# Appendix for "Testing functional inequalities conditional on estimated functions"

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**Abstract**

This appendix outlines the steps of the econometric proofs for our main results along with additional details and results referenced throughout the paper. Step-by-step details are included in the **Econometric Supplement** of the paper, available online at `http://www.personal.psu.edu/aza12/condit-ineq-functions-supplement.pdf`

## A1 Propositions 1A and 1B

The step-by-step derivations are included in Section A1 of the online Econometric Supplement. We summarize the main steps here.

### A1.1 Two key preliminary results for $\widehat{Q}(x,t,\widehat{\theta}) - Q_F(x,t,\theta_F^*)$

Let $\Omega_{f_g}^{d,\ell}$, $\Omega_{R_p,1}^{d,\ell}$, $\Omega_{R_p,2}$ and $\Omega_{R_p,3}^{\ell}$ be as defined in Assumption 2. For a given $(x,t) \in \mathcal{X} \times \mathcal{T}$, let

$$
\begin{aligned}
\Xi_{\ell,f_g}(x,\theta_F^*) &\equiv \sum_{d=1}^{D}\left(\frac{\partial g_d(x,\theta_F^*)}{\partial\theta_\ell}\cdot\frac{\partial f_g(g(x,\theta_F^*))}{\partial g_d} - \frac{\partial\left[\Omega_{f_g}^{d,\ell}(g(x,\theta_F^*))\cdot f_g(g(x,\theta_F^*))\right]}{\partial g_d}\right),\\
\underbrace{\Xi_{f_g}(x,\theta_F^*)}_{1\times k} &\equiv \Big(\Xi_{1,f_g}(x,\theta_F^*),\ldots,\Xi_{k,f_g}(x,\theta_F^*)\Big),\\
\Xi_{\ell,R_p}(x,t,\theta_F^*) &\equiv \sum_{d=1}^{D}\left(\frac{\partial\left[\Omega_{R_p,2}(g(x,\theta_F^*),t)f_g(g(x,\theta_F^*))\right]}{\partial g_d}\cdot\frac{\partial g_d(x,\theta_F^*)}{\partial\theta_\ell} - \frac{\partial\left[\Omega_{R_p,1}^{d,\ell}(g(x,\theta_F^*),t)f_g(g(x,\theta_F^*))\right]}{\partial g_d}\right.\\
&\qquad\left. +\Omega_{R_p,3}^{\ell}(g(x,\theta_F^*),t)\cdot f_g(g(x,\theta_F^*))\right),\\
\underbrace{\Xi_{R_p}(x,t,\theta_F^*)}_{1\times k} &\equiv \Big(\Xi_{1,R_p}(x,t,\theta_F^*),\ldots,\Xi_{k,R_p}(x,t,\theta_F^*)\Big).
\end{aligned}
\tag{A-1}
$$

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Note that, for each $p$, $d$, $\ell$, $\Omega^{d,\ell}_{R_p,1}(g,t) = \Omega_{R_p,2}(g,t) = \Omega^{\ell}_{R_p,3}(g,t) = 0 \ \forall \ g \notin \mathcal{G}$, and therefore,

$$\Xi_{R_p}(x,t,\theta^*_F) = 0 \quad \forall \ (x,t) : g(x,\theta^*_F) \notin \mathcal{G}. \tag{A-2}$$

Let $\Xi_{f_g}(x,\theta^*_F)$ and $\Xi_{R_p}(x,t,\theta^*_F)$ be as described in (A-1) and, for each $p$, define

$$\underbrace{\Xi_{Q_p}(x,t,\theta^*_F)}_{1\times k} \equiv \frac{\Xi_{R_p}(x,t,\theta^*_F) - Q_{p,F}(x,t,\theta^*_F)\cdot \Xi_{f_g}(x,\theta^*_F)}{f_g(g(x,\theta^*_F))} \tag{A-3}$$

Note from (A-2) and the definition of $Q_{p,F}$ that

$$\Xi_{Q_p}(x,t,\theta^*_F) = 0 \quad \forall \ (x,t) : g(x,\theta^*_F) \notin \mathcal{G}. \tag{A-4}$$

Let,

$$\begin{aligned}
\psi^{Q_p}_F(V_i,x,t,\theta^*_F,h_n) \equiv & \frac{1}{h_n^D}\left\{\left(\frac{S_p(Y_i,t)-\Gamma_{p,F}(x,t,\theta^*_F)}{f_g(g(x,\theta^*_F))}\right)\cdot \omega_p(g(X_i,\theta^*_F))\cdot K\left(\frac{\Delta g(X_i,x,\theta^*_F)}{h_n}\right)\right.\\
& \left. - E_F\left[\left(\frac{S_p(Y_i,t)-\Gamma_{p,F}(x,t,\theta^*_F)}{f_g(g(x,\theta^*_F))}\right)\cdot \omega_p(g(X_i,\theta^*_F))\cdot K\left(\frac{\Delta g(X_i,x,\theta^*_F)}{h_n}\right)\right]\right\} \\
& + \Xi_{Q_p}(x,t,\theta^*_F)\psi^{\theta}_F(Z_i), \quad \text{with} \\
\psi^{Q}_F(V_i,x,t,\theta^*_F,h_n) \equiv & \left(\psi^{Q_1}_F(V_i,x,t,\theta^*_F,h_n),\ldots,\psi^{Q_P}_F(V_i,x,t,\theta^*_F,h_n)\right)'.
\end{aligned} \tag{A-5}$$

Proposition S1 in the online Econometric Supplement shows that, under Assumptions 1-4,

$$\begin{aligned}
&\widehat{Q}(x,t,\widehat{\theta}) = Q_F(x,t,\theta^*_F) + \frac{1}{n}\sum_{i=1}^{n}\psi^{Q}_F(V_i,x,t,\theta^*_F,h_n) + \zeta^{Q}_n(x,t), \quad \text{where} \\
&\left.\begin{aligned}
&\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left\|\zeta^{Q}_n(x,t)\right\| = o_p\left(\tfrac{1}{n^{1/2+\epsilon}}\right), \\
&\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left\|\widehat{Q}(x,t,\widehat{\theta}) - Q_F(x,t,\theta^*_F)\right\| = o_p\left(\tfrac{1}{n^{1/4+\epsilon/2}}\right)
\end{aligned}\right\}\ \text{uniformly over } \mathcal{F},
\end{aligned} \tag{A-6}$$

where $\epsilon > 0$ is the constant described in Assumption 4. In addition, we also show that

$$\sup_{F\in\mathcal{F}} P_F\left(\sup_{(x,t)\in\mathcal{X}}\left|\widehat{Q}(x,t,\widehat{\theta}) - Q_F(x,t,\theta^*_F)\right| \geq b_n\right) \longrightarrow 0. \tag{A-7}$$

