a) "p known" $\boxed{\text{Imbens \& Manski}}$ ②

$$\hat{\theta}^l = p \cdot \hat{\eta} \quad ; \quad \hat{\theta}^u = p \cdot \hat{\eta} + (1-p)$$

$$\hat{\eta} = \frac{\frac{1}{N}\sum_{i=1}^{N} Y_i \cdot W_i}{\frac{1}{N}\sum_{i=1}^{N} W_i} \equiv \frac{\hat{g}}{\hat{p}} = \eta + \frac{1}{p}\cdot(\hat{g}-g) - \frac{\eta}{p}\cdot(\hat{p}-p) + O_p\left(\tfrac{1}{N}\right)$$

$$= \eta + \frac{\hat{g}}{p} - \cancel{\eta} - \frac{\eta}{p}\cdot\hat{p} + \cancel{\eta} + O_p\left(\tfrac{1}{N}\right)$$

$$= \eta + \frac{\hat{g} - \eta\cdot\hat{p}}{p} + O_p\left(\tfrac{1}{N}\right)$$

$$= \eta + \frac{1}{p}\cdot\frac{1}{N}\sum_{i=1}^{N}(Y_i \cdot W_i - \eta\cdot W_i) + O_p\left(\tfrac{1}{N}\right)$$

$$= \eta + \frac{1}{p}\cdot\frac{1}{N}\sum_{i=1}^{N}(Y_i - \eta)\cdot W_i + O_p\left(\tfrac{1}{N}\right)$$

$$E[(Y_i - \eta)\cdot W_i] = E_W[(E[Y|W=1] - \eta)\cdot W] = 0$$

$$E[(Y_i - \eta)^2\cdot W_i] = E_W[Var[Y|W=1]\cdot W]$$

$$= p \cdot Var(Y|W=1) \equiv p\cdot\sigma^2$$

$\Rightarrow$

$$\sqrt{N}(\hat{\eta} - \eta) = \frac{1}{p}\cdot\frac{1}{N}\sum_{i=1}^{N}(Y_i - \eta)\cdot W_i + O_p\left(\tfrac{1}{\sqrt{N}}\right)$$

$$\xrightarrow{d} N(0, \sigma^2/p)$$

$$\sqrt{N}(\hat{\theta}^{l} - \theta^{l}) \xrightarrow{d} N(0, \rho \cdot \sigma^2)$$

$$\sqrt{N}(\hat{\theta}^{u} - \theta^{u}) \xrightarrow{d} N(0, \rho \cdot \sigma^2)$$

a) Confidence interval that contains the identified set $[\theta_l, \theta_u]$ with pre-specified prob $\alpha\%$

$$CI_{\alpha}^{[\theta^l, \theta^u]} = [\hat{\theta}^{l} - c_N^{l}, \; \hat{\theta}^{u} + c_N^{u}] \quad \text{with: } [c_N^{l} > 0, \; c_N^{u} > 0]$$

$$[\theta^l, \theta^u] \in CI_{\alpha}^{[\theta^l, \theta^u]} \iff \begin{cases} \hat{\theta}^{l} - c_N^{l} \leq \theta^{l} \\ \text{and} \\ \hat{\theta}^{u} + c_N^{u} \geq \theta^{u} \end{cases}$$

i.e, $\hat{\theta}^{l} - \theta^{l} \leq c_N^{l}$ and $\hat{\theta}^{u} - \theta^{u} \geq -c_N^{u}$

$$\Pr[\hat{\theta}^{l} - \theta^{l} \leq c_N^{l} \text{ and } \hat{\theta}^{u} - \theta^{u} \geq -c_N^{u}]$$

$$= 1 - \Pr[\hat{\theta}^{l} - \theta^{l} > c_N^{l} \text{ } \underline{\text{or}} \text{ } \hat{\theta}^{u} - \theta^{u} < -c_N^{u}]$$

$$= 1 - \Big[\Pr[\hat{\theta}^{l} - \theta^{l} > c_N^{l}] + \Pr[\hat{\theta}^{u} - \theta^{u} < -c_N^{u}]$$
$$- \Pr[\hat{\theta}^{l} - \theta^{l} > c_N^{l} \text{ and } \hat{\theta}^{u} - \theta^{u} < -c_N^{u}]\Big]$$

$$\hat{\theta}^l - \theta^l = p \cdot (\hat{\eta} - \eta)$$

$$= \frac{1}{N} \sum_{i=1}^{N} (Y_i - \eta) \cdot W_i + O_p\left(\frac{1}{N}\right)$$

with known

$$\hat{\theta}^u - \theta^u = p \cdot (\hat{\eta} - \eta) \quad \text{same}$$

$$= \frac{1}{N} \sum_{i=1}^{N} (Y_i - \eta) \cdot W_i + O_p\left(\frac{1}{N}\right)$$

$$\Rightarrow \lim_{N \to \infty} \Pr\left[\hat{\theta} - \theta^l > c_N^l \text{ and } \hat{\theta}^u - \theta^u < -c_N^u\right]$$

$$= 0$$

$$\forall \; c_N^l > 0, \; c_N^u > 0$$

$$\Rightarrow \lim_{N \to \infty} \Pr\left[\hat{\theta}^l - \theta^l \le c_N^l \text{ and } \hat{\theta}^u - \theta^u \ge -c_N^u\right]$$

$$= 1 - \lim_{N \to \infty} \Pr\left[\hat{\theta}^l - \theta^l > c_N^l\right] - \lim_{N \to \infty} \Pr\left[\hat{\theta}^u - \theta^u < -c_N^u\right]$$

$$= 1 - \lim_{N \to \infty} \Pr\left[\frac{\sqrt{N}\,(\hat{\theta}^l - \theta^l)}{p \cdot \hat{\sigma}} > \frac{\sqrt{N} \cdot c_N^l}{p \cdot \hat{\sigma}}\right]$$

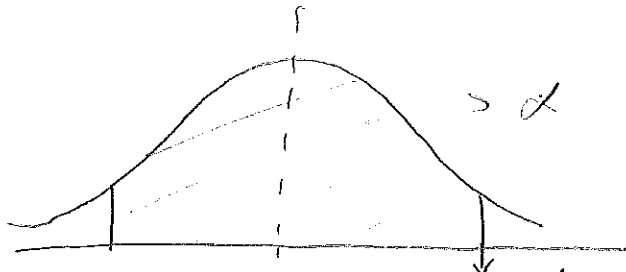
$$- \lim_{N \to \infty} \Pr\left[\frac{\sqrt{N}\,(\hat{\theta}^u - \theta^u)}{p \cdot \hat{\sigma}} < -\frac{\sqrt{N} \cdot c_N^u}{p \cdot \hat{\sigma}}\right]$$

$$\approx 1 - \left[\left(1 - \Phi\left(\frac{\sqrt{N} \cdot c_N^l}{p \cdot \hat{\sigma}}\right)\right) - \Phi\left(-\frac{\sqrt{N} \cdot c_N^u}{p \cdot \hat{\sigma}}\right)\right]$$

$\Rightarrow$

$$\approx \Phi\left(\frac{\sqrt{N}\cdot C_N^{\ell}}{\rho\cdot\hat{\sigma}}\right) - \Phi\left(\frac{-\sqrt{N}\cdot C_N^{u}}{\rho\cdot\hat{\sigma}}\right)$$

since it's the same variance, we can set $C_N^{\ell} = C_N^{u}$ such that



$$\frac{\sqrt{N}\cdot C_N^{\ell}}{\rho\cdot\hat{\sigma}} = z_{\frac{(1+\alpha)}{2}}$$

$$C_N = z_{\frac{(1+\alpha)}{2}} \cdot \frac{\rho\cdot\hat{\sigma}}{\sqrt{N}} = C_N^{u} = C_N^{\ell}$$

$$CI_{\alpha}^{[\theta^{\ell},\theta^{u}]} = \left[\hat{\theta}^{\ell} - \frac{\rho\cdot\hat{\sigma}}{\sqrt{N}}\cdot z_{\frac{(1+\alpha)}{2}} \; , \; \hat{\theta}^{u} + \frac{\rho\cdot\hat{\sigma}}{\sqrt{N}}\cdot z_{\frac{(1+\alpha)}{2}}\right]$$

as $\rho \to 1$, $CI_{\alpha}^{[\theta^{\ell},\theta^{u}]} \longrightarrow \left[\hat{\mu} - \frac{\hat{\sigma}}{\sqrt{N}}\cdot z_{\frac{(1+\alpha)}{2}} \; , \; \hat{\mu} + \frac{\hat{\sigma}}{\sqrt{N}}\cdot z_{\frac{(1+\alpha)}{2}}\right]$

the correct α% confidence interval for $\hat{\mu}$ in the identified case.

