Econ 589

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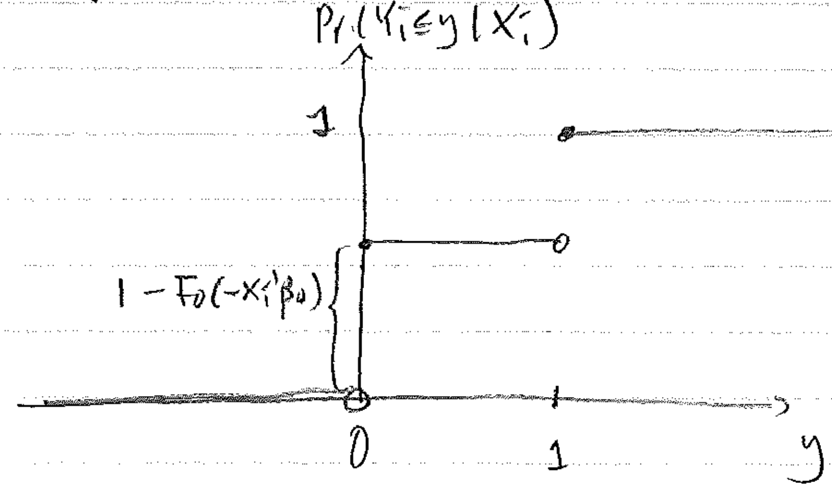
Maximum Score and showing $N^{\frac{1}{3}}$-consistency

see top of page ③ for the model where $Y_i = \mathbb{1}\{X_i'\beta_0 + \varepsilon_i \geq 0\}$

$$Y_i = \mathbb{1}\{X_i'\beta_0 + \varepsilon_i \leq 0\}$$

$$\varepsilon_i | X_i \sim F_0 \quad ; \quad \overbrace{\text{median}(\varepsilon_i | X_i) = 0}^{\text{only restriction made}} \quad w.p.1$$

and $F_0(\cdot)$ strictly increasing in a nbd. of zero.

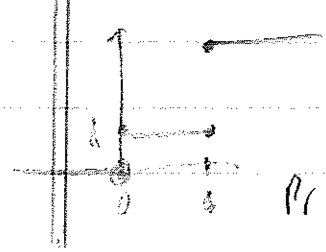


if $1 - F_0(-X_i'\beta_0) \geq \frac{1}{2} \Rightarrow \text{median}(Y_i | X_i) = 0$

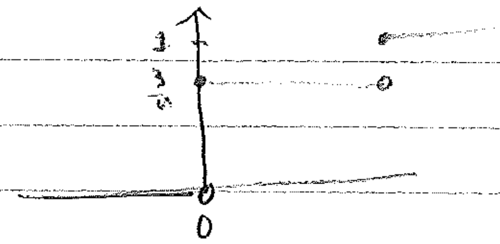
if $1 - F_0(-X_i'\beta_0) < \frac{1}{2} \Rightarrow \text{median}(Y_i | X_i) = 1$

$\text{median}(X) = \inf\{x : F_X(x) \geq \frac{1}{2}\}$

$\text{median}(X) = m \Leftrightarrow Pr(X \leq m) \geq \frac{1}{2}$ and $Pr(X \geq m) \geq \frac{1}{2}$



$Pr(X \leq 0) = 1/4$, $Pr(X \geq 0) = \frac{3}{4}$, $Pr(X \leq 1) = 1$, $Pr(X \geq 1) = 1$



$Pr(X \leq 0) = \frac{3}{4}$

$Pr(X \geq 0) = 1$

Quantile function:

$Q(p) = \inf\{x \in \mathbb{R} : p \leq F(x)\}$

$\Rightarrow$ median $(Y) = \inf\{y \in \mathbb{R} : F(x) \geq \frac{1}{2}\}$ ✓

OK:

$$\text{median}(Y_i | X_i) = \begin{cases} 0 & \text{if } 1 - F_0(-X_i'\beta_0) \geq \frac{1}{2} \\ 1 & \text{if } 1 - F_0(-X_i'\beta_0) < \frac{1}{2} \end{cases}$$

But $1 - F_0(-X_i'\beta_0) \geq \frac{1}{2} \iff F_0(-X_i'\beta_0) \leq \frac{1}{2}$

$\iff -X_i'\beta_0 \leq 0 \iff X_i'\beta_0 \geq 0$

so,

$$\text{median}(Y_i | X_i) = \begin{cases} 0 & \text{if } X_i'\beta_0 \geq 0 \\ 1 & \text{if } X_i'\beta_0 < 0 \end{cases}$$

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*Note: If the model is described as $Y_i = \mathbb{1}\{X_i'\beta_0 + \varepsilon_i \geq 0\}$, then we would maximize this objective function.