From here, combining Assumptions 1-5, Proposition S1 and Section S3.5 in the online Econometric Supplement show that,

$$\sup_{F\in\mathcal{F}} P_F\left(\sup_{(x,t)\in\mathcal{X}}\left|\mathcal{B}(\widehat{Q}(x,t,\widehat{\theta}))-\mathcal{B}(Q_F(x,t,\theta_F^*))\right|\geq b_n\right)\longrightarrow 0, \tag{A-8}$$

and,

$$\begin{aligned}
&\mathcal{B}(\widehat{Q}(x,t,\widehat{\theta}))=\mathcal{B}(Q_F(x,t,\theta_F^*))+\frac{1}{n}\sum_{i=1}^{n}\psi_F^{\mathcal{B}}(V_i,x,t,\theta_F^*,h_n)+\zeta_n^{\mathcal{B}}(x,t), \quad \text{where,}\\
&\psi_F^{\mathcal{B}}(V_i,x,t,\theta_F^*,h_n)\equiv\nabla_Q\mathcal{B}(Q_F(x,t,\theta_F^*))\psi_F^{Q}(V_i,x,t,\theta_F^*,h_n)=\sum_{p=1}^{P}\frac{\partial\mathcal{B}\left(Q_F(x,t,\theta_F^*)\right)}{\partial Q_p}\psi_F^{Q_p}(V_i,x,t,\theta_F^*,h_n),\\
&\left.\begin{aligned}&\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left\|\zeta_n^{\mathcal{B}}(x,t)\right\|=o_p\left(\tfrac{1}{n^{1/2+\epsilon}}\right), \text{ and}\\
&\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\mathcal{B}(\widehat{Q}(x,t,\widehat{\theta}))-\mathcal{B}(Q_F(x,t,\theta_F^*))\right|=o_p\left(\tfrac{1}{n^{1/4+\epsilon/2}}\right)\end{aligned}\right\}\text{ uniformly over } \mathcal{F},
\end{aligned} \tag{A-9}$$

where $\epsilon>0$ is the constant described in Assumption 4 and $\psi_F^{Q}$, $\psi_F^{Q_p}$ are as described in (A-5). The results in (A-8) and (A-9) are key steps leading to Propiositions 1A and 1B.

## A1.2 Steps of the proof of Proposition 1A

Proposition S1 in the Econometric Supplement and the results in (A-8)-(A-9) are the first building blocks of the proof of Proposition 1A. We outline the main steps of the proof here, with all the details included in Section S3 (in particular, Section S3.6) of the Econometric Supplement. Let

$$\widetilde{T}_{1,F}\equiv\frac{1}{n}\sum_{i=1}^{n}\mathcal{B}\left(\widehat{Q}(X_i,t_i,\widehat{\theta})\right)\mathbb{1}\left\{\mathcal{B}\left(Q_F(X_i,t_i,\theta_F^*)\right)\geq 0\right\}\phi(X_i,t_i)$$

Note that $\widetilde{T}_{1,F}$ replaces the indicator function $\mathbb{1}\left\{\mathcal{B}\left(\widehat{Q}(X_i,t_i,\widehat{\theta})\right)\geq -b_n\right\}$ with $\mathbb{1}\left\{\mathcal{B}\left(Q_F(X_i,t_i,\theta_F^*)\right)\geq 0\right\}$. Our first step is to analyze $\widehat{T}_1-\widetilde{T}_{1,F}\equiv\xi_{T_1,n}^{a}$. We have

$$\xi_{T_1,n}^{a}\equiv\widehat{T}_1-\widetilde{T}_{1,F}=\sum_{i=1}^{n}\mathcal{B}\left(\widehat{Q}(X_i,t_i,\widehat{\theta})\right)\phi(X_i,t_i)\Big[\mathbb{1}\left\{\mathcal{B}\left(\widehat{Q}(X_i,t_i,\widehat{\theta})\right)\geq -b_n\right\}-\mathbb{1}\left\{\mathcal{B}\left(Q_F(X_i,t_i,\theta_F^*)\right)\geq 0\right\}\Big]$$

Therefore,

$$\left|\xi_{T_1,n}^{a}\right|\leq\sum_{i=1}^{n}\left|\mathcal{B}\left(\widehat{Q}(X_i,t_i,\widehat{\theta})\right)\right|\phi(X_i,t_i)\left|\mathbb{1}\left\{\mathcal{B}\left(\widehat{Q}(X_i,t_i,\widehat{\theta})\right)\geq -b_n\right\}-\mathbb{1}\left\{\mathcal{B}\left(Q_F(X_i,t_i,\theta_F^*)\right)\geq 0\right\}\right|$$

We have

$$
\begin{aligned}
&\left|\mathbb{1}\left\{\mathcal{B}\left(\widehat{Q}(X_i,t_i,\widehat{\theta})\right) \geq -b_n\right\} - \mathbb{1}\left\{\mathcal{B}\left(Q_F(X_i,t_i,\theta_F^*)\right) \geq 0\right\}\right| \\
=&\mathbb{1}\left\{\mathcal{B}\left(\widehat{Q}(X_i,t_i,\widehat{\theta})\right) \geq -b_n\ ,\ -2b_n \leq \mathcal{B}\left(Q_F(X_i,t_i,\theta_F^*)\right) < 0\right\} + \mathbb{1}\left\{\mathcal{B}\left(\widehat{Q}(X_i,t_i,\widehat{\theta})\right) \geq -b_n\ ,\ \mathcal{B}\left(Q_F(X_i,t_i,\theta_F^*)\right) < -2b_n\right\} \\
+&\mathbb{1}\left\{\mathcal{B}\left(\widehat{Q}(X_i,t_i,\widehat{\theta})\right) < -b_n\ ,\ \mathcal{B}\left(Q_F(X_i,t_i,\theta_F^*)\right) \geq 0\right\} \\
\leq&\mathbb{1}\left\{-2b_n \leq \mathcal{B}\left(Q_F(X_i,t_i,\theta_F^*)\right) < 0\right\} + \mathbb{1}\left\{\left|\mathcal{B}\left(\widehat{Q}(X_i,t_i,\widehat{\theta})\right) - \mathcal{B}\left(Q_F(X_i,t_i,\theta_F^*)\right)\right| \geq b_n\right\}
\end{aligned}
$$