Key Width of identified set:

$$\theta^{u} - \theta^{\ell} = \rho\cdot\mu + (1-\rho) - \rho\cdot\mu = 1-\rho$$

$\Rightarrow$ if $\rho = 1$, width = zero (point identification)

Next, a confidence interval that includes the true value θ with the pre-spec. prob $\alpha\%$:

- Three cases:

a) $\theta = \theta_\ell$, b) $\theta = \theta^u$, c) $\theta \in (\theta_\ell, \theta_u)$

- CI_α^θ is of the form:

$$CI_\alpha^\theta = [\hat{\theta}^\ell - D_N^\ell, \ \hat{\theta}^u + D_N^u]$$

a) $\underline{\theta = \theta_\ell}$:

$$Pr[\theta \in CI_\alpha^\theta] = Pr[\hat{\theta}^\ell - D_N^\ell \le \theta_\ell \le \hat{\theta}^u + D_N^u]$$

$$= Pr[\theta_\ell \le \hat{\theta}^u + D_N^u] - Pr[\theta_\ell \le \hat{\theta}^\ell - D_N^\ell]$$

$$= Pr[\theta_\ell - D_N^u \le \hat{\theta}^u] - Pr[\theta_\ell + D_N^\ell \le \hat{\theta}^\ell]$$

$$= Pr\left[\sqrt{N}\frac{(\theta_\ell - \theta_u)}{P \cdot \hat{\sigma}} - \sqrt{N} \cdot \frac{D_N^u}{P \cdot \hat{\sigma}} \le \sqrt{N}\frac{(\hat{\theta}^u - \theta^u)}{P \cdot \hat{\sigma}}\right]$$

$$- Pr\left[\frac{D_N^\ell \cdot \sqrt{N}}{P \cdot \hat{\sigma}} \le \sqrt{N}\frac{(\hat{\theta}^\ell - \theta^\ell)}{P \cdot \hat{\sigma}}\right]$$

$$\approx 1 - \Phi\left(\sqrt{N}\,\frac{(\theta_e - \theta_u)}{\rho \cdot \hat{\sigma}} - \sqrt{N} \cdot \frac{D_N^u}{\rho \cdot \hat{\sigma}} \right)$$

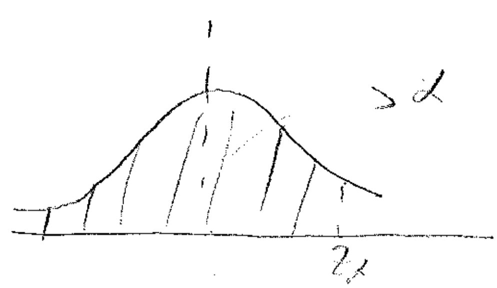
$$- \left(1 - \Phi\left(\frac{D_N^l \cdot \sqrt{N}}{\rho \cdot \hat{\sigma}} \right) \right)$$

$$= \Phi\left(\frac{\sqrt{N} \cdot D_N^l}{\rho \cdot \hat{\sigma}} \right) - \Phi\left(\sqrt{N}\,\frac{(\rho - 1)}{\rho \cdot \hat{\sigma}} - \sqrt{N} \cdot \frac{D_N^u}{\rho \cdot \hat{\sigma}} \right)$$

- If we set $D_N^l = O_p\left(\frac{1}{\sqrt{N}}\right)$, then the second term converges to $\Phi(-\infty) = 0$ ~~is true~~ (if $\rho < 1$)

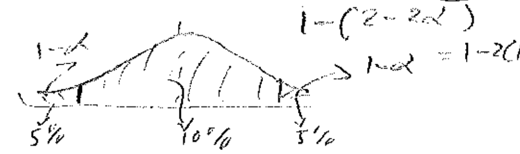
and the whole thing becomes

$$\approx \Phi\left(\frac{\sqrt{N} \cdot D_N^l}{\rho \cdot \hat{\sigma}} \right)$$



need: $\frac{\sqrt{N} \cdot D_N^l}{\rho \cdot \hat{\sigma}} = z_\alpha \iff D_N^l = \frac{\rho \cdot \hat{\sigma}}{\sqrt{N}} \cdot z_\alpha$

$1-2(1-\alpha)$ $\qquad$ $\Phi(z_\alpha) = \Phi(-z_\alpha)$ $\qquad$ $1-(2-2\alpha)$



$\Rightarrow$ for a) $\theta = \theta_\ell$, we need:

$$\boxed{D_N^\ell = \frac{\rho\cdot\hat{\sigma}}{\sqrt{N}}\cdot z_\alpha}$$

b) $\underline{\theta = \theta_u}$:

$$\Pr[\theta \in CI_\alpha^\theta] = \Pr\left[\hat{\theta}^\ell - D_N^\ell \le \theta_u \le \hat{\theta}^u + D_N^u\right]$$

$$= \Pr\left[-D_N^u \le \hat{\theta}^u - \theta_u\right] - \Pr\left[\theta_u \le \hat{\theta}^\ell - D_N^\ell\right]$$

$$= \Pr\left[-\frac{\sqrt{N}\,D_N^u}{\rho\cdot\hat{\sigma}} \le \sqrt{N}\,\frac{(\hat{\theta}^u - \theta_u)}{\rho\cdot\hat{\sigma}}\right] - \Pr\left[\sqrt{N}\,\frac{(\theta_u - \theta_\ell)}{\rho\cdot\hat{\sigma}} + \frac{\sqrt{N}\cdot D_N^\ell}{\rho\cdot\hat{\sigma}} \le \sqrt{N}\,\frac{(\hat{\theta}^\ell - \theta_\ell)}{\rho\cdot\hat{\sigma}}\right]$$

$$\simeq \left[1 - \Phi\left(-\frac{\sqrt{N}\,D_N^u}{\rho\cdot\hat{\sigma}}\right)\right] - \left[1 - \Phi\left(\sqrt{N}\,\frac{(\theta_u - \theta_\ell)}{\rho\cdot\hat{\sigma}} + \frac{\sqrt{N}\,D_N^\ell}{\rho\cdot\hat{\sigma}}\right)\right]$$

$$= \Phi\left(\sqrt{N}\,\frac{(1-\rho)}{\rho\cdot\hat{\sigma}} + z_\alpha\right) - \Phi\left(-\frac{\sqrt{N}\cdot D_N^u}{\rho\cdot\hat{\sigma}}\right)$$

If $\rho < 1$, first term converges to 1, so

$$1 - \Phi\left(-\frac{\sqrt{N}\cdot D_N^u}{\rho\cdot\hat{\sigma}}\right) = \alpha \iff \frac{\sqrt{N}\cdot D_N^u}{\rho\cdot\hat{\sigma}} = z_\alpha$$

$\Rightarrow$ need $\boxed{D_N^u = \frac{\rho \cdot \hat{\sigma}}{\sqrt{N}} \cdot z_\alpha}$

$\Rightarrow$

$$CI_\alpha^\theta = \left[\hat{\theta}_l - \frac{\rho \cdot \hat{\sigma}}{\sqrt{N}} \cdot z_\alpha \, , \; \hat{\theta}_u + \frac{\rho \cdot \hat{\sigma}}{\sqrt{N}}\right.$$

Problem: ~~if ... $p \approx 1$~~ $\Rightarrow$

c) $\theta \in (\theta^l, \theta^u)$ [interior]

$$Pr[\theta \in CI_\alpha^\theta] = Pr[\hat{\theta}^l - D_N^l \leq \theta \leq \hat{\theta}^u + D$$

$$= Pr[\theta \leq \hat{\theta}^u + D_N^u] - Pr[\theta \leq \hat{\theta}^l - D_N^l]$$

$$= Pr\left[\sqrt{N}\frac{(\theta - \theta^u)}{\rho \cdot \hat{\sigma}} - \frac{\sqrt{N} \cdot D_N^u}{\rho \cdot \hat{\sigma}} \leq \sqrt{N}\frac{(\hat{\theta}_u - \theta_u)}{\rho \cdot \hat{\sigma}}\right]$$

$$- Pr\left[\sqrt{N}\frac{(\theta - \theta^l)}{\rho \cdot \hat{\sigma}} + \frac{\sqrt{N} \cdot D_N^l}{\rho \cdot \hat{\sigma}} \leq \sqrt{N}\frac{(\hat{\theta}^l - \theta^l}{\rho \cdot \hat{\sigma}}\right.$$

$$\approx 1 - \Phi\left(\sqrt{N}\frac{(\theta - \theta^u)}{\rho \cdot \hat{\sigma}} - z_\alpha\right)$$

$$- \left[1 - \Phi\left(\sqrt{N}\frac{(\theta - \theta^l)}{\rho \cdot \hat{\sigma}} + z_\alpha\right)\right] \longrightarrow$$

$$\longrightarrow \Phi\left(\sqrt{N}\frac{(\theta - \theta^l)}{\rho \cdot \hat{\sigma}}\right) - \Phi\left(\sqrt{N}\frac{(\theta - \theta^u)}{\rho \cdot \hat{\sigma}} - z_\alpha\right) \rightarrow \Phi(\; - \Phi$$

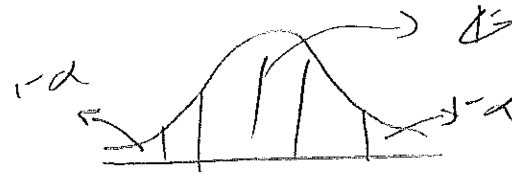
Problem: What if $p \simeq 1$?
(i.e, almost point-identified)

$\Rightarrow$ at θ^{l}, coverage probability is

$$\approx \Phi(z_\alpha) - \Phi(-z_\alpha) = 1 - 2\cdot\alpha$$

$\Rightarrow$ at θ^{u}, it becomes

$$\approx \Phi(z_1) - \Phi(-z_\alpha) = 1 - 2(1-\alpha) = 2\alpha - 1$$



we don't achieve the coverage probability! (it's strictly smaller)

$\therefore$ we need to set (D_N^{l}, D_N^{u}) based on the worst-case scenario i.e: $p = 1$

Suppose we set ~~for $\theta = \theta_e$, we need~~ at $\rho = 1$

$$\Phi\left(\sqrt{N}\cdot\frac{D_N^l}{\hat\sigma}\right) - \Phi\left(\sqrt{N}\cdot\frac{D_N^u}{\hat\sigma}\right) = \alpha$$

$\Rightarrow$ $D_N^l = D_N^u = z_{\frac{1+\alpha}{2}}\cdot\frac{\hat\sigma}{\sqrt{N}}$ would satisfy

PCB.

But we want it to be (adaptive.)

