1,\dots,k$ and $d = 1,\dots,D$, let

$$
\Omega_{f_g}^{d,\ell}(g) = E_F\left[\frac{\partial g_d(X,\theta_F^*)}{\partial \theta_\ell}\middle| g(X,\theta_F^*) = g\right].
$$

Recall that, from the kernel properties described in Assumption 4, we have

$$
\int \cdots \int \nabla_d K(\psi_1,\dots,\psi_D)d\psi_1\cdots d\psi_D = 0, \quad \int \cdots \int \psi_d \nabla_d K(\psi_1,\dots,\psi_D)d\psi_1\cdots d\psi_D = -1,
$$

$$
\int \cdots \int \psi_1^{j_1}\cdots\psi_D^{j_D}\nabla_d K(\psi_1,\dots,\psi_D)d\psi_1\cdots d\psi_D = 0 \quad \forall\ (j_1,\dots,j_D): \begin{cases} \sum_{s=1}^D j_s \leq M, \\ j_\ell \neq 0 \text{ for some } \ell \neq d, \text{ or } j_d \neq 1 \end{cases}
$$

$$
\int \cdots \int \left|\psi_1\right|^{j_1}\cdots\left|\psi_D\right|^{j_D} \cdot \left|\nabla_d K(\psi_1,\dots,\psi_D)\right| d\psi_1\cdots d\psi_D < \infty \quad \forall\ (j_1,\dots,j_D): \sum_{s=1}^D j_s = M+1.
$$

For a given $x \in \mathcal{X}$ let

$$
\begin{aligned}
&\Xi_{\ell,f_g}(x,\theta_F^*) \equiv \sum_{d=1}^D \left(\frac{\partial g_d(x,\theta_F^*)}{\partial \theta_\ell} \cdot \frac{\partial f_g(g(x,\theta_F^*))}{\partial g_d} - \frac{\partial\left[\Omega_{f_g}^{d,\ell}(g(x,\theta_F^*)) \cdot f_g(g(x,\theta_F^*))\right]}{\partial g_d}\right), \\
&\underbrace{\Xi_{f_g}(x,\theta_F^*)}_{1\times k} \equiv \left(\Xi_{1,f_g}(x,\theta_F^*),\dots,\Xi_{k,f_g}(x,\theta_F^*)\right)
\end{aligned}
$$

From the smoothness conditions described in Assumption 2, an $(M+1)^{th}$-order approximation yields that there exists a constant $\overline{B}_{2,f} < \infty$ such that

$$
\begin{aligned}
&\frac{1}{h_n^{D+1}} \cdot E_F\left[\Lambda_{f_g}(X,x,\theta_F^*,h_n)\right] = \Xi_{f_g}(x,\theta_F^*) + B_n^{\Xi_{f_g}}(x), \quad \text{where} \\
&\sup_{x\in\mathcal{X}} \left\|B_n^{\Xi_{f_g}}(x)\right\| \leq \overline{B}_{2,f} \cdot h_n^M \quad \forall\ F \in \mathcal{F}.
\end{aligned} \tag{S3.1.49}
$$

By Assumption 2, $\exists\ \overline{\mu}_{\Xi_{f_g}} < \infty$ such that

$$\sup_{x\in\mathcal{X}} \left\|\Xi_{f_g}(x,\theta_F^*)\right\| \leq \overline{\mu}_{\Xi_{f_g}} \quad \forall\ F \in \mathcal{F}. \tag{S3.1.50}$$

Plugging (S3.1.49) into (S3.1.47),

$$\xi_{a,n}^{f_g}(x,\theta_F^*) = \left(\frac{n-1}{n}\right)\cdot\left(\Xi_{f_g}(x,\theta_F^*) + B_n^{\Xi_{f_g}}(x)\right)\frac{1}{n}\sum_{i=1}^{n}\psi_F^{\theta}(Z_i) + \left(\Xi_{f_g}(x,\theta_F^*) + B_n^{\Xi_{f_g}}(x)\right)\varepsilon_n^{\theta} + \varsigma_{1,n}^{f_g}(x).$$

From here we can express

$$\xi_{a,n}^{f_g}(x,\theta_F^*) = \Xi_{f_g}(x,\theta_F^*)\frac{1}{n}\sum_{i=1}^{n}\psi_F^{\theta}(Z_i) + \varsigma_{2,n}^{f_g}(x) + \varsigma_{1,n}^{f_g}(x), \tag{S3.1.51}$$

where

$$\varsigma_{2,n}^{f_g}(x) \equiv \left(\left(\frac{n-1}{n}\right)\cdot B_n^{\Xi_{f_g}}(x) - \frac{1}{n}\right)\cdot\frac{1}{n}\sum_{i=1}^{n}\psi_F^{\theta}(Z_i) + \left(\Xi_{f_g}(x,\theta_F^*) + B_n^{\Xi_{f_g}}(x)\right)\varepsilon_n^{\theta}$$

Let $\tau > 0$ be the constant described in Assumption 1. From the conditions described there and the results in (S3.1.49) and (S3.1.50), we have

$$\begin{aligned}\sup_{x\in\mathcal{X}}\left|\varsigma_{2,n}^{f_g}(x)\right| &= \left(O\left(h_n^M\right) + O\left(\frac{1}{n}\right)\right)\cdot O_p\left(\frac{1}{n^{1/2}}\right) + \left(O(1) + O\left(h_n^M\right)\right)\cdot o_p\left(\frac{1}{n^{1/2+\tau}}\right)\\ &= O_p\left(\max\left\{\frac{h_n^M}{n^{1/2}}\,,\,\frac{1}{n^{3/2}}\,,\,\frac{1}{n^{1/2+\tau}}\right\}\right) \quad \text{uniformly over } \mathcal{F}\end{aligned} \tag{S3.1.52}$$

Combining (S3.1.48) and (S3.1.52) and defining $\varsigma_n^{f_g}(x) \equiv \varsigma_{1,n}^{f_g}(x) + \varsigma_{2,n}^{f_g}(x)$, (S3.1.51) becomes

$$\begin{aligned}&\xi_{a,n}^{f_g}(x,\theta_F^*) = \Xi_{f_g}(x,\theta_F^*)\cdot\frac{1}{n}\sum_{i=1}^{n}\psi_F^{\theta}(Z_i) + \varsigma_n^{f_g}(x),\\ &\text{where} \quad \sup_{x\in\mathcal{X}}\left|\varsigma_n^{f_g}(x)\right| = O_p\left(\max\left\{\frac{h_n^M}{n^{1/2}}\,,\,\frac{1}{n^{3/2}}\,,\,\frac{1}{n^{1/2+\tau}}\,,\,\frac{1}{n\cdot h_n^{D+1}}\right\}\right) \quad \text{uniformly over } \mathcal{F}\end{aligned} \tag{S3.1.53}$$

S3.1.2 A uniform linear representation result for $\widehat{f_g}(g(x,\widehat{\theta})) - f_g(g(x,\theta_F^*))$

Let

$$\psi_F^{f_g}(V_i,x,\theta_F^*,h_n) \equiv \frac{1}{h_n^D}\cdot\left(K\left(\frac{\Delta g(X_i,x,\theta_F^*)}{h_n}\right) - E_F\left[K\left(\frac{\Delta g(X_i,x,\theta_F^*)}{h_n}\right)\right]\right) + \Xi_{f_g}(x,\theta_F^*)\psi_F^{\theta}(Z_i). \tag{S3.1.54}$$

Combining (S3.1.41) and (S3.1.53), we have

$$\begin{aligned}&\widehat{f_g}(g(x,\widehat{\theta})) = f_g(g(x,\theta_F^*)) + \frac{1}{n}\sum_{i=1}^{n}\psi_F^{f_g}(V_i,x,\theta_F^*,h_n) + \zeta_n^{f_g}(x),\\ &\text{where} \quad \zeta_n^{f_g}(x) \equiv B_n^{f_g}(x) + \varsigma_n^{f_g}(x) + \xi_{b,n}^{f_g}(x,\overline{\theta}_x)\end{aligned} \tag{S3.1.55}$$

Let $\epsilon > 0$ be the constant described in Assumption 4. Applying the results from (S3.1.19), (S3.1.42) and (S3.1.53),

$$\begin{aligned}\sup_{x\in\mathcal{X}}\left|\zeta_n^{f_g}(x)\right| &= O\left(h_n^M\right) + O_p\left(\max\left\{\frac{h_n^M}{n^{1/2}}, \frac{1}{n^{3/2}}, \frac{1}{n^{1/2+\tau}}\right\}\right) + O_p\left(\frac{1}{n\cdot h_n^{D+2}}\right)\\ &= O_p\left(\max\left\{h_n^M, \frac{1}{n^{3/2}}, \frac{1}{n^{1/2+\tau}}, \frac{1}{n\cdot h_n^{D+2}}\right\}\right) = o_p\left(\frac{1}{n^{1/2+\epsilon}}\right) \quad \text{uniformly over } \mathcal{F}\end{aligned} \tag{S3.1.56}$$

By the conditions in Assumptions 1 and 4 along with the results in (S3.1.40) and (S3.1.50), we can once again invoke Result S1 to show that

$$\sup_{x\in\mathcal{X}}\left|\frac{1}{n}\sum_{i=1}^{n}\psi_F^{f_g}(V_i,x,\theta_F^*,h_n)\right| = O_p\left(\frac{1}{n^{1/2}\cdot h_n^D}\right) \quad \text{uniformly over } \mathcal{F}. \tag{S3.1.57}$$

And therefore,

$$\sup_{x\in\mathcal{X}}\left|\widehat{f_g}(g(x,\widehat{\theta})) - f_g(g(x,\theta_F^*))\right| = O_p\left(\frac{1}{n^{1/2}\cdot h_n^D}\right) + o_p\left(\frac{1}{n^{1/2+\epsilon}}\right) = O_p\left(\frac{1}{n^{1/2}\cdot h_n^D}\right) \quad \text{uniformly over } \mathcal{F}. \tag{S3.1.58}$$

From our previous results and Assumption 2, it also follows that

$$\sup_{x\in\mathcal{X}}\left|\frac{1}{\widehat{f_g}(g(x,\widehat{\theta}))}\right| = O_p(1) \quad \text{uniformly over } \mathcal{F}. \tag{S3.1.59}$$

To see why, recall from Assumption 2 that $\inf_{x\in\mathcal{X}} f_g(g(x,\theta_F^*)) \geq \underline{f}_g > 0$ for all $F\in\mathcal{F}$. Take any $\delta\in(0,1)$. Note that, for any $x\in\mathcal{X}$, $\widehat{f_g}(g(x,\widehat{\theta})) < (1-\delta)\cdot\underline{f}_g$ only if $\left|\widehat{f_g}(g(x,\widehat{\theta})) - f_g(g(x,\theta_F^*))\right| > \delta\cdot\underline{f}_g$. Therefore,

$$\sup_{F\in\mathcal{F}} P_F\left(\sup_{x\in\mathcal{X}}\left|\frac{1}{\widehat{f_g}(g(x,\widehat{\theta}))}\right| > \frac{1}{(1-\delta)\cdot\underline{f}_g}\right) \leq \sup_{F\in\mathcal{F}} P_F\left(\sup_{x\in\mathcal{X}}\left|\widehat{f_g}(g(x,\widehat{\theta})) - f_g(g(x,\theta_F^*))\right| > \delta\cdot\underline{f}_g\right)$$

From (S3.1.45), there $\exists\ \overline{M}_{f_g} > 0$, $K_1 > 0$, $K_2 > 0$ and $K_3 > 0$ such that, for any $c > 0$, there exists n_c such that

$$\sup_{F\in\mathcal{F}} P_F\left(\sup_{x\in\mathcal{X}}\left|\widehat{f_g}(g(x,\widehat{\theta}))-f_g(g(x,\theta_F^*))\right|\geq c\right)$$
$$\leq \frac{\overline{M}_{f_g}}{\left((n^{1/2}\wedge r_n)\cdot h_n^{D+1}\cdot\left(\left(K_1\cdot c-K_2\cdot h_n^M-\frac{K_3}{n\cdot h_n^{D+1}}\right)\wedge\left(K_1\cdot c-K_2\cdot h_n^M-\frac{K_3}{n\cdot h_n^{D+1}}\right)^{1/4}\right)\right)^q}\quad\forall\ n>n_c$$

Thus, for any $\delta\in(0,1)$ there exists n_δ such that

$$\sup_{F\in\mathcal{F}} P_F\left(\sup_{x\in\mathcal{X}}\left|\frac{1}{\widehat{f_g}(g(x,\widehat{\theta}))}\right|>\frac{1}{(1-\delta)\cdot\underline{f}_g}\right)$$
$$\leq \frac{\overline{M}_{f_g}}{\left((n^{1/2}\wedge r_n)\cdot h_n^{D+1}\cdot\left(\left(K_1\cdot\delta\cdot\underline{f}_g-K_2\cdot h_n^M-\frac{K_3}{n\cdot h_n^{D+1}}\right)\wedge\left(K_1\cdot\delta\cdot\underline{f}_g-K_2\cdot h_n^M-\frac{K_3}{n\cdot h_n^{D+1}}\right)^{1/4}\right)\right)^q}\quad\forall\ n>n_\delta$$

Now take any $\epsilon>0$ and let $n_{\delta,\epsilon}$ be the smallest integer such that

$$\frac{\overline{M}_{f_g}}{\left((n^{1/2}\wedge r_n)\cdot h_n^{D+1}\cdot\left(\left(K_1\cdot\delta\cdot\underline{f}_g-K_2\cdot h_n^M-\frac{K_3}{n\cdot h_n^{D+1}}\right)\wedge\left(K_1\cdot\delta\cdot\underline{f}_g-K_2\cdot h_n^M-\frac{K_3}{n\cdot h_n^{D+1}}\right)^{1/4}\right)\right)^q}<\epsilon.$$

Thus, for any $\epsilon>0$ and $\delta\in(0,1)$, there exists $n_{\delta,\epsilon}$ such that

$$\sup_{F\in\mathcal{F}} P_F\left(\sup_{x\in\mathcal{X}}\left|\frac{1}{\widehat{f_g}(g(x,\widehat{\theta}))}\right|>\frac{1}{(1-\delta)\cdot\underline{f}_g}\right)\leq\epsilon\quad\forall\ n>n_{\delta,\epsilon},$$

which proves (S3.1.59).

S3.2 Asymptotic properties of $\widehat{R}_p(x,t,\widehat{\theta})$

Recall that we have defined,

$$\widehat{R}_p(x,t,\theta)=\frac{1}{n\cdot h_n^D}\sum_{i=1}^n S_p(Y_i,t)\omega_p(g(X_i,\theta))K\left(\frac{\Delta g(X_i,x,\theta)}{h_n}\right),$$

which is an estimator for the functional

$$R_{p,F}(x,t,\theta_F^*)\equiv\Gamma_{p,F}(x,t,\theta_F^*)\cdot\omega_p(g(x,\theta_F^*))\cdot f_g(g(x,\theta_F^*)).$$

Our analysis of $\widehat{R}_p(x,t,\widehat{\theta})$ will follow parallel steps to that of $\widehat{f}_g(g(x,\widehat{\theta}))$ in the previous sections. As we did there, we begin as in equation (S3.1.10), with a second-order approximation,

$$\widehat{R}_p(x,t,\widehat{\theta}) = \widehat{R}_p(x,t,\theta_F^*) + \frac{\partial \widehat{R}_p(x,t,\theta_F^*)}{\partial \theta}\left(\widehat{\theta}-\theta_F^*\right) + \frac{1}{2}\left(\widehat{\theta}-\theta_F^*\right)' \frac{\partial^2 \widehat{R}_p(x,t,\overline{\theta}_x)}{\partial\theta\partial\theta'}\left(\widehat{\theta}-\theta_F^*\right) \tag{S3.2.1}$$

where $\overline{\theta}_x$ belongs in the line segment connecting $\widehat{\theta}$ and θ_F^*. Thus, since Θ is taken to be convex, we have $\overline{\theta}_x \in \Theta$. Parallel to the definitions in (S3.1.5), let

$$\begin{aligned}
\Upsilon_{R_p}^{\ell,m}(V_i,x,t,\theta,h) &\equiv S_p(Y_i,t)\omega_p(g(X_i,\theta))\sum_{d=1}^{D}\sum_{s=1}^{D}\nabla_{ds}K\left(\frac{\Delta g(X_i,x,\theta)}{h}\right)\frac{\partial \Delta g_d(X_i,x,\theta)}{\partial\theta_\ell}\frac{\partial\Delta g_s(X_i,x,\theta)}{\partial\theta_m},\\
\Phi_{R_p}^{\ell,m}(V_i,x,t,\theta,h) &\equiv S_p(Y_i,t)\sum_{d=1}^{D}\nabla_d K\left(\frac{\Delta g(X_i,x,\theta)}{h}\right)\times\left\{\omega_p(g(X_i,\theta))\frac{\partial^2\Delta g_d(X_i,x,\theta)}{\partial\theta_m\partial\theta_\ell}\right.\\
&\qquad\left.+\frac{\partial\omega_p(g(X_i,\theta)}{\partial\theta_\ell}\frac{\partial\Delta g_d(X_i,x,\theta)}{\partial\theta_m}+\frac{\partial\omega_p(g(X_i,\theta)}{\partial\theta_m}\frac{\partial\Delta g_d(X_i,x,\theta)}{\partial\theta_\ell}\right\},\\
\rho_{R_p}^{\ell,m}(V_i,x,t,\theta,h) &\equiv S_p(Y_i,t)\frac{\partial^2\omega_p(g(X_i,\theta))}{\partial\theta_m\partial\theta_\ell}K\left(\frac{\Delta g(X_i,x,\theta)}{h_n}\right).
\end{aligned}$$

By the conditions described in Assumptions 3 and 4, there exist $\overline{\eta}_\Upsilon < \infty$, $\overline{\eta}_\Phi < \infty$ and $\overline{\eta}_\rho < \infty$ such that, for all $F \in \mathcal{F}$,

$$\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}}\left|E_F\left[\Upsilon_{R_p}^{\ell,m}(V,x,t,\theta,h)\right]\right| \le \overline{\eta}_\Upsilon \qquad \sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}}\left|E_F\left[\Phi_{R_p}^{\ell,m}(V,x,t,\theta,h)\right]\right| \le \overline{\eta}_\Phi$$

$$\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}}\left|E_F\left[\rho_{R_p}^{\ell,m}(V,x,t,\theta,h)\right]\right| \le \overline{\eta}_\rho$$

As we stated in the paragraph following Assumption 3, the conditions there, combined with Lemma 2.13 in Pakes and Pollard (1989) imply that, for each $d = 1,\ldots,D$ and $\ell,m \in 1,\ldots,k$, the classes of functions

$$\mathscr{H}_{1d}^{\ell} = \left\{ m : \mathcal{S}_X \longrightarrow \mathbb{R}\colon\ m(x) = \frac{\partial \Delta g_d(x,s,\theta)}{\partial \theta_\ell} \text{ for some } s \in \mathcal{S}_X,\ \theta \in \Theta \right\},$$

$$\mathscr{H}_{2d}^{\ell,m} = \left\{ m : \mathcal{S}_X \longrightarrow \mathbb{R}\colon\ m(x) = \frac{\partial^2 \Delta g_d(x,s,\theta)}{\partial \theta_\ell \partial \theta_m} \text{ for some } s \in \mathcal{S}_X,\ \theta \in \Theta \right\},$$

$$\mathscr{H}_{3}^{\ell} = \left\{ m : \mathcal{S}_X \longrightarrow \mathbb{R}\colon\ m(x) = \frac{\partial \omega_p(x,s,\theta)}{\partial \theta_\ell} \text{ for some } s \in \mathcal{S}_X,\ \theta \in \Theta \right\},$$

$$\mathscr{H}_{4}^{\ell,m} = \left\{ m : \mathcal{S}_X \longrightarrow \mathbb{R}\colon\ m(x) = \frac{\partial^2 \omega_p(x,s,\theta)}{\partial \theta_\ell \partial \theta_m} \text{ for some } s \in \mathcal{S}_X,\ \theta \in \Theta \right\}$$

are Euclidean for an envelope $G_1(x)$ that satisfies $E_F\left[G_1(X)^{4q}\right] \le \overline{\mu}_{G_1} < \infty$ for all $F \in \mathcal{F}$, with q being the integer described in Assumption 1. Next, consider the following three classes of functions

$$\mathscr{G}_{1,p}^{\ell,m} = \left\{ r : \mathcal{S}_V \longrightarrow \mathbb{R}\colon\ r(v) = \Upsilon_{R_p}^{\ell,m}(v,u,t,\theta,h) \text{ for some } u \in \mathcal{S}_X,\ t \in \mathcal{T},\ \theta \in \Theta,\ h > 0 \right\},$$

$$\mathscr{G}_{2,p}^{\ell,m} = \left\{ r : \mathcal{S}_V \longrightarrow \mathbb{R}\colon\ r(v) = \Phi_{R_p}^{\ell,m}(v,u,t,\theta,h) \text{ for some } u \in \mathcal{S}_X,\ t \in \mathcal{T},\ \theta \in \Theta,\ h > 0 \right\},$$

$$\mathscr{G}_{3,p}^{\ell,m} = \left\{ r : \mathcal{S}_V \longrightarrow \mathbb{R}\colon\ r(v) = \rho_{R_p}^{\ell,m}(v,u,t,\theta,h) \text{ for some } u \in \mathcal{S}_X,\ t \in \mathcal{T},\ \theta \in \Theta,\ h > 0 \right\}.$$

By the Euclidean properties of the classes $\mathscr{H}_{1d}^{\ell}$, $\mathscr{H}_{2d}^{\ell,m}$, $\mathscr{H}_{3}^{\ell}$ and $\mathscr{H}_{4}^{\ell,m}$, and by the conditions described in Assumptions 3 and 4, the Euclidean-preserving properties described, e.g, in Lemma 2.14 in Pakes and Pollard (1989), yield that $\mathscr{G}_{1,p}^{\ell,m}$, $\mathscr{G}_{2,p}^{\ell,m}$ and $\mathscr{G}_{3,p}^{\ell,m}$ are Euclidean classes for an envelope $\overline{G}(\cdot)$ for which there exists $\overline{\eta}_{\overline{G}}$ such that $E_F\left[\overline{G}(V)^{4q}\right] \le \overline{\eta}_{\overline{G}}$ for all $F \in \mathcal{F}$. Let

$$\nu_{\Upsilon_{R_p},n}^{\ell,m}(x,t,\theta,h) = \frac{1}{n}\sum_{i=1}^{n}\left(\Upsilon_{R_p}^{\ell,m}(V_i,x,t,\theta,h) - E_F\left[\Upsilon_{R_p}^{\ell,m}(V_i,x,t,\theta,h)\right]\right),$$

$$\nu_{\Phi_{R_p},n}^{\ell,m}(x,t,\theta,h) = \frac{1}{n}\sum_{i=1}^{n}\left(\Phi_{R_p}^{\ell,m}(V_i,x,t,\theta,h) - E_F\left[\Phi_{R_p}^{\ell,m}(V_i,x,t,\theta,h)\right]\right),$$

$$\nu_{\rho_{R_p},n}^{\ell,m}(x,t,\theta,h) = \frac{1}{n}\sum_{i=1}^{n}\left(\rho_{R_p}^{\ell,m}(V_i,x,t,\theta,h) - E_F\left[\rho_{R_p}^{\ell,m}(V_i,x,t,\theta,h)\right]\right)$$

The Euclidean properties and the existence-of-moment features of the envelope for the classes of functions $\mathscr{G}_{1,p}^{\ell,m}$, $\mathscr{G}_{2,p}^{\ell,m}$ and $\mathscr{G}_{3,p}^{\ell,m}$ allow us to apply Result S1 and we obtain that there exists $\overline{M}_1 < \infty$

such that, for any $b>0$,

$$\begin{aligned}
\sup_{F\in\mathcal{F}} P_F\left(\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}}\left|\nu^{\ell,m}_{\Upsilon_{R_p},n}(x,t,\theta,h)\right|\geq b\right) &\leq \frac{\overline{M}_1}{\left(n^{1/2}\cdot b\right)^q} = O\left(\frac{1}{\left(n^{1/2}\cdot b\right)^q}\right)\\
\sup_{F\in\mathcal{F}} P_F\left(\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}}\left|\nu^{\ell,m}_{\Phi_{R_p},n}(x,t,\theta,h)\right|\geq b\right) &\leq \frac{\overline{M}_1}{\left(n^{1/2}\cdot b\right)^q} = O\left(\frac{1}{\left(n^{1/2}\cdot b\right)^q}\right),\\
\sup_{F\in\mathcal{F}} P_F\left(\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}}\left|\nu^{\ell,m}_{\rho_{R_p},n}(x,t,\theta,h)\right|\geq b\right) &\leq \frac{\overline{M}_1}{\left(n^{1/2}\cdot b\right)^q} = O\left(\frac{1}{\left(n^{1/2}\cdot b\right)^q}\right)
\end{aligned} \tag{S3.2.2}$$

Note that (S3.2.2) implies, in particular, that

$$\left.\begin{aligned}
\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}}\left|\nu^{\ell,m}_{\Upsilon_{R_p},n}(x,t,\theta,h)\right| &= O_p\left(\tfrac{1}{n^{1/2}}\right)\\
\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}}\left|\nu^{\ell,m}_{\Phi_{R_p},n}(x,t,\theta,h)\right| &= O_p\left(\tfrac{1}{n^{1/2}}\right)\\
\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}}\left|\nu^{\ell,m}_{\rho_{R_p},n}(x,t,\theta,h)\right| &= O_p\left(\tfrac{1}{n^{1/2}}\right)
\end{aligned}\right\}\quad \text{uniformly over } \mathcal{F}. \tag{S3.2.3}$$

The results in (S3.2.2) and (S3.2.3) are analogous to those in (S3.1.8) and (S3.1.9), respectively. With these results at hand, let us proceed by analyzing the quadratic term in (S3.2.1). For a given $(x,t)\in\mathcal{S}_X\times\mathcal{T}$, $\theta\in\Theta$ and $F\in\mathcal{F}$, let

$$\xi^{R_p}_{b,n}(x,t,\theta) = \left(\widehat{\theta}-\theta^*_F\right)'\,\frac{\partial^2\widehat{R}_p(x,t,\theta)}{\partial\theta\partial\theta'}\left(\widehat{\theta}-\theta^*_F\right).$$

We have

$$
\begin{aligned}
\xi_{b,n}^{R_p}(x,t,\theta) &= \sum_{\ell=1}^{k}\sum_{m=1}^{k}\frac{\partial^2\widehat{R}_p(x,t,\theta)}{\partial\theta_\ell\theta_m}\cdot\left(\widehat{\theta}_m-\theta^*_{m,F}\right)\cdot\left(\widehat{\theta}_\ell-\theta^*_{\ell,F}\right)\\
&= \sum_{\ell=1}^{k}\sum_{m=1}^{k}\frac{1}{h_n^{D+2}}\cdot E_F\left[\Upsilon_{R_p}^{\ell,m}(V,x,t,\theta,h_n)\right]\cdot\left(\widehat{\theta}_m-\theta^*_{m,F}\right)\cdot\left(\widehat{\theta}_\ell-\theta^*_{\ell,F}\right)\\
&+ \sum_{\ell=1}^{k}\sum_{m=1}^{k}\frac{1}{h_n^{D+2}}\cdot \nu_{\Upsilon_{R_p},n}^{\ell,m}(x,t,\theta,h_n)\cdot\left(\widehat{\theta}_m-\theta^*_{m,F}\right)\cdot\left(\widehat{\theta}_\ell-\theta^*_{\ell,F}\right)\\
&+ \sum_{\ell=1}^{k}\sum_{m=1}^{k}\frac{1}{h_n^{D+1}}\cdot E_F\left[\Phi_{R_p}^{\ell,m}(V,x,t,\theta,h_n)\right]\cdot\left(\widehat{\theta}_m-\theta^*_{m,F}\right)\cdot\left(\widehat{\theta}_\ell-\theta^*_{\ell,F}\right) \qquad \text{(S3.2.4)}\\
&+ \sum_{\ell=1}^{k}\sum_{m=1}^{k}\frac{1}{h_n^{D+1}}\cdot \nu_{\Phi_{R_p},n}^{\ell,m}(x,t,\theta,h_n)\cdot\left(\widehat{\theta}_m-\theta^*_{m,F}\right)\cdot\left(\widehat{\theta}_\ell-\theta^*_{\ell,F}\right)\\
&+ \sum_{\ell=1}^{k}\sum_{m=1}^{k}\frac{1}{h_n^{D}}\cdot E_F\left[\rho_{R_p}^{\ell,m}(V,x,t,\theta,h_n)\right]\cdot\left(\widehat{\theta}_m-\theta^*_{m,F}\right)\cdot\left(\widehat{\theta}_\ell-\theta^*_{\ell,F}\right)\\
&+ \sum_{\ell=1}^{k}\sum_{m=1}^{k}\frac{1}{h_n^{D}}\cdot \nu_{\rho_{R_p},n}^{\ell,m}(x,t,\theta,h_n)\cdot\left(\widehat{\theta}_m-\theta^*_{m,F}\right)\cdot\left(\widehat{\theta}_\ell-\theta^*_{\ell,F}\right).
\end{aligned}
$$

Take any $b > 0$. From (S3.2.4), we have

$$P_F\left(\sup_{(x,\theta)\in\mathcal{S}_X\times\Theta}\left|\xi^{f_g}_{b,n}(x,\theta)\right|\geq b\right)\leq\underbrace{\sum_{\ell=1}^{k}\sum_{m=1}^{k}P_F\left(\frac{\overline{\eta}_{\Upsilon}}{h_n^{D+2}}\cdot\left|\widehat{\theta}_m-\theta^*_{m,F}\right|\cdot\left|\widehat{\theta}_\ell-\theta^*_{\ell,F}\right|\geq\frac{b}{6k^2}\right)}_{(A)}$$

$$+\underbrace{\sum_{\ell=1}^{k}\sum_{m=1}^{k}P_F\left(\frac{1}{h_n^{D+2}}\cdot\sup_{\substack{(x,\theta)\in\mathcal{S}_X\times\Theta\\h>0}}\left|\nu^{\ell,m}_{\Upsilon_{R_p},n}(x,\theta,h)\right|\cdot\left|\widehat{\theta}_m-\theta^*_{m,F}\right|\cdot\left|\widehat{\theta}_\ell-\theta^*_{\ell,F}\right|\geq\frac{b}{6k^2}\right)}_{(B)}$$

$$+\underbrace{\sum_{\ell=1}^{k}\sum_{m=1}^{k}P_F\left(\frac{\overline{\eta}_{\Phi}}{h_n^{D+1}}\cdot\left|\widehat{\theta}_m-\theta^*_{m,F}\right|\cdot\left|\widehat{\theta}_\ell-\theta^*_{\ell,F}\right|\geq\frac{b}{6k^2}\right)}_{(C)}$$

$$+\underbrace{\sum_{\ell=1}^{k}\sum_{m=1}^{k}P_F\left(\frac{1}{h_n^{D+1}}\cdot\sup_{\substack{(x,\theta)\in\mathcal{S}_X\times\Theta\\h>0}}\left|\nu^{\ell,m}_{\Phi_{R_p},n}(x,\theta,h)\right|\cdot\left|\widehat{\theta}_m-\theta^*_{m,F}\right|\cdot\left|\widehat{\theta}_\ell-\theta^*_{\ell,F}\right|\geq\frac{b}{6k^2}\right)}_{(D)}$$

$$+\underbrace{\sum_{\ell=1}^{k}\sum_{m=1}^{k}P_F\left(\frac{\overline{\eta}_{\rho}}{h_n^{D}}\cdot\left|\widehat{\theta}_m-\theta^*_{m,F}\right|\cdot\left|\widehat{\theta}_\ell-\theta^*_{\ell,F}\right|\geq\frac{b}{6k^2}\right)}_{(E)}$$

$$+\underbrace{\sum_{\ell=1}^{k}\sum_{m=1}^{k}P_F\left(\frac{1}{h_n^{D}}\cdot\sup_{\substack{(x,\theta)\in\mathcal{S}_X\times\Theta\\h>0}}\left|\nu^{\ell,m}_{\rho_{R_p},n}(x,\theta,h)\right|\cdot\left|\widehat{\theta}_m-\theta^*_{m,F}\right|\cdot\left|\widehat{\theta}_\ell-\theta^*_{\ell,F}\right|\geq\frac{b}{6k^2}\right)}_{(F)}$$

From here, using the results in (S3.2.2) and (S3.2.3) and the conditions in Assumption 1, parallel steps to those we used in equations (S3.1.14)-(S3.1.17) lead us to the following result, which is analogous to that in (S3.1.19)

$$
\begin{aligned}
\sup_{F\in\mathcal{F}} P_F\left(\sup_{(x,t)\in\mathcal{S}_X\times\mathcal{T}}\left|\xi^{R_p}_{b,n}(x,t,\overline{\theta}_x)\right|\geq b\right) &\leq \sup_{F\in\mathcal{F}} P_F\left(\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}}}\left|\xi^{R_p}_{b,n}(x,t,\theta)\right|\geq b\right)\\
&= O\left(\frac{1}{\left(h_n^{\frac{D+2}{2}}\cdot(n^{1/2}\wedge r_n)\cdot(b^{1/2}\wedge b^{1/4})\right)^q}\right)\quad \forall\ b>0,\quad \text{and}\\
\sup_{(x,t)\in\mathcal{S}_X\times\mathcal{T}}\left|\xi^{R_p}_{b,n}(x,t,\overline{\theta}_x)\right| &\leq \sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}}}\left|\xi^{R_p}_{b,n}(x,t,\theta)\right|\\
&= O_p\left(\frac{1}{n\cdot h_n^{D+2}}\right)\quad \text{uniformly over } \mathcal{F}.
\end{aligned}
\tag{S3.2.5}
$$

Now let us focus on the linear term in the second order approximation (S3.2.1). Let

$$
\xi^{R_p}_{a,n}(x,t,\theta)=\frac{\partial\widehat{R}_p(x,t,\theta)}{\partial\theta}\left(\widehat{\theta}-\theta^*_F\right).
$$

The decomposition of $\xi^{R_p}_{a,n}(x,t,\theta)$ will be described in detail in equation (S3.2.18). First, we need to introduce each one of the terms that appear there, as well as their relevant asymptotic properties. For each $p=1,\ldots,P$ define

$$
\begin{aligned}
\Lambda^{\ell}_{R_p}(V_i,x,t,\theta,h) &\equiv S_p(Y_i,t)\omega_p(g(X_i,\theta))\sum_{d=1}^{D}\nabla_d K\left(\frac{\Delta g(X_i,x,\theta)}{h}\right)\frac{\partial\Delta g_d(X_i,x,\theta)}{\partial\theta_\ell},\\
\beta^{\ell}_{R_p}(V_i,x,t,\theta,h) &\equiv S_p(Y_i,t)\frac{\partial\omega_p(g(X_i,\theta))}{\partial\theta_\ell}K\left(\frac{\Delta g(X_i,x,\theta)}{h}\right),\\
\underbrace{\Lambda_{R_p}(V_i,x,t,\theta,h)}_{1\times k} &\equiv \left(\Lambda^1_{R_p}(V_i,x,t,\theta,h),\ldots,\Lambda^k_{R_p}(V_i,x,t,\theta,h)\right),\\
\underbrace{\beta_{R_p}(V_i,x,t,\theta,h)}_{1\times k} &\equiv \left(\beta^1_{R_p}(V_i,x,t,\theta,h),\ldots,\beta^k_{R_p}(V_i,x,t,\theta,h)\right).
\end{aligned}
$$

By the conditions described in Assumptions 3 and 4, there exist $\overline{\eta}_\Lambda<\infty$ and $\overline{\eta}_\beta<\infty$ such that, for all $F\in\mathcal{F}$,

$$
\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}}\left\|E_F\left[\Lambda_{R_p}(V,x,t,\theta,h)\right]\right\|\leq\overline{\eta}_\Lambda\qquad \sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}}\left\|E_F\left[\beta_{R_p}(V,x,t,\theta,h)\right]\right\|\leq\overline{\eta}_\beta. \tag{S3.2.6}
$$

By the arguments we used above to establish the Euclidean properties of the classes of functions $\mathscr{G}_{1,p}^{\ell,m}$, $\mathscr{G}_{2,p}^{\ell,m}$ and $\mathscr{G}_{3,p}^{\ell,m}$, the following classes are also Euclidean,

$$\mathscr{M}_{4,p}^{\ell} = \left\{ r : \mathcal{S}_V \longrightarrow \mathbb{R} : r(v) = \Lambda_{R_p}^{\ell}(v,u,t,\theta,h) \ \text{ for some } u \in \mathcal{S}_X,\, t \in \mathcal{T},\, \theta \in \Theta,\, h > 0 \right\},$$
$$\mathscr{M}_{5,p}^{\ell} = \left\{ r : \mathcal{S}_V \longrightarrow \mathbb{R} : r(v) = \beta_{R_p}^{\ell}(v,u,t,\theta,h) \ \text{ for some } u \in \mathcal{S}_X,\, t \in \mathcal{T},\, \theta \in \Theta,\, h > 0 \right\}$$

for an envelope $\overline{M}_1(\cdot)$ such that $E_F\left[\overline{M}_1(V)^{4q}\right] \leq \overline{\eta}_{\overline{M}_1}$ for all $F \in \mathcal{F}$. From here, if we define

$$\begin{aligned}
\nu_{\Lambda_{R_p},n}^{\ell}(x,t,\theta,h) &= \frac{1}{n}\sum_{i=1}^{n}\left(\Lambda_{R_p}^{\ell}(V_i,x,t,\theta,h) - E_F\left[\Lambda_{R_p}^{\ell}(V_i,x,t,\theta,h)\right]\right),\\
\nu_{\beta_{R_p},n}^{\ell}(x,t,\theta,h) &= \frac{1}{n}\sum_{i=1}^{n}\left(\Lambda_{R_p}^{\ell}(V_i,x,t,\theta,h) - E_F\left[\Lambda_{R_p}^{\ell}(V_i,x,t,\theta,h)\right]\right),\\
\underbrace{\nu_{\Lambda_{R_p},n}(x,t,\theta,h)}_{1\times k} &= \left(\nu_{\Lambda_{R_p},n}^{1}(x,t,\theta,h),\ldots,\nu_{\Lambda_{R_p},n}^{k}(x,t,\theta,h)\right),\\
\underbrace{\nu_{\beta_{R_p},n}(x,t,\theta,h)}_{1\times k} &= \left(\nu_{\beta_{R_p},n}^{1}(x,t,\theta,h),\ldots,\nu_{\beta_{R_p},n}^{k}(x,t,\theta,h)\right),
\end{aligned}$$

then, as we have done before, Result S1 yields that, for any $b > 0$,

$$\begin{aligned}
\sup_{F\in\mathcal{F}} P_F\left(\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}} \left\|\nu_{\Lambda_{R_p},n}(x,t,\theta,h)\right\| \geq b\right) &= O\left(\frac{1}{\left(n^{1/2}\cdot b\right)^{q}}\right),\\
\sup_{F\in\mathcal{F}} P_F\left(\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}} \left\|\nu_{\beta_{R_p},n}(x,t,\theta,h)\right\| \geq b\right) &= O\left(\frac{1}{\left(n^{1/2}\cdot b\right)^{q}}\right)
\end{aligned} \tag{S3.2.7}$$

and

$$\left.\begin{aligned} \sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}} \left\|\nu_{\Lambda_{R_p},n}(x,t,\theta,h)\right\| &= O_p\left(\tfrac{1}{n^{1/2}}\right)\\ \sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}} \left\|\nu_{\beta_{R_p},n}(x,t,\theta,h)\right\| &= O_p\left(\tfrac{1}{n^{1/2}}\right) \end{aligned}\right\} \text{ uniformly over } \mathcal{F}. \tag{S3.2.8}$$

Note that, under our assumptions, there exist $\overline{\eta}_{m,\Lambda} < \infty$ and $\overline{\eta}_{m,\beta} < \infty$ such that, for all $F \in \mathcal{F}$,

$$\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}} \left|E_F\left[\Lambda_{R_p}(V_i,x,t,\theta,h)\psi_F^{\theta}(Z_i)\right]\right| \le \overline{\eta}_{m,\Lambda}, \qquad \sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}} \left|E_F\left[\Lambda_{R_p}(V_i,x,t,\theta,h)\psi_F^{\theta}(Z_i)\right]\right| \le \overline{\eta}_{m,\beta} \tag{S3.2.9}$$

Recall that we have defined $v \equiv (y,x,z)$. Take the classes of functions

$$\begin{aligned} \mathscr{M}_{6,F}^{p} &= \left\{m : \mathcal{S}_V \longrightarrow \mathbb{R} \colon m(v_1) = \Lambda_{R_p}(v_1,u,t,\theta,h)\psi_F^{\theta}(z_1) \text{ for some } u\in\mathcal{S}_X,\, t\in\mathcal{T},\, \theta\in\Theta,\, h>0\right\},\\ \mathscr{M}_{7,F}^{p} &= \left\{m : \mathcal{S}_V \longrightarrow \mathbb{R} \colon m(v_1) = \beta_{R_p}(v_1,u,t,\theta,h)\psi_F^{\theta}(z_1) \text{ for some } u\in\mathcal{S}_X,\, t\in\mathcal{T},\, \theta\in\Theta,\, h>0\right\} \end{aligned}$$

By the properties of the classes $\mathscr{M}_{4,p}^{\ell}$ and $\mathscr{M}_{5,p}^{\ell}$ described above, and the integrability properties of the influence function $\psi_F^{\theta}(Z)$ described in Assumption 1, the Euclidean-preserving result in Lemma 2.14 in Pakes and Pollard (1989) yields that both $\mathscr{M}_{6,F}^{p}$ and $\mathscr{M}_{7,F}^{p}$ are Euclidean classes for an envelope $\overline{M}_2(\cdot)$ such that $E_F\left[\overline{M}_2(V)^{4q}\right] \le \overline{\eta}_{\overline{M}_2}$ for all $F \in \mathcal{F}$. Next, let

$$\begin{aligned} m_{\Lambda,n}^{R_p}(x,t,\theta,h) &= \frac{1}{n}\sum_{i=1}^{n}\left(\Lambda_{R_p}(V_i,x,t,\theta,h)\psi_F^{\theta}(Z_i) - E_F\left[\Lambda_{R_p}(V_i,x,t,\theta,h)\psi_F^{\theta}(Z_i)\right]\right),\\ m_{\beta,n}^{R_p}(x,t,\theta,h) &= \frac{1}{n}\sum_{i=1}^{n}\left(\beta_{R_p}(V_i,x,t,\theta,h)\psi_F^{\theta}(Z_i) - E_F\left[\beta_{R_p}(V_i,x,t,\theta,h)\psi_F^{\theta}(Z_i)\right]\right), \end{aligned}$$

By the Euclidean properties of the classes $\mathscr{M}^{p}_{6,F}$ and $\mathscr{M}^{p}_{7,F}$ described above, once again Result S1 yields that, for any $b > 0$,

$$\sup_{F\in\mathcal{F}} P_F\left(\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}} \left|m^{R_p}_{\Lambda,n}(x,t,\theta,h)\right| \geq b\right) = O\left(\frac{1}{\left(n^{1/2}\cdot b\right)^q}\right),$$

$$\sup_{F\in\mathcal{F}} P_F\left(\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}} \left|m^{R_p}_{\beta,n}(x,t,\theta,h)\right| \geq b\right) = O\left(\frac{1}{\left(n^{1/2}\cdot b\right)^q}\right), \tag{S3.2.10}$$

and

$$\left.\begin{array}{ll}\sup\limits_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}} \left|m^{R_p}_{\Lambda,n}(x,t,\theta,h)\right| & = O_p\left(\frac{1}{n^{1/2}}\right)\\ \sup\limits_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}} \left|m^{R_p}_{\beta,n}(x,t,\theta,h)\right| & = O_p\left(\frac{1}{n^{1/2}}\right)\end{array}\right\} \text{ uniformly over } \mathcal{F}.$$

Now, consider the following classes of functions on $\mathcal{S}^2_V$, which will be relevant for the U-processes that will appear in our analysis,

$$\mathscr{G}^{p}_{4,F} = \left\{m : \mathcal{S}^2_V \longrightarrow \mathbb{R}\colon\ m(v_1,v_2) = \Lambda_{R_p}(v_1,s,t,\theta,h)\psi^{\theta}_F(z_2) \ \text{ for some } s\in\mathcal{S}_X,\ \theta\in\Theta,\ h>0\right\},$$
$$\mathscr{G}^{p}_{5,F} = \left\{m : \mathcal{S}^2_V \longrightarrow \mathbb{R}\colon\ m(v_1,v_2) = \beta_{R_p}(v_1,s,t,\theta,h)\psi^{\theta}_F(z_2) \ \text{ for some } s\in\mathcal{S}_X,\ \theta\in\Theta,\ h>0\right\}.$$

Similar to the case of $\mathscr{M}^{p}_{6,F}$ and $\mathscr{M}^{p}_{7,F}$, the Euclidean properties of the classes $\mathscr{M}^{\ell}_{4,p}$ and $\mathscr{M}^{\ell}_{5,p}$ described above, and the integrability properties of the influence function $\psi^{\theta}_F(Z)$ described in Assumption 1, the Euclidean-preserving result in Lemma 2.14 in Pakes and Pollard (1989) yields that both $\mathscr{G}^{p}_{4,F}$ and $\mathscr{G}^{p}_{5,F}$ are Euclidean classes for an envelope $\overline{G}_4(\cdot)$ for which there exists $\overline{\mu}_{\overline{G}_4}$ such that $E_F\left[\overline{G}_4(V_1,Z_2)^{4q}\right] \leq \overline{\eta}_{\overline{G}_4} < \infty$ for all $F\in\mathcal{F}$ (with $(X_1,Z_2)\sim F\otimes F$), with q being the integer described in Assumption 1. Next, define the following U-statistics,

$$U^{\Lambda}_{R_p,n}(x,t,\theta,h) = \frac{1}{n(n-1)}\sum_{i=1}^{n}\sum_{j\neq i}\Lambda_{R_p}(V_i,x,t,\theta,h)\psi^{\theta}_F(Z_j),$$
$$U^{\beta}_{R_p,n}(x,t,\theta,h) = \frac{1}{n(n-1)}\sum_{i=1}^{n}\sum_{j\neq i}\beta_{R_p}(V_i,x,t,\theta,h)\psi^{\theta}_F(Z_j), \tag{S3.2.11}$$

Let

$$\begin{aligned}
\varphi^{R_p}_{\Lambda,F}(Z_i,x,t,\theta,h) &= E_F\left[\Lambda_{R_p}(V,x,t,\theta,h)\right]\psi^{\theta}_F(Z_i),\\
\vartheta^{R_p}_{\Lambda,F}(V_i,V_j,x,t,\theta,h) &= \Lambda_{R_p}(V_i,x,t,h)\psi^{\theta}_F(Z_j)+\Lambda_{R_p}(V_j,x,t,h)\psi^{\theta}_F(Z_i)-\varphi^{R_p}_{\Lambda,F}(Z_i,x,t,\theta,h)-\varphi^{R_p}_{\Lambda,F}(Z_j,x,t,\theta,h),\\
\varphi^{R_p}_{\beta,F}(Z_i,x,t,\theta,h) &= E_F\left[\beta_{R_p}(V,x,t,\theta,h)\right]\psi^{\theta}_F(Z_i),\\
\vartheta^{R_p}_{\beta,F}(V_i,V_j,x,t,\theta,h) &= \beta_{R_p}(V_i,x,t,h)\psi^{\theta}_F(Z_j)+\beta_{R_p}(V_j,x,t,h)\psi^{\theta}_F(Z_i)-\varphi^{R_p}_{\beta,F}(Z_i,x,t,\theta,h)-\varphi^{R_p}_{\beta,F}(Z_j,x,t,\theta,h),
\end{aligned} \tag{S3.2.12}$$

Note that $\vartheta^{R_p}_{\Lambda,F}(v_1,v_2,x,t,\theta,h)=\vartheta^{R_p}_{\Lambda,F}(v_2,v_1,x,t,\theta,h)$ and $\vartheta^{R_p}_{\beta,F}(v_1,v_2,x,t,\theta,h)=\vartheta^{R_p}_{\beta,F}(v_2,v_1,x,t,\theta,h)$ (they are both symmetric in their first two arguments), and

$$\begin{aligned}
E_F\Big[\vartheta^{R_p}_{\Lambda,F}(V_i,V_j,x,t,\theta,h)\Big|V_i\Big] = E_F\Big[\vartheta^{R_p}_{\Lambda,F}(V_i,V_j,x,t,\theta,h)\Big|V_j\Big] = 0,\\
E_F\Big[\vartheta^{R_p}_{\beta,F}(V_i,V_j,x,t,\theta,h)\Big|V_i\Big] = E_F\Big[\vartheta^{R_p}_{\beta,F}(V_i,V_j,x,t,\theta,h)\Big|V_j\Big] = 0
\end{aligned}$$

From the functionals in (S3.2.12), define the following degenerate U-statistics

$$\begin{aligned}
\widetilde{U}^{\Lambda}_{R_p,n}(x,t,\theta,h) &= \frac{1}{n(n-1)}\sum_{i=1}^{n}\sum_{j\neq i}\vartheta^{R_p}_{\Lambda,F}(V_i,V_j,x,t,\theta,h),\\
\widetilde{U}^{\beta}_{R_p,n}(x,t,\theta,h) &= \frac{1}{n(n-1)}\sum_{i=1}^{n}\sum_{j\neq i}\vartheta^{R_p}_{\beta,F}(V_i,V_j,x,t,\theta,h)
\end{aligned}$$

From the Euclidean properties of the classes of functions $\mathscr{G}^p_{4,F}$ and $\mathscr{G}^p_{5,F}$ described above, Lemma 20 in Nolan and Pollard (1987) (or Lemma 5 in Sherman (1994)) along (once again) with the Euclidean-preserving properties described in Lemma 2.14 in Pakes and Pollard (1989) yield that the classes of functions

$$\begin{aligned}
\mathscr{G}^p_{6,F} &= \Big\{m:\mathcal{S}^2_V\longrightarrow\mathbb{R}\colon\ m(v_1,v_2)=\vartheta^{R_p}_{\Lambda,F}(v_1,v_2,x,t,\theta,h)\ \text{ for some } x\in\mathcal{S}_X,\ t\in\mathcal{T},\ \theta\in\Theta,\ h>0\Big\},\\
\mathscr{G}^p_{7,F} &= \Big\{m:\mathcal{S}^2_V\longrightarrow\mathbb{R}\colon\ m(v_1,v_2)=\vartheta^{R_p}_{\beta,F}(v_1,v_2,x,t,\theta,h)\ \text{ for some } x\in\mathcal{S}_X,\ t\in\mathcal{T},\ \theta\in\Theta,\ h>0\Big\}
\end{aligned}$$

are Euclidean for an envelope $\overline{G}_5(\cdot)$ that satisfies $E_F\Big[\overline{G}_5(V_1,V_2)^{4q}\Big]\leq\overline{\eta}_{\overline{G}_5}<\infty$ for all $F\in\mathcal{F}$ (with $(V_1,V_2)\sim F\otimes F$) and q being the integer described in Assumption 1. From here, applying once again Result S1 we obtain that, for any $b > 0$,

$$\sup_{F\in\mathcal{F}} P_F\left(\sup_{\substack{\theta\in\Theta \\ (x,t)\in\mathcal{S}_X\times\mathcal{T} \\ h>0}} \left|\widetilde{U}^{\Lambda}_{R_p,n}(x,t,\theta,h)\right| \geq b \right) = O\left(\frac{1}{(n\cdot b)^q}\right),$$

$$\sup_{F\in\mathcal{F}} P_F\left(\sup_{\substack{\theta\in\Theta \\ (x,t)\in\mathcal{S}_X\times\mathcal{T} \\ h>0}} \left|\widetilde{U}^{\beta}_{R_p,n}(x,t,\theta,h)\right| \geq b \right) = O\left(\frac{1}{(n\cdot b)^q}\right), \tag{S3.2.13}$$

and

$$\left.\begin{aligned} \sup_{\substack{\theta\in\Theta \\ (x,t)\in\mathcal{S}_X\times\mathcal{T} \\ h>0}} \left|\widetilde{U}^{\Lambda}_{R_p,n}(x,t,\theta,h)\right| &= O_p\left(\tfrac{1}{n}\right) \\ \sup_{\substack{\theta\in\Theta \\ (x,t)\in\mathcal{S}_X\times\mathcal{T} \\ h>0}} \left|\widetilde{U}^{\beta}_{R_p,n}(x,t,\theta,h)\right| &= O_p\left(\tfrac{1}{n}\right) \end{aligned}\right\} \text{ uniformly over } \mathcal{F}$$

Note from the definitions in (S3.2.12) that

$$E_F\left[\varphi^{R_p}_{\Lambda,F}(Z_i,x,t,\theta,h)\right] = E_F\left[\varphi^{R_p}_{\beta,F}(Z_i,x,t,\theta,h)\right] = 0.$$

Define

$$\nu^{\Lambda}_{\varphi_{R_p},n}(x,t,\theta,h) = \frac{1}{n}\sum_{i=1}^{n}\varphi^{R_p}_{\Lambda,F}(Z_i,x,t,\theta,h), \quad \text{and} \quad \nu^{\beta}_{\varphi_{R_p},n}(x,t,\theta,h) = \frac{1}{n}\sum_{i=1}^{n}\varphi^{R_p}_{\beta,F}(Z_i,x,t,\theta,h)$$

And consider the classes of functions,

$$\begin{aligned} \mathscr{G}^p_{8,F} &= \left\{ m : \mathcal{S}_Z \longrightarrow \mathbb{R} \colon m(z) = \varphi^{R_p}_{\Lambda,F}(z,x,t,\theta,h) \text{ for some } x\in\mathcal{S}_X,\, t\in\mathcal{T},\, \theta\in\Theta,\, h>0 \right\}, \\ \mathscr{G}^p_{9,F} &= \left\{ m : \mathcal{S}_Z \longrightarrow \mathbb{R} \colon m(z) = \varphi^{R_p}_{\beta,F}(z,x,t,\theta,h) \text{ for some } x\in\mathcal{S}_X,\, t\in\mathcal{T},\, \theta\in\Theta,\, h>0 \right\}, \end{aligned}$$

By the same arguments used for the classes $\mathscr{G}^p_{6,F}$ and $\mathscr{G}^p_{7,F}$ (the Euclidean properties of the classes $\mathscr{G}^p_{4,F}$ and $\mathscr{G}^p_{5,F}$ combined with Lemma 20 in Nolan and Pollard (1987) or Lemma 5 in Sherman (1994)) we obtain that both $\mathscr{G}^p_{8,F}$ and $\mathscr{G}^p_{9,F}$ are Euclidean classes of functions for an envelope $\overline{G}_6(\cdot)$ that satisfies $E_F\left[\overline{G}_6(Z)^{4q}\right] \leq \overline{\eta}_{\overline{G}_6} < \infty$ for all $F\in\mathcal{F}$ and q being the integer described in Assumption

1. From here, once again, applying Result S1 we obtain that, for any $b > 0$,

$$\sup_{F\in\mathcal{F}} P_F\left(\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}} \left|\nu^{\Lambda}_{\varphi_{R_p},n}(x,t,\theta,h)\right| \geq b\right) = O\left(\frac{1}{\left(n^{1/2}\cdot b\right)^q}\right),$$
$$\sup_{F\in\mathcal{F}} P_F\left(\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}} \left|\nu^{\beta}_{\varphi_{R_p},n}(x,t,\theta,h)\right| \geq b\right) = O\left(\frac{1}{\left(n^{1/2}\cdot b\right)^q}\right), \tag{S3.2.14}$$

and

$$\left.\begin{aligned} \sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}} \left|\nu^{\Lambda}_{\varphi_{R_p},n}(x,t,\theta,h)\right| &= O_p\left(\tfrac{1}{n^{1/2}}\right)\\ \sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}} \left|\nu^{\beta}_{\varphi_{R_p},n}(x,t,\theta,h)\right| &= O_p\left(\tfrac{1}{n^{1/2}}\right) \end{aligned}\right\} \text{ uniformly over } \mathcal{F}$$

Equipped with the results in (S3.2.13) and (S3.2.14), let us look at the Hoeffding decomposition of the U-statistics $U^{\Lambda}_{R_p,n}(x,t,\theta,h)$ and $U^{\beta}_{R_p,n}(x,t,\theta,h)$ defined in (S3.2.11). These are given by (see Serfling (1980, pages 177-178) or Sherman (1994, equations (6)-(7))),

$$\begin{aligned} U^{\Lambda}_{R_p,n}(x,t,\theta,h) &= \nu^{\Lambda}_{\varphi_{R_p},n}(x,t,\theta,h) + \frac{1}{2}\cdot\widetilde{U}^{\Lambda}_{R_p,n}(x,t,\theta,h),\\ U^{\beta}_{R_p,n}(x,t,\theta,h) &= \nu^{\beta}_{\varphi_{R_p},n}(x,t,\theta,h) + \frac{1}{2}\cdot\widetilde{U}^{\beta}_{R_p,n}(x,t,\theta,h). \end{aligned} \tag{S3.2.15}$$

Therefore, from (S3.2.13) and (S3.2.14), for any $b > 0$ we have

$$\begin{aligned}
&\sup_{F\in\mathcal{F}} P_F\left(\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}}\left|U^{\Lambda}_{R_p,n}(x,t,\theta,h)\right|\geq b\right)\\
\leq&\sup_{F\in\mathcal{F}} P_F\left(\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}}\left|\nu^{\Lambda}_{\varphi_{R_p,n}}(x,t,\theta,h)\right|\geq\frac{b}{2}\right)+\sup_{F\in\mathcal{F}} P_F\left(\frac{1}{2}\cdot\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}}\left|\widetilde{U}^{\Lambda}_{R_p,n}(x,t,\theta,h)\right|\geq\frac{b}{2}\right)\\
=&O\left(\frac{1}{(n\cdot b)^q}\right)+O\left(\frac{1}{\left(n^{1/2}\cdot b\right)^q}\right)=O\left(\frac{1}{\left(n^{1/2}\cdot b\right)^q}\right)
\end{aligned}\tag{S3.2.16}$$

and

$$\begin{aligned}
&\sup_{F\in\mathcal{F}} P_F\left(\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}}\left|U^{\beta}_{R_p,n}(x,t,\theta,h)\right|\geq b\right)\\
\leq&\sup_{F\in\mathcal{F}} P_F\left(\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}}\left|\nu^{\beta}_{\varphi_{R_p,n}}(x,t,\theta,h)\right|\geq\frac{b}{2}\right)+\sup_{F\in\mathcal{F}} P_F\left(\frac{1}{2}\cdot\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}}\left|\widetilde{U}^{\beta}_{R_p,n}(x,t,\theta,h)\right|\geq\frac{b}{2}\right)\\
=&O\left(\frac{1}{(n\cdot b)^q}\right)+O\left(\frac{1}{\left(n^{1/2}\cdot b\right)^q}\right)=O\left(\frac{1}{\left(n^{1/2}\cdot b\right)^q}\right)
\end{aligned}\tag{S3.2.17}$$

Equipped with the results in (S3.2.6)-(S3.2.17), we are ready to analyze the components of $\xi^{R_p}_{a,n}(x,t,\theta)$.