$\Rightarrow$ Look for β that minimizes:

$$E\left[|Y_i - 0| \cdot \mathbb{1}\{X_i'\beta \geq 0\} + |Y_i - 1| \cdot \mathbb{1}\{X_i'\beta < 0\}\right]$$

$$= E\left[Y_i \cdot \mathbb{1}\{X_i'\beta \geq 0\} + (1-Y_i) \cdot \mathbb{1}\{X_i'\beta < 0\}\right]$$

$$= E\left[Y_i \cdot \mathbb{1}\{X_i'\beta \geq 0\} + (1-Y_i) \cdot (1 - \mathbb{1}\{X_i'\beta \geq 0\})\right]$$

$$= E\left[2Y_i \cdot \mathbb{1}\{X_i'\beta \geq 0\} + (1-Y_i) - \mathbb{1}\{X_i'\beta \geq 0\}\right]$$

$$= E\left[2Y_i \cdot \mathbb{1}\{X_i'\beta \geq 0\} - \mathbb{1}\{X_i'\beta \geq 0\}\right] + E\left[(1-Y_i)\right]$$

$$= E\left[(2Y_i - 1) \cdot \mathbb{1}\{X_i'\beta \geq 0\}\right] + E\left[(1-Y_i)\right]$$

$$= 2\underbrace{E\left[(Y_i - \tfrac{1}{2}) \cdot \mathbb{1}\{X_i'\beta \geq 0\}\right]}_{\text{Let this be our objective function.}} + E\left[(1-Y_i)\right]$$

Let

$$G(\beta) = E\left[(Y_i - \tfrac{1}{2}) \cdot \mathbb{1}\{X_i'\beta \geq 0\}\right]$$

$$= E\left[(\mathbb{1}\{X_i'\beta_0 + \varepsilon_i \geq 0\} - \tfrac{1}{2}) \cdot \mathbb{1}\{X_i'\beta \geq 0\}\right]$$

$$m(Y_i, X_i; \beta) = (Y_i - \tfrac{1}{2}) \cdot \mathbb{1}\{X_i'\beta \geq 0\}$$

- I need to look at $Var\left(m(Y_i, X_i; \beta) - m(Y_i, X_i\right.$

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$m(Y_i, X_i; \beta) - m(Y_i, X_i; \beta_0) =$

$(Y_i - \frac{1}{2}) \cdot (1\!\!1\{X_i'\beta \geq 0\} - 1\!\!1\{X_i'\beta_0 \geq 0\})$

$E[(m(Y_i, X_i; \beta) - m(Y_i, X_i; \beta_0))^2]$

$= E[(Y_i - \frac{1}{2})^2 \cdot (1\!\!1\{X_i'\beta \geq 0\} - 1\!\!1\{X_i'\beta_0 \geq 0\})^2]$

$(Y_i - \frac{1}{2})^2 = \underbrace{Y_i^2 - Y_i} + \frac{1}{4} = \frac{1}{4}$ } since $Y_i^2 = Y_i$

$= \frac{1}{4} E[(1\!\!1\{X_i'\beta \geq 0\} - 1\!\!1\{X_i'\beta_0 \geq 0\})^2]$

$= \frac{1}{4} E[1\!\!1\{X_i'\beta \geq 0\} - 2\, 1\!\!1\{X_i'\beta \geq 0\} \cdot 1\!\!1\{X_i'\beta_0 \geq 0\}$

$+ 1\!\!1\{X_i'\beta_0 \geq 0\}]$

$= \frac{1}{4} \cdot Pr(X_i'\beta \geq 0) - \frac{1}{2} Pr(X_i'\beta \geq 0, X_i'\beta_0 \geq 0)$

$+ \frac{1}{4} Pr(X_i'\beta_0 \geq 0)$

$= \frac{1}{4} \cdot Pr(X_i'\beta \geq 0) - \frac{1}{2} Pr(X_i'\beta \geq 0 \mid X_i'\beta_0 \geq 0) \cdot Pr(X_i'\beta_0 \geq$