From here, we have

$$
\begin{aligned}
&\left|\xi^a_{T_1,n}\right| \\
\leq&\frac{1}{n}\sum_{i=1}^{n}\left|\mathcal{B}\left(\widehat{Q}(X_i,t_i,\widehat{\theta})\right)\right|\phi(X_i,t_i)\mathbb{1}\left\{-2b_n \leq \mathcal{B}\left(Q_F(X_i,t_i,\theta_F^*)\right) < 0\right\} \\
+&\frac{1}{n}\sum_{i=1}^{n}\left|\mathcal{B}\left(\widehat{Q}(X_i,t_i,\widehat{\theta})\right)\right|\phi(X_i,t_i)\mathbb{1}\left\{\left|\mathcal{B}\left(\widehat{Q}(X_i,t_i,\widehat{\theta})\right) - \mathcal{B}\left(Q_F(X_i,t_i,\theta_F^*)\right)\right| \geq b_n\right\} \\
\leq&\frac{1}{n}\sum_{i=1}^{n}\left(\left|\mathcal{B}\left(Q_F(X_i,t_i,\theta_F^*)\right)\right| + \left|\mathcal{B}\left(\widehat{Q}(X_i,t_i,\widehat{\theta})\right) - \mathcal{B}\left(Q_F(X_i,t_i,\theta_F^*)\right)\right|\right)\phi(X_i,t_i)\mathbb{1}\left\{-2b_n \leq \mathcal{B}\left(Q_F(X_i,t_i,\theta_F^*)\right) < 0\right\} \\
+&\frac{1}{n}\sum_{i=1}^{n}\left|\mathcal{B}\left(\widehat{Q}(X_i,t_i,\widehat{\theta})\right)\right|\phi(X_i,t_i)\mathbb{1}\left\{\left|\mathcal{B}\left(\widehat{Q}(X_i,t_i,\widehat{\theta})\right) - \mathcal{B}\left(Q_F(X_i,t_i,\theta_F^*)\right)\right| \geq b_n\right\} \\
\leq&\left(2b_n + \sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\mathcal{B}\left(\widehat{Q}(x,t,\widehat{\theta})\right) - \mathcal{B}\left(Q_F(x,t,\theta_F^*)\right)\right|\right) \times \frac{1}{n}\sum_{i=1}^{n}\phi(X_i,t_i)\mathbb{1}\left\{-2b_n \leq \mathcal{B}\left(Q_F(X_i,t_i,\theta_F^*)\right) < 0\right\} \\
+&\frac{1}{n}\sum_{i=1}^{n}\left|\mathcal{B}\left(\widehat{Q}(X_i,t_i,\widehat{\theta})\right)\right|\phi(X_i,t_i)\mathbb{1}\left\{\left|\mathcal{B}\left(\widehat{Q}(X_i,t_i,\widehat{\theta})\right) - \mathcal{B}\left(Q_F(X_i,t_i,\theta_F^*)\right)\right| \geq b_n\right\}
\end{aligned}
$$

From (A-8), we have $\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\mathcal{B}\left(\widehat{Q}(x,t,\widehat{\theta})\right) - \mathcal{B}\left(Q(x,t,\theta_F^*)\right)\right| = o_p\left(\frac{1}{n^{1/4+\epsilon/2}}\right)$ uniformly over $\mathcal{F}$, where $\epsilon > 0$ is the constant described in Assumption 4. Therefore, uniformly over $\mathcal{F}$ we have

$$
\begin{aligned}
\left|\xi^a_{T_1,n}\right| \leq& \left(2b_n + o_p\left(\frac{1}{n^{1/4+\epsilon/2}}\right)\right) \times \frac{1}{n}\sum_{i=1}^{n}\phi(X_i,t_i)\mathbb{1}\left\{-2b_n \leq \mathcal{B}\left(Q(X_i,t_i,\theta_F^*)\right) < 0\right\} \\
&+ \frac{1}{n}\sum_{i=1}^{n}\left|\mathcal{B}\left(\widehat{Q}(X_i,t_i,\widehat{\theta})\right)\right|\phi(X_i,t_i)\mathbb{1}\left\{\left|\mathcal{B}\left(\widehat{Q}(X_i,t_i,\widehat{\theta})\right) - \mathcal{B}\left(Q(X_i,t_i,\theta_F^*)\right)\right| \geq b_n\right\}.
\end{aligned}
$$

For a given $b > 0$, let

$$
m^a_{T_1,n}(b) \equiv \frac{1}{n}\sum_{i=1}^{n}\left(\phi(X_i,t_i)\mathbb{1}\left\{-b \leq \mathcal{B}\left(Q_F(X_i,t_i,\theta_F^*)\right) < 0\right\} - E_F\left[\phi(X_i,t_i)\mathbb{1}\left\{-b \leq \mathcal{B}\left(Q_F(X_i,t_i,\theta_F^*)\right) < 0\right\}\right]\right)
$$

From Assumption 6, there exist constants $(\overline{A}, \overline{V})$ such that, for each $F \in \mathcal{F}$, the following class of indicator functions is Euclidean $(\overline{A}, \overline{V})$ for the constant envelope 1,

$$\Big\{m : \mathcal{X} \times \mathcal{T} \longrightarrow \mathbb{R} : m(x,t) = \mathbb{1}\Big\{-b \leq \mathcal{B}\big(Q_F(x,t,\theta_F^*)\big) < 0\Big\} \text{ for some } 0 < b \leq b_0\Big\}.$$

From here, as we show in the Econometric Supplement, our assumptions and Sherman (1994, Corollary 4A) yield,

$$\sup_{0<b<b_0} \left| m_{T_1,n}^a(b) \right| = O_p\left(\frac{1}{n^{1/2}}\right) \quad \text{uniformly over } \mathcal{F}.$$