Make D_N^l & D_N^u solve:

$$\Phi\left(\frac{\sqrt{N}\cdot D_N^l}{\rho\cdot\hat\sigma}\right) - \Phi\left(\sqrt{N}\,\frac{(\rho-1)}{\rho\cdot\hat\sigma} - \sqrt{N}\,\frac{\cdot D_N^u}{\rho\cdot\hat\sigma}\right) = \alpha$$

That is, make them functions of ρ

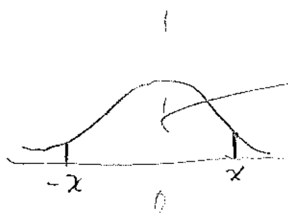
so as to avoid being conservative if $\rho<1$

- For a symmetric interval: $D_N^u = D_N^l \equiv D_N$

Steps:

a) Solve [for R_N]: $\Phi(R_N) - \Phi\left(\frac{\sqrt{N}(\rho-1)}{\rho\cdot\hat\sigma} - R_N\right) = \alpha$

b) $\boxed{D_N = \frac{\rho\cdot\hat\sigma}{\sqrt{N}}\cdot R_N}$



$\Phi(-x) = 1 - \Phi(x)$

Letting $R_N = Q_N$ =>

Steps:

~~a) Solve [for Q_N]: $\Phi(-Q_N) - \Phi($~~

$$\Phi\left(\sqrt{N}\,\frac{(\rho-1)}{\rho\cdot\hat{\sigma}} - R_N\right) = \Phi\left(-\sqrt{N}\,\frac{(1-\rho)}{\rho\cdot\hat{\sigma}} - R_N\right)$$

$$= 1 - \Phi\left(\sqrt{N}\,\frac{(1-\rho)}{\rho\cdot\hat{\sigma}} + R_N\right)$$

$$\Phi(R_N) = 1 - \Phi(-R_N)$$

=> Steps:

a) Solve [for R_N]:

$$\Phi\left(\sqrt{N}\,\frac{(1-\rho)}{\rho\cdot\hat{\sigma}} + R_N\right) - \Phi(-R_N) = \alpha$$

b) Let $D_N = \frac{\rho\cdot\hat{\sigma}}{\sqrt{N}} \cdot R_N$

=>

$$CI^{\theta}_{\alpha} = \left[\hat{\theta}_e - D_N,\ \hat{\theta}_e + D_N\right]$$

General Case [Section 4 in Paper]

- Identified set: $[\theta_\ell, \theta_u]$ $\boxed{\Delta \equiv \theta_u - \theta_\ell}$
- We have estimators $\hat{\theta}_\ell, \hat{\theta}_u$ which are $\sqrt{N}$-consistent, asymptotically normal.

$$\sqrt{N}\begin{pmatrix}\hat{\theta}_\ell-\theta_\ell \\ \hat{\theta}_u-\theta_u\end{pmatrix} \xrightarrow{d} N\left(\begin{pmatrix}0\\0\end{pmatrix}\begin{pmatrix}\sigma_\ell^2, & \rho\sigma_\ell\sigma_u \\ \rho,\sigma_\ell\sigma_u, & \sigma_u^2\end{pmatrix}\right)$$

(we can have $\rho = 1$ eh e.g., if width of ident. region $\Delta \equiv \theta_u - \theta_\ell$ is known).

$$CI_\alpha^\theta = [\hat{\theta}_\ell - D_N^\ell, \hat{\theta}_u + D_N^u]$$

such that

$$Pr(\hat{\theta}_\ell - D_N^\ell \leq \theta \leq \hat{\theta}_u + D_N^u) \quad \forall\, \theta \in [\theta_\ell, \theta_u]$$

uniformly.

a) $\theta = \theta^u$:

$$Pr(\hat{\theta}_\ell - D_N^\ell \leq \theta_u \leq \hat{\theta}_u + D_N^u)$$

$$= Pr(\theta_u \leq \hat{\theta}_u + D_N^u) - Pr(\theta_u < \hat{\theta}_\ell - D_N^\ell)$$

INC = 40,000
POP = 30,000

$= \Pr($ [illegible] $- D_N^u \le$ [illegible] $)$

$$= \Pr\left(-\frac{\sqrt{N}\, D_N^u}{\hat{\sigma}_u} \le \sqrt{N}\left(\frac{\hat{\theta}_u-\theta_u}{\hat{\sigma}_u}\right)\right)$$

$$- \Pr\left(\sqrt{N}\overbrace{\left(\frac{\theta_u-\theta_l}{\hat{\sigma}_l}\right)}^{=\Delta} + \sqrt{N}\,\frac{D_N^l}{\hat{\sigma}_l} < \sqrt{N}\left(\frac{\hat{\theta}_l-\theta_l}{\hat{\sigma}_l}\right)\right)$$

$$= 1 - \Pr\left(\sqrt{N}\left(\frac{\hat{\theta}_u-\theta_u}{\hat{\sigma}_u}\right) \le -\frac{\sqrt{N}\, D_N^u}{\hat{\sigma}_u}\right)$$

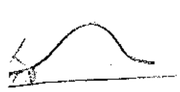
$$- \left[1 - \Pr\left(\sqrt{N}\left(\frac{\hat{\theta}_l-\theta_l}{\hat{\sigma}_l}\right) \le \frac{\sqrt{N}\cdot\Delta}{\hat{\sigma}_l} + \sqrt{N}\,\frac{D_N^l}{\hat{\sigma}_l}\right)\right]$$

$$= \Pr\left(\sqrt{N}\left(\frac{\hat{\theta}_l-\theta_l}{\hat{\sigma}_l}\right) \le \frac{\sqrt{N}\cdot\Delta}{\hat{\sigma}_l} + \sqrt{N}\,\frac{D_N^l}{\hat{\sigma}_l}\right)$$

Φ $$- \Pr\left(\sqrt{N}\left(\frac{\hat{\theta}_u-\theta_u}{\hat{\sigma}_u}\right) \le -\frac{\sqrt{N}\, D_N^u}{\hat{\sigma}_u}\right)$$

$2\alpha-1 \ge \alpha \iff$

$\alpha \ge 1 \to \Leftarrow$

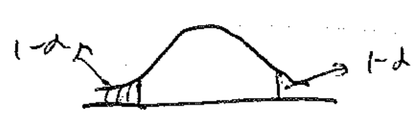


as $N \to \infty$, this becomes

$-\Phi\left(-\frac{\sqrt{N}\, D_N^u}{\hat{\sigma}_u}\right)$

$= 1-\alpha$

$$\boxed{\approx \Phi\left(\frac{\sqrt{N}\cdot\Delta}{\hat{\sigma}_l} + \sqrt{N}\,\frac{D_N^l}{\hat{\sigma}_l}\right) - \Phi\left(-\frac{\sqrt{N}\, D_N^u}{\hat{\sigma}_u}\right)}$$



$\alpha - (1-\alpha) = 2\alpha - 1$

b) $\theta = \theta_\ell$:

$$Pr(\hat{\theta}_\ell - D_N^\ell \le \theta_\ell \le \hat{\theta}_u + D_N^u)$$

$$= Pr(\theta_\ell \le \hat{\theta}_u + D_N^u) - Pr(\theta_\ell < \hat{\theta}_\ell - D_N^\ell)$$

$$= Pr\left(\sqrt{N}\,\frac{(\theta_\ell - \theta_u)}{\hat{\sigma}_u} - \sqrt{N}\cdot\frac{D_N^u}{\hat{\sigma}_u} \le \sqrt{N}\,\frac{(\hat{\theta}_u - \theta_u)}{\hat{\sigma}_u}\right)$$

$$- Pr\left(\sqrt{N}\cdot\frac{D_N^\ell}{\hat{\sigma}_\ell} < \sqrt{N}\,\frac{(\hat{\theta}_\ell - \theta_\ell)}{\hat{\sigma}_\ell}\right)$$

$$= 1 - Pr\left(\sqrt{N}\,\frac{(\hat{\theta}_u - \theta_u)}{\hat{\sigma}_u} \le -\sqrt{N}\cdot\frac{\Delta}{\hat{\sigma}_u} - \sqrt{N}\cdot\frac{D_N^u}{\hat{\sigma}_u}\right)$$

$$- \left(1 - Pr\left(\sqrt{N}\,\frac{(\hat{\theta}_\ell - \theta_\ell)}{\hat{\sigma}_\ell} \le \sqrt{N}\cdot\frac{D_N^\ell}{\hat{\sigma}_\ell}\right)\right)$$

$$= Pr\left(\sqrt{N}\,\frac{(\hat{\theta}_\ell - \theta_\ell)}{\hat{\sigma}_\ell} \le \sqrt{N}\cdot\frac{D_N^\ell}{\hat{\sigma}_\ell}\right)$$

$$- Pr\left(\sqrt{N}\,\frac{(\hat{\theta}_u - \theta_u)}{\hat{\sigma}_u} \le -\sqrt{N}\cdot\frac{\Delta}{\hat{\sigma}_u} - \sqrt{N}\cdot\frac{D_N^u}{\hat{\sigma}_u}\right)$$

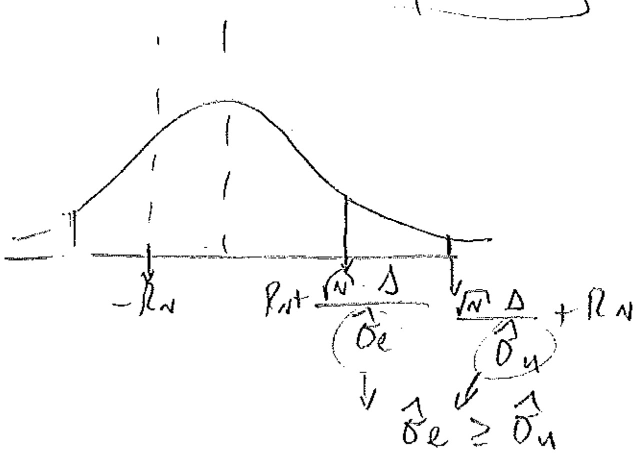
$$\approx \bar{\Phi}\left(\sqrt{N}\cdot\frac{D_N^{\ell}}{\hat{\sigma}_e}\right)-\bar{\Phi}\left(-\sqrt{N}\cdot\frac{\Delta}{\hat{\sigma}_u}-\sqrt{N}\cdot\frac{D_N^{u}}{\hat{\sigma}_u}\right)$$