$+ \frac{1}{4} Pr(X_i'\beta_0 \geq 0)$

$$E[m(Y_i, X_i; \beta) - m(Y_i, X_i; \beta_0)]$$

$$= E[(Y_i - \tfrac{1}{2}) \cdot (\mathbb{1}\{X_i'\beta \geq 0\} - \mathbb{1}\{X_i'\beta_0 \geq 0\})]$$

$$= E[(\tfrac{1}{2} - F_0(X_i'\beta_0)) \cdot (\mathbb{1}\{X_i'\beta \geq 0\} - \mathbb{1}\{X_i'\beta_0 \geq 0\})]$$

need conditions under which

$$\beta \neq \beta_0 \Rightarrow E[m(Y_i, X_i; \beta) - m(Y_i, X_i; \beta_0)] > 0$$

Intuitively:

$$\mathbb{1}\{X_i'\beta \geq 0\} \text{ and } \underline{\mathbb{1}\{X_i'\beta_0 < 0\}} \Rightarrow \tfrac{1}{2} - F_0(X_i'\beta_0) > 0$$

$$\mathbb{1}\{X_i'\beta < 0\} \text{ and } \underline{\mathbb{1}\{X_i'\beta_0 \geq 0\}} \Rightarrow \tfrac{1}{2} - F_0(X_i'\beta_0) \leq 0$$

- Let $H(\beta) = P(X'\beta \geq 0)$

in an $O_p(1)$ nbd. of β_0

- Assume $\|H(\beta) - H(\beta_0)\| = O(\|\beta - \beta_0\|)$
- Let $J(\beta) = P(X'\beta \geq 0 \mid X'\beta_0 \geq 0)$
- Assume $\|J(\beta) - \underbrace{J(\beta_0)}_{1}\| = O(\|\beta - \beta_0\|)$

Let:

$$E\left[\left(\tfrac{1}{2}-F_0(X_i'\beta_0)\right)\cdot \mathbb{1}\{X_i'\beta \geq 0\}\right] \equiv T(\beta)$$

$$\Rightarrow \|T(\beta)-T(\beta_0)\| = O(\|\beta-\beta_0\|)$$

∘ In fact, suppose these objects are smooth in a nbd. of β_0, so:

$$H(\beta)-H(\beta_0) = \nabla_\beta H(\beta_0)\cdot(\beta-\beta_0) + O(\|\beta-\beta_0\|^2)$$

$$J(\beta)-J(\beta_0) = \nabla_\beta J(\beta_0)\cdot(\beta-\beta_0) + O(\|\beta-\beta_0\|^2)$$

$$T(\beta)-T(\beta_0) = \nabla_\beta T(\beta_0)\cdot(\beta-\beta_0) + O(\|\beta-\beta_0\|^2)$$

$$\Rightarrow \left(E[m(Y_i,X_i;\beta)-m(Y_i,X_i;\beta_0)]\right)^2$$

$$= 0 + 2\,E[m(Y_i,X_i;\beta)-m(Y_i,X_i;\beta_0)] \times \underbrace{\nabla_\beta E[m(Y_i,X_i;\beta)]}_{\nabla_\beta J(\beta_0)}(\beta-\beta_0)$$

zero at $\beta=\beta_0$

$\nabla[f(\beta)^2] = 2f(\beta)\,\nabla_\beta f(\beta)$ $\quad + O(\|\beta-\beta_0\|^2)$

$$E[(m(Y_i,X_i;\beta)-m(Y_i,X_i;\beta_0))^2] = 0$$

$$+\left[\tfrac{1}{4}\cdot\nabla_\beta H(\beta_0) - \tfrac{1}{2}\nabla_\beta J(\beta_0)\cdot P_1(X'\beta_0\geq 0)\right](\beta-\beta_0)$$
$$+ O(\|\beta-\beta_0\|^2)$$

$\Rightarrow$

$$\mathrm{Var}\left[\left(m(Y_i,X_i;\beta)-m(Y_i,X_i;\beta_0)\right)\right]$$

$$= \left[\frac{1}{4}\cdot\nabla_\beta H(\beta_0)-\frac{1}{2}\nabla_\beta J(\beta_0)\cdot \Pr(X'\beta_0\geq 0)\right]'(\beta-\beta_0)$$

$$+\,O(\|\beta-\beta_0\|^2)$$

$$= O(\|\beta-\beta_0\|) \quad \text{unless} \quad \frac{1}{4}\nabla_\beta H(\beta_0)-\frac{1}{2}\nabla_\beta J(\beta_0)\Pr(X'\beta_0\geq 0) = 0$$

◦ Therefore, we have:

squared root of this

$$G_N(\beta)-G_N(\beta_0) = G(\beta)-G(\beta_0)+O_p\left(\frac{\|\beta-\beta_0\|^{\frac{1}{2}}}{\sqrt{N}}\right)$$

- $N^{\frac{1}{3}}$-consistency follows from here and our theorem on M-estimators' rate of convergence.