For $n$ large enough we have $0 < 2b_n \leq b_0$. Therefore, for $n$ large enough,

$$\left| m_{T_1,n}^a(2b_n) \right| \leq \sup_{0<b<b_0} \left| m_{T_1,n}^a(b) \right| = O_p\left(\frac{1}{n^{1/2}}\right) \quad \text{uniformly over } \mathcal{F}. \tag{A-10}$$

We have

$$\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{n} \phi(X_i,t_i)\mathbb{1}\Big\{-2b_n \leq \mathcal{B}\big(Q_F(X_i,t_i,\theta_F^*)\big) < 0\Big\} \\
=& m_{T_1,n}^a(2b_n) + E_F\Big[\mathbb{1}\Big\{-2b_n \leq \mathcal{B}\big(Q_F(X,t,\theta_F^*)\big) < 0\Big\} \cdot \mathbb{1}\Big\{(X,t) \in \mathcal{X} \times \mathcal{T}\Big\}\Big].
\end{aligned}$$

From Assumption 7A, there exist finite constants $\underline{b}_1 > 0$ and $\overline{C}_{\mathcal{B},1} > 0$ such that, for all $0 < b \leq \underline{b}_1$,

$$E_F\Big[\mathbb{1}\Big\{-b \leq \mathcal{B}\big(Q_F(X,t,\theta_F^*)\big) < 0\Big\} \cdot \mathbb{1}\Big\{(X,t) \in \mathcal{X} \times \mathcal{T}\Big\}\Big] \leq \overline{C}_{\mathcal{B},1} \cdot b \quad \forall\, F \in \mathcal{F},$$

For $n$ large enough, we have $0 < 2b_n \leq \underline{b}_1 \wedge b_0$, and from Assumption 4, we have $n^{1/2} \cdot b_n \longrightarrow \infty$. This, combined with equation (A-10) and Assumption 7A yields,

$$\begin{aligned}
\frac{1}{n}\sum_{i=1}^{n} \phi(X_i,t_i)\mathbb{1}\Big\{-2b_n \leq \mathcal{B}\big(Q_F(X_i,t_i,\theta_F^*)\big) < 0\Big\} &\leq O_p\left(\frac{1}{n^{1/2}}\right) + O(b_n) \\
&= b_n \cdot \left(O_p\left(\frac{1}{b_n \cdot n^{1/2}}\right) + O(1)\right) \\
&= b_n \cdot \big(o_p(1) + O(1)\big) \\
&= O_p(b_n) \quad \text{uniformly over } \mathcal{F}.
\end{aligned}$$

Next, note from Assumption 5 and equation (A-9), we have $\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\mathcal{B}\left(\widehat{Q}(x,t,\widehat{\theta})\right)\right| = O_p(1)$ uniformly over $\mathcal{F}$. Therefore,

$$\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{n}\left|\mathcal{B}\left(\widehat{Q}(X_i,t_i,\widehat{\theta})\right)\right|\phi(X_i,t_i)\mathbb{1}\left\{\left|\mathcal{B}\left(\widehat{Q}(X_i,t_i,\widehat{\theta})\right)-\mathcal{B}\left(Q(X_i,t_i,\theta_F^*)\right)\right|\geq b_n\right\}\\
\leq&\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\mathcal{B}\left(\widehat{Q}(x,t,\widehat{\theta})\right)\right|\times\frac{1}{n}\sum_{i=1}^{n}\phi(X_i,t_i)\mathbb{1}\left\{\left|\mathcal{B}\left(\widehat{Q}(X_i,t_i,\widehat{\theta})\right)-\mathcal{B}\left(Q(X_i,t_i,\theta_F^*)\right)\right|\geq b_n\right\}\\
=&O_p(1)\times\frac{1}{n}\sum_{i=1}^{n}\phi(X_i,t_i)\mathbb{1}\left\{\left|\mathcal{B}\left(\widehat{Q}(X_i,t_i,\widehat{\theta})\right)-\mathcal{B}\left(Q(X_i,t_i,\theta_F^*)\right)\right|\geq b_n\right\},\quad\text{uniformly over }\mathcal{F}.
\end{aligned}$$

Recall that, since $\phi(\cdot)\geq 0$, we have

$$\begin{aligned}
&\left|\frac{1}{n}\sum_{i=1}^{n}\phi(X_i,t_i)\mathbb{1}\left\{\left|\mathcal{B}\left(\widehat{Q}(X_i,t_i,\widehat{\theta})\right)-\mathcal{B}\left(Q(X_i,t_i,\theta_F^*)\right)\right|\geq b_n\right\}\right|\\
=&\frac{1}{n}\sum_{i=1}^{n}\phi(X_i,t_i)\mathbb{1}\left\{\left|\mathcal{B}\left(\widehat{Q}(X_i,t_i,\widehat{\theta})\right)-\mathcal{B}\left(Q(X_i,t_i,\theta_F^*)\right)\right|\geq b_n\right\}.
\end{aligned}$$

Next, note that

$$\begin{aligned}
&\sup_{F\in\mathcal{F}}P_F\left(\frac{1}{n}\sum_{i=1}^{n}\phi(X_i,t_i)\mathbb{1}\left\{\left|\mathcal{B}\left(\widehat{Q}(X_i,t_i,\widehat{\theta})\right)-\mathcal{B}\left(Q(X_i,t_i,\theta_F^*)\right)\right|\geq b_n\right\}\neq 0\right)\\
\leq&\sup_{F\in\mathcal{F}}P_F\left(\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\mathcal{B}(\widehat{Q}(x,t,\widehat{\theta}))-\mathcal{B}(Q_F(x,t,\theta_F^*))\right|\geq b_n\right)\longrightarrow 0
\end{aligned}$$

where the last result follows from (A-8). In particular, for any $\delta>0$ and $\Delta>0$,

$$\sup_{F\in\mathcal{F}}P_F\left(\frac{1}{n}\sum_{i=1}^{n}\phi(X_i,t_i)\mathbb{1}\left\{\left|\mathcal{B}\left(\widehat{Q}(X_i,t_i,\widehat{\theta})\right)-\mathcal{B}\left(Q(X_i,t_i,\theta_F^*)\right)\right|\geq b_n\right\}\geq\frac{\delta}{n^{1/2+\Delta}}\right)\longrightarrow 0.$$