We have,

$$
\begin{aligned}
\xi_{a,n}^{R_p}(x,t,\theta) &\equiv \frac{\partial \widehat{R}_p(x,t,\theta)}{\partial \theta}\left(\widehat{\theta}-\theta_F^*\right) \\
&= \frac{1}{n^2\cdot h_n^{D+1}}\sum_{i=1}^{n}\sum_{j=1}^{n}\Lambda_{R_p}(V_i,x,t,\theta,h_n)\psi_F^{\theta}(Z_j)+\left(\frac{1}{n\cdot h_n^{D+1}}\sum_{i=1}^{n}\Lambda_{R_p}(V_i,x,t,\theta,h_n)\right)\varepsilon_n^{\theta} \\
&+ \frac{1}{n^2\cdot h_n^{D}}\sum_{i=1}^{n}\sum_{j=1}^{n}\beta_{R_p}(V_i,x,t,\theta,h_n)\psi_F^{\theta}(Z_j)+\left(\frac{1}{n\cdot h_n^{D}}\sum_{i=1}^{n}\beta_{R_p}(V_i,x,t,\theta,h_n)\right)\varepsilon_n^{\theta} \\
&= \frac{1}{h_n^{D+1}}\cdot\left(\frac{n-1}{n}\right)\cdot U_{R_p,n}^{\Lambda}(x,t,\theta,h_n)+\frac{1}{h_n^{D}}\cdot\left(\frac{n-1}{n}\right)\cdot U_{R_p,n}^{\beta}(x,t,\theta,h_n) \\
&+ \frac{1}{n\cdot h_n^{D+1}}\cdot E_F\left[\Lambda_{R_p}(V,x,t,\theta,h_n)\psi_F^{\theta}(Z)\right]+\frac{1}{n\cdot h_n^{D+1}}\cdot m_{\Lambda,n}^{R_p}(x,t,\theta,h_n) \\
&+ \frac{1}{h_n^{D+1}}\cdot E_F\left[\Lambda_{R_p}(V,x,t,\theta,h_n)\right]\varepsilon_n^{\theta}+\frac{1}{h_n^{D+1}}\cdot \nu_{\Lambda_{R_p},n}(x,t,\theta,h_n)\varepsilon_n^{\theta} \\
&+ \frac{1}{n\cdot h_n^{D}}\cdot E_F\left[\beta_{R_p}(V,x,t,\theta,h_n)\psi_F^{\theta}(Z)\right]+\frac{1}{n\cdot h_n^{D}}\cdot m_{\beta,n}^{R_p}(x,t,\theta,h_n) \\
&+ \frac{1}{h_n^{D}}\cdot E_F\left[\beta_{R_p}(V,x,t,\theta,h_n)\right]\varepsilon_n^{\theta}+\frac{1}{h_n^{D}}\cdot \nu_{\beta_{R_p},n}(x,t,\theta,h_n)\varepsilon_n^{\theta}
\end{aligned}
\tag{S3.2.18}
$$

From (S3.2.6), (S3.2.9) and (S3.2.18),

$$
\begin{aligned}
\left|\xi_{a,n}^{R_p}(x,t,\theta)\right| &\leq \frac{1}{h_n^{D+1}}\cdot\left(\frac{n-1}{n}\right)\cdot\left|U_{R_p,n}^{\Lambda}(x,t,\theta,h_n)\right|+\frac{1}{h_n^{D}}\cdot\left(\frac{n-1}{n}\right)\cdot\left|U_{R_p,n}^{\beta}(x,t,\theta,h_n)\right| \\
&+ \frac{1}{n\cdot h_n^{D+1}}\cdot\left|m_{\Lambda,n}^{R_p}(x,t,\theta,h_n)\right| \\
&+ \frac{1}{h_n^{D+1}}\cdot\left\|\nu_{\Lambda_{R_p},n}(x,t,\theta,h_n)\right\|\cdot\left\|\varepsilon_n^{\theta}\right\|+\frac{\overline{\eta}_{\Lambda}}{h_n^{D+1}}\cdot\left\|\varepsilon_n^{\theta}\right\|+\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}, \\
&+ \frac{1}{n\cdot h_n^{D}}\cdot\left|m_{\beta,n}^{R_p}(x,t,\theta,h_n)\right| \\
&+ \frac{1}{h_n^{D}}\cdot\left\|\nu_{\beta_{R_p},n}(x,t,\theta,h_n)\right\|\cdot\left\|\varepsilon_n^{\theta}\right\|+\frac{\overline{\eta}_{\beta}}{h_n^{D}}\cdot\left\|\varepsilon_n^{\theta}\right\|+\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^{D}}
\end{aligned}
$$

Thus, for any $b > 0$,

$$
\begin{aligned}
&\sup_{F\in\mathcal{F}} P_F\left(\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}}}\left|\xi^{R_p}_{a,n}(x,t,\theta)\right| \geq b\right)\\
\leq&\sup_{F\in\mathcal{F}} P_F\left(\frac{1}{h_n^{D+1}}\cdot\left(\frac{n-1}{n}\right)\cdot\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}}\left|U^{\Lambda}_{R_p,n}(x,t,\theta,h)\right| \geq \frac{1}{8}\cdot\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^{D}}\right)\right)\\
+&\sup_{F\in\mathcal{F}} P_F\left(\frac{1}{h_n^{D}}\cdot\left(\frac{n-1}{n}\right)\cdot\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}}\left|U^{\beta}_{R_p,n}(x,t,\theta,h)\right| \geq \frac{1}{8}\cdot\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^{D}}\right)\right)\\
+&\sup_{F\in\mathcal{F}} P_F\left(\frac{1}{n\cdot h_n^{D+1}}\cdot\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}}\left|m^{R_p}_{\Lambda,n}(x,t,\theta,h)\right| \geq \frac{1}{8}\cdot\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^{D}}\right)\right)\\
+&\sup_{F\in\mathcal{F}} P_F\left(\frac{1}{h_n^{D+1}}\cdot\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}}\left\|\nu_{\Lambda_{R_p},n}(x,t,\theta,h)\right\|\cdot\left\|\varepsilon^{\theta}_{n}\right\| \geq \frac{1}{8}\cdot\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^{D}}\right)\right)\\
+&\sup_{F\in\mathcal{F}} P_F\left(\frac{\overline{\eta}_{\Lambda}}{h_n^{D+1}}\cdot\left\|\varepsilon^{\theta}_{n}\right\| \geq \frac{1}{8}\cdot\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^{D}}\right)\right)\\
+&\sup_{F\in\mathcal{F}} P_F\left(\frac{1}{n\cdot h_n^{D}}\cdot\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}}\left|m^{R_p}_{\beta,n}(x,t,\theta,h)\right| \geq \frac{1}{8}\cdot\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^{D}}\right)\right)\\
+&\sup_{F\in\mathcal{F}} P_F\left(\frac{1}{h_n^{D}}\cdot\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}}\left\|\nu_{\beta_{R_p},n}(x,t,\theta,h)\right\|\cdot\left\|\varepsilon^{\theta}_{n}\right\| \geq \frac{1}{8}\cdot\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^{D}}\right)\right)\\
+&\sup_{F\in\mathcal{F}} P_F\left(\frac{\overline{\eta}_{\beta}}{h_n^{D}}\cdot\left\|\varepsilon^{\theta}_{n}\right\| \geq \frac{1}{8}\cdot\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^{D}}\right)\right)
\end{aligned}
\tag{S3.2.19}
$$

Using our previous results we can analyze each of the terms on the right hand side of (S3.2.19). If $b > 0$ is fixed[5], there exists n_0 such that $b - \frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}} - \frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^{D}} > 0\ \forall\ n > n_0$. From (S3.2.16), it follows that

[5]As before, what follows is true for any sequence $b_n > 0$ such that $b_n \cdot n \cdot h_n^{D+1} \longrightarrow \infty$.

$$\sup_{F\in\mathcal{F}} P_F\left(\frac{1}{h_n^{D+1}}\cdot\left(\frac{n-1}{n}\right)\cdot \sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}}\left|U^{\Lambda}_{R_p,n}(x,t,\theta,h)\right|\geq\frac{1}{8}\cdot\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^{D}}\right)\right)$$

$$=\sup_{F\in\mathcal{F}} P_F\left(\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}}\left|U^{\Lambda}_{R_p,n}(x,t,\theta,h)\right|\geq\frac{1}{8}\cdot\left(\frac{n}{n-1}\right)\cdot h_n^{D+1}\cdot\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^{D}}\right)\right) \tag{S3.2.20A}$$

$$=O\left(\frac{1}{\left(n^{1/2}\cdot h_n^{D+1}\cdot\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^{D}}\right)\right)^q}\right)$$

From (S3.2.17),

$$\sup_{F\in\mathcal{F}} P_F\left(\frac{1}{h_n^{D}}\cdot\left(\frac{n-1}{n}\right)\cdot \sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}}\left|U^{\beta}_{R_p,n}(x,t,\theta,h)\right|\geq\frac{1}{8}\cdot\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^{D}}\right)\right)$$

$$=\sup_{F\in\mathcal{F}} P_F\left(\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}}\left|U^{\beta}_{R_p,n}(x,t,\theta,h)\right|\geq\frac{1}{8}\cdot\left(\frac{n}{n-1}\right)\cdot h_n^{D+1}\cdot\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^{D}}\right)\right) \tag{S3.2.20B}$$

$$=O\left(\frac{1}{\left(n^{1/2}\cdot h_n^{D}\cdot\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^{D}}\right)\right)^q}\right)$$

From (S3.2.10),

$$
\begin{aligned}
&\sup_{F\in\mathcal{F}} P_F\left(\frac{1}{n\cdot h_n^{D+1}}\cdot\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}}\left|m_{\Lambda,n}^{R_p}(x,t,\theta,h)\right|\geq\frac{1}{8}\cdot\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^{D}}\right)\right)\\
=&\sup_{F\in\mathcal{F}} P_F\left(\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}}\left|m_{\Lambda,n}^{R_p}(x,t,\theta,h)\right|\geq\frac{1}{8}\cdot n\cdot h_n^{D+1}\cdot\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^{D}}\right)\right)\\
=&O\left(\frac{1}{\left(n^{3/2}\cdot h_n^{D+1}\cdot\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^{D}}\right)\right)^q}\right)
\end{aligned}
\tag{S3.2.20C}
$$

Next, from (S3.2.7) and Assumption 1,

$$
\begin{aligned}
&\sup_{F\in\mathcal{F}} P_F\left(\frac{1}{h_n^{D+1}}\cdot\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}}\left\|\nu_{\Lambda_{R_p},n}(x,t,\theta,h)\right\|\cdot\left\|\varepsilon_n^{\theta}\right\|\geq\frac{1}{8}\cdot\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^{D}}\right)\right)\\
\leq&\sup_{F\in\mathcal{F}} P_F\left(\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}}\left\|\nu_{\Lambda_{R_p},n}(x,t,\theta,h)\right\|\geq\left(\frac{1}{8}\cdot h_n^{D+1}\cdot\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^{D}}\right)\right)^{1/2}\right)\\
+&\sup_{F\in\mathcal{F}} P_F\left(\left\|\varepsilon_n^{\theta}\right\|\geq\left(\frac{1}{8}\cdot h_n^{D+1}\cdot\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^{D}}\right)\right)^{1/2}\right)\\
=&O\left(\frac{1}{\left(n^{1/2}\cdot h_n^{\frac{D+1}{2}}\cdot\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^{D}}\right)^{1/2}\right)^q}\right)+O\left(\frac{1}{\left(r_n\cdot h_n^{\frac{D+1}{2}}\cdot\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^{D}}\right)^{1/2}\right)^q}\right)\\
=&O\left(\frac{1}{\left((n^{1/2}\wedge r_n)\cdot h_n^{\frac{D+1}{2}}\cdot\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^{D}}\right)^{1/2}\right)^q}\right)
\end{aligned}
\tag{S3.2.20D}
$$

Next, also from Assumption 1,

$$\sup_{F\in\mathcal{F}} P_F\left(\frac{\overline{\eta}_\Lambda}{h_n^{D+1}}\cdot\left\|\varepsilon_n^\theta\right\|\geq\frac{1}{4}\cdot\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^D}\right)\right)=\sup_{F\in\mathcal{F}} P_F\left(\left\|\varepsilon_n^\theta\right\|\geq\frac{h_n^{D+1}}{4\overline{\eta}_\Lambda}\cdot\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^D}\right)\right)$$
$$=O\left(\frac{1}{\left(r_n\cdot h_n^{D+1}\cdot\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^D}\right)\right)^q}\right)\tag{S3.2.20E}$$

From (S3.2.10),

$$\sup_{F\in\mathcal{F}} P_F\left(\frac{1}{n\cdot h_n^D}\cdot\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}}\left|m_{\beta,n}^{R_p}(x,t,\theta,h)\right|\geq\frac{1}{8}\cdot\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^D}\right)\right)$$
$$=\sup_{F\in\mathcal{F}} P_F\left(\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}}\left|m_{\beta,n}^{R_p}(x,t,\theta,h)\right|\geq\frac{1}{8}\cdot n\cdot h_n^D\cdot\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^D}\right)\right)\tag{S3.2.20F}$$
$$=O\left(\frac{1}{\left(n^{3/2}\cdot h_n^D\cdot\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^D}\right)\right)^q}\right)$$

From (S3.2.7) and Assumption 1,

$$\begin{aligned}
&\sup_{F\in\mathcal{F}} P_F\left(\frac{1}{h_n^D}\cdot \sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}}\left\|\nu_{\beta_{R_p},n}(x,t,\theta,h)\right\|\cdot\left\|\varepsilon_n^\theta\right\|\geq\frac{1}{8}\cdot\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^D}\right)\right)\\
\leq&\sup_{F\in\mathcal{F}} P_F\left(\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}}\left\|\nu_{\beta_{R_p},n}(x,t,\theta,h)\right\|\geq\left(\frac{1}{8}\cdot h_n^D\cdot\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^D}\right)\right)^{1/2}\right)\\
+&\sup_{F\in\mathcal{F}} P_F\left(\left\|\varepsilon_n^\theta\right\|\geq\left(\frac{1}{8}\cdot h_n^D\cdot\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^D}\right)\right)^{1/2}\right)\\
=&O\left(\frac{1}{\left(n^{1/2}\cdot h_n^{\frac{D}{2}}\cdot\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^D}\right)^{1/2}\right)^q}\right)+O\left(\frac{1}{\left(r_n\cdot h_n^{\frac{D}{2}}\cdot\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^D}\right)^{1/2}\right)^q}\right)\\
=&O\left(\frac{1}{\left((n^{1/2}\wedge r_n)\cdot h_n^{\frac{D}{2}}\cdot\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^D}\right)^{1/2}\right)^q}\right)
\end{aligned}\tag{S3.2.20G}$$

Also from Assumption 1,

$$\begin{aligned}
&\sup_{F\in\mathcal{F}} P_F\left(\frac{\overline{\eta}_\Lambda}{h_n^D}\cdot\left\|\varepsilon_n^\theta\right\|\geq\frac{1}{4}\cdot\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^D}\right)\right)=\sup_{F\in\mathcal{F}} P_F\left(\left\|\varepsilon_n^\theta\right\|\geq\frac{h_n^D}{4\overline{\eta}_\Lambda}\cdot\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^D}\right)\right)\\
=&O\left(\frac{1}{\left(r_n\cdot h_n^D\cdot\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^D}\right)\right)^q}\right)
\end{aligned}\tag{S3.2.20H}$$

For large enough n, we have[6] $b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^D}>0$ and $h_n<1$ (and therefore $h_n^{D+1}<h_n^{\frac{D+1}{2}}<h_n^{\frac{D}{2}}$). Combining (S3.2.20A)-(S3.2.20H) with (S3.2.19) we obtain,

$$\sup_{F\in\mathcal{F}} P_F\left(\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}}}\left|\xi_{a,n}^{R_p}(x,t,\theta)\right|\geq b\right)=O\left(\frac{1}{\left((n^{1/2}\wedge r_n)\cdot h_n^{D+1}\cdot\left(\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^D}\right)\wedge\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^D}\right)^{1/2}\right)\right)^q}\right)$$

[6]See footnote 5.

Going back to the second-order approximation in (S3.2.1),

$$\widehat{R}_p(x,t,\widehat{\theta}) = \widehat{R}_p(x,t,\theta_F^*) + \xi_{a,n}^{R_p}(x,t,\theta_F^*) + \xi_{b,n}^{R_p}(x,t,\overline{\theta}_x), \tag{S3.2.21}$$

where, for any $b>0$

$$\begin{aligned}
&\sup_{F\in\mathcal{F}} P_F\left(\sup_{(x,t)\in\mathcal{S}_X\times\mathcal{T}}\left|\xi_{a,n}^{R_p}(x,t,\theta_F^*)\right|\geq b\right) \leq \sup_{F\in\mathcal{F}} P_F\left(\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}}}\left|\xi_{a,n}^{R_p}(x,t,\theta)\right|\geq b\right)\\
&= O\left(\frac{1}{\left((n^{1/2}\wedge r_n)\cdot h_n^{D+1}\cdot\left(\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^{D}}\right)\wedge\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^{D}}\right)^{1/2}\right)\right)^q}\right)
\end{aligned} \tag{S3.2.22}$$

As we have done before, note that for any $c>0$, we have $\min\left\{c,\, c^{1/2}\,,\, c^{1/4}\right\} = \min\left\{c\,,\, c^{1/4}\right\}$. Using this and combining (S3.2.5) with (S3.2.22), we have that for any $b>0$,

$$\begin{aligned}
&\sup_{F\in\mathcal{F}} P_F\left(\sup_{(x,t)\in\mathcal{S}_X\times\mathcal{T}}\left|\xi_{a,n}^{R_p}(x,t,\theta_F^*)+\xi_{b,n}^{R_p}(x,t,\overline{\theta}_x)\right|\geq b\right)\\
\leq&\sup_{F\in\mathcal{F}} P_F\left(\sup_{(x,t)\in\mathcal{S}_X\times\mathcal{T}}\left|\xi_{a,n}^{R_p}(x,t,\theta_F^*)\right|\geq \frac{b}{2}\right)+\sup_{F\in\mathcal{F}} P_F\left(\sup_{(x,t)\in\mathcal{S}_X\times\mathcal{T}}\left|\xi_{b,n}^{R_p}(x,t,\overline{\theta}_x)\right|\geq \frac{b}{2}\right)\\
=&O\left(\frac{1}{\left((n^{1/2}\wedge r_n)\cdot h_n^{D+1}\cdot\left(\left(\frac{b}{2}-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^{D}}\right)\wedge\left(\frac{b}{2}-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^{D}}\right)^{1/4}\right)\right)^q}\right)
\end{aligned} \tag{S3.2.23}$$

Now let us analyze $\widehat{R}_p(x,t,\theta_F^*)$. For a given $(x,t)\in\mathcal{S}_X\times\mathcal{T}$, $\theta\in\Theta$ and $h>0$ let

$$\begin{aligned}
&\nu_n^{R_p}(x,t,\theta,h)\\
&=\frac{1}{n}\sum_{i=1}^n\left(S_p(Y_i,t)\omega_p(g(X_i,\theta))K\left(\frac{\Delta g(X_i,x,\theta)}{h}\right)-E_F\left[S_p(Y_i,t)\omega_p(g(X_i,\theta))K\left(\frac{\Delta g(X_i,x,\theta)}{h}\right)\right]\right)
\end{aligned}$$

From Assumptions 3 and 4, the class of functions

$$\mathscr{G}_{10}^p=\left\{m:\mathcal{S}_V\longrightarrow\mathbb{R}\colon\, m(v)=S_p(y,u,t)\omega_p(g(x,\theta))K\left(\frac{\Delta g(x,u,\theta)}{h}\right)\ \text{for some } u\in\mathcal{S}_X,\, t\in\mathcal{T},\, \theta\in\Theta,\, h>0\right\}$$

is Euclidean for an envelope $\overline{\omega}\cdot\overline{K}\cdot\overline{G}_7(Y)$ such that $E_F\left[\overline{G}_7(Y)^{4q}\right]\leq\overline{\eta}_{\overline{G}_7}<\infty$ for all $F\in\mathcal{F}$, with q being the integer described in Assumption 1. From here, applying Result S1 once again, we obtain that, for any $b > 0$,

$$\sup_{F\in\mathcal{F}} P_F\left(\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}} \left|\nu_n^{R_p}(x,t,\theta,h)\right| \geq b\right) = O\left(\frac{1}{\left(n^{1/2}\cdot b\right)^q}\right), \text{ and therefore,}$$
$$\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}} \left|\nu_n^{R_p}(x,t,\theta,h)\right| = O_p\left(\frac{1}{n^{1/2}}\right) \quad \text{uniformly over } \mathcal{F}. \tag{S3.2.24}$$

Recall that we have defined

$$R_{p,F}(x,t,\theta_F^*) \equiv \Gamma_{p,F}(x,t,\theta_F^*)\cdot\omega_p(g(x,\theta_F^*))\cdot f_g(g(x,\theta_F^*)).$$

For a given $(x,t)\in\mathcal{S}_X\times\mathcal{T}$ let

$$B_n^{R_p}(x,t) \equiv \underbrace{\frac{1}{h_n^D}\cdot E_F\left[S_p(Y,t)\omega_p(g(X,\theta))K\left(\frac{\Delta g(X,x,\theta_F^*)}{h_n}\right)\right] - R_{p,F}(x,t,\theta_F^*)}_{\text{bias}}.$$

We have,

$$\widehat{R}_p(x,t,\theta_F^*) = R_{p,F}(x,t,\theta_F^*) + \frac{1}{h_n^D}\cdot\nu_n^{R_p}(x,t,\theta_F^*,h_n) + B_n^{R_p}(x,t).$$

From here, (S3.2.21) yields,

$$\widehat{R}_p(x,t,\widehat{\theta}) = R_{p,F}(x,t,\theta_F^*) + \frac{1}{h_n^D}\cdot\nu_n^{R_p}(x,t,\theta_F^*,h_n) + B_n^{R_p}(x,t) + \xi_{a,n}^{R_p}(x,t,\theta_F^*) + \xi_{b,n}^{R_p}(x,t,\overline{\theta}_x), \tag{S3.2.25}$$

From Assumptions 4 and the smoothness conditions described in Assumption 2, an M^{th}-order approximation implies that there exists a constant $\overline{B}_{1,f} < \infty$ such that

$$\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left|B_n^{R_p}(x,t)\right| \leq \overline{B}_{1,R}\cdot h_n^M \quad \forall\ F\in\mathcal{F}. \tag{S3.2.26}$$

From here,[7]

$$\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left|\widehat{R}_p(x,t,\widehat{\theta}) - R_{p,F}(x,t,\theta_F^*)\right| \leq \frac{1}{h_n^D}\cdot\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\nu_n^{R_p}(x,t,\theta_F^*,h_n)\right| + \overline{B}_{1,R}\cdot h_n^M$$
$$+ \sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\xi_{a,n}^{R_p}(x,t,\theta_F^*) + \xi_{b,n}^{R_p}(x,t,\overline{\theta}_x)\right|$$

[7] Note that here we are now focusing on our testing range $\mathcal{X}$.

Thus, for any $b > 0$,

$$
\begin{aligned}
&\sup_{F\in\mathcal{F}} P_F\left(\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left|\widehat{R}_p(x,t,\widehat{\theta}) - R_{p,F}(x,t,\theta_F^*)\right| \geq b\right) \\
\leq &\sup_{F\in\mathcal{F}} P_F\left(\frac{1}{h_n^D}\cdot \sup_{(x,t)\in\in\mathcal{X}\times\mathcal{T}} \left|\nu_n^{R_p}(x,t,\theta_F^*,h_n)\right| + \sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left|\xi_{a,n}^{R_p}(x,t,\theta_F^*) + \xi_{b,n}^{R_p}(x,t,\overline{\theta}_x)\right| \geq b - \overline{B}_{1,R}\cdot h_n^M\right) \\
\leq &\sup_{F\in\mathcal{F}} P_F\left(\frac{1}{h_n^D}\cdot \sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left|\nu_n^{R_p}(x,t,\theta_F^*,h_n)\right| \geq \left(\frac{b-\overline{B}_{1,R}\cdot h_n^M}{2}\right)\right) \\
+ &\sup_{F\in\mathcal{F}} P_F\left(\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left|\xi_{a,n}^{R_p}(x,t,\theta_F^*) + \xi_{b,n}^{R_p}(x,t,\overline{\theta}_x)\right| \geq \left(\frac{b-\overline{B}_{1,R}\cdot h_n^M}{2}\right)\right)
\end{aligned}
\tag{S3.2.27}
$$

For a given $b > 0$, there exists an n_0 such that[8]

$$
\frac{b}{4} - \frac{\overline{B}_{1,R}}{4}\cdot h_n^M - \frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}} - \frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^D} > 0, \quad \text{and} \quad h_n < 1 \quad \forall\, n > n_0
$$

From (S3.2.23) and (S3.2.24), the inequality in (S3.2.27) yields

$$
\begin{aligned}
&\sup_{F\in\mathcal{F}} P_F\left(\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left|\widehat{R}_p(x,t,\widehat{\theta}) - R_{p,F}(x,t,\theta_F^*)\right| \geq b\right) \\
&= O\left(\frac{1}{\left(n^{1/2}\cdot h_n^D\cdot\left(\frac{b}{2} - \frac{\overline{B}_{1,R}}{2}\cdot h_n^M\right)\right)^q}\right) \\
&+ O\left(\frac{1}{\left((n^{1/2}\wedge r_n)\cdot h_n^{D+1}\cdot\left(\left(\frac{b}{4} - \frac{\overline{B}_{1,R}}{4}\cdot h_n^M - \frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}} - \frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^D}\right)\wedge\left(\frac{b}{4} - \frac{\overline{B}_{1,R}}{4}\cdot h_n^M - \frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}} - \frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^D}\right)^{1/4}\right)\right)^q}\right) \\
&= O\left(\frac{1}{\left((n^{1/2}\wedge r_n)\cdot h_n^{D+1}\cdot\left(\left(\frac{b}{4} - \frac{\overline{B}_{1,R}}{2}\cdot h_n^M - \frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}} - \frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^D}\right)\wedge\left(\frac{b}{4} - \frac{\overline{B}_{1,R}}{2}\cdot h_n^M - \frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}} - \frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^D}\right)^{1/4}\right)\right)^q}\right)
\end{aligned}
\tag{S3.2.28}
$$

[8]What follows is true more generally if we replace the constant b with a sequence $s_n > 0$ which may converge to zero as long as $h_n^M/s_n \longrightarrow 0$ and $s_n\cdot n\cdot h_n^{D+1} \longrightarrow 0$. See footnote 4.

S3.2.1 A general result for $\sup_{F\in\mathcal{F}} P_F\left(\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left|\widehat{R}_p(x,t,\widehat{\theta})-R_{p,F}(x,t,\theta_F^*)\right| \geq s_n\right)$

We can generalize the result in (S3.2.28) as follows. There exist constants $K_4>0$, $K_5>0$, $K_6>0$ and $C_3>0$ such that, for any sequence $s_n>0$ (possibly converging to zero) such that

$$\frac{h_n^M}{s_n} \longrightarrow 0 \quad \text{and} \quad s_n \cdot n \cdot h_n^{D+1} \longrightarrow \infty,$$

we have

$$\begin{aligned}
&\sup_{F\in\mathcal{F}} P_F\left(\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left|\widehat{R}_p(x,t,\widehat{\theta})-R_{p,F}(x,t,\theta_F^*)\right| \geq s_n\right) \\
&= O\left(\frac{1}{\left((n^{1/2}\wedge r_n)\cdot h_n^{D+1}\cdot\left(\left(K_4\cdot s_n-K_5\cdot h_n^M-\frac{K_6}{n\cdot h_n^{D+1}}-\frac{C_3}{n\cdot h_n^D}\right)\wedge\left(K_4\cdot s_n-K_5\cdot h_n^M-\frac{K_6}{n\cdot h_n^{D+1}}-\frac{C_3}{n\cdot h_n^D}\right)^{1/4}\right)\right)^q}\right)
\end{aligned} \tag{S3.2.29}$$

S3.2.2 A uniform linear representation result for $\widehat{R}_p(x,t,\widehat{\theta})-R_{p,F}(x,t,\theta_F^*)$

Next, we want to obtain a linear representation for $\widehat{R}_p(x,t,\widehat{\theta})-R_{p,F}(x,t,\theta_F^*)$ over $(x,t)\in\mathcal{X}\times\mathcal{T}$. To this end let us go back to (S3.2.21) and recall that

$$\widehat{R}_p(x,t,\widehat{\theta}) = \widehat{R}_p(x,t,\theta_F^*) + \xi_{a,n}^{R_p}(x,t,\theta_F^*) + \xi_{b,n}^{R_p}(x,t,\overline{\theta}_x).$$

From (S3.2.5),

$$\sup_{(x,t)\in\mathcal{S}_X\times\mathcal{T}} \left|\xi_{b,n}^{R_p}(x,t,\overline{\theta}_x)\right| = O_p\left(\frac{1}{n\cdot h_n^{D+2}}\right) \quad \text{uniformly over } \mathcal{F}.$$

To derive the linear representation we will focus on more detail on the term $\xi^{R_p}_{a,n}(x,t,\theta^*_F)$. From (S3.2.18) and the Hoeffding decompositions of $U^{\Lambda}_{R_p,n}$ and $U^{\beta}_{R_p,n}$ in (S3.2.15) we have,

$$\begin{aligned}
\xi^{R_p}_{a,n}(x,t,\theta^*_F) = &\left(\frac{n-1}{n}\right)\cdot\left(\frac{1}{h_n^{D+1}}\cdot E_F\left[\Lambda_{R_p}(V,x,t,\theta^*_F,h_n)\right]+\frac{1}{h_n^{D}}\cdot E_F\left[\beta_{R_p}(V,x,t,\theta^*_F,h_n)\right]\right)\cdot\frac{1}{n}\sum_{i=1}^{n}\psi^{\theta}_F(Z_i)\\
&+\left(\frac{1}{h_n^{D+1}}\cdot E_F\left[\Lambda_{R_p}(V,x,t,\theta^*_F,h_n)\right]+\frac{1}{h_n^{D+1}}\cdot E_F\left[\beta_{R_p}(V,x,t,\theta^*_F,h_n)\right]\right)\varepsilon^{\theta}_n\\
&+\left(\frac{n-1}{2n}\right)\cdot\left(\frac{1}{h_n^{D+1}}\cdot\widetilde{U}^{\Lambda}_{R_p,n}(x,t,\theta^*_F,h_n)+\frac{1}{h_n^{D}}\cdot\widetilde{U}^{\beta}_{R_p,n}(x,t,\theta^*_F,h_n)\right)\\
&+\frac{1}{n}\cdot\left(\frac{1}{h_n^{D+1}}\cdot E_F\left[\Lambda_{R_p}(V,x,t,\theta^*_F,h_n)\psi^{\theta}_F(Z)\right]+\frac{1}{h_n^{D}}\cdot E_F\left[\beta_{R_p}(V,x,t,\theta^*_F,h_n)\psi^{\theta}_F(Z)\right]\right)\\
&+\frac{1}{n\cdot h_n^{D+1}}\cdot m^{R_p}_{\Lambda,n}(x,t,\theta^*_F,h_n)+\frac{1}{h_n^{D+1}}\cdot\nu_{\Lambda_{R_p},n}(x,t,\theta^*_F,h_n)\varepsilon^{\theta}_n\\
&+\frac{1}{n\cdot h_n^{D}}\cdot m^{R_p}_{\beta,n}(x,t,\theta^*_F,h_n)+\frac{1}{h_n^{D}}\cdot\nu_{\beta_{R_p},n}(x,t,\theta^*_F,h_n)\varepsilon^{\theta}_n
\end{aligned}\tag{S3.2.30}$$

Therefore we can express

$$\begin{aligned}
\xi^{R_p}_{a,n}(x,t,\theta^*_F) = &\left(\frac{n-1}{n}\right)\cdot\left(\frac{1}{h_n^{D+1}}\cdot E_F\left[\Lambda_{R_p}(V,x,t,\theta^*_F,h_n)\right]+\frac{1}{h_n^{D}}\cdot E_F\left[\beta_{R_p}(V,x,t,\theta^*_F,h_n)\right]\right)\cdot\frac{1}{n}\sum_{i=1}^{n}\psi^{\theta}_F(Z_i)\\
&+\left(\frac{1}{h_n^{D+1}}\cdot E_F\left[\Lambda_{R_p}(V,x,t,\theta^*_F,h_n)\right]+\frac{1}{h_n^{D+1}}\cdot E_F\left[\beta_{R_p}(V,x,t,\theta^*_F,h_n)\right]\right)\varepsilon^{\theta}_n\\
&+\varsigma^{R_p}_{1,n}(x,t)
\end{aligned}\tag{S3.2.31}$$

where

$$\begin{aligned}
\varsigma^{R_p}_{1,n}(x,t)\equiv &\left(\frac{n-1}{2n}\right)\cdot\left(\frac{1}{h_n^{D+1}}\cdot\widetilde{U}^{\Lambda}_{R_p,n}(x,t,\theta^*_F,h_n)+\frac{1}{h_n^{D}}\cdot\widetilde{U}^{\beta}_{R_p,n}(x,t,\theta^*_F,h_n)\right)\\
&+\frac{1}{n}\cdot\left(\frac{1}{h_n^{D+1}}\cdot E_F\left[\Lambda_{R_p}(V,x,t,\theta^*_F,h_n)\psi^{\theta}_F(Z)\right]+\frac{1}{h_n^{D}}\cdot E_F\left[\beta_{R_p}(V,x,t,\theta^*_F,h_n)\psi^{\theta}_F(Z)\right]\right)\\
&+\frac{1}{n\cdot h_n^{D+1}}\cdot m^{R_p}_{\Lambda,n}(x,t,\theta^*_F,h_n)+\frac{1}{h_n^{D+1}}\cdot\nu_{\Lambda_{R_p},n}(x,t,\theta^*_F,h_n)\varepsilon^{\theta}_n\\
&+\frac{1}{n\cdot h_n^{D}}\cdot m^{R_p}_{\beta,n}(x,t,\theta^*_F,h_n)+\frac{1}{h_n^{D}}\cdot\nu_{\beta_{R_p},n}(x,t,\theta^*_F,h_n)\varepsilon^{\theta}_n
\end{aligned}$$

Let us analyze $\varsigma_{1,n}^{R_p}(x,t)$. From (S3.2.13),

$$\left(\frac{n-1}{2n}\right)\cdot\left(\frac{1}{h_n^{D+1}}\cdot\sup_{(x,t)\in\mathcal{S}_X\times\mathcal{T}}\left|\widetilde{U}^{\Lambda}_{R_p,n}(x,t,\theta_F^*,h_n)\right|+\frac{1}{h_n^D}\cdot\sup_{(x,t)\in\mathcal{S}_X\times\mathcal{T}}\left|\widetilde{U}^{\beta}_{R_p,n}(x,t,\theta_F^*,h_n)\right|\right)$$
$$=O_p\left(\frac{1}{n\cdot h_n^{D+1}}\right)+O_p\left(\frac{1}{n\cdot h_n^{D}}\right)=O_p\left(\frac{1}{n\cdot h_n^{D+1}}\right).$$

From (S3.2.9),

$$\left.\begin{array}{l}\frac{1}{n\cdot h_n^{D+1}}\cdot\sup_{(x,t)\in\mathcal{S}_X\times\mathcal{T}}\left|E_F\left[\Lambda_{R_p}(V,x,t,\theta_F^*,h_n)\psi_F^{\theta}(Z)\right]\right|\leq\frac{1}{n\cdot h_n^{D+1}}\cdot\overline{\eta}_{m,\Lambda}\\ \frac{1}{n\cdot h_n^{D}}\cdot\sup_{(x,t)\in\mathcal{S}_X\times\mathcal{T}}\left|E_F\left[\beta_{R_p}(V,x,t,\theta_F^*,h_n)\psi_F^{\theta}(Z)\right]\right|\leq\frac{1}{n\cdot h_n^{D}}\cdot\overline{\eta}_{m,\beta}\end{array}\right\}\quad\forall\ F\in\mathcal{F}$$

From (S3.2.10),

$$\left.\begin{array}{l}\frac{1}{n\cdot h_n^{D+1}}\cdot\sup_{(x,t)\in\mathcal{S}_X\times\mathcal{T}}\left|m^{R_p}_{\Lambda,n}(x,t,\theta_F^*,h_n)\right|=\frac{1}{n\cdot h_n^{D+1}}\cdot O_p\left(\frac{1}{n^{1/2}}\right)=O_p\left(\frac{1}{n^{3/2}\cdot h_n^{D+1}}\right)\\ \frac{1}{n\cdot h_n^{D}}\cdot\sup_{(x,t)\in\mathcal{S}_X\times\mathcal{T}}\left|m^{R_p}_{\beta,n}(x,t,\theta_F^*,h_n)\right|=\frac{1}{n\cdot h_n^{D}}\cdot O_p\left(\frac{1}{n^{1/2}}\right)=O_p\left(\frac{1}{n^{3/2}\cdot h_n^{D}}\right)\end{array}\right\}\quad\text{uniformly over }\mathcal{F}$$

Let $\tau>0$ be the constant described in Assumption 1. From the conditions described there and the result in (S3.2.8),

$$\left.\begin{array}{l}\frac{1}{h_n^{D+1}}\cdot\sup_{(x,t)\in\mathcal{S}_X\times\mathcal{T}}\left\|\nu_{\Lambda_{R_p},n}(x,t,\theta_F^*,h_n)\right\|\cdot\left\|\varepsilon_n^{\theta}\right\|=\frac{1}{h_n^{D+1}}\cdot O_p\left(\frac{1}{n^{1/2}}\right)\cdot o_p\left(\frac{1}{n^{1/2+\tau}}\right)=o_p\left(\frac{1}{n^{1+\tau}\cdot h_n^{D+1}}\right)\\ \frac{1}{h_n^{D}}\cdot\sup_{(x,t)\in\mathcal{S}_X\times\mathcal{T}}\left\|\nu_{\beta_{R_p},n}(x,t,\theta_F^*,h_n)\right\|\cdot\left\|\varepsilon_n^{\theta}\right\|=\frac{1}{h_n^{D}}\cdot O_p\left(\frac{1}{n^{1/2}}\right)\cdot o_p\left(\frac{1}{n^{1/2+\tau}}\right)=o_p\left(\frac{1}{n^{1+\tau}\cdot h_n^{D}}\right)\end{array}\right\}\ \text{uniformly over }\mathcal{F}$$

Therefore,

$$\begin{aligned}\sup_{(x,t)\in\mathcal{S}_X\times\mathcal{T}}\left|\varsigma_{1,n}^{R_p}(x,t)\right|&=O_p\left(\frac{1}{n\cdot h_n^{D+1}}\right)+O\left(\frac{1}{n\cdot h_n^{D+1}}\right)+O_p\left(\frac{1}{n^{3/2}\cdot h_n^{D+1}}\right)+o_p\left(\frac{1}{n^{1+\tau}\cdot h_n^{D+1}}\right)\\&\quad+O_p\left(\frac{1}{n\cdot h_n^{D}}\right)+O\left(\frac{1}{n\cdot h_n^{D}}\right)+O_p\left(\frac{1}{n^{3/2}\cdot h_n^{D}}\right)+o_p\left(\frac{1}{n^{1+\tau}\cdot h_n^{D}}\right)\\&=O_p\left(\frac{1}{n\cdot h_n^{D+1}}\right)\quad\text{uniformly over }\mathcal{F}\end{aligned}\tag{S3.2.32}$$

With this result at hand, going back to (S3.2.30) we see that we need to analyze the term

$$\frac{1}{h_n^{D+1}}\cdot E_F\left[\Lambda_{R_p}(V,x,t,\theta_F^*,h_n)\right]+\frac{1}{h_n^D}\cdot E_F\left[\beta_{R_p}(V,x,t,\theta_F^*,h_n)\right].$$

As defined in Assumption 2, for each $p = 1, \ldots, P$, $\ell = 1, \ldots, k$ and $d = 1, \ldots, D$, let

$$\begin{aligned}
\Omega^{d,\ell}_{R_p,1}(g,t) &= E_F\left[S_p(Y,t)\omega_p(g(X,\theta^*_F))\frac{\partial g_d(X,\theta^*_F)}{\partial\theta_\ell}\middle|g(X,\theta^*_F)=g\right],\\
\Omega_{R_p,2}(g,t) &= E_F\left[S_p(Y,t)\omega_p(g(X,\theta^*_F))\middle|g(X,\theta^*_F)=g\right],\\
\Omega^{\ell}_{R_p,3}(g,t) &= E_F\left[S_p(Y,t)\frac{\partial\omega_p(g(X,\theta^*_F))}{\partial\theta_\ell}\middle|g(X,\theta^*_F)=g\right].
\end{aligned}$$

As we defined in equation (A-1), for a given $F \in \mathcal{F}$ and $(x,t) \in \mathcal{X}\times\mathcal{T}$ let

$$\begin{aligned}
\Xi_{\ell,R_p}(x,t,\theta^*_F) &\equiv \sum_{d=1}^{D}\left(\frac{\partial\left[\Omega_{R_p,2}(g(x,\theta^*_F),t)f_g(g(x,\theta^*_F))\right]}{\partial g_d}\cdot\frac{\partial g_d(x,\theta^*_F)}{\partial\theta_\ell}-\frac{\partial\left[\Omega^{d,\ell}_{R_p,1}(g(x,\theta^*_F),t)f_g(g(x,\theta^*_F))\right]}{\partial g_d}\right.\\
&\qquad\left.+\Omega^{\ell}_{R_p,3}(g(x,\theta^*_F),t)\cdot f_g(g(x,\theta^*_F))\right),\\
\underbrace{\Xi_{R_p}(x,t,\theta^*_F)}_{1\times k} &\equiv \left(\Xi_{1,R_p}(x,t,\theta^*_F),\ldots,\Xi_{k,R_p}(x,t,\theta^*_F)\right)
\end{aligned}$$

From the smoothness conditions described in Assumption 2, there exists a constant $\overline{B}_{2,R} < \infty$ such that, an $(M+1)^{th}$-order approximation for $\frac{1}{h_n^{D+1}}\cdot E_F\left[\Lambda_{R_p}(V,x,t,\theta^*_F,h_n)\right]$, and an M^{th}-order approximation for $\frac{1}{h_n^{D}}\cdot E_F\left[\beta_{R_p}(V,x,t,\theta^*_F,h_n)\right]$ yield

$$\begin{aligned}
&\frac{1}{h_n^{D+1}}\cdot E_F\left[\Lambda_{R_p}(V,x,t,\theta^*_F,h_n)\right]+\frac{1}{h_n^{D}}\cdot E_F\left[\beta_{R_p}(V,x,t,\theta^*_F,h_n)\right]=\Xi_{R_p}(x,t,\theta^*_F)+B_n^{\Xi_{R_p}}(x,t),\quad\text{where}\\
&\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left\|B_n^{\Xi_{R_p}}(x,t)\right\|\le\overline{B}_{2,R}\cdot h_n^M\quad\forall\,F\in\mathcal{F}.
\end{aligned}\tag{S3.2.33}$$

By Assumption 2, $\exists\ \overline{\eta}_{\Xi_{R_p}} < \infty$ such that

$$\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left\|\Xi_{R_p}(x,t,\theta^*_F)\right\|\le\overline{\eta}_{\Xi_{R_p}}\quad\forall\,F\in\mathcal{F}.\tag{S3.2.34}$$

Plugging (S3.2.33) into (S3.2.31),

$$\xi^{R_p}_{a,n}(x,t,\theta^*_F)=\left(\frac{n-1}{n}\right)\cdot\left(\Xi_{R_p}(x,t,\theta^*_F)+B_n^{\Xi_{R_p}}(x,t)\right)\frac{1}{n}\sum_{i=1}^{n}\psi^{\theta}_F(Z_i)+\left(\Xi_{R_p}(x,t,\theta^*_F)+B_n^{\Xi_{R_p}}(x,t)\right)\varepsilon^{\theta}_n+\varsigma^{R_p}_{1,n}(x,t).$$

From here we can express

$$\xi^{R_p}_{a,n}(x,t,\theta^*_F) = \Xi_{R_p}(x,t,\theta^*_F)\frac{1}{n}\sum_{i=1}^{n}\psi^{\theta}_F(Z_i) + \varsigma^{R_p}_{2,n}(x,t) + \varsigma^{R_p}_{1,n}(x,t), \tag{S3.2.35}$$

where

$$\varsigma^{R_p}_{2,n}(x,t) \equiv \left(\left(\frac{n-1}{n}\right)\cdot B^{\Xi_{R_p}}_n(x,t) - \frac{1}{n}\right)\cdot\frac{1}{n}\sum_{i=1}^{n}\psi^{\theta}_F(Z_i) + \left(\Xi_{R_p}(x,t,\theta^*_F) + B^{\Xi_{R_p}}_n(x,t)\right)\varepsilon^{\theta}_n$$

Let $\tau > 0$ be the constant described in Assumption 1. From the conditions described there and the results in (S3.2.33) and (S3.2.34), we have

$$\begin{aligned}\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\varsigma^{R_p}_{2,n}(x,t)\right| &= \left(O\left(h^M_n\right) + O\left(\frac{1}{n}\right)\right)\cdot O_p\left(\frac{1}{n^{1/2}}\right) + \left(O(1) + O\left(h^M_n\right)\right)\cdot o_p\left(\frac{1}{n^{1/2+\tau}}\right)\\ &= O_p\left(\max\left\{\frac{h^M_n}{n^{1/2}}, \frac{1}{n^{3/2}}, \frac{1}{n^{1/2+\tau}}\right\}\right) \quad \text{uniformly over } \mathcal{F}\end{aligned}$$

Combining (S3.2.32) with the previous expression and defining $\varsigma^{R_p}_n(x,t) \equiv \varsigma^{R_p}_{1,n}(x,t) + \varsigma^{R_p}_{2,n}(x,t)$, (S3.2.35) becomes

$$\begin{aligned}&\xi^{R_p}_{a,n}(x,t,\theta^*_F) = \Xi_{R_p}(x,t,\theta^*_F)\cdot\frac{1}{n}\sum_{i=1}^{n}\psi^{\theta}_F(Z_i) + \varsigma^{R_p}_n(x,t),\\ &\text{where} \quad \sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\varsigma^{R_p}_n(x,t)\right| = O_p\left(\max\left\{\frac{h^M_n}{n^{1/2}}, \frac{1}{n^{3/2}}, \frac{1}{n^{1/2+\tau}}, \frac{1}{n\cdot h^{D+1}_n}\right\}\right) \quad \text{uniformly over } \mathcal{F}\end{aligned} \tag{S3.2.36}$$

Let

$$\begin{aligned}&\psi^{R_p}_F(V_i,x,t,\theta^*_F,h_n)\\ &\equiv \frac{1}{h^D_n}\cdot\left(S_p(Y_i,t)\omega_p(g(X_i,\theta^*_F))K\left(\frac{\Delta g(X_i,x,\theta^*_F)}{h_n}\right) - E_F\left[S_p(Y_i,t)\omega_p(g(X_i,\theta^*_F))K\left(\frac{\Delta g(X_i,x,\theta^*_F)}{h_n}\right)\right]\right)\\ &\quad + \Xi_{R_p}(x,t,\theta^*_F)\psi^{\theta}_F(Z_i).\end{aligned} \tag{S3.2.37}$$

Combining (S3.2.25) and (S3.2.36), we have

$$\begin{aligned}&\widehat{R}_p(x,t,\widehat{\theta}) = R_{p,F}(x,t,\theta^*_F) + \frac{1}{n}\sum_{i=1}^{n}\psi^{R_p}_F(V_i,x,t,\theta^*_F,h_n) + \zeta^{R_p}_n(x,t),\\ &\text{where} \quad \zeta^{R_p}_n(x,t) \equiv B^{R_p}_n(x,t) + \varsigma^{R_p}_n(x,t) + \xi^{R_p}_{b,n}(x,t,\overline{\theta}_x)\end{aligned} \tag{S3.2.38}$$

Let $\epsilon > 0$ be the constant described in Assumption 4. Applying the results from (S3.2.5), (S3.2.26) and (S3.2.36),

$$\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\zeta_n^{R_p}(x,t)\right| = O\left(h_n^M\right) + O_p\left(\max\left\{\frac{h_n^M}{n^{1/2}}\,,\,\frac{1}{n^{3/2}}\,,\,\frac{1}{n^{1/2+\tau}}\right\}\right) + O_p\left(\frac{1}{n\cdot h_n^{D+2}}\right)$$
$$= O_p\left(\max\left\{h_n^M\,,\,\frac{1}{n^{3/2}}\,,\,\frac{1}{n^{1/2+\tau}}\,,\,\frac{1}{n\cdot h_n^{D+2}}\right\}\right) = o_p\left(\frac{1}{n^{1/2+\epsilon}}\right) \quad \text{uniformly over } \mathcal{F} \tag{S3.2.39}$$

By the conditions in Assumptions 1, 2, 3 and 4 along with the results in (S3.2.24) and (S3.2.34), we can once again invoke Result S1 to show that

$$\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\frac{1}{n}\sum_{i=1}^{n}\psi_F^{R_p}(V_i,x,t,\theta_F^*,h_n)\right| = O_p\left(\frac{1}{n^{1/2}\cdot h_n^D}\right) \quad \text{uniformly over } \mathcal{F}. \tag{S3.2.40}$$

And therefore,

$$\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\widehat{R}_p(x,t,\widehat{\theta}) - R_{p,F}(x,t,\theta_F^*)\right| = O_p\left(\frac{1}{n^{1/2}\cdot h_n^D}\right) + o_p\left(\frac{1}{n^{1/2+\epsilon}}\right) = O_p\left(\frac{1}{n^{1/2}\cdot h_n^D}\right) \quad \text{uniformly over } \mathcal{F}. \tag{S3.2.41}$$

S3.3 Asymptotic properties of $\widehat{Q}_p(x,t,\widehat{\theta})$

Combining our results for $\widehat{f}_g(g(x,\widehat{\theta}))$ and $\widehat{R}_p(x,t,\widehat{\theta})$ we can obtain the relevant asymptotic properties of $\widehat{Q}_p(x,t,\widehat{\theta})$. Recall that

$$\widehat{Q}_p(x,t,\widehat{\theta}) = \frac{\widehat{R}_p(x,t,\widehat{\theta})}{\widehat{f}_g(g(x,\widehat{\theta}))}$$

which is an estimator for

$$Q_{p,F}(x,t,\theta_F^*) \equiv \Gamma_{p,F}(x,t,\theta_F^*)\cdot\omega_p(g(x,\theta_F^*)) = \frac{R_{p,F}(x,t,\theta_F^*)}{f_g(g(x,\theta_F^*))}.$$

Proposition S1 *Under Assumptions 1-4, the following results hold.*

(i) There exist finite constants $A_1 > 0$, $A_2 > 0$, $C_1 > 0$, $C_2 > 0$ and $C_3 > 0$ such that, for any sequence $s_n > 0$ such that $s_n \longrightarrow 0$, with

$$\frac{h_n^M}{s_n} \longrightarrow 0 \quad \textit{and} \quad s_n\cdot n\cdot h_n^{D+1} \longrightarrow \infty,$$

we have,

$$\sup_{F\in\mathcal{F}} P_F\left(\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left\|\widehat{Q}(x,t,\widehat{\theta})-Q_F(x,t,\theta_F^*)\right\|\geq s_n\right)$$
$$=O\left(\frac{1}{\left((n^{1/2}\wedge r_n)\cdot h_n^{D+1}\cdot\left(A_1\cdot s_n-C_1\cdot h_n^M-\frac{C_2}{n\cdot h_n^{D+1}}-\frac{C_3}{n\cdot h_n^D}\right)\right)^q}\right)$$

(ii) Let $\Xi_{f_g}(x,\theta_F^*)$ and $\Xi_{R_p}(x,t,\theta_F^*)$ be as described in (A-1) *and, for each p, define*

$$\underbrace{\Xi_{Q_p}(x,t,\theta_F^*)}_{1\times k}\equiv\frac{\Xi_{R_p}(x,t,\theta_F^*)-Q_{p,F}(x,t,\theta_F^*)\cdot\Xi_{f_g}(x,\theta_F^*)}{f_g(g(x,\theta_F^*))} \tag{S3.3.1}$$

and

$$\begin{aligned}\psi_F^{Q_p}(V_i,x,t,\theta_F^*,h_n)\equiv&\frac{1}{h_n^D}\left\{\left(\frac{S_p(Y_i,t)-\Gamma_{p,F}(x,t,\theta_F^*)}{f_g(g(x,\theta_F^*))}\right)\cdot\omega_p(g(X_i,\theta_F^*))\cdot K\left(\frac{\Delta g(X_i,x,\theta_F^*)}{h_n}\right)\right.\\&\left.-E_F\left[\left(\frac{S_p(Y_i,t)-\Gamma_{p,F}(x,t,\theta_F^*)}{f_g(g(x,\theta_F^*))}\right)\cdot\omega_p(g(X_i,\theta_F^*))\cdot K\left(\frac{\Delta g(X_i,x,\theta_F^*)}{h_n}\right)\right]\right\}\\&+\Xi_{Q_p}(x,t,\theta_F^*)\psi_F^{\theta}(Z_i).\end{aligned}$$

And let

$$\psi_F^{Q}(V_i,x,t,\theta_F^*,h_n)\equiv\left(\psi_F^{Q_1}(V_i,x,t,\theta_F^*,h_n),\ldots,\psi_F^{Q_P}(V_i,x,t,\theta_F^*,h_n)\right)'.$$

We have

$$\widehat{Q}(x,t,\widehat{\theta})=Q_F(x,t,\theta_F^*)+\frac{1}{n}\sum_{i=1}^{n}\psi_F^{Q}(V_i,x,t,\theta_F^*,h_n)+\zeta_n^Q(x,t),\quad\textit{where}$$
$$\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left\|\zeta_n^Q(x,t)\right\|=o_p\left(\frac{1}{n^{1/2+\epsilon}}\right)\quad\textit{uniformly over }\mathcal{F}.$$

where $\epsilon>0$ is the constant described in Assumption 4. And we have,

$$\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left\|\widehat{Q}(x,t,\widehat{\theta})-Q_F(x,t,\theta_F^*)\right\|=o_p\left(\frac{1}{n^{1/4+\epsilon/2}}\right)\quad\textit{uniformly over }\mathcal{F}.$$

S3.3.1 Proof of part (i) of Proposition S1

We begin by stating a useful result which follows immediately from (S3.1.45) and (S3.2.29). Let K_1, K_2 and K_3 be the constants described in (S3.1.45) and let K_4, K_5, K_6 and C_3 be the constants described in (S3.2.29), and define $C_0\equiv K_1\wedge K_4$, $C_1\equiv K_2\vee K_5$ and $C_2\equiv K_3\vee K_6$. Then, for any sequence $s_n > 0$ (possibly converging to zero) such that

$$\frac{h_n^M}{s_n} \longrightarrow 0 \quad \text{and} \quad s_n \cdot n \cdot h_n^{D+1} \longrightarrow \infty,$$

we have

$$\begin{aligned}
&\sup_{F\in\mathcal{F}} P_F\left(\sup_{x\in\mathcal{X}}\left|\widehat{f}_g(g(x,\widehat{\theta})) - f_g(g(x,\theta_F^*))\right| \geq s_n\right) + \sup_{F\in\mathcal{F}} P_F\left(\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\widehat{R}_{p,F}(x,t,\widehat{\theta}) - R_{p,F}(x,t,\theta_F^*)\right| \geq s_n\right)\\
=&O\left(\frac{1}{\left((n^{1/2}\wedge r_n)\cdot h_n^{D+1}\cdot\left(\left(C_0\cdot s_n - C_1\cdot h_n^M - \frac{C_2}{n\cdot h_n^{D+1}} - \frac{C_3}{n\cdot h_n^D}\right)\wedge\left(C_0\cdot s_n - C_1\cdot h_n^M - \frac{C_2}{n\cdot h_n^{D+1}} - \frac{C_3}{n\cdot h_n^D}\right)^{1/4}\right)\right)^q}\right)
\end{aligned} \tag{S3.3.2}$$

Recall that

$$\begin{aligned}
R_{p,F}(x,t,\theta_F^*) &\equiv \Gamma_{p,F}(x,t,\theta_F^*)\cdot\omega_p(g(x,\theta_F^*))\cdot f_g(g(x,\theta_F^*)),\\
Q_{p,F}(x,t,\theta_F^*) &\equiv \Gamma_{p,F}(x,t,\theta_F^*)\cdot\omega_p(g(x,\theta_F^*)) = \frac{R_{p,F}(x,t,\theta_F^*)}{f_g(g(x,\theta_F^*))},
\end{aligned}$$

and

$$\widehat{Q}_p(x,t,\widehat{\theta}) = \frac{\widehat{R}_p(x,t,\widehat{\theta})}{\widehat{f}_g(g(x,\widehat{\theta}))}.$$

We have

$$\widehat{Q}_p(x,t,\widehat{\theta}) = Q_{p,F}(x,t,\theta_F^*) + Q_{p,F}(x,t,\theta_F^*)\cdot\frac{\left(f_g(g(x,\theta_F^*)) - \widehat{f}_g(g(x,\widehat{\theta}))\right)}{\widehat{f}_g(g(x,\widehat{\theta}))} + \frac{\left(\widehat{R}_p(x,t,\widehat{\theta}) - R_{p,F}(x,t,\theta_F^*)\right)}{\widehat{f}_g(g(x,\widehat{\theta}))}$$

Recall from Assumption 2 that there exist constants $\underline{f}_g > 0$ and $\overline{\Gamma} < \infty$ such that, for each $F \in \mathcal{F}$,

$$\inf_{x\in\mathcal{X}} f_g(g(x,\theta_F^*)) \geq \underline{f}_g, \quad \text{and} \quad \sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\Gamma_{p,F}(x,t,\theta_F^*)\right| \leq \overline{\Gamma}, \quad p = 1,\ldots,P.$$

and the nonnegative weight function $\omega_p(\cdot)$ is bounded above by $\overline{\omega}$. Therefore,

$$\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|Q_{p,F}(x,t,\theta_F^*)\right| \leq \overline{\Gamma}\cdot\overline{\omega} \equiv \overline{Q}.$$

Thus,

$$\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\widehat{Q}_p(x,t,\widehat{\theta}) - Q_{p,F}(x,t,\theta_F^*)\right| \leq \frac{\overline{Q}\cdot\sup_{x\in\mathcal{X}}\left|\widehat{f}_g(g(x,\widehat{\theta})) - f_g(g(x,\theta_F^*))\right|}{\inf_{x\in\mathcal{X}}\left|\widehat{f}_g(g(x,\widehat{\theta}))\right|} + \frac{\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\widehat{R}_p(x,t,\widehat{\theta}) - R_{p,F}(x,t,\theta_F^*)\right|}{\inf_{x\in\mathcal{X}}\left|\widehat{f}_g(g(x,\widehat{\theta}))\right|}.$$

Recall from (S3.1.59) that $\sup_{x\in\mathcal{X}}\left|\frac{1}{\widehat{f}_g(g(x,\widehat{\theta}))}\right| = \frac{1}{\inf_{x\in\mathcal{X}}\left|\widehat{f}_g(g(x,\widehat{\theta}))\right|} = O_p(1)$ uniformly over $\mathcal{F}$. Take any $s>0$. From here and the previous expression,

$$\mathbb{1}\left\{\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\widehat{Q}_p(x,t,\widehat{\theta})-Q_{p,F}(x,t,\theta_F^*)\right|\geq s\right\}\leq \underbrace{\mathbb{1}\left\{\sup_{x\in\mathcal{X}}\left|\widehat{f}_g(g(x,\widehat{\theta}))-f_g(g(x,\theta_F^*))\right|\geq\left(\frac{s}{2\overline{Q}}\right)\cdot\inf_{x\in\mathcal{X}}\left|\widehat{f}_g(g(x,\widehat{\theta}))\right|\right\}}_{(A)}$$
$$+\underbrace{\mathbb{1}\left\{\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\widehat{R}_p(x,t,\widehat{\theta})-R_{p,F}(x,t,\theta_F^*)\right|\geq\left(\frac{s}{2}\right)\cdot\inf_{x\in\mathcal{X}}\left|\widehat{f}_g(g(x,\widehat{\theta}))\right|\right\}}_{(B)} \tag{S3.3.3}$$

Let us analyze the indicator function (A) in (S3.3.3). We have

$$\begin{aligned}
&\mathbb{1}\left\{\sup_{x\in\mathcal{X}}\left|\widehat{f}_g(g(x,\widehat{\theta}))-f_g(g(x,\theta_F^*))\right|\geq\left(\frac{s}{2\overline{Q}}\right)\cdot\inf_{x\in\mathcal{X}}\left|\widehat{f}_g(g(x,\widehat{\theta}))\right|\right\}\\
&=\underbrace{\mathbb{1}\left\{\sup_{x\in\mathcal{X}}\left|\widehat{f}_g(g(x,\widehat{\theta}))-f_g(g(x,\theta_F^*))\right|\geq\left(\frac{s}{2\overline{Q}}\right)\cdot\inf_{x\in\mathcal{X}}\left|\widehat{f}_g(g(x,\widehat{\theta}))\right|\right\}\cdot\mathbb{1}\left\{\inf_{x\in\mathcal{X}}\left|\widehat{f}_g(g(x,\widehat{\theta}))\right|\geq\frac{1}{2}\cdot\underline{f}_g\right\}}_{\leq\mathbb{1}\left\{\sup_{x\in\mathcal{X}}\left|\widehat{f}_g(g(x,\widehat{\theta}))-f_g(g(x,\theta_F^*))\right|\geq\left(\frac{\underline{f}_g}{4\overline{Q}}\right)\cdot s\right\}}\\
&+\underbrace{\mathbb{1}\left\{\sup_{x\in\mathcal{X}}\left|\widehat{f}_g(g(x,\widehat{\theta}))-f_g(g(x,\theta_F^*))\right|\geq\left(\frac{s}{2\overline{Q}}\right)\cdot\inf_{x\in\mathcal{X}}\left|\widehat{f}_g(g(x,\widehat{\theta}))\right|\right\}\cdot\mathbb{1}\left\{\inf_{x\in\mathcal{X}}\left|\widehat{f}_g(g(x,\widehat{\theta}))\right|<\frac{1}{2}\cdot\underline{f}_g\right\}}_{\leq 1\times\mathbb{1}\left\{\sup_{x\in\mathcal{X}}\left|\widehat{f}_g(g(x,\widehat{\theta}))-f_g(g(x,\theta_F^*))\right|\geq\frac{1}{2}\cdot\underline{f}_g\right\}}\\
&\leq\mathbb{1}\left\{\sup_{x\in\mathcal{X}}\left|\widehat{f}_g(g(x,\widehat{\theta}))-f_g(g(x,\theta_F^*))\right|\geq\left(\frac{\underline{f}_g}{4\overline{Q}}\cdot s\right)\wedge\frac{\underline{f}_g}{2}\right\}.
\end{aligned} \tag{S3.3.4A}$$

Next, the indicator function (B) in (S3.3.3). We have

$$\begin{aligned}
&\mathbb{1}\left\{\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\widehat{R}_p(x,t,\widehat{\theta})-R_{p,F}(x,t,\theta_F^*)\right|\geq\left(\frac{s}{2}\right)\cdot\inf_{x\in\mathcal{X}}\left|\widehat{f}_g(g(x,\widehat{\theta}))\right|\right\}\\
=&\underbrace{\mathbb{1}\left\{\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\widehat{R}_p(x,t,\widehat{\theta})-R_{p,F}(x,t,\theta_F^*)\right|\geq\left(\frac{s}{2}\right)\cdot\inf_{x\in\mathcal{X}}\left|\widehat{f}_g(g(x,\widehat{\theta}))\right|\right\}\cdot\mathbb{1}\left\{\inf_{x\in\mathcal{X}}\left|\widehat{f}_g(g(x,\widehat{\theta}))\right|\geq\frac{1}{2}\cdot\underline{f}_g\right\}}_{\mathbb{1}\left\{\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\widehat{R}_p(x,t,\widehat{\theta})-R_{p,F}(x,t,\theta_F^*)\right|\geq\left(\frac{\underline{f}_g}{4}\right)\cdot s\right\}}\\
&+\underbrace{\mathbb{1}\left\{\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\widehat{R}_p(x,t,\widehat{\theta})-R_{p,F}(x,t,\theta_F^*)\right|\geq\left(\frac{s}{2}\right)\cdot\inf_{x\in\mathcal{X}}\left|\widehat{f}_g(g(x,\widehat{\theta}))\right|\right\}\cdot\mathbb{1}\left\{\inf_{x\in\mathcal{X}}\left|\widehat{f}_g(g(x,\widehat{\theta}))\right|<\frac{1}{2}\cdot\underline{f}_g\right\}}_{\leq 1\times\mathbb{1}\left\{\sup_{x\in\mathcal{X}}\left|\widehat{f}_g(g(x,\widehat{\theta}))-f_g(g(x,\theta_F^*))\right|\geq\frac{1}{2}\cdot\underline{f}_g\right\}}\\
\leq&\mathbb{1}\left\{\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\widehat{R}_p(x,t,\widehat{\theta})-R_{p,F}(x,t,\theta_F^*)\right|\geq\left(\frac{\underline{f}_g}{4}\right)\cdot s\right\}\vee\mathbb{1}\left\{\sup_{x\in\mathcal{X}}\left|\widehat{f}_g(g(x,\widehat{\theta}))-f_g(g(x,\theta_F^*))\right|\geq\frac{1}{2}\cdot\underline{f}_g\right\}
\end{aligned}\tag{S3.3.4B}$$