Let $R_N^{\ell} \equiv \sqrt{N}\cdot\frac{D_N^{\ell}}{\hat{\sigma}_e}$ & $R_N^{u} \equiv \sqrt{N}\,\frac{D_N^{u}}{\hat{\sigma}_u}$

$\Rightarrow$ need:

$$\begin{cases} \theta=\theta_u & \bar{\Phi}\left(\frac{\sqrt{N}\cdot\Delta}{\hat{\sigma}_e}+R_N^{\ell}\right)-\bar{\Phi}\left(-R_N^{u}\right)=\alpha \\ \theta=\theta_\ell & \bar{\Phi}\left(\frac{\sqrt{N}\cdot\Delta}{\hat{\sigma}_u}+R_N^{u}\right)-\bar{\Phi}\left(-R_N^{\ell}\right)=\alpha. \end{cases}$$

Can we make $R_N^{\ell}=R_N^{u}\equiv R_N$?
It would have to satisfy at least one of these two as equality, and the other one as $\boxed{\geq\alpha}$



If Δ were known:

=> Steps:

a) Solve (in R_N):

$$\Phi\left(\frac{\sqrt{N}\cdot\Delta}{\max\{\hat{\sigma}_l,\hat{\sigma}_u\}}+R_N\right)-\Phi(-R_N)=\alpha$$

b) Let $D_N^l=\frac{\hat{\sigma}_l}{\sqrt{N}}\cdot R_N$ & $D_N^u=\frac{\hat{\sigma}_u}{\sqrt{N}}\cdot R_N$

and

$$\boxed{CI_\alpha^\theta=\left[\hat{\theta}_l-D_N^l,\ \hat{\theta}_u+D_N^u\right]}$$

Δ unknown:

-> Proceed analogously: Let $\bar{R}_N$ solve

a) $$\Phi\left(\frac{\sqrt{N}\cdot\hat{\Delta}}{\max\{\hat{\sigma}_l,\hat{\sigma}_u\}}+\bar{R}_N\right)-\Phi(-\bar{R}_N)=\alpha$$

b) $$\overline{CI}_\alpha^\theta=\left[\hat{\theta}_l-\bar{D}_N^l,\ \hat{\theta}_u+\bar{D}_N^u\right)$$

where $\boxed{\bar{D}_N^l=\frac{\hat{\sigma}_l}{\sqrt{N}}\cdot\bar{R}_N \ \& \ \bar{D}_N^u=\frac{\hat{\sigma}_u}{\sqrt{N}}\cdot\bar{R}_N}$

ojo:

$$\Phi\left(\frac{\sqrt{N}\cdot\Delta}{\max\{\sigma_e,\sigma_u\}}+R_N\right)-\Phi\left(\frac{\sqrt{N}\cdot\hat{\Delta}}{\max\{\sigma_e,\sigma_u\}}+R_N\right)$$

$$=\phi\left(\frac{\sqrt{N}\,\tilde{\Delta}}{\max\{\sigma_e,\sigma_u\}}+R_N\right)\cdot\underbrace{\sqrt{N}\left(\hat{\Delta}-\Delta\right)}_{O_p(1)}\Big/\max\{\sigma_e,\sigma_u\}\ \max$$

$\longrightarrow 0$ if $\Delta>0$, but if $\Delta=0$ it ~~equals~~ converges in probability to

$N(\hat{\Delta}-\Delta)^2 = \frac{1}{\sqrt{N}}$

$\phi(R_N)\cdot\sqrt{N}(\hat{\Delta}-\Delta)$ + if $\Delta=0$

but $R_N = z_{\frac{(1+\alpha)}{2}}$, so this ~~equals~~ converges to

$$\boxed{\underbrace{\phi\left(z_{\frac{(1+\alpha)}{2}}\right)\cdot\sqrt{N}\left(\hat{\Delta}-\Delta\right)}_{\text{not zero}}}\longrightarrow\phi\left(z_{\frac{(1+\alpha)}{2}}\right)\cdot N(0,\Sigma)$$

if it has an asympt. Normal distribution

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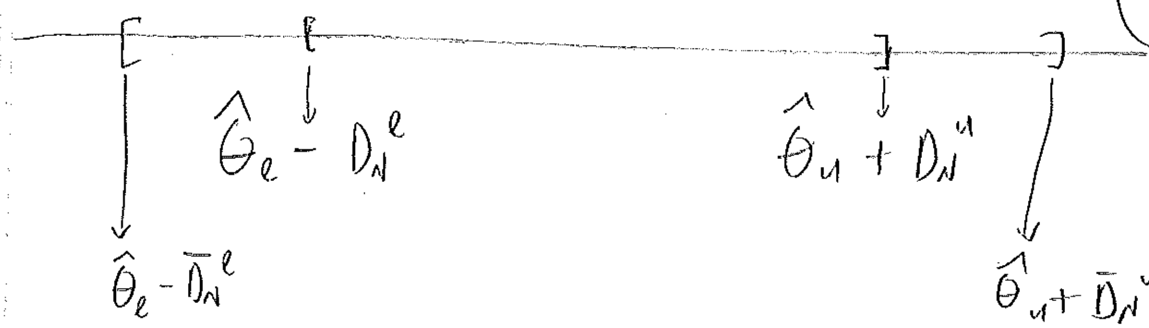
Cases:

A) $[\hat{\theta}_\ell - D_N^\ell,\ \hat{\theta}_u + D_N^u] \subseteq [\hat{\theta}_\ell - \bar{D}_N^\ell,\ \hat{\theta}_u + \bar{D}_N^u]$



$$\sqrt{N}\cdot\frac{\hat{\Delta}}{\max\{\hat{\sigma}_\ell,\hat{\sigma}_u\}} = \sqrt{N}\cdot\frac{\hat{\Delta}}{\max\{\sigma_\ell,\sigma_u\}}\cdot\left(\frac{\max\{\sigma_\ell,\sigma_u\}}{\max\{\hat{\sigma}_\ell,\hat{\sigma}_u\}}\right)$$

$$= \sqrt{N}\cdot\frac{\Delta}{\max\{\sigma_\ell,\sigma_u\}}\cdot\left(\frac{\max\{\sigma_\ell,\sigma_u\}}{\max\{\hat{\sigma}_\ell,\hat{\sigma}_u\}}\right)$$

$$+ \sqrt{N}\cdot\frac{(\hat{\Delta}-\Delta)}{\max\{\sigma_\ell,\sigma_u\}}\cdot\left(\frac{\max\{\sigma_\ell,\sigma_u\}}{\max\{\hat{\sigma}_\ell,\hat{\sigma}_u\}}\right)$$

$$\approx \sqrt{N}\cdot\frac{\Delta}{\max\{\sigma_\ell,\sigma_u\}} + \boxed{\sqrt{N}\cdot\frac{(\hat{\Delta}-\Delta)}{\max\{\sigma_\ell,\sigma_u\}}} + o_p(1)$$

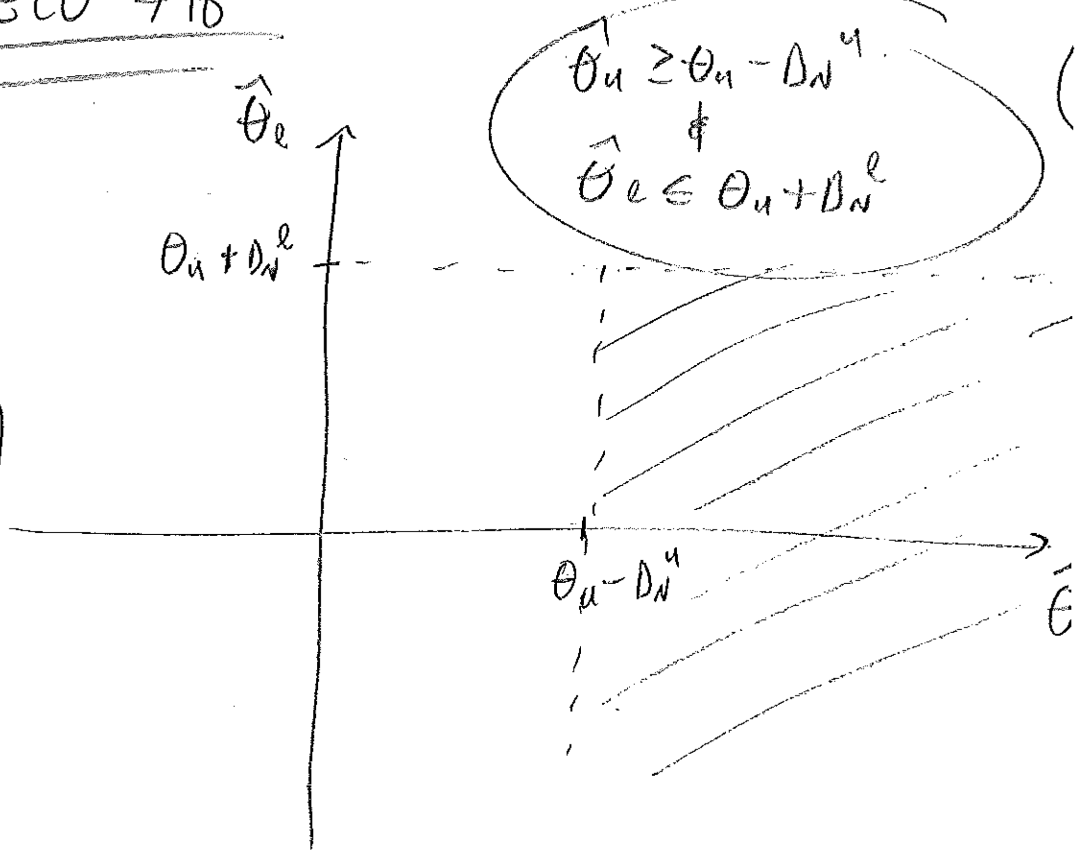
Shrinkage:

$$\tilde{\Delta} = \begin{cases} \hat{\Delta}, & \text{if } \hat{\Delta} \geq b_N \\ 0 & \text{otherwise} \end{cases}$$

where $b_N \to 0$ is a sequence ; $\sqrt{N}\cdot b_N \to +\infty$

ECO 716

[Imbens + Manski (2004) Basics]



$$= Pr(\hat\theta_\ell \leq \theta_u + D_N^\ell) - Pr(\hat\theta_\ell \leq \theta_u + D_N^\ell \text{ and } \hat\theta_u \leq \theta_u - D$$

$$= Pr(\hat\theta_\ell \leq \theta_u + D_N^\ell) - Pr(\hat\theta_u \leq \theta_u - D_N^u) \cdot Pr(\hat\theta_\ell \leq \theta_u + D_N^\ell$$