That is,

$$\frac{1}{n}\sum_{i=1}^{n}\phi(X_i,t_i)\mathbb{1}\left\{\left|\mathcal{B}\left(\widehat{Q}(X_i,t_i,\widehat{\theta})\right)-\mathcal{B}\left(Q(X_i,t_i,\theta_F^*)\right)\right|\geq b_n\right\}=o_p\left(\frac{1}{n^{1/2+\Delta}}\right)\quad\forall\ \Delta>0,\ \text{uniformly over }\mathcal{F}.$$

Therefore,

$$\frac{1}{n}\sum_{i=1}^{n}\left|\mathcal{B}\left(\widehat{Q}(X_i,t_i,\widehat{\theta})\right)\right|\phi(X_i,t_i)\mathbb{1}\left\{\left|\mathcal{B}\left(\widehat{Q}(X_i,t_i,\widehat{\theta})\right)-\mathcal{B}\left(Q(X_i,t_i,\theta_F^*)\right)\right|\geq b_n\right\}$$
$$= o_p\left(\frac{1}{n^{1/2+\Delta}}\right) \quad \forall\, \Delta>0, \text{ uniformly over } \mathcal{F}.$$

Combining the previous results we have that for any $\Delta > 0$,

$$\begin{aligned}\left|\xi^a_{T_1,n}\right| &\leq \left(2b_n + o_p\left(\frac{1}{n^{1/4+\epsilon/2}}\right)\right)\times O_p\left(b_n\right) + o_p\left(\frac{1}{n^{1/2+\Delta}}\right)\\ &= O_p\left(b_n^2\right) + o_p\left(\frac{b_n}{n^{1/4+\epsilon/2}}\right) + o_p\left(\frac{1}{n^{1/2+\Delta}}\right) \quad \text{uniformly over } \mathcal{F}.\end{aligned}$$

Take any $\Delta > 0$ and note that $\left(\frac{b_n}{n^{1/4+\epsilon/2}}\right)\cdot n^{1/2+\Delta} = \left(n^{1/2+2\Delta-\epsilon}\cdot b_n^2\right)^{1/2}$. In Assumption 4 we stated that there exists $\delta_0 > 0$ such that $n^{1/2+\delta_0}\cdot b_n^2 \longrightarrow 0$. Therefore,

$$\frac{b_n}{n^{1/4+\epsilon/2}} = o\left(\frac{1}{n^{1/2+\Delta}}\right) \quad \forall\; 0<\Delta\leq\frac{\delta_0}{2}.$$

From here, we obtain

$$\left|\xi^a_{T_1,n}\right| = o_p\left(\frac{1}{n^{1/2+\delta_0/2}}\right) \quad \text{uniformly over } \mathcal{F}.$$

Therefore, using the linear representation result in (A-9),

$$\begin{aligned}\widehat{T_1} &= \widetilde{T}_{1,F} + \xi^a_{T_1,n}\\ &= \frac{1}{n}\sum_{i=1}^{n}\mathcal{B}\left(\widehat{Q}(X_i,t_i,\widehat{\theta})\right)\mathbb{1}\left\{\mathcal{B}\left(Q_F(X_i,t_i,\theta_F^*)\right)\geq 0\right\}\phi(X_i,t_i) + \xi^a_{T_1,n}\\ &= \frac{1}{n}\sum_{i=1}^{n}\left(\mathcal{B}(Q_F(X_i,t_i,\theta_F^*)) + \frac{1}{n}\sum_{j=1}^{n}\psi_F^{\mathcal{B}}(V_j,X_i,t_i,\theta_F^*,h_n) + \zeta_n^{\mathcal{B}}(X_i,t_i)\right)\mathbb{1}\left\{\mathcal{B}\left(Q_F(X_i,t_i,\theta_F^*)\right)\geq 0\right\}\phi(X_i,t_i) + \xi^a_{T_1,n}\\ &= \frac{1}{n}\sum_{i=1}^{n}\left(\mathcal{B}\left(Q_F(X_i,t_i,\theta_F^*)\right)\right)_+\phi(X_i,t_i) + \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\psi_F^{\mathcal{B}}(V_j,X_i,t_i,\theta_F^*,h_n)\mathbb{1}\left\{\mathcal{B}\left(Q_F(X_i,t_i,\theta_F^*)\right)\geq 0\right\}\phi(X_i,t_i)\\ &+ \frac{1}{n}\sum_{i=1}^{n}\zeta_n^{\mathcal{B}}(X_i,t_i)\mathbb{1}\left\{\mathcal{B}\left(Q_F(X_i,t_i,\theta_F^*)\right)\geq 0\right\}\phi(X_i,t_i) + \xi^a_{T_1,n},\end{aligned}$$

where $\psi_F^{\mathcal{B}}$ and $\zeta_n^{\mathcal{B}}$ are as described in (A-9). Adding and subtracting $T_{1,F}$ on the right hand side,

$$\begin{aligned}\widehat{T_1} =& T_{1,F} + \frac{1}{n}\sum_{i=1}^{n}\Big(\left(\mathcal{B}\left(Q_F(X_i,t_i,\theta_F^*)\right)\right)_+ \phi(X_i,t_i) - T_{1,F}\Big)\\ &+ \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\psi_F^{\mathcal{B}}(V_j,X_i,t_i,\theta_F^*,h_n)\mathbb{1}\left\{\mathcal{B}\left(Q_F(X_i,t_i,\theta_F^*)\right)\geq 0\right\}\phi(X_i,t_i)\\ &+ \underbrace{\frac{1}{n}\sum_{i=1}^{n}\zeta_n^{\mathcal{B}}(X_i,t_i)\mathbb{1}\left\{\mathcal{B}\left(Q_F(X_i,t_i,\theta_F^*)\right)\geq 0\right\}\phi(X_i,t_i)}_{\equiv \xi^b_{T_1,n}} + \xi^a_{T_1,n}.\end{aligned} \tag{A-11}$$

From the results summarized in (A-9), we obtain

$$\begin{aligned}\left|\xi^b_{T_1,n}\right| \equiv \left|\frac{1}{n}\sum_{i=1}^{n}\zeta_n^{\mathcal{B}}(X_i,t_i)\mathbb{1}\left\{\mathcal{B}\left(Q_F(X_i,t_i,\theta_F^*)\right)\geq 0\right\}\phi(X_i,t_i)\right| &\leq \overline{\phi}\cdot \sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\zeta_n^{\mathcal{B}}(x,t)\right|\\ &= o_p\left(\frac{1}{n^{1/2+\epsilon}}\right) \quad \text{uniformly over } \mathcal{F}\end{aligned}$$