Let $D_1\equiv\frac{\underline{f}_g}{4}\wedge\frac{\underline{f}_g}{4Q}$ and $D_2\equiv\frac{\underline{f}_g}{2}$. Combining (S3.3.4A) and (S3.3.4B) with (S3.3.3), for any $s>0$ we have,

$$\begin{aligned}
\mathbb{1}\left\{\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\widehat{Q}_p(x,t,\widehat{\theta})-Q_{p,F}(x,t,\theta_F^*)\right|\geq s\right\}\leq&\ \mathbb{1}\left\{\sup_{x\in\mathcal{X}}\left|\widehat{f}_g(g(x,\widehat{\theta}))-f_g(g(x,\theta_F^*))\right|\geq(D_1\cdot s)\wedge D_2\right\}\\
&+\mathbb{1}\left\{\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\widehat{R}_p(x,t,\widehat{\theta})-R_{p,F}(x,t,\theta_F^*)\right|\geq(D_1\cdot s)\wedge D_2\right\}
\end{aligned}$$

Let $\widetilde{A}_1\equiv C_0\cdot D_1$ and $A_2\equiv C_0\cdot D_2$. From here and (S3.3.2), we have that for any $s>0$,

$$\begin{aligned}
&\sup_{F\in\mathcal{F}}P_F\left(\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\widehat{Q}_p(x,t,\widehat{\theta})-Q_{p,F}(x,t,\theta_F^*)\right|\geq s\right)\\
=&O\left(\frac{1}{\left((n^{1/2}\wedge r_n)\cdot h_n^{D+1}\cdot\left(\left(\left(\left(\widetilde{A}_1\cdot s\right)\wedge A_2\right)-C_1\cdot h_n^M-\frac{C_2}{n\cdot h_n^{D+1}}-\frac{C_3}{n\cdot h_n^D}\right)\wedge\left(\left(\left(\widetilde{A}_1\cdot s\right)\wedge A_2\right)-C_1\cdot h_n^M-\frac{C_2}{n\cdot h_n^{D+1}}-\frac{C_3}{n\cdot h_n^D}\right)^{1/4}\right)\right)^q}\right)
\end{aligned}$$

Next, let us stack

$$\widehat{Q}(x,t,\widehat{\theta})\equiv\left(\widehat{Q}_1(x,t,\widehat{\theta}),\ldots,\widehat{Q}_P(x,t,\widehat{\theta})\right)',\quad\text{and}\quad Q_F(x,t,\theta_F^*)\equiv\left(Q_{1,F}(x,t,\theta_F^*),\ldots,Q_{P,F}(x,t,\theta_F^*)\right)'$$

and note that, by the equivalence of Norms in Euclidean space, for some constant $m > 0$ that depends only on P we have

$$\sup_{F\in\mathcal{F}} P_F\left(\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left\|\widehat{Q}(x,t,\widehat{\theta}) - Q_F(x,t,\theta_F^*)\right\| \geq s\right) \leq \sum_{p=1}^{P} \sup_{F\in\mathcal{F}} P_F\left(\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left|\widehat{Q}_p(x,t,\widehat{\theta}) - Q_{p,F}(x,t,\theta_F^*)\right| \geq m\cdot s\right)$$

for any $s > 0$. Therefore, if we let $A_1 \equiv m \cdot \widetilde{A}_1$, the above result yields,

$$\begin{aligned}
&\sup_{F\in\mathcal{F}} P_F\left(\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left\|\widehat{Q}(x,t,\widehat{\theta}) - Q_F(x,t,\theta_F^*)\right\| \geq s\right) \\
=&O\left(\frac{1}{\left((n^{1/2}\wedge r_n)\cdot h_n^{D+1}\cdot\left(\left(((A_1\cdot s)\wedge A_2) - C_1\cdot h_n^M - \frac{C_2}{n\cdot h_n^{D+1}} - \frac{C_3}{n\cdot h_n^D}\right)\wedge\left(((A_1\cdot s)\wedge A_2) - C_1\cdot h_n^M - \frac{C_2}{n\cdot h_n^{D+1}} - \frac{C_3}{n\cdot h_n^D}\right)^{1/4}\right)\right)^q}\right)
\end{aligned}$$

In particular, consider a sequence $s_n > 0$ such that $s_n \longrightarrow 0$ and

$$\frac{h_n^M}{s_n} \longrightarrow 0 \quad \text{and} \quad s_n\cdot n\cdot h_n^{D+1} \longrightarrow \infty.$$

Since $s_n \longrightarrow 0$, $\exists\ n_0$ such that

$$A_1\cdot s_n < A_2 \quad \text{and} \quad A_1\cdot s_n - C_1\cdot h_n^M - \frac{C_2}{n\cdot h_n^{D+1}} - \frac{C_3}{n\cdot h_n^D} < 1 \quad \forall\ n > n_0.$$

Note that for any such s_n,

$$\begin{aligned}
&\left(((A_1\cdot s_n)\wedge A_2) - C_1\cdot h_n^M - \frac{C_2}{n\cdot h_n^{D+1}} - \frac{C_3}{n\cdot h_n^D}\right)\wedge\left(((A_1\cdot s_n)\wedge A_2) - C_1\cdot h_n^M - \frac{C_2}{n\cdot h_n^{D+1}} - \frac{C_3}{n\cdot h_n^D}\right)^{1/4} \\
&= A_1\cdot s_n - C_1\cdot h_n^M - \frac{C_2}{n\cdot h_n^{D+1}} - \frac{C_3}{n\cdot h_n^D} \quad \forall\ n > n_0.
\end{aligned}$$

Therefore, for any such sequence s_n we have

$$\begin{aligned}
&\sup_{F\in\mathcal{F}} P_F\left(\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left\|\widehat{Q}(x,t,\widehat{\theta}) - Q_F(x,t,\theta_F^*)\right\| \geq s_n\right) \\
&\qquad = O\left(\frac{1}{\left((n^{1/2}\wedge r_n)\cdot h_n^{D+1}\cdot\left(A_1\cdot s_n - C_1\cdot h_n^M - \frac{C_2}{n\cdot h_n^{D+1}} - \frac{C_3}{n\cdot h_n^D}\right)\right)^q}\right)
\end{aligned} \tag{S3.3.5}$$

This proves part (i) of Proposition S1, ■

S3.4 Proof of part (ii) of Proposition S1

A second-order approximation yields

$$\widehat{Q}_p(x,t,\widehat{\theta}) = Q_{p,F}(x,t,\theta_F^*) + \frac{1}{f_g(g(x,\theta_F^*)} \cdot \left(\widehat{R}_p(x,t,\widehat{\theta}) - R_{p,F}(x,t,\theta_F^*)\right) - \frac{Q_{p,F}(x,t,\theta_F^*)}{f_g(g(x,\theta_F^*))} \cdot \left(\widehat{f}_g(g(x,\widehat{\theta})) - f_g(x,\theta_F^*)\right)$$
$$- \frac{\left(\widehat{R}_p(x,t,\widehat{\theta}) - R_{p,F}(x,t,\theta_F^*)\right) \cdot \left(\widehat{f}_g(g(x,\widehat{\theta})) - f_g(x,\theta_F^*)\right)}{\widetilde{f}_g(x)^2} + \frac{\widetilde{R}_p(x,t) \cdot \left(\widehat{f}_g(g(x,\widehat{\theta})) - f_g(x,\theta_F^*)\right)^2}{\widetilde{f}_g(x)^3} \tag{S3.4.1}$$

where $\widetilde{f}_g(x)$ is an intermediate point between $\widehat{f}_g(g(x,\widehat{\theta}))$ and $f_g(x,\theta_F^*)$, and $\widetilde{R}_p(x,t)$ is an intermediate point between $\widehat{R}_p(x,t,\widehat{\theta})$ and $R_{p,F}(x,t,\theta_F^*)$. In (S3.1.59) we showed that

$$\sup_{x\in\mathcal{X}} \left| \frac{1}{\widehat{f}_g(g(x,\widehat{\theta}))} \right| = O_p(1) \quad \text{uniformly over } \mathcal{F}.$$

We showed that this follows because, for any $\delta \in (0,1)$,

$$\sup_{F\in\mathcal{F}} P_F\left(\sup_{x\in\mathcal{X}} \left| \frac{1}{\widehat{f}_g(g(x,\widehat{\theta}))} \right| > \frac{1}{(1-\delta)\cdot \underline{f}_g} \right) \leq \sup_{F\in\mathcal{F}} P_F\left(\sup_{x\in\mathcal{X}} \left| \widehat{f}_g(g(x,\widehat{\theta})) - f_g(g(x,\theta_F^*)) \right| > \delta \cdot \underline{f}_g \right)$$

and, from (S3.1.45), for any $\delta \in (0,1)$ and $\epsilon > 0$, $\exists\ n_{\delta,\epsilon}$ such that

$$\sup_{F\in\mathcal{F}} P_F\left(\sup_{x\in\mathcal{X}} \left| \widehat{f}_g(g(x,\widehat{\theta})) - f_g(g(x,\theta_F^*)) \right| > \delta \cdot \underline{f}_g \right) < \epsilon \quad \forall\ n > n_{\delta,\epsilon}.$$

Similarly, we have

$$\sup_{F\in\mathcal{F}} P_F\left(\sup_{x\in\mathcal{X}} \left| \frac{1}{\widetilde{f}_g(g(x)} \right| > \frac{1}{(1-\delta)\cdot \underline{f}_g} \right) \leq \sup_{F\in\mathcal{F}} P_F\left(\sup_{x\in\mathcal{X}} \left| \widetilde{f}_g(x) - f_g(g(x,\theta_F^*)) \right| > \delta \cdot \underline{f}_g \right).$$

Since $\widetilde{f}_g(x)$ is an intermediate value between $\widehat{f}_g(g(x,\widehat{\theta}))$ and $f_g(x,\theta_F^*)$, we have $\sup_{x\in\mathcal{X}} \left|\widetilde{f}_g(x) - f_g(g(x,\theta_F^*))\right| \leq \sup_{x\in\mathcal{X}} \left|\widehat{f}_g(g(x,\widehat{\theta})) - f_g(g(x,\theta_F^*))\right|$. Therefore,

$$\sup_{F\in\mathcal{F}} P_F\left(\sup_{x\in\mathcal{X}} \left| \frac{1}{\widetilde{f}_g(g(x)} \right| > \frac{1}{(1-\delta)\cdot \underline{f}_g} \right) \leq \sup_{F\in\mathcal{F}} P_F\left(\sup_{x\in\mathcal{X}} \left| \widehat{f}_g(g(x,\widehat{\theta})) - f_g(g(x,\theta_F^*)) \right| > \delta \cdot \underline{f}_g \right)$$

and thus,

$$\sup_{x\in\mathcal{X}} \left| \frac{1}{\widetilde{f}_g(x)} \right| = O_p(1) \quad \text{uniformly over } \mathcal{F}. \tag{S3.4.2}$$

From Assumption 2, $\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|R_{p,F}(x,t,\theta_F^*)\right| \leq \overline{\Gamma}\cdot\overline{\omega}\cdot\overline{f}_g \equiv \overline{R}$, and from (S3.2.41),

$$\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\widehat{R}_p(x,t,\widehat{\theta}) - R_{p,F}(x,t,\theta_F^*)\right| = O_p\left(\frac{1}{n^{1/2}\cdot h_n^D}\right) \quad \text{uniformly over } \mathcal{F}.$$

Therefore, since $\widetilde{R}_p(x,t)$ is an intermediate point between $\widehat{R}_p(x,t,\widehat{\theta})$ and $R_{p,F}(x,t,\theta_F^*)$,

$$\begin{aligned}
\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\widetilde{R}_p(x,t)\right| &\leq \overline{R} + \sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\widetilde{R}_p(x,t) - R_{p,F}(x,t,\theta_F^*)\right| \\
&\leq \overline{R} + \sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\widehat{R}_p(x,t,\widehat{\theta}) - R_{p,F}(x,t,\theta_F^*)\right| \\
&= O(1) + O_p\left(\frac{1}{n^{1/2}\cdot h_n^D}\right) = O_p(1) \quad \text{uniformly over } \mathcal{F}.
\end{aligned} \tag{S3.4.3}$$

Let $\psi_F^{f_g}(V_i,x,\theta_F^*,h_n)$ and $\psi_F^{R_p}(V_i,x,t,\theta_F^*,h_n)$ be as described in (S3.1.54) and (S3.2.37) and define

$$\begin{aligned}
\psi_F^{Q_p}(V_i,x,t,\theta_F^*,h_n) \equiv{}& \frac{1}{f_g(g(x,\theta_F^*))}\cdot\psi_F^{R_p}(V_i,x,t,\theta_F^*,h_n) - \frac{Q_{p,F}(x,t,\theta_F^*)}{f_g(g(x,\theta_F^*))}\cdot\psi_F^{f_g}(V_i,x,\theta_F^*,h_n) \\
={}& \frac{1}{h_n^D}\left\{\left(\frac{S_p(Y_i,t)-\Gamma_{p,F}(x,t,\theta_F^*)}{f_g(g(x,\theta_F^*))}\right)\cdot\omega_p(g(X_i,\theta_F^*))\cdot K\left(\frac{\Delta g(X_i,x,\theta_F^*)}{h_n}\right)\right. \\
&\left. - E_F\left[\left(\frac{S_p(Y_i,t)-\Gamma_{p,F}(x,t,\theta_F^*)}{f_g(g(x,\theta_F^*))}\right)\cdot\omega_p(g(X_i,\theta_F^*))\cdot K\left(\frac{\Delta g(X_i,x,\theta_F^*)}{h_n}\right)\right]\right\} \\
&+\left(\frac{\Xi_{R_p}(x,t,\theta_F^*) - Q_{p,F}(x,t,\theta_F^*)\cdot\Xi_{f_g}(x,\theta_F^*)}{f_g(g(x,\theta_F^*))}\right)\psi_F^{\theta}(Z_i).
\end{aligned}$$

Defining

$$\underbrace{\Xi_{Q_p}(x,t,\theta_F^*)}_{1\times k} \equiv \frac{\Xi_{R_p}(x,t,\theta_F^*) - Q_{p,F}(x,t,\theta_F^*)\cdot\Xi_{f_g}(x,\theta_F^*)}{f_g(g(x,\theta_F^*))},$$

we can re-write $\psi_F^{Q_p}(V_i,x,t,\theta_F^*,h_n)$ as,

$$\begin{aligned}
\psi_F^{Q_p}(V_i,x,t,\theta_F^*,h_n) \equiv{}& \frac{1}{h_n^D}\left\{\left(\frac{S_p(Y_i,t)-\Gamma_{p,F}(x,t,\theta_F^*)}{f_g(g(x,\theta_F^*))}\right)\cdot\omega_p(g(X_i,\theta_F^*))\cdot K\left(\frac{\Delta g(X_i,x,\theta_F^*)}{h_n}\right)\right. \\
&\left. - E_F\left[\left(\frac{S_p(Y_i,t)-\Gamma_{p,F}(x,t,\theta_F^*)}{f_g(g(x,\theta_F^*))}\right)\cdot\omega_p(g(X_i,\theta_F^*))\cdot K\left(\frac{\Delta g(X_i,x,\theta_F^*)}{h_n}\right)\right]\right\} \\
&+\Xi_{Q_p}(x,t,\theta_F^*)\psi_F^{\theta}(Z_i),
\end{aligned} \tag{S3.4.4}$$

Using the uniform linear representation results in (S3.1.55) and (S3.2.38), equation (S3.4.1) becomes

$$\begin{aligned}\widehat{Q}_p(x,t,\widehat{\theta}) &= Q_{p,F}(x,t,\theta^*_F) + \frac{1}{n}\sum_{i=1}^{n}\psi_F^{Q_p}(V_i,x,t,\theta^*_F,h_n) + \zeta_n^{Q_p}(x,t), \quad \text{where}\\ \zeta_n^{Q_p}(x,t) &\equiv \frac{1}{f_g(g(x,\theta^*_F))}\cdot\zeta_n^{R_p}(x,t) - \frac{Q_{p,F}(x,t,\theta^*_F)}{f_g(g(x,\theta^*_F))}\cdot\zeta_n^{f_g}(x)\\ &\quad - \frac{\left(\widehat{R}_p(x,t,\widehat{\theta}) - R_{p,F}(x,t,\theta^*_F)\right)\cdot\left(\widehat{f}_g(g(x,\widehat{\theta})) - f_g(x,\theta^*_F)\right)}{\widetilde{f}_g(x)^2}\\ &\quad + \frac{\widetilde{R}_p(x,t)\cdot\left(\widehat{f}_g(g(x,\widehat{\theta})) - f_g(x,\theta^*_F)\right)^2}{\widetilde{f}_g(x)^3}\end{aligned} \tag{S3.4.5}$$

Let $\epsilon > 0$ be the constant described in Assumption 4. From (S3.1.56) and (S3.2.39),

$$\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\zeta_n^{R_p}(x,t)\right| = o_p\left(\frac{1}{n^{1/2+\epsilon}}\right), \quad \text{uniformly over } \mathcal{F}$$

And from (S3.1.58) and (S3.2.41),

$$\begin{aligned}\sup_{x\in\mathcal{X}}\left|\widehat{f}_g(x,\widehat{\theta}) - f_g(x,\theta^*_F)\right| &= O_p\left(\frac{1}{n^{1/2}\cdot h_n^D}\right) \quad \text{uniformly over } \mathcal{F}.\\ \sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\widehat{R}_p(x,t,\widehat{\theta}) - R_{p,F}(x,t,\theta^*_F)\right| &= O_p\left(\frac{1}{n^{1/2}\cdot h_n^D}\right) \quad \text{uniformly over } \mathcal{F}.\end{aligned}$$

In Assumption 4 we stated that the constant $\epsilon > 0$ satisfies $n^{1/2-\epsilon}\cdot h_n^{2D} \longrightarrow \infty$. These results, combined with (S3.4.2) and (S3.4.3) yield,

$$\begin{aligned}\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\zeta_n^{Q_p}(x,t)\right| &\leq \frac{1}{\underline{f}_g}\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\zeta_n^{R_p}(x,t)\right| + \frac{\overline{Q}}{\underline{f}_g}\sup_{x\in\mathcal{X}}\left|\zeta_n^{f_g}(x)\right|\\ &\quad + \sup_{x\in\mathcal{X}}\left|\frac{1}{\widetilde{f}_g(x)^2}\right|\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\widehat{R}_p(x,t,\widehat{\theta}) - R_{p,F}(x,t,\theta^*_F)\right| \times \sup_{x\in\mathcal{X}}\left|\widehat{f}_g(x,\widehat{\theta}) - f_g(x,\theta^*_F)\right|\\ &\quad + \sup_{x\in\mathcal{X}}\left|\frac{1}{\widetilde{f}_g(x)^3}\right|\left(\sup_{x\in\mathcal{X}}\left|\widehat{f}_g(x,\widehat{\theta}) - f_g(x,\theta^*_F)\right|\right)^2\\ &= o_p\left(\frac{1}{n^{1/2+\epsilon}}\right) + O_p\left(\frac{1}{n\cdot h_n^{2D}}\right) \quad \text{uniformly over } \mathcal{F}\\ &= o_p\left(\frac{1}{n^{1/2+\epsilon}}\right) \quad \text{uniformly over } \mathcal{F}.\end{aligned} \tag{S3.4.6}$$

From (S3.1.57), (S3.2.40) and the conditions in Assumption 2 which assert that, for each $F \in \mathcal{F}$,

$$\inf_{x\in\mathcal{X}} f_g(g(x,\theta_F^*)) \geq \underline{f}_g, \quad \sup_{x\in\mathcal{X}} f_g(g(x,\theta_F^*)) \leq \overline{f}_g, \text{ and } \sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left|Q_{p,F}(x,t,\theta_F^*)\right| \leq \overline{Q}, \quad p = 1,\ldots,P.$$

we have that

$$\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left|\frac{1}{n}\sum_{i=1}^{n} \psi_F^{Q_p}(V_i,x,t,\theta_F^*,h_n)\right| = O_p\left(\frac{1}{n^{1/2}\cdot h_n^D}\right) \quad \text{uniformly over } \mathcal{F}. \tag{S3.4.7}$$

and therefore, from (S3.4.5) and (S3.4.6),

$$\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left|\widehat{Q}_p(x,t,\widehat{\theta}) - Q_{p,F}(x,t,\theta_F^*)\right| = O_p\left(\frac{1}{n^{1/2}\cdot h_n^D}\right) + o_p\left(\frac{1}{n^{1/2+\epsilon}}\right) = O_p\left(\frac{1}{n^{1/2}\cdot h_n^D}\right) \quad \text{uniformly over } \mathcal{F}.$$

In particular, since $n^{1/2-\epsilon}\cdot h_n^{2D} \longrightarrow \infty$ by Assumption 4, the previous result implies that

$$\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left|\widehat{Q}_p(x,t,\widehat{\theta}) - Q_{p,F}(x,t,\theta_F^*)\right| = o_p\left(\frac{1}{n^{1/4+\epsilon/2}}\right) \quad \text{uniformly over } \mathcal{F}.$$

Let

$$\psi_F^{Q}(V_i,x,t,\theta_F^*,h_n) \equiv \left(\psi_F^{Q_1}(V_i,x,t,\theta_F^*,h_n),\ldots,\psi_F^{Q_P}(V_i,x,t,\theta_F^*,h_n)\right)',$$
$$\zeta_n^{Q}(x,t) \equiv \left(\zeta_n^{Q_1}(x,t),\ldots,\zeta_n^{Q_P}(x,t)\right)'$$

From (S3.4.5) and (S3.4.6), we have

$$\widehat{Q}(x,t,\widehat{\theta}) = Q_F(x,t,\theta_F^*) + \frac{1}{n}\sum_{i=1}^{n}\psi_F^{Q}(V_i,x,t,\theta_F^*,h_n) + \zeta_n^{Q}(x,t), \quad \text{where}$$
$$\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left\|\zeta_n^{Q}(x,t)\right\| = o_p\left(\frac{1}{n^{1/2+\epsilon}}\right) \quad \text{uniformly over } \mathcal{F}.$$

where $\epsilon > 0$ is the constant described in Assumption 4. And we have,

$$\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left\|\widehat{Q}(x,t,\widehat{\theta}) - Q_F(x,t,\theta_F^*)\right\| = o_p\left(\frac{1}{n^{1/4+\epsilon/2}}\right) \quad \text{uniformly over } \mathcal{F}.$$

The previous two results prove part (ii) of Proposition S1. ■

S3.5 Asymptotic properties of $\mathcal{B}(\widehat{Q}(x,t,\widehat{\theta}))-\mathcal{B}(Q_F(x,t,\theta_F^*))$

From Assumption 5, the following second-order approximation is valid,

$$
\begin{aligned}
\mathcal{B}(\widehat{Q}(x,t,\widehat{\theta})) = \mathcal{B}(Q_F(x,t,\theta_F^*)) + \nabla_Q \mathcal{B}(Q_F(x,t,\theta_F^*))\Big(\widehat{Q}(x,t,\widehat{\theta}) - Q_F(x,t,\theta_F^*)\Big) \\
+ \frac{1}{2}\Big(\widehat{Q}(x,t,\widehat{\theta}) - Q_F(x,t,\theta_F^*)\Big)' \nabla_{QQ'} \mathcal{B}(\widetilde{Q}(x,t))\Big(\widehat{Q}(x,t,\widehat{\theta}) - Q_F(x,t,\theta_F^*)\Big),
\end{aligned}
$$

where $\widetilde{Q}(x,t)$ belongs in the line segment connecting $\widehat{Q}(x,t,\widehat{\theta})$ and $Q_F(x,t,\theta_F^*)$, and thus

$$
\Big\|\widetilde{Q}(x,t) - Q_F(x,t,\theta_F^*)\Big\| \leq \Big\|\widehat{Q}(x,t,\widehat{\theta}) - Q_F(x,t,\theta_F^*)\Big\|.
$$

From here, the results in Proposition S1 yield

$$
\begin{aligned}
\mathcal{B}(\widehat{Q}(x,t,\widehat{\theta})) &= \mathcal{B}(Q_F(x,t,\theta_F^*)) + \nabla_Q \mathcal{B}(Q_F(x,t,\theta_F^*))\frac{1}{n}\sum_{i=1}^{n} \psi_F^{Q}(V_i,x,t,\theta_F^*,h_n) \\
&+ \nabla_Q \mathcal{B}(Q_F(x,t,\theta_F^*))\zeta_n^{Q}(x,t) \\
&+ \frac{1}{2}\Big(\widehat{Q}(x,t,\widehat{\theta}) - Q_F(x,t,\theta_F^*)\Big)' \nabla_{QQ'} \mathcal{B}(\widetilde{Q}(x,t))\Big(\widehat{Q}(x,t,\widehat{\theta}) - Q_F(x,t,\theta_F^*)\Big) \\
&\equiv \mathcal{B}(Q_F(x,t,\theta_F^*)) + \frac{1}{n}\sum_{i=1}^{n} \psi_F^{\mathcal{B}}(V_i,x,t,\theta_F^*,h_n) + \zeta_n^{\mathcal{B}}(x,t),
\end{aligned}
\tag{S3.5.1}
$$

where,

$$
\begin{aligned}
\psi_F^{\mathcal{B}}(V_i,x,t,\theta_F^*,h_n) &\equiv \nabla_Q \mathcal{B}(Q_F(x,t,\theta_F^*))\psi_F^{Q}(V_i,x,t,\theta_F^*,h_n) \\
&= \sum_{p=1}^{P} \frac{\partial \mathcal{B}\Big(Q_F(x,t,\theta_F^*)\Big)}{\partial Q_p}\psi_F^{Q_p}(V_i,x,t,\theta_F^*,h_n), \quad \text{and} \\
\zeta_n^{\mathcal{B}}(x,t) &\equiv \nabla_Q \mathcal{B}(Q_F(x,t,\theta_F^*))\zeta_n^{Q}(x,t) \\
&+ \frac{1}{2}\Big(\widehat{Q}(x,t,\widehat{\theta}) - Q_F(x,t,\theta_F^*)\Big)' \nabla_{QQ'} \mathcal{B}(\widetilde{Q}(x,t))\Big(\widehat{Q}(x,t,\widehat{\theta}) - Q_F(x,t,\theta_F^*)\Big).
\end{aligned}
$$

We proceed by noting that, under the conditions of Assumption 5 and the results in Proposition S1, we have

$$
\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \Big\|\nabla_{QQ'} \mathcal{B}(\widetilde{Q}(x,t))\Big\| = O_p(1) \quad \text{uniformly over } \mathcal{F}.
$$

to see this, recall that $\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left\| Q_F(x,t,\theta_F^*) \right\| \leq \overline{\Gamma}\cdot\overline{\omega} \equiv \overline{Q}$, where $\overline{\Gamma}$ is as described in Assumption 2 and recall that $\left\| \widetilde{Q}(x,t) - Q_F(x,t,\theta_F^*) \right\| \leq \left\| \widehat{Q}(x,t,\widehat{\theta}) - Q_F(x,t,\theta_F^*) \right\|$. Therefore,

$$
\begin{aligned}
\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left\| \widetilde{Q}(x,t) \right\| &\leq \sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left\| \widetilde{Q}(x,t) - Q_F(x,t,\theta_F^*) \right\| + \overline{Q} \\
&\leq \sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left\| \widehat{Q}(x,t,\widehat{\theta}) - Q_F(x,t,\theta_F^*) \right\| + \overline{Q}
\end{aligned}
$$

Therefore, from the conditions described in Assumption 5 and the results in Proposition S1, we obtain

$$
\begin{aligned}
\sup_{F\in\mathcal{F}} P_F\left(\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left\| \nabla_{QQ'}\mathcal{B}(\widetilde{\Gamma}(x,t)) \right\| > \overline{H}_Q \right) &\leq \sup_{F\in\mathcal{F}} P_F\left(\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left\| \widetilde{Q}(x,t) \right\| > \overline{Q} + C_Q \right) \\
&\leq \sup_{F\in\mathcal{F}} P_F\left(\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left\| \widehat{Q}(x,t,\widehat{\theta}) - Q_F(x,t,\theta_F^*) \right\| > C_Q \right) \longrightarrow 0,
\end{aligned}
$$

with the last result following from Proposition S1. Thus, $\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left\| \nabla_{QQ'}\mathcal{B}(\widetilde{Q}(x,t)) \right\| = O_p(1)$ uniformly over $\mathcal{F}$, as claimed. From here and the results in Proposition S1, we have

$$
\begin{aligned}
&\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left| \left(\widehat{Q}(x,t,\widehat{\theta}) - Q_F(x,t,\theta_F^*) \right)' \nabla_{QQ'}\mathcal{B}(\widetilde{Q}(x,t)) \left(\widehat{Q}(x,t,\widehat{\theta}) - Q_F(x,t,\theta_F^*) \right) \right| \\
&\leq \sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left\| \nabla_{QQ'}\mathcal{B}(\widetilde{Q}(x,t)) \right\| \times \left(\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left\| \widehat{Q}(x,t,\widehat{\theta}) - Q_F(x,t,\theta_F^*) \right\| \right)^2 \\
&= O_p(1) \times \left(o_p\left(\frac{1}{n^{1/4+\epsilon/2}} \right) \right)^2 = o_p\left(\frac{1}{n^{1/2+\epsilon}} \right) \quad \text{uniformly over } \mathcal{F},
\end{aligned}
$$

where $\epsilon > 0$ is the constant described in Assumption 4. Next, by Assumption 5 and the results in Proposition S1,

$$
\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left| \nabla_Q\mathcal{B}(Q_F(x,t,\theta_F^*))\zeta_n^Q(x,t) \right| \leq \overline{H}_Q \cdot \sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left\| \zeta_n^Q(x,t) \right\| = o_p\left(\frac{1}{n^{1/2+\epsilon}} \right) \quad \text{uniformly over } \mathcal{F}.
$$

Therefore, from the conditions described in Assumption 5 and the results in Proposition S1,

$$
\begin{aligned}
\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left| \zeta_n^{\mathcal{B}}(x,t) \right| &\leq \sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left| \nabla_Q\mathcal{B}(Q_F(x,t,\theta_F^*))\zeta_n^Q(x,t) \right| \\
&\quad + \sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left| \left(\widehat{Q}(x,t,\widehat{\theta}) - Q_F(x,t,\theta_F^*) \right)' \nabla_{QQ'}\mathcal{B}(\widetilde{Q}(x,t)) \left(\widehat{Q}(x,t,\widehat{\theta}) - Q_F(x,t,\theta_F^*) \right) \right| \\
&= o_p\left(\frac{1}{n^{1/2+\epsilon}} \right) \quad \text{uniformly over } \mathcal{F}.
\end{aligned} \tag{S3.5.2}
$$

Together, (S3.5.1) and (S3.5.2) yield

$$\mathcal{B}(\widehat{Q}(x,t,\widehat{\theta})) = \mathcal{B}(Q_F(x,t,\theta_F^*)) + \frac{1}{n}\sum_{i=1}^{n}\psi_F^{\mathcal{B}}(V_i,x,t,\theta_F^*,h_n) + \zeta_n^{\mathcal{B}}(x,t), \quad \text{where}$$
$$\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left\|\zeta_n^{\mathcal{B}}(x,t)\right\| = o_p\left(\frac{1}{n^{1/2+\epsilon}}\right) \quad \text{uniformly over } \mathcal{F},$$

where $\epsilon > 0$ is the constant described in Assumption 4. Finally, recall from (S3.4.7) that

$$\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left\|\frac{1}{n}\sum_{i=1}^{n}\psi_F^{Q}(V_i,x,t,\theta_F^*,h_n)\right\| = O_p\left(\frac{1}{n^{1/2}\cdot h_n^D}\right) \quad \text{uniformly over } \mathcal{F}.$$

Therefore, from here and Assumption 5,

$$\begin{aligned}\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left|\frac{1}{n}\sum_{i=1}^{n}\psi_F^{\mathcal{B}}(V_i,x,t,\theta_F^*,h_n)\right| &\leq \sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left\|\nabla_Q\mathcal{B}(Q_F(x,t,\theta_F^*))\right\|\cdot \sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left\|\frac{1}{n}\sum_{i=1}^{n}\psi_F^{Q}(V_i,x,t,\theta_F^*,h_n)\right\| \\ &\leq \overline{H}_Q\cdot O_p\left(\frac{1}{n^{1/2}\cdot h_n^D}\right) \quad \text{uniformly over } \mathcal{F}.\end{aligned}$$

Since $n^{1/2-\epsilon}\cdot h_n^{2D} \longrightarrow \infty$ by Assumption 4, the above result combined with (S3.5.1) and (S3.5.2) implies,

$$\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left|\mathcal{B}(\widehat{Q}(x,t,\widehat{\theta})) - \mathcal{B}(Q_F(x,t,\theta_F^*))\right| = o_p\left(\frac{1}{n^{1/4+\epsilon/2}}\right) \quad \text{uniformly over } \mathcal{F}. \tag{S3.5.3}$$

Next, note from Assumption 5 that, for any $s > 0$ we have

$$\begin{aligned}&\sup_{F\in\mathcal{F}} P_F\left(\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left|\mathcal{B}(\widehat{Q}(x,t,\widehat{\theta})) - \mathcal{B}(Q_F(x,t,\theta_F^*))\right| \geq s\right) \\ &\qquad\qquad \leq \sup_{F\in\mathcal{F}} P_F\left(\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left\|\widehat{Q}(x,t,\widehat{\theta}) - Q_F(x,t,\theta_F^*)\right\| \geq \left(\frac{s}{M_1}\right)\wedge M_2\right)\end{aligned}$$

In particular, from Proposition S1, if we take any positive sequence $s_n > 0$ such that $s_n \longrightarrow 0$, with

$$\frac{h_n^M}{s_n} \longrightarrow 0 \quad \text{and} \quad s_n\cdot n\cdot h_n^{D+1} \longrightarrow \infty,$$

we have

$$\sup_{F\in\mathcal{F}} P_F\left(\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\mathcal{B}(\widehat{Q}(x,t,\widehat{\theta}))-\mathcal{B}(Q_F(x,t,\theta_F^*))\right|\geq s_n\right)$$
$$=O\left(\frac{1}{\left((n^{1/2}\wedge r_n)\cdot h_n^{D+1}\cdot\left(\left(\frac{A_1}{M_1}\right)\cdot s_n-C_1\cdot h_n^M-\frac{C_2}{n\cdot h_n^{D+1}}-\frac{C_3}{n\cdot h_n^D}\right)\right)^q}\right)$$

In particular, take the sequence b_n used in the construction of $\widehat{T}_1$ and $\widehat{T}_2$. By the bandwidth convergence restrictions described in Assumption 4, we have $\frac{h_n^M}{b_n}\longrightarrow 0$ and $b_n\cdot n\cdot h_n^{D+1}\longrightarrow\infty$, and therefore,

$$\sup_{F\in\mathcal{F}} P_F\left(\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\mathcal{B}(\widehat{Q}(x,t,\widehat{\theta}))-\mathcal{B}(Q_F(x,t,\theta_F^*))\right|\geq b_n\right)\longrightarrow 0. \tag{S3.5.4}$$

S3.6 Proof of Proposition 1A

S3.6.1 Asymptotic properties of $\widehat{T}_1$

Let

$$\widetilde{T}_{1,F}\equiv\frac{1}{n}\sum_{i=1}^n\mathcal{B}\left(\widehat{Q}(X_i,t_i,\widehat{\theta})\right)\mathbb{1}\left\{\mathcal{B}\left(Q_F(X_i,t_i,\theta_F^*)\right)\geq 0\right\}\phi(X_i,t_i)$$

Note that $\widetilde{T}_{1,F}$ replaces the indicator function $\mathbb{1}\left\{\mathcal{B}\left(\widehat{Q}(X_i,t_i,\widehat{\theta})\right)\geq -b_n\right\}$ with $\mathbb{1}\left\{\mathcal{B}\left(Q_F(X_i,t_i,\theta_F^*)\right)\geq 0\right\}$. Our first step is to analyze $\widehat{T}_1-\widetilde{T}_{1,F}\equiv\xi^a_{T_1,n}$. We have

$$\xi^a_{T_1,n}\equiv\widehat{T}_1-\widetilde{T}_{1,F}=\sum_{i=1}^n\mathcal{B}\left(\widehat{Q}(X_i,t_i,\widehat{\theta})\right)\phi(X_i,t_i)\left[\mathbb{1}\left\{\mathcal{B}\left(\widehat{Q}(X_i,t_i,\widehat{\theta})\right)\geq -b_n\right\}-\mathbb{1}\left\{\mathcal{B}\left(Q_F(X_i,t_i,\theta_F^*)\right)\geq 0\right\}\right]$$

Therefore,

$$\left|\xi^a_{T_1,n}\right|\leq\sum_{i=1}^n\left|\mathcal{B}\left(\widehat{Q}(X_i,t_i,\widehat{\theta})\right)\right|\phi(X_i,t_i)\left|\mathbb{1}\left\{\mathcal{B}\left(\widehat{Q}(X_i,t_i,\widehat{\theta})\right)\geq -b_n\right\}-\mathbb{1}\left\{\mathcal{B}\left(Q_F(X_i,t_i,\theta_F^*)\right)\geq 0\right\}\right|$$

We have

$$\begin{aligned}
&\left|\mathbb{1}\left\{\mathcal{B}\left(\widehat{Q}(X_i,t_i,\widehat{\theta})\right)\geq -b_n\right\}-\mathbb{1}\left\{\mathcal{B}\left(Q_F(X_i,t_i,\theta_F^*)\right)\geq 0\right\}\right|\\
=&\mathbb{1}\left\{\mathcal{B}\left(\widehat{Q}(X_i,t_i,\widehat{\theta})\right)\geq -b_n\ ,\ -2b_n\leq\mathcal{B}\left(Q_F(X_i,t_i,\theta_F^*)\right)<0\right\}+\mathbb{1}\left\{\mathcal{B}\left(\widehat{Q}(X_i,t_i,\widehat{\theta})\right)\geq -b_n\ ,\ \mathcal{B}\left(Q_F(X_i,t_i,\theta_F^*)\right)<-2b_n\right\}\\
+&\mathbb{1}\left\{\mathcal{B}\left(\widehat{Q}(X_i,t_i,\widehat{\theta})\right)<-b_n\ ,\ \mathcal{B}\left(Q_F(X_i,t_i,\theta_F^*)\right)\geq 0\right\}\\
\leq&\mathbb{1}\left\{-2b_n\leq\mathcal{B}\left(Q_F(X_i,t_i,\theta_F^*)\right)<0\right\}+\mathbb{1}\left\{\left|\mathcal{B}\left(\widehat{Q}(X_i,t_i,\widehat{\theta})\right)-\mathcal{B}\left(Q_F(X_i,t_i,\theta_F^*)\right)\right|\geq b_n\right\}
\end{aligned} \tag{S3.6.1}$$

From here, we have

$$\begin{aligned}
&\left|\xi^a_{T_1,n}\right| \\
\leq&\frac{1}{n}\sum_{i=1}^n \left|\mathcal{B}\left(\widehat{Q}(X_i,t_i,\widehat{\theta})\right)\right|\phi(X_i,t_i)\mathbb{1}\left\{-2b_n \leq \mathcal{B}\left(Q_F(X_i,t_i,\theta^*_F)\right) < 0\right\} \\
+&\frac{1}{n}\sum_{i=1}^n \left|\mathcal{B}\left(\widehat{Q}(X_i,t_i,\widehat{\theta})\right)\right|\phi(X_i,t_i)\mathbb{1}\left\{\left|\mathcal{B}\left(\widehat{Q}(X_i,t_i,\widehat{\theta})\right)-\mathcal{B}\left(Q_F(X_i,t_i,\theta^*_F)\right)\right| \geq b_n\right\} \\
\leq&\frac{1}{n}\sum_{i=1}^n \left(\left|B\left(Q_F(X_i,t_i,\theta^*_F)\right)\right| + \left|\mathcal{B}\left(\widehat{Q}(X_i,t_i,\widehat{\theta})\right)-\mathcal{B}\left(Q_F(X_i,t_i,\theta^*_F)\right)\right|\right)\phi(X_i,t_i)\mathbb{1}\left\{-2b_n \leq \mathcal{B}\left(Q_F(X_i,t_i,\theta^*_F)\right) < 0\right\} \\
+&\frac{1}{n}\sum_{i=1}^n \left|\mathcal{B}\left(\widehat{Q}(X_i,t_i,\widehat{\theta})\right)\right|\phi(X_i,t_i)\mathbb{1}\left\{\left|\mathcal{B}\left(\widehat{Q}(X_i,t_i,\widehat{\theta})\right)-\mathcal{B}\left(Q_F(X_i,t_i,\theta^*_F)\right)\right| \geq b_n\right\} \\
\leq&\left(2b_n + \sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\mathcal{B}\left(\widehat{Q}(x,t,\widehat{\theta})\right)-\mathcal{B}\left(Q_F(x,t,\theta^*_F)\right)\right|\right)\times\frac{1}{n}\sum_{i=1}^n \phi(X_i,t_i)\mathbb{1}\left\{-2b_n \leq \mathcal{B}\left(Q_F(X_i,t_i,\theta^*_F)\right) < 0\right\} \\
+&\frac{1}{n}\sum_{i=1}^n \left|\mathcal{B}\left(\widehat{Q}(X_i,t_i,\widehat{\theta})\right)\right|\phi(X_i,t_i)\mathbb{1}\left\{\left|\mathcal{B}\left(\widehat{Q}(X_i,t_i,\widehat{\theta})\right)-\mathcal{B}\left(Q_F(X_i,t_i,\theta^*_F)\right)\right| \geq b_n\right\}
\end{aligned} \tag{S3.6.2}$$

From (S3.5.3), we have $\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\mathcal{B}\left(\widehat{Q}(x,t,\widehat{\theta})\right)-\mathcal{B}\left(Q(x,t,\theta^*_F)\right)\right| = o_p\left(\frac{1}{n^{1/4+\epsilon/2}}\right)$ uniformly over $\mathcal{F}$, where $\epsilon > 0$ is the constant described in Assumption 4. Therefore, uniformly over $\mathcal{F}$ we have

$$\begin{aligned}
\left|\xi^a_{T_1,n}\right| \leq& \left(2b_n + o_p\left(\frac{1}{n^{1/4+\epsilon/2}}\right)\right)\times\frac{1}{n}\sum_{i=1}^n \phi(X_i,t_i)\mathbb{1}\left\{-2b_n \leq \mathcal{B}\left(Q(X_i,t_i,\theta^*_F)\right) < 0\right\} \\
&+\frac{1}{n}\sum_{i=1}^n \left|\mathcal{B}\left(\widehat{Q}(X_i,t_i,\widehat{\theta})\right)\right|\phi(X_i,t_i)\mathbb{1}\left\{\left|\mathcal{B}\left(\widehat{Q}(X_i,t_i,\widehat{\theta})\right)-\mathcal{B}\left(Q(X_i,t_i,\theta^*_F)\right)\right| \geq b_n\right\}.
\end{aligned} \tag{S3.6.3}$$

We will analyze each of the two summands in (S3.6.3). For a given $b > 0$, let

$$m^a_{T_1,n}(b) \equiv \frac{1}{n}\sum_{i=1}^n \left(\phi(X_i,t_i)\mathbb{1}\left\{-b \leq \mathcal{B}\left(Q_F(X_i,t_i,\theta^*_F)\right) < 0\right\} - E_F\left[\phi(X_i,t_i)\mathbb{1}\left\{-b \leq \mathcal{B}\left(Q_F(X_i,t_i,\theta^*_F)\right) < 0\right\}\right]\right)$$

From Assumption 6, there exist constants $(\overline{A},\overline{V})$ such that, for each $F \in \mathcal{F}$, the following class of indicator functions is Euclidean $(\overline{A},\overline{V})$ for the constant envelope 1,

$$\left\{m : \mathcal{X}\times\mathcal{T} \longrightarrow \mathbb{R} : m(x,t) = \mathbb{1}\left\{-b \leq \mathcal{B}\left(Q_F(x,t,\theta^*_F)\right) < 0\right\} \text{ for some } 0 < b \leq b_0\right\}.$$

From here, Result S1 yields,

$$\sup_{0<b<b_0} \left| m^a_{T_1,n}(b) \right| = O_p\left(\frac{1}{n^{1/2}}\right) \quad \text{uniformly over } \mathcal{F}.$$

For n large enough we have $0 < 2b_n \le b_0$. Therefore, for n large enough,

$$\left| m^a_{T_1,n}(2b_n) \right| \le \sup_{0<b<b_0} \left| m^a_{T_1,n}(b) \right| = O_p\left(\frac{1}{n^{1/2}}\right) \quad \text{uniformly over } \mathcal{F}. \tag{S3.6.4}$$

We have

$$\begin{aligned} &\frac{1}{n}\sum_{i=1}^{n} \phi(X_i,t_i)\mathbb{1}\Big\{-2b_n \le \mathcal{B}\left(Q_F(X_i,t_i,\theta^*_F)\right) < 0\Big\} \\ =& m^a_{T_1,n}(2b_n) + E_F\left[\mathbb{1}\Big\{-2b_n \le \mathcal{B}\left(Q_F(X,t,\theta^*_F)\right) < 0\Big\} \cdot \mathbb{1}\Big\{(X,t) \in \mathcal{X}\times\mathcal{T}\Big\}\right]. \end{aligned}$$

From Assumption 7A, there exist finite constants $\underline{b}_1 > 0$ and $\overline{C}_{\mathcal{B},1} > 0$ such that, for all $0 < b \le \underline{b}_1$,

$$E_F\left[\mathbb{1}\Big\{-b \le \mathcal{B}\left(Q_F(X,t,\theta^*_F)\right) < 0\Big\} \cdot \mathbb{1}\Big\{(X,t) \in \mathcal{X}\times\mathcal{T}\Big\}\right] \le \overline{C}_{\mathcal{B},1} \cdot b \quad \forall\ F \in \mathcal{F},$$

For n large enough, we have $0 < 2b_n \le \underline{b}_1 \wedge b_0$, and from Assumption 4, we have $n^{1/2} \cdot b_n \longrightarrow \infty$. This, combined with equation (S3.6.4) and Assumption 7A yields,

$$\begin{aligned} \frac{1}{n}\sum_{i=1}^{n} \phi(X_i,t_i)\mathbb{1}\Big\{-2b_n \le \mathcal{B}\left(Q_F(X_i,t_i,\theta^*_F)\right) < 0\Big\} &\le O_p\left(\frac{1}{n^{1/2}}\right) + O(b_n) \\ &= b_n \cdot \left(O_p\left(\frac{1}{b_n \cdot n^{1/2}}\right) + O(1)\right) \\ &= b_n \cdot \left(o_p(1) + O(1)\right) \\ &= O_p(b_n) \quad \text{uniformly over } \mathcal{F}. \end{aligned} \tag{S3.6.5}$$

Next, note from Assumption 5 and equation (S3.5.3), we have $\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left|\mathcal{B}\left(\widehat{Q}(x,t,\widehat{\theta})\right)\right| = O_p(1)$ uniformly over $\mathcal{F}$. Therefore,

$$\begin{aligned} &\frac{1}{n}\sum_{i=1}^{n} \left|\mathcal{B}\left(\widehat{Q}(X_i,t_i,\widehat{\theta})\right)\right| \phi(X_i,t_i)\mathbb{1}\left\{\left|\mathcal{B}\left(\widehat{Q}(X_i,t_i,\widehat{\theta})\right) - \mathcal{B}\left(Q(X_i,t_i,\theta^*_F)\right)\right| \ge b_n\right\} \\ \le& \sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left|\mathcal{B}\left(\widehat{Q}(x,t,\widehat{\theta})\right)\right| \times \frac{1}{n}\sum_{i=1}^{n} \phi(X_i,t_i)\mathbb{1}\left\{\left|\mathcal{B}\left(\widehat{Q}(X_i,t_i,\widehat{\theta})\right) - \mathcal{B}\left(Q(X_i,t_i,\theta^*_F)\right)\right| \ge b_n\right\} \\ =& O_p(1) \times \frac{1}{n}\sum_{i=1}^{n} \phi(X_i,t_i)\mathbb{1}\left\{\left|\mathcal{B}\left(\widehat{Q}(X_i,t_i,\widehat{\theta})\right) - \mathcal{B}\left(Q(X_i,t_i,\theta^*_F)\right)\right| \ge b_n\right\}, \quad \text{uniformly over } \mathcal{F}. \end{aligned}$$

Recall that, since $\phi(\cdot) \geq 0$, we have

$$\begin{aligned}
&\left|\frac{1}{n}\sum_{i=1}^{n}\phi(X_i,t_i)\mathbb{1}\left\{\left|\mathcal{B}\left(\widehat{Q}(X_i,t_i,\widehat{\theta})\right)-\mathcal{B}\left(Q(X_i,t_i,\theta_F^*)\right)\right|\geq b_n\right\}\right| \\
=&\frac{1}{n}\sum_{i=1}^{n}\phi(X_i,t_i)\mathbb{1}\left\{\left|\mathcal{B}\left(\widehat{Q}(X_i,t_i,\widehat{\theta})\right)-\mathcal{B}\left(Q(X_i,t_i,\theta_F^*)\right)\right|\geq b_n\right\}.
\end{aligned}$$

Next, note that

$$\begin{aligned}
&\sup_{F\in\mathcal{F}} P_F\left(\frac{1}{n}\sum_{i=1}^{n}\phi(X_i,t_i)\mathbb{1}\left\{\left|\mathcal{B}\left(\widehat{Q}(X_i,t_i,\widehat{\theta})\right)-\mathcal{B}\left(Q(X_i,t_i,\theta_F^*)\right)\right|\geq b_n\right\}\neq 0\right) \\
\leq&\sup_{F\in\mathcal{F}}\ P_F\left(\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\mathcal{B}(\widehat{Q}(x,t,\widehat{\theta}))-\mathcal{B}(Q_F(x,t,\theta_F^*))\right|\geq b_n\right)\longrightarrow 0
\end{aligned}$$

where the last result follows from (S3.5.4). In particular, for any $\delta > 0$ and $\Delta > 0$,

$$\sup_{F\in\mathcal{F}} P_F\left(\frac{1}{n}\sum_{i=1}^{n}\phi(X_i,t_i)\mathbb{1}\left\{\left|\mathcal{B}\left(\widehat{Q}(X_i,t_i,\widehat{\theta})\right)-\mathcal{B}\left(Q(X_i,t_i,\theta_F^*)\right)\right|\geq b_n\right\}\geq\frac{\delta}{n^{1/2+\Delta}}\right)\longrightarrow 0.$$

That is,

$$\frac{1}{n}\sum_{i=1}^{n}\phi(X_i,t_i)\mathbb{1}\left\{\left|\mathcal{B}\left(\widehat{Q}(X_i,t_i,\widehat{\theta})\right)-\mathcal{B}\left(Q(X_i,t_i,\theta_F^*)\right)\right|\geq b_n\right\}=o_p\left(\frac{1}{n^{1/2+\Delta}}\right)\quad\forall\ \Delta>0,\ \text{ uniformly over } \mathcal{F}.$$

Therefore,

$$\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{n}\left|\mathcal{B}\left(\widehat{Q}(X_i,t_i,\widehat{\theta})\right)\right|\phi(X_i,t_i)\mathbb{1}\left\{\left|\mathcal{B}\left(\widehat{Q}(X_i,t_i,\widehat{\theta})\right)-\mathcal{B}\left(Q(X_i,t_i,\theta_F^*)\right)\right|\geq b_n\right\} \\
&=o_p\left(\frac{1}{n^{1/2+\Delta}}\right)\quad\forall\ \Delta>0,\ \text{ uniformly over } \mathcal{F}.
\end{aligned}$$

Plugging the results in (S3.6.5) and the previous expression into (S3.6.3), for any $\Delta > 0$ we have

$$\begin{aligned}
\left|\xi_{T_1,n}^a\right| &\leq \left(2b_n+o_p\left(\frac{1}{n^{1/4+\epsilon/2}}\right)\right)\times O_p\left(b_n\right)+o_p\left(\frac{1}{n^{1/2+\Delta}}\right) \\
&=O_p\left(b_n^2\right)+o_p\left(\frac{b_n}{n^{1/4+\epsilon/2}}\right)+o_p\left(\frac{1}{n^{1/2+\Delta}}\right)\quad \text{uniformly over } \mathcal{F}.
\end{aligned}$$

Take any $\Delta > 0$ and note that $\left(\frac{b_n}{n^{1/4+\epsilon/2}}\right) \cdot n^{1/2+\Delta} = \left(n^{1/2+2\Delta-\epsilon} \cdot b_n^2\right)^{1/2}$. In Assumption 4 we stated that there exists $\delta_0 > 0$ such that $n^{1/2+\delta_0} \cdot b_n^2 \longrightarrow 0$. Therefore,

$$\frac{b_n}{n^{1/4+\epsilon/2}} = o\left(\frac{1}{n^{1/2+\Delta}}\right) \quad \forall\ 0 < \Delta \leq \frac{\delta_0}{2}.$$

From here, we obtain

$$\left|\xi^a_{T_1,n}\right| = o_p\left(\frac{1}{n^{1/2+\delta_0/2}}\right) \quad \text{uniformly over } \mathcal{F}. \tag{S3.6.6}$$

Therefore, using the linear representation result in (S3.5.1),

$$\begin{aligned}
\widehat{T}_1 &= \widetilde{T}_{1,F} + \xi^a_{T_1,n} \\
&= \frac{1}{n}\sum_{i=1}^{n} \mathcal{B}\left(\widehat{Q}(X_i,t_i,\widehat{\theta})\right) \mathbb{1}\left\{\mathcal{B}\left(Q_F(X_i,t_i,\theta^*_F)\right) \geq 0\right\} \phi(X_i,t_i) + \xi^a_{T_1,n} \\
&= \frac{1}{n}\sum_{i=1}^{n} \left(\mathcal{B}(Q_F(X_i,t_i,\theta^*_F)) + \frac{1}{n}\sum_{j=1}^{n} \psi^{\mathcal{B}}_F(V_j,X_i,t_i,\theta^*_F,h_n) + \zeta^{\mathcal{B}}_n(X_i,t_i)\right) \mathbb{1}\left\{\mathcal{B}\left(Q_F(X_i,t_i,\theta^*_F)\right) \geq 0\right\} \phi(X_i,t_i) + \xi^a_{T_1,n} \\
&= \frac{1}{n}\sum_{i=1}^{n} \left(\mathcal{B}\left(Q_F(X_i,t_i,\theta^*_F)\right)\right)_+ \phi(X_i,t_i) + \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} \psi^{\mathcal{B}}_F(V_j,X_i,t_i,\theta^*_F,h_n) \mathbb{1}\left\{\mathcal{B}\left(Q_F(X_i,t_i,\theta^*_F)\right) \geq 0\right\} \phi(X_i,t_i) \\
&+ \frac{1}{n}\sum_{i=1}^{n} \zeta^{\mathcal{B}}_n(X_i,t_i) \mathbb{1}\left\{\mathcal{B}\left(Q_F(X_i,t_i,\theta^*_F)\right) \geq 0\right\} \phi(X_i,t_i) + \xi^a_{T_1,n}.
\end{aligned}$$

Recall that our population statistic $T_{1,F}$ was defined in equation (11A) as

$$T_{1,F} \equiv E_F\left[\left(\mathcal{B}\left(Q_F(X,t,\theta^*_F)\right)\right)_+ \phi(X,t)\right],$$

with the expectation taken with respect to (X,t). Therefore, from the above expression we have,

$$\begin{aligned}
\widehat{T}_1 = &T_{1,F} + \frac{1}{n}\sum_{i=1}^{n}\left(\left(\mathcal{B}\left(Q_F(X_i,t_i,\theta^*_F)\right)\right)_+ \phi(X_i,t_i) - T_{1,F}\right) \\
&\underbrace{+ \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} \psi^{\mathcal{B}}_F(V_j,X_i,t_i,\theta^*_F,h_n) \mathbb{1}\left\{\mathcal{B}\left(Q_F(X_i,t_i,\theta^*_F)\right) \geq 0\right\} \phi(X_i,t_i) + \frac{1}{n}\sum_{i=1}^{n} \zeta^{\mathcal{B}}_n(X_i,t_i) \mathbb{1}\left\{\mathcal{B}\left(Q_F(X_i,t_i,\theta^*_F)\right) \geq 0\right\} \phi(X_i,t_i) + \xi^a_{T_1,n}}_{\equiv \xi^b_{T_1,n}}.
\end{aligned} \tag{S3.6.7}$$

Let us analyze each of the terms in (S3.6.7). From (S3.5.2) we have,

$$\left|\xi^b_{T_1,n}\right| \equiv \left|\frac{1}{n}\sum_{i=1}^{n}\zeta^{\mathcal{B}}_n(X_i,t_i)\mathbb{1}\left\{\mathcal{B}\left(Q_F(X_i,t_i,\theta^*_F)\right)\geq 0\right\}\phi(X_i,t_i)\right| \leq \overline{\phi}\cdot \sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\zeta^{\mathcal{B}}_n(x,t)\right|$$
$$= o_p\left(\frac{1}{n^{1/2+\epsilon}}\right) \quad \text{uniformly over } \mathcal{F} \tag{S3.6.8}$$

Next, recall that

$$\begin{aligned}\psi^{\mathcal{B}}_F(V_j,x,t,\theta^*_F,h_n) &\equiv \nabla_Q\mathcal{B}(Q_F(x,t,\theta^*_F))\psi^Q_F(V_j,x,t,\theta^*_F,h_n)\\ &= \sum_{p=1}^{P}\frac{\partial\mathcal{B}\left(Q_F(x,t,\theta^*_F)\right)}{\partial Q_p}\psi^{Q_p}_F(V_j,x,t,\theta^*_F,h_n).\end{aligned}$$

For $h>0$, $x\in\mathcal{S}_X$ and $t\in\mathcal{T}$, let

$$\begin{aligned}\Lambda^{Q_p}_F(V_j,x,t,\theta^*_F,h) \equiv &\left(\frac{S_p(Y_j,t)-\Gamma_{p,F}(x,t,\theta^*_F)}{f_g(g(x,\theta^*_F))}\right)\omega_p(g(X_j,\theta^*_F))K\left(\frac{\Delta g(X_j,x,\theta^*_F)}{h}\right)\\ &- E_F\left[\left(\frac{S_p(Y_j,t)-\Gamma_{p,F}(x,t,\theta^*_F)}{f_g(g(x,\theta^*_F))}\right)\omega_p(g(X_j,\theta^*_F))K\left(\frac{\Delta g(X_j,x,\theta^*_F)}{h}\right)\right].\end{aligned}$$

From the definition in (S3.4.4), we have

$$\psi^{Q_p}_F(V_j,x,t,\theta^*_F,h_n) = \frac{1}{h^D_n}\Lambda^{Q_p}_F(V_j,x,t,\theta^*_F) + \Xi_{Q_p}(x,t,\theta^*_F)\psi^{\theta}_F(Z_j).$$

For $h>0$ and $v_1\in\mathcal{S}_V$, $v_2\in\mathcal{S}_V$ let

$$\begin{aligned}\Lambda^a_{T_1,F}(v_1,v_2,h) &\equiv \sum_{p=1}^{P}\frac{\partial\mathcal{B}\left(Q_F(x_1,t_1,\theta^*_F)\right)}{\partial Q_p}\Lambda^{Q_p}_F(v_2,x_1,t_1,\theta^*_F,h)\mathbb{1}\left\{\mathcal{B}\left(Q_F(x_1,t_1,\theta^*_F)\right)\geq 0\right\}\phi(x_1,t_1),\\ \Lambda^b_{T_1,F}(v_1,v_2) &\equiv \left(\sum_{p=1}^{P}\frac{\partial\mathcal{B}\left(Q_F(x_1,t_1,\theta^*_F)\right)}{\partial Q_p}\mathbb{1}\left\{\mathcal{B}\left(Q_F(x_1,t_1,\theta^*_F)\right)\geq 0\right\}\phi(x_1,t_1)\Xi_{Q_p}(x_1,t_1,\theta^*_F)\right)\psi^{\theta}_F(z_2)\end{aligned}$$

and define

$$\begin{aligned}U^a_{T_1,n}(h) &\equiv \frac{1}{n\cdot(n-1)}\sum_{i=1}^{n}\sum_{j\neq i}\Lambda^a_{T_1,F}(V_i,V_j,h), \quad \nu^a_{T_1,n}(h) \equiv \frac{1}{n}\sum_{i=1}^{n}\left(\Lambda^a_{T_1,F}(V_i,V_i,h) - E_F\left[\Lambda^a_{T_1,F}(V,V,h)\right]\right),\\ U^b_{T_1,n} &\equiv \frac{1}{n\cdot(n-1)}\sum_{i=1}^{n}\sum_{j\neq i}\Lambda^b_{T_1,F}(V_i,V_j), \qquad \nu^b_{T_1,n} \equiv \frac{1}{n}\sum_{i=1}^{n}\left(\Lambda^b_{T_1,F}(V_i,V_i) - E_F\left[\Lambda^b_{T_1,F}(V,V)\right]\right).\end{aligned} \tag{S3.6.9}$$

We have,

$$
\begin{aligned}
&\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\psi_F^{\mathcal{B}}(V_j,X_i,t_i,\theta_F^*,h_n)\mathbb{1}\{\mathcal{B}(Q_F(X_i,t_i,\theta_F^*))\geq 0\}\phi(X_i,t_i)\\
=&\left(\frac{n-1}{n}\right)\cdot\frac{1}{h_n^D}\cdot U_{T_1,n}^a(h_n)+\left(\frac{n-1}{n}\right)\cdot U_{T_1,n}^b+\frac{1}{n\cdot h_n^D}\cdot \nu_{T_1,n}^a(h_n)+\frac{1}{n}\cdot\nu_{T_1,n}^b\\
&+\frac{1}{n\cdot h_n^D}\cdot E_F\left[\Lambda_{T_1,F}^a(V,V,h_n)\right]+\frac{1}{n}\cdot E_F\left[\Lambda_{T_1,F}^b(V,V)\right]
\end{aligned}
\tag{S3.6.10}
$$