If $\hat\theta_\ell \leq \hat\theta_u$ w.p. 1, then:

$$Pr(\hat\theta_\ell \leq \theta_u + D_N^\ell \mid \hat\theta_u \leq \theta_u - D_N^u) = 1$$

$\Rightarrow$ Queda:

$$Pr(\hat\theta_\ell \leq \theta_u + D_N^\ell) - Pr(\hat\theta_u \leq \theta_u - D_N^u)$$

$$= Pr\left(\frac{\sqrt{N}(\hat\theta_\ell - \theta_\ell)}{\sigma_\ell} \leq \frac{\sqrt{N}(\theta_u - \theta_\ell)}{\sigma_\ell} + \sqrt{N}\frac{D_N^\ell}{\sigma_\ell}\right) \qquad - \Phi\left(-\sqrt{N}\frac{D_N}{\sigma}\right.$$

$$- Pr\left(\frac{\sqrt{N}(\hat\theta_u - \theta_u)}{\sigma_u} \leq \sqrt{N}\frac{D_N^u}{\sigma_u}\right) \approx \Phi\left(\frac{\sqrt{N}\cdot\Delta}{\sigma_\ell} + \sqrt{N}\frac{D_N^\ell}{\sigma_\ell}\right.$$

$\Phi(-c) = 1 - \Phi(c)$

(2)

if $\theta_0 = \theta_u$

$$\boxed{\Rightarrow Pr\left(\hat{\theta}_l - D_N^l \le \theta_u \le \hat{\theta}_u + D_N^u\right) \simeq \Phi\left(\frac{\sqrt{N}\cdot\Delta}{\sigma_l} + \sqrt{N}\frac{D_N^l}{\sigma_l}\right) - \Phi\left(-\sqrt{N}\frac{D_N^u}{\sigma_u}\right)}$$

if $\Delta > 0 \Rightarrow$ & this becomes

$$\simeq 1 - \Phi\left(-\sqrt{N}\frac{D_N^u}{\sigma_u}\right) = 1 - \left[1 - \Phi\left(\sqrt{N}\frac{D_N^u}{\sigma_u}\right)\right]$$

$$\boxed{= \Phi\left(\sqrt{N}\frac{D_N^u}{\sigma_u}\right)}$$

Next:

$$Pr\left(\hat{\theta}_l - D_N^l \le \theta_l \le \hat{\theta}_u + D_N^u\right)$$

$$= Pr\left(\hat{\theta}_l \le \theta_l + D_N^l\right) - Pr\left(\hat{\theta}_l \le \theta_l + D_N^l \text{ and } \hat{\theta}_u \le \theta_l - D_N^u\right)$$

$$= Pr\left(\sqrt{N}\frac{(\hat{\theta}_l - \theta_l)}{\sigma_l} \le \frac{\sqrt{N} D_N^l}{\sigma_l}\right) - Pr\left(\sqrt{N}\frac{(\hat{\theta}_u - \theta_u)}{\sigma_u} \le \sqrt{N}\frac{(\theta_l - \theta_u)}{\sigma_u} - \sqrt{N}\frac{D_N^u}{\sigma_u}\right)$$

if $\theta_0 = \theta_l$

$$\simeq \boxed{\Phi\left(\sqrt{N}\frac{D_N^l}{\sigma_l}\right) - \Phi\left(-\frac{\sqrt{N}\Delta}{\sigma_u} - \sqrt{N}\frac{D_N^u}{\sigma_u}\right)}$$

$$= \Phi\left(\sqrt{N}\frac{D_N^l}{\sigma_l}\right) - \left[1 - \Phi\left(\frac{\sqrt{N}\Delta}{\sigma_u} + \frac{\sqrt{N} D_N^u}{\sigma_u}\right)\right]$$

if $\Delta > 0$, then this is $\simeq \boxed{\Phi\left(\sqrt{N}\frac{D_N^l}{\sigma_l}\right)}$

(3)

<u>Next:</u> $\theta_0 \in (\theta_\ell, \theta_u)$ [interior of Θ_I]

- For any $\theta \in (\theta_\ell, \theta_u)$

$$\Pr(\hat\theta_\ell - D_N^\ell \le \theta \le \hat\theta_u + D_N^u)$$

$$= \Pr(\hat\theta_\ell \le \theta + D_N^\ell) - \Pr(\hat\theta_u \le \theta - D_N^u)$$

$$= \Pr\left(\sqrt{N}\frac{(\hat\theta_\ell - \theta_\ell)}{\sigma_\ell} \le \sqrt{N}\frac{(\theta-\theta_\ell)}{\sigma_\ell} + \sqrt{N}\frac{D_N^\ell}{\sigma_\ell}\right)$$

Resumirlo así

$$- \Pr\left(\sqrt{N}\frac{(\hat\theta_u - \theta_u)}{\sigma_u} \le \sqrt{N}\frac{(\theta-\theta_u)}{\sigma_u} - \sqrt{N}\frac{D_N^u}{\sigma_u}\right)$$

$$\boxed{\approx \Phi\left(\sqrt{N}\frac{(\theta-\theta_\ell)}{\sigma_\ell} + \sqrt{N}\frac{D_N^\ell}{\sigma_\ell}\right) - \left[1 - \Phi\left(\sqrt{N}\frac{(\theta_u-\theta)}{\sigma_u} + \sqrt{N}\frac{D_N^u}{\sigma_u}\right)\right]} \to 1 - [1-1] = 1$$

+ if $\theta = \theta_\ell \to \Phi\left(\frac{\sqrt{N}D_N^\ell}{\sigma_\ell}\right) - \left[1 - \Phi\left(\frac{\sqrt{N}\Delta}{\sigma_u} + \frac{\sqrt{N}D_N^u}{\sigma_u}\right)\right]$

+ if $\theta = \theta_u \to \Phi\left(\frac{\sqrt{N}\Delta}{\sigma_\ell} + \frac{\sqrt{N}D_N^\ell}{\sigma_\ell}\right) - \left[1 - \Phi\left(\frac{\sqrt{N}D_N^u}{\sigma_u}\right)\right]$

$$= \Phi\left(\frac{\sqrt{N}D_N^u}{\sigma_u}\right) - \left[1 - \Phi\left(\frac{\sqrt{N}\Delta}{\sigma_\ell} + \frac{\sqrt{N}D_N^\ell}{\sigma_\ell}\right)\right]$$

* Therefore: If $\Delta > 0$,

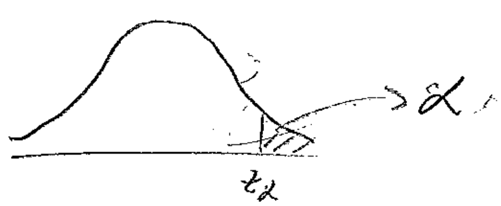
$$\Pr\left(\hat{\theta}_l - D_N^l \le \theta_0 \le \hat{\theta}_u + D_N^u\right) \simeq$$

$$\begin{cases} \Phi\left(\dfrac{\sqrt{N}\cdot D_N^l}{\sigma_l}\right), & \text{if } \theta_0 = \theta_l \\ \Phi\left(\dfrac{\sqrt{N}\cdot D_N^u}{\sigma_u}\right), & \text{if } \theta_0 = \theta_u \\ 1, & \text{if } \theta_0 \in (\theta_l, \theta_u) \end{cases}$$

$\Rightarrow$ choose D_N^l, D_N^u to solve:

$$\Phi\left(\frac{\sqrt{N}\cdot D_N^l}{\sigma_l}\right) = 1-\alpha$$

$$\Phi\left(\frac{\sqrt{N}\cdot D_N^u}{\sigma_u}\right) = 1-\alpha$$



$$\Rightarrow \sqrt{N}\cdot\frac{D_N^l}{\sigma_l} = z_\alpha \; ; \quad \sqrt{N}\cdot\frac{D_N^u}{\sigma_u} = z_\alpha$$

$$\Rightarrow \boxed{D_N^l = \frac{\sigma_l\cdot z_\alpha}{\sqrt{N}}} \qquad \boxed{D_N^u = \frac{\sigma_u\cdot z_\alpha}{\sqrt{N}}}$$