The proof proceeds from here by computing the Hoeffding decomposition (see Serfling (1980, pages 177-178) or Sherman (1994, equations (6)-(7))) of the U-process on the right-hand side of (A-11). From the smoothness conditions in Assumption 7A,

$$\begin{aligned}&\frac{1}{n}\sum_{i=1}^{n}E_F\left[\psi_F^{\mathcal{B}}(V_i,X_j,t_j,\theta_F^*,h_n)\mathbb{1}\left\{\mathcal{B}\left(Q_F(X_j,t_j,\theta_F^*)\right)\geq 0\right\}\phi(X_j,t_j)\;\Big|\; V_i\right]\\ &+\frac{1}{n}\sum_{i=1}^{n}E_F\left[\psi_F^{\mathcal{B}}(V_j,X_i,t_i,\theta_F^*,h_n)\mathbb{1}\left\{\mathcal{B}\left(Q_F(X_i,t_i,\theta_F^*)\right)\geq 0\right\}\phi(X_i,t_i)\;\Big|\; V_i\right]\\ &=\frac{1}{n}\sum_{i=1}^{n}\left[\sum_{p=1}^{P}\Omega^p_{T_1}(Y_i,g(X_i,\theta_F^*))\cdot\omega_p(g(X_i,\theta_F^*)) + \Xi_{T_1,F}\psi_F^{\theta}(Z_i)\right] + \overline{B}_{T_1,n},\\ &\quad\text{where}\quad \sup_{F\in\mathcal{F}}\left|\overline{B}_{T_1,n}\right| = O\left(h_n^M\right) = o\left(n^{1/2+\epsilon}\right),\end{aligned}$$

and where $\epsilon > 0$ is the positive constant described in Assumption 4. The above expression is the leading term in the Hoeffding decomposition of the U-process. The result in Proposition 1A follows from here using the maximal inequality results in Sherman (1994, Corollary 4A) to analyze the degenerate U-process that appears as the second term of the decomposition. The conditions in Sherman (1994, Corollary 4A) are satisfied by the restrictions in Assumptions 1-6 and 7A. Let $\Xi_{Q_p}(x,t,\theta_F^*)$ be as defined in equation (A-3) and define,

$$\Xi^p_{T_1,F} \equiv E_F\left[\frac{\partial\mathcal{B}\Big(Q_F(X,t,\theta_F^*)\Big)}{\partial Q_p}\mathbb{1}\left\{\mathcal{B}\left(Q_F(X,t,\theta_F^*)\right)\geq 0\right\}\phi(X,t)\Xi_{Q_p}(X,t,\theta_F^*)\right], \quad \Xi_{T_1,F}\equiv\sum_{p=1}^{P}\Xi^p_{T_1,F}, \tag{A-12}$$

where the above expectation is taken with respect to $(X,t)$. Let

$$\psi_F^{T_1}(V_i)\equiv\Big(\left(\mathcal{B}\left(Q_F(X_i,t_i,\theta_F^*)\right)\right)_+\phi(X_i,t_i)-T_{1,F}\Big)+\sum_{p=1}^{P}\Omega^p_{T_1}(Y_i,g(X_i,\theta_F^*))\cdot\omega_p(g(X_i,\theta_F^*))+\Xi_{T_1,F}\psi_F^{\theta}(Z_i). \tag{A-13}$$

Let $\Delta\equiv\epsilon\wedge(\delta_0/2)$, where $\epsilon$ and $\delta_0$ are as described in Assumption 4. If Assumptions 1-6 and 7A hold,

$$\widehat{T_1}=T_{1,F}+\frac{1}{n}\sum_{i=1}^{n}\psi_F^{T_1}(V_i)+\varepsilon_n^{T_1}, \quad \text{where} \quad |\varepsilon_n^{T_1}|=o_p\left(\frac{1}{n^{1/2+\Delta}}\right) \quad \text{uniformly over } \mathcal{F}. \tag{A-14}$$

This is the statement in Proposition 1A. The details of every step leading to this result can be found in Section S3 of the online Econometric Supplement. ■

### A1.2.1 Properties of the influence function $\psi_F^{T_1}(V)$

The following discussion is a summary from Section S3.6.2 of the online Econometric Supplement. The *influence function* $\psi_F^{T_1}(V)$ has two key features,

$$\begin{aligned}&(i) \quad E_F\left[\psi_F^{T_1}(V)\right]=0 \quad \forall\, F\in\mathcal{F},\\ &(ii) \quad P_F\Big(\mathcal{B}\left(\Gamma_F(X,t,\theta_F^*)\right)<0 \,\Big|\, (X,t)\in\mathcal{X}_F^*\times\mathcal{T}\Big)=1 \implies P_F\Big(\psi_F^{T_1}(V)=0\Big)=1.\end{aligned} \tag{A-15}$$

To see part (i) of (A-15), note first that, by construction, $E_F\left[\Big(\mathcal{B}\Big(Q_F(X_i,t_i,\theta_F^*)\Big)\Big)_+\phi(X_i,t_i)-T_{1,F}\right]=0$ (since $T_{1,F}\equiv E_F\left[\Big(\mathcal{B}\Big(Q_F(X,t,\theta_F^*)\Big)\Big)_+\phi(X,t)\right]$, as we defined in (11A)). Next, since $E_F\left[\psi_F^{\theta}(Z)\right]=0$, we have $E_F\left[\Xi_{T_1,F}\psi_F^{\theta}(Z)\right]=\Xi_{T_1,F}E_F\left[\psi_F^{\theta}(Z)\right]=0$. Finally, as we show in equation (S3.6.21) of the online Econometric Supplement, we have $E_F\left[\Omega^p_{T_1}(Y,g(X,\theta_F^*))\cdot\omega_p(g(X,\theta_F^*)\right]=0$, and these results hold for all $F\in\mathcal{F}$, therefore establishing part (i) of (A-15). Next, let us show part (ii). First, recall that we defined our target testing range $\mathcal{X}_F^*$ for $X$ as $\mathcal{X}_F^*=\left\{x\in\mathcal{S}_X : x\in\mathcal{X} \text{ and } g(x,\theta_F^*)\in\mathcal{G}.\right\}$. Next, recall that $\omega_p(\cdot)\geq 0$ with $\omega_p(g)>0$ if and only if $g\in\mathcal{G}$, and $\phi(\cdot)\geq 0$ with $\phi(x,t)>0$ if and only if