We will proceed by analyzing the U-statistic $U_{T_1,n}^a(h_n)$. First, let us study

$$
\varphi_{T_1,F}^a(V_i,h_n)\equiv E_F\left[\Lambda_{T_1,F}^a(V_i,V_j,h_n)+\Lambda_{T_1,F}^a(V_j,V_i,h_n)\Big|V_i\right].
$$

For each $F\in\mathcal{F}$ and a given $(x,t)\in\mathcal{X}\times\mathcal{T}$, let

$$
\mu_{F,n}^{S_p}(x,t)\equiv\frac{1}{h_n^D}\cdot E_F\left[\Big(S_p(Y,t)-\Gamma_{p,F}(x,t,\theta_F^*)\Big)\omega_p(g(X,\theta_F^*))K\left(\frac{\Delta g(X,x,\theta_F^*)}{h_n}\right)\right].
\tag{S3.6.11}
$$

Recall that in Assumption 2 we defined, for each $F\in\mathcal{F}$,

$$
\Omega_{R_p,0}(g,t)\equiv E_F\left[S_p(Y,t)\Big|g(X,\theta_F^*)=g\right].
$$

By iterated expectations,

$$
\mu_{F,n}^{S_p}(x,t)=\frac{1}{h_n^D}\int_u\Big(\Omega_{R_p,0}(u,t)-\Gamma_{p,F}(x,t,\theta_F^*)\Big)\omega_p(u)K\left(\frac{u-g(x,\theta_F^*)}{h_n}\right)f_g(u)du.
$$

From here, using the smoothness properties described in Assumption 2, performing an M^{th}–order approximation and noting that $\Omega_{R_p,0}(g(x,\theta_F^*),t)=\Gamma_{p,F}(x,t,\theta_F^*)$, there exists a finite constant $\overline{B}_{\mu_S}>0$ such that,

$$
\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\mu_{F,n}^{S_p}(x,t)\right|\leq\overline{B}_{\mu_S}\cdot h_n^M\quad\forall\ F\in\mathcal{F}.
\tag{S3.6.12}
$$

We have,

$$
\begin{aligned}
\frac{1}{h_n^D}\Lambda_F^{Q_p}(V_j,x,t,\theta_F^*,h_n)=&\frac{1}{h_n^D}\left(\frac{S_p(Y_j,t)-\Gamma_{p,F}(x,t,\theta_F^*)}{f_g(g(x,\theta_F^*))}\right)\cdot\omega_p(g(X_j,\theta_F^*))\cdot K\left(\frac{\Delta g(X_j,x,\theta_F^*)}{h_n}\right)\\
&-\frac{\mu_{F,n}^{S_p}(x,t)}{f_g(g(x,\theta_F^*))}
\end{aligned}
$$

And, from here,

$$
\begin{aligned}
&\frac{1}{h_n^D}\cdot E_F\left[\frac{\partial\mathcal{B}\left(Q_F(X_i,t_i,\theta_F^*)\right)}{\partial Q_p}\Lambda_F^{Q_p}(V_j,X_i,t_i,\theta_F^*,h_n)\mathbb{1}\left\{\mathcal{B}\left(Q_F(X_i,t_i,\theta_F^*)\right)\geq 0\right\}\phi(X_i,t_i)\middle|V_i\right]\\
&=\frac{\partial\mathcal{B}\left(Q_F(X_i,t_i,\theta_F^*)\right)}{\partial Q_p}\frac{1}{f_g(g(X_i,\theta_F^*))}\mathbb{1}\left\{\mathcal{B}\left(Q_F(X_i,t_i,\theta_F^*)\right)\geq 0\right\}\phi(X_i,t_i)\underbrace{\left(\mu_{F,n}^{S_p}(X_i,t_i)-\mu_{F,n}^{S_p}(X_i,t_i)\right)}_{=0}\\
&=0\quad\forall\ F\in\mathcal{F}.
\end{aligned}
\tag{S3.6.13}
$$

From Assumptions 2 and 5

$$
\begin{aligned}
&\sup_{(x,t)\in\mathcal{S}_{X,t}}\left|\frac{\partial\mathcal{B}\left(Q_F(x,t,\theta_F^*)\right)}{\partial Q_p}\frac{1}{f_g(g(x,\theta_F^*))}\mathbb{1}\left\{\mathcal{B}\left(Q_F(x,t,\theta_F^*)\right)\geq 0\right\}\phi(x,t)\right|\\
&\leq\overline{\phi}\cdot\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\frac{\partial\mathcal{B}\left(Q_F(x,t,\theta_F^*)\right)}{\partial Q_p}\frac{1}{f_g(g(x,\theta_F^*))}\right|\leq\overline{\phi}\cdot\frac{\overline{H}_Q}{\underline{f}_g}\quad\forall\ F\in\mathcal{F}
\end{aligned}
$$

Also from Assumption 2, there exists a finite constant $\overline{C}_{\Xi_{Q_p}}>0$ such that

$$
\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left\|\Xi_{Q_p}(x,t,\theta_F^*)\right\|\leq\overline{C}_{\Xi_{Q_p}}\quad\forall\ F\in\mathcal{F}.
$$

Therefore, from Assumptions 2 and 5,

$$
\begin{aligned}
&\sup_{(x,t)\in\mathcal{S}_{X,t}}\left\|\frac{\partial\mathcal{B}\left(Q_F(x,t,\theta_F^*)\right)}{\partial Q_p}\mathbb{1}\left\{\mathcal{B}\left(Q_F(x,t,\theta_F^*)\right)\geq 0\right\}\phi(x,t)\Xi_{Q_p}(x,t,\theta_F^*)\right\|\\
&\leq\overline{\phi}\cdot\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left\|\frac{\partial\mathcal{B}\left(Q_F(x,t,\theta_F^*)\right)}{\partial Q_p}\Xi_{Q_p}(x,t,\theta_F^*)\right\|\leq\overline{\phi}\cdot\overline{H}_Q\cdot\overline{C}_{\Xi_{Q_p}}\equiv\overline{C}_{\Xi_{T_1}}\quad\forall\ F\in\mathcal{F}.
\end{aligned}
\tag{S3.6.14}
$$

Thus, from (S3.6.13),

$$
\frac{1}{h_n^D}\cdot E_F\left[\frac{\partial\mathcal{B}\left(Q_F(X_i,t_i,\theta_F^*)\right)}{\partial Q_p}\Lambda_F^{Q_p}(V_j,X_i,t_i,\theta_F^*,h_n)\mathbb{1}\left\{\mathcal{B}\left(Q_F(X_i,t_i,\theta_F^*)\right)\geq 0\right\}\phi(X_i,t_i)\middle|V_i\right]=0,\ \text{ and therefore,}
$$

$$
E_F\left[\Lambda_{T_1,F}^{a}(V_j,V_i,\theta_F^*,h_n)\right]=0\quad\forall\ F\in\mathcal{F}
\tag{S3.6.15}
$$

Next, let us analyze

$$
\frac{1}{h_n^D}\cdot E_F\left[\frac{\partial\mathcal{B}\left(Q_F(X_i,t_i,\theta_F^*)\right)}{\partial Q_p}\Lambda_F^{Q_p}(V_j,X_i,t_i,\theta_F^*,h_n)\mathbb{1}\left\{\mathcal{B}\left(Q_F(X_i,t_i,\theta_F^*)\right)\geq 0\right\}\phi(X_i,t_i)\middle|V_j\right].
$$

Recall from Assumption 7A that we defined

$$\Omega^p_{T_1}(y,g) = E_F\left[\left(S_p(y,t) - \Gamma_{p,F}(X,t,\theta^*_F)\right)\frac{\partial \mathcal{B}\left(Q_F(X,t,\theta^*_F)\right)}{\partial Q_p}\phi(X,t)\mathbb{1}\left\{\mathcal{B}\left(Q_F(X,t,\theta^*_F)\right) \geq 0\right\}\middle| g(X,\theta^*_F) = g\right]$$

Using iterated expectations, we have

$$\begin{aligned}
&\frac{1}{h_n^D}\cdot E_F\left[\frac{\partial \mathcal{B}\left(Q_F(X_i,t_i,\theta^*_F)\right)}{\partial Q_p}\Lambda_F^{Q_p}(V_j,X_i,t_i,\theta^*_F,h_n)\mathbb{1}\left\{\mathcal{B}\left(Q_F(X_i,t_i,\theta^*_F)\right) \geq 0\right\}\phi(X_i,t_i)\middle| V_j\right]\\
&= \frac{1}{h_n^D}E_F\left[\frac{\Omega^p_{T_1}(Y_j,g(X_i,\theta^*_F))}{f_g(g(X_i,\theta^*_F))}K\left(\frac{\Delta g(X_i,X_j,\theta^*_F)}{h_n}\right)\middle| V_j\right]\cdot\omega_p(g(X_j,\theta^*_F))\\
&- E_F\left[\frac{\mu^{S_p}_{F,n}(X_i,t_i)}{f_g(g(X_i,\theta^*_F))}\frac{\partial \mathcal{B}\left(Q_F(X_i,t_i,\theta^*_F)\right)}{\partial Q_p}\mathbb{1}\left\{\mathcal{B}\left(Q_F(X_i,t_i,\theta^*_F)\right) \geq 0\right\}\phi(X_i,t_i)\right]
\end{aligned}$$

We have,

$$\begin{aligned}
&\frac{1}{h_n^D}E_F\left[\frac{\Omega^p_{T_1}(Y_j,g(X_i,\theta^*_F))}{f_g(g(X_i,\theta^*_F))}K\left(\frac{\Delta g(X_i,X_j,\theta^*_F)}{h_n}\right)\middle| V_j\right]\cdot\omega_p(g(X_j,\theta^*_F))\\
=&\frac{1}{h_n^D}\int\frac{\Omega^p_{T_1}(Y_j,u)}{f_g(u)}K\left(\frac{u-g(X_j,\theta^*_F)}{h_n}\right)f_g(u)du\cdot\omega_p(g(X_j,\theta^*_F))
\end{aligned}$$

Since $\omega_p(g) \neq 0$ if and only if $g \in \mathcal{G}$, the above expression is nonzero only if $g(X_j,\theta^*_F) \in \mathcal{G}$. Since the kernel K has bounded support such that $K(\psi) \neq 0$ if and only if $\|\psi\| \leq S$ and since $h_n \to 0$, for large enough n it follows that $\left\|\frac{u-g}{h_n}\right\| \leq S$ for $g \in \mathcal{G}$ implies $u \in \mathcal{G}'$, where $\mathcal{G}'$ is the set that contains $\mathcal{G}$ and is described in Assumption 7A. In particular, by the conditions described there, for large enough n, $\left\|\frac{u-g}{h_n}\right\| \leq S$ for $g \in \mathcal{G}$ implies $f_g(u) \geq \underline{f}_g > 0$. Therefore, for large enough n the terms $f_g(u)$ in the numerator and in the denominator of the previous expression can cancel each other out (since they are nonzero) and we have

$$\begin{aligned}
&\frac{1}{h_n^D}E_F\left[\frac{\Omega^p_{T_1}(Y_j,g(X_i,\theta^*_F))}{f_g(g(X_i,\theta^*_F))}K\left(\frac{\Delta g(X_i,X_j,\theta^*_F)}{h_n}\right)\middle| V_j\right]\cdot\omega_p(g(X_j,\theta^*_F))\\
=&\frac{1}{h_n^D}\int\Omega^p_{T_1}(Y_j,u)K\left(\frac{u-g(X_j,\theta^*_F)}{h_n}\right)du\cdot\omega_p(g(X_j,\theta^*_F)) \quad \forall\ F \in \mathcal{F}.
\end{aligned}$$

From here, the smoothness conditions described in Assumption 7A and an M^{th}–order approximation imply that there exists a finite constant $\overline{B}_{\Omega_{T_1}} > 0$ such that

$$\begin{aligned}
&\frac{1}{h_n^D} E_F\left[\frac{\Omega_{T_1}^p(Y_j, g(X_i,\theta_F^*))}{f_g(g(X_i,\theta_F^*))} K\left(\frac{\Delta g(X_i,X_j,\theta_F^*)}{h_n}\right)\middle| V_j\right]\cdot \omega_p(g(X_j,\theta_F^*))\\
=&\Omega_{T_1}^p(Y_j, g(X_j,\theta_F^*))\cdot \omega_p(g(X_j,\theta_F^*)) + B_{\Omega_{T_1},n}^p(Y_j,X_j)\cdot \omega_p(g(X_j,\theta_F^*)),\\
&\quad\text{where} \quad \sup_{(y,x)\in\mathcal{S}_{Y,X}}\left|B_{\Omega_{T_1},n}^p(y,x)\cdot \omega_p(g(x,\theta_F^*))\right| \le \overline{B}_{\Omega_{T_1}}\cdot h_n^M \quad \forall\, F\in\mathcal{F}.
\end{aligned}$$

Next, from Assumptions 2 and 5 and from the result in (S3.6.12),

$$\left|E_F\left[\frac{\mu_{F,n}^{S_p}(X_i,t_i)}{f_g(g(X_i,\theta_F^*))}\frac{\partial \mathcal{B}\big(Q_F(X_i,t_i,\theta_F^*)\big)}{\partial Q_p}\mathbb{1}\left\{\mathcal{B}\left(Q_F(X_i,t_i,\theta_F^*)\right)\ge 0\right\}\phi(X_i,t_i)\right]\right| \le \overline{\phi}\cdot\frac{\overline{H}_Q}{\underline{f}_g}\cdot \overline{B}_{\mu_S}\cdot h_n^M \quad \forall\, F\in\mathcal{F}.$$

These results combined yield,

$$\begin{aligned}
&\frac{1}{h_n^D}\cdot E_F\left[\frac{\partial\mathcal{B}\big(Q_F(X_i,t_i,\theta_F^*)\big)}{\partial Q_p}\Lambda_F^{Q_p}(V_j,X_i,t_i,\theta_F^*,h_n)\mathbb{1}\left\{\mathcal{B}\left(Q_F(X_i,t_i,\theta_F^*)\right)\ge 0\right\}\phi(X_i,t_i)\middle| V_j\right]\\
&= \Omega_{T_1}^p(Y_j,g(X_j,\theta_F^*))\cdot\omega_p(g(X_j,\theta_F^*)) + B_{T_1,n}^p(Y_j,X_j), \qquad\qquad \text{(S3.6.16)}\\
&\quad\text{where} \quad \sup_{(y,x)\in\mathcal{S}_{Y,X}}\left|B_{T_1,n}^p(y,x)\right| \le \left(\overline{B}_{\Omega_{T_1}} + \overline{\phi}\cdot\frac{\overline{H}_Q}{\underline{f}_g}\cdot \overline{B}_{\mu_S}\right)\cdot h_n^M \equiv \overline{C}_{T_1}^a\cdot h_n^M \quad \forall\, F\in\mathcal{F}.
\end{aligned}$$

Denote

$$\sum_{p=1}^{P} B_{T_1,n}^p(Y_i,X_i) \equiv \overline{B}_{T_1,n}(Y_i,X_i).$$

Combining (S3.6.15) and (S3.6.16), we have

$$\begin{aligned}
\frac{1}{h_n^D}\cdot\varphi_{T_1,F}^a(V_i,h_n) &\equiv \frac{1}{h_n^D}\cdot E_F\left[\Lambda_{T_1,F}^a(V_i,V_j,h_n) + \Lambda_{T_1,F}^a(V_j,V_i,h_n)\middle| V_i\right]\\
&\equiv \sum_{p=1}^{P}\Omega_{T_1}^p(Y_i,g(X_i,\theta_F^*))\cdot\omega_p(g(X_i,\theta_F^*)) + \overline{B}_{T_1,n}(Y_i,X_i), \qquad \text{(S3.6.17)}\\
&\quad\text{where} \quad \sup_{(y,x)\in\mathcal{S}_{Y,X}}\left|\overline{B}_{T_1,n}(y,x)\right| \le P\cdot\overline{C}_{T_1}^a\cdot h_n^M \equiv \overline{C}_{T_1}^b\cdot h_n^M \quad \forall\, F\in\mathcal{F}.
\end{aligned}$$

Next, let

$$\vartheta_{T_1,F}^a(V_i,V_j,h) \equiv \Lambda_{T_1,F}^a(V_i,V_j,h) + \Lambda_{T_1,F}^a(V_j,V_i,h) - \varphi_{T_1,F}^a(V_i,h) - \varphi_{T_1,F}^a(V_j,h)$$

Note that $\vartheta_{T_1,F}^a(V_i,V_j,h) = \vartheta_{T_1,F}^a(V_j,V_i,h)$ and $E_F\left[\vartheta_{T_1,F}^a(V_i,V_j,h)|V_i\right] = E_F\left[\vartheta_{T_1,F}^a(V_i,V_j,h)|V_j\right] = 0$. Define the following degenerate U-statistic of order 2,

$$\widetilde{U}^a_{T_1,n}(h) = \frac{1}{n(n-1)}\sum_{i=1}^{n}\sum_{j\neq i}\vartheta^a_{T_1,F}(V_i,V_j,h).$$

As we have invoked previously, from the bounded-variation properties of the kernel K and the conditions stated in Assumptions 2-5 and 7A, from Example 2.10 in Pakes and Pollard (1989) and Lemma 20 in Nolan and Pollard (1987) (or Lemma 5 in Sherman (1994)) combined with the Euclidean-preserving properties in Lemma 2.14 in Pakes and Pollard (1989), the class of functions

$$\left\{m : \mathcal{S}^2_V \longrightarrow \mathbb{R}\text{: } m(v_1,v_2) = \vartheta^a_{T_1,F}(v_1,v_2,h) \text{ for some } h>0\right\}$$

is Euclidean for an envelope $\overline{G}^a_{T_1}(\cdot)$ that satisfies $E_F\left[\overline{G}^a_{T_1}(V_1,V_2)^{4q}\right]$ for all $F\in\mathcal{F}$ (with $(V_1,V_2)\sim F\otimes F$) and q being the integer described in Assumption 1. From here, applying Result S1 we obtain

$$\sup_{h>0}\left|\widetilde{U}^a_{T_1,n}(h)\right| = O_p\left(\frac{1}{n}\right) \quad \text{uniformly over } \mathcal{F}$$

Therefore,

$$\left|\widetilde{U}^a_{T_1,n}(h_n)\right| \le \sup_{h>0}\left|\widetilde{U}^a_{T_1,n}(h)\right| = O_p\left(\frac{1}{n}\right) \quad \text{uniformly over } \mathcal{F} \tag{S3.6.18}$$

The Hoeffding decomposition of $U^a_{T_1,n}(h_n)$ (see Serfling (1980, pages 177-178) or Sherman (1994, equations (6)-(7))) yields,

$$\begin{aligned}\frac{1}{h_n^D}\cdot U^a_{T_1,n}(h_n) &= \frac{1}{n}\sum_{i=1}^{n}\frac{1}{h_n^D}\cdot\varphi^a_{T_1,F}(V_i,h_n) + \frac{1}{2h_n^D}\cdot\widetilde{U}^a_{T_1,n}(h_n)\\ &= \frac{1}{n}\sum_{i=1}^{n}\left(\sum_{p=1}^{P}\Omega^p_{T_1}(Y_i,g(X_i,\theta^*_F))\cdot\omega_p(g(X_i,\theta^*_F))\right) + \frac{1}{n}\sum_{i=1}^{n}\overline{B}_{T_1,n}(Y_i,X_i) + \frac{1}{2h_n^D}\cdot\widetilde{U}^a_{T_1,n}(h_n),\end{aligned} \tag{S3.6.19}$$

where, from (S3.6.17) and (S3.6.18),

$$\begin{aligned}\left|\frac{1}{n}\sum_{i=1}^{n}\overline{B}_{T_1,n}(Y_i,X_i) + \frac{1}{2h_n^D}\cdot\widetilde{U}^a_{T_1,n}(h_n)\right| &\le \sup_{(y,x)\in\mathcal{S}_{Y,X}}\left|\overline{B}_{T_1,n}(y,x)\right| + \sup_{h>0}\left|\widetilde{U}^a_{T_1,n}(h)\right|\\ &= O(h_n^M) + O_p\left(\frac{1}{n\cdot h_n^D}\right) = o_p\left(\frac{1}{n^{1/2+\epsilon}}\right) \quad \forall\ F\in\mathcal{F}\end{aligned} \tag{S3.6.20}$$

where (as throughout), $\epsilon>0$ is the constant described in Assumption 4. Next, note that

$$E_F\left[\Omega^p_{T_1}(Y,g(X,\theta^*_F))\cdot\omega_p(g(X,\theta^*_F))\right] = 0 \quad \forall\ F\in\mathcal{F} \tag{S3.6.21}$$

for each $p = 1,\ldots,P$. To see this, let $(V_1, V_2) \sim F \otimes F$. Then, by the definition of $\Omega^p_{T_1}$, we have

$$\Omega^p_{T_1}(Y_1, g(X_1, \theta^*_F)) \cdot \omega_p(g(X_1, \theta^*_F)) =$$
$$E_F\left[\left(S_p(Y_1, X_2, t_2) - \Gamma_{p,F}(X_2, t_2, \theta^*_F)\right) \frac{\partial \mathcal{B}\left(Q_F(X_2, t_2, \theta^*_F)\right)}{\partial Q_p} \phi(X_2, t_2) \mathbb{1}\{\mathcal{B}(Q_F(X_2, t_2, \theta^*_F)) \geq 0\} \middle| g(X_2, \theta^*_F) = g(X_1, \theta^*_F), V_1\right]$$
$$\cdot\, \omega_p(g(X_1, \theta^*_F))$$

Therefore, by iterated expectations, we have

$$E_F\left[\Omega^p_{T_1}(Y_1, g(X_1, \theta^*_F)) \cdot \omega_p(g(X_1, \theta^*_F))\right] =$$
$$E_F\Bigg[E_F\left[\left(S_p(Y_1, X_2, t_2) - \Gamma_{p,F}(X_2, t_2, \theta^*_F)\right) \cdot \omega_p(g(X_1, \theta^*_F)) \middle| g(X_1, \theta^*_F) = g(X_2, \theta^*_F), V_2\right]$$
$$\cdot \frac{\partial \mathcal{B}\left(Q_F(X_2, t_2, \theta^*_F)\right)}{\partial Q_p} \phi(X_2, t_2) \mathbb{1}\{\mathcal{B}(Q_F(X_2, t_2, \theta^*_F)) \geq 0\}\Bigg]$$
$$= E_F\left[\left(\underbrace{\Gamma_{p,F}(X_2, t_2, \theta^*_F) - \Gamma_{p,F}(X_2, t_2, \theta^*_F)}_{=0}\right) \cdot \omega_p(g(X_2, \theta^*_F)) \frac{\partial \mathcal{B}\left(Q_F(X_2, t_2, \theta^*_F)\right)}{\partial Q_p} \phi(X_2, t_2) \mathbb{1}\{\mathcal{B}(Q_F(X_2, t_2, \theta^*_F)) \geq 0\}\right]$$
$$= 0.$$

By Assumption 7A, there exists a finite constant $\overline{\eta}_{\Omega_{T_1}} > 0$ such that $E_F\left[\left|\Omega^p_{T_1}(Y, g(X, \theta^*_F)) \cdot \omega_p(g(X, \theta^*_F)\right|^2\right] \leq \overline{\eta}_{\Omega_{T_1}}$ for all $F \in \mathcal{F}$. From here, a Chebyshev inequality yields

$$\left|\frac{1}{n}\sum_{i=1}^n\left(\sum_{p=1}^P \Omega^p_{T_1}(Y_i, g(X_i, \theta^*_F)) \cdot \omega_p(g(X_i, \theta^*_F))\right)\right| = O_p\left(\frac{1}{n^{1/2}}\right) \quad \text{uniformly over } \mathcal{F}.$$

(S3.6.19), (S3.6.20) and the previous expression yield,

$$\frac{1}{h_n^D} \cdot U^a_{T_1,n}(h_n) = O_p\left(\frac{1}{n^{1/2}}\right) \quad \text{uniformly over } \mathcal{F}. \tag{S3.6.22}$$

Next, we turn attention to the U-statistic $U^b_{T_1,n}$, which we defined in (S3.6.9) as

$$U^b_{T_1,n} \equiv \frac{1}{n \cdot (n-1)} \sum_{i=1}^n \sum_{j \neq i} \Lambda^b_{T_1,F}(V_i, V_j),$$

where

$$\Lambda^b_{T_1,F}(v_1,v_2) \equiv \left(\sum_{p=1}^{P} \frac{\partial \mathcal{B}\left(Q_F(x_1,t_1,\theta^*_F)\right)}{\partial Q_p} \mathbb{1}\left\{\mathcal{B}\left(Q_F(x_1,t_1,\theta^*_F)\right) \geq 0\right\} \phi(x_1,t_1) \Xi_{Q_p}(x_1,t_1,\theta^*_F)\right) \psi^\theta_F(z_2).$$

Define,

$$\begin{aligned} \Xi^p_{T_1,F} &\equiv E_F\left[\frac{\partial \mathcal{B}\left(Q_F(X,t,\theta^*_F)\right)}{\partial Q_p} \mathbb{1}\left\{\mathcal{B}\left(Q_F(X,t,\theta^*_F)\right) \geq 0\right\} \phi(X,t) \Xi_{Q_p}(X,t,\theta^*_F)\right], \\ \Xi_{T_1,F} &\equiv \sum_{p=1}^{P} \Xi^p_{T_1,F} \end{aligned} \tag{S3.6.23}$$

where the above expectation is taken with respect to (X,t). Let

$$\varphi^b_{T_1,F}(V_i) \equiv E_F\left[\Lambda^b_{T_1,F}(V_i,V_j) + \Lambda^b_{T_1,F}(V_j,V_i) \middle| V_i\right].$$

Then,

$$\varphi^b_{T_1,F}(V_i) = \Xi_{T_1,F} \psi^\theta_F(Z_i).$$

Note that

$$E_F\left[\varphi^b_{T_1,F}(V_i)\right] = 0, \quad \text{and therefore,} \quad E_F\left[\Lambda^b_{T_1,F}(V_i,V_j)\right] = 0.$$

Next, let

$$\vartheta^b_{T_1,F}(V_i,V_j) \equiv \Lambda^b_{T_1,F}(V_i,V_j) + \Lambda^b_{T_1,F}(V_j,V_i) - \varphi^b_{T_1,F}(V_i) - \varphi^b_{T_1,F}(V_j)$$

and define the degenerate U-statistic

$$\widetilde{U}^b_{T_1,n} = \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} \vartheta^b_{T_1,F}(V_i,V_j).$$

From Assumption 1, $E_F\left[\left\|\psi^\theta_F(Z)\right\|^{4q}\right] \leq \overline{\mu}_\psi \ \forall\, F \in \mathcal{F}$, where q is the integer described there. Also, as we stated in (S3.6.14), Assumptions 2 and 5 imply $\left\|\Xi_{T_1,F}\right\| \leq P \cdot \overline{C}_{\Xi_{T_1}} \ \forall\, F \in \mathcal{F}$. Therefore, there exists a finite constant $\overline{\mu}_{T_1^b} > 0$ such that $E_F\left[\vartheta^b_{T_1,F}(V_1,V_2)^{4q}\right] \leq \overline{\mu}_{T_1^b}$ for all $F \in \mathcal{F}$ (with $(V_1,V_2) \sim F \otimes F$). From here, applying Result S1 we obtain

$$\left|\widetilde{U}^b_{T_1,n}\right| = O_p\left(\frac{1}{n}\right) \quad \text{uniformly over } \mathcal{F}. \tag{S3.6.24}$$

The Hoeffding decomposition of $U^b_{T_1,n}$ is,

$$\begin{aligned} U^b_{T_1,n} &= \frac{1}{n}\sum_{i=1}^{n}\varphi^b_{T_1}(V_i) + \frac{1}{2}\widetilde{U}^b_{T_1,n} \\ &= \Xi_{T_1,F}\frac{1}{n}\sum_{i=1}^{n}\psi^\theta_F(Z_i) + O_p\left(\frac{1}{n}\right) \quad \text{uniformly over } \mathcal{F}. \end{aligned} \tag{S3.6.25}$$

From Assumption 1,

$$\left\|\frac{1}{n}\sum_{i=1}^{n}\psi^\theta_F(Z_i)\right\| = O_p\left(\frac{1}{n^{1/2}}\right) \quad \text{uniformly over } \mathcal{F}.$$

Combined with (S3.6.14),

$$\left\|\Xi_{T_1,F}\frac{1}{n}\sum_{i=1}^{n}\psi^\theta_F(Z_i)\right\| \leq \overline{C}_{\Xi_{T_1}} \cdot \left\|\frac{1}{n}\sum_{i=1}^{n}\psi^\theta_F(Z_i)\right\| = O_p\left(\frac{1}{n^{1/2}}\right) \quad \text{uniformly over } \mathcal{F}.$$

(S3.6.25) and the previous expression yield

$$\left|U^b_{T_1,n}\right| = O_p\left(\frac{1}{n^{1/2}}\right) \quad \text{uniformly over } \mathcal{F}. \tag{S3.6.26}$$

Let us continue with the process $\nu^a_{T_1,n}(h)$ defined in (S3.6.9). The same arguments that led to (S3.6.18) yield

$$\left|\nu^a_{T_1,n}(h_n)\right| \leq \sup_{h>0}\left|\nu^a_{T_1,n}(h)\right| = O_p\left(\frac{1}{n^{1/2}}\right) \quad \text{uniformly over } \mathcal{F}. \tag{S3.6.27}$$

And from (S3.6.14), we have

$$\left|E_F\left[\Lambda^a_{T_1,F}(V,V,h_n)\right]\right| \leq \sup_{h>0}\left|E_F\left[\Lambda^a_{T_1,F}(V,V,h)\right]\right| \leq P\cdot\overline{C}_{\Xi_{T_1}} \quad \forall\, F\in\mathcal{F} \tag{S3.6.28}$$

Next, take the process $\nu^b_{T_2,n}$ defined in (S3.6.9). The same arguments that led to (S3.6.24) yield

$$\left|\nu^b_{T_1,n}\right| = O_p\left(\frac{1}{n^{1/2}}\right) \quad \text{uniformly over } \mathcal{F}. \tag{S3.6.29}$$

By Assumption 1, there exists a finite constant $\overline{\eta}_\psi > 0$ such that $E_F\left[\left\|\psi^\theta_F(Z)\right\|\right] \leq \overline{\eta}_\psi$ for all $F\in\mathcal{F}$. Combined with the result in (S3.6.14), this yields

$$\left|E_F\left[\Lambda^b_{T_1,F}(V,V)\right]\right| \leq P\cdot\overline{C}_{\Xi_{T_1}}\cdot\overline{\eta}_\psi \quad \forall\, F\in\mathcal{F}. \tag{S3.6.30}$$

From (S3.6.10), we have

$$\begin{aligned}
&\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\psi_F^{\mathcal{B}}(V_j,X_i,t_i,\theta_F^*,h_n)\mathbb{1}\{\mathcal{B}(Q_F(X_i,t_i,\theta_F^*))\geq 0\}\phi(X_i,t_i)\\
=&\frac{1}{h_n^D}\cdot U^a_{T_1,n}(h_n)+U^b_{T_1,n}-\frac{1}{n}\cdot\frac{1}{h_n^D}\cdot U^a_{T_1,n}(h_n)-\frac{1}{n}\cdot U^b_{T_1,n}+\frac{1}{n\cdot h_n^D}\cdot \nu^a_{T_1,n}(h_n)+\frac{1}{n}\cdot\nu^b_{T_1,n}\\
&+\frac{1}{n\cdot h_n^D}\cdot E_F\left[\Lambda^a_{T_1,F}(V,V,h_n)\right]+\frac{1}{n}\cdot E_F\left[\Lambda^b_{T_1,F}(V,V)\right]
\end{aligned}$$

Using (S3.6.19) and (S3.6.25), the above expression becomes

$$\begin{aligned}
&\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\psi_F^{\mathcal{B}}(V_j,X_i,t_i,\theta_F^*,h_n)\mathbb{1}\{\mathcal{B}(Q_F(X_i,t_i,\theta_F^*))\geq 0\}\phi(X_i,t_i)\\
=&\frac{1}{n}\sum_{i=1}^{n}\left(\left(\sum_{p=1}^{P}\Omega^p_{T_1}(Y_i,g(X_i,\theta_F^*))\cdot\omega_p(g(X_i,\theta_F^*))\right)+\Xi_{T_1,F}\psi_F^{\theta}(Z_i)\right)+\xi^c_{T_1,n},
\end{aligned}\tag{S3.6.31}$$

where

$$\begin{aligned}
\xi^c_{T_1,n}\equiv&\frac{1}{n}\sum_{i=1}^{n}\overline{B}_{T_1,n}(Y_i,X_i)+\frac{1}{2h_n^D}\cdot\widetilde{U}^a_{T_1,n}(h_n)+\frac{1}{2}\cdot\widetilde{U}^b_{T_1,n}-\frac{1}{n}\cdot\frac{1}{h_n^D}\cdot U^a_{T_1,n}(h_n)-\frac{1}{n}\cdot U^b_{T_1,n}\\
&+\frac{1}{n\cdot h_n^D}\cdot\nu^a_{T_1,n}(h_n)+\frac{1}{n}\cdot\nu^b_{T_1,n}+\frac{1}{n\cdot h_n^D}\cdot E_F\left[\Lambda^a_{T_1,F}(V,V,h_n)\right]+\frac{1}{n}\cdot E_F\left[\Lambda^b_{T_1,F}(V,V)\right]
\end{aligned}$$

From equations (S3.6.20), (S3.6.24), (S3.6.22), (S3.6.26), (S3.6.27), (S3.6.29), (S3.6.28) and (S3.6.30), we have

$$\begin{aligned}
\left|\xi^c_{T_1,n}\right|\leq\,& o_p\left(\frac{1}{n^{1/2+\epsilon}}\right)+O_p\left(\frac{1}{n\cdot h_n^D}\right)+O_p\left(\frac{1}{n}\right)+O_p\left(\frac{1}{n^{3/2}\cdot h_n^D}\right)+O_p\left(\frac{1}{n^{3/2}}\right)\\
&+O_p\left(\frac{1}{n^{3/2}\cdot h_n^D}\right)+O_p\left(\frac{1}{n^{3/2}}\right)+O\left(\frac{1}{n\cdot h_n^D}\right)+O\left(\frac{1}{n}\right)\\
=\,& o_p\left(\frac{1}{n^{1/2+\epsilon}}\right)\quad\text{uniformly over }\mathcal{F}.
\end{aligned}\tag{S3.6.32}$$

where $\epsilon>0$ is as described in Assumption 4. Let

$$\psi_F^{T_1}(V_i)\equiv\left(\left(\mathcal{B}(Q_F(X_i,t_i,\theta_F^*))\right)_+\phi(X_i,t_i)-T_{1,F}\right)+\sum_{p=1}^{P}\Omega^p_{T_1}(Y_i,g(X_i,\theta_F^*))\cdot\omega_p(g(X_i,\theta_F^*))+\Xi_{T_1,F}\psi_F^{\theta}(Z_i).\tag{S3.6.33}$$

Let $\Delta \equiv \epsilon \wedge (\delta_0/2)$. Plugging (S3.6.31), (S3.6.32), (S3.6.8) and (S3.6.6) into (S3.6.7), we have

$$\widehat{T_1} = T_{1,F} + \frac{1}{n}\sum_{i=1}^{n} \psi_F^{T_1}(V_i) + \varepsilon_n^{T_1}, \quad \text{where} \quad \left|\varepsilon_n^{T_1}\right| = o_p\left(\frac{1}{n^{1/2+\Delta}}\right) \quad \text{uniformly over } \mathcal{F}. \tag{S3.6.34}$$

The results in (S3.6.33) and (S3.6.34) prove the linear representation result in Proposition 1A. ■

S3.6.2 Properties of the influence function $\psi_F^{T_1}(V)$

The *influence function* $\psi_F^{T_1}(V)$ has two key features,

$$\begin{aligned} &(i) \quad E_F\left[\psi_F^{T_1}(V)\right] = 0 \quad \forall\, F \in \mathcal{F}, \\ &(ii) \quad P_F\left(\mathcal{B}(\Gamma_F(X,t,\theta_F^*)) < 0 \,\middle|\, (X,t) \in \mathcal{X}_F^* \times \mathcal{T}\right) = 1 \implies P_F\left(\psi_F^{T_1}(V) = 0\right) = 1. \end{aligned} \tag{S3.6.35}$$

To see part (i) of (S3.6.35), note first that, by construction, $E_F\left[\left(\mathcal{B}\left(Q_F(X_i,t_i,\theta_F^*)\right)\right)_+ \phi(X_i,t_i) - T_{1,F}\right] = 0$ (since $T_{1,F} \equiv E_F\left[\left(\mathcal{B}\left(Q_F(X,t,\theta_F^*)\right)\right)_+ \phi(X,t)\right]$, as we defined in (11A)). Next, since $E_F\left[\psi_F^{\theta}(Z)\right] = 0$, we have $E_F\left[\Xi_{T_1,F}\psi_F^{\theta}(Z)\right] = \Xi_{T_1,F}E_F\left[\psi_F^{\theta}(Z)\right] = 0$. Finally, as we showed in equation (S3.6.21), we have $E_F\left[\Omega_{T_1}^{p}(Y,g(X,\theta_F^*)) \cdot \omega_p(g(X,\theta_F^*)\right] = 0$, and these results hold for all $F \in \mathcal{F}$, therefore establishing part (i) of (S3.6.35). Next, let us show part (ii). First, recall that we defined our target testing range $\mathcal{X}_F^*$ for X as

$$\mathcal{X}_F^* = \{x \in \mathcal{S}_X : x \in \mathcal{X} \text{ and } g(x,\theta_F^*) \in \mathcal{G}.\}$$

Next, recall that $\omega_p(\cdot) \geq 0$ with $\omega_p(g) > 0$ if and only if $g \in \mathcal{G}$, and $\phi(\cdot) \geq 0$ with $\phi(x,t) > 0$ if and only if $(x,t) \in \mathcal{X} \times \mathcal{T}$. Next, recall from (7) that,

$$\begin{aligned} &\mathcal{B}(Q_F(x,t,\theta_F^*)) = \mathcal{B}(\Gamma_F(x,t,\theta_F^*)) \cdot \mathcal{H}(\omega(g(x,\theta_F^*))), \\ &\text{where} \begin{cases} \mathcal{H}(\cdot) \geq 0, \\ \mathcal{H}(\omega(g(x,\theta_F^*)) > 0 \iff g(x,\theta_F^*) \in \mathcal{G}. \end{cases} \end{aligned}$$

Therefore,

$$(\mathcal{B}(Q_F(x,t,\theta_F^*)))_+ \phi(x,t) = (\mathcal{B}(\Gamma_F(x,t,\theta_F^*)))_+ \cdot \mathcal{H}(\omega(g(x,\theta_F^*)))\, \phi(x,t)$$

with $\mathcal{H}\left(\omega(g(x,\theta_F^*))\right)\phi(x,t) \neq 0$ only if $(x,t) \in \mathcal{X}_F^* \times \mathcal{T}$. Thus, $P_F\left(\mathcal{B}\left(\Gamma_F(X,t,\theta_F^*)\right) < 0 \middle| (X,t) \in \mathcal{X}_F^* \times \mathcal{T}\right) = 1$ implies

$$P_F\left((\mathcal{B}(Q_F(X,t,\theta_F^*)))_+ \phi(X,t) = 0\right) = P_F\left((\mathcal{B}(\Gamma_F(X,t,\theta_F^*)))_+ \cdot \mathcal{H}(\omega(g(X,\theta_F^*)))\, \phi(X,t) = 0\right) = 1,$$

and $T_{1,F}=0$. Also, if $P_F\left(\mathcal{B}\left(\Gamma_F(X,t,\theta_F^*)\right)<0\middle|(X,t)\in\mathcal{X}_F^*\times\mathcal{T}\right)=1$, then

$$\begin{aligned}&P_F\left(\phi(X,t)\mathbb{1}\left\{\mathcal{B}\left(Q_F(X,t,\theta_F^*)\right)\geq 0\right\}=0\;\middle|\;g(X,\theta_F^*)\in\mathcal{G}\right)\\=&P_F\left(\phi(X,t)\mathbb{1}\left\{\mathcal{B}\left(\Gamma_F(X,t,\theta_F^*)\right)\cdot\mathcal{H}\left(\omega(g(X,\theta_F^*))\right)\geq 0\right\}=0\;\middle|\;g(X,\theta_F^*)\in\mathcal{G}\right)=1.\end{aligned}$$

Thus, for any (y,g), the above result implies

$$\begin{aligned}&\Omega_{T_1}^p(y,g)\cdot\omega_p(g)\\=&E_F\left[\left(S_p(y,t)-\Gamma_{p,F}(X,t,\theta_F^*)\right)\frac{\partial\mathcal{B}\left(Q_F(X,t,\theta_F^*)\right)}{\partial Q_p}\underbrace{\phi(X,t)\mathbb{1}\left\{\mathcal{B}\left(Q_F(X,t,\theta_F^*)\right)\geq 0\right\}}_{\substack{=0\ F\text{−a.s if } g(X,\theta_F^*)=g\in\mathcal{G}\\ \text{(i.e, if } \omega_p(g)\neq 0)}}\;\middle|\;g(X,\theta_F^*)=g\right]\cdot\underbrace{\omega_p(g)}_{\substack{=0\text{ if}\\ g\notin\mathcal{G}}}\\=&0.\end{aligned}$$

Therefore,

$$P_F\left(\mathcal{B}\left(\Gamma_F(X,t,\theta_F^*)\right)<0\middle|(X,t)\in\mathcal{X}_F^*\times\mathcal{T}\right)=1\implies P_F\left(\Omega_{T_1}^p\left(Y,g(X,\theta_F^*)\right)\cdot\omega_p(g(X,\theta_F^*))=0\right)=1.$$

Finally, recall from (A-4) that $\Xi_{Q_p}(x,t,\theta_F^*)=0\;\;\forall\ x:g(x,\theta_F^*)\notin\mathcal{G}$. Therefore,

$$\phi(x,t)\Xi_{Q_p}(x,t,\theta_F^*)=0\quad\forall\ (x,t)\notin\mathcal{X}_F^*\times\mathcal{T}.$$

And from our definition of $\Xi_{T_1,F}^p$ and $\Xi_{T_1,F}$ in (S3.6.23), if $P_F\left(\mathcal{B}\left(\Gamma_F(X,t,\theta_F^*)\right)<0\middle|(X,t)\in\mathcal{X}_F^*\times\mathcal{T}\right)=1$, then

$$\begin{aligned}\Xi_{T_1,F}^p&\equiv E_F\left[\frac{\partial\mathcal{B}\left(Q_F(X,t,\theta_F^*)\right)}{\partial Q_p}\mathbb{1}\left\{\mathcal{B}\left(Q_F(X,t,\theta_F^*)\right)\geq 0\right\}\phi(X,t)\Xi_{Q_p}(X,t,\theta_F^*)\right]\\&=E_F\left[\frac{\partial\mathcal{B}\left(Q_F(X,t,\theta_F^*)\right)}{\partial Q_p}\underbrace{\mathbb{1}\left\{\mathcal{B}\left(\Gamma_F(X,t,\theta_F^*)\right)\cdot\mathcal{H}\left(\omega(g(X,\theta_F^*))\right)\geq 0\right\}}_{=0\text{ if }(X,t)\in\mathcal{X}_F^*\times\mathcal{T}}\underbrace{\phi(X,t)\Xi_{Q_p}(X,t,\theta_F^*)}_{\substack{=0\\ \text{if }(X,t)\notin\mathcal{X}_F^*\times\mathcal{F}}}\right]\\&=0\quad\forall\ p=1,\ldots,P.\\\implies\quad\Xi_{T_1,F}&\equiv\sum_{p=1}^{P}\Xi_{T_1,F}^p=0.\end{aligned}$$

Therefore, $\Xi_{T_1,F}\psi_F^\theta(Z)=0$. Combined, these results yield

$$P_F\left(\mathcal{B}\left(\Gamma_F(X,t,\theta_F^*)\right)<0\;\middle|\;(X,t)\in\mathcal{X}_F^*\times\mathcal{T}\right)=1\implies P_F\left(\psi_F^{T_1}(V)=0\right)=1,$$

which establishes part (ii) of (S3.6.35).

S3.7 Proof of Proposition 1B

S3.7.1 Asymptotic properties of $\widehat{T}_2$

Recall that

$$T_{2,F} \equiv \int_t T_{0,F}(t)d\mathcal{W}(t), \quad \text{where} \quad T_{0,F}(t) \quad \equiv E_F\left[(\mathcal{B}(Q_F(X,t,\theta_F^*)))_+ \phi(X,t)\right],$$

and

$$\widehat{T}_2 \equiv \int_t \widehat{T}_0(t)d\mathcal{W}(t), \quad \text{where} \quad \widehat{T}_0(t) \equiv \frac{1}{n}\sum_{i=1}^n \mathcal{B}\left(\widehat{Q}(X_i,t,\widehat{\theta})\right)\mathbb{1}\left\{\mathcal{B}\left(\widehat{Q}(X_i,t,\widehat{\theta})\right) \geq -b_n\right\}\phi(X_i),$$

To establish the asymptotic properties of $\widehat{T}_2 - T_{2,F}$, we analyze those of $\widehat{T}_0(t) - T_{0,F}(t)$ following the same steps we took in our study of $\widehat{T}_1 - T_{1,F}$. For a given t let

$$\widetilde{T}_{0,F} \equiv \frac{1}{n}\sum_{i=1}^n \mathcal{B}\left(\widehat{Q}(X_i,t,\widehat{\theta})\right)\mathbb{1}\left\{\mathcal{B}\left(Q_F(X_i,t,\theta_F^*)\right) \geq 0\right\}\phi(X_i).$$

$\widetilde{T}_{0,F}$ replaces the indicator function $\mathbb{1}\left\{\mathcal{B}\left(\widehat{Q}(X_i,t,\widehat{\theta})\right) \geq -b_n\right\}$ with $\mathbb{1}\left\{\mathcal{B}\left(Q_F(X_i,t,\theta_F^*)\right) \geq 0\right\}$. Our first step will be to analyze $\widehat{T}_0(t) - T_{0,F}(t) \equiv \xi^a_{T_0,n}(t)$. Using the same decomposition as in (S3.6.1), we have

$$\begin{aligned}
&\left|\xi^a_{T_0,n}(t)\right| \\
&\leq \left(2b_n + \sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\mathcal{B}\left(\widehat{Q}(x,t,\widehat{\theta})\right) - \mathcal{B}\left(Q_F(x,t,\theta_F^*)\right)\right|\right) \times \frac{1}{n}\sum_{i=1}^n \phi(X_i,t)\mathbb{1}\left\{-2b_n \leq \mathcal{B}\left(Q_F(X_i,t,\theta_F^*)\right) < 0\right\} \\
&+\frac{1}{n}\sum_{i=1}^n \left|\mathcal{B}\left(\widehat{Q}(X_i,t,\widehat{\theta})\right)\right|\phi(X_i,t)\mathbb{1}\left\{\left|\mathcal{B}\left(\widehat{Q}(X_i,t,\widehat{\theta})\right) - \mathcal{B}\left(Q_F(X_i,t,\theta_F^*)\right)\right| \geq b_n\right\}
\end{aligned}$$

The above is analogous to (S3.6.2). From (S3.5.3), we have $\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\mathcal{B}\left(\widehat{Q}(x,t,\widehat{\theta})\right) - \mathcal{B}\left(Q(x,t,\theta_F^*)\right)\right| = o_p\left(\frac{1}{n^{1/4+\epsilon/2}}\right)$ uniformly over $\mathcal{F}$, where $\epsilon > 0$ is the constant described in Assumption 4. Therefore, uniformly over $\mathcal{F}$ we have

$$\begin{aligned}
\left|\xi^a_{T_0,n}(t)\right| \leq &\left(2b_n + o_p\left(\frac{1}{n^{1/4+\epsilon/2}}\right)\right) \times \frac{1}{n}\sum_{i=1}^n \phi(X_i,t)\mathbb{1}\left\{-2b_n \leq \mathcal{B}\left(Q(X_i,t,\theta_F^*)\right) < 0\right\} \\
&+ \frac{1}{n}\sum_{i=1}^n \left|\mathcal{B}\left(\widehat{Q}(X_i,t,\widehat{\theta})\right)\right|\phi(X_i)\mathbb{1}\left\{\left|\mathcal{B}\left(\widehat{Q}(X_i,t,\widehat{\theta})\right) - \mathcal{B}\left(Q(X_i,t,\theta_F^*)\right)\right| \geq b_n\right\}.
\end{aligned} \tag{S3.7.1}$$

The above is analogous to (S3.6.3). We will analyze each of the two summands in (S3.7.1). For a given $b > 0$ and $t \in \mathcal{T}$, let

$$m^a_{T_0,n}(b,t) \equiv \frac{1}{n}\sum_{i=1}^{n}\Big(\phi(X_i)\mathbb{1}\Big\{-b \le \mathcal{B}(Q_F(X_i,t,\theta^*_F)) < 0\Big\} - E_F\Big[\phi(X_i,t)\mathbb{1}\Big\{-b \le \mathcal{B}(Q_F(X_i,t,\theta^*_F)) < 0\Big\}\Big]\Big)$$

From Assumption 6, there exist constants $(\overline{A},\overline{V})$ such that, for each $F \in \mathcal{F}$, the following class of indicator functions is Euclidean $(\overline{A},\overline{V})$ for the constant envelope 1,

$$\Big\{m : \mathcal{X} \longrightarrow \mathbb{R} : m(x) = \mathbb{1}\Big\{-b \le \mathcal{B}(Q_F(x,t,\theta^*_F)) < 0\Big\} \text{ for some } 0 < b \le b_0 \text{ and } t \in \mathcal{T}\Big\}.$$

From here, Result S1 yields,

$$\sup_{\substack{0<b<b_0\\ t\in\mathcal{T}}} \Big|m^a_{T_0,n}(b,t)\Big| = O_p\left(\frac{1}{n^{1/2}}\right) \quad \text{uniformly over } \mathcal{F}.$$

For n large enough we have $0 < 2b_n \le b_0$. Therefore, for n large enough,

$$\Big|m^a_{T_0,n}(2b_n,t)\Big| \le \sup_{\substack{0<b<b_0\\ t\in\mathcal{T}}} \Big|m^a_{T_0,n}(b,t)\Big| = O_p\left(\frac{1}{n^{1/2}}\right) \quad \text{uniformly over } \mathcal{F}. \tag{S3.7.2}$$

We have

$$\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{n}\phi(X_i)\mathbb{1}\Big\{-2b_n \le \mathcal{B}(Q_F(X_i,t,\theta^*_F)) < 0\Big\}\\
=&m^a_{T_0,n}(2b_n,t) + E_F\Big[\mathbb{1}\Big\{-2b_n \le \mathcal{B}(Q_F(X,t,\theta^*_F)) < 0\Big\}\cdot\mathbb{1}\Big\{X \in \mathcal{X}\Big\}\Big].
\end{aligned}$$

From Assumption 7B, there exist finite constants $\underline{b}_2 > 0$ and $\overline{C}_{\mathcal{B},2} > 0$ such that, for all $0 < b \le \underline{b}_2$,

$$\sup_{t\in\mathcal{T}} E_F\Big[\mathbb{1}\Big\{-b \le \mathcal{B}(Q_F(X,t,\theta^*_F)) < 0\Big\}\cdot\mathbb{1}\Big\{X \in \mathcal{X}\Big\}\Big] \le \overline{C}_{\mathcal{B},2}\cdot b \quad \forall\ F \in \mathcal{F},$$

For n large enough, we have $0 < 2b_n \le \underline{b}_2 \wedge b_0$, and from Assumption 4, we have $n^{1/2}\cdot b_n \longrightarrow \infty$. This, combined with equation (S3.7.2) and Assumption 7B yields,

$$\begin{aligned}
\sup_{t\in\mathcal{T}} \frac{1}{n}\sum_{i=1}^{n}\phi(X_i)\mathbb{1}\Big\{-2b_n \le \mathcal{B}(Q_F(X_i,t,\theta^*_F)) < 0\Big\} &\le O_p\left(\frac{1}{n^{1/2}}\right) + O(b_n)\\
&= b_n\cdot\left(O_p\left(\frac{1}{b_n\cdot n^{1/2}}\right) + O(1)\right)\\
&= b_n\cdot\Big(o_p(1) + O(1)\Big)\\
&= O_p(b_n) \quad \text{uniformly over } \mathcal{F}.
\end{aligned} \tag{S3.7.3}$$

Next, note from Assumption 5 and equation (S3.5.3), we have $\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left|\mathcal{B}\left(\widehat{Q}(x,t,\widehat{\theta})\right)\right| = O_p(1)$ uniformly over $\mathcal{F}$. Therefore,

$$\begin{aligned}
&\sup_{t\in\mathcal{T}} \frac{1}{n}\sum_{i=1}^{n} \left|\mathcal{B}\left(\widehat{Q}(X_i,t,\widehat{\theta})\right)\right| \phi(X_i)\mathbb{1}\left\{\left|\mathcal{B}\left(\widehat{Q}(X_i,t,\widehat{\theta})\right)-\mathcal{B}\left(Q(X_i,t,\theta_F^*)\right)\right| \geq b_n\right\} \\
\leq &\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left|\mathcal{B}\left(\widehat{Q}(x,t,\widehat{\theta})\right)\right| \times \sup_{t\in\mathcal{T}} \frac{1}{n}\sum_{i=1}^{n} \phi(X_i)\mathbb{1}\left\{\left|\mathcal{B}\left(\widehat{Q}(X_i,t,\widehat{\theta})\right)-\mathcal{B}\left(Q(X_i,t,\theta_F^*)\right)\right| \geq b_n\right\} \\
=&O_p(1)\times\left(\sup_{t\in\mathcal{T}} \frac{1}{n}\sum_{i=1}^{n} \phi(X_i)\mathbb{1}\left\{\left|\mathcal{B}\left(\widehat{Q}(X_i,t,\widehat{\theta})\right)-\mathcal{B}\left(Q(X_i,t,\theta_F^*)\right)\right| \geq b_n\right\}\right), \quad \text{uniformly over } \mathcal{F}.
\end{aligned}$$

Next, note that

$$\begin{aligned}
&\sup_{F\in\mathcal{F}} P_F\left(\sup_{t\in\mathcal{T}} \frac{1}{n}\sum_{i=1}^{n} \phi(X_i)\mathbb{1}\left\{\left|\mathcal{B}\left(\widehat{Q}(X_i,t,\widehat{\theta})\right)-\mathcal{B}\left(Q(X_i,t,\theta_F^*)\right)\right| \geq b_n\right\} \neq 0\right) \\
\leq &\sup_{F\in\mathcal{F}} P_F\left(\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left|\mathcal{B}(\widehat{Q}(x,t,\widehat{\theta}))-\mathcal{B}(Q_F(x,t,\theta_F^*))\right| \geq b_n\right) \longrightarrow 0
\end{aligned}$$

where the last equality follows from (S3.5.4). In particular, for any $\delta>0$ and $\Delta>0$,

$$\sup_{F\in\mathcal{F}} P_F\left(\sup_{t\in\mathcal{T}} \frac{1}{n}\sum_{i=1}^{n} \phi(X_i)\mathbb{1}\left\{\left|\mathcal{B}\left(\widehat{Q}(X_i,t,\widehat{\theta})\right)-\mathcal{B}\left(Q(X_i,t,\theta_F^*)\right)\right| \geq b_n\right\} \geq \frac{\delta}{n^{1/2+\Delta}}\right) \longrightarrow 0.$$

That is,

$$\sup_{t\in\mathcal{T}} \frac{1}{n}\sum_{i=1}^{n} \phi(X_i)\mathbb{1}\left\{\left|\mathcal{B}\left(\widehat{Q}(X_i,t,\widehat{\theta})\right)-\mathcal{B}\left(Q(X_i,t,\theta_F^*)\right)\right| \geq b_n\right\} = o_p\left(\frac{1}{n^{1/2+\Delta}}\right) \quad \forall\ \Delta>0, \ \text{uniformly over } \mathcal{F}.$$

Therefore,

$$\begin{aligned}
&\sup_{t\in\mathcal{T}} \frac{1}{n}\sum_{i=1}^{n} \left|\mathcal{B}\left(\widehat{Q}(X_i,t,\widehat{\theta})\right)\right| \phi(X_i)\mathbb{1}\left\{\left|\mathcal{B}\left(\widehat{Q}(X_i,t,\widehat{\theta})\right)-\mathcal{B}\left(Q(X_i,t,\theta_F^*)\right)\right| \geq b_n\right\} \\
&= o_p\left(\frac{1}{n^{1/2+\Delta}}\right) \quad \forall\ \Delta>0, \ \text{uniformly over } \mathcal{F}.
\end{aligned}$$

Plugging the results in (S3.7.3) and in the previous expression into (S3.7.1), for any $\Delta>0$ we have

$$\begin{aligned}
\sup_{t\in\mathcal{T}} \left|\xi^a_{T_0,n}(t)\right| &\leq \left(2b_n + o_p\left(\frac{1}{n^{1/4+\epsilon/2}}\right)\right)\times O_p\left(b_n\right) + o_p\left(\frac{1}{n^{1/2+\Delta}}\right) \\
&= O_p\left(b_n^2\right) + o_p\left(\frac{b_n}{n^{1/4+\epsilon/2}}\right) + o_p\left(\frac{1}{n^{1/2+\Delta}}\right) \quad \text{uniformly over } \mathcal{F}.
\end{aligned}$$

Take any $\Delta > 0$ and note that $\left(\frac{b_n}{n^{1/4+\epsilon/2}}\right) \cdot n^{1/2+\Delta} = \left(n^{1/2+2\Delta-\epsilon} \cdot b_n^2\right)^{1/2}$. In Assumption 4 we stated that there exists $\delta_0 > 0$ such that $n^{1/2+\delta_0} \cdot b_n^2 \longrightarrow 0$. Therefore,

$$\frac{b_n}{n^{1/4+\epsilon/2}} = o\left(\frac{1}{n^{1/2+\Delta}}\right) \quad \forall\ 0 < \Delta \le \frac{\delta_0}{2}.$$

From here, we obtain

$$\sup_{t \in \mathcal{T}} \left|\xi^a_{T_0,n}(t)\right| = o_p\left(\frac{1}{n^{1/2+\delta_0/2}}\right) \quad \text{uniformly over } \mathcal{F}. \tag{S3.7.4}$$

Therefore, using the linear representation result in (S3.5.1),

$$\begin{aligned}
&\widehat{T_0}(t) = \widetilde{T}_{0,F}(t) + \xi^a_{T_0,n}(t)\\
&= \frac{1}{n}\sum_{i=1}^n \mathcal{B}\left(\widehat{Q}(X_i,t,\widehat{\theta})\right) \mathbb{1}\left\{\mathcal{B}\left(Q_F(X_i,t,\theta_F^*)\right) \ge 0\right\} \phi(X_i) + \xi^a_{T_0,n}(t)\\
&= \frac{1}{n}\sum_{i=1}^n \Bigg(\mathcal{B}(Q_F(X_i,t,\theta_F^*)) + \frac{1}{n}\sum_{j=1}^n \psi_F^{\mathcal{B}}(V_j,X_i,t,\theta_F^*,h_n) + \zeta_n^{\mathcal{B}}(X_i,t)\Bigg) \mathbb{1}\left\{\mathcal{B}\left(Q_F(X_i,t,\theta_F^*)\right) \ge 0\right\} \phi(X_i) + \xi^a_{T_0,n}(t)\\
&= \frac{1}{n}\sum_{i=1}^n \left(\mathcal{B}\left(Q_F(X_i,t,\theta_F^*)\right)\right)_+ \phi(X_i) + \frac{1}{n^2}\sum_{i=1}^n\sum_{j=1}^n \psi_F^{\mathcal{B}}(V_j,X_i,t,\theta_F^*,h_n) \mathbb{1}\left\{\mathcal{B}\left(Q_F(X_i,t,\theta_F^*)\right) \ge 0\right\} \phi(X_i,t)\\
&+ \frac{1}{n}\sum_{i=1}^n \zeta_n^{\mathcal{B}}(X_i,t) \mathbb{1}\left\{\mathcal{B}\left(Q_F(X_i,t,\theta_F^*)\right) \ge 0\right\} \phi(X_i,t) + \xi^a_{T_0,n}(t).
\end{aligned}$$

Recall that $T_{0,F}(t)$ was defined as $E_F\left[\left(\mathcal{B}\left(Q_F(X,t,\theta_F^*)\right)\right)_+ \phi(X)\right]$, with the expectation taken with respect to (X,t). Therefore, from the above expression we have,

$$\begin{aligned}
\widehat{T_0}(t) =& T_{0,F}(t) + \frac{1}{n}\sum_{i=1}^n \Big(\left(\mathcal{B}\left(Q_F(X_i,t,\theta_F^*)\right)\right)_+ \phi(X_i) - T_{0,F}(t)\Big)\\
&\underbrace{+ \frac{1}{n^2}\sum_{i=1}^n\sum_{j=1}^n \psi_F^{\mathcal{B}}(V_j,X_i,t,\theta_F^*,h_n) \mathbb{1}\left\{\mathcal{B}\left(Q_F(X_i,t,\theta_F^*)\right) \ge 0\right\} \phi(X_i)
+ \frac{1}{n}\sum_{i=1}^n \zeta_n^{\mathcal{B}}(X_i,t) \mathbb{1}\left\{\mathcal{B}\left(Q_F(X_i,t,\theta_F^*)\right) \ge 0\right\} \phi(X_i)}_{\equiv \xi^b_{T_0,n}(t)} + \xi^a_{T_0,n}(t).
\end{aligned} \tag{S3.7.5}$$

Let us analyze each of the terms in (S3.7.5). From (S3.5.2) we have,

$$\sup_{t\in\mathcal{T}}\left|\xi^{b}_{T_1,n}(t)\right| \equiv \left|\frac{1}{n}\sum_{i=1}^{n}\zeta^{\mathcal{B}}_{n}(X_i,t)\mathbb{1}\left\{\mathcal{B}\left(Q_F(X_i,t,\theta^*_F)\right)\geq 0\right\}\phi(X_i)\right| \leq \overline{\phi}\cdot\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\zeta^{\mathcal{B}}_{n}(x,t)\right|$$
$$= o_p\left(\frac{1}{n^{1/2+\epsilon}}\right) \quad \text{uniformly over } \mathcal{F} \tag{S3.7.6}$$