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- But what if $\Delta = 0$? $\Rightarrow$

$$Pr(\hat{\theta}_L - \Delta_N^l \leq \theta_0 \leq \hat{\theta}_u + \Delta_N^u)$$

$$\approx \Phi(z_\alpha) - [1 - \Phi(z_\alpha)]$$

$$= 2 \cdot \Phi(z_\alpha) - 1 = 2 \cdot (1-\alpha) - 1$$

$$= 2 - 2\alpha - 1 = \boxed{1 - 2\alpha}$$

$$\boxed{1 - 2\alpha < 1 - \alpha \; !}$$

under-coverage

Econ 589 — Imbens & Manski (~~complement~~ supplement) (1)

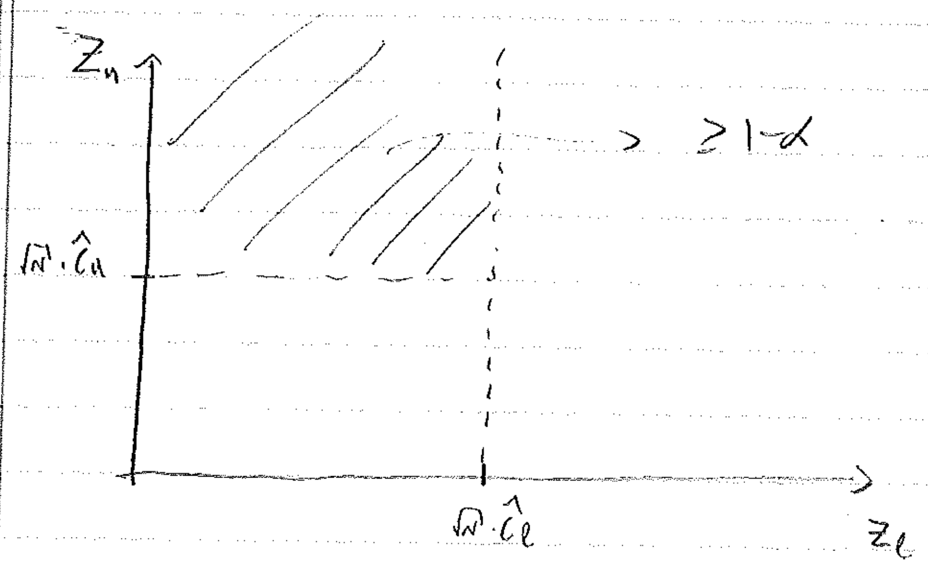
$\theta_0 \in [\theta_l, \theta_u]$

$CS_{\theta_I} = [\hat{\theta}_l - \hat{c}_l, \hat{\theta}_u + \hat{c}_u]$ such that:

$$\lim_{N \to \infty} \Pr(\hat{\theta}_l - \hat{c}_l \le \theta_l \text{ and } \hat{\theta}_u + \hat{c}_u \ge \theta_u) \ge 1-\alpha$$

$$= \lim_{N \to \infty} \Pr(\underbrace{\sqrt{N}(\hat{\theta}_l - \theta_l)}_{z_l} \le \sqrt{N}\cdot\hat{c}_l \text{ and } \underbrace{\sqrt{N}(\hat{\theta}_u - \theta_u)}_{z_u} \ge \sqrt{N}\hat{c}_u) \ge 1-\alpha$$

$$\begin{matrix} \sqrt{N}(\hat{\theta}_l - \theta_l) \\ \sqrt{N}(\hat{\theta}_u - \theta_u) \end{matrix} \xrightarrow{d} N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \Sigma_l, & \Sigma_{ul} \\ \Sigma_{ul}, & \Sigma_u \end{pmatrix}\right)$$



(a)

Assumption:

$\exists\ \hat{\theta}_l, \hat{\theta}_u$ such that:

$$\sqrt{N}\begin{pmatrix}\hat{\theta}_l - \theta_l \\ \hat{\theta}_u - \theta_u\end{pmatrix} \xrightarrow{d} N\left(\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}\sigma_l^2, & \rho\sigma_l\sigma_u \\ \rho\sigma_l\sigma_u, & \sigma_u^2\end{bmatrix}\right)$$

uniformly in $P \in \mathcal{P}$,

$$\underline{\sigma}^2 \leq \sigma_l^2 \leq \bar{\sigma}^2, \quad \underline{\sigma}^2 \leq \sigma_u^2 \leq \bar{\sigma}^2$$

$$\begin{pmatrix}Z_l\\Z_u\end{pmatrix} \sim N\left(\begin{bmatrix}0\\0\end{bmatrix}, \begin{pmatrix}\sigma_l^2, & \rho\sigma_l\sigma_u \\ \rho\sigma_l\sigma_u, & \sigma_u^2\end{pmatrix}\right)$$

$$Z_l \mid Z_u = z \sim N\left(\frac{\sigma_l}{\sigma_u}\rho\cdot z,\ (1-\rho^2)\cdot\sigma_l^2\right)$$

~~Assume~~

Imbens & Manski CI:

$$CS_{\theta_0}(1-\alpha) = \left(\hat{\theta}_l - \frac{c_\alpha \hat{\sigma}_l}{\sqrt{N}},\ \hat{\theta}_u + \frac{c_\alpha\cdot\hat{\sigma}_u}{\sqrt{N}}\right)$$

where c_α solves:

$$\Phi\left(c_\alpha + \frac{\sqrt{N}\cdot\hat{\Delta}}{\max\{\hat{\sigma}_l,\hat{\sigma}_u\}}\right) - \Phi(-c_\alpha) = 1-\alpha$$

6

If $1-\alpha = .95$

- If $\hat{\Delta} \approx 0$, $c_\alpha \approx \Phi^{-1}(0.975) \approx 1.96$
- If $\hat{\Delta} > 0$, $c_\alpha \approx \Phi^{-1}(0.95) \approx 1.64$

Assumption*: Need super-efficiency of $\hat{\Delta}$ at $\Delta = 0$

$$\lim_{N \to \infty} \inf_{\substack{\theta \in \Theta_I, \\ P \in \mathcal{P}: \\ \theta_0(P) = \theta}} \Pr(\theta \in CS_{\theta_0}^{1-\alpha}) \geq 1-\alpha$$

Stoye (2009)

$\exists$ sequence $\{a_N\}$: $a_N \to 0$ and $a_N \cdot \sqrt{N} \to \infty$ and
$\sqrt{N}\,|\hat{\Delta} - \Delta_N| \xrightarrow{P} 0$ for all $\{P_N\} \in \mathcal{P}$: $\Delta(P_N) \leq a_N$

Lemma 3 (Stoye)

- If $\Pr(\hat{\theta}_u \geq \hat{\theta}_\ell) = 1$ $\forall P \in \mathcal{P}$, then Assumption* is satisfied

Otherwise, force super-efficiency:
$b_N \to 0$: $b_N \sqrt{N} \to \infty$. Then let

$$\Delta^* = \begin{cases} \hat{\Delta} & \text{if } \hat{\Delta} > b_N \\ 0, & \text{otherwise} \end{cases}$$

- Let (c_ℓ^3, c_u^3) minimize

$$(\hat{\sigma}_\ell \cdot c_\ell + \hat{\sigma}_u \cdot c_u)$$

don't calibrate through a single equation

subject to the constraint:

$$\Pr\left(-c_\ell \leq z_1,\ \hat{\rho} z_1 \leq c_u + \frac{\sqrt{N}\Delta^*}{\hat{\sigma}_u} + \sqrt{1-\hat{\rho}^2}\cdot z_2\right.$$

$$\Pr\left(-c_\ell - \frac{\sqrt{N}\Delta^*}{\hat{\sigma}_\ell} + \sqrt{1-\hat{\rho}^2}\cdot z_2 \leq \hat{\rho} z_1,\ z_1 \leq c\right.$$

And let

$$CS_{\theta_0}^{1-\alpha} = \begin{cases} \left[\hat{\theta}_\ell - \frac{\hat{\sigma}_\ell \cdot c_\ell^3}{\sqrt{N}},\ \hat{\theta}_u + \frac{\hat{\sigma}_u c_u^3}{\sqrt{N}}\right] \text{ if } \\ \phi \quad \text{otherwise} \end{cases}$$

Proposition 3:

$$\lim_{N\to\infty} \inf_{\substack{\theta \in \Theta \\ \text{st} \\ \theta_0(P)=0 \\ P\in\mathcal{P}}} \Pr(\theta_0 \in CS_{\theta_0}^{1-\alpha}) = 1-\alpha$$

Chernozhukov, Hong & Tamer (2007) $Q_N(\theta)$ (CHT) → ~~ECO 716~~

$(x)_+ = \max\{x, 0\}$

$$E[m(x,\theta)] \le 0$$

general ~~extremum~~ estimator problem.

$$Q(\theta) = E[m(X,\theta)]_+' \, W(\theta) \, E[m(X,\theta)]_+$$

$W(\cdot) \rightarrow$ diagonal matrix with $>>0$ diag.

$\underline{\Theta}_I$ = set of minimizers of $Q(\theta)$

$\rightarrow \| E[m(X,\theta)]' W^{\frac{1}{2}}(\theta) \|_+^2$

In the usual

$(x)_+ = \max\{x, 0\}$

$E[m(X,\theta)] = 0$ case,

rep $Q(\theta) =$ (without the "+")

and $\underline{\Theta}_I$ = set of minimizers

Proposal:

- Estimate $\underline{\Theta}_I$ consistently ~~then~~ as contour sets of level c, denoted as

$$C_n(c) = \{\theta \in \underline{\Theta} : a_N \cdot Q_N(\theta) \le c\}$$

→ choose c so that $C_n(c)$ converges at fastest possible rate

for some normalizing sequence "a_N", selected so that ~~last~~ is stochastically bounded with nondeg. assumpt.

$$c_N := \sup_{\theta \in \underline{\Theta}_I} a_N \cdot Q_N(\theta)$$

(ii)

- The <u>level</u> $c = \hat{c}$ (possibly data-dependent) should be selected efficiently, so that
  $C_N(\hat{c})$ converges to Θ_I at fastest possible rate.