$(x,t) \in \mathcal{X} \times \mathcal{T}$. Next, recall from (7) that,

$$\mathcal{B}\left(Q_F(x,t,\theta_F^*)\right) = \mathcal{B}\left(\Gamma_F(x,t,\theta_F^*)\right) \cdot \mathcal{H}\left(\omega(g(x,\theta_F^*))\right),$$
$$\text{where } \begin{cases} \mathcal{H}(\cdot) \geq 0, \\ \mathcal{H}(\omega(g(x,\theta_F^*)) > 0 \iff g(x,\theta_F^*) \in \mathcal{G}. \end{cases}$$

From here, it follows that $\left(\mathcal{B}\left(Q_F(x,t,\theta_F^*)\right)\right)_+ \phi(x,t) = \left(\mathcal{B}\left(\Gamma_F(x,t,\theta_F^*)\right)\right)_+ \cdot \mathcal{H}\left(\omega(g(x,\theta_F^*))\right)\phi(x,t)$, and $\mathcal{H}\left(\omega(g(x,\theta_F^*))\right)\phi(x,t) \neq 0$ only if $(x,t) \in \mathcal{X}_F^* \times \mathcal{T}$. Thus, $P_F\left(\mathcal{B}\left(\Gamma_F(X,t,\theta_F^*)\right) < 0 \middle| (X,t) \in \mathcal{X}_F^* \times \mathcal{T}\right) = 1$ implies $P_F\left(\left(\mathcal{B}\left(Q_F(X,t,\theta_F^*)\right)\right)_+ \phi(X,t) = 0\right) = P_F\left(\left(\mathcal{B}\left(\Gamma_F(X,t,\theta_F^*)\right)\right)_+ \cdot \mathcal{H}\left(\omega(g(X,\theta_F^*))\right)\phi(X,t) = 0\right) = 1$, and $T_{1,F} = 0$. Also, if $P_F\left(\mathcal{B}\left(\Gamma_F(X,t,\theta_F^*)\right) < 0 \middle| (X,t) \in \mathcal{X}_F^* \times \mathcal{T}\right) = 1$, then

$$\begin{aligned} & P_F\left(\phi(X,t)\mathbb{1}\left\{\mathcal{B}\left(Q_F(X,t,\theta_F^*)\right) \geq 0\right\} = 0 \;\middle|\; g(X,\theta_F^*) \in \mathcal{G}\right) \\ =& P_F\left(\phi(X,t)\mathbb{1}\left\{\mathcal{B}\left(\Gamma_F(X,t,\theta_F^*)\right) \cdot \mathcal{H}\left(\omega(g(X,\theta_F^*))\right) \geq 0\right\} = 0 \;\middle|\; g(X,\theta_F^*) \in \mathcal{G}\right) = 1. \end{aligned}$$

Thus, for any $(y,g)$, the above result implies

$$\begin{aligned} & \Omega_{T_1}^p(y,g) \cdot \omega_p(g) \\ =& E_F\left[\left(S_p(y,t) - \Gamma_{p,F}(X,t,\theta_F^*)\right)\frac{\partial \mathcal{B}\left(Q_F(X,t,\theta_F^*)\right)}{\partial Q_p} \underbrace{\phi(X,t)\mathbb{1}\left\{\mathcal{B}\left(Q_F(X,t,\theta_F^*)\right) \geq 0\right\}}_{\substack{=0\ F-\text{a.s if } g(X,\theta_F^*) = g \in \mathcal{G} \\ \text{(i.e, if } \omega_p(g) \neq 0)}} \;\middle|\; g(X,\theta_F^*) = g\right] \cdot \underbrace{\omega_p(g)}_{\substack{=0 \text{ if} \\ g \notin \mathcal{G}}} \\ =& 0. \end{aligned}$$

Thus, $P_F\left(\mathcal{B}\left(\Gamma_F(X,t,\theta_F^*)\right) < 0 \middle| (X,t) \in \mathcal{X}_F^* \times \mathcal{T}\right) = 1 \implies P_F\left(\Omega_{T_1}^p(Y, g(X,\theta_F^*)) \cdot \omega_p(g(X,\theta_F^*)) = 0\right) = 1$. Finally, recall from (A-4) that $\Xi_{Q_p}(x,t,\theta_F^*) = 0 \quad \forall\, x : g(x,\theta_F^*) \notin \mathcal{G}$. Therefore, $\phi(x,t)\Xi_{Q_p}(x,t,\theta_F^*) = 0 \quad \forall\, (x,t) \notin \mathcal{X}_F^* \times \mathcal{T}$, and from our definition of $\Xi_{T_1,F}^p$ and $\Xi_{T_1,F}$ in (A-12), it follows by inspection that, if $P_F\left(\mathcal{B}\left(\Gamma_F(X,t,\theta_F^*)\right)<0\middle|(X,t)\in\mathcal{X}_F^*\times\mathcal{T}\right)=1$, then

$$\begin{aligned}
\Xi^p_{T_1,F} &\equiv E_F\left[\frac{\partial\mathcal{B}\left(Q_F(X,t,\theta_F^*)\right)}{\partial Q_p}\mathbb{1}\{\mathcal{B}(Q_F(X,t,\theta_F^*))\geq 0\}\phi(X,t)\Xi_{Q_p}(X,t,\theta_F^*)\right]\\
&= E_F\left[\frac{\partial\mathcal{B}\left(Q_F(X,t,\theta_F^*)\right)}{\partial Q_p}\underbrace{\mathbb{1}\{\mathcal{B}(\Gamma_F(X,t,\theta_F^*))\cdot\mathcal{H}(\omega(g(X,\theta_F^*)))\geq 0\}}_{=0\text{ if }(X,t)\in\mathcal{X}_F^*\times\mathcal{T}}\underbrace{\phi(X,t)\Xi_{Q_p}(X,t,\theta_F^*)}_{\substack{=0\\ \text{if }(X,t)\notin\mathcal{X}_F^*\times\mathcal{F}}}\right]\\
&= 0 \quad \forall\ p=1,\ldots,P.\\
\implies \quad \Xi_{T_1,F} &\equiv \sum_{p=1}^{P}\Xi^p_{T_1,F}=0.
\end{aligned}$$

Therefore, $\Xi_{T_1,F}\psi_F^\theta(Z)=0$. Combined, these results yield

$$P_F\left(\mathcal{B}(\Gamma_F(X,t,\theta_F^*))<0\ \middle|\ (X,t)\in\mathcal{X}_F^*\times\mathcal{T}\right)=1 \implies P_F\left(\psi_F^{T_1}(V)=0\right)=1,$$

which establishes part (ii) of (A-15).