Next, as in our analysis of $\widehat{T}_1$, recall that

$$\begin{aligned}\psi^{\mathcal{B}}_{F}(V_j,x,t,\theta^*_F,h_n) &\equiv \nabla_Q\mathcal{B}(Q_F(x,t,\theta^*_F))\psi^{Q}_{F}(V_j,x,t,\theta^*_F,h_n)\\ &= \sum_{p=1}^{P}\frac{\partial\mathcal{B}\Big(Q_F(x,t,\theta^*_F)\Big)}{\partial Q_p}\psi^{Q_p}_{F}(V_j,x,t,\theta^*_F,h_n).\end{aligned}$$

As we did in our analysis of $\widehat{T}_1$, for $h>0$, $x\in\mathcal{S}_X$ and $t\in\mathcal{T}$, let

$$\begin{aligned}\Lambda^{Q_p}_{F}(V_j,x,t,\theta^*_F,h) \equiv{}& \left(\frac{S_p(Y_j,t)-\Gamma_{p,F}(x,t,\theta^*_F)}{f_g(g(x,\theta^*_F))}\right)\omega_p(g(X_j,\theta^*_F))K\left(\frac{\Delta g(X_j,x,\theta^*_F)}{h}\right)\\ &- E_F\left[\left(\frac{S_p(Y_j,t)-\Gamma_{p,F}(x,t,\theta^*_F)}{f_g(g(x,\theta^*_F))}\right)\omega_p(g(X_j,\theta^*_F))K\left(\frac{\Delta g(X_j,x,\theta^*_F)}{h}\right)\right].\end{aligned}$$

From the definition in (S3.4.4), we have

$$\psi^{Q_p}_{F}(V_j,x,t,\theta^*_F,h_n) = \frac{1}{h_n^D}\Lambda^{Q_p}_{F}(V_j,x,t,\theta^*_F) + \Xi_{Q_p}(x,t,\theta^*_F)\psi^{\theta}_{F}(Z_j).$$

For $t\in\mathcal{T}$, $h>0$ and $v_1\in\mathcal{S}_V$, $v_2\in\mathcal{S}_V$ let

$$\begin{aligned}\Lambda^{a}_{T_0,F}(v_1,v_2,t,h) &\equiv \sum_{p=1}^{P}\frac{\partial\mathcal{B}\Big(Q_F(x_1,t,\theta^*_F)\Big)}{\partial Q_p}\Lambda^{Q_p}_{F}(v_2,x_1,t,\theta^*_F,h)\mathbb{1}\left\{\mathcal{B}\left(Q_F(x_1,t,\theta^*_F)\right)\geq 0\right\}\phi(x_1),\\ \Lambda^{b}_{T_0,F}(v_1,v_2,t) &\equiv \left(\sum_{p=1}^{P}\frac{\partial\mathcal{B}\Big(Q_F(x_1,t,\theta^*_F)\Big)}{\partial Q_p}\mathbb{1}\left\{\mathcal{B}\left(Q_F(x_1,t,\theta^*_F)\right)\geq 0\right\}\phi(x_1)\Xi_{Q_p}(x_1,t,\theta^*_F)\right)\psi^{\theta}_{F}(z_2)\end{aligned}$$

and define

$$\begin{aligned}U^{a}_{T_0,n}(t,h) &\equiv \frac{1}{n\cdot(n-1)}\sum_{i=1}^{n}\sum_{j\neq i}\Lambda^{a}_{T_0,F}(V_i,V_j,t,h), & \nu^{a}_{T_0,n}(t,h) &\equiv \frac{1}{n}\sum_{i=1}^{n}\Big(\Lambda^{a}_{T_0,F}(V_i,V_i,t,h)-E_F\Big[\Lambda^{a}_{T_0,F}(V,V,t,h)\Big]\Big),\\ U^{b}_{T_0,n}(t) &\equiv \frac{1}{n\cdot(n-1)}\sum_{i=1}^{n}\sum_{j\neq i}\Lambda^{b}_{T_0,F}(V_i,V_j,t), & \nu^{b}_{T_0,n}(t) &\equiv \frac{1}{n}\sum_{i=1}^{n}\Big(\Lambda^{b}_{T_0,F}(V_i,V_i,t)-E_F\Big[\Lambda^{b}_{T_0,F}(V,V,t)\Big]\Big).\end{aligned} \tag{S3.7.7}$$

We have,

$$
\begin{aligned}
&\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\psi_F^{\mathcal{B}}(V_j,X_i,t,\theta_F^*,h_n)\mathbb{1}\{\mathcal{B}(Q_F(X_i,t,\theta_F^*))\geq 0\}\phi(X_i)\\
=&\left(\frac{n-1}{n}\right)\cdot\frac{1}{h_n^D}\cdot U_{T_0,n}^a(t,h_n)+\left(\frac{n-1}{n}\right)\cdot U_{T_0,n}^b(t)+\frac{1}{n\cdot h_n^D}\cdot \nu_{T_0,n}^a(t,h_n)+\frac{1}{n}\cdot\nu_{T_0,n}^b(t)\\
&+\frac{1}{n\cdot h_n^D}\cdot E_F\Big[\Lambda_{T_0,F}^a(V,V,t,h_n)\Big]+\frac{1}{n}\cdot E_F\Big[\Lambda_{T_0,F}^b(V,V,t)\Big]
\end{aligned}
\tag{S3.7.8}
$$

Let us analyze the U-statistic $U_{T_0,n}^a(t,h_n)$. First, let us study

$$
\varphi_{T_0,F}^a(V_i,t,h_n)\equiv E_F\Big[\Lambda_{T_0,F}^a(V_i,V_j,t,h_n)+\Lambda_{T_0,F}^a(V_j,V_i,t,h_n)\Big|V_i\Big].
$$

As we did in our analysis of $\widehat{T}_1$, for each $F\in\mathcal{F}$ and a given $(x,t)\in\mathcal{X}\times\mathcal{T}$, let

$$
\mu_{F,n}^{S_p}(x,t)\equiv\frac{1}{h_n^D}\cdot E_F\left[\Big(S_p(Y,t)-\Gamma_{p,F}(x,t,\theta_F^*)\Big)\omega_p(g(X,\theta_F^*))K\left(\frac{\Delta g(X,x,\theta_F^*)}{h_n}\right)\right].
$$

Recall that in Assumption 2 we defined, for each $F\in\mathcal{F}$,

$$
\Omega_{R_p,0}(g,t)\equiv E_F\Big[S_p(Y,t)\Big|g(X,\theta_F^*)=g\Big].
$$

By iterated expectations,

$$
\mu_{F,n}^{S_p}(x,t)=\frac{1}{h_n^D}\int_u\Big(\Omega_{R_p,0}(u,t)-\Gamma_{p,F}(x,t,\theta_F^*)\Big)\omega_p(u)K\left(\frac{u-g(x,\theta_F^*)}{h_n}\right)f_g(u)du.
$$

From here, using the smoothness properties described in Assumption 2, performing an M^{th}–order approximation and noting that $\Omega_{R_p,0}(g(x,\theta_F^*),t)=\Gamma_{p,F}(x,t,\theta_F^*)$, there exists a finite constant $\overline{B}_{\mu_S}>0$ such that,

$$
\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\mu_{F,n}^{S_p}(x,t)\right|\leq\overline{B}_{\mu_S}\cdot h_n^M\quad\forall\ F\in\mathcal{F}.
\tag{S3.7.9}
$$

We have,

$$
\begin{aligned}
\frac{1}{h_n^D}\Lambda_F^{Q_p}(V_j,x,t,\theta_F^*,h_n)=&\frac{1}{h_n^D}\left(\frac{S_p(Y_j,t)-\Gamma_{p,F}(x,t,\theta_F^*)}{f_g(g(x,\theta_F^*))}\right)\cdot\omega_p(g(X_j,\theta_F^*))\cdot K\left(\frac{\Delta g(X_j,x,\theta_F^*)}{h_n}\right)\\
&-\frac{\mu_{F,n}^{S_p}(x,t)}{f_g(g(x,\theta_F^*))}
\end{aligned}
$$

And, from here,

$$\frac{1}{h_n^D}\cdot E_F\left[\frac{\partial\mathcal{B}\left(Q_F(X_i,t,\theta_F^*)\right)}{\partial Q_p}\Lambda_F^{Q_p}(V_j,X_i,t,\theta_F^*,h_n)\mathbb{1}\left\{\mathcal{B}\left(Q_F(X_i,t,\theta_F^*)\right)\geq 0\right\}\phi(X_i)\middle|V_i\right]$$

$$=\frac{\partial\mathcal{B}\left(Q_F(X_i,t,\theta_F^*)\right)}{\partial Q_p}\frac{1}{f_g(g(X_i,\theta_F^*))}\mathbb{1}\left\{\mathcal{B}\left(Q_F(X_i,t,\theta_F^*)\right)\geq 0\right\}\phi(X_i)\underbrace{\left(\mu_{F,n}^{S_p}(X_i,t)-\mu_{F,n}^{S_p}(X_i,t)\right)}_{=0} \quad \text{(S3.7.10)}$$

$$=0 \quad \forall\ F\in\mathcal{F}.$$

From Assumptions 2 and 5

$$\sup_{(x,t)\in\mathcal{S}_X\times\mathcal{T}}\left|\frac{\partial\mathcal{B}\left(Q_F(x,t,\theta_F^*)\right)}{\partial Q_p}\frac{1}{f_g(g(x,\theta_F^*))}\mathbb{1}\left\{\mathcal{B}\left(Q_F(x,t,\theta_F^*)\right)\geq 0\right\}\phi(x)\right|$$
$$\leq\overline{\phi}\cdot\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\frac{\partial\mathcal{B}\left(Q_F(x,t,\theta_F^*)\right)}{\partial Q_p}\frac{1}{f_g(g(x,\theta_F^*))}\right|\leq\overline{\phi}\cdot\frac{\overline{H}_Q}{\underline{f}_g}\quad\forall\ F\in\mathcal{F}$$

Also from Assumption 2, there exists a finite constant $\overline{C}_{\Xi_{Q_p}}>0$ such that

$$\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left\|\Xi_{Q_p}(x,t,\theta_F^*)\right\|\leq\overline{C}_{\Xi_{Q_p}}\quad\forall\ F\in\mathcal{F}.$$

Therefore, from Assumptions 2 and 5,

$$\sup_{(x,t)\in\mathcal{S}_X\times\mathcal{T}}\left\|\frac{\partial\mathcal{B}\left(Q_F(x,t,\theta_F^*)\right)}{\partial Q_p}\mathbb{1}\left\{\mathcal{B}\left(Q_F(x,t,\theta_F^*)\right)\geq 0\right\}\phi(x)\Xi_{Q_p}(x,t,\theta_F^*)\right\|$$
$$\leq\overline{\phi}\cdot\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left\|\frac{\partial\mathcal{B}\left(Q_F(x,t,\theta_F^*)\right)}{\partial Q_p}\Xi_{Q_p}(x,t,\theta_F^*)\right\|\leq\overline{\phi}\cdot\overline{H}_Q\cdot\overline{C}_{\Xi_{Q_p}}\equiv\overline{C}_{\Xi_{T_1}}\quad\forall\ F\in\mathcal{F}. \quad \text{(S3.7.11)}$$

Thus, from (S3.7.10),

$$\frac{1}{h_n^D}\cdot E_F\left[\frac{\partial\mathcal{B}\left(Q_F(X_i,t,\theta_F^*)\right)}{\partial Q_p}\Lambda_F^{Q_p}(V_j,X_i,t,\theta_F^*,h_n)\mathbb{1}\left\{\mathcal{B}\left(Q_F(X_i,t,\theta_F^*)\right)\geq 0\right\}\phi(X_i)\middle|V_i\right]=0,\ \text{ and therefore,}$$

$$E_F\left[\Lambda_{T_0,F}^{a}(V_j,V_i,\theta_F^*,t,h_n)\right]=0\quad\forall\ t\in\mathcal{T},\ \forall\ F\in\mathcal{F} \quad \text{(S3.7.12)}$$

Next, let us analyze

$$\frac{1}{h_n^D}\cdot E_F\left[\frac{\partial\mathcal{B}\left(Q_F(X_i,t,\theta_F^*)\right)}{\partial Q_p}\Lambda_F^{Q_p}(V_j,X_i,t,\theta_F^*,h_n)\mathbb{1}\left\{\mathcal{B}\left(Q_F(X_i,t,\theta_F^*)\right)\geq 0\right\}\phi(X_i)\middle|V_j\right].$$

Recall from Assumption 7B that we defined

$$\Omega^p_{T_0}(y,t,g) = E_F\left[\left(S_p(y,t) - \Gamma_{p,F}(X,t,\theta^*_F)\right)\frac{\partial\mathcal{B}\left(Q_F(X,t,\theta^*_F)\right)}{\partial Q_p}\phi(X)\mathbb{1}\left\{\mathcal{B}\left(Q_F(X,t,\theta^*_F)\right) \geq 0\right\}\middle| g(X,\theta^*_F) = g\right]$$

Using iterated expectations, we have

$$\begin{aligned}
&\frac{1}{h_n^D}\cdot E_F\left[\frac{\partial\mathcal{B}\left(Q_F(X_i,t,\theta^*_F)\right)}{\partial Q_p}\Lambda^{Q_p}_F(V_j,X_i,t,\theta^*_F,h_n)\mathbb{1}\left\{\mathcal{B}\left(Q_F(X_i,t,\theta^*_F)\right)\geq 0\right\}\phi(X_i)\middle| V_j\right]\\
&= \frac{1}{h_n^D}E_F\left[\frac{\Omega^p_{T_0}(Y_j,t,g(X_i,\theta^*_F))}{f_g(g(X_i,\theta^*_F))}K\left(\frac{\Delta g(X_i,X_j,\theta^*_F)}{h_n}\right)\middle| V_j\right]\cdot\omega_p(g(X_j,\theta^*_F))\\
&- E_F\left[\frac{\mu^{S_p}_{F,n}(X_i,t)}{f_g(g(X_i,\theta^*_F))}\frac{\partial\mathcal{B}\left(Q_F(X_i,t,\theta^*_F)\right)}{\partial Q_p}\mathbb{1}\left\{\mathcal{B}\left(Q_F(X_i,t,\theta^*_F)\right)\geq 0\right\}\phi(X_i)\right]
\end{aligned}$$

We have,

$$\begin{aligned}
&\frac{1}{h_n^D}E_F\left[\frac{\Omega^p_{T_0}(Y_j,t,g(X_i,\theta^*_F))}{f_g(g(X_i,\theta^*_F))}K\left(\frac{\Delta g(X_i,X_j,\theta^*_F)}{h_n}\right)\middle| V_j\right]\cdot\omega_p(g(X_j,\theta^*_F))\\
=&\frac{1}{h_n^D}\int\frac{\Omega^p_{T_0}(Y_j,t,u)}{f_g(u)}K\left(\frac{u-g(X_j,\theta^*_F)}{h_n}\right)f_g(u)du\cdot\omega_p(g(X_j,\theta^*_F))
\end{aligned}$$

As we pointed out in our analysis of $\widehat{T_1}$, since $\omega_p(g)\neq 0$ if and only if $g\in\mathcal{G}$, the above expression is nonzero only if $g(X_j,\theta^*_F)\in\mathcal{G}$, and since the kernel K has bounded support such that $K(\psi)\neq 0$ if and only if $\|\psi\|\leq S$ and since $h_n\to 0$, for large enough n, $\left\|\frac{u-g}{h_n}\right\|\leq S$ for $g\in\mathcal{G}$ implies $f_g(u)\geq \underline{f}_g > 0$. Therefore, for large enough n the terms $f_g(u)$ in the numerator and in the denominator of the previous expression can cancel each other out (since they are nonzero) and we have

$$\begin{aligned}
&\frac{1}{h_n^D}E_F\left[\frac{\Omega^p_{T_0}(Y_j,t,g(X_i,\theta^*_F))}{f_g(g(X_i,\theta^*_F))}K\left(\frac{\Delta g(X_i,X_j,\theta^*_F)}{h_n}\right)\middle| V_j\right]\cdot\omega_p(g(X_j,\theta^*_F))\\
=&\frac{1}{h_n^D}\int\Omega^p_{T_0}(Y_j,t,u)K\left(\frac{u-g(X_j,\theta^*_F)}{h_n}\right)du\cdot\omega_p(g(X_j,\theta^*_F))\quad\forall\ t\in\mathcal{T},\ \forall\ F\in\mathcal{F}
\end{aligned}$$

From here, the smoothness conditions described in Assumption 7B and an M^{th}–order approximation imply that there exists a finite constant $\overline{B}_{\Omega_{T_0}} > 0$ such that

$$
\begin{aligned}
&\frac{1}{h_n^D} E_F\left[\frac{\Omega^p_{T_0}(Y_j, t, g(X_i, \theta^*_F))}{f_g(g(X_i, \theta^*_F))} K\left(\frac{\Delta g(X_i, X_j, \theta^*_F)}{h_n}\right)\middle| V_j\right] \cdot \omega_p(g(X_j, \theta^*_F)) \\
=&\Omega^p_{T_0}(Y_j, t, g(X_j, \theta^*_F)) \cdot \omega_p(g(X_j, \theta^*_F)) + B^p_{\Omega_{T_0},n}(Y_j, X_j, t) \cdot \omega_p(g(X_j, \theta^*_F)), \\
&\quad \text{where} \quad \sup_{\substack{(y,x)\in\mathcal{S}_{Y,X} \\ t\in\mathcal{T}}} \left|B^p_{\Omega_{T_0},n}(y, x, t) \cdot \omega_p(g(x, \theta^*_F))\right| \leq \overline{B}_{\Omega_{T_0}} \cdot h_n^M \quad \forall\, F \in \mathcal{F}.
\end{aligned}
$$

Next, from Assumptions 2 and 5 and from the result in (S3.7.9),

$$
\sup_{t\in\mathcal{T}} \left| E_F\left[\frac{\mu^{S_p}_{F,n}(X_i, t)}{f_g(g(X_i, \theta^*_F))} \frac{\partial\mathcal{B}\left(Q_F(X_i, t, \theta^*_F)\right)}{\partial Q_p} \mathbb{1}\left\{\mathcal{B}\left(Q_F(X_i, t, \theta^*_F)\right) \geq 0\right\} \phi(X_i)\right]\right| \leq \overline{\phi} \cdot \frac{\overline{H}_Q}{\underline{f}_g} \cdot \overline{B}_{\mu_S} \cdot h_n^M \quad \forall\, F \in \mathcal{F}.
$$

These results combined yield,

$$
\begin{aligned}
&\frac{1}{h_n^D} \cdot E_F\left[\frac{\partial\mathcal{B}\left(Q_F(X_i, t, \theta^*_F)\right)}{\partial Q_p} \Lambda^{Q_p}_F(V_j, X_i, t, \theta^*_F, h_n) \mathbb{1}\left\{\mathcal{B}\left(Q_F(X_i, t, \theta^*_F)\right) \geq 0\right\} \phi(X_i)\middle| V_j\right] \\
&= \Omega^p_{T_0}(Y_j, t, g(X_j, \theta^*_F)) \cdot \omega_p(g(X_j, \theta^*_F)) + B^p_{T_0,n}(Y_j, X_j, t), \\
&\quad \text{where} \quad \sup_{\substack{(y,x)\in\mathcal{S}_{Y,X} \\ t\in\mathcal{T}}} \left|B^p_{T_0,n}(y, x, t)\right| \leq \left(\overline{B}_{\Omega_{T_0}} + \overline{\phi} \cdot \frac{\overline{H}_Q}{\underline{f}_g} \cdot \overline{B}_{\mu_S}\right) \cdot h_n^M \equiv \overline{C}^a_{T_0} \cdot h_n^M \quad \forall\, F \in \mathcal{F}.
\end{aligned}
\tag{S3.7.13}
$$

Denote

$$
\sum_{p=1}^{P} B^p_{T_0,n}(Y_i, X_i, t) \equiv \overline{B}_{T_0,n}(Y_i, X_i, t).
$$

Combining (S3.7.12) and (S3.7.13), we have

$$
\begin{aligned}
\frac{1}{h_n^D} \cdot \varphi^a_{T_0,F}(V_i, t, h_n) &\equiv \frac{1}{h_n^D} \cdot E_F\left[\Lambda^a_{T_0,F}(V_i, V_j, t, h_n) + \Lambda^a_{T_0,F}(V_j, V_i, t, h_n)\middle| V_i\right] \\
&\equiv \sum_{p=1}^{P} \Omega^p_{T_0}(Y_i, t, g(X_i, \theta^*_F)) \cdot \omega_p(g(X_i, \theta^*_F)) + \overline{B}_{T_0,n}(Y_i, X_i, t), \\
&\quad \text{where} \quad \sup_{\substack{(y,x)\in\mathcal{S}_{Y,X} \\ t\in\mathcal{T}}} \left|\overline{B}_{T_0,n}(y, x, t)\right| \leq P \cdot \overline{C}^a_{T_0} \cdot h_n^M \equiv \overline{C}^b_{T_0} \cdot h_n^M \quad \forall\, F \in \mathcal{F}.
\end{aligned}
\tag{S3.7.14}
$$

Next, let

$$\vartheta^a_{T_0,F}(V_i,V_j,t,h) \equiv \Lambda^a_{T_0,F}(V_i,V_j,t,h) + \Lambda^a_{T_0,F}(V_j,V_i,t,h) - \varphi^a_{T_0,F}(V_i,t,h) - \varphi^a_{T_0,F}(V_j,t,h)$$

Note that $\vartheta^a_{T_0,F}(V_i,V_j,t,h) = \vartheta^a_{T_0,F}(V_j,V_i,t,h)$ and $E_F\left[\vartheta^a_{T_0,F}(V_i,V_j,t,h)|V_i\right] = E_F\left[\vartheta^a_{T_0,F}(V_i,V_j,t,h)|V_j\right] = 0$. Define the following degenerate U-statistic of order 2,

$$\widetilde{U}^a_{T_0,n}(t,h) = \frac{1}{n(n-1)}\sum_{i=1}^{n}\sum_{j\neq i}\vartheta^a_{T_0,F}(V_i,V_j,t,h).$$

From the conditions described in Assumption 6, there exist constants $(\overline{A},\overline{V})$ such that, for each $F\in\mathcal{F}$, the following class of indicator functions is Euclidean $(\overline{A},\overline{V})$ for the constant envelope 1,

$$\left\{m:\mathcal{X}\longrightarrow\mathbb{R}: m(x) = \mathbb{1}\left\{\mathcal{B}\left(Q_F(x,t,\theta^*_F)\right)\geq 0\right\}\ \text{ for some } t\in\mathcal{T}\right\}.$$

This fact, combined with the bounded-variation properties of the kernel K and the conditions stated in Assumptions 2-5 and 7B, from Example 2.10 in Pakes and Pollard (1989) and Lemma 20 in Nolan and Pollard (1987) (or Lemma 5 in Sherman (1994)) combined with the Euclidean-preserving properties in Lemma 2.14 in Pakes and Pollard (1989), the class of functions

$$\left\{m:\mathcal{S}^2_V\longrightarrow\mathbb{R}\text{: } m(v_1,v_2) = \vartheta^a_{T_0,F}(v_1,v_2,t,h)\ \text{ for some } h>0\right\}$$

is Euclidean for an envelope $\overline{G}^a_{T_0}(\cdot)$ that satisfies $E_F\left[\overline{G}^a_{T_0}(V_1,V_2)^{4q}\right]$ for all $F\in\mathcal{F}$ (with $(V_1,V_2)\sim F\otimes F$) and q being the integer described in Assumption 1. From here, applying Result S1 we obtain

$$\sup_{t\in\mathcal{T},h>0}\left|\widetilde{U}^a_{T_0,n}(t,h)\right| = O_p\left(\frac{1}{n}\right)\quad\text{uniformly over } \mathcal{F}$$

Therefore,

$$\sup_{t\in\mathcal{T}}\left|\widetilde{U}^a_{T_0,n}(t,h_n)\right| \leq \sup_{t\in\mathcal{T},h>0}\left|\widetilde{U}^a_{T_0,n}(t,h)\right| = O_p\left(\frac{1}{n}\right)\quad\text{uniformly over } \mathcal{F} \tag{S3.7.15}$$

The Hoeffding decomposition of $U^{a}_{T_0,n}(t,h_n)$ (see Serfling (1980, pages 177-178) or Sherman (1994, equations (6)-(7))) yields,

$$\begin{aligned}\frac{1}{h_n^D}\cdot U^{a}_{T_0,n}(t,h_n) &= \frac{1}{n}\sum_{i=1}^{n}\frac{1}{h_n^D}\cdot \varphi^{a}_{T_0,F}(V_i,t,h_n)+\frac{1}{2h_n^D}\cdot \widetilde{U}^{a}_{T_0,n}(t,h_n)\\ &= \frac{1}{n}\sum_{i=1}^{n}\left(\sum_{p=1}^{P}\Omega^{p}_{T_0}(Y_i,t,g(X_i,\theta^*_F))\cdot\omega_p(g(X_i,\theta^*_F))\right)+\frac{1}{n}\sum_{i=1}^{n}\overline{B}_{T_0,n}(Y_i,X_i,t)+\frac{1}{2h_n^D}\cdot\widetilde{U}^{a}_{T_0,n}(t,h_n),\end{aligned} \tag{S3.7.16}$$

where, from (S3.7.14) and (S3.7.15),

$$\begin{aligned}\left|\frac{1}{n}\sum_{i=1}^{n}\overline{B}_{T_0,n}(Y_i,X_i,t)+\frac{1}{2h_n^D}\cdot\widetilde{U}^{a}_{T_0,n}(t,h_n)\right| &\leq \sup_{\substack{(y,x)\in\mathcal{S}_{Y,X}\\ t\in\mathcal{T}}}\left|\overline{B}_{T_0,n}(y,x,t)\right|+\sup_{t\in\mathcal{T},h>0}\left|\widetilde{U}^{a}_{T_0,n}(t,h)\right|\\ &= O(h_n^M)+O_p\left(\frac{1}{n\cdot h_n^D}\right)=o_p\left(\frac{1}{n^{1/2+\epsilon}}\right)\quad \forall\ F\in\mathcal{F}\end{aligned} \tag{S3.7.17}$$

where (once again), $\epsilon>0$ is the constant described in Assumption 4. Next, note that

$$E_F\left[\Omega^{p}_{T_0}(Y,t,g(X,\theta^*_F))\cdot\omega_p(g(X,\theta^*_F)\right]=0\quad \forall\ t\in\mathcal{T},\ \forall\ F\in\mathcal{F} \tag{S3.7.18}$$

for each $p=1,\ldots,P$. The proof is parallel to that of (S3.6.21). Let $(V_1,V_2)\sim F\otimes F$. Then, by the definition of $\Omega^{p}_{T_0}$, we have

$$\begin{aligned}&\Omega^{p}_{T_0}(Y_1,t,g(X_1,\theta^*_F))\cdot\omega_p(g(X_1,\theta^*_F))=\\ &E_F\left[\left(S_p(Y_1,X_2,t)-\Gamma_{p,F}(X_2,t,\theta^*_F)\right)\frac{\partial\mathcal{B}\left(Q_F(X_2,t,\theta^*_F)\right)}{\partial Q_p}\phi(X_2)\mathbb{1}\left\{\mathcal{B}\left(Q_F(X_2,t,\theta^*_F)\right)\geq 0\right\}\middle|g(X_2,\theta^*_F)=g(X_1,\theta^*_F),V_1\right]\\ &\cdot\omega_p(g(X_1,\theta^*_F))\end{aligned}$$

Therefore, by iterated expectations, we have

$$
\begin{aligned}
&E_F\left[\Omega^p_{T_0}(Y_1,t,g(X_1,\theta^*_F))\cdot\omega_p(g(X_1,\theta^*_F))\right]=\\
&E_F\Bigg[E_F\left[\left(S_p(Y_1,X_2,t)-\Gamma_{p,F}(X_2,t,\theta^*_F)\right)\cdot\omega_p(g(X_1,\theta^*_F))\Big|g(X_1,\theta^*_F)=g(X_2,\theta^*_F),V_2\right]\\
&\quad\cdot\frac{\partial\mathcal{B}\left(Q_F(X_2,t,\theta^*_F)\right)}{\partial Q_p}\phi(X_2)\mathbb{1}\left\{\mathcal{B}\left(Q_F(X_2,t,\theta^*_F)\right)\geq 0\right\}\Bigg]\\
&=E_F\left[\underbrace{\left(\Gamma_{p,F}(X_2,t,\theta^*_F)-\Gamma_{p,F}(X_2,t,\theta^*_F)\right)}_{=0}\cdot\omega_p(g(X_2,\theta^*_F))\frac{\partial\mathcal{B}\left(Q_F(X_2,t,\theta^*_F)\right)}{\partial Q_p}\phi(X_2)\mathbb{1}\left\{\mathcal{B}\left(Q_F(X_2,t,\theta^*_F)\right)\geq 0\right\}\right]\\
&=0.
\end{aligned}
$$

By Assumption 7B, there exists a finite constant $\overline{\eta}_{\Omega_{T_0}}>0$ such that

$$\sup_{t\in\mathcal{T}}E_F\left[\left|\Omega^p_{T_0}(Y,t,g(X,\theta^*_F))\cdot\omega_p(g(X,\theta^*_F)\right|^2\right]\leq\overline{\eta}_{\Omega_{T_0}}\quad\forall\ F\in\mathcal{F}.$$

From here, a Chebyshev inequality yields

$$\sup_{t\in\mathcal{T}}\left|\frac{1}{n}\sum_{i=1}^{n}\left(\sum_{p=1}^{P}\Omega^p_{T_0}(Y_i,t,g(X_i,\theta^*_F))\cdot\omega_p(g(X_i,\theta^*_F))\right)\right|=O_p\left(\frac{1}{n^{1/2}}\right)\quad\text{uniformly over }\mathcal{F}.$$

(S3.7.16), (S3.7.17) and the previous expression yield,

$$\sup_{t\in\mathcal{T}}\frac{1}{h_n^D}\cdot U^a_{T_0,n}(t,h_n)=O_p\left(\frac{1}{n^{1/2}}\right)\quad\text{uniformly over }\mathcal{F}.\tag{S3.7.19}$$

Next, we turn attention to the U-statistic $U^b_{T_0,n}(t)$, which we defined in (S3.7.7) as

$$U^b_{T_0,n}(t)\equiv\frac{1}{n\cdot(n-1)}\sum_{i=1}^{n}\sum_{j\neq i}\Lambda^b_{T_0,F}(V_i,V_j,t),$$

where

$$\Lambda^b_{T_0,F}(v_1,v_2,t)\equiv\left(\sum_{p=1}^{P}\frac{\partial\mathcal{B}\left(Q_F(x_1,t,\theta^*_F)\right)}{\partial Q_p}\mathbb{1}\left\{\mathcal{B}\left(Q_F(x_1,t,\theta^*_F)\right)\geq 0\right\}\phi(x_1)\Xi_{Q_p}(x_1,t,\theta^*_F)\right)\psi^\theta_F(z_2).$$

Define,

$$
\begin{aligned}
\Xi^{p}_{T_0,F}(t) &\equiv E_F\left[\frac{\partial \mathcal{B}\left(Q_F(X,t,\theta^*_F)\right)}{\partial Q_p}\mathbb{1}\{\mathcal{B}\left(Q_F(X,t,\theta^*_F)\right)\geq 0\}\phi(X)\Xi_{Q_p}(X,t,\theta^*_F)\right],\\
\Xi_{T_0,F}(t) &\equiv \sum_{p=1}^{P}\Xi^{p}_{T_0,F}(t)
\end{aligned}
\tag{S3.7.20}
$$

where the above expectation is taken with respect to X. Let

$$
\varphi^{b}_{T_0,F}(V_i,t) \equiv E_F\left[\Lambda^{b}_{T_0,F}(V_i,V_j,t)+\Lambda^{b}_{T_0,F}(V_j,V_i,t)\middle| V_i\right].
$$

Then,

$$
\varphi^{b}_{T_0,F}(V_i,t) = \Xi_{T_0,F}(t)\psi^{\theta}_{F}(Z_i).
$$

Note that

$$
E_F\left[\varphi^{b}_{T_0,F}(V_i,t)\right]=0\ \forall\ t\in\mathcal{T},\ \forall\ F\in\mathcal{F},\quad \text{and therefore,}\quad E_F\left[\Lambda^{b}_{T_0,F}(V_i,V_j)\right]=0\ \forall\ t\in\mathcal{T},\ \forall\ F\in\mathcal{F}
$$

Next, let

$$
\vartheta^{b}_{T_0,F}(V_i,V_j,t)\equiv \Lambda^{b}_{T_0,F}(V_i,V_j,t)+\Lambda^{b}_{T_0,F}(V_j,V_i,t)-\varphi^{b}_{T_0,F}(V_i,t)-\varphi^{b}_{T_0,F}(V_j,t)
$$

and define the degenerate U-statistic

$$
\widetilde{U}^{b}_{T_0,n}(t)=\frac{1}{n(n-1)}\sum_{i=1}^{n}\sum_{j\neq i}\vartheta^{b}_{T_0,F}(V_i,V_j,t).
$$

From Assumption 1, $E_F\left[\left\|\psi^{\theta}_{F}(Z)\right\|^{4q}\right]\leq\overline{\mu}_{\psi}\ \forall\ F\in\mathcal{F}$, where q is the integer described there. Also, as we stated in (S3.7.11), Assumptions 2 and 5 imply $\sup_{t\in\mathcal{T}}\left\|\Xi_{T_0,F}(t)\right\|\leq P\cdot\overline{C}_{\Xi_{T_1}}\ \forall\ F\in\mathcal{F}$. Therefore, there exists an envelope $\overline{G}_{T_0}(\cdot)$ and a finite constant $\overline{\mu}_{T_0^b}>0$ such that $\sup_{t\in\mathcal{T}}\left|\vartheta^{b}_{T_0,F}(v_1,v_2,t)\right|\leq\overline{G}_{T_0}(v_1,v_2)$ and $E_F\left[\overline{G}_{T_0}(V_1,V_2)^{4q}\right]\leq\overline{\mu}_{T_1^b}$ for all $F\in\mathcal{F}$ (with $(V_1,V_2)\sim F\otimes F$). From here, applying Result S1 we obtain

$$
\sup_{t\in\mathcal{T}}\left|\widetilde{U}^{b}_{T_0,n}(t)\right|=O_p\left(\frac{1}{n}\right)\quad\text{uniformly over }\mathcal{F}. \tag{S3.7.21}
$$

The Hoeffding decomposition of $U^b_{T_0,n}(t)$ is,

$$
\begin{aligned}
U^b_{T_0,n}(t) &= \frac{1}{n}\sum_{i=1}^{n}\varphi^b_{T_0}(V_i,t) + \frac{1}{2}\widetilde{U}^b_{T_0,n}(t) \\
&= \Xi_{T_0,F}(t)\frac{1}{n}\sum_{i=1}^{n}\psi^\theta_F(Z_i) + O_p\left(\frac{1}{n}\right) \quad \text{uniformly over } \mathcal{F}.
\end{aligned}
\tag{S3.7.22}
$$

From Assumption 1,

$$
\left\|\frac{1}{n}\sum_{i=1}^{n}\psi^\theta_F(Z_i)\right\| = O_p\left(\frac{1}{n^{1/2}}\right) \quad \text{uniformly over } \mathcal{F}.
$$

Combined with (S3.7.11),

$$
\sup_{t\in\mathcal{T}}\left\|\Xi_{T_0,F}(t)\frac{1}{n}\sum_{i=1}^{n}\psi^\theta_F(Z_i)\right\| \le \overline{C}_{\Xi_{T_0}}\cdot\left\|\frac{1}{n}\sum_{i=1}^{n}\psi^\theta_F(Z_i)\right\| = O_p\left(\frac{1}{n^{1/2}}\right) \quad \text{uniformly over } \mathcal{F}. \tag{S3.7.23}
$$

(S3.7.22) and (S3.7.23) yield

$$
\sup_{t\in\mathcal{T}}\left|U^b_{T_0,n}(t)\right| = O_p\left(\frac{1}{n^{1/2}}\right) \quad \text{uniformly over } \mathcal{F}. \tag{S3.7.24}
$$

Let us continue with the process $\nu^a_{T_0,n}(t,h)$ defined in (S3.7.7). The same arguments that led to (S3.7.15) yield

$$
\sup_{t\in\mathcal{T}}\left|\nu^a_{T_0,n}(t,h_n)\right| \le \sup_{t\in\mathcal{T},h>0}\left|\nu^a_{T_0,n}(t,h)\right| = O_p\left(\frac{1}{n^{1/2}}\right) \quad \text{uniformly over } \mathcal{F}. \tag{S3.7.25}
$$

And from (S3.7.11), we have

$$
\sup_{t\in\mathcal{T}}\left|E_F\left[\Lambda^a_{T_0,F}(V,V,t,h_n)\right]\right| \le \sup_{t\in\mathcal{T},h>0}\left|E_F\left[\Lambda^a_{T_0,F}(V,V,t,h)\right]\right| \le P\cdot\overline{C}_{\Xi_{T_0}} \quad \forall\ F\in\mathcal{F} \tag{S3.7.26}
$$

Next, take the process $\nu^b_{T_0,n}(t)$ defined in (S3.7.7). The same arguments that led to (S3.7.21) yield

$$
\sup_{t\in\mathcal{T}}\left|\nu^b_{T_0,n}(t)\right| = O_p\left(\frac{1}{n^{1/2}}\right) \quad \text{uniformly over } \mathcal{F}. \tag{S3.7.27}
$$

By Assumption 1, there exists a finite constant $\overline{\eta}_\psi > 0$ such that $E_F\left[\left\|\psi^\theta_F(Z)\right\|\right] \le \overline{\eta}_\psi$ for all $F\in\mathcal{F}$. Combined with the result in (S3.7.11), this yields

$$
\sup_{t\in\mathcal{T}}\left|E_F\left[\Lambda^b_{T_1,F}(V,V,t)\right]\right| \le P\cdot\overline{C}_{\Xi_{T_0}}\cdot\overline{\eta}_\psi \quad \forall\ F\in\mathcal{F}. \tag{S3.7.28}
$$

S3.7.2 Proof of Proposition 1B

From (S3.7.8), we have

$$
\begin{aligned}
&\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\psi_F^{\mathcal{B}}(V_j,X_i,t,\theta_F^*,h_n)\mathbb{1}\left\{\mathcal{B}\left(Q_F(X_i,t,\theta_F^*)\right)\geq 0\right\}\phi(X_i)\\
=&\frac{1}{h_n^D}\cdot U_{T_0,n}^{a}(t,h_n)+U_{T_0,n}^{b}(t)-\frac{1}{n}\cdot\frac{1}{h_n^D}\cdot U_{T_0,n}^{a}(t,h_n)-\frac{1}{n}\cdot U_{T_0,n}^{b}(t)+\frac{1}{n\cdot h_n^D}\cdot \nu_{T_0,n}^{a}(t,h_n)+\frac{1}{n}\cdot\nu_{T_0,n}^{b}(t)\\
&+\frac{1}{n\cdot h_n^D}\cdot E_F\left[\Lambda_{T_0,F}^{a}(V,V,t,h_n)\right]+\frac{1}{n}\cdot E_F\left[\Lambda_{T_0,F}^{b}(V,V,t)\right]
\end{aligned}
$$

Using (S3.7.16) and (S3.7.22), the above expression becomes

$$
\begin{aligned}
&\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\psi_F^{\mathcal{B}}(V_j,X_i,t,\theta_F^*,h_n)\mathbb{1}\left\{\mathcal{B}\left(Q_F(X_i,t,\theta_F^*)\right)\geq 0\right\}\phi(X_i)\\
=&\frac{1}{n}\sum_{i=1}^{n}\left(\left(\sum_{p=1}^{P}\Omega_{T_0}^{p}(Y_i,t,g(X_i,\theta_F^*))\cdot\omega_p(g(X_i,\theta_F^*))\right)+\Xi_{T_0,F}(t)\psi_F^{\theta}(Z_i)\right)+\xi_{T_0,n}^{c}(t),
\end{aligned}
\tag{S3.7.29}
$$

where

$$
\begin{aligned}
\xi_{T_0,n}^{c}(t)\equiv&\frac{1}{n}\sum_{i=1}^{n}\overline{B}_{T_0,n}(Y_i,X_i,t)+\frac{1}{2h_n^D}\cdot\widetilde{U}_{T_0,n}^{a}(t,h_n)+\frac{1}{2}\cdot\widetilde{U}_{T_0,n}^{b}(t)-\frac{1}{n}\cdot\frac{1}{h_n^D}\cdot U_{T_0,n}^{a}(t,h_n)-\frac{1}{n}\cdot U_{T_0,n}^{b}(t)\\
&+\frac{1}{n\cdot h_n^D}\cdot\nu_{T_0,n}^{a}(t,h_n)+\frac{1}{n}\cdot\nu_{T_0,n}^{b}(t)+\frac{1}{n\cdot h_n^D}\cdot E_F\left[\Lambda_{T_0,F}^{a}(V,V,t,h_n)\right]+\frac{1}{n}\cdot E_F\left[\Lambda_{T_0,F}^{b}(V,V,t)\right]
\end{aligned}
$$

From equations (S3.7.17), (S3.7.21), (S3.7.19), (S3.7.24), (S3.7.25), (S3.7.27), (S3.7.26) and (S3.7.28), we have

$$
\begin{aligned}
\sup_{t\in\mathcal{T}}\left|\xi_{T_0,n}^{c}(t)\right|\leq& o_p\left(\frac{1}{n^{1/2+\epsilon}}\right)+O_p\left(\frac{1}{n\cdot h_n^D}\right)+O_p\left(\frac{1}{n}\right)+O_p\left(\frac{1}{n^{3/2}\cdot h_n^D}\right)+O_p\left(\frac{1}{n^{3/2}}\right)\\
&+O_p\left(\frac{1}{n^{3/2}\cdot h_n^D}\right)+O_p\left(\frac{1}{n^{3/2}}\right)+O\left(\frac{1}{n\cdot h_n^D}\right)+O\left(\frac{1}{n}\right)\\
=&o_p\left(\frac{1}{n^{1/2+\epsilon}}\right)\quad\text{uniformly over }\mathcal{F}.
\end{aligned}
\tag{S3.7.30}
$$

where $\epsilon>0$ is as described in Assumption 4. Let

$$
\psi_F^{T_0}(V_i,t)\equiv\left(\left(\mathcal{B}\left(Q_F(X_i,t,\theta_F^*)\right)\right)_+\phi(X_i)-T_{0,F}(t)\right)+\sum_{p=1}^{P}\Omega_{T_0}^{p}(Y_i,t,g(X_i,\theta_F^*))\cdot\omega_p(g(X_i,\theta_F^*))+\Xi_{T_0,F}(t)\psi_F^{\theta}(Z_i).
\tag{S3.7.31}
$$

Let $\Delta \equiv \epsilon \wedge (\delta_0/2)$. Plugging (S3.7.29), (S3.7.30), (S3.7.6) and (S3.7.4) into (S3.7.5), we have

$$\widehat{T_0}(t) = T_{0,F}(t) + \frac{1}{n}\sum_{i=1}^{n} \psi_F^{T_0}(V_i,t) + \varepsilon_n^{T_0}(t), \quad \text{where} \quad \sup_{t\in\mathcal{T}} \left|\varepsilon_n^{T_0}(t)\right| = o_p\left(\frac{1}{n^{1/2+\Delta}}\right) \quad \text{uniformly over } \mathcal{F}. \tag{S3.7.32}$$

The influence function $\psi_F^{T_0}(V,t)$ has two key features,

$$\begin{aligned} &(i) \quad E_F\left[\psi_F^{T_0}(V,t)\right] = 0 \quad \forall\, t \in \mathcal{T},\ \forall\, F \in \mathcal{F}, \\ &(ii) \quad P_F\left(\mathcal{B}(\Gamma_F(X,t,\theta_F^*)) < 0 \;\middle|\; X \in \mathcal{X}_F^*\right) = 1 \implies P_F\left(\psi_F^{T_0}(V,t) = 0\right) = 1. \end{aligned} \tag{S3.7.33}$$

Part (i) of (S3.7.33) follows directly from (S3.7.18) and the facts that $E_F\left[\psi_F^{\theta}(Z)\right] = 0$ and, by construction, $E_F\left[\left(\mathcal{B}\left(Q_F(X_i,t,\theta_F^*)\right)\right)_+ \phi(X_i) - T_{0,F}(t)\right] = 0$ for each $t \in \mathcal{T}$. To show part (ii), note that, from (7), $P_F\left(\mathcal{B}\left(\Gamma_F(X,t,\theta_F^*)\right) < 0 \middle| X \in \mathcal{X}_F^*\right) = 1$ implies

$$P_F\left((\mathcal{B}(Q_F(X,t,\theta_F^*)))_+ \phi(X) = 0\right) = P_F\left((\mathcal{B}(\Gamma_F(X,t,\theta_F^*)))_+ \cdot \mathcal{H}(\omega(g(X,\theta_F^*)))\,\phi(X) = 0\right) = 1,$$

and $T_{0,F}(t) = 0$. Also, if $P_F\left(\mathcal{B}\left(\Gamma_F(X,t,\theta_F^*)\right) < 0 \middle| X \in \mathcal{X}_F^*\right) = 1$, then

$$\begin{aligned} &P_F\left(\phi(X)\mathbb{1}\left\{\mathcal{B}(Q_F(X,t,\theta_F^*)) \geq 0\right\} = 0 \;\middle|\; g(X,\theta_F^*) \in \mathcal{G}\right) \\ =&P_F\left(\phi(X)\mathbb{1}\left\{\mathcal{B}(\Gamma_F(X,t,\theta_F^*)) \cdot \mathcal{H}(\omega(g(X,\theta_F^*))) \geq 0\right\} = 0 \;\middle|\; g(X,\theta_F^*) \in \mathcal{G}\right) = 1. \end{aligned}$$

Thus, for any (y,t,g), the above result implies

$$\begin{aligned} &\Omega_{T_0}^{p}(y,t,g) \cdot \omega_p(g) \\ =&E_F\left[\left(S_p(y,t) - \Gamma_{p,F}(X,t,\theta_F^*)\right) \frac{\partial \mathcal{B}\left(Q_F(X,t,\theta_F^*)\right)}{\partial Q_p} \underbrace{\phi(X)\mathbb{1}\left\{\mathcal{B}(Q_F(X,t,\theta_F^*)) \geq 0\right\}}_{\substack{=0\ F\text{-a.s if } g(X,\theta_F^*) = g \in \mathcal{G} \\ \text{(i.e, if } \omega_p(g) \neq 0)}} \;\middle|\; g(X,\theta_F^*) = g\right] \cdot \underbrace{\omega_p(g)}_{\substack{=0 \text{ if} \\ g \notin \mathcal{G}}} \\ =&0. \end{aligned}$$

Therefore,

$$P_F\left(\mathcal{B}(\Gamma_F(X,t,\theta_F^*)) < 0 \middle| X \in \mathcal{X}_F^*\right) = 1 \implies P_F\left(\Omega_{T_0}^{p}(Y,t,g(X,\theta_F^*)) \cdot \omega_p(g(X,\theta_F^*)) = 0\right) = 1.$$

Finally, recall from (A-4) that $\Xi_{Q_p}(x,t,\theta_F^*) = 0 \ \ \forall\, x : g(x,\theta_F^*) \notin \mathcal{G}$. Therefore, for each $t \in \mathcal{T}$,

$$\phi(x)\Xi_{Q_p}(x,t,\theta_F^*) = 0 \quad \forall\, x \notin \mathcal{X}_F^*.$$

And from our definition of $\Xi^{p}_{T_0,F}(t)$ and $\Xi_{T_0,F}(t)$ in (S3.7.20), if $P_F\left(\mathcal{B}\left(\Gamma_F(X,t,\theta^*_F)\right) < 0 \middle| X \in \mathcal{X}^*_F\right) = 1$, then

$$\begin{aligned}
\Xi^{p}_{T_0,F}(t) &\equiv E_F\left[\frac{\partial \mathcal{B}\left(Q_F(X,t,\theta^*_F)\right)}{\partial Q_p} \mathbb{1}\left\{\mathcal{B}\left(Q_F(X,t,\theta^*_F)\right) \geq 0\right\} \phi(X) \Xi_{Q_p}(X,t,\theta^*_F)\right] \\
&= E_F\left[\frac{\partial \mathcal{B}\left(Q_F(X,t,\theta^*_F)\right)}{\partial Q_p} \underbrace{\mathbb{1}\left\{\mathcal{B}\left(\Gamma_F(X,t,\theta^*_F)\right) \cdot \mathcal{H}\left(\omega(g(X,\theta^*_F))\right) \geq 0\right\}}_{=0 \text{ if } X \in \mathcal{X}^*_F} \underbrace{\phi(X) \Xi_{Q_p}(X,t,\theta^*_F)}_{\substack{=0 \\ \text{if } X \notin \mathcal{X}^*_F}}\right] \\
&= 0 \quad \forall\, p = 1, \ldots, P. \\
\implies \quad \Xi_{T_0,F}(t) &\equiv \sum_{p=1}^{P} \Xi^{p}_{T_0,F}(t) = 0.
\end{aligned}$$

Therefore, $\Xi_{T_0,F}(t)\psi^{\theta}_F(Z) = 0$. Combined, these results yield

$$P_F\left(\mathcal{B}\left(\Gamma_F(X,t,\theta^*_F)\right) < 0 \;\middle|\; X \in \mathcal{X}^*_F\right) = 1 \implies P_F\left(\psi^{T_0}_F(V,t) = 0\right) = 1,$$

which establishes part (ii) of (S3.7.33). Equipped with (S3.7.32), we can characterize a linear representation for $\widehat{T_2}$ by recalling that, by definition,

$$\widehat{T_2} \equiv \int_t \widehat{T_0}(t) d\mathcal{W}(t), \quad T_{2,F} \equiv \int_t T_{0,F}(t) d\mathcal{W}(t).$$

Recall that we have normalized $\int_t d\mathcal{W}(t) = 1$ for simplicity. Let

$$\psi^{T_2}_F(V_i) \equiv \int_t \psi^{T_0}_F(V_i,t) d\mathcal{W}(t), \quad \varepsilon^{T_2}_n \equiv \int_t \varepsilon^{T_0}_n(t) d\mathcal{W}(t). \tag{S3.7.34}$$

From (S3.7.32), we have

$$\begin{aligned}
&\widehat{T_2} = T_{2,F} + \frac{1}{n}\sum_{i=1}^{n} \psi^{T_2}_F(V_i) + \varepsilon^{T_2}_n, \\
&\text{where} \quad \left|\varepsilon^{T_2}_n\right| \leq \sup_{t \in \mathcal{T}} \left|\varepsilon^{T_0}_n(t)\right| \cdot \underbrace{\int_t d\mathcal{W}(t)}_{=1} = o_p\left(\frac{1}{n^{1/2+\Delta}}\right) \quad \text{uniformly over } \mathcal{F}.
\end{aligned} \tag{S3.7.35}$$

Equations (S3.7.31), (S3.7.32), (S3.7.34) and (S3.7.35) prove the linear representation result in Proposition 1B. ■

S3.7.3 Properties of the influence function $\psi_F^{T_2}(V)$

From (S3.7.33), we have the following properties for the influence function $\psi_F^{T_2}(V)$,

$$\begin{aligned}
&(i) \quad E_F\left[\psi_F^{T_2}(V)\right] = 0 \quad \forall\ F \in \mathcal{F}, \\
&(ii) \quad P_F\left(\mathcal{B}\left(\Gamma_F(X,t,\theta_F^*)\right) < 0 \;\middle|\; X \in \mathcal{X}_F^*\right) = 1 \ \text{ for } \mathcal{W}\text{–a.e } t \in \mathcal{T} \implies P_F\left(\psi_F^{T_2}(V) = 0\right) = 1.
\end{aligned} \tag{S3.7.36}$$

Part (i) of (S3.7.36) follows directly from part (i) of (S3.7.33) since,

$$E_F\left[\psi_F^{T_2}(V)\right] = E_F\left[\int_t \psi_F^{T_0}(V,t)d\mathcal{W}(t)\right] = \int_t \underbrace{\left(E_F\left[\psi_F^{T_0}(V,t)\right]\right)}_{=0\ \forall\ t\in\mathcal{T}} d\mathcal{W}(t) = 0.$$

Similarly, part (ii) of (S3.7.36) follows directly from part (ii) of (S3.7.33) since,

$$P_F\left(\mathcal{B}\left(\Gamma_F(X,t,\theta_F^*)\right) < 0 \;\middle|\; X \in \mathcal{X}_F^*\right) = 1 \ \text{ for } \mathcal{W}\text{–a.e } t \in \mathcal{T} \implies P_F\left(\psi_F^{T_0}(V,t) = 0 \ \text{ for } \mathcal{W}\text{–a.e } t \in \mathcal{T}\ \right) = 1.$$

And,

$$P_F\left(\psi_F^{T_0}(V,t) = 0 \ \text{ for } \mathcal{W}\text{–a.e } t \in \mathcal{T}\ \right) = 1 \implies P_F\left(\psi_F^{T_2}(V) = 0\right) = P_F\left(\int_t \psi_F^{T_0}(V,t)d\mathcal{W}(t) = 0\right) = 1.$$

S4 Estimators for the influence functions $\psi_F^{T_1}(V)$ and $\psi_F^{T_2}(V)$

Here we describe how we obtained the influence function estimators discussed in Section 3.5.1 of the paper and in Appendix A2, which are used in the construction of $\widehat{\sigma}_1^2$ and $\widehat{\sigma}_2^2$.