How? → Select $\hat{c}$ as small as possible (but not smaller than C_N, and let it grow very slowly with N ($\hat{c} \propto \ln(N)$))

- If degeneracy property holds, $\hat{c}$ can be set as a constant

- Hausdorff distance:

$$d_H(A,B) = \max\left\{\sup_{a\in A} d(a,B),\ \sup_{b\in B} d(b,A)\right\}$$

with $d(b,A) = \inf_{a\in A} \|b-a\|$

$d_H(A,B) = \infty$ if A or B are empty

- Consistency means:

$$d_H(C_N(\hat{c}), \Theta_I) \xrightarrow{P} 0$$

will follow from uniform convergence of Q_N to Q over Θ.

rates of convergence

$$d_H\left(C_N(\hat{c}), \Theta_I\right) = O_p\left(\sqrt{\frac{\max(\hat{c},1)}{N}}\right)$$

if degeneracy $\longrightarrow$ this is $O_p\left(\frac{1}{\sqrt{N}}\right)$

- $C_N(\hat{c})$ also needs to be a confidence region for Θ_I. That is,

$$\lim_{N\to\infty} P\left(\Theta_I \subseteq C_N(c)\right) = \alpha$$

$\alpha \in (0,1)$

Note: $\{\Theta_I \subseteq C_N(c)\} \Longleftrightarrow \{C_N \leq c\}$

therefore, $\hat{c}$ should be a consistent estimator of the α^{th} quantile of C_N.

- How can $\hat{c}$ be obtained?
  - By characterizing the analytical limiting distribution of C_N
  - Or by subsampling.

- Nonequicontinuity of the empirical process $a_N Q_N(\theta)$ is obstacle. Replace it with

(iv)

Pointwise approach. —

Pointwise confidence regions

— Inversion of hypotheses tests

Suppose

$$C_N(\theta) = a_N \cdot Q_N(\theta)$$

and $\Pr[C_N(\theta) \leq c] \rightarrow \Pr(C(\theta) \leq c)$ for each $c \geq 0$ $\forall\ \theta \in \Theta_I$

• Let $c(\alpha, \theta)$ denote the α^{th} quantile of $C(\theta)$.

• Let $\hat{c}(\theta) = c(\alpha, \theta) + o_p(1)$ and
~~$\hat{c}$ - estimate of~~
$\hat{c} \geq \sup_{\theta \in \Theta_I} c(\alpha, \theta) + o_p(1)$

such that $\hat{c} = O_p(1)$. Let

$\cancel{c(\theta) = \hat{c}(\theta)\ \max}\ \{\hat{c}$

$$\bar{c}(\theta) = \min \{\hat{c}(\theta), \hat{c}\}$$

$$C_N(\bar{c}(\cdot)) = \{\theta \in \Theta : a_N Q_N(\theta) \leq \bar{c}(\theta)\}$$

V

$\Rightarrow \forall \theta^* \in \overline{\Theta}_I$:

$$\liminf_{N \to \infty} \Pr[\theta^* \in C_N(\hat{c})] \geq$$

$$\liminf_{N \to \infty} \Pr[\theta^* \in C_N(\bar{c}(\cdot))] \geq \alpha$$

———o———o———

Econ 589

(1)

Andrews & Soares [GMS] (Econometrica 2010)

• Model: $\theta_0 \in \Theta \subset \mathbb{R}^d$ is assumed to satisfy:

$$E_{F_0}[m_j(W_i;\theta_0)] \begin{cases} \geq 0 & \text{for } j=1,\ldots,p \\ = 0 & \text{for } j=p+1,\ldots,p+v \end{cases}$$

→ They propose confidence sets based on the inversion of a test for $H_0: \theta=\theta_0$. The nominal level $1-\alpha$ CS for θ is:

$$CS_n = \{\theta \in \Theta : T_n(\theta) \leq \underbrace{c_{n,\alpha}(\theta)}_{\text{critical value}}\}$$

• They consider:
- GMS
- subsampling
- plug-in asymptotic

• Uniformity: → Required for the asymptotic size to give a good approximation to the finite sample size of CS.

- Exact confidence size of CS_n is:

$$ExCS_n = \inf_{(\theta,F) \in \mathcal{F}} P_F(T_n(\theta) \leq c_{1-\alpha}(\theta))$$

- Asymptotic confidence size of CS_n is:

$$AsyCS = \lim_{n \to \infty} ExCS_n$$

* Uniformity

- Let $\bar{m}_N(\theta)$ denote the sample moment functions

$$\bar{m}_N(\theta) = \frac{1}{N} \sum_{i=1}^{N} m_j(W_i;\theta)$$

- $\hat{\Sigma}_N(\theta) \rightarrow$ estimator of the asymptotic variance of $n^{\frac{1}{2}} \bar{m}_n(\theta)$.

- They look at test stm statistics of the form

$$T_N(\theta) = S(N^{\frac{1}{2}} \bar{m}_N(\theta), \hat{\Sigma}_N(\theta))$$

where:

a) $S((m_1, m_2), \Sigma)$ is nonincreasing in m_1 $\forall\ m_1 \in \mathbb{R}^p$

b) $S(m, \Sigma) = S(\Delta m, \Delta \Sigma \Delta)$

c) $S(m, \Omega) \geq 0$ $\forall\ m \in \mathbb{R}^k$

d) $S(m, \Omega)$ is continuous at all $m \in \mathbb{R}^p$

Example:

a) $S_1(m, \Sigma) = \sum_{j=1}^{p} \left[\frac{m_j}{\sigma_j} \right]_-^2 + \sum_{j=p+1}^{p+v} \left(\frac{m_j}{\sigma_j} \right)^2$

$$[x]_- = \begin{cases} x & \text{if } x < 0 \\ 0 & \text{if } x \geq 0 \end{cases}$$

σ_j^2 is j^{th} diagonal element of Σ.

b) $S_2(m, \Sigma) = \inf_{t = (t_1, 0) : t_1 \in \mathbb{R}^p_{+, \infty}} (m - t)' \Sigma^{-1} (m - t)$

(5)

(*)

$$T_N(\theta) = S(N^{\frac{1}{2}} \bar{m}_N(\theta), \hat{\Sigma}_N(\theta))$$

$$= S(\hat{D}_N^{-\frac{1}{2}}(\theta) N^{\frac{1}{2}} \bar{m}_N(\theta), \hat{\Omega}_N(\theta)),$$

where $\hat{D}_N(\theta) := \mathrm{Diag}(\hat{\Sigma}_N(\theta))$

$$\hat{\Omega}_N(\theta) := \hat{D}_N^{-\frac{1}{2}}(\theta) \hat{\Sigma}_N(\theta) \hat{D}_N^{-\frac{1}{2}}(\theta)$$

• Under appropriate sequence of null distributions $\{F_N, N \geq 1\}$, the asymptotic Null distribution of $T_N(\theta_0)$ is that of:

$$S(\Omega_0^{\frac{1}{2}} Z^* + (h_1, 0_v), \Omega_0), \; Z^* \sim N(0_K, I_K)$$

$h_1 \in \mathbb{R}^p_{+,\infty}$ → degree of slackness

$\mathbb{R}^p_{[+,\infty]}$ → vectors which can be either real, or $+\infty$.

• GMS critical value → the $1-\alpha^{th}$ quantile of a DGP of the asymptotic Null distribution of $S(\Omega_0^{\frac{1}{2}} Z^* + (h_1, 0_v), \Omega_0)$, where Ω_0 is replaced with $\hat{\Omega}$ and h_1 with a p-vector in $\mathbb{R}^p_{+,\infty}$ whose value depends on measure of slackness → of the moment inequalities

$$\xi_N(\theta) = \left(\frac{1}{K_N}\right) \cdot N^{\frac{1}{2}} \cdot \hat{D}_N^{-\frac{1}{2}}(\theta) \bar{m}_N(\theta) \in \mathbb{R}^K$$

where $\{k_n : n \geq 1\}$, $k_n \to +\infty$ slowly

- Law of iterated logarithm → operates in-between the LLN and the CLT

$\underbrace{\quad\quad}_{\text{iid}}$

Let $S_N = Y_1 + \cdots + Y_N$. Then,

$$\lim_{N \to \infty} \sup \frac{S_N}{\sqrt{N \cdot \log \log(N)}} = \sqrt{2} \quad a.s$$

$$\frac{S_N}{\sqrt{N \cdot \log \log N}} \xrightarrow{P} 0, \text{ but } \frac{S_N}{\sqrt{N \log \log N}} \not\to_{a.s} 0$$

✓ Law of iterated logarithm:

$$k_N = \sqrt{\log \log N}$$

- or BIC choice: } preferable in simulations

$$k_N = \sqrt{\log(N)}$$

BIC → model selection

Replace $h_{1,j}$ with: $\varphi_j(\xi_N(\theta_0), \hat{\Omega}_N(\theta_0))$, where:

(i) $\varphi_j(\xi, \Omega) = 0 \quad \forall \ \xi = (\xi_1, \dots, \xi_k)'$ where $\xi_j = 0$ and all Ω

(ii) $\varphi_j(\xi, \Omega) \to \infty$ as $(\xi, \Omega) \to (\xi_*, \Omega_*)$, $\forall \ \xi_* = (\xi_{*,1}, \dots, \xi_{*,k})'$ where $\xi_{*,j} = \infty$

Example:

$$\varphi_j(\xi, \Omega) = \begin{cases} 0 & \text{if } \xi_j \leq 1 \\ \infty & \text{if } \xi_j > 1 \end{cases}$$

Note: $\xi_j > 1$ implies:

$$\left| \frac{N^{\frac{1}{2}} \bar{m}_{N,j}(\theta_0)}{\hat{\sigma}_{N,j}(\theta_0)} > (\kappa_N) \right| \quad \to \infty$$