S4.1 Construction of our estimators

Fix g, x and t. We will first decompose the functionals defined in Assumption 2 as follows,

$$
\begin{aligned}
&(1)\quad \Omega^{d,\ell}_{f_g}(g) \equiv E_F\left[\frac{\partial g_d(X,\theta^*_F)}{\partial \theta_\ell}\middle| g(X,\theta^*_F)=g\right] = \frac{A^{d,\ell}_{f_g}(g)}{f_g(g)},\\
&\qquad \text{where}\quad A^{d,\ell}_{f_g}(g) \equiv \Omega^{d,\ell}_{f_g}(g)\cdot f_g(g),\\
&(2)\quad \Omega^{d,\ell}_{R_p,1}(g,t) \equiv E_F\left[S_p(Y,t)\omega_p(g(X,\theta^*_F))\frac{\partial g_d(X,\theta^*_F)}{\partial \theta_\ell}\middle| g(X,\theta^*_F)=g\right] = \frac{A^{d,\ell}_{R_p,1}(g,t)}{f_g(g)},\\
&\qquad \text{where}\quad A^{d,\ell}_{R_p,1}(g,t) \equiv \Omega^{d,\ell}_{R_p,1}(g,t)\cdot f_g(g),\\
&(3)\quad \Omega_{R_p,2}(g,t) \equiv E_F\left[S_p(Y,t)\omega_p(g(X,\theta^*_F))\middle| g(X,\theta^*_F)=g\right] = \frac{A_{R_p,2}(g,t)}{f_g(g)},\\
&\qquad \text{where}\quad A_{R_p,2}(g,t) \equiv \Omega_{R_p,2}(g,t)\cdot f_g(g),\\
&(4)\quad \Omega^{\ell}_{R_p,3}(g,t) \equiv E_F\left[S_p(Y,t)\frac{\partial \omega_p(g(X,\theta^*_F))}{\partial \theta_\ell}\middle| g(X,\theta^*_F)=g\right] = \frac{A^{\ell}_{R_p,3}(g,t)}{f_g(g)},\\
&\qquad \text{where}\quad A^{\ell}_{R_p,3}(g,t) \equiv \Omega^{\ell}_{R_p,3}(g,t)\cdot f_g(g).
\end{aligned}
\tag{S4.1.1}
$$

In addition to the above functionals, the following derivatives will be relevant,

$$
\frac{\partial \Omega^{d,\ell}_{f_g}(g)}{\partial g_d},\quad \frac{\partial \Omega^{d,\ell}_{R_p,1}(g,t)}{\partial g_d},\quad \text{and}\quad \frac{\partial \Omega_{R_p,2}(g,t)}{\partial g_d}.
$$

From the expressions in (S4.1.1), we have

$$
\begin{aligned}
\frac{\partial \Omega^{d,\ell}_{f_g}(g)}{\partial g_d} &= \frac{\partial A^{d,\ell}_{f_g}(g)}{\partial g_d}\cdot\frac{1}{f_g(g)} - \frac{A_{f_g}(g)}{f_g(g)^2}\cdot\frac{\partial f_g(g)}{\partial g_d},\\
\frac{\partial \Omega^{d,\ell}_{R_p,1}(g,t)}{\partial g_d} &= \frac{\partial A^{d,\ell}_{R_p,1}(g,t)}{\partial g_d}\cdot\frac{1}{f_g(g)} - \frac{A^{d,\ell}_{R_p,1}(g,t)}{f_g(g)^2}\cdot\frac{\partial f_g(g)}{\partial g_d},\\
\frac{\partial \Omega_{R_p,2}(g,t)}{\partial g_d} &= \frac{\partial A_{R_p,2}(g,t)}{\partial g_d}\cdot\frac{1}{f_g(g)} - \frac{A_{R_p,2}(g,t)}{f_g(g)^2}\cdot\frac{\partial f_g(g)}{\partial g_d},
\end{aligned}
$$

We will estimate the above functionals and the derivatives described using the same type of kernel estimators we employed to construct $\widehat{T}_1$. Our estimator of $f_g(g)$ is

$$
\widehat{f_g}(g) = \frac{1}{n\cdot h_n^D}\sum_{i=1}^{n} K\left(\frac{g(X_i,\widehat{\theta})-g}{h_n}\right),
$$

From here, we estimate

$$(1)\quad \widehat{\Omega}^{d,\ell}_{f_g}(g)=\frac{\widehat{A}^{d,\ell}_{f_g}(g)}{\widehat{f_g}(g)},\quad \widehat{A}^{d,\ell}_{f_g}(g)\equiv\frac{1}{nh_n^D}\sum_{i=1}^{n}\frac{\partial g_d(X_i,\widehat{\theta})}{\partial\theta_\ell}K\left(\frac{g(X_i,\widehat{\theta})-g}{h_n}\right),$$

$$(2)\quad \widehat{\Omega}^{d,\ell}_{R_p,1}(g,t)=\frac{\widehat{A}^{d,\ell}_{R_p,1}(g,t)}{\widehat{f_g}(g)},\quad \widehat{A}^{d,\ell}_{R_p,1}(g,t)\equiv\frac{1}{n\cdot h_n^D}\sum_{i=1}^{n}S_p(Y_i,t)\omega_p(g(X_i,\widehat{\theta}))\frac{\partial g_d(X_i,\widehat{\theta})}{\partial\theta_\ell}K\left(\frac{g(X_i,\widehat{\theta})-g}{h_n}\right),$$

$$(3)\quad \widehat{\Omega}_{R_p,2}(g,t)=\frac{\widehat{A}_{R_p,2}(g,t)}{\widehat{f_g}(g)},\quad \widehat{A}_{R_p,2}(g,t)\equiv\frac{1}{n\cdot h_n^D}\sum_{i=1}^{n}S_p(Y_i,t)\omega_p(g(X_i,\widehat{\theta}))K\left(\frac{g(X_i,\widehat{\theta})-g}{h_n}\right),$$

$$(4)\quad \widehat{\Omega}^{\ell}_{R_p,3}(g,t)=\frac{\widehat{A}^{\ell}_{R_p,3}(g,t)}{\widehat{f_g}(g)},\quad \widehat{A}^{\ell}_{R_p,3}(g,t)\equiv\frac{1}{n\cdot h_n^D}\sum_{i=1}^{n}S_p(Y_i,t)\frac{\partial\omega_p(g(X_i,\widehat{\theta}))}{\partial\theta_\ell}K\left(\frac{g(X_i,\widehat{\theta})-g}{h_n}\right),$$

From here, we have

$$\begin{aligned}
\frac{\partial\widehat{\Omega}^{d,\ell}_{f_g}(g)}{\partial g_d}&=\frac{\partial\widehat{A}^{d,\ell}_{f_g}(g)}{\partial g_d}\cdot\frac{1}{\widehat{f_g}(g)}-\frac{\widehat{A}_{f_g}(g)}{\widehat{f_g}(g)^2}\cdot\frac{\partial\widehat{f_g}(g)}{\partial g_d},\\
\frac{\partial\widehat{\Omega}^{d,\ell}_{R_p,1}(g,t)}{\partial g_d}&=\frac{\partial\widehat{A}^{d,\ell}_{R_p,1}(g,t)}{\partial g_d}\cdot\frac{1}{\widehat{f_g}(g)}-\frac{\widehat{A}^{d,\ell}_{R_p,1}(g,t)}{\widehat{f_g}(g)^2}\cdot\frac{\partial\widehat{f_g}(g)}{\partial g_d},\\
\frac{\partial\widehat{\Omega}_{R_p,2}(g,t)}{\partial g_d}&=\frac{\partial\widehat{A}_{R_p,2}(g,t)}{\partial g_d}\cdot\frac{1}{\widehat{f_g}(g)}-\frac{\widehat{A}_{R_p,2}(g,t)}{\widehat{f_g}(g)^2}\cdot\frac{\partial\widehat{f_g}(g)}{\partial g_d},
\end{aligned}$$

Under Assumptions 1-4, for each d, ℓ and p we have that, uniformly over $\mathcal{F}$,

$$\begin{aligned}
\sup_{x\in\mathcal{X}}\left|\widehat{\Omega}^{d,\ell}_{f_g}(g(x,\widehat{\theta}))-\Omega^{d,\ell}_{f_g}(g(x,\theta^*_F))\right|&=o_p(1),\\
\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\widehat{\Omega}^{d,\ell}_{R_p,1}(g(x,\widehat{\theta}),t)-\Omega^{d,\ell}_{R_p,1}(g(x,\theta^*_F),t)\right|&=o_p(1),\\
\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\widehat{\Omega}_{R_p,2}(g(x,\widehat{\theta}),t)-\Omega_{R_p,2}(g(x,\theta^*_F),t)\right|&=o_p(1),\\
\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\widehat{\Omega}^{\ell}_{R_p,3}(g(x,\widehat{\theta}),t)-\Omega^{\ell}_{R_p,3}(g(x,\theta^*_F),t)\right|&=o_p(1),
\end{aligned}\tag{S4.1.2}$$

and,

$$
\begin{aligned}
\sup_{x\in\mathcal{X}}\left|\frac{\partial\widehat{\Omega}^{d,\ell}_{f_g}(g(x,\widehat{\theta}))}{\partial g_d}-\frac{\partial\Omega^{d,\ell}_{f_g}(g(x,\theta^*_F))}{\partial g_d}\right| &= o_p(1),\\
\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\frac{\partial\widehat{\Omega}^{d,\ell}_{R_p,1}(g(x,\widehat{\theta}),t)}{\partial g_d}-\frac{\partial\Omega^{d,\ell}_{R_p,1}(g(x,\theta^*_F),t)}{\partial g_d}\right| &= o_p(1),\\
\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\frac{\partial\widehat{\Omega}_{R_p,2}(g(x,\widehat{\theta}),t)}{\partial g_d}-\frac{\partial\Omega_{R_p,2}(g(x,\theta^*_F),t)}{\partial g_d}\right| &= o_p(1).
\end{aligned}
\tag{S4.1.3}
$$

Next, we estimate $\Xi_{f_g}(x,\theta^*_F)$ and $\Xi_{R_p}(x,t,\theta^*_F)$, where these functionals are described in (A-1) and $\Xi_{Q_p}(x,t,\theta^*_F)$, where this functional is described in equation (A-3). Our estimators are,

$$
\begin{aligned}
\widehat{\Xi}_{\ell,f_g}(x,\widehat{\theta}) &\equiv \sum_{d=1}^{D}\left(\frac{\partial g_d(x,\widehat{\theta})}{\partial\theta_\ell}\cdot\frac{\partial\widehat{f}_g(g(x,\widehat{\theta}))}{\partial g_d}-\frac{\partial\left[\widehat{\Omega}^{d,\ell}_{f_g}(g(x,\widehat{\theta}))\widehat{f}_g(g(x,\widehat{\theta}))\right]}{\partial g_d}\right),\\
\widehat{\Xi}_{f_g}(x,\widehat{\theta}) &\equiv \left(\widehat{\Xi}_{1,f_g}(x,\widehat{\theta}),\ldots,\widehat{\Xi}_{k,f_g}(x,\widehat{\theta})\right),\\
\widehat{\Xi}_{\ell,R_p}(x,t,\widehat{\theta}) &\equiv \sum_{d=1}^{D}\left(\frac{\partial\left[\widehat{\Omega}_{R_p,2}(g(x,\widehat{\theta}),t)\widehat{f}_g(g(x,\widehat{\theta}))\right]}{\partial g_d}\cdot\frac{\partial g_d(x,\widehat{\theta})}{\partial\theta_\ell}-\frac{\partial\left[\widehat{\Omega}^{d,\ell}_{R_p,1}(g(x,\widehat{\theta}),t)\widehat{f}_g(g(x,\widehat{\theta}))\right]}{\partial g_d}\right.\\
&\qquad\left.+\widehat{\Omega}^{\ell}_{R_p,3}(g(x,\widehat{\theta}),t)\cdot\widehat{f}_g(g(x,\widehat{\theta}))\right),\\
\widehat{\Xi}_{R_p}(x,t,\widehat{\theta}) &\equiv \left(\widehat{\Xi}_{1,R_p}(x,t,\widehat{\theta}),\ldots,\widehat{\Xi}_{k,R_p}(x,t,\widehat{\theta})\right),\\
\widehat{\Xi}_{Q_p}(x,t,\widehat{\theta}) &\equiv \frac{\widehat{\Xi}_{R_p}(x,t,\widehat{\theta})-\widehat{Q}_p(x,t,\widehat{\theta})\cdot\widehat{\Xi}_{f_g}(x,\widehat{\theta})}{\widehat{f}_g(g(x,\widehat{\theta}))},
\end{aligned}
$$

where $\widehat{Q}_p(x,t,\widehat{\theta})$ is the estimator we described in (12). Under the conditions in Assumptions 1-4 (see (S4.1.2)-(S4.1.3)), we obtain that, for each p and uniformly over $\mathcal{F}$,

$$
\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left\|\widehat{\Xi}_{Q_p}(x,t,\widehat{\theta})-\Xi_{Q_p}(x,t,\theta^*_F)\right\| = o_p(1). \tag{S4.1.4}
$$

From here we are ready to estimate the functional $\Xi_{T_1,F}$ described in Proposition 1A, equation (A-12)

$$
\begin{aligned}
\widehat{\Xi}^p_{T_1} &\equiv \frac{1}{n}\sum_{i=1}^{n}\frac{\partial\mathcal{B}\left(\widehat{Q}(X_i,t_i,\widehat{\theta})\right)}{\partial\widehat{Q}_p}\mathbb{1}\left\{\mathcal{B}\left(\widehat{Q}(X_i,t_i,\widehat{\theta})\right)\geq -b_n\right\}\phi(X_i,t_i)\widehat{\Xi}_{Q_p}(X_i,t_i,\widehat{\theta}),\\
\widehat{\Xi}_{T_1} &\equiv \sum_{p=1}^{P}\widehat{\Xi}^p_{T_1},
\end{aligned}
$$

And for a given t we estimate the functional $\Xi_{T_0,F}(t)$ described in Proposition 1B, equation (A-16) with

$$\widehat{\Xi}^{p}_{T_0}(t) \equiv \frac{1}{n}\sum_{i=1}^{n}\frac{\partial\mathcal{B}\left(\widehat{Q}(X_i,t,\widehat{\theta})\right)}{\partial\widehat{Q}_p}\mathbb{1}\left\{\mathcal{B}\left(\widehat{Q}(X_i,t,\widehat{\theta})\right)\geq -b_n\right\}\phi(X_i,t)\widehat{\Xi}_{Q_p}(X_i,t,\widehat{\theta}),$$

$$\widehat{\Xi}_{T_0}(t) \equiv \sum_{p=1}^{P}\widehat{\Xi}^{p}_{T_0}(t),$$

Under the conditions in Assumptions 1-4 (see equations (S4.1.2)-(S4.1.4)), uniformly over $\mathcal{F}$ we have

$$\begin{aligned}\left\|\widehat{\Xi}_{T_1}-\Xi_{T_1,F}\right\| &= o_p(1),\\ \sup_{t\in\mathcal{T}}\left\|\widehat{\Xi}_{T_0}(t)-\Xi_{T_0,F}(t)\right\| &= o_p(1),\\ \left\|\int_t\widehat{\Xi}_{T_0}(t)d\mathcal{W}(t)-\int_t\Xi_{T_0,F}(t)d\mathcal{W}(t)\right\| &= o_p(1)\end{aligned} \tag{S4.1.5}$$

Next, for a given x, t, we estimate

$$\widehat{\Gamma}_p(x,t,\widehat{\theta}) = \frac{\frac{1}{n\cdot h_n^D}\sum_{i=1}^{n}S_p(Y_i,t)K\left(\frac{\Delta g(X_i,x,\widehat{\theta})}{h_n}\right)}{\widehat{f_g}(g(x,\widehat{\theta}))}.$$

For a given (y,g), we estimate the functional $\Omega^{p}_{T_1}(y,g)$ described in Assumption 7A, equation (17) with

$$\widehat{\Omega}^{p}_{T_1}(y,g) = \frac{\frac{1}{n\cdot h_n^D}\sum_{i=1}^{n}\left(S_p(y,X_i,t_i)-\widehat{\Gamma}_p(X_i,t_i,\widehat{\theta})\right)\frac{\partial\mathcal{B}\left(\widehat{Q}(X_i,t_i,\widehat{\theta})\right)}{\partial Q_p}\phi(X_i,t_i)\mathbb{1}\left\{\mathcal{B}\left(\widehat{Q}(X_i,t_i,\widehat{\theta})\right)\geq -b_n\right\}K\left(\frac{g(X_i,\widehat{\theta})-g}{h_n}\right)}{\widehat{f_g}(g)}$$

And for a given (y,g,t), we estimate the functional $\Omega^{p}_{T_0}(y,t,g)$ described in Assumption 7B, equation (21) with

$$\widehat{\Omega}^{p}_{T_0}(y,t,g) = \frac{\frac{1}{n\cdot h_n^D}\sum_{i=1}^{n}\left(S_p(y,X_i,t)-\widehat{\Gamma}_p(X_i,t,\widehat{\theta})\right)\frac{\partial\mathcal{B}\left(\widehat{Q}(X_i,t,\widehat{\theta})\right)}{\partial Q_p}\phi(X_i)\mathbb{1}\left\{\mathcal{B}\left(\widehat{Q}(X_i,t,\widehat{\theta})\right)\geq -b_n\right\}K\left(\frac{g(X_i,\widehat{\theta})-g}{h_n}\right)}{\widehat{f_g}(g)}$$

Under the conditions in Assumptions 1-6 and 7A (i.e, the conditions of Proposition 1A), for each

p and uniformly over $\mathcal{F}$ we have

$$\sup_{(y,x)\in\mathcal{S}_{Y,X}}\left|\widehat{\Omega}^{p}_{T_1}\left(y,g(x,\widehat{\theta})\right)\cdot\omega_p(g(x,\widehat{\theta}))-\Omega^{p}_{T_1}\left(y,g(x,\theta^*_F)\right)\cdot\omega_p(g(x,\theta^*_F))\right|=o_p(1),$$

$$\sup_{(x,t)\in\mathcal{S}_X\times\mathcal{T}}\left|\mathcal{B}\left(\widehat{Q}(x,t,\widehat{\theta})\right)\mathbb{1}\left\{\mathcal{B}\left(\widehat{Q}(x,t,\widehat{\theta})\right)\geq -b_n\right\}\phi(x,t)-\left(\mathcal{B}\left(\widehat{Q}(x,t,\theta^*_F)\right)\right)_+\phi(x,t)\right|=o_p(1), \tag{S4.1.6}$$

$$\left|\widehat{T}_1-T_{1,F}\right|=o_p(1)$$

(the last line follows directly from equation (18) in Proposition 1A). And under the conditions in Assumptions 1-6 and 7B (i.e, the conditions of Proposition 1B), for each p and uniformly over $\mathcal{F}$ we have

$$\sup_{\substack{(y,x)\in\mathcal{S}_{Y,X}\\ t\in\mathcal{T}}}\left|\widehat{\Omega}^{p}_{T_0}\left(y,t,g(x,\widehat{\theta})\right)\cdot\omega_p(g(x,\widehat{\theta}))-\Omega^{p}_{T_1}\left(y,t,g(x,\theta^*_F)\right)\cdot\omega_p(g(x,\theta^*_F))\right|=o_p(1),$$

$$\sup_{(y,x)\in\mathcal{S}_{Y,X}}\left|\int_t\widehat{\Omega}^{p}_{T_0}\left(y,t,g(x,\widehat{\theta})\right)d\mathcal{W}(t)\cdot\omega_p(g(x,\widehat{\theta}))-\int_t\Omega^{p}_{T_0}\left(y,t,g(x,\theta^*_F)\right)d\mathcal{W}(t)\cdot\omega_p(g(x,\theta^*_F))\right|=o_p(1),$$

$$\sup_{(x,t)\in\mathcal{S}_X\times\mathcal{T}}\left|\mathcal{B}\left(\widehat{Q}(x,t,\widehat{\theta})\right)\mathbb{1}\left\{\mathcal{B}\left(\widehat{Q}(x,t,\widehat{\theta})\right)\geq -b_n\right\}\phi(x)-\left(\mathcal{B}\left(\widehat{Q}(x,t,\theta^*_F)\right)\right)_+\phi(x)\right|=o_p(1),$$

$$\sup_{x\in\mathcal{S}_X}\left|\int_t\mathcal{B}\left(\widehat{Q}(x,t,\widehat{\theta})\right)\mathbb{1}\left\{\mathcal{B}\left(\widehat{Q}(x,t,\widehat{\theta})\right)\geq -b_n\right\}d\mathcal{W}(t)\phi(x)-\int_t\left(\mathcal{B}\left(\widehat{Q}(x,t,\theta^*_F)\right)\right)_+d\mathcal{W}(t)\phi(x)\right|=o_p(1),$$

$$\left|\widehat{T}_2-T_{2,F}\right|=o_p(1) \tag{S4.1.7}$$

(the last line follows directly from equation (22) in Proposition 1B). The last component for our estimators for the influence functions $\psi^{T_1}_F(Z)$ and $\psi^{T_2}_F(Z)$ is an estimator of the influence function $\psi^{\theta}_F(Z)$ for the estimator $\widehat{\theta}$. From Assumption 8, we have an estimator $\widehat{\psi}^{\theta}(Z)$ for the influence function $\psi^{\theta}_F(Z)$ that satisfies,

$$\frac{1}{n}\sum_{i=1}^{n}\left\|\widehat{\psi}^{\theta}(Z_i)-\psi^{\theta}_F(Z_i)\right\|^2=o_p(1)\quad\text{uniformly over }\mathcal{F}.$$

Our estimator for the influence function $\psi^{T_1}_F(V)$ of the statistic $\widehat{T}_1$ described in equation (A-13) is,

$$\widehat{\psi}^{T_1}(V_i)\equiv\left(\mathcal{B}\left(\widehat{Q}(X_i,t_i,\widehat{\theta})\right)\mathbb{1}\left\{\mathcal{B}\left(\widehat{Q}(X_i,t_i,\widehat{\theta})\right)\geq -b_n\right\}\phi(X_i,t_i)-\widehat{T}_1\right)+\sum_{p=1}^{P}\widehat{\Omega}^{p}_{T_1}(Y_i,g(X_i,\widehat{\theta}))\cdot\omega(g(X_i,\widehat{\theta}))$$
$$+\widehat{\Xi}_{T_1}\widehat{\psi}^{\theta}(Z_i)$$

And our estimator for the influence function $\psi^{T_2}_F(V)$ of the statistic $\widehat{T}_2$ described in equation (A-17) is,

$$\widehat{\psi}^{T_2}(V_i) \equiv \left(\int_t \mathcal{B}\left(\widehat{Q}(X_i,t,\widehat{\theta})\right) \mathbb{1}\left\{\mathcal{B}\left(\widehat{Q}(X_i,t,\widehat{\theta})\right) \geq -b_n\right\} d\mathcal{W}(t)\phi(X_i) - \widehat{T}_2 \right)$$
$$+ \sum_{p=1}^{P} \int_t \widehat{\Omega}^p_{T_0}(Y_i,t,g(X_i,\widehat{\theta}))d\mathcal{W}(t)\cdot\omega(g(X_i,\widehat{\theta})) + \int_t \widehat{\Xi}_{T_0}(t)d\mathcal{W}(t)\widehat{\psi}^{\theta}(Z_i)$$

From the results shown in equations (S4.1.5) and (S4.1.6), we have that under the conditions in Assumptions 1-6, 7A and 8,

$$\frac{1}{n}\sum_{i=1}^{n}\left|\widehat{\psi}^{T_1}(V_i) - \psi_F^{T_1}(V_i)\right|^2 = o_p(1) \quad \text{uniformly over } \mathcal{F}.$$

And from the results shown in equations (S4.1.5) and (S4.1.7), we have that under the conditions in Assumptions 1-6, 7B and 8,

$$\frac{1}{n}\sum_{i=1}^{n}\left|\widehat{\psi}^{T_2}(V_i) - \psi_F^{T_2}(V_i)\right|^2 = o_p(1) \quad \text{uniformly over } \mathcal{F}.$$

S5 Examples of estimators that satisfy Assumption 1

In the examples that follow, all expectations are taken with respect to a generic distribution $F \in \mathcal{F}$. To simplify our exposition we omit denoting explicitly the dependence of these functionals on F.

S5.1 A convenient definition

Take a collection of column vectors $(v_\ell)_{\ell=1}^d$ where $v_\ell \in \mathbb{R}^d$ for each ℓ, and let

$$\underbrace{v \equiv (v_1', v_2', \ldots, v_d')'}_{d^2\times 1}.$$

For any such v we will define

$$\underbrace{H_d(v)}_{d\times d} \equiv \begin{pmatrix} v_1' \\ v_2' \\ \vdots \\ v_d' \end{pmatrix} \text{ and, when it exists, we will denote } M_d(v) \equiv H_d(v)^{-1}. \tag{S5.1.1}$$

S5.2 An OLS estimator (proof of Result OLS in Appendix A4.1)

Consider an iid sample $(Z_{1i}, Z_{2i})_{i=1}^n$ where $Z_i \equiv (Z_{1i}, Z_{2i}) \sim F$, with $Z_{1i} \in \mathbb{R}$ and $Z_{2i} \in \mathbb{R}^k$. Denote the ℓ^{th} element in Z_{2i} as $Z_{2i,\ell}$. Define

$$\underbrace{\overline{G}_\ell}_{k\times 1} \equiv \frac{1}{n}\sum_{i=1}^n Z_{2i}Z_{2i,\ell}, \quad \underbrace{\lambda_{\ell,F}}_{k\times 1} \equiv E_F[Z_2 Z_{2,\ell}],$$
$$\overline{G} \equiv \underbrace{\left(\overline{G}_1', \overline{G}_2', \ldots, \overline{G}_k'\right)'}_{k^2\times 1} \quad \text{and} \quad \lambda_F \equiv \underbrace{\left(\lambda_{1,F}', \lambda_{2,F}', \ldots, \lambda_{k,F}'\right)'}_{k^2\times 1}.$$

Assumption LS *$\exists$ $\overline{M}_{z_2z_2}$ such that $\left\|(E_F[Z_2Z_2'])^{-1}\right\| \le \overline{M}_{z_2z_2}$ $\forall$ $F \in \mathcal{F}$. For some $q \ge 2$, there exist $\overline{\mu}_{z_2z_2}$ and $\overline{\mu}_{z_2\nu}$ such that, for each ℓ, m,*

$$E_F\left[\left|Z_{2,\ell}Z_{2,m} - E_F[Z_{2,\ell}Z_{2,m}]\right|^q\right] \le \overline{\mu}_{z_2z_2} \quad \text{and} \quad E_F\left[\left|Z_{2,\ell}\nu\right|^q\right] \le \overline{\mu}_{z_2\nu} \quad \forall\ F \in \mathcal{F}.$$

There exists $\overline{M}_\lambda$ such that $\|M_k(\lambda_F)\| \le \overline{M}_\lambda$ for all $F \in \mathcal{F}$ and there exist $K_1 > 0$, $K_2 > 0$ and $\alpha > 0$ such that, for any $F \in \mathcal{F}$ and $v \in \mathbb{R}^{k^2}$,

$$\|v - \lambda_F\| \le K_1 \quad \Longrightarrow \quad \|M_k(v) - M_k(\lambda_F)\| \le K_2 \cdot \|v - \lambda_F\|^\alpha,$$

and there exists $K_3 < \infty$ such that

$$\sup_{v:\|v-\lambda_F\|\le K_1}\left\{\|M_k(v) - M_k(\lambda_F)\|\right\} \le K_3 \quad \forall\ F \in \mathcal{F}$$

Consider the OLS estimator

$$\widehat{\theta} = \left(\frac{1}{n}\sum_{i=1}^n Z_{2i}Z_{2i}'\right)^{-1}\frac{1}{n}\sum_{i=1}^n Z_{2i}Z_{1i},$$

and let
$$\theta_F^* \equiv (E_F[Z_2Z_2'])^{-1} \cdot E_F[Z_2Z_1],$$

and let us express $Z_1 = Z_2'\theta_F^* + \left(Z_1 - Z_2'\theta_F^*\right) \equiv Z_2'\theta_F^* + \nu$, where $\nu \equiv \left(Z_1 - Z_2'\theta_F^*\right)$. Note that $E_F[Z_2\nu] = 0$ by the definition of θ_F^*. In the usual linear regression model where we assume a structural relationship given by $Z_1 = Z_2'\beta_0 + \varepsilon$ with $E_F[Z_2\varepsilon] = 0$ $\forall$ $F \in \mathcal{F}$, we would have $\nu = \varepsilon$ and $\theta_F^* = \beta_0$ for all $F \in \mathcal{F}$. Let $M_k(\cdot)$ be as defined in (S5.1.1) and note that $M_k(\lambda_F) = \left(E_F[Z_2 Z_2']\right)^{-1}$. We have

$$\begin{aligned} \widehat{\theta} &= \theta_F^* + M_k(\lambda_F)\frac{1}{n}\sum_{i=1}^n Z_{2i}\nu_i + \left(M_k(\overline{G}) - M_k(\lambda_F)\right)\cdot\frac{1}{n}\sum_{i=1}^n Z_{2i}\nu_i \\ &\equiv \theta_F^* + \frac{1}{n}\sum_{i=1}^n \psi_F^\theta(Z_i) + \varepsilon_n^\theta, \quad \text{where} \\ \psi_F^\theta(Z_i) &\equiv M_k(\lambda_F)\cdot Z_{2i}\nu_i = (E_F[Z_2Z_2'])^{-1}\cdot Z_{2i}\nu_i, \\ \varepsilon_n^\theta &\equiv \left(M_k(\overline{G}) - M_k(\lambda_F)\right)\cdot\frac{1}{n}\sum_{i=1}^n Z_{2i}\nu_i. \end{aligned} \tag{S5.2.1}$$

Note that $E_F\left[\psi_F(Z_i)^\theta\right] = 0$. Take any $c > 0$. Using the conditions in Assumption LS,

$$\begin{aligned} \mathbb{1}\left\{\|\varepsilon_n^\theta\| \geq c\right\} &\leq \mathbb{1}\left\{\left\|M_k(\overline{G}) - M_k(\lambda_F)\right\|\cdot\left\|\frac{1}{n}\sum_{i=1}^n Z_{2i}\nu_i\right\| \geq c\right\} \\ &= \underbrace{\mathbb{1}\left\{\left\|M_k(\overline{G}) - M_k(\lambda_F)\right\|\cdot\left\|\frac{1}{n}\sum_{i=1}^n Z_{2i}\nu_i\right\| \geq c\right\}\cdot\mathbb{1}\left\{\left\|M_k(\overline{G}) - M_k(\lambda_F)\right\| \leq K_3\right\}}_{\leq \mathbb{1}\left\{K_3\cdot\left\|\frac{1}{n}\sum_{i=1}^n Z_{2i}\nu_i\right\|\geq c\right\}} \\ &\quad + \underbrace{\mathbb{1}\left\{\left\|M_k(\overline{G}) - M_k(\lambda_F)\right\|\cdot\left\|\frac{1}{n}\sum_{i=1}^n Z_{2i}\nu_i\right\| \geq c\right\}\cdot\mathbb{1}\left\{\left\|M_k(\overline{G}) - M_k(\lambda_F)\right\| > K_3\right\}}_{\leq \mathbb{1}\left\{\|\overline{G} - \lambda_F\| \geq K_1\right\}} \\ &\leq \mathbb{1}\left\{\left\|\frac{1}{n}\sum_{i=1}^n Z_{2i}\nu_i\right\| \geq \frac{c}{K_3}\right\} + \mathbb{1}\left\{\left\|\overline{G} - \lambda_F\right\| \geq K_1\right\} \\ &\leq \mathbb{1}\left\{\left\|\frac{1}{n}\sum_{i=1}^n Z_{2i}\nu_i\right\| \geq \left(\frac{c}{K_3}\right)\wedge K_1\right\} + \mathbb{1}\left\{\left\|\overline{G} - \lambda_F\right\| \geq \left(\frac{c}{K_3}\right)\wedge K_1\right\} \end{aligned} \tag{S5.2.2}$$

Take $b > 0$. Assumption LS and Chebyshev's inequality imply that, for all ℓ, m in $1, \ldots, k$,

$$\sup_{F\in\mathcal{F}} P_F\left(\left|\frac{1}{n}\sum_{i=1}^n\left(Z_{2i,\ell}Z_{2i,m} - E_F\left[Z_{2i,\ell}Z_{2i,m}\right]\right)\right| \geq b\right) \leq \frac{\overline{\mu}_{z_2z_2}}{\left(n^{1/2}\cdot b\right)^q}$$

Therefore,

$$\sup_{F\in\mathcal{F}} P_F\left(\left\|\overline{G} - \lambda_F\right\| \geq b\right) \leq \frac{\overline{M}_1}{\left(n^{1/2}\cdot b\right)^q} \tag{S5.2.3}$$

where $\overline{M}_1$ depends only on $\overline{\mu}_{z_2z_2}$ and k. Similarly, Assumption LS also implies that there exists a constant $\overline{M}_2$ which depends only on $\overline{\mu}_{z_2\nu}$ and k such that, for any $b>0$

$$\sup_{F\in\mathcal{F}} P_F\left(\left\|\frac{1}{n}\sum_{i=1}^{n} Z_{2i}\nu_i\right\| \geq b\right) \leq \frac{\overline{M}_2}{\left(n^{1/2}\cdot b\right)^q} \tag{S5.2.4}$$

Combining (S5.2.3) and (S5.2.4) with (S5.2.2), we have that for any $c>0$,

$$\sup_{F\in\mathcal{F}} P_F\left(\left\|\varepsilon_n^{\theta}\right\| \geq c\right) \leq \frac{\overline{M}_1+\overline{M}_2}{\left(n^{1/2}\cdot\left(\left(\frac{c}{K_3}\right)\wedge K_1\right)\right)^q} = o\left(\frac{1}{n^{1/2+\delta}}\right) \quad \forall\ 0<\delta<\frac{q-1}{2} \tag{S5.2.5}$$

Take $0<\delta<\frac{q-1}{2}$ and consider a sequence $c_n>0$ such that $n^{\frac{q-1-2\delta}{2q}}\cdot c_n \longrightarrow \infty$. Then, the result in (S5.2.5) would still hold for c_n. Thus,

$$\sup_{F\in\mathcal{F}} P_F\left(\left\|\varepsilon_n^{\theta}\right\| \geq c_n\right) = o\left(\frac{1}{n^{1/2+\delta}}\right) \quad \forall\ c_n : n^{\frac{q-1-2\delta}{2q}}\cdot c_n \longrightarrow \infty, \quad 0<\delta<\frac{q-1}{2}$$

Take any $b>0$. From our previous results,

$$\sup_{F\in\mathcal{F}} P_F\left(\left\|\frac{1}{n}\sum_{i=1}^{n}\psi_F^{\theta}(Z_i)\right\| \geq b\right) \leq \sup_{F\in\mathcal{F}} P_F\left(\left\|\frac{1}{n}\sum_{i=1}^{n} Z_{2i}\nu_i\right\| \geq \frac{b}{\overline{M}_\lambda}\right) \leq \frac{\overline{M}_2}{\left(n^{1/2}\cdot\left(\frac{b}{\overline{M}_\lambda}\right)\right)^q}$$

Thus, going back to the linear representation result in (S5.2.1), we have that for any $c>0$,

$$\begin{aligned}\sup_{F\in\mathcal{F}} P_F\left(\left\|\widehat{\theta}-\theta_F^*\right\| \geq c\right) &\leq \sup_{F\in\mathcal{F}} P_F\left(\left\|\frac{1}{n}\sum_{i=1}^{n}\psi_F^{\theta}(Z_i)\right\| \geq \frac{c}{2}\right) + \sup_{F\in\mathcal{F}} P_F\left(\left\|\varepsilon_n^{\theta}\right\| \geq \frac{c}{2}\right)\\ &\leq \frac{\overline{M}_2}{\left(n^{1/2}\cdot\left(\frac{c}{2\overline{M}_\lambda}\right)\right)^q} + \frac{\overline{M}_1+\overline{M}_2}{\left(n^{1/2}\cdot\left(\left(\frac{c}{2K_3}\right)\wedge K_1\right)\right)^q}\end{aligned}$$

And so,

$$\sup_{F\in\mathcal{F}} P_F\left(\left\|\widehat{\theta}-\theta_F^*\right\| \geq c\right) \longrightarrow 0 \quad \forall\ c>0.$$

Thus $\left\|\widehat{\theta}-\theta_F^*\right\| = o_p(1)$ uniformly over $\mathcal{F}$. Recall from its definition in (S5.2.1) that

$$\left\|\varepsilon_n^{\theta}\right\| \leq \left\|M_k(\overline{G})-M_k(\lambda_F)\right\|\cdot\left\|\frac{1}{n}\sum_{i=1}^{n} Z_{2i}\nu_i\right\|.$$

From (S5.2.4), we have

$$\left\| \frac{1}{n} \sum_{i=1}^{n} Z_{2i} v_i \right\| = O_p\left(n^{-1/2}\right), \quad \text{uniformly over } \mathcal{F}. \tag{S5.2.6}$$

Now let us analyze $\left\| M_k(\overline{G}) - M_k(\lambda_F) \right\|$. Take any $b > 0$. From the conditions in Assumption LS,

$$\begin{aligned}
\mathbb{1}\left\{ \left\| M_k(\overline{G}) - M_k(\lambda_F) \right\| \geq b \right\} &\leq \max\left(\mathbb{1}\left\{ K_2 \cdot \left\| \overline{G} - \lambda_F \right\|^{\alpha} \geq b \right\}, \mathbb{1}\left\{ \left\| \overline{G} - \lambda_F \right\| \geq K_1 \right\} \right) \\
&\leq \mathbb{1}\left\{ \left\| \overline{G} - \lambda_F \right\| \geq \left(\frac{b}{K_2} \right)^{1/\alpha} \wedge K_1 \right\}
\end{aligned}$$

Take $\tau > 0$. Then, from (S5.2.3),

$$\sup_{F \in \mathcal{F}} P_F\left(\left\| M_k(\overline{G}) - M_k(\lambda_F) \right\| \geq n^{-\tau} \cdot b \right) \leq \frac{\overline{M}_1}{\left(n^{1/2} \cdot \left(\left(\frac{n^{-\tau} \cdot b}{K_2} \right)^{1/\alpha} \wedge K_1 \right) \right)^{q}} \longrightarrow 0 \quad \forall\, \tau < \frac{\alpha}{2},$$

which means,

$$\left\| M_k(\overline{G}) - M_k(\lambda_F) \right\| = o_p\left(n^{-\tau}\right) \quad \forall\, \tau < \frac{\alpha}{2}, \quad \text{uniformly over } \mathcal{F}.$$

Combining (S5.2.6) and the previous expression, we have that for any $0 < \tau < \frac{\alpha}{2}$,

$$\left\| \varepsilon_n^{\theta} \right\| = o_p\left(\frac{1}{n^{1/2+\tau}} \right), \quad \text{uniformly over } \mathcal{F}$$

Together, (S5.2.1), (S5.2.5) and the previous expression show that the conditions in Assumption 1 are satisfied, with $\psi_F^{\theta}(Z_i) = \left(E_F[Z_2 Z_2'] \right)^{-1} \cdot Z_{2i} v_i$, $r_n = n^{1/2}$, $0 < \tau < \alpha/2$, and $0 < \overline{\delta} < (q-1)/2$. **This proves Result OLS in Appendix A4.1 of the paper.** ■

S5.3 GMM

Consider an iid sample $(Z_i)_{i=1}^n$ where $Z_i \sim F$. Let $\mathcal{S}_Z$ denote the support of Z and assume for simplicity that $\mathcal{S}_Z$ is the same for each $F \in \mathcal{F}$. Let us focus on an exactly-identified GMM model where $\theta \in \mathbb{R}^k$ and

$$\underbrace{g(Z_i, \theta)}_{k \times 1} \equiv \left(g_1(Z_i, \theta), g_2(Z_i, \theta), \ldots, g_k(Z_i, \theta) \right)'$$

is a collection of parametric moment functions satisfying $E_F[g(Z, \theta_F^*)] = 0$. We denote θ_F^* possibly as a functional of F for generality (to include, e.g, the OLS example described above). For simplicity we focus on an exactly-identified GMM model with as many moment restrictions as parameters which includes, e.g, MLE and NLS as special cases. Let Θ denote the parameter space and let

$$\overline{g}_\ell(\theta) \equiv \frac{1}{n}\sum_{i=1}^{n} g_\ell(Z_i,\theta), \quad \text{and} \quad \underbrace{\overline{g}(\theta)}_{k\times 1} \equiv \left(\overline{g}_1(\theta), \overline{g}_2(\theta), \ldots, \overline{g}_k(\theta)\right)'$$

denote the sample moments. Suppose the GMM estimator $\widehat{\theta} \in \Theta$ is characterized by the condition $\overline{g}(\widehat{\theta}) = 0$. Suppose the moment functions are differentiable with respect to θ and for the ℓ^{th} moment function g_ℓ denote

$$\underbrace{G_\ell(z,\theta)}_{k\times 1} \equiv \left(\frac{\partial g_\ell(z,\theta)}{\partial \theta_1} \quad \frac{\partial g_\ell(z,\theta)}{\partial \theta_2} \quad \cdots \quad \frac{\partial g_\ell(z,\theta)}{\partial \theta_k}\right)',$$

$$\underbrace{\overline{G}_\ell(\theta)}_{k\times 1} \equiv \frac{1}{n}\sum_{i=1}^{n} G_\ell(Z_i,\theta) \quad \text{and} \quad \underbrace{\lambda_{\ell,F}(\theta)}_{k\times 1} = E_F\left[G_\ell(Z,\theta)\right]].$$

From here and the definition of $\widehat{\theta}$ we obtain the following mean value expression for the ℓ^{th} sample moment

$$0 = \overline{g}_\ell(\widehat{\theta}) = \overline{g}_\ell(\theta_F^*) + \overline{G}_\ell(\overline{\theta}_\ell)'(\widehat{\theta} - \theta_F^*) \quad \ell = 1,\ldots,k, \tag{S5.3.1}$$

where $\overline{\theta}_\ell$ belongs in the line segment connecting $\widehat{\theta}$ and θ_F^*. Take a collection $(\theta_\ell)_{\ell=1}^k$ where each $\theta_\ell \in \Theta$. This is a collection of k points in the parameter space Θ. For any such collection we will denote

$$\underset{\sim}{\theta} \equiv (\theta_1', \theta_2', \ldots, \theta_k')' \in \underbrace{\Theta \times \Theta \times \cdots \times \Theta}_{k \text{ products}} \equiv \Theta^k$$

In particular, we will denote $\underset{\sim}{\overline{\theta}} \equiv (\overline{\theta}_1', \overline{\theta}_2', \ldots, \overline{\theta}_k')'$, where $\overline{\theta}_\ell$ is as described in the mean-value approximation (S5.3.1), and $\underset{\sim}{\theta}_F^* \equiv \left(\theta_F^{*\prime}, \theta_F^{*\prime}, \ldots, \theta_F^{*\prime}\right)'$. For a given $\underset{\sim}{\theta} \equiv (\theta_1', \theta_2', \ldots, \theta_k')' \in \Theta^k$ let

$$\underbrace{\overline{G}(\underset{\sim}{\theta})}_{k^2\times 1} \equiv \left(\overline{G}_1(\theta_1)', \overline{G}_2(\theta_2)', \ldots, \overline{G}_k(\theta_k)'\right)', \quad \underbrace{\lambda_F(\underset{\sim}{\theta})}_{k^2\times 1} = (\lambda_{1,F}(\theta_1)', \lambda_{2,F}(\theta_2)', \ldots, \lambda_{k,F}(\theta_k)')'.$$

Assumption GMM

(i) There exists an integer $q \geq 2$ and a constant $\overline{\mu}_g < \infty$ such that, for each $\ell = 1,\ldots,k$ and each $F \in \mathcal{F}$, $E_F\left[g_\ell(Z,\theta_F^*)^q\right] \leq \overline{\mu}_g$. There exists a nonnegative function $\overline{V}(\cdot)$ such that, for each $\ell = 1,\ldots,k$ and $m = 1,\ldots,k$,

$$\left\|\frac{\partial g_\ell(z,\theta)}{\partial \theta_m} - \frac{\partial g_\ell(z,\theta')}{\partial \theta_m}\right\| \leq \overline{V}(z)\cdot\left\|\theta - \theta'\right\| \quad \forall\, z \in \mathcal{S}_Z \quad \text{and} \quad \theta, \theta' \in \Theta,$$

and there exists $\overline{\mu}_{\overline{V}}$ such that $E_F\left[\overline{V}(Z)^{4q}\right] \leq \overline{\mu}_{\overline{V}}$ $\forall\, F \in \mathcal{F}$, where q is the integer described above.

(ii) Let H_k and M_k be as defined in (S5.1.1). *$\exists$ $\underline{d} > 0$, $\overline{M}_\lambda$, $K_3 > 0$ and $K_4 > 0$ and $\alpha_1 > 0$ such that, for every $F \in \mathcal{F}$,*

$$\inf_{\underset{\sim}{\theta} \in \Theta^k} \left| det(H_k(\lambda_F(\underset{\sim}{\theta}))) \right| \geq \underline{d} \quad \sup_{\underset{\sim}{\theta} \in \Theta^k} \left\| M_k(\lambda_F(\underset{\sim}{\theta})) \right\| \leq \overline{M}_\lambda$$

$$\forall\ \underset{\sim}{\theta} \in \Theta^k, \quad \|v - \lambda_F(\underset{\sim}{\theta})\| \leq K_3 \quad \Longrightarrow \quad \left\| M_k(\lambda_F(\underset{\sim}{\theta})) - M_k(v) \right\| \leq K_4 \cdot \|v - \lambda_F(\underset{\sim}{\theta})\|^{\alpha_1}.$$

And,

$$\sup_{\substack{v: \|v - \lambda_F(\underset{\sim}{\theta})\| \leq K_3 \\ \underset{\sim}{\theta} \in \Theta^k}} \left\{ \left\| M_k(\lambda_F(\underset{\sim}{\theta})) - M_k(v) \right\| \right\} \leq K_5 < \infty$$

(iii) $\exists$ $K_6 > 0$, $K_7 > 0$ and $\alpha_2 > 0$ such that, for every $F \in \mathcal{F}$,

$$\left\| \lambda_F(\underset{\sim}{\theta}) - \lambda_F(\underset{\sim}{\theta}^*_F) \right\| \leq K_7 \cdot \left\| \underset{\sim}{\theta} - \underset{\sim}{\theta}^*_F \right\|^{\alpha_2} \quad \forall\ \underset{\sim}{\theta} \in \Theta^k : \left\| \underset{\sim}{\theta} - \underset{\sim}{\theta}^*_F \right\| \leq K_6$$

As defined above, let $\underset{\sim}{\theta}^*_F \equiv \left(\theta^{*\prime}_F, \theta^{*\prime}_F, \ldots, \theta^{*\prime}_F \right)'$ and note that $M_k(\lambda_F(\underset{\sim}{\theta}^*_F)) = \left(E_F\left[\frac{\partial g(Z, \theta^*_F)}{\partial\theta\partial\theta'} \right] \right)^{-1}$. Combining the mean-value expressions in (S5.3.1) for each of the $\ell = 1, \ldots, k$ sample moments, we have

$$\overline{g}(\theta^*_F) + H_k\left(\overline{G}(\underset{\sim}{\overline{\theta}}) \right)\left(\widehat{\theta} - \theta^*_F \right) = 0.$$

From here,

$$\begin{aligned}
\widehat{\theta} =& \theta^*_F - M_k(\overline{G}(\underset{\sim}{\overline{\theta}})) \cdot \overline{g}(\theta^*_F) \\
\widehat{\theta} =& \theta^*_F - M_k(\lambda_F(\underset{\sim}{\theta}^*_F)) \cdot \overline{g}(\theta^*_F) - \left[M_k(\lambda_F(\underset{\sim}{\overline{\theta}})) - M_k(\lambda_F(\underset{\sim}{\theta}^*_F)) \right] \cdot \overline{g}(\theta^*_F) \\
& + \left[M_k(\lambda_F(\underset{\sim}{\overline{\theta}})) - M_k(\overline{G}(\underset{\sim}{\overline{\theta}})) \right] \cdot \overline{g}(\theta^*_F) \\
\equiv& \theta^*_F + \frac{1}{n} \sum_{i=1}^{n} \psi^\theta_F(Z_i) + \varepsilon^\theta_n, \quad \text{where}
\end{aligned} \tag{S5.3.2}$$

$$\begin{aligned}
\psi^\theta_F(Z_i) \equiv& - M_k(\lambda_F(\underset{\sim}{\theta}^*_F)) \cdot g(Z_i, \theta^*_F) = -\left(E_F\left[\frac{\partial g(Z, \theta^*_F)}{\partial\theta\partial\theta'} \right] \right)^{-1} \cdot g(Z_i, \theta^*_F), \\
\varepsilon^\theta_n \equiv& \left[M_k(\lambda_F(\underset{\sim}{\overline{\theta}})) - M_k(\lambda_F(\underset{\sim}{\theta}^*_F)) \right] \cdot \overline{g}(\theta^*_F) + \left[M_k(\lambda_F(\underset{\sim}{\overline{\theta}})) - M_k(\overline{G}(\underset{\sim}{\overline{\theta}})) \right] \cdot \overline{g}(\theta^*_F)
\end{aligned}$$

Consider the class of functions

$$\mathscr{G}_{\ell,m} = \left\{ f : \mathcal{S}_Z \to \mathbb{R} : f(z) = \frac{\partial g_\ell(z, \theta)}{\partial \theta_m} \text{ for some } \theta \in \Theta \right\}$$

By part (i) of Assumption GMM and Lemma 2.13 in Pakes and Pollard (1989), there exist positive constants A and V such that, for every the class $\mathscr{G}_{\ell,m}$ is Euclidean (A, V) for an envelope $\overline{W}(z)$

for which $\exists\ \overline{\mu}_{\overline{W}} < \infty$ such that $E_F\left[\overline{W}(Z)^{4q}\right] \leq \overline{\mu}_{\overline{W}}$ (where q is the integer described in Assumption GMM). The conditions in Result S1 are satisfied for the integer q described in Assumption GMM and there exists a constant $\overline{D} < \infty$ such that, for each $\ell, m \in 1,\ldots,k$ and for any $b > 0$,

$$\sup_{F\in\mathcal{F}} P_F\left(\sup_{\theta\in\Theta}\left|\frac{1}{n}\sum_{i=1}^{n}\left(\frac{\partial g_\ell(Z_i,\theta)}{\partial\theta_m} - E_F\left[\frac{\partial g_\ell(Z,\theta)}{\partial\theta_m}\right]\right)\right| \geq b\right) \leq \frac{\overline{D}}{\left(n^{1/2}\cdot b\right)^q} \tag{S5.3.3}$$

Next, note that for any $b > 0$, we have

$$\mathbb{1}\left\{\sup_{\underset{\sim}{\theta}\in\Theta^k}\left\|\overline{G}(\underset{\sim}{\theta}) - \lambda_F(\underset{\sim}{\theta})\right\| \geq b\right\} \leq \sum_{\ell=1}^{k}\sum_{m=1}^{k}\mathbb{1}\left\{\sup_{\theta\in\Theta}\left|\frac{1}{n}\sum_{i=1}^{n}\left(\frac{\partial g_\ell(Z_i,\theta)}{\partial\theta_m} - E_F\left[\frac{\partial g_\ell(Z,\theta)}{\partial\theta_m}\right]\right)\right| \geq m_k\cdot b\right\},$$

where m_k is a constant that depends only on k. Thus, from (S5.3.3) we have that there exists a constant $\overline{M}_1 < \infty$ such that, for all $b > 0$,

$$\sup_{F\in\mathcal{F}} P_F\left(\sup_{\underset{\sim}{\theta}\in\Theta^k}\left\|\overline{G}(\underset{\sim}{\theta}) - \lambda_F(\underset{\sim}{\theta})\right\| \geq b\right) \leq \frac{\overline{M}_1}{\left(n^{1/2}\cdot b\right)^q}, \tag{S5.3.4}$$

where q is the integer described in Assumption GMM. Using Chebyshev's inequality, part (i) of Assumption GMM also yields the following result for each $\ell = 1,\ldots,k$,

$$\sup_{F\in\mathcal{F}} P_F\left(\left|\frac{1}{n}\sum_{i=1}^{n} g_\ell(Z_i,\theta_F^*)\right| \geq b\right) \leq \frac{\overline{\mu}_g}{\left(n^{1/2}\cdot b\right)^q} \quad \forall\ b > 0. \tag{S5.3.5}$$

Next, note that

$$\mathbb{1}\left\{\left\|\frac{1}{n}\sum_{i=1}^{n} g(Z_i,\theta_F^*)\right\| \geq b\right\} \leq \sum_{\ell=1}^{n}\mathbb{1}\left\{\left|\frac{1}{n}\sum_{i=1}^{n} g_\ell(Z_i,\theta_F^*)\right| \geq c_k\cdot b\right\},$$

where c_k is a constant that depends only on k. By the conditions in Assumption GMM we have

$$\begin{aligned}\mathbb{1}\left\{\left\|\frac{1}{n}\sum_{i=1}^{n}\psi_F^{\theta}(Z_i)\right\| \geq b\right\} &\leq \mathbb{1}\left\{\left\|\frac{1}{n}\sum_{i=1}^{n} g(Z_i,\theta_F^*)\right\| \geq \frac{b}{\overline{M}_\lambda}\right\}\\ &\leq \sum_{\ell=1}^{n}\mathbb{1}\left\{\left|\frac{1}{n}\sum_{i=1}^{n} g_\ell(Z_i,\theta_F^*)\right| \geq c_k\cdot\left(\frac{b}{\overline{M}_\lambda}\right)\right\},\end{aligned}$$

From here and (S5.3.5) we have that there exist constants $\overline{M}_2 < \infty$ and $\overline{M}_3 < \infty$ such that, for all

$b > 0$,

$$\sup_{F\in\mathcal{F}} P_F\left(\left\|\frac{1}{n}\sum_{i=1}^{n} g(Z_i,\theta_F^*)\right\| \geq b\right) \leq \frac{\overline{M}_2}{\left(n^{1/2}\cdot b\right)^q}, \quad \text{and} \quad \sup_{F\in\mathcal{F}} P_F\left(\left\|\frac{1}{n}\sum_{i=1}^{n} \psi_F^{\theta}(Z_i)\right\| \geq b\right) \leq \frac{\overline{M}_3}{\left(n^{1/2}\cdot b\right)^q}, \tag{S5.3.6}$$

where q is the integer described in Assumption GMM. Note that the above result implies that

$$\left\|\overline{g}(\theta_F^*)\right\| \equiv \left\|\frac{1}{n}\sum_{i=1}^{n} g(Z_i,\theta_F^*)\right\| = O_p\left(n^{-1/2}\right), \quad \text{and} \quad \left\|\frac{1}{n}\sum_{i=1}^{n} \psi_F^{\theta}(Z_i)\right\| = O_p\left(n^{-1/2}\right), \quad \text{uniformly over } \mathcal{F}. \tag{S5.3.7}$$

Now, take any $c > 0$ and note from Assumption GMM that

$$\left\|\varepsilon_n^{\theta}\right\| \leq 2\overline{M}_{\lambda}\cdot\left\|\overline{g}(\theta_F^*)\right\| + \left\|M_k(\lambda_F(\overline{\underset{\sim}{\theta}})) - M_k(\overline{G}(\overline{\underset{\sim}{\theta}}))\right\|\cdot\left\|\overline{g}(\theta_F^*)\right\|.$$

Therefore,

$$\begin{aligned}
\mathbb{1}\left\{\|\varepsilon_n^{\theta}\| \geq c\right\} &\leq \max\left(\mathbb{1}\left\{2\overline{M}_{\lambda}\left\|\overline{g}(\theta_F^*)\right\| \geq \frac{c}{2}\right\}, \mathbb{1}\left\{\left\|M_k(\lambda_F(\overline{\underset{\sim}{\theta}})) - M_k(\overline{G}(\overline{\underset{\sim}{\theta}}))\right\|\cdot\left\|\overline{g}(\theta_F^*)\right\| \geq \frac{c}{2}\right\}\right) \\
&= \max\left(\mathbb{1}\left\{\left\|\overline{g}(\theta_F^*)\right\| \geq \frac{c}{4\overline{M}_{\lambda}}\right\}, \underbrace{\mathbb{1}\left\{\left\|M_k(\lambda_F(\overline{\underset{\sim}{\theta}})) - M_k(\overline{G}(\overline{\underset{\sim}{\theta}}))\right\|\cdot\left\|\overline{g}(\theta_F^*)\right\| \geq \frac{c}{2}\right\}}_{(III)}\right)
\end{aligned} \tag{S5.3.8}$$

Let us examine the term (III) in (S5.3.8). From the conditions in Assumption GMM, we have

$$\begin{aligned}
&\mathbb{1}\left\{\left\|M_k(\lambda_F(\overline{\underset{\sim}{\theta}})) - M_k(\overline{G}(\overline{\underset{\sim}{\theta}}))\right\|\cdot\left\|\overline{g}(\theta_F^*)\right\| \geq \frac{c}{2}\right\} = \\
&\underbrace{\mathbb{1}\left\{\left\|M_k(\lambda_F(\overline{\underset{\sim}{\theta}})) - M_k(\overline{G}(\overline{\underset{\sim}{\theta}}))\right\|\cdot\left\|\overline{g}(\theta_F^*)\right\| \geq \frac{c}{2}\right\}\cdot\mathbb{1}\left\{\left\|M_k(\lambda_F(\overline{\underset{\sim}{\theta}})) - M_k(\overline{G}(\overline{\underset{\sim}{\theta}}))\right\| \leq K_5\right\}}_{\leq \mathbb{1}\left\{\|\overline{g}(\theta_F^*)\| \geq \frac{c}{2K_5}\right\}} \\
&+ \underbrace{\mathbb{1}\left\{\left\|M_k(\lambda_F(\overline{\underset{\sim}{\theta}})) - M_k(\overline{G}(\overline{\underset{\sim}{\theta}}))\right\|\cdot\left\|\overline{g}(\theta_F^*)\right\| \geq \frac{c}{2}\right\}\cdot\mathbb{1}\left\{\left\|M_k(\lambda_F(\overline{\underset{\sim}{\theta}})) - M_k(\overline{G}(\overline{\underset{\sim}{\theta}}))\right\| > K_5\right\}}_{\leq \mathbb{1}\left\{\|\overline{G}(\overline{\underset{\sim}{\theta}}) - \lambda_F(\overline{\underset{\sim}{\theta}})\| \geq K_3\right\}}
\end{aligned}$$

Therefore,

$$
\begin{aligned}
\mathbb{1}\left\{\left\|M_k(\lambda_F(\overline{\underset{\sim}{\theta}}))-M_k(\overline{G}(\overline{\underset{\sim}{\theta}}))\right\|\cdot\left\|\overline{g}(\theta_F^*)\right\|\geq\frac{c}{2}\right\} &\leq \mathbb{1}\left\{\left\|\overline{g}(\theta_F^*)\right\|\geq\frac{c}{2K_5}\right\}+\mathbb{1}\left\{\left\|\overline{G}(\overline{\underset{\sim}{\theta}})-\lambda_F(\overline{\underset{\sim}{\theta}})\right\|\geq K_3\right\}\\
&\leq \mathbb{1}\left\{\left\|\overline{g}(\theta_F^*)\right\|\geq\left(\frac{c}{2K_5}\right)\wedge K_3\right\}+\mathbb{1}\left\{\left\|\overline{G}(\overline{\underset{\sim}{\theta}})-\lambda_F(\overline{\underset{\sim}{\theta}})\right\|\geq\left(\frac{c}{2K_5}\right)\wedge K_3\right\}\\
&\leq \mathbb{1}\left\{\left\|\overline{g}(\theta_F^*)\right\|\geq\left(\frac{c}{2K_5}\right)\wedge K_3\right\}+\mathbb{1}\left\{\sup_{\underset{\sim}{\theta}\in\Theta^k}\left\|\overline{G}(\underset{\sim}{\theta})-\lambda_F(\underset{\sim}{\theta})\right\|\geq\left(\frac{c}{2K_5}\right)\wedge K_3\right\}
\end{aligned}
\tag{S5.3.9}
$$

Combining (S5.3.8) and (S5.3.9), we have

$$
\mathbb{1}\left\{\left\|\varepsilon_n^\theta\right\|\geq c\right\}\leq\mathbb{1}\left\{\left\|\overline{g}(\theta_F^*)\right\|\geq\left(K_3\wedge\left(\frac{1}{2K_5}\wedge\frac{1}{4\overline{M}_\lambda}\right)\cdot c\right)\right\}+\mathbb{1}\left\{\sup_{\underset{\sim}{\theta}\in\Theta^k}\left\|\overline{G}(\underset{\sim}{\theta})-\lambda_F(\underset{\sim}{\theta})\right\|\geq\left(K_3\wedge\left(\frac{1}{2K_5}\wedge\frac{1}{4\overline{M}_\lambda}\right)\cdot c\right)\right\}
$$

From here, combining (S5.3.4) and (S5.3.6), we have that for any $c>0$,

$$
\sup_{F\in\mathcal{F}}P_F\left(\left\|\varepsilon_n^\theta\right\|\geq c\right)\leq\frac{\overline{M}_1+\overline{M}_2}{\left(n^{1/2}\cdot\left(K_3\wedge\left(\frac{1}{2K_5}\wedge\frac{1}{4\overline{M}_\lambda}\right)\cdot c\right)\right)^q}=o\left(\frac{1}{n^{1/2+\delta}}\right)\quad\forall\ 0<\delta<\frac{q-1}{2}
$$

Take $0<\delta<\frac{q-1}{2}$ and consider a sequence $c_n>0$ such that $n^{\frac{q-1-2\delta}{2q}}\cdot c_n\longrightarrow\infty$. Then, the result in the previous expression would still hold for c_n. Thus,

$$
\sup_{F\in\mathcal{F}}P_F\left(\left\|\varepsilon_n^\theta\right\|\geq c_n\right)=o\left(\frac{1}{n^{1/2+\delta}}\right)\quad\forall\ c_n: n^{\frac{q-1-2\delta}{2q}}\cdot c_n\longrightarrow\infty,\quad 0<\delta<\frac{q-1}{2}
\tag{S5.3.10}
$$

From our previous results we have

$$
\begin{aligned}
\sup_{F\in\mathcal{F}}P_F\left(\left\|\widehat{\theta}-\theta_F^*\right\|\geq c\right)&\leq\sup_{F\in\mathcal{F}}P_F\left(\left\|\frac{1}{n}\sum_{i=1}^n\psi_F^\theta(Z_i)\right\|\geq\frac{c}{2}\right)+\sup_{F\in\mathcal{F}}P_F\left(\left\|\varepsilon_n^\theta\right\|\geq\frac{c}{2}\right)\\
&\leq\frac{\overline{M}_3}{\left(n^{1/2}\cdot\frac{c}{2}\right)^q}+\frac{\overline{M}_1+\overline{M}_2}{\left(n^{1/2}\cdot\left(K_3\wedge\left(\frac{1}{2K_5}\wedge\frac{1}{4\overline{M}_\lambda}\right)\cdot\frac{c}{2}\right)\right)^q}\longrightarrow 0\quad\forall\ c>0
\end{aligned}
\tag{S5.3.11}
$$

Thus, $\left\|\widehat{\theta}-\theta_F^*\right\|=o_p(1)$ uniformly over $\mathcal{F}$. Next, from the definition of ε_n^θ in (S5.3.2), we have

$$
\left\|\varepsilon_n^\theta\right\|\leq\left\|M_k(\lambda_F(\overline{\underset{\sim}{\theta}}))-M_k(\lambda_F(\underset{\sim}{\theta}_F^*))\right\|\cdot\left\|\overline{g}(\theta_F^*)\right\|+\left\|M_k(\lambda_F(\overline{\underset{\sim}{\theta}}))-M_k(\overline{G}(\overline{\underset{\sim}{\theta}}))\right\|\cdot\left\|\overline{g}(\theta_F^*)\right\|
\tag{S5.3.12}
$$

As we stated in (S5.3.7), $\left\|\overline{g}(\theta_F^*)\right\|=O_p\left(n^{-1/2}\right)$ uniformly over $\mathcal{F}$. Let us examine the asymptotic properties of $\left\|M_k(\lambda_F(\overline{\underset{\sim}{\theta}}))-M_k(\lambda_F(\underset{\sim}{\theta}_F^*))\right\|$ and $\left\|M_k(\lambda_F(\overline{\underset{\sim}{\theta}}))-M_k(\overline{G}(\overline{\underset{\sim}{\theta}}))\right\|$ under Assumption GMM.

Take any $b > 0$, then from Assumption GMM,

$$\mathbb{1}\left\{\sup_{\underset{\sim}{\theta}\in\Theta^k}\left\|M_k(\overline{G}(\underset{\sim}{\theta})) - M_k(\lambda_F(\underset{\sim}{\theta}))\right\| \geq b\right\} \leq \mathbb{1}\left\{\sup_{\underset{\sim}{\theta}\in\Theta^k}\left(K_4\cdot\left\|\overline{G}(\underset{\sim}{\theta}) - \lambda_F(\underset{\sim}{\theta})\right\|^{\alpha_1}\right) \geq b\right\} + \mathbb{1}\left\{\sup_{\underset{\sim}{\theta}\in\Theta^k}\left\|\overline{G}(\underset{\sim}{\theta}) - \lambda_F(\underset{\sim}{\theta})\right\| \geq K_3\right\}$$
$$\leq \mathbb{1}\left\{\sup_{\underset{\sim}{\theta}\in\Theta^k}\left\|\overline{G}(\underset{\sim}{\theta}) - \lambda_F(\underset{\sim}{\theta})\right\| \geq K_3 \wedge \left(\frac{b}{K_4}\right)^{1/\alpha_1}\right\}$$

Therefore, from (S5.3.4),

$$\sup_{F\in\mathcal{F}} P_F\left(\sup_{\underset{\sim}{\theta}\in\Theta^k}\left\|M_k(\overline{G}(\underset{\sim}{\theta})) - M_k(\lambda_F(\underset{\sim}{\theta}))\right\| \geq n^{-\tau}\cdot b\right) \leq \frac{\overline{M}_1}{\left(n^{1/2}\cdot\left(K_3 \wedge \left(\frac{n^{-\tau}\cdot b}{K_4}\right)^{1/\alpha_1}\right)\right)^q} \longrightarrow 0 \quad \forall\, \tau < \frac{\alpha_1}{2}$$

which means

$$\sup_{\underset{\sim}{\theta}\in\Theta^k}\left\|M_k(\overline{G}(\underset{\sim}{\theta})) - M_k(\lambda_F(\underset{\sim}{\theta}))\right\| = o_p\left(n^{-\tau}\right) \quad \forall\, \tau < \frac{\alpha_1}{2}, \quad \text{uniformly over } \mathcal{F}.$$

In particular,

$$\left\|M_k(\overline{G}(\overline{\underset{\sim}{\theta}})) - M_k(\lambda_F(\overline{\underset{\sim}{\theta}}))\right\| = o_p\left(n^{-\tau}\right) \quad \forall\ \ \tau < \frac{\alpha_1}{2}, \quad \text{uniformly over } \mathcal{F}. \tag{S5.3.13}$$

Next, recall from the definitions of $\overline{\underset{\sim}{\theta}}$ and $\underset{\sim}{\theta}^*_F$ that, for any $\delta > 0$, we have $\mathbb{1}\left\{\left\|\overline{\underset{\sim}{\theta}} - \underset{\sim}{\theta}^*_F\right\| \geq \delta\right\} \leq \mathbb{1}\left\{\left\|\widehat{\theta} - \theta^*_F\right\| \geq d_k\cdot\delta\right\}$, where d_k is a constant that depends only on k. From here and Assumption GMM we have that, for any $\eta > 0$,

$$\mathbb{1}\left\{\left\|\lambda_F(\overline{\underset{\sim}{\theta}}) - \lambda_F(\underset{\sim}{\theta}^*_F)\right\| \geq \eta\right\} \leq \mathbb{1}\left\{K_7\cdot\left\|\overline{\underset{\sim}{\theta}} - \underset{\sim}{\theta}^*_F\right\|^{\alpha_2} \geq \eta\right\} + \mathbb{1}\left\{\left\|\overline{\underset{\sim}{\theta}} - \underset{\sim}{\theta}^*_F\right\| \geq K_6\right\}$$
$$\leq \mathbb{1}\left\{\left\|\overline{\underset{\sim}{\theta}} - \underset{\sim}{\theta}^*_F\right\| \geq K_6 \wedge \left(\frac{\eta}{K_7}\right)^{1/\alpha_2}\right\}$$
$$\leq \mathbb{1}\left\{\left\|\widehat{\theta} - \theta^*_F\right\| \geq d_k\cdot\left(K_6 \wedge \left(\frac{\eta}{K_7}\right)^{1/\alpha_2}\right)\right\}$$

Take $c > 0$. Using Assumption GMM and the result in the previous expression,

$$\begin{aligned}
\mathbb{1}\left\{\left\|M_k(\lambda_F(\underset{\sim}{\overline{\theta}})) - M_k(\lambda_F(\underset{\sim}{\theta}^*_F))\right\| \geq c\right\} &\leq \mathbb{1}\left\{K_4 \cdot \left\|\lambda_F(\underset{\sim}{\overline{\theta}}) - \lambda_F(\underset{\sim}{\theta}^*_F)\right\|^{\alpha_1} \geq c\right\} + \mathbb{1}\left\{\left\|\lambda_F(\underset{\sim}{\overline{\theta}}) - \lambda_F(\underset{\sim}{\theta}^*_F)\right\| \geq K_3\right\} \\
&\leq \mathbb{1}\left\{\left\|\lambda_F(\underset{\sim}{\overline{\theta}}) - \lambda_F(\underset{\sim}{\theta}^*_F)\right\| \geq K_3 \wedge \left(\frac{c}{K_4}\right)^{1/\alpha_1}\right\} \\
&\leq \mathbb{1}\left\{\left\|\widehat{\theta} - \theta^*_F\right\| \geq d_k \cdot \left(K_6 \wedge \left(\frac{K_3 \wedge \left(\frac{c}{K_4}\right)^{1/\alpha_1}}{K_7}\right)^{1/\alpha_2}\right)\right\} \\
&\leq \mathbb{1}\left\{\left\|\widehat{\theta} - \theta^*_F\right\| \geq d_k \cdot \left(K_6 \wedge \left[\left(K_3 \wedge (c/K_4)^{1/\alpha_1}\right)\big/K_7\right]^{1/\alpha_2}\right)\right\}
\end{aligned}$$

From here, using (S5.3.11), for any $b > 0$ we have

$$\begin{aligned}
&\sup_{F\in\mathcal{F}} P_F\left(\left\|M_k(\lambda_F(\underset{\sim}{\overline{\theta}})) - M_k(\lambda_F(\underset{\sim}{\theta}^*_F))\right\| \geq b\right) \\
&\qquad \leq \frac{\overline{M}_3}{\left(n^{1/2} \cdot \frac{1}{2} \cdot d_k \cdot \left(K_6 \wedge \left[\left(K_3 \wedge (b/K_4)^{1/\alpha_1}\right)\big/K_7\right]^{1/\alpha_2}\right)\right)^q} \\
&\qquad + \frac{\overline{M}_1 + \overline{M}_2}{\left(n^{1/2} \cdot \left(K_3 \wedge \left(\frac{1}{2K_5} \wedge \frac{1}{4\overline{M}_\lambda}\right) \cdot \frac{1}{2} \cdot d_k \cdot \left(K_6 \wedge \left[\left(K_3 \wedge (b/K_4)^{1/\alpha_1}\right)\big/K_7\right]^{1/\alpha_2}\right)\right)\right)^q}
\end{aligned}$$

Therefore,

$$\sup_{F\in\mathcal{F}} P_F\left(\left\|M_k(\lambda_F(\underset{\sim}{\overline{\theta}})) - M_k(\lambda_F(\underset{\sim}{\theta}^*_F))\right\| \geq n^{-\tau} \cdot b\right) \longrightarrow 0 \quad \forall\ \tau < \frac{\alpha_1 \cdot \alpha_2}{2}$$

which means,

$$\left\|M_k(\lambda_F(\underset{\sim}{\overline{\theta}})) - M_k(\lambda_F(\underset{\sim}{\theta}^*_F))\right\| = o_p\left(n^{-\tau}\right) \quad \forall\ \tau < \frac{\alpha_1 \cdot \alpha_2}{2}$$

Thus, combining (S5.3.7), (S5.3.12), (S5.3.13) and the previous expression, we have that for any $0 < \tau < \frac{\alpha_1}{2} \wedge \frac{\alpha_1 \cdot \alpha_2}{2}$,

$$\left\|\varepsilon^\theta_n\right\| = o_p\left(\frac{1}{n^{1/2+\tau}}\right) \quad \text{uniformly over } \mathcal{F}.$$

Together, (S5.3.2), (S5.3.10) and the result in the previous expression show that the conditions in Assumption 1 are satisfied, with $\psi^\theta_F(Z_i) = -\left(E_F\left[\frac{\partial g(Z,\theta^*_F)}{\partial\theta\partial\theta'}\right]\right)^{-1} \cdot g(Z_i, \theta^*_F)$, $r_n = n^{1/2}$, $0 < \tau < \frac{\alpha_1}{2} \wedge \frac{\alpha_1 \cdot \alpha_2}{2}$, and $0 < \overline{\delta} < \frac{q-1}{2}$. ■

S5.4 Density-weighted average derivatives

Consider an iid sample $(Z_{1i}, Z_{2i})_{i=1}^n$ where $Z_{1i} \in \mathbb{R}$, $Z_{2i} \in \mathbb{R}^d$ and $Z_i \equiv (Z_{1i}, Z_{2i}) \sim F \in \mathcal{F}$. As in our previous discussions, we will let $\mathcal{S}_\xi$ denote the support of the r.v ξ. We will group $Z \equiv (Z_1, Z_2)$ and we will assume that $\mathcal{S}_Z$ is the same for all $F \in \mathcal{F}$. Suppose that, for each $F \in \mathcal{F}$, we have

$E_F[Z_1|Z_2] \equiv \mu_F(Z_2) = G_F(Z_2'\beta_0)$, where G_F is unknown but smooth as described in Powell, Stock, and Stoker (1989) (we will be precise about these smoothness conditions below). Let f_{z_2} denote the density of Z_2, assumed to be absolutely continuous with respect to Lebesgue measure, and denote $\delta_F \equiv E_F\left[f_{z_2}(Z_2)G_F'(Z_2'\beta_0)\right]$ and $\theta_F^* \equiv \delta_F \cdot \beta_0$. Using integration by parts, under the conditions described in Powell, Stock, and Stoker (1989), we have

$$\theta_F^* = -2 \cdot E_F\left[Z_1 \cdot \frac{\partial f_{z_2}(Z_2)}{\partial Z_2}\right].$$

Let $K : \mathbb{R}^d \to \mathbb{R}$ be a kernel function (whose conditions we will describe below) and let $\sigma > 0$ be a strictly positive scalar. For a pair of observations $i \neq j$ in the sample let

$$p(Z_i, Z_j; \sigma) \equiv \left(Z_{1i} - Z_{1j}\right) \cdot K^{(1)}\left(\frac{Z_{2j} - Z_{2i}}{\sigma}\right).$$

Let $\sigma_n \to 0$ be a bandwidth sequence. The estimator for θ_F^* proposed in Powell, Stock, and Stoker (1989) is of the form

$$\widehat{\theta} = \binom{n}{2}^{-1} \frac{1}{\sigma_n^{d+1}} \sum_{i<j} p(Z_i, Z_j; \sigma_n). \tag{S5.4.1}$$

Assumption DWAD

(i) There exists an integer $q \geq 2$ and a constant $\overline{\mu}_{z_1} < \infty$ such that $E_F\left[Z_1^{4q}\right] \leq \overline{\mu}_{z_1}$ for all $F \in \mathcal{F}$.

(ii) The kernel $K : \mathbb{R}^d \to \mathbb{R}$ is a multiplicative kernel of the form $K(\psi) = \prod_{\ell=1}^d \kappa(\psi_\ell)$ (with $\psi \equiv (\psi_1, \ldots, \psi_k)$). $\kappa(\cdot)$ is a bounded kernel function satisfying $|\kappa(v)| \leq \overline{\kappa} < \infty$ for all v. The kernel function $\kappa(\cdot)$ is also of bounded variation and it has support of the form $[-S, S]$. $\kappa(\cdot)$ is symmetric around zero and has the properties of a bias-reducing kernel of order L: $\int_{-S}^{S} v^j \kappa(v)dv = 0$ for $j = 1, \ldots, L-1$ and $\int_{-S}^{S} |v|^L \kappa(v)dv < \infty$. In addition, $\kappa(\cdot)$ is differentiable, with first derivative denoted as $\kappa'(\cdot)$. The function $\kappa^{'}(\cdot)$ is bounded, satisfying $|k'(v)| \leq \overline{\kappa}_1 < \infty$ for all v, and it is also of bounded variation. Since $\kappa(\cdot)$ is symmetric around zero, $\kappa'(\cdot)$ is antisymmetric around zero, satisfying $\kappa'(v) = -\kappa'(-v)$ for all $v \in [-S, S]$. Thus, if we let $K^{(1)}$ denote the Jacobian of K, then $K^{(1)}(\psi) = -K^{(1)}(-\psi)$ for all $\psi \in \mathbb{R}^d$. We have $\left|K(\psi)\right| \leq \overline{K} < \infty$ and $\left\|K^{(1)}(\psi)\right\| \leq \overline{K}_1 < \infty$ for all $\psi \in \mathbb{R}^d$.

(iii) Let L be the constant described above. Then, both $f_{z_2}(z_2)$ and $\mu_F(z_2)$ are L-times continuously differentiable with respect to z_2 for F–a.e $z_2 \in \mathcal{S}_Z$, with derivatives that are uniformly bounded over $\mathcal{S}_Z$ for all $F \in \mathcal{F}$.

(iv) Let L be as described above. The bandwidth sequence $\sigma_n > 0$ satisfies $\sigma_n \longrightarrow 0$ and is such that $n^{1/2-\Delta} \cdot \sigma_n^{d+1} \longrightarrow \infty$, $n^{1/2+\Delta} \cdot \sigma_n^L \longrightarrow 0$ and $n^{1+\Delta} \cdot \sigma_n^{d+1} \cdot \sigma_n^L \longrightarrow 0$ for some $0 < \Delta < 1/2$. In addition, the integer q described above and Δ are such that $q\Delta > \frac{1}{2}$.

For a given $\sigma > 0$ let

$$
\begin{aligned}
r_{1,F}(Z_i;\sigma) &= E_F\left[p(Z_i,Z_j;\sigma)|Z_i\right] - E_F\left[p(Z_i,Z_j;\sigma)\right],\\
r_{2,F}(Z_i,Z_j;\sigma) &= \left(p(Z_i,Z_j;\sigma) - E_F\left[p(Z_i,Z_j;\sigma)\right]\right) - r_{1,F}(Z_i;\sigma) - r_{1,F}(Z_j;\sigma),\\
U_{2,n}(\sigma) &= \binom{n}{2}^{-1}\sum_{i<j} r_{2,F}(Z_i,Z_j;\sigma).
\end{aligned}
$$

$U_{2,n}(\sigma)$ is a degenerate U-statistic of order 2 and $\{U_{2,n}(\sigma) : \sigma > 0\}$ is a degenerate U-process of order 2. Going forward we will denote $U_{2,n}(\sigma_n) \equiv U_{2,n}$. A Hoeffding decomposition (see Serfling (1980, pages 177-178) or Sherman (1994, equations (6)-(7))) of the U-statistic in (S5.4.1) yields

$$
\widehat{\theta} = \frac{1}{\sigma_n^{d+1}}\cdot E_F\left[p(Z_i,Z_j;\sigma_n)\right] + \frac{2}{n\cdot\sigma_n^{d+1}}\sum_{i=1}^{n} r_{1,F}(Z_i;\sigma_n) + \frac{1}{\sigma_n^{d+1}}\cdot U_{2,n}. \tag{S5.4.2}
$$

For a given $(z_1,z_2) \in \mathcal{S}_Z$, let $\varphi_F(z_1,z_2) \equiv (z_1 - \mu_F(z_2))\cdot f_{z_2}(z_2)$. Under the smoothness conditions and the higher-order properties of the kernel described in Assumption DWAD, an M^{th}–order approximation yields the following re-expression of (S5.4.2),

$$
\begin{aligned}
\widehat{\theta} =& \theta_F^* - \frac{2}{n}\sum_{i=1}^{n}\left(\frac{\varphi_F(Z_{1i},Z_{2i})}{\partial Z_2} - E_F\left[\frac{\varphi_F(Z_{1i},Z_{2i})}{\partial Z_2}\right]\right) + \frac{1}{\sigma_n^{d+1}}\cdot U_{2,n} + B_{n,F}\\
\equiv& \theta_F^* + \frac{1}{n}\sum_{i=1}^{n}\psi_F^{\theta}(Z_i) + \varepsilon_n^{\theta}, \quad \text{where}\\
\psi_F^{\theta}(Z_i) \equiv& -2\cdot\left(\frac{\varphi_F(Z_{1i},Z_{2i})}{\partial Z_2} - E_F\left[\frac{\varphi_F(Z_{1i},Z_{2i})}{\partial Z_2}\right]\right),\\
\varepsilon_n^{\theta} \equiv& \frac{1}{\sigma_n^{d+1}}\cdot U_{2,n} + B_{n,F},
\end{aligned} \tag{S5.4.3}
$$

where $B_{n,F}$ is a bias aggregate term which, by the smoothness conditions and the higher-order properties of the kernel described in Assumption DWAD, is such that there exists a constant $\overline{C}_H > 0$ such that

$$
||B_{n,F}|| \leq \overline{C}_H\cdot\sigma_n^L \quad \forall\ F \in \mathcal{F}. \tag{S5.4.4}
$$

Let us examine the properties of the degenerate U-process $\{U_{2,n}(\sigma) : \sigma > 0\}$ under Assumption DWAD. By Lemma 22 in Nolan and Pollard (1987), if $\lambda(\cdot)$ is a real-valued function of bounded variation on $\mathbb{R}$, the class of all functions of the form $z_2 \to \lambda(\gamma' z_2)$ with γ ranging over $\mathbb{R}^d$ is Euclidean for the constant envelope $\overline{\lambda} \equiv \sup_{b\in\mathbb{R}}|\lambda(b)|$. Combining this with the closure properties of Euclidean classes described in Lemma 2.14 in Pakes and Pollard (1989), the conditions in

Assumption DWAD imply that the class of functions

$$\mathscr{H} = \left\{ f : \mathcal{S}_Z^2 \longrightarrow \mathbb{R} : f(z_a, z_b) = (z_{1a} - z_{1b}) \cdot K^{(1)}\left(\frac{z_{2b} - z_{2a}}{\sigma}\right) \text{ for some } \sigma > 0 \right\}$$

is Euclidean for envelope $G(z_a, z_b) = \overline{K}_1 \cdot (|z_{1a}| + |z_{1b}|)$. By Assumption DWAD, there exists $\overline{\mu}_G < \infty$ such that $E_F\left[G(Z_{1i}, Z_{1j})^{4q}\right] \leq \overline{\mu}_G$. From here, the conditions in Result S1 are satisfied for the integer q described in Assumption DWAD and we have that, for all $b > 0$, there exists $\overline{D}_H < \infty$ such that

$$\sup_{F \in \mathcal{F}} P_F\left(\sup_{\sigma > 0} \left|U_{2,n}(\sigma)\right| \geq b\right) \leq \frac{\overline{D}_H}{(n \cdot b)^q}$$

From here it follows, in particular,

$$\sup_{F \in \mathcal{F}} P_F\left(\left|U_{2,n}\right| \geq b\right) \leq \frac{\overline{D}_H}{(n \cdot b)^q} \quad \forall\, b > 0 \tag{S5.4.5}$$

Note that (S5.4.5) implies

$$\left|U_{2,n}\right| = O_p\left(\frac{1}{n}\right) \quad \text{uniformly over } \mathcal{F}. \tag{S5.4.6}$$

Take any $c > 0$. From (S5.4.4), we have

$$\mathbb{1}\left\{\left\|\varepsilon_n^\theta\right\| \geq c\right\} \leq \mathbb{1}\left\{\sup_{\sigma > 0}\left\|U_{2,n}(\sigma)\right\| \geq \sigma_n^{d+1} \cdot \left(c - \overline{C}_H \cdot \sigma_n^L\right)\right\}$$

Let n_0 be the smallest integer such that $\overline{C}_H \cdot \sigma_n^L < c$. Then, from the previous expression and (S5.4.5),

$$\sup_{F \in \mathcal{F}} P_F\left(\left\|\varepsilon_n^\theta\right\| \geq c\right) \leq \frac{\overline{D}_H}{\left(n \cdot \sigma_n^{d+1} \cdot \left(c - \overline{C}_H \cdot \sigma_n^L\right)\right)^q} \quad \forall\, n \geq n_0 \tag{S5.4.7}$$

Note from Assumption DWAD(iv) that $\frac{n^{1/2+\overline{\delta}}}{(n \cdot \sigma_n^{d+1})} \to 0$ for any $0 < \overline{\delta} < q\Delta - \frac{1}{2}$. Next, combining (S5.4.4) and (S5.4.6), we have

$$\left\|\varepsilon_n^\theta\right\| = O_p\left(\frac{1}{n \cdot \sigma_n^{d+1}}\right) + O\left(\sigma_n^L\right) \quad \text{uniformly over } \mathcal{F}.$$

Therefore, by the conditions in Assumption DWAD, for any $0 < \tau < \Delta$,

$$\left\|\varepsilon_n^\theta\right\| = o_p\left(\frac{1}{n^{1/2+\tau}}\right), \quad \text{uniformly over } \mathcal{F}.$$

Together, (S5.4.3), (S5.4.7) and the previous expression show that the conditions in Assumption 1 are satisfied, with $\psi_F^\theta(Z_i) = -2 \cdot \left(\frac{\varphi_F(Z_{1i}, Z_{2i})}{\partial Z_2} - E_F\left[\frac{\varphi_F(Z_{1i}, Z_{2i})}{\partial Z_2}\right]\right)$, $r_n = n \cdot \sigma_n^{d+1}$, $0 < \tau < \Delta$, and $0 < \overline{\delta} < q\Delta - \frac{1}{2}$.

■

S5.5 A semiparametric, multiple-index estimator (proof of Result SMIM in Appendix A4.2)

Consider a collection of d single-valued indices $(m_\ell(W_\ell,\theta_\ell))_{\ell=1}^d$ where $\theta_\ell \in \mathbb{R}^{k_\ell}$. Each m_ℓ has a known parametric functional form (e.g, $m_\ell(W_\ell,\theta_\ell) = W_\ell'\theta_\ell$). Group $\cup_{\ell=1}^d W_\ell \equiv Z_2$ and let $\theta \equiv \left(\theta_1',\theta_2',\ldots,\theta_d'\right)' \in \mathbb{R}^k$ and denote

$$m(Z_2,\theta) \equiv (m_1(W_1,\theta_1), m_2(W_2,\theta_2),\ldots,m_d(W_d,\theta_d))' \in \mathbb{R}^d.$$

For simplicity let us focus on the case where Z_2 is a vector of jointly continuously distributed random variables. Let Z_1 be a scalar random variable and group $Z \equiv (Z_1,Z_2) \sim F \in \mathcal{F}$. Let Θ denote the parameter space for θ, assume Θ to be bounded and consider a model where there exists a $\theta^* \in \Theta$ such that

$$E_F[Z_1|Z_2] = E_F\left[Z_1\middle|m(Z_2,\theta^*)\right] \quad \forall\ F \in \mathcal{F}$$

For a given $\theta \in \Theta$ let $\mu_F(m(Z_2,\theta)) \equiv E_F[Z_1|m(Z_2,\theta)]$. Our model therefore assumes $E_F[Z_1|Z_2] = \mu_F(m(Z_2,\theta^*))$. Let $\phi \in \mathbb{R}^k$ denote a vector of pre-specified instrument functions and consider an estimator based on the moment conditions

$$E_F\left[\phi(Z_2)\cdot\left(Z_1 - \mu_F(m(Z_2,\theta^*))\right)\right] = 0 \qquad \text{(S5.5.1)}$$

Suppose we have a random sample $(Z_{1i},Z_{2i})_{i=1}^n$ where $Z_i \equiv (Z_{1i},Z_{2i}) \sim F \in \mathcal{F}$. Let $\mathcal{S}_\xi$ denote the support of the r.v ξ and for simplicity assume throughout that $\mathcal{S}_Z$ is the same for all $F \in \mathcal{F}$. Suppose that the instrument functions are designed such that $\phi(z_2) = 0\ \forall\ z_2 \notin \mathcal{Z}_2$, where $\mathcal{Z}_2 \subset \mathcal{S}_{Z_2}$ is a pre-specified set belonging in the interior of $\mathcal{S}_{Z_2}$ for all $F \in \mathcal{F}$. We refer to $\mathcal{Z}_2$ as our *inference range*. Thus, the instrument functions also serve as trimming functions to keep inference confined to the set $\mathcal{Z}_2$. Finally, suppose $\|\phi(z_2)\| \leq \overline{\phi}\ \forall\ z_2$. Let

$$\mathcal{M} \equiv \left\{m \in \mathbb{R}^d\colon\ m = m(z_2,\theta)\ \text{ for some } (z_2,\theta) \in \mathcal{Z}_2 \times \Theta\right\}.$$

$\mathcal{M}$ is the range of all possible values of the index $m(z_2,\theta)$ over our inference range and the parameter space. Let $\sigma_n \to 0$ denote a bandwidth sequence and let K denote a kernel function. For a given $\theta \in \Theta$ and $z_2 \in \mathcal{Z}_2$, let $f_m(m(Z_2,\theta))$ denote the density of $m(Z_2,\theta)$. Consider a kernel-based estimator of $\mu_F(m(z_2,\theta))$ of the form

$$\widehat{\mu}(m(z_2,\theta)) = \frac{\widehat{R}(m(z_2,\theta))}{\widehat{f_m}(m(z_2,\theta))}, \quad \text{where}$$
$$\widehat{R}(m(z_2,\theta)) = \frac{1}{n\cdot\sigma_n^d}\sum_{i=1}^n Z_{1i}K\left(\frac{m(Z_{2i},\theta)-m(z_2,\theta)}{\sigma_n}\right),$$
$$\widehat{f_m}(m(z_2,\theta)) = \frac{1}{n\cdot\sigma_n^d}\sum_{i=1}^n K\left(\frac{m(Z_{2i},\theta)-m(z_2,\theta)}{\sigma_n}\right).$$

Consider an estimator $\widehat{\theta}$ defined by the sample analog moment conditions to (S5.5.1),

$$\frac{1}{n}\sum_{i=1}^n \phi(Z_{2i})\cdot\left(Z_{1i}-\widehat{\mu}(m(Z_{2i},\widehat{\theta}))\right) = 0. \tag{S5.5.2}$$

Assumption SMIM1 *For some $q \geq 2$, we have $E_F[Z_1^{4q}] \leq \overline{\mu}_{4q} < \infty$ for all $F \in \mathcal{F}$. Also, there exist constants $\underline{f}_m > 0$, $\overline{f}_m < \infty$ and $\overline{\mu} < \infty$ such that $\overline{f}_m \geq f_m(m) \geq \underline{f}_m$ and $|\mu_F(m)| \leq \overline{\mu}$ $\forall$ $m \in \mathcal{M}$ and all $F \in \mathcal{F}$. Also assume that both $f_m(m)$ and $\mu_F(m)$ are L-times continuously differentiable with respect to m for F–a.e $m \in \mathcal{M}$, with derivatives that are uniformly bounded over $\mathcal{M}$ for all $F \in \mathcal{F}$. The kernel $K : \mathbb{R}^d \to \mathbb{R}$ is a multiplicative kernel of the form $K(\psi) = \prod_{\ell=1}^d \kappa(\psi_\ell)$ (with $\psi \equiv (\psi_1,\ldots,\psi_d)$), where $\kappa(\cdot)$ is a function of bounded-variation, a bias-reducing kernel of order L with support of the form $[-S, S]$ (i.e, $\int_{-S}^S v^j\kappa(v)dv = 0$ for $j = 1,\ldots,L-1$ and $\int_{-S}^S |v|^L\kappa(v)dv < \infty$) and symmetric around zero. We have $\sup_{\psi\in\mathbb{R}^d}\left|K(\psi)\right| \leq \overline{K}$. The bandwidth sequence $\sigma_n > 0$ satisfies $\sigma_n \longrightarrow 0$, with $n^{1/2+\Delta}\cdot\sigma_n^L \longrightarrow 0$ and $n^{1/2-\Delta}\cdot\sigma_n^d \longrightarrow \infty$ for some $0 < \Delta < 1/2$. q and Δ are such that $q\Delta > \frac{1}{2}$.*

Denote $R_F(m) \equiv \mu_F(m)\cdot f_m(m)$ and note from Assumption SMIM1 that $|R_F(m)| \leq \overline{\mu}\cdot\overline{f}_m \equiv \overline{R}$ $\forall$ $m \in \mathcal{M}$ and all $F \in \mathcal{F}$. Fix $m \in \mathcal{M}$. A second order aproximation yields

$$\widehat{\mu}(m) = \mu_F(m) + \frac{1}{f_m(m)}\cdot\left(\widehat{R}(m)-R_F(m)\right) - \frac{\mu_F(m)}{f_m(m)}\cdot\left(\widehat{f_m}(m)-f_m(m)\right)$$
$$-\frac{\left(\widehat{R}(m)-R_F(m)\right)\cdot\left(\widehat{f_m}(m)-f_m(m)\right)}{\widetilde{f_m}(m)^2} + \frac{\widetilde{R}(m)\cdot\left(\widehat{f_m}(m)-f_m(m)\right)^2}{\widetilde{f_m}(m)^3},$$

where $\widetilde{f_m}(m)$ is an intermediate point between $\widehat{f_m}(m)$ and $f_m(m)$, and $\widetilde{R}(m)$ is an intermediate point between $\widehat{R}(m)$ and $R_F(m)$. From here, we have that, for any given $(z_2,\theta) \in \mathcal{Z}_2 \times \Theta$,

$$\widehat{\mu}(m(z_2,\theta)) - \mu_F(m(z_2,\theta)) = \frac{1}{n \cdot \sigma_n^d} \sum_{i=1}^{n} \frac{(Z_{1i} - \mu_F(m(z_2,\theta)))}{f_m(m(z_2,\theta))} \cdot K\left(\frac{m(Z_{2i},\theta) - m(z_2,\theta)}{\sigma_n}\right) + \varepsilon_n^{\mu}(m(z_2,\theta)),$$

where

$$\begin{aligned}\varepsilon_n^{\mu}(m(z_2,\theta)) \equiv &- \frac{\left(\widehat{R}(m(z_2,\theta)) - R_F(m(z_2,\theta))\right) \cdot \left(\widehat{f_m}(m(z_2,\theta)) - f_m(m(z_2,\theta))\right)}{\widetilde{f_m}(m(z_2,\theta))^2} \\ &+ \frac{\widetilde{R}(m(z_2,\theta)) \cdot \left(\widehat{f_m}(m(z_2,\theta)) - f_m(m(z_2,\theta))\right)^2}{\widetilde{f_m}(m(z_2,\theta))^3}\end{aligned} \tag{S5.5.3}$$

For the next few lines let us omit the dependence of $\widehat{f_m}$, $\widehat{R}$, $\widetilde{f_m}$, $\widetilde{R}$, ε_n^{μ}, f_m and R_F on $m(z_2,\theta)$ to simplify the exposition. Note first that

$$\begin{aligned}\left|\varepsilon_n^{\mu}\right| &\leq \frac{|\widehat{R} - R_F| \cdot |\widehat{f_m} - f_m|}{|\widetilde{f_m}|^2} + \frac{\overline{R} \cdot |\widehat{f_m} - f_m|^2}{|\widetilde{f_m}|^3} + \frac{|\widetilde{R} - R_F| \cdot |\widehat{f_m} - f_m|^2}{|\widetilde{f_m}|^3} \\ &\leq \frac{|\widehat{R} - R_F| \cdot |\widehat{f_m} - f_m|}{|\widetilde{f_m}|^2} + \frac{\overline{R} \cdot |\widehat{f_m} - f_m|^2}{|\widetilde{f_m}|^3} + \frac{|\widehat{R} - R_F| \cdot |\widehat{f_m} - f_m|^2}{|\widetilde{f_m}|^3}\end{aligned}$$

where the last inequality follows because, by definition, $|\widetilde{R} - R_F| \leq |\widehat{R} - R_F|$. Next, note that, $\forall\ c > 0$, we have $\mathbb{1}\{|\xi_1| + |\xi_2| + |\xi_3| \geq c\} \leq \max\left(\mathbb{1}\{|\xi_1| \geq \frac{c}{3}\}\ ,\ \mathbb{1}\{|\xi_2| \geq \frac{c}{3}\}\ ,\ \mathbb{1}\{|\xi_3| \geq \frac{c}{3}\}\right)$. Therefore, for any $c > 0$,

$$\begin{aligned}\mathbb{1}\left\{\left|\varepsilon_n^{\mu}\right| \geq c\right\} \leq \max\Bigg(&\underbrace{\mathbb{1}\left\{\left|\widehat{R} - R_F\right| \cdot \left|\widehat{f_m} - f_m\right| \geq \frac{|\widetilde{f_m}|^2 \cdot c}{3}\right\}}_{(IV)}\ ,\ \underbrace{\mathbb{1}\left\{\left|\widehat{f_m} - f_m\right|^2 \geq \frac{|\widetilde{f_m}|^3 \cdot c}{3\overline{R}}\right\}}_{(V)} \\ &,\ \underbrace{\mathbb{1}\left\{\left|\widehat{R} - R_F\right| \cdot \left|\widehat{f_m} - f_m\right|^2 \geq \frac{|\widetilde{f_m}|^3 \cdot c}{3}\right\}}_{(VI)}\Bigg)\end{aligned} \tag{S5.5.4}$$

Next, note that, $\forall\ c > 0$, we have $\mathbb{1}\{|\xi_1| \cdot |\xi_2| \geq c\} \leq \max\left(\mathbb{1}\{|\xi_1| \geq c^{1/2}\}\ ,\ \mathbb{1}\{|\xi_2| \geq c^{1/2}\}\right)$. Therefore,

$$\begin{aligned}&\underbrace{\mathbb{1}\left\{\left|\widehat{R} - R_F\right| \cdot \left|\widehat{f_m} - f_m\right| \geq \frac{|\widetilde{f_m}|^2 \cdot c}{3}\right\}}_{(IV)} \\ &\leq \max\left(\mathbb{1}\left\{\left|\widehat{R} - R_F\right| \geq \left(\frac{|\widetilde{f_m}|^2 \cdot c}{3}\right)^{1/2}\right\}\ ,\ \mathbb{1}\left\{\left|\widehat{f_m} - f_m\right| \geq \left(\frac{|\widetilde{f_m}|^2 \cdot c}{3}\right)^{1/2}\right\}\right)\end{aligned} \tag{S5.5.5A}$$

$$\underbrace{\mathbb{1}\left\{\left|\widehat{f}_m - f_m\right|^2 \geq \frac{|\widetilde{f}_m|^3 \cdot c}{3\overline{R}}\right\}}_{(V)} = \mathbb{1}\left\{\left|\widehat{f}_m - f_m\right| \geq \left(\frac{|\widetilde{f}_m|^3 \cdot c}{3\overline{R}}\right)^{1/2}\right\} \tag{S5.5.5B}$$

$$\begin{aligned}&\underbrace{\mathbb{1}\left\{\left|\widehat{R} - R_F\right| \cdot \left|\widehat{f}_m - f_m\right|^2 \geq \frac{|\widetilde{f}_m|^3 \cdot c}{3}\right\}}_{(VI)} \\ &\leq \max\left(\mathbb{1}\left\{\left|\widehat{R} - R_F\right| \geq \left(\frac{|\widetilde{f}_m|^3 \cdot c}{3}\right)^{1/2}\right\}, \mathbb{1}\left\{\left|\widehat{f}_m - f_m\right| \geq \left(\frac{|\widetilde{f}_m|^3 \cdot c}{3}\right)^{1/4}\right\}\right)\end{aligned} \tag{S5.5.5C}$$

Note that $\min\left\{|\widetilde{f}_m|, |\widetilde{f}_m|^{3/2}, |\widetilde{f}_m|^{3/4}\right\} = \min\left\{|\widetilde{f}_m|^{3/2}, |\widetilde{f}_m|^{3/4}\right\}$, and $\min\left\{c^{1/4}, c^{1/2}\right\} \leq \min\left\{c^{1/4}, c\right\}$ for all $c > 0$. Given this, let

$$\varphi^{\mu}(\widetilde{f}_m, c) \equiv \frac{1}{\sqrt{3}} \cdot \min\left\{\frac{1}{\overline{R}}, 1\right\} \cdot \min\left\{|\widetilde{f}_m|^{3/2}, |\widetilde{f}_m|^{3/4}\right\} \cdot \min\left\{c^{1/4}, c\right\}.$$

Combining (S5.5.5A), (S5.5.5B) and (S5.5.5C) with (S5.5.4), we have

$$\mathbb{1}\left\{\left|\varepsilon_n^{\mu}\right| \geq c\right\} \leq \max\Big(\underbrace{\mathbb{1}\left\{\left|\widehat{R} - R_F\right| \geq \varphi^{\mu}(\widetilde{f}_m, c)\right\}}_{(VII)}, \underbrace{\mathbb{1}\left\{\left|\widehat{f}_m - f_m\right| \geq \varphi^{\mu}(\widetilde{f}_m, c)\right\}}_{(VIII)}\Big) \tag{S5.5.6}$$

Let us analyze term (VII) in (S5.5.6) first. Begin by expressing it as

$$\begin{aligned}\mathbb{1}\left\{\left|\widehat{R} - R_F\right| \geq \varphi^{\mu}(\widetilde{f}_m, c)\right\} &= \mathbb{1}\left\{\left|\widehat{R} - R_F\right| \geq \varphi^{\mu}(\widetilde{f}_m, c)\right\} \cdot \mathbb{1}\left\{\left|\widetilde{f}_m\right| \geq \left|f_m\right| - \frac{1}{2} \cdot \underline{f}_m\right\} \\ &+ \mathbb{1}\left\{\left|\widehat{R} - R_F\right| \geq \varphi^{\mu}(\widetilde{f}_m, c)\right\} \cdot \mathbb{1}\left\{\left|\widetilde{f}_m\right| < \left|f_m\right| - \frac{1}{2} \cdot \underline{f}_m\right\}\end{aligned} \tag{S5.5.7}$$

We begin with the first term on the right-hand side of (S5.5.7). Note first that $|f_m| - \frac{1}{2} \cdot \underline{f}_m \geq \frac{1}{2} \cdot \underline{f}_m$ and therefore $\mathbb{1}\left\{|\widetilde{f}_m| \geq |f_m| - \frac{1}{2} \cdot \underline{f}_m\right\} \leq \mathbb{1}\left\{|\widetilde{f}_m| \geq \frac{1}{2} \cdot \underline{f}_m\right\}$. Define

$$D^{\varepsilon^{\mu}} \equiv \frac{1}{\sqrt{3}} \cdot \min\left\{\frac{1}{\overline{R}}, 1\right\} \cdot \min\left\{\left(\frac{1}{2} \cdot \underline{f}_m\right)^{3/2}, \left(\frac{1}{2} \cdot \underline{f}_m\right)^{3/4}\right\}$$

Then, the first term on the right-hand side of (S5.5.7) satisfies

$$\mathbb{1}\left\{\left|\widehat{R} - R_F\right| \geq \varphi^{\mu}(\widetilde{f}_m, c)\right\} \cdot \mathbb{1}\left\{\left|\widetilde{f}_m\right| \geq \left|f_m\right| - \frac{1}{2} \cdot \underline{f}_m\right\} \leq \mathbb{1}\left\{\left|\widehat{R} - R_F\right| \geq D^{\varepsilon^{\mu}} \cdot \min\left\{c^{1/4}, c\right\}\right\} \tag{S5.5.8A}$$

Next, we move on to the second term on the right-hand side of (S5.5.7). Recall that, by definition, we have $|\widehat{f}_m - f_m| \geq |\widetilde{f}_m - f_m|$. Therefore, $|\widehat{f}_m - f_m| \geq |\widetilde{f}_m - f_m| \geq |f_m| - |\widetilde{f}_m|$ and thus, $\mathbb{1}\{|\widetilde{f}_m| < |f_m| - \frac{1}{2} \cdot \underline{f}_m\} = \mathbb{1}\{|f_m| - |\widetilde{f}_m| > \frac{1}{2} \cdot \underline{f}_m\} \leq \mathbb{1}\{|\widetilde{f}_m - f_m| > \frac{1}{2} \cdot \underline{f}_m\} \leq \mathbb{1}\{|\widehat{f}_m - f_m| \geq \frac{1}{2} \cdot \underline{f}_m\}$. Therefore, the second term on the right-hand side of (S5.5.7) satisfies,

$$\begin{aligned}
&\mathbb{1}\left\{\left|\widehat{R} - R_F\right| \geq \varphi^\mu(\widetilde{f}_m, c)\right\} \cdot \mathbb{1}\left\{\left|\widetilde{f}_m\right| < \left|f_m\right| - \frac{1}{2} \cdot \underline{f}_m\right\} \\
\leq &\mathbb{1}\left\{\left|\widehat{R} - R_F\right| \geq \varphi^\mu(\widetilde{f}_m, c)\right\} \cdot \mathbb{1}\left\{\left|\widehat{f}_m - f_m\right| \geq \frac{1}{2} \cdot \underline{f}_m\right\} \leq \mathbb{1}\left\{\left|\widehat{f}_m - f_m\right| \geq \frac{1}{2} \cdot \underline{f}_m\right\}
\end{aligned} \tag{S5.5.8B}$$

Combining (S5.5.8A) and (S5.5.8B) with (S5.5.7), we have that the term (VII) in equation (S5.5.6) satisfies,

$$\begin{aligned}
\underbrace{\mathbb{1}\left\{\left|\widehat{R} - R_F\right| \geq \varphi^\mu(\widetilde{f}_m, c)\right\}}_{(VII)} \leq \max\Bigg(&\mathbb{1}\left\{\left|\widehat{R} - R_F\right| \geq \min\left\{\frac{\underline{f}_m}{2}\ ,\ D^{\varepsilon^\mu} c^{1/4}\ ,\ D^{\varepsilon^\mu} c\right\}\right\} \\
&,\ \mathbb{1}\left\{\left|\widehat{f}_m - f_m\right| \geq \min\left\{\frac{\underline{f}_m}{2}\ ,\ D^{\varepsilon^\mu} c^{1/4}\ ,\ D^{\varepsilon^\mu} c\right\}\right\}\Bigg)
\end{aligned} \tag{S5.5.9A}$$

Next, we analyze the term (VIII) in equation (S5.5.6). Similar to (S5.5.7), let us write it as

$$\begin{aligned}
\mathbb{1}\left\{\left|\widehat{f}_m - f_m\right| \geq \varphi^\mu(\widetilde{f}_m, c)\right\} = &\mathbb{1}\left\{\left|\widehat{f}_m - f_m\right| \geq \varphi^\mu(\widetilde{f}_m, c)\right\} \cdot \mathbb{1}\left\{\left|\widetilde{f}_m\right| \geq \left|f_m\right| - \frac{1}{2} \cdot \underline{f}_m\right\} \\
&+ \mathbb{1}\left\{\left|\widehat{f}_m - f_m\right| \geq \varphi^\mu(\widetilde{f}_m, c)\right\} \cdot \mathbb{1}\left\{\left|\widetilde{f}_m\right| < \left|f_m\right| - \frac{1}{2} \cdot \underline{f}_m\right\}
\end{aligned}$$

Parallel steps to those leading to (S5.5.8A) and (S5.5.8B) now yield,

$$\underbrace{\mathbb{1}\left\{\left|\widehat{f}_m - f_m\right| \geq \varphi^\mu(\widetilde{f}_m, c)\right\}}_{(VIII)} \leq \mathbb{1}\left\{\left|\widehat{f}_m - f_m\right| \geq \min\left\{\frac{\underline{f}_m}{2}\ ,\ D^{\varepsilon^\mu} c^{1/4}\ ,\ D^{\varepsilon^\mu} c\right\}\right\} \tag{S5.5.9B}$$

Denote

$$\underline{\varphi}^\mu(c) \equiv \min\left\{\frac{\underline{f}_m}{2}\ ,\ D^{\varepsilon^\mu} c\ ,\ D^{\varepsilon^\mu} c^{1/4}\right\}. \tag{S5.5.10}$$

Combining (S5.5.9A) and (S5.5.9B) with (S5.5.6), we have that, for any $c > 0$,

$$\begin{aligned}
&\mathbb{1}\left\{\left|\varepsilon_n^{\mu}(m(z_2,\theta))\right| \geq c\right\} \\
&\leq \max\left(\mathbb{1}\left\{\left|\widehat{R}(m(z_2,\theta)) - R_F(m(z_2,\theta))\right| \geq \underline{\varphi}^{\mu}(c)\right\},\ \mathbb{1}\left\{\left|\widehat{f}_m(m(z_2,\theta)) - f_m(m(z_2,\theta))\right| \geq \underline{\varphi}^{\mu}(c)\right\}\right) \\
&\leq \mathbb{1}\left\{\left|\widehat{R}(m(z_2,\theta)) - R_F(m(z_2,\theta))\right| \geq \underline{\varphi}^{\mu}(c)\right\} + \mathbb{1}\left\{\left|\widehat{f}_m(m(z_2,\theta)) - f_m(m(z_2,\theta))\right| \geq \underline{\varphi}^{\mu}(c)\right\} \\
&\quad \forall\ (z_2,\theta) \in \mathcal{Z}_2 \times \Theta\ \forall\ F \in \mathcal{F}
\end{aligned} \tag{S5.5.11}$$

For $(z_2,\theta) \in \mathcal{Z}_2 \times \Theta$ and $\sigma > 0$, let

$$\begin{aligned}
p^R(Z_i,z_2,\theta,\sigma) &\equiv Z_{1i} \cdot K\left(\frac{m(Z_{2i},\theta) - m(z_2,\theta)}{\sigma}\right),\ p^{f_m}(Z_i,z_2,\theta,\sigma) \equiv K\left(\frac{m(Z_{2i},\theta) - m(z_2,\theta)}{\sigma}\right), \\
\nu_n^R(z_2,\theta,\sigma) &\equiv \frac{1}{n}\sum_{i=1}^{n}\left(p^R(Z_i,z_2,\theta,\sigma) - E_F\left[p^R(Z,z_2,\theta,\sigma)\right]\right), \quad \text{and} \\
\nu_n^{f_m}(z_2,\theta,\sigma) &\equiv \frac{1}{n}\sum_{i=1}^{n}\left(p^{f_m}(Z_i,z_2,\theta,\sigma) - E_F\left[p^{f_m}(Z,z_2,\theta,\sigma)\right]\right).
\end{aligned}$$

Assumption SMIM2 *Consider the following class of functions defined on $\mathcal{S}_{Z_2}$*

$$\mathscr{G}_1 = \left\{g : \mathcal{S}_{Z_2} \to \mathbb{R} : g(z_2) = K\left(\alpha \cdot m(z_2,\theta) + \beta \cdot m(v,\theta)\right)\ \text{for some}\ v \in \mathcal{S}_{Z_2},\ \theta \in \Theta,\ \alpha,\beta \in \mathbb{R}\right\}$$

Then, $\mathscr{G}_1$ is Euclidean for the constant envelope $\overline{K}$.

For indices of the form $m(z_2,\theta) = z_2'\theta$, the condition in Assumption SMIM2 follows immediately from Lemma 22 in Nolan and Pollard (1987), who showed that if $\lambda(\cdot)$ is a real-valued function of bounded variation on $\mathbb{R}$, the class of all functions of the form $x \to \lambda(\gamma' x + \tau)$ with γ ranging over $\mathbb{R}^d$ and τ ranging over $\mathbb{R}$ is Euclidean for a constant envelope. By Assumption SMIM2, the class of functions

$$\left\{g : \mathcal{S}_{Z_2} \to \mathbb{R} : g(z_2) = K\left(\frac{m(z_2,\theta) - m(v,\theta)}{\sigma}\right)\ \text{for some}\ v \in \mathcal{S}_{Z_2},\ \theta \in \Theta,\ \sigma > 0\right\}$$

is Euclidean for the constant envelope $\overline{K}$ and the conditions in Result S1 are satisfied for any integer q (due to the constant nature of the envelope) and we have that, for all $b > 0$ and any integer q, there exists $\overline{M}_1 < \infty$ such that

$$\sup_{F \in \mathcal{F}} P_F\left(\sup_{\substack{(z_2,\theta) \in \mathcal{Z}_2 \times \Theta \\ \sigma > 0}} \left|\nu_n^{f_m}(z_2,\theta,\sigma)\right| > b\right) \leq \frac{\overline{M}_1}{\left(n^{1/2} \cdot b\right)^q} \tag{S5.5.12}$$

From here it follows that, for any $\varepsilon > 0$, there exists a finite $\Delta_\varepsilon > 0$ such that

$$\sup_{F\in\mathcal{F}} P_F\left(\sup_{\substack{(z_2,\theta)\in\mathcal{Z}_2\times\Theta\\ \sigma>0}} \left|n^{1/2}\nu_n^{f_m}(z_2,\theta,\sigma)\right| > \Delta_\varepsilon\right) \leq \varepsilon,$$

which means that

$$\sup_{(z_2,\theta)\in\mathcal{Z}_2\times\Theta} \left|\nu_n^{f_m}(z_2,\theta,\sigma)\right| = O_p\left(\frac{1}{n^{1/2}}\right) \text{ uniformly over } \mathcal{F} \tag{S5.5.13}$$

By Pakes and Pollard (1989, Lemma 2.14), Assumption SMIM2 implies that the class of functions

$$\mathscr{G}_2 = \left\{g : \mathcal{S}_{Z_1}\times\mathcal{S}_{Z_2} \to \mathbb{R} : g(z_1,z_2) = z_1\cdot K\left(\frac{m(z_2,\theta)-m(v,\theta)}{\sigma}\right) \text{ for some } v\in\mathcal{S}_{Z_2},\, \theta\in\Theta,\, \sigma>0\right\}$$

is also Euclidean for the envelope $G(z_1) = |z_1|\cdot\overline{K}$. By Assumption SMIM1, $E_F[G(Z_1)^{4q}] = \overline{K}^{4q}\cdot E_F[Z_1^{4q}] \leq \overline{K}^{4q}\cdot\overline{\mu}_{4q} < \infty$ for all $F\in\mathcal{F}$, from here, Result S1 implies the existence of $\overline{M}_2 < \infty$ such that, for the integer q described in Assumption SMIM1,

$$\sup_{F\in\mathcal{F}} P_F\left(\sup_{\substack{(z_2,\theta)\in\mathcal{Z}_2\times\Theta\\ \sigma>0}} \left|\nu_n^{R}(z_2,\theta,\sigma)\right| > b\right) \leq \frac{\overline{M}_2}{\left(n^{1/2}\cdot b\right)^q} \tag{S5.5.14}$$

which in turn also implies that

$$\sup_{(z_2,\theta)\in\mathcal{Z}_2\times\Theta} \left|\nu_n^{R}(z_2,\theta,\sigma)\right| = O_p\left(\frac{1}{n^{1/2}}\right) \text{ uniformly over } \mathcal{F}. \tag{S5.5.15}$$

We have

$$\begin{aligned}
\widehat{R}(m(z_2,\theta)) &= R_F(m(z_2,\theta)) + \frac{1}{\sigma_n^d}\cdot\nu_n^R(z_2,\theta,\sigma_n) + B_{n,F}^R(z_2,\theta),\\
\widehat{f_m}(m(z_2,\theta)) &= f_m(m(z_2,\theta)) + \frac{1}{\sigma_n^d}\cdot\nu_n^{f_m}(z_2,\theta,\sigma_n) + B_{n,F}^{f_m}(z_2,\theta),
\end{aligned}$$

where

$$B_{n,F}^R(z_2,\theta) \equiv E_F\left[\frac{p^R(Z,z_2,\theta,\sigma_n)}{\sigma_n^d}\right] - R_F(z_2,\theta), \quad B_{n,F}^{f_m}(z_2,\theta) \equiv E_F\left[\frac{p^{f_m}(Z,z_2,\theta,\sigma_n)}{\sigma_n^d}\right] - f_m(z_2,\theta)$$

are the corresponding bias terms. By the smoothness conditions described above and the bias-reducing nature of the kernel K, there exists a constant $C_B^{\mu_a}$ such that

$$\sup_{(z_2,\theta)\in\mathcal{Z}_2\times\Theta}\left|B_{n,F}^{R}(z_2,\theta)\right| \le C_B^{\mu_a}\cdot\sigma_n^L, \quad \sup_{(z_2,\theta)\in\mathcal{Z}_2\times\Theta}\left|B_{n,F}^{f_m}(z_2,\theta)\right| \le C_B^{\mu_a}\cdot\sigma_n^L \quad \forall\ F\in\mathcal{F} \tag{S5.5.16}$$

Define

$$s_{1,n} \equiv C_B^{\mu_a}\cdot\sigma_n^L. \tag{S5.5.17}$$

Then, from (S5.5.13), (S5.5.15) and (S5.5.16)

$$\begin{aligned}
\sup_{(z_2,\theta)\in\mathcal{Z}_2\times\Theta}\left|\widehat{R}(m(z_2,\theta))-R_F(m(z_2,\theta))\right| &\le \frac{1}{\sigma_n^d}\cdot\sup_{(z_2,\theta)\in\mathcal{Z}_2\times\Theta}\left|\nu_n^R(z_2,\theta,\sigma_n)\right|+s_{1,n} = O_p\left(\frac{1}{n^{1/2}\cdot\sigma_n^d}\right)+s_{1,n}\\
\sup_{(z_2,\theta)\in\mathcal{Z}_2\times\Theta}\left|\widehat{f_m}(m(z_2,\theta))-f_m(m(z_2,\theta))\right| &\le \frac{1}{\sigma_n^d}\cdot\sup_{(z_2,\theta)\in\mathcal{Z}_2\times\Theta}\left|\nu_n^{f_m}(z_2,\theta,\sigma_n)\right|+s_{1,n} = O_p\left(\frac{1}{n^{1/2}\cdot\sigma_n^d}\right)+s_{1,n},
\end{aligned} \tag{S5.5.18}$$

uniformly over $\mathcal{F}$. Take any $b>0$ and let n_0 be the smallest integer such that $s_{1,n}<b$. Combining (S5.5.12), (S5.5.14) and (S5.5.18),

$$\begin{aligned}
\sup_{F\in\mathcal{F}} P_F\left(\sup_{(z_2,\theta)\in\mathcal{Z}_2\times\Theta}\left|\widehat{f_m}(m(z_2,\theta))-f_m(m(z_2,\theta))\right|>b\right) &\le \frac{\overline{M}_1}{\left(n^{1/2}\cdot\sigma_n^d\cdot(b-s_{1,n})\right)^q} \quad \forall\ n\ge n_0,\\
\sup_{F\in\mathcal{F}} P_F\left(\sup_{(z_2,\theta)\in\mathcal{Z}_2\times\Theta}\left|\widehat{R}(m(z_2,\theta))-R_F(m(z_2,\theta))\right|>b\right) &\le \frac{\overline{M}_2}{\left(n^{1/2}\cdot\sigma_n^d\cdot(b-s_{1,n})\right)^q} \quad \forall\ n\ge n_0.
\end{aligned} \tag{S5.5.19}$$

Going back to the definition of $\varepsilon_n^{\mu}(m(z_2,\theta))$ in (S5.5.3), recall that

$$\begin{aligned}
\left|\varepsilon_n^{\mu}(m(z_2,\theta))\right| \le &\frac{\left|\widehat{R}(m(z_2,\theta))-R_F(m(z_2,\theta))\right|\cdot\left|\widehat{f_m}(m(z_2,\theta))-f_m(m(z_2,\theta))\right|}{\left|\widetilde{f_m}(m(z_2,\theta))\right|^2}\\
&+\frac{\overline{R}\cdot\left|\widehat{f_m}(m(z_2,\theta))-f_m(m(z_2,\theta))\right|^2}{\left|\widetilde{f_m}(m(z_2,\theta))\right|^3}\\
&+\frac{\left|\widehat{R}(m(z_2,\theta))-R_F(m(z_2,\theta))\right|\cdot\left|\widehat{f_m}(m(z_2,\theta))-f_m(m(z_2,\theta))\right|^2}{\left|\widetilde{f_m}(m(z_2,\theta))\right|^3}
\end{aligned} \tag{S5.5.20}$$

where $\widetilde{f_m}(m(z_2,\theta)$ is an intermediate point between $\widehat{f_m}(m(z_2,\theta)$ and $f_m(m(z_2,\theta)$ and $\overline{R}\equiv\overline{\mu}\cdot\overline{f}_m$. From (S5.5.18) and Assumption SMIM1, it immediately follows that $\sup_{(z_2,\theta)\in\mathcal{Z}_2\times\Theta}\left|\widetilde{R}(m(z_2,\theta))\right| = O_p(1)$ uniformly over $\mathcal{F}$. Assumption SMIM1 and the result in (S5.5.18) also imply that $\sup_{(z_2,\theta)\in\mathcal{Z}_2\times\Theta}\left|\frac{1}{\widetilde{f_m}(m(z_2,\theta))}\right| =$

$O_p(1)$ uniformly over $\mathcal{F}$. To see why, take any $\delta > 0$ and note that

$$P_F\left(\sup_{(z_2,\theta)\in\mathcal{Z}_2\times\Theta}\left|\frac{1}{\widehat{f}_m(m(z_2,\theta))}\right| > \frac{1}{(1-\delta)\cdot \underline{f}_m}\right) \leq P_F\left(\sup_{(z_2,\theta)\in\mathcal{Z}_2\times\Theta}\left|\widehat{f}_m(m(z_2,\theta)) - f_m(m(z_2,\theta))\right| > \delta\cdot \underline{f}_m\right)$$

Let n_0 be the smallest n such that $s_{1,n} < \delta\cdot \underline{f}_m$. Then,

$$\sup_{F\in\mathcal{F}} P_F\left(\sup_{(z_2,\theta)\in\mathcal{Z}_2\times\Theta}\left|\frac{1}{\widehat{f}_m(m(z_2,\theta))}\right| > \frac{1}{(1-\delta)\cdot \underline{f}_m}\right) \leq \frac{\overline{M}_1}{\left(n^{1/2}\cdot\sigma_n^d\cdot(\delta\cdot f_m - s_{1,n})\right)^q} \quad \forall\ n \geq n_0$$

Therefore, for any $\varepsilon > 0$ there exists a small enough δ_ϵ and n_ε such that

$$\sup_{F\in\mathcal{F}} P_F\left(\sup_{(z_2,\theta)\in\mathcal{Z}_2\times\Theta}\left|\frac{1}{\widehat{f}_m(m(z_2,\theta))}\right| > \frac{1}{(1-\delta)\cdot \underline{f}_m}\right) \leq \varepsilon \quad \forall\ n \geq n_\varepsilon,$$

and so $\sup_{(z_2,\theta)\in\mathcal{Z}_2\times\Theta}\left|\frac{1}{\widehat{f}_m(m(z_2,\theta))}\right| = O_p(1)$ uniformly over $\mathcal{F}$. From here, (S5.5.18) and (S5.5.20) yield

$$\sup_{(z_2,\theta)\in\mathcal{Z}_2\times\Theta}\left|\varepsilon_n^\mu(m(z_2,\theta))\right| = O_p\left(\left(\frac{1}{n^{1/2}\cdot\sigma_n^d} + s_{1,n}\right)^2\right), \quad \text{uniformly over } \mathcal{F}. \tag{S5.5.21}$$

And going back to (S5.5.11) we have that, for any $c > 0$,

$$\begin{aligned}\mathbb{1}\left\{\left|\varepsilon_n^\mu(m(z_2,\theta))\right| \geq c\right\} &\leq \mathbb{1}\left\{\left|\nu_n^R(z_2,\theta,\sigma_n)\right| \geq \sigma_n^d\cdot\left(\underline{\varphi}^\mu(c) - s_{1,n}\right)\right\}\\ &\quad + \mathbb{1}\left\{\left|\nu_n^{f_m}(z_2,\theta,\sigma_n)\right| \geq \sigma_n^d\cdot\left(\underline{\varphi}^\mu(c) - s_{1,n}\right)\right\}\\ &\quad \forall\ (z_2,\theta)\in\mathcal{Z}_2\times\Theta,\ \forall\ F\in\mathcal{F}.\end{aligned}$$

where $\underline{\varphi}^\mu(c)$ is defined in (S5.5.10). Thus, if we take any $c > 0$ and we let n_0 be the smallest integer such that $s_{1,n} < \underline{\varphi}^\mu(c)$, (S5.5.19) and the previous expression yield

$$\sup_{F\in\mathcal{F}} P_F\left(\sup_{(z_2,\theta)\in\mathcal{Z}_2\times\Theta}\left|\varepsilon_n^\mu(m(z_2,\theta))\right| \geq c\right) \leq \frac{\overline{M}_1 + \overline{M}_2}{\left(n^{1/2}\cdot\sigma_n^d\cdot(\underline{\varphi}^\mu(c) - s_{1,n})\right)^q} \quad \forall\ n \geq n_0. \tag{S5.5.22}$$

For a given $\theta\in\Theta$ define

$$\nu_n^\mu(\theta) \equiv \frac{1}{n}\sum_{i=1}^n \phi(Z_{2i})\cdot\left(\widehat{\mu}(m(Z_{2i},\theta)) - \mu_F(m(Z_{2i},\theta))\right). \tag{S5.5.23}$$

For a given $\sigma > 0$, let

$$
\begin{aligned}
m_F^{\mu}(Z_i, Z_j, \theta) &\equiv \frac{1}{2}\left(\phi(Z_{2j}) \cdot \frac{\big(Z_{1i} - \mu_F(m(Z_{2j}, \theta))\big)}{f_m(m(Z_{2j}, \theta))} + \phi(Z_{2i}) \cdot \frac{\big(Z_{1j} - \mu_F(m(Z_{2i}, \theta))\big)}{f_m(m(Z_{2i}, \theta))}\right), \\
p_F^{\mu}(Z_i, Z_j; \theta, \sigma) &\equiv m_F^{\mu}(Z_i, Z_j, \theta) \cdot K\left(\frac{m(Z_{2i}, \theta) - m(Z_{2j}, \theta)}{\sigma}\right), \\
U_{2,n}^{\mu}(\theta, \sigma) &= \binom{2}{n}^{-1} \sum_{i<j} p_F^{\mu}(Z_i, Z_j; \theta, \sigma),
\end{aligned}
\tag{S5.5.24}
$$

and denote $U_{2,n}^{\mu}(\theta, \sigma_n) \equiv U_{2,n}^{\mu}(\theta)$. Let

$$
Q_F(z, \theta) \equiv \phi(z_2) \cdot \frac{(z_1 - \mu_F(m(z_2, \theta)))}{f_m(m(z_2, \theta)}.
$$

From (S5.5.3), we have

$$
\nu_n^{\mu}(\theta) = \frac{1}{\sigma_n^d} \cdot U_{2,n}^{\mu}(\theta) + \left(\frac{K(0)}{n \cdot \sigma_n^d}\right) \cdot \frac{1}{n} \sum_{i=1}^{n} Q_F(Z_i, \theta) + \frac{1}{n} \sum_{i=1}^{n} \phi(Z_{2i}) \cdot \varepsilon_n^{\mu}(m(Z_{2i}, \theta)) \tag{S5.5.25}
$$

Recall that we have assumed that there exist constants $\underline{f}_m > 0$, $\overline{f}_m < \infty$ and $\overline{\mu} < \infty$ such that $\overline{f}_m \geq f_m(m) \geq \underline{f}_m$ and $|\mu_F(m)| \leq \overline{\mu}$ $\forall\, m \in \mathcal{M}$ and all $F \in \mathcal{F}$. We have also assumed that both $f_m(m)$ and $\mu_F(m)$ are L-times continuously differentiable with respect to m for F–a.e $m \in \mathcal{M}$, with derivatives that are uniformly bounded over $\mathcal{M}$ for all $F \in \mathcal{F}$. Let

$$
\eta_F(m(Z_2, \theta)) \equiv E_F[\phi(Z_2) \mid m(Z_2, \theta)],
$$

and assume that, like the other functionals analyzed before, $\eta_F(m)$ is also L-times continuously differentiable with respect to m for F–a.e $m \in \mathcal{M}$ with derivatives that are uniformly bounded over $\mathcal{M}$ for all $F \in \mathcal{F}$. For a given $\theta \in \Theta$ and $\sigma > 0$, let

$$
\begin{aligned}
r_{1,F}^{\mu}(Z_i; \theta, \sigma) &= E_F\left[p_F^{\mu}(Z_i, Z_j; \theta, \sigma) \middle| Z_i\right] - E_F\left[p_F^{\mu}(Z_i, Z_j; \theta, \sigma)\right], \\
r_{2,F}^{\mu}(Z_i, Z_j; \theta, \sigma) &= \left(p_F^{\mu}(Z_i, Z_j; \theta, \sigma) - E_F\left[p_F^{\mu}(Z_i, Z_j; \theta, \sigma)\right]\right) - r_{1,F}^{\mu}(Z_i; \theta, \sigma) - r_{1,F}^{\mu}(Z_j; \theta, \sigma), \\
V_{2,n}^{\mu}(\theta, \sigma) &\equiv \binom{2}{n}^{-1} \sum_{i<j} r_{2,F}^{\mu}(Z_i, Z_j; \theta, \sigma)
\end{aligned}
$$

$V_{2,n}^{\mu}(\theta, \sigma)$ is a degenerate U-statistic of order 2 and $\left\{V_{2,n}^{\mu}(\theta, \sigma) : \theta \in \Theta,\, \sigma > 0\right\}$ is a degenerate U-process of order 2, and therefore compatible with the conditions for Result S1 under the assumptions we will describe below.