GMS quantiles ~~come from~~ for $H_0: \theta = \theta_0$:

(i) Compute $m_N(\theta_0)$ and $\hat{\Sigma}_N(\theta_0)$

(ii) Compute $T_N(\theta_0) = S(N^{\frac{1}{2}} m_N(\theta_0), \hat{\Sigma}_N(\theta_0))$

(iii) $\xi_N(\theta_0) = \kappa_N^{-1} N^{\frac{1}{2}} D_{N}^{-\frac{1}{2}}(\hat{\Sigma}_N(\theta_0)) \bar{m}_N(\theta_0)$ where $\kappa_N = (\lambda_N(N))^{\frac{1}{2}}$. Compute $\varphi(\xi_N(\theta_0), \hat{\Omega}_N(\theta_0))$

(iv) Simulate R iid random vectors $\{Z_r^*: r = 1, \dots, R\}$, $Z_r^* \sim N(0_k, I_k)$

(v) Take $\hat{c}_N(\theta_0, 1-\alpha)$ as the $1-\alpha$ sample quantile of:

$$S\left(\hat{\Lambda}^{\frac{1}{2}}(\theta_0) \cdot Z_r^* + \varphi(\xi_N(\theta_0), \hat{\Omega}_N(\theta_0)), \hat{\Omega}_N\right)$$

Reject $H_0: \theta = \theta_0$ if $T_N(\theta_0) > \hat{c}_N(\theta_0, 1-\alpha)$

m Andrews & Shi (I)

$E[m_j(W;\theta_0)|X] \leq 0$ a.s , $j=1,\dots,p$

- Space of functions G : $g \geq 0$ $\forall g \in G$

$\Rightarrow E[m_j(W;\theta_0)\cdot g_j(X_i)] \leq 0$ $\forall g_j \in G$

- Let: $g=(g_1,\dots,g_p)$

$$\bar{m}_N(\theta,g) = \frac{1}{N}\sum_{i=1}^{N} m(W_i,\theta,g) \text{ for } g \in G$$

where

$$m(W_i,\theta,g) = \begin{bmatrix} m_1(W_i,\theta)g_1(X_i) \\ m_2(W_i,\theta)g_2(X_i) \\ \vdots \\ m_K(W_i,\theta)g_K(X_i) \end{bmatrix} \text{ for } g \in G$$

- Sample var-cov of ~~...~~ $N^{\frac{1}{2}}\bar{m}_N(\theta,g)$ is:

$$\hat{\Sigma}_N(\theta,g) = \frac{1}{N}\sum_{i=1}^{N}(m(W_i,\theta,g)-\bar{m}_N(\theta,g))\cdot(m(W_i,\theta,g)-\bar{m}_N(\theta,g))'$$

use:

$$\bar{\Sigma}_N(\theta,g) = \hat{\Sigma}_N(\theta,g) + \varepsilon\cdot \mathrm{Diag}(\hat{\Sigma}_N(\theta,1_K))$$

for $g \in G$

(II)

They propose

Cramér-von Mises - type (CvM) statistics:

$$T_N(\theta) = \int S(N^{1/2}\bar{m}_N(\theta,g), \bar{\Sigma}_N(\theta,g))\, dQ(g)$$

where S is a non-negative function, Q is a weight function

or KS-type:

$$T_N(\theta) = \sup_{g \in \mathcal{G}} S(N^{1/2}\bar{m}_N(\theta,g), \bar{\Sigma}_N(\theta,g))$$

Ex. of S:

$$S(m,\Sigma) = \sum_{j=1}^{p}\left[\frac{m_j}{\sigma_j}\right]_+^2 = \sum_{j=1}^{m}\left(\max\left\{\frac{m_j}{\sigma_j}, 0\right\}\right)$$

and σ_j^2 is the j^{th} diagonal element of Σ

Let

$$v_N(\theta,g) = N^{\frac{1}{2}}(\bar{m}_N(\theta,g) - E[\bar{m}_N(\theta,g)])$$

$\Rightarrow$

$$S(N^{1/2}\bar{m}_N(\theta,g), \bar{\Sigma}_N(\theta,g)) =$$

$$S(v_N + N^{\frac{1}{2}}\bar{m}_N(\theta,g), \bar{\Sigma}_N(\theta,g))$$

presence of this term is key for uniform Asymptotic Coverage probabilities
- It is a measure of "slackness"

- Proposal:

* To compute critical values, look at those of

$$\int S(v_N(\theta,g) + \varphi_{1N}(\theta), \bar{\Sigma}_N(\theta,g))$$

where φ_{1N} is constructed such that

$\varphi_N(\theta) \leq N^{\frac{1}{2}}\bar{m}_N(\theta,g) \quad \forall g \in G$ w.p.a.1 uniformly over θ

IV

- Let $\{B_N\}$ be a non-decreasing sequence of ≥ 0 constants, and let κ_N be a sequence:

$$\kappa_N \to \infty \quad \text{and} \quad \frac{B_N}{\kappa_N} \to 0$$

let $\xi_N(\theta,g) = N^{\frac{1}{2}} \bar{D}_N^{\frac{1}{2}} \; N^{\frac{1}{2}} \bar{m}_N(\theta,g)$,

then

$$\varphi_N(\theta,g) = B_N \cdot \mathbb{1}\{\xi_N(\theta,g) > \kappa_N\}$$

~~* Uniform convergence~~

* Weak convergence of

$$\nu_N(\cdot) \Rightarrow \nu(\cdot) \quad \text{over} \quad \begin{array}{l}\theta \in \Theta \\ g \in G\end{array}$$

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(1)

Chernozhukov, Lee & Rosen (2013)

- Their paper introduces an inferential procedure for cases where a parameter of interest, θ^* belongs within bounds $[\theta^l(v), \theta^u(v)]$, where $v \in \mathcal{V}$. Specifically, models where

$$\theta^* \in [\theta^l(v), \theta^u(v)] \quad \forall\, v \in \mathcal{V}$$

- And therefore, the identified set is:

$$\Theta_I = \bigcap_{v \in \mathcal{V}} [\theta^l(v), \theta^u(v)] = [\sup_{v\in\mathcal{V}} \theta^l(v), \inf_{v\in\mathcal{V}} \theta^u(v)]$$

- They analyze both the case where $\theta^l(\cdot)$ and $\theta^u(\cdot)$ have parametric or nonparametric estimators.

- They develop methods to construct confidence regions that achieve a desired asymptotic level

- Their approach extends to models of the type:

$$\left.\sup_{v\in\mathcal{V}} \theta^l(x,v) \leq \theta^*(x) \leq \inf_{v\in\mathcal{V}} \theta^u(x,v)\right\}$$

useful for conditional ~~moment~~ inequalities

Example: Conditional moment inequalities

- Consider a model that predicts:

$$E[m_j(X, \eta_0) | Z = z] \geq 0 \quad \forall j = 1, \ldots, J, \; z \in \mathcal{Z}_j$$

- Here, η_0 represents the parameter of interest.

- Define $v = (z, j)$, $\mathcal{V} = \{(z, j) : z \in \mathcal{Z}_j, j \in \{1, \ldots, J\}\}$

- Let: $\theta(\eta, v) = E[m_j(X, \eta) | Z = z]$,

- Suppose we want to test:

$$E[m_j(X, \eta) | Z = z] \geq 0 \quad \forall j = 1, \ldots, J, \; z \in \mathcal{Z}_j$$

- This becomes the problem of testing whether:

$$\inf_{v \in \mathcal{V}} \theta(\eta, v) \geq 0$$

versus:

$$\inf_{v \in \mathcal{V}} \theta(\eta, v) < 0$$

- The method in this paper can be applied to this problem.

Outline of their inferential approach

- Denote ~~a~~ an upper bound θ_0 on θ^* as:

$$\theta^* \leq \theta_0 = \inf_{v \in \mathcal{V}} \theta(v)$$

- Their goal is to construct estimators and confidence regions for θ_0.

- The naive estimator $\inf_{v \in \mathcal{V}} \hat{\theta}(v)$

performs badly in practice, ~~typ~~ tends to be downwards-biased in finite samples.

- Instead, they propose precision-corrected estimates for θ_0 of the form:

$$\boxed{\hat{\theta}_0(p) = \inf_{v \in \mathcal{V}} [\hat{\theta}(v) + k(p) \cdot s(v)],}$$

where $s(v)$ is the standard error of $\hat{\theta}(v)$ and $k(p)$ is an estimator of the pth quantile of the standardized process, $\sup_{v \in \mathcal{V}} Z_N(v)$, where

$$Z_N(v) = \frac{\theta(v) - \hat{\theta}(v)}{\sigma(v)}$$

[where $\frac{\sigma(v)}{s(v)} \xrightarrow{p} 1$ uniformly in $\mathcal{V}$].

- The goal ~~rel~~ is for $\hat{\theta}_0(p)$ to satisfy:

$$\boxed{\lim_{N \to \infty} \Pr[\theta_0 \le \hat{\theta}_0(p)] \ge p}$$

- They propose to approximate the quantiles of $Z_N(v)$ by using a sequence of Gaussian processes such that, for an appropriate sequence $\{a_N\}$,

$$a_N \sup_{v \in V} |Z_N(v) - Z_N^*(v)| = O_p(1)$$

- $K(p)$ is constructed by approximating the quantiles of $\sup_{v \in \hat{V}} |Z_N^*(v)|$ over a set $\hat{V}$ such that it contains [w.p.a.1], the argmin set

They propose using preliminary estimator $\hat{V}_N$ for V_0.

$$\boxed{V_0 = \arg\inf_{v \in V} \theta(v)}$$

- The paper is devoted to showing how to construct $K(p)$ either by analytical methods or by simulation.