Let us denote $V^{\mu}_{2,n}(\theta,\sigma_n) \equiv V^{\mu}_{2,n}(\theta)$. A Hoeffding decomposition (see Serfling (1980, pages 177-178) or Sherman (1994, equations (6)-(7))) of the U-statistic $U^{\mu}_{2,n}(\theta)$ in (S5.5.24), combined with the higher-order kernel properties and the smoothness conditions described above yield the following result,

$$\frac{1}{\sigma_n^d} \cdot U^{\mu}_{2,n}(\theta) = \frac{1}{n}\sum_{i=1}^{n} \eta_F(m(Z_{2i},\theta)) \cdot \Big(Z_{1i} - \mu_F(m(Z_{2i},\theta))\Big) + \frac{1}{\sigma_n^d} \cdot V^{\mu}_{2,n}(\theta) + B^{\mu}_{n,F}(\theta) \tag{S5.5.26}$$

where $B_{n,\mu}(\theta)$ is a bias term which, by our smoothness assumptions, is such that there exists a constant $C_B^{\mu_b}$ such that

$$\sup_{\theta\in\Theta}\left\|B^{\mu}_{n,F}(\theta)\right\| \leq C_B^{\mu_b} \cdot \sigma_n^L \quad \forall\, F \in \mathcal{F} \tag{S5.5.27}$$

Plugging (S5.5.26) into (S5.5.25), we have

$$\begin{aligned} \nu^{\mu}_n(\theta) &= \frac{1}{n}\sum_{i=1}^{n} \eta_F(m(Z_{2i},\theta)) \cdot \Big(Z_{1i} - \mu_F(m(Z_{2i},\theta))\Big) + \varepsilon_n^{\nu^{\mu}}(\theta), \quad \text{where} \\ \varepsilon_n^{\nu^{\mu}}(\theta) &\equiv \frac{1}{\sigma_n^d} \cdot V^{\mu}_{2,n}(\theta) + \left(\frac{K(0)}{n\cdot\sigma_n^d}\right) \cdot \frac{1}{n}\sum_{i=1}^{n}\Big(Q_F(Z_i,\theta) - E_F\left[Q_F(Z,\theta)\right]\Big) \\ &\quad + \frac{1}{n}\sum_{i=1}^{n} \phi(Z_{2i}) \cdot \varepsilon^{\mu}_n(m(Z_{2i},\theta)) + \left(\frac{K(0)}{n\cdot\sigma_n^d}\right) \cdot E_F\left[Q_F(Z,\theta)\right] + B^{\mu}_{n,F}(\theta) \end{aligned} \tag{S5.5.28}$$

From our assumptions, it follows that there exists a constant $C_Q < \infty$ such that

$$\sup_{\theta\in\Theta}\Big|E_F\left[Q_F(Z,\theta)\right]\Big| \leq C_Q \quad \forall\, F \in \mathcal{F}$$

Define

$$s_{2,n} \equiv \frac{|K(0)|\cdot C_Q}{n\cdot\sigma_n^d} + C_B^{\mu_b} \cdot \sigma_n^L. \tag{S5.5.29}$$

Thus, from (S5.5.27) and the previous expression,

$$\begin{aligned} \left\|\varepsilon_n^{\nu^{\mu}}(\theta)\right\| \leq & \frac{1}{\sigma_n^d} \cdot \left\|V^{\mu}_{2,n}(\theta)\right\| + \left|\frac{K(0)}{n\cdot\sigma_n^d}\right| \cdot \left\|\frac{1}{n}\sum_{i=1}^{n}\Big(Q_F(Z_i,\theta) - E_F\left[Q_F(Z,\theta)\right]\Big)\right\| \\ & + \left\|\frac{1}{n}\sum_{i=1}^{n} \phi(Z_{2i}) \cdot \varepsilon^{\mu}_n(m(Z_{2i},\theta))\right\| + s_{2,n} \end{aligned} \tag{S5.5.30}$$

Assumption SMIM3 *The index $m(z_2,\theta)$ is smooth with respect to θ and, for every $F \in \mathcal{F}$, the following*

Jacobians are well-defined for F−a.e $z_2 \in \mathcal{S}_{Z_2}$ and for all $\theta \in \Theta$,

$$\underbrace{\nabla_\theta \mu_F(m(z_2;\theta))}_{1\times k} \equiv \begin{pmatrix} \frac{\partial \mu_F(m(z_2;\theta)'}{\partial \theta_1} & \frac{\partial \mu_F(m(z_2;\theta)'}{\partial \theta_2} & \cdots & \frac{\partial \mu_F(m(z_2;\theta)'}{\partial \theta_d} \end{pmatrix},$$

$$\underbrace{\nabla_\theta f_m(m(z_2;\theta))}_{1\times k} \equiv \begin{pmatrix} \frac{\partial f_m(m(z_2;\theta)'}{\partial \theta_1} & \frac{\partial f_m(m(z_2;\theta)'}{\partial \theta_2} & \cdots & \frac{\partial f_m(m(z_2;\theta)'}{\partial \theta_d} \end{pmatrix}$$

There exists a nonnegative function $\overline{H}_1(\cdot)$ such that, for each $F \in \mathcal{F}$,

$$\sup_{\theta\in\Theta} \left\|\nabla_\theta \mu_F(m(z_2;\theta))\right\| \leq \overline{H}_1(z_2) \quad \forall\ z_2 \in \mathcal{Z}_2,$$
$$\sup_{\theta\in\Theta} \|\nabla_\theta f_m(m(z_2;\theta))\| \leq \overline{H}_1(z_2) \quad \forall\ z_2 \in \mathcal{Z}_2,$$

and there exists $\overline{\mu}_{\overline{H}_1} < \infty$ such that $E_F[\overline{H}_1(Z_2)^{4q}] \leq \overline{\mu}_{\overline{H}_1}$ $\forall\ F \in \mathcal{F}$, where q is the integer described in Assumption SMIM1.

If we let m_F^μ be as defined in (S5.5.24), and

$$\overline{G}_1(Z_i, Z_j) \equiv \left(\frac{(|Z_{1i}| + \overline{f}_m + \overline{\mu})}{2\underline{f}_m^2} \cdot \|\phi(Z_{2j})\| \cdot \overline{H}_1(Z_{2j}) + \frac{(|Z_{1j}| + \overline{f}_m + \overline{\mu})}{2\underline{f}_m^2} \cdot \|\phi(Z_{2i})\| \cdot \overline{H}_1(Z_{2i}) \right)$$

where $\underline{f}_m$, $\overline{f}_m$ and $\overline{\mu}$ are as defined in Assumption SMIM1, then, for each $F \in \mathcal{F}$,

$$\left\|m_F^\mu(Z_i, Z_j, \theta) - m_F^\mu(Z_i, Z_j, \theta')\right\| \leq \overline{G}_1(Z_i, Z_j) \cdot \left\|\theta - \theta'\right\| \quad \forall\ \theta, \theta' \in \Theta$$

Let q be the integer described in Assumption SMIM1. By Assumption SMIM3, there exists $\overline{\mu}_{\overline{G}_1} < \infty$ such that,

$$E_F\left[\overline{G}_1(Z_i, Z_j)^{4q}\right] \leq \overline{\mu}_{\overline{G}_1} \quad \forall\ F \in \mathcal{F}$$

For the ℓ^{th} component ($\phi_\ell(Z_2)$) of $\phi(Z_2)$ and for each F let

$$\mathscr{G}_{3,F}^\ell = \Bigg\{ g : \mathcal{S}_Z^2 \to \mathbb{R} : g(z_a, z_b) = \frac{1}{2}\left(\phi_\ell(z_{2b}) \cdot \frac{(z_{1a} - \mu_F(m(z_{2b}, \theta)))}{f_m(m(z_{2b}, \theta))} + \phi_\ell(z_{2a}) \cdot \frac{(z_{1b} - \mu_F(m(z_{2a}, \theta)))}{f_m(m(z_{2a}, \theta))} \right)$$
$$\times K\left(\frac{m(z_{2a}, \theta) - m(z_{2b}, \theta)}{\sigma} \right) \quad \text{for some } \ \theta \in \Theta,\ \sigma > 0 \Bigg\}$$

By Assumptions SMIM2 and SMIM3, Lemmas 2.13 and 2.14 in Pakes and Pollard (1989), there exist positive constants A_3 and V_3 such that, for every $F \in \mathcal{F}$, the class $\mathscr{G}_{3,F}^\ell$ is Euclidean (A_3, V_3)

for the envelope

$$G_3(z_a,z_b)=\frac{1}{2\underline{f}_m}\left(\left\|\phi(z_{2b})\right\|\cdot\left|(z_{1a}-\mu_F(m(z_{2b},\theta_0)))\right|+\left\|\phi(z_{2a})\right\|\cdot\left|(z_{1b}-\mu_F(m(z_{2a},\theta_0)))\right|\right)+M_0\cdot\overline{G}_1(z_a,z_b)$$

where θ_0 is an arbitrary point of Θ and $M_0^\ell\equiv 2\sqrt{k}\sup_\Theta\|\theta-\theta_0\|$. Let q be the integer described in Assumption SMIM1. By the conditions described in AssumptionsSMIM1 and SMIM3, there exists $\overline{\mu}_{G_3}<\infty$ such that $E_F\left[G_3(Z_i,Z_j)^{4q}\right]\le\overline{\mu}_{G_3}$ for all $F\in\mathcal{F}$. Next, let

$$\mathscr{G}_{4,F}^\ell=\left\{f:\mathcal{S}_Z\to\mathbb{R}:f(z)=E_F\left[g(z,Z)\right]\ \text{ for some }\ g\in\mathscr{G}_{3,F}^\ell\right\}$$

By Lemma 20 in Nolan and Pollard (1987) (or Lemma 5 in Sherman (1994)), Assumptions SMIM2 and SMIM3 imply that there exist positive constants A_4 and V_4 such that $\mathscr{G}_{4,F}^\ell$ is Euclidean (A_4,V_4) for the envelope

$$G_4(z)=\sqrt{E_F\left[G_3(z,Z)^2\right]}$$

Let q be any positive integer. By Jensen's inequality, $G_4(z)^{4q}=\left(E_F\left[G_3(z,Z)^2\right]\right)^{2q}\le E_F\left[G_3(z,Z)^{4q}\right]$. Therefore, $E_F\left[G_4(Z_i)^{4q}\right]\le E_F\left[G_3(Z_i,Z_j)^{4q}\right]\le\overline{\mu}_{G_3}$. The conditions in Result S1 are satisfied for the integer q described in Assumption SMIM2 and there exists a constant $\overline{M}_3<\infty$ such that, for all $b>0$,

$$\sup_{F\in\mathcal{F}}P_F\left(\sup_{\theta\in\Theta}\left\|V_{2,n}^\mu(\theta)\right\|>b\right)\le\sup_{F\in\mathcal{F}}P_F\left(\sup_{\substack{\theta\in\Theta\\h>0}}\left\|V_{2,n}^\mu(\theta)\right\|>b\right)\le\frac{\overline{M}_3}{(n\cdot b)^q}\tag{S5.5.31}$$

which in turn also implies that

$$\sup_{\theta\in\Theta}\left\|V_{2,n}^\mu(\theta)\right\|=O_p\left(\frac{1}{n}\right)\ \text{ uniformly over }\mathcal{F}.\tag{S5.5.32}$$

For ϕ_ℓ, the ℓ^{th} component of ϕ, let

$$\mathscr{G}_{5,F}^\ell=\left\{g:\mathcal{S}_Z\to\mathbb{R}:g(z)=\phi_\ell(z_2)\cdot\frac{(z_1-\mu_F(m(z_2,\theta)))}{f_m(m(z_2,\theta))}\quad\text{for some }\ \theta\in\Theta\right\}$$

Let

$$\overline{G}_2(z)\equiv\frac{(|z_1|+\overline{f}_m+\overline{\mu})}{2\underline{f}_m^2}\cdot\|\phi(z_2)\|\cdot\overline{H}_1(z_2),$$

where $\overline{H}_1(\cdot)$ is as described in Assumption SMIM3. By the conditions described there, for any

$F \in \mathcal{F}$, we have

$$\left|\phi_\ell(z_2)\cdot\frac{(z_1-\mu_F(m(z_2,\theta)))}{f_m(m(z_2,\theta))}-\phi_\ell(z_2)\cdot\frac{(z_1-\mu_F(m(z_2,\theta')))}{f_m(m(z_2,\theta'))}\right|\leq\overline{G}_2(z)\cdot\left\|\theta-\theta'\right\|\quad\forall\ \theta,\theta'\in\Theta.$$

By Assumptions SMIM2 and SMIM3, Lemmas 2.13 and 2.14 in Pakes and Pollard (1989), there exist positive constants A_5 and V_5 such that, for every $F \in \mathcal{F}$, the class $\mathcal{G}^{\ell}_{5,F}$ is Euclidean (A_5, V_5) for the envelope

$$G_5(z)=\frac{1}{\underline{f}_m}\cdot\left\|\phi(z_2)\right\|\cdot\left|(z_1-\mu_F(m(z_2,\theta_0)))\right|+M_0\cdot\overline{G}_2(z)$$

where θ_0 is an arbitrary point of Θ and $M_0 \equiv 2\sqrt{k}\sup_{\Theta}\|\theta-\theta_0\|$. By the conditions in Assumption SMIM3, there exists $\overline{\mu}_{G_5} < \infty$ such that $E_F\left[G_5(Z)^{4q}\right] \leq \overline{\mu}_{G_5}$ for all $F \in \mathcal{F}$. Thus, conditions in Result S1 are satisfied for the integer q described in Assumption SMIM2 and there exists a constant $\overline{M}_4 < \infty$ such that, for all $b > 0$,

$$\sup_{F\in\mathcal{F}}\ P_F\left(\sup_{\theta\in\Theta}\left\|\frac{1}{n}\sum_{i=1}^{n}\Big(Q_F(Z_i,\theta)-E_F\left[Q_F(Z,\theta)\right]\Big)\right\|>b\right)\leq\frac{\overline{M}_4}{\left(n^{1/2}\cdot b\right)^q}\tag{S5.5.33}$$

which in turn also implies that

$$\sup_{\theta\in\Theta}\left\|\frac{1}{n}\sum_{i=1}^{n}\Big(Q_F(Z_i,\theta)-E_F\left[Q_F(Z,\theta)\right]\Big)\right\|=O_p\left(\frac{1}{n^{1/2}}\right)\text{ uniformly over }\mathcal{F}.\tag{S5.5.34}$$

Now, going back to (S5.5.30), we have

$$\begin{aligned}\sup_{\theta\in\Theta}\left\|\varepsilon_n^{\nu^\mu}(\theta)\right\|\leq&\frac{1}{\sigma_n^d}\cdot\sup_{\theta\in\Theta}\left\|V_{2,n}^{\mu}(\theta)\right\|+\left|\frac{K(0)}{n\cdot\sigma_n^d}\right|\cdot\sup_{\theta\in\Theta}\left\|\frac{1}{n}\sum_{i=1}^{n}\Big(Q_F(Z_i,\theta)-E_F\left[Q_F(Z,\theta)\right]\Big)\right\|\\&+\overline{\phi}\cdot\sup_{(x,\theta)\in\mathcal{X}\times\Theta}\left|\varepsilon_n^{\mu}(m(x,\theta))\right|+s_{2,n}\end{aligned}$$

And so, from (S5.5.21), (S5.5.32) and (S5.5.34), we have that, uniformly over $\mathcal{F}$,

$$\sup_{\theta\in\Theta}\left\|\varepsilon_n^{\nu^\mu}(\theta)\right\|=O_p\left(\frac{1}{n\cdot\sigma_n^d}\right)+O_p\left(\frac{1}{n^{3/2}\cdot\sigma_n^d}\right)+O_p\left(\left(\frac{1}{n^{1/2}\cdot\sigma_n^d}+s_{1,n}\right)^2\right)+s_{2,n}\tag{S5.5.35}$$

Take any $b > 0$ and let n_0 be the smallest integer such that

$$s_{2,n}<b\quad\text{and}\quad s_{1,n}<\underbrace{\min\left\{\frac{\underline{f}_m}{2},\,D^{\varepsilon^\mu}\cdot\left(\frac{b-s_{2,n}}{3\overline{\phi}}\right),\,D^{\varepsilon^\mu}\cdot\left(\frac{b-s_{2,n}}{3\overline{\phi}}\right)^{1/4}\right\}}_{=\underline{\varphi}^{\mu}\left(\frac{b-s_{2,n}}{3\overline{\phi}}\right)\ \text{(see (S5.5.10))}}$$

Then, from (S5.5.22), (S5.5.31) and (S5.5.33),

$$\begin{aligned}\sup_{F\in\mathcal{F}} P_F\left(\sup_{\theta\in\Theta}\left\|\varepsilon_n^{\nu^\mu}(\theta)\right\|>b\right)\leq&\frac{\overline{M}_3}{\left(n\cdot\sigma_n^d\cdot\left(\frac{b-s_{2,n}}{3}\right)\right)^q}+\frac{\overline{M}_4}{\left(n^{3/2}\cdot\sigma_n^d\cdot\left(\frac{b-s_{2,n}}{3|K(0)|}\right)\right)^q}\\&+\frac{\overline{M}_1+\overline{M}_2}{\left(n^{1/2}\cdot\sigma_n^d\cdot\left(\underline{\varphi}^\mu\left(\frac{b-s_{2,n}}{3\overline{\phi}}\right)-s_{1,n}\right)\right)^q}\qquad\forall\ n\geq n_0\end{aligned}\tag{S5.5.36}$$

Equipped with the result in (S5.5.36), let us go back to the analysis of the estimator $\widehat{\theta}$ described by (S5.5.2). Recall that we defined

$$\underbrace{\eta_F(m(Z_2,\theta))}_{k\times1}\equiv E_F\left[\phi(Z_2)|m(Z_2,\theta)\right]=\begin{pmatrix}E_F\left[\phi_1(Z_2)|m(Z_2,\theta)\right]\\E_F\left[\phi_2(Z_2)|m(Z_2,\theta)\right]\\\vdots\\E_F\left[\phi_k(Z_2)|m(Z_2,\theta)\right]\end{pmatrix}\equiv\begin{pmatrix}\eta_{1,F}(m(Z_2,\theta))\\\eta_{2,F}(m(Z_2,\theta))\\\vdots\\\eta_{k,F}(m(Z_2,\theta))\end{pmatrix}$$

We add the following smoothness conditions to those described in Assumption SMIM3.

Assumption SMIM4 *For every $F\in\mathcal{F}$, the following Jacobians are well-defined for $F-$a.e $z_2\in\mathcal{S}_{Z_2}$ and everywhere on Θ,*

$$\underbrace{\nabla_\theta\eta_{\ell,F}(m(z_2,\theta))}_{1\times k}\equiv\left(\frac{\partial\eta_{\ell,F}(m(z_2,\theta)'}{\partial\theta_1}\quad\frac{\partial\eta_{\ell,F}(m(z_2,\theta)'}{\partial\theta_2}\quad\cdots\quad\frac{\partial\eta_{\ell,F}(m(z_2,\theta)'}{\partial\theta_d}\right),\quad\ell=1,\ldots,k\tag{S5.5.37}$$

Let

$$\underbrace{\nabla_\theta\eta_F(m(z_2,\theta)}_{k\times k}\equiv\begin{pmatrix}\nabla_\theta\eta_{1,F}(m(z_2,\theta))\\\nabla_\theta\eta_{2,F}(m(z_2,\theta))\\\vdots\\\nabla_\theta\eta_{k,F}(m(z_2,\theta))\end{pmatrix}$$

and express $\phi(z_2) \equiv (\phi_1(z_2), \phi_2(z_2), \ldots, \phi_k(z_2))' \in \mathbb{R}^k$. *For* $\ell = 1, \ldots, k$, *define*

$$
\begin{aligned}
\underbrace{T_{\ell,F}(Z,\theta)}_{1\times k} &\equiv \Big(\phi_\ell(Z_2) - \eta_{\ell,F}(m(Z_2,\theta))\Big) \cdot \nabla_\theta \mu_F(m(Z_2,\theta)) + \nabla_\theta \eta_{\ell,F}(m(Z_2,\theta)) \cdot \Big(Z_1 - \mu_F(m(Z_2,\theta))\Big), \\
\underbrace{T_F(Z,\theta)}_{k^2\times 1} &\equiv \Big(T_{1,F}(Z,\theta) \quad T_{2,F}(Z,\theta) \quad \cdots \quad T_{k,F}(Z,\theta)\Big)', \\
\underbrace{\lambda_{\ell,F}(\theta)}_{1\times k} &\equiv E_F\left[T_{\ell,F}(Z,\theta)\right], \\
\underbrace{\lambda_F(\theta)}_{k^2\times 1} &\equiv E\left[T_F(Z,\theta)\right] = \Big(\lambda_{1,F}(\theta) \quad \lambda_{2,F}(\theta) \quad \cdots \quad \lambda_{k,F}(\theta)\Big)'
\end{aligned}
$$

(i) *There exists a nonnegative function* $\overline{H}_2(\cdot)$ *such that, for each* $F \in \mathcal{F}$,

$$\sup_{\theta\in\Theta} \left\|\nabla_\theta \eta_F(m(z_2,\theta))\right\| \leq \overline{H}_2(z_2) \quad \forall\, z_2 \in \mathcal{Z}_2$$

and there exists $\overline{\mu}_{\overline{H}_6} < \infty$ *such that* $E_F\left[\overline{H}_2(Z_2)^{4q}\right] \leq \overline{\mu}_{\overline{H}_6}$ *for all* $F \in \mathcal{F}$, *where* q *is the integer described in Assumption SMIM1. Note that this condition, combined with Assumptions SMIM1 and SMIM3 imply that there exists a nonnegative function* $\overline{G}_6(\cdot)$ *such that, for all* $F \in \mathcal{F}$,

$$\left\|T_F(z,\theta) - T_F(z,\theta')\right\| \leq \overline{G}_6(z) \cdot \left\|\theta - \theta'\right\| \quad \forall\, z \in \mathcal{S}_Z \quad and \quad \theta, \theta' \in \Theta,$$

and there exists $\overline{\mu}_{\overline{G}_6} < \infty$ *such that* $E_F\left[\overline{G}_6(Z)^{4q}\right] \leq \overline{\mu}_{\overline{G}_6}$ $\forall\, F \in \mathcal{F}$, *where* q *is the integer described in Assumption SMIM1.*

(ii) *Let* H_k *and* M_k *be as defined in* (S5.1.1). *Assume that* $\exists\ \underline{d} > 0$, $\overline{M}_\lambda$, $K_5 > 0$, $K_6 > 0$ *and* $\alpha_1 > 0$ *such that, for every* $F \in \mathcal{F}$,

$$
\begin{gathered}
\inf_{\theta\in\Theta} \left|det\left(H_k(\lambda_F(\theta))\right)\right| \geq \underline{d} \quad \sup_{\theta\in\Theta} \left\|M_k(\lambda_F(\theta))\right\| \leq \overline{M}_\lambda \\
\left\|M_k(\lambda_F(\theta)) - M_k(v)\right\| \leq K_6 \cdot \|\lambda_F(\theta) - v\|^{\alpha_1} \quad \forall\, v, \theta : \|v - \lambda_F(\theta)\| \leq K_5,\ \theta \in \Theta.
\end{gathered}
\tag{S5.5.38}
$$

And,

$$\sup_{\substack{v:\|v-\lambda_F(\theta)\|\leq K_5 \\ \theta\in\Theta}} \left\{\left\|M_k(\lambda_F(\theta)) - M_k(v)\right\|\right\} \leq K_7 < \infty$$

(iii) $\exists\ K_8 > 0$, $K_9 > 0$ *and* $\alpha_2 > 0$ *such that, for every* $F \in \mathcal{F}$,

$$\left\|\lambda_F(\theta) - \lambda_F(\theta^*)\right\| \leq K_9 \cdot \|\theta - \theta^*\|^{\alpha_2} \quad \forall\, \theta : \|\theta - \theta^*\| \leq K_8$$

Let us go back to (S5.5.2), which defines the estimator $\widehat{\theta}$ by the sample-analog moment condition $\frac{1}{n}\sum_{i=1}^{n}\phi(Z_{2i})\cdot\Big(Z_{1i}-\widehat{\mu}(m(Z_{2i},\widehat{\theta}))\Big)=0$. From here we obtain,

$$\begin{aligned}0&=\frac{1}{n}\sum_{i=1}^{n}\phi(Z_{2i})\cdot\Big(Z_{1i}-\mu_F\Big(m(Z_{2i},\theta^*)\Big)\Big)+\frac{1}{n}\sum_{i=1}^{n}\phi(Z_{2i})\cdot\Big(\mu_F\Big(m(Z_{2i},\theta^*)\Big)-\mu_F\Big(m(Z_{2i},\widehat{\theta})\Big)\Big)\\&\quad+\underbrace{\frac{1}{n}\sum_{i=1}^{n}\phi(Z_{2i})\cdot\Big(\mu_F\Big(m(Z_{2i},\widehat{\theta})\Big)-\widehat{\mu}\Big(m(Z_{2i},\widehat{\theta})\Big)\Big)}_{-\nu_n^{\mu}(\widehat{\theta})\text{ (see (S5.5.23))}}\end{aligned}\tag{S5.5.39}$$

From our result in (S5.5.28), we have

$$\nu_n^{\mu}(\widehat{\theta})=\frac{1}{n}\sum_{i=1}^{n}\eta_F(m(Z_{2i},\widehat{\theta}))\cdot\Big(Z_{1i}-\mu_F\Big(m(Z_{2i},\widehat{\theta})\Big)\Big)+\varepsilon_n^{\nu^{\mu}}(\widehat{\theta}).$$

Thus, (S5.5.39) becomes

$$\begin{aligned}0&=\frac{1}{n}\sum_{i=1}^{n}\phi(Z_{2i})\cdot\Big(Z_{1i}-\mu_F\Big(m(Z_{2i},\theta^*)\Big)\Big)+\frac{1}{n}\sum_{i=1}^{n}\phi(Z_{2i})\cdot\mu_F\Big(m(Z_{2i},\theta^*)\Big)\\&\quad-\frac{1}{n}\sum_{i=1}^{n}\Big[\phi(Z_{2i})\cdot\mu_F\Big(m(Z_{2i},\widehat{\theta})\Big)+\eta_F\Big(m(Z_{2i},\widehat{\theta})\Big)\cdot\Big(Z_{1i}-\mu_F\Big(m(Z_{2i},\widehat{\theta})\Big)\Big)\Big]\\&\quad-\varepsilon_n^{\nu^{\mu}}(\widehat{\theta})\end{aligned}$$

And from here, using the Jacobians defined in (S5.5.37) and the Mean Value Theorem, the previous expression becomes,

$$\begin{aligned}0&=\frac{1}{n}\sum_{i=1}^{n}\Big(\phi(Z_{2i})-\eta_F\Big(m(Z_{2i},\theta^*)\Big)\Big)\cdot\Big(Z_{1i}-\mu_F\Big(m(Z_{2i},\theta^*)\Big)\Big)\\&-\left\{\frac{1}{n}\sum_{i=1}^{n}\Big[\Big(\phi(Z_{2i})-\eta_F\Big(m(Z_{2i},\overline{\theta})\Big)\Big)\cdot\nabla_\theta\mu_F\Big(m(Z_{2i},\overline{\theta})\Big)+\nabla_\theta\eta_F\Big(m(Z_{2i},\overline{\theta})\Big)\cdot\Big(Z_{1i}-\mu_F\Big(m(Z_{2i},\overline{\theta})\Big)\Big)\Big]\right\}\\&\quad\times\Big(\widehat{\theta}-\theta^*\Big)\\&-\varepsilon_n^{\nu^{\mu}}(\widehat{\theta})\end{aligned}\tag{S5.5.40}$$

where $\overline{\theta}$ belongs in the line segment connecting $\widehat{\theta}$ and θ^* (thus $\overline{\theta}\in\Theta$). Let $T_{\ell,F}$ and T_F be as defined in Assumption SMIM4 and let

$$\underbrace{\overline{T}_{\ell}(\theta)}_{1\times k} \equiv \frac{1}{n}\sum_{i=1}^{n} T_{\ell,F}(Z_i,\theta),$$
$$\underbrace{\overline{T}(\theta)}_{k^2\times 1} \equiv \frac{1}{n}\sum_{i=1}^{n} T_F(Z_i,\theta) = \Big(\overline{T}_1(\theta) \quad \overline{T}_2(\theta) \quad \cdots \quad \overline{T}_k(\theta)\Big)'$$

Using our definition of M_k in (S5.1.1), the expression in (S5.5.40) yields,

$$\begin{aligned}\widehat{\theta} &= \theta^* + M_k\left(\overline{T}(\overline{\theta})\right)\cdot\frac{1}{n}\sum_{i=1}^{n}\Big(\phi(Z_{2i}) - \eta_F\Big(m(Z_{2i},\theta^*)\Big)\Big)\cdot\Big(Z_{1i} - \mu_F\Big(m(Z_{2i},\theta^*)\Big)\Big)\\ &- M_k\left(\overline{T}(\overline{\theta})\right)\cdot\varepsilon_n^{\nu^\mu}(\widehat{\theta})\end{aligned} \tag{S5.5.41}$$

Define

$$\begin{aligned}\zeta_F(Z_i) &\equiv \Big(\phi(Z_{2i}) - \eta_F\Big(m(Z_{2i},\theta^*)\Big)\Big)\cdot\Big(Z_{1i} - \mu_F\Big(m(Z_{2i},\theta^*)\Big)\Big),\\ \psi_F^{\theta}(Z_i) &\equiv M_k\left(\lambda_F(\theta^*)\right)\cdot\zeta_F(Z_i).\end{aligned} \tag{S5.5.42}$$

Note that $E_F[\zeta_F(Z)] = E_F[\psi_F^{\theta}(Z)] = 0$. We can re-express (S5.5.41) as,

$$\begin{aligned}\widehat{\theta} &= \theta^* + \frac{1}{n}\sum_{i=1}^{n}\psi_F^{\theta}(Z_i) + \varepsilon_n^{\theta}, \quad \text{where}\\ \varepsilon_n^{\theta} &\equiv \Big(M_k\left(\overline{T}(\overline{\theta})\right) - M_k\left(\lambda_F(\overline{\theta})\right)\Big)\cdot\frac{1}{n}\sum_{i=1}^{n}\zeta_F(Z_i)\\ &\quad + \Big(M_k\left(\lambda_F(\overline{\theta})\right) - M_k\left(\lambda_F(\theta^*)\right)\Big)\cdot\frac{1}{n}\sum_{i=1}^{n}\zeta_F(Z_i)\\ &\quad - \Big(M_k\left(\overline{T}(\overline{\theta})\right) - M_k\left(\lambda_F(\overline{\theta})\right)\Big)\cdot\varepsilon_n^{\nu^\mu}(\widehat{\theta})\\ &\quad - M_k\left(\lambda_F(\overline{\theta})\right)\cdot\varepsilon_n^{\nu^\mu}(\widehat{\theta})\end{aligned} \tag{S5.5.43}$$

Recall from the conditions in (S5.5.38) that $\sup_{\theta\in\Theta}\left\|M_k(\lambda_F(\theta))\right\| \leq \overline{M}_\lambda$. Therefore,

$$\begin{aligned}\left\|\varepsilon_n^{\theta}\right\| \leq &\left\|M_k\left(\overline{T}(\overline{\theta})\right) - M_k\left(\lambda_F(\overline{\theta})\right)\right\|\cdot\left\|\frac{1}{n}\sum_{i=1}^{n}\zeta_F(Z_i)\right\| + 2\overline{M}_\lambda\cdot\left\|\frac{1}{n}\sum_{i=1}^{n}\zeta_F(Z_i)\right\|\\ &+ \left\|M_k\left(\overline{T}(\overline{\theta})\right) - M_k\left(\lambda_F(\overline{\theta})\right)\right\|\cdot\left\|\varepsilon_n^{\nu^\mu}(\widehat{\theta})\right\| + \overline{M}_\lambda\cdot\left\|\varepsilon_n^{\nu^\mu}(\widehat{\theta})\right\|\end{aligned}$$

Therefore, for any $c > 0$,

$$\begin{aligned}\mathbb{1}\left\{\left\|\varepsilon_n^{\theta}\right\| \geq c\right\} \leq \mathbb{1}&\left\{\left\|M_k\left(\overline{T}(\overline{\theta})\right) - M_k\left(\lambda_F(\overline{\theta})\right)\right\| \cdot \left\|\frac{1}{n}\sum_{i=1}^{n}\zeta_F(Z_i)\right\| \geq \frac{c}{4}\right\}\\ &+ \mathbb{1}\left\{2\overline{M}_{\lambda} \cdot \left\|\frac{1}{n}\sum_{i=1}^{n}\zeta_F(Z_i)\right\| \geq \frac{c}{4}\right\}\\ &+ \mathbb{1}\left\{\left\|M_k\left(\overline{T}(\overline{\theta})\right) - M_k\left(\lambda_F(\overline{\theta})\right)\right\| \cdot \left\|\varepsilon_n^{\nu^{\mu}}(\widehat{\theta})\right\| \geq \frac{c}{4}\right\}\\ &+ \mathbb{1}\left\{\overline{M}_{\lambda} \cdot \left\|\varepsilon_n^{\nu^{\mu}}(\widehat{\theta})\right\| \geq \frac{c}{4}\right\}\end{aligned} \tag{S5.5.44}$$

Let K_5 and K_7 be as described in (S5.5.38) and note that by the conditions described there,

$$\begin{aligned}&\mathbb{1}\left\{\left\|M_k\left(\overline{T}(\overline{\theta})\right) - M_k\left(\lambda_F(\overline{\theta})\right)\right\| \cdot \left\|\frac{1}{n}\sum_{i=1}^{n}\zeta_F(Z_i)\right\| \geq \frac{c}{4}\right\}\\ &= \underbrace{\mathbb{1}\left\{\left\|M_k\left(\overline{T}(\overline{\theta})\right) - M_k\left(\lambda_F(\overline{\theta})\right)\right\| \cdot \left\|\frac{1}{n}\sum_{i=1}^{n}\zeta_F(Z_i)\right\| \geq \frac{c}{4}\right\} \times \mathbb{1}\left\{\left\|M_k\left(\overline{T}(\overline{\theta})\right) - M_k\left(\lambda_F(\overline{\theta})\right)\right\| \leq K_7\right\}}_{\leq \mathbb{1}\left\{\left\|\frac{1}{n}\sum_{i=1}^{n}\zeta_F(Z_i)\right\| \geq \frac{c}{4K_7}\right\}}\\ &+ \underbrace{\mathbb{1}\left\{\left\|M_k\left(\overline{T}(\overline{\theta})\right) - M_k\left(\lambda_F(\overline{\theta})\right)\right\| \cdot \left\|\frac{1}{n}\sum_{i=1}^{n}\zeta_F(Z_i)\right\| \geq \frac{c}{4}\right\} \times \mathbb{1}\left\{\left\|M_k\left(\overline{T}(\overline{\theta})\right) - M_k\left(\lambda_F(\overline{\theta})\right)\right\| > K_7\right\}}_{\leq \mathbb{1}\left\{\left\|\overline{T}(\overline{\theta}) - \lambda_F(\overline{\theta})\right\| \geq K_5\right\}}\\ &\leq \mathbb{1}\left\{\left\|\frac{1}{n}\sum_{i=1}^{n}\zeta_F(Z_i)\right\| \geq \frac{c}{4K_7}\right\} + \mathbb{1}\left\{\left\|\overline{T}(\overline{\theta}) - \lambda_F(\overline{\theta})\right\| \geq K_5\right\}.\end{aligned} \tag{S5.5.45A}$$

Similarly,

$$\begin{aligned}&\mathbb{1}\left\{\left\|M_k\left(\overline{T}(\overline{\theta})\right) - M_k\left(\lambda_F(\overline{\theta})\right)\right\| \cdot \left\|\varepsilon_n^{\nu^{\mu}}(\widehat{\theta})\right\| \geq \frac{c}{4}\right\}\\ &= \underbrace{\mathbb{1}\left\{\left\|M_k\left(\overline{T}(\overline{\theta})\right) - M_k\left(\lambda_F(\overline{\theta})\right)\right\| \cdot \left\|\varepsilon_n^{\nu^{\mu}}(\widehat{\theta})\right\| \geq \frac{c}{4}\right\} \times \mathbb{1}\left\{\left\|M_k\left(\overline{T}(\overline{\theta})\right) - M_k\left(\lambda_F(\overline{\theta})\right)\right\| \leq K_7\right\}}_{\leq \mathbb{1}\left\{\left\|\varepsilon_n^{\nu^{\mu}}(\widehat{\theta})\right\| \geq \frac{c}{4K_7}\right\}}\\ &+ \underbrace{\mathbb{1}\left\{\left\|M_k\left(\overline{T}(\overline{\theta})\right) - M_k\left(\lambda_F(\overline{\theta})\right)\right\| \cdot \left\|\varepsilon_n^{\nu^{\mu}}(\widehat{\theta})\right\| \geq \frac{c}{4}\right\} \times \mathbb{1}\left\{\left\|M_k\left(\overline{T}(\overline{\theta})\right) - M_k\left(\lambda_F(\overline{\theta})\right)\right\| > K_7\right\}}_{\leq \mathbb{1}\left\{\left\|\overline{T}(\overline{\theta}) - \lambda_F(\overline{\theta})\right\| \geq K_5\right\}}\\ &\leq \mathbb{1}\left\{\left\|\varepsilon_n^{\nu^{\mu}}(\widehat{\theta})\right\| \geq \frac{c}{4K_7}\right\} + \mathbb{1}\left\{\left\|\overline{T}(\overline{\theta}) - \lambda_F(\overline{\theta})\right\| \geq K_5\right\}.\end{aligned} \tag{S5.5.45B}$$

Combining (S5.5.45A) and (S5.5.45B) with (S5.5.44), for any $c > 0$ we have

$$\begin{aligned}\mathbb{1}\left\{\left\|\varepsilon_n^{\theta}\right\| \geq c\right\} \leq \mathbb{1}&\left\{\left\|\frac{1}{n}\sum_{i=1}^{n}\zeta_F(Z_i)\right\| \geq \left(K_5 \wedge \left(\frac{1}{8\overline{M}_\lambda} \wedge \frac{1}{4K_7}\right)\cdot c\right)\right\} \\ &+ \mathbb{1}\left\{\sup_{\theta\in\Theta}\left\|\frac{1}{n}\sum_{i=1}^{n}\Big(T_F(Z_i,\theta) - E_F\left[T_F(Z,\theta)\right]\Big)\right\| \geq \left(K_5 \wedge \left(\frac{1}{8\overline{M}_\lambda} \wedge \frac{1}{4K_7}\right)\cdot c\right)\right\} \\ &+ \mathbb{1}\left\{\sup_{\theta\in\Theta}\left\|\varepsilon_n^{\nu^\mu}(\theta)\right\| \geq \left(K_5 \wedge \left(\frac{1}{8\overline{M}_\lambda} \wedge \frac{1}{4K_7}\right)\cdot c\right)\right\}\end{aligned} \tag{S5.5.46}$$

Take the ℓ^{th} component ($T_F^{\ell}(Z,\theta)$) of $T_F(Z,\theta)$ let

$$\mathscr{G}_{6,F}^{\ell} = \left\{g : \mathcal{S}_Z \to \mathbb{R} : g(z) = T_F^{\ell}(z,\theta) \text{ for some } \theta \in \Theta\right\}$$

By Assumptions SMIM2, SMIM3 and SMIM4, and Lemmas 2.13 and 2.14 in Pakes and Pollard (1989), there exist positive constants A_6 and V_6 such that, for every $F \in \mathcal{F}$, the class $\mathscr{G}_{6,F}^{\ell}$ is Euclidean (A_6, V_6) for an envelope $G_6(z)$ for which $\exists\ \overline{\mu}_{G_6} < \infty$ such that $E_F\left[G_6(Z)^{4q}\right] \leq \overline{\mu}_{G_6}$ (where q is the integer described in Assumption SMIM1). The conditions in Result S1 are satisfied for the integer q described in Assumption SMIM2 and there exists a constant $\overline{M}_5 < \infty$ such that, for all $b > 0$,

$$\sup_{F\in\mathcal{F}} P_F\left(\sup_{\theta\in\Theta}\left\|\frac{1}{n}\sum_{i=1}^{n}\Big(T_F(Z_i,\theta) - E_F\left[T_F(Z,\theta)\right]\Big)\right\| > b\right) \leq \frac{\overline{M}_5}{\left(n^{1/2}\cdot b\right)^q} \tag{S5.5.47}$$

Next, note from Assumption SMIM1 that there exists $\overline{\mu}_\zeta < \infty$ such that $E_F\left[\zeta_F(Z)^{4q}\right] \leq \overline{\mu}_\zeta$ for all $F \in \mathcal{F}$. From here, a straightforward Chebyshev inequality implies that there exists a constant $\overline{M}_6$ such that, for any $b > 0$,

$$\sup_{F\in\mathcal{F}} P_F\left(\left\|\frac{1}{n}\sum_{i=1}^{n}\zeta_F(Z_i)\right\| \geq b\right) \leq \frac{\overline{M}_6}{\left(n^{1/2}\cdot b\right)^q} \tag{S5.5.48}$$

in both instances (S5.5.47 and S5.5.48), q is the integer described in Assumption SMIM1. Take any $c > 0$ and let n_0 be the smallest integer such that

$$\begin{aligned}&s_{2,n} < \left(K_5 \wedge \left(\frac{1}{8\overline{M}_\lambda} \wedge \frac{1}{4K_7}\right)\cdot c\right) \quad \text{and} \\ &s_{1,n} < \underbrace{\min\left\{\frac{\underline{f}_m}{2}\ ,\ D^{\varepsilon^\mu}\cdot\left(\frac{\left(K_5 \wedge \left(\frac{1}{8\overline{M}_\lambda} \wedge \frac{1}{4K_7}\right)\cdot c\right) - s_{2,n}}{3\overline{\phi}}\right)\ ,\ D^{\varepsilon^\mu}\cdot\left(\frac{\left(K_5 \wedge \left(\frac{1}{8\overline{M}_\lambda} \wedge \frac{1}{4K_7}\right)\cdot c\right) - s_{2,n}}{3\overline{\phi}}\right)^{1/4}\right\}}_{=\underline{\varphi}^\mu\left(\left(K_5\wedge\left(\frac{1}{8\overline{M}_\lambda}\wedge\frac{1}{4K_7}\right)\cdot c\right) - s_{2,n}\right)\ \text{(see (S5.5.10))}}\end{aligned}$$

From (S5.5.36), (S5.5.47) and (S5.5.48), the inequality in (S5.5.46) implies

$$
\begin{aligned}
\sup_{F\in\mathcal{F}} P_F\left(\left\|\varepsilon_n^\theta\right\| \geq c\right) \leq & \frac{\overline{M}_5+\overline{M}_6}{\left(n^{1/2}\cdot\left(\left(K_5\wedge\left(\frac{1}{8\overline{M}_\lambda}\wedge\frac{1}{4K_7}\right)\cdot c\right)\right)\right)^q} \\
& + \frac{\overline{M}_3}{\left(n\cdot\sigma_n^d\cdot\left(\frac{\left(K_5\wedge\left(\frac{1}{8\overline{M}_\lambda}\wedge\frac{1}{4K_7}\right)\cdot c\right)-s_{2,n}}{3}\right)\right)^q} + \frac{\overline{M}_4}{\left(n^{3/2}\cdot\sigma_n^d\cdot\left(\frac{\left(K_5\wedge\left(\frac{1}{8\overline{M}_\lambda}\wedge\frac{1}{4K_7}\right)\cdot c\right)-s_{2,n}}{3|K(0)|}\right)\right)^q} \\
& + \frac{\overline{M}_1+\overline{M}_2}{\left(n^{1/2}\cdot\sigma_n^d\cdot\left(\underline{\varphi}^\mu\left(\frac{\left(K_5\wedge\left(\frac{1}{8\overline{M}_\lambda}\wedge\frac{1}{4K_7}\right)\cdot c\right)-s_{2,n}}{3\overline{\phi}}\right)-s_{1,n}\right)\right)^q} \qquad \forall\ n\geq n_0
\end{aligned}
$$

We can obtain a simplified bound from the previous expression. Take any positive constants A_1, A_2 such that

$$
A_1 \leq \frac{1}{3}\cdot\min\left\{\frac{1}{\underline{\phi}}, \frac{1}{|K(0)|}, 1\right\}\times K_5, \quad A_2 \leq \frac{1}{3}\cdot\min\left\{\frac{1}{\underline{\phi}}, \frac{1}{|K(0)|}, 1\right\}\left(\frac{1}{8\overline{M}_\lambda}\wedge\frac{1}{4K_7}\right),
$$

and let $M^{SE} \equiv \sum_{j=1}^{7}\overline{M}_j$. Take any positive sequence $c>0$ and let n_0 be the smallest integer such that

$$
s_{2,n} < A_1\wedge A_2\cdot c, \quad \text{and} \quad s_{1,n} < \min\left\{A_1 - s_{2,n}\ ,\left(A_2\cdot c - s_{2,n}\right), \left(A_2\cdot c - s_{2,n}\right)^{1/4}\right\}
$$

Let

$$
\Lambda_n^{SE}(c) \equiv n^{1/2}\cdot\sigma_n^d\cdot\left(\min\left\{A_1 - s_{2,n}\ ,\left(A_2\cdot c - s_{2,n}\right), \left(A_2\cdot c - s_{2,n}\right)^{1/4}\right\} - s_{1,n}\right).
$$

Then,

$$
\sup_{F\in\mathcal{F}} P_F\left(\left\|\varepsilon_n^\theta\right\|\geq c\right) \leq \frac{M^{SE}}{\left(\Lambda_n^{SE}(c)\right)^q} \quad \forall\ n\geq n_0 \tag{S5.5.49}
$$

where q is the integer described in Assumption SMIM1. Recall from Assumption SMIM1 that the bandwidth sequence $\sigma_n \longrightarrow 0$ satisfies $n^{1/2+\Delta}\cdot\sigma_n^L \longrightarrow 0$ and $n^{1/2-\Delta}\cdot\sigma_n^d \longrightarrow \infty$ for some $0<\Delta<1/2$. Next recall that the sequences $s_{1,n}$ and $s_{2,n}$ are defined in (S5.5.17) and (S5.5.29) as $s_{1,n} \equiv C_B^{\mu_a}\cdot\sigma_n^L$ and $s_{2,n} \equiv \frac{|K(0)|\cdot C_Q}{n\cdot\sigma_n^d} + C_B^{\mu_b}\cdot\sigma_n^L$. Therefore, $n^{1/2+\Delta}\cdot s_{1,n} \longrightarrow 0$ and $n^{1/2+\Delta}\cdot s_{2,n} \longrightarrow 0$ and $\Lambda_n(c) \longrightarrow \infty$ for all $c>0$, and thus from (S5.5.49) we have

$$
\sup_{F\in\mathcal{F}} P_F\left(\left\|\varepsilon_n^\theta\right\|\geq c\right) = O\left(\frac{1}{\left(n^{1/2}\cdot\sigma_n^d\right)^q}\right) \quad \forall\ c>0
$$

Furthermore, recall that Assumption SMIM1 states that Δ and q are such that $q\Delta > 1/2$. From here we have that for any $0 < \delta < q\Delta - \frac{1}{2}$, we have

$$\sup_{F\in\mathcal{F}} P_F\left(\left\|\varepsilon_n^{\theta}\right\| \geq c\right) = o\left(\frac{1}{n^{1/2+\delta}}\right) \tag{S5.5.50}$$

More generally, suppose c_n is a sequence such that $n^{1/2-\Delta}\cdot\sigma_n^d\cdot c_n \longrightarrow \infty$. Then, the result in (S5.5.50) would still hold for c_n. Thus,

$$\sup_{F\in\mathcal{F}} P_F\left(\left\|\varepsilon_n^{\theta}\right\| \geq c_n\right) = o\left(\frac{1}{n^{1/2+\delta}}\right) \quad \forall\ c_n : n^{1/2-\Delta}\cdot\sigma_n^d\cdot c_n \longrightarrow \infty, \text{ and } 0 < \delta < q\Delta - \frac{1}{2}$$

Next, by Assumptions SMIM1 and SMIM4, there exists $\overline{\mu}_\psi < \infty$ such that $E_F\left[\left(\psi_F^{\theta}(Z)\right)^{4q}\right] \leq \overline{\mu}_\psi$ for all $F \in \mathcal{F}$. From here, a straightforward Chebyshev inequality implies that there exists a constant $\overline{M}_8$ such that, for any $b > 0$,

$$\sup_{F\in\mathcal{F}} P_F\left(\left\|\frac{1}{n}\sum_{i=1}^{n}\psi_F^{\theta}(Z_i)\right\| \geq b\right) \leq \frac{\overline{M}_8}{\left(n^{1/2}\cdot b\right)^q}$$

Thus, going back to the linear representation in (S5.5.43), we have that for any $c > 0$ there exists n_0 such that

$$\begin{aligned}\sup_{F\in\mathcal{F}} P_F\left(\left\|\widehat{\theta}-\theta^*\right\| \geq c\right) &\leq \sup_{F\in\mathcal{F}} P_F\left(\left\|\frac{1}{n}\sum_{i=1}^{n}\psi_F^{\theta}(Z_i)\right\| \geq \frac{c}{2}\right) + \sup_{F\in\mathcal{F}} P_F\left(\left\|\varepsilon_n^{\theta}\right\| \geq \frac{c}{2}\right)\\ &\leq \frac{\overline{M}_8}{\left(n^{1/2}\cdot c/2\right)^q} + \frac{M^{SE}}{\left(\Lambda_n^{SE}(c/2)\right)^q} \quad \forall\ n \geq n_0.\end{aligned} \tag{S5.5.51}$$

And so,

$$\sup_{F\in\mathcal{F}} P_F\left(\left\|\widehat{\theta}-\theta^*\right\| \geq c\right) \longrightarrow 0 \quad \forall\ c > 0$$

Thus, $\left\|\widehat{\theta}-\theta^*\right\| = o_p(1)$ uniformly over $\mathcal{F}$. Equipped with the previous expression we can obtain a more precise asymptotic result for ε_n^{θ}. Recall from (S5.5.43) that it is defined as

$$\begin{aligned}\varepsilon_n^{\theta} \equiv &\left(M_k\left(\overline{T}(\overline{\theta})\right) - M_k\left(\lambda_F(\overline{\theta})\right)\right)\cdot\frac{1}{n}\sum_{i=1}^{n}\zeta_F(Z_i) + \left(M_k\left(\lambda_F(\overline{\theta})\right) - M_k\left(\lambda_F(\theta^*)\right)\right)\cdot\frac{1}{n}\sum_{i=1}^{n}\zeta_F(Z_i)\\ &-\left(M_k\left(\overline{T}(\overline{\theta})\right) - M_k\left(\lambda_F(\overline{\theta})\right)\right)\cdot\varepsilon_n^{\nu^\mu}(\widehat{\theta}) - M_k\left(\lambda_F(\overline{\theta})\right)\cdot\varepsilon_n^{\nu^\mu}(\widehat{\theta})\end{aligned}$$

Take any $c > 0$. By Assumption SMIM4,

$$\mathbb{1}\left\{\left\|M_k\Big(\overline{T}(\overline{\theta})\Big) - M_k\Big(\lambda_F(\overline{\theta})\Big)\right\| \geq c\right\} \leq \max\Bigg(\mathbb{1}\left\{K_6 \cdot \left\|\overline{T}(\overline{\theta}) - \lambda_F(\overline{\theta})\right\|^{\alpha_1} \geq c\right\}\ ,\ \mathbb{1}\left\{\left\|\overline{T}(\overline{\theta}) - \lambda_F(\overline{\theta})\right\| \geq K_5\right\}\Bigg)$$
$$\leq \mathbb{1}\left\{\left\|\overline{T}(\overline{\theta}) - \lambda_F(\overline{\theta})\right\| \geq K_5 \wedge \left(\frac{c}{K_6}\right)^{1/\alpha_1}\right\}$$

Thus, from (S5.5.47), for any $c > 0$ there exists n_0 such that

$$\sup_{F\in\mathcal{F}}\ P_F\left(\left\|M_k\Big(\overline{T}(\overline{\theta})\Big) - M_k\Big(\lambda_F(\overline{\theta})\Big)\right\| \geq c\right) \leq \sup_{F\in\mathcal{F}}\ P_F\left(\sup_{\theta\in\Theta}\left\|\frac{1}{n}\sum_{i=1}^{n}\Big(T_F(Z_i,\theta) - E_F\left[T_F(Z,\theta)\right]\Big)\right\| \geq K_5 \wedge \left(\frac{c}{K_6}\right)^{1/\alpha_1}\right)$$
$$\leq \frac{\overline{M}_5}{\left(n^{1/2}\cdot\left[K_5 \wedge \left(\frac{c}{K_6}\right)^{1/\alpha_1}\right]\right)^q} \quad \forall\ n \geq n_0$$

Take any $c > 0$ and $\tau > 0$. Then, there exists n_* such that $K_5 \wedge \left(\frac{n^{-\tau}\cdot c}{K_6}\right)^{1/\alpha_1} = \left(\frac{n^{-\tau}\cdot c}{K_6}\right)^{1/\alpha_1}$ for all $n \geq n_*$. Therefore, there exists n_0 such that

$$\sup_{F\in\mathcal{F}}\ P_F\left(\left\|M_k\Big(\overline{T}(\overline{\theta})\Big) - M_k\Big(\lambda_F(\overline{\theta})\Big)\right\| \geq n^{-\tau}\cdot c\right) \leq \frac{\overline{M}_5}{\left(n^{1/2-\tau/\alpha_1}\cdot(c/K_6)^{1/\alpha_1}\right)^q} \quad \forall\ n \geq n_0$$

Therefore,

$$\sup_{F\in\mathcal{F}}\ P_F\left(\left\|M_k\Big(\overline{T}(\overline{\theta})\Big) - M_k\Big(\lambda_F(\overline{\theta})\Big)\right\| \geq n^{-\tau}\cdot c\right) \longrightarrow 0 \quad \forall\ c > 0,\ \tau < \frac{\alpha_1}{2}$$

which means,

$$\left\|M_k\Big(\overline{T}(\overline{\theta})\Big) - M_k\Big(\lambda_F(\overline{\theta})\Big)\right\| = o_p\left(n^{-\tau}\right) \quad \forall\ \tau < \frac{\alpha_1}{2}, \quad \text{uniformly over } \mathcal{F}. \tag{S5.5.52}$$

Take any $c > 0$. Once again from Assumption SMIM4, we have

$$\mathbb{1}\left\{\left\|M_k\left(\lambda_F(\overline{\theta})\right) - M_k\left(\lambda_F(\theta^*)\right)\right\| \geq c\right\} \leq \mathbb{1}\left\{\left\|\lambda_F(\overline{\theta}) - \lambda_F(\theta^*)\right\| \geq K_5 \wedge \left(\frac{c}{K_6}\right)^{1/\alpha_1}\right\}$$
$$\leq \max\left(\mathbb{1}\left\{K_9\cdot\left\|\overline{\theta} - \theta^*\right\|^{\alpha_2} \geq K_5 \wedge \left(\frac{c}{K_6}\right)^{1/\alpha_1}\right\}\ ,\ \mathbb{1}\left\{\left\|\overline{\theta} - \theta^*\right\| \geq K_8\right\}\right)$$
$$\leq \mathbb{1}\left\{\left\|\overline{\theta} - \theta^*\right\| \geq \left[\frac{1}{K_9}\cdot\left(K_5 \wedge \left(\frac{c}{K_6}\right)^{1/\alpha_1}\right)\right]^{1/\alpha_2} \wedge K_8\right\}$$

Recall that $\overline{\theta}$ belongs in the line segment connecting $\widehat{\theta}$ and θ^* and therefore $\|\overline{\theta} - \theta^*\| \leq \|\widehat{\theta} - \theta^*\|$.

Thus, from the above result and (S5.5.51) we have that for each $c > 0$, there exists n_0 such that

$$\begin{aligned}
&\sup_{F\in\mathcal{F}} P_F\left(\left\|M_k\left(\lambda_F(\overline{\theta})\right) - M_k\left(\lambda_F(\theta^*)\right)\right\| \geq c\right) \\
&\leq \sup_{F\in\mathcal{F}} P_F\left(\left\|\widehat{\theta} - \theta^*\right\| \geq \left[\frac{1}{K_9}\cdot\left(K_5 \wedge \left(\frac{c}{K_6}\right)^{1/\alpha_1}\right)\right]^{1/\alpha_2} \wedge K_8\right) \\
&\leq \frac{\overline{M}_8}{\left(n^{1/2}\cdot\frac{1}{2}\cdot\left(\left[\frac{1}{K_9}\cdot\left(K_5\wedge\left(\frac{c}{K_6}\right)^{1/\alpha_1}\right)\right]^{1/\alpha_2}\wedge K_8\right)\right)^q} + \frac{M^{SE}}{\left(\Lambda_n^{SE}\left(\frac{1}{2}\cdot\left(\left[\frac{1}{K_9}\cdot\left(K_5\wedge\left(\frac{c}{K_6}\right)^{1/\alpha_1}\right)\right]^{1/\alpha_2}\wedge K_8\right)\right)\right)^q} \\
&\quad \forall\ n \geq n_0.
\end{aligned}$$

Recall that

$$\Lambda_n^{SE}(c) \equiv n^{1/2}\cdot\sigma_n^d\cdot\left(\min\left\{A_1 - s_{2,n}\ ,\ \left(A_2\cdot c - s_{2,n}\right), \left(A_2\cdot c - s_{2,n}\right)^{1/4}\right\} - s_{1,n}\right).$$

And recall from Assumption SMIM1 that the bandwidth sequence $\sigma_n \longrightarrow 0$ satisfies $n^{1/2+\Delta}\cdot\sigma_n^L \longrightarrow 0$ and $n^{1/2-\Delta}\cdot\sigma_n^d \longrightarrow \infty$ for some $0 < \Delta < 1/2$. Take any $c > 0$ and $\tau > 0$. Then, there exists n_* such that

$$\begin{aligned}
&\frac{1}{2}\cdot\left(\left[\frac{1}{K_9}\cdot\left(K_5\wedge\left(\frac{c}{K_6}\right)^{1/\alpha_1}\right)\right]^{1/\alpha_2}\wedge K_8\right) = \frac{1}{2\cdot K_9^{1/\alpha_2}}\cdot\left(\frac{c}{K_6}\right)^{1/(\alpha_1\cdot\alpha_2)}\cdot n^{-\tau/(\alpha_1\cdot\alpha_2)}, \\
&\Lambda_n^{SE}\left(\frac{1}{2}\cdot\left(\left[\frac{1}{K_9}\cdot\left(K_5\wedge\left(\frac{n^{-\tau}\cdot c}{K_6}\right)^{1/\alpha_1}\right)\right]^{1/\alpha_2}\wedge K_8\right)\right) = n^{1/2-\tau/(\alpha_1\cdot\alpha_2)}\cdot\sigma_n^d\cdot\frac{A_2}{2\cdot K_9^{1/\alpha_2}}\cdot\left(\frac{c}{K_6}\right)^{1/(\alpha_1\cdot\alpha_2)} + o(1) \\
&\forall\ n \geq n_*
\end{aligned}$$

Therefore, for each $c > 0$ there exists n_0 such that

$$\begin{aligned}
&\sup_{F\in\mathcal{F}} P_F\left(\left\|M_k\left(\lambda_F(\overline{\theta})\right) - M_k\left(\lambda_F(\theta^*)\right)\right\| \geq n^{-\tau}\cdot c\right) \\
&\leq \frac{\overline{M}_8}{\left(n^{1/2-\tau/(\alpha_1\cdot\alpha_2)}\cdot\frac{1}{2\cdot K_9^{1/\alpha_2}}\cdot\left(\frac{c}{K_6}\right)^{1/(\alpha_1\cdot\alpha_2)}\right)^q} + \frac{M^{SE}}{\left(n^{1/2-\tau/(\alpha_1\cdot\alpha_2)}\cdot\sigma_n^d\cdot\frac{A_2}{2\cdot K_9^{1/\alpha_2}}\cdot\left(\frac{c}{K_6}\right)^{1/(\alpha_1\cdot\alpha_2)} + o(1)\right)^q} \\
&\quad \forall\ n \geq n_0.
\end{aligned}$$

Therefore,

$$\sup_{F\in\mathcal{F}} P_F\left(\left\|M_k\left(\lambda_F(\overline{\theta})\right) - M_k\left(\lambda_F(\theta^*)\right)\right\| \geq n^{-\tau}\cdot c\right) \longrightarrow 0 \quad \forall\ c > 0, \tau < \alpha_1\cdot\alpha_2\cdot\Delta.$$

which means,

$$\left\|M_k\left(\lambda_F(\overline{\theta})\right)-M_k\left(\lambda_F(\theta^*)\right)\right\|=o_p\left(n^{-\tau}\right)\quad\forall\ \tau<\alpha_1\cdot\alpha_2\cdot\Delta,\quad\text{uniformly over }\mathcal{F}.\tag{S5.5.53}$$

Next, recall from (S5.5.35) that, uniformly over $\mathcal{F}$,

$$\sup_{\theta\in\Theta}\left\|\varepsilon_n^{\nu^\mu}(\theta)\right\|=O_p\left(\frac{1}{n\cdot\sigma_n^d}\right)+O_p\left(\frac{1}{n^{3/2}\cdot\sigma_n^d}\right)+O_p\left(\left(\frac{1}{n^{1/2}\cdot\sigma_n^d}+s_{1,n}\right)^2\right)+s_{2,n}=o_p\left(\frac{1}{n^{1/2+\Delta}}\right)$$

and from (S5.5.48) we also have that, uniformly over $\mathcal{F}$,

$$\left\|\frac{1}{n}\sum_{i=1}^{n}\zeta_F(Z_i)\right\|=O_p\left(\frac{1}{n^{1/2}}\right).$$

From here, (S5.5.52) and (S5.5.53) we have that, uniformly over $\mathcal{F}$, and for any $0<\tau<\left(\frac{\alpha_1}{2}\right)\wedge(\alpha_1\cdot\alpha_2\cdot\Delta)$,

$$\begin{aligned}\left\|\varepsilon_n^{\theta}\right\|\leq&\underbrace{\left\|M_k\left(\overline{T}(\overline{\theta})\right)-M_k\left(\lambda_F(\overline{\theta})\right)\right\|}_{=o_p(n^{-\tau})}\cdot\underbrace{\left\|\frac{1}{n}\sum_{i=1}^{n}\zeta_F(Z_i)\right\|}_{=O_p(n^{-1/2})}+\underbrace{\left\|M_k\left(\lambda_F(\overline{\theta})\right)-M_k\left(\lambda_F(\theta^*)\right)\right\|}_{=o_p(n^{-\tau})}\cdot\underbrace{\left\|\frac{1}{n}\sum_{i=1}^{n}\zeta_F(Z_i)\right\|}_{=O_p(n^{-1/2})}\\&+\underbrace{\left\|M_k\left(\overline{T}(\overline{\theta})\right)-M_k\left(\lambda_F(\overline{\theta})\right)\right\|}_{=o_p(n^{-\tau})}\cdot\underbrace{\sup_{\theta\in\Theta}\left\|\varepsilon_n^{\nu^\mu}(\theta)\right\|}_{=o_p(n^{-1/2-\Delta})}+\overline{M}_\lambda\cdot\underbrace{\sup_{\theta\in\Theta}\left\|\varepsilon_n^{\nu^\mu}(\theta)\right\|}_{=o_p(n^{-1/2-\Delta})}\end{aligned}$$

Therefore, for any $0<\tau<\min\left\{\left(\frac{\alpha_1}{2}\right),(\alpha_1\cdot\alpha_2\cdot\Delta),\Delta\right\}$,

$$\left\|\varepsilon_n^{\theta}\right\|=o_p\left(\frac{1}{n^{1/2+\tau}}\right),\quad\text{uniformly over }\mathcal{F}.\tag{S5.5.54}$$

Together, (S5.5.42), (S5.5.49) and (S5.5.54) show that the conditions in Assumption 1 are satisfied, with $\psi_F^{\theta}(Z_i)=M_k\left(\lambda_F(\theta^*)\right)\cdot\zeta_F(Z_i)$, $r_n=n^{1/2}\cdot\sigma_n^d$, $0<\tau<\min\left\{\left(\frac{\alpha_1}{2}\right),(\alpha_1\cdot\alpha_2\cdot\Delta),\Delta\right\}$, and $0<\overline{\delta}<q\Delta-\frac{1}{2}$. **This proves Result SMIM in Appendix A4.2 of the paper.** ■

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