### A1.3 Steps of the proof of Proposition 1B

As with Proposition 1A, the first building blocks are Proposition S1 in the Econometric Supplement and the results in (A-8)-(A-9). Since the steps are analogous to those outlined above, we refer the reader to Section S3 (in particular, Section S3.7.2) of the online Econometric Supplement for all the details, and we will focus on describing the influence function $\psi_F^{T_2}(V)$ here. Let $\Xi_{Q_p}(x,t,\theta_F^*)$ be as defined in equation (A-3) and define,

$$\Xi^p_{T_0,F}(t)\equiv E_F\left[\frac{\partial\mathcal{B}\left(Q_F(X,t,\theta_F^*)\right)}{\partial Q_p}\mathbb{1}\{\mathcal{B}(Q_F(X,t,\theta_F^*))\geq 0\}\phi(X)\Xi_{Q_p}(X,t,\theta_F^*)\right],\quad \Xi_{T_0,F}(t)\equiv\sum_{p=1}^{P}\Xi^p_{T_0,F}(t) \tag{A-16}$$

where the above expectation is taken with respect to $X$. Let

$$\begin{aligned}
\psi_F^{T_0}(V_i,t) &\equiv \left((\mathcal{B}(Q_F(X_i,t,\theta_F^*)))_+\phi(X_i)-T_{0,F}(t)\right)+\sum_{p=1}^{P}\Omega^p_{T_0}(Y_i,t,g(X_i,\theta_F^*))\cdot\omega_p(g(X_i,\theta_F^*))+\Xi_{T_0,F}(t)\psi_F^\theta(Z_i),\\
\psi_F^{T_2}(V_i) &\equiv \int_t \psi_F^{T_0}(V_i,t)d\mathcal{W}(t)
\end{aligned} \tag{A-17}$$

$\psi_F^{T_2}(V_i)$ is the influence function in Proposition 1B.

### A1.3.1 Properties of the influence function $\psi_F^{T_2}(V)$

Sections S3.7.2 and S3.7.3 of the Econometric Supplement derive the two key properties of the influence function $\psi_F^{T_2}(V)$ stated in Proposition 1B. We will summarize the results here and we refer the reader to the Econometric Supplement for all the details. First, we show that the influence function $\psi_F^{T_0}(V,t)$ has two key features,

$$\begin{aligned}
&(i) \quad E_F\left[\psi_F^{T_0}(V,t)\right] = 0 \quad \forall\ t \in \mathcal{T},\ \forall\ F \in \mathcal{F}, \\
&(ii) \quad P_F\left(\mathcal{B}(\Gamma_F(X,t,\theta_F^*)) < 0 \,\middle|\, X \in \mathcal{X}_F^*\right) = 1 \implies P_F\left(\psi_F^{T_0}(V,t) = 0\right) = 1.
\end{aligned} \tag{A-18}$$

From (A-18), we have the following properties for the influence function $\psi_F^{T_2}(V)$,

$$\begin{aligned}
&(i) \quad E_F\left[\psi_F^{T_2}(V)\right] = 0 \quad \forall\ F \in \mathcal{F}, \\
&(ii) \quad P_F\left(\mathcal{B}(\Gamma_F(X,t,\theta_F^*)) < 0 \,\middle|\, X \in \mathcal{X}_F^*\right) = 1 \text{ for } \mathcal{W}\text{-a.e } t \in \mathcal{T} \implies P_F\left(\psi_F^{T_2}(V) = 0\right) = 1.
\end{aligned} \tag{A-19}$$

Part (i) of (A-19) follows directly from part (i) of (A-18) since,

$$E_F\left[\psi_F^{T_2}(V)\right] = E_F\left[\int_t \psi_F^{T_0}(V,t)d\mathcal{W}(t)\right] = \int_t \underbrace{\left(E_F\left[\psi_F^{T_0}(V,t)\right]\right)}_{=0\ \forall\ t\in\mathcal{T}} d\mathcal{W}(t) = 0.$$

Similarly, part (ii) of (A-19) follows directly from part (ii) of (A-18) since,

$$P_F\left(\mathcal{B}(\Gamma_F(X,t,\theta_F^*)) < 0 \,\middle|\, X \in \mathcal{X}_F^*\right) = 1 \text{ for } \mathcal{W}\text{-a.e } t \in \mathcal{T} \implies P_F\left(\psi_F^{T_0}(V,t) = 0 \text{ for } \mathcal{W}\text{-a.e } t \in \mathcal{T}\right) = 1.$$

And,

$$P_F\left(\psi_F^{T_0}(V,t) = 0 \text{ for } \mathcal{W}\text{-a.e } t \in \mathcal{T}\right) = 1 \implies P_F\left(\psi_F^{T_2}(V) = 0\right) = P_F\left(\int_t \psi_F^{T_0}(V,t)d\mathcal{W}(t) = 0\right) = 1.$$

The two properties in (A-19) are those described in the statement of Proposition 1B.

## A2 Estimators used for the influence functions $\psi_F^{T_1}(V)$ and $\psi_F^{T_2}(V)$

Here we describe the estimators discussed in Section 3.5.1 and used in the construction of $\widehat{\sigma}_1^2$ and $\widehat{\sigma}_2^2$. The details of the construction are included in the online Econometric Supplement. We have,

$$(1)\quad \widehat{\Omega}_{f_g}^{d,\ell}(g) = \frac{\widehat{A}_{f_g}^{d,\ell}(g)}{\widehat{f}_g(g)}, \quad \widehat{A}_{f_g}^{d,\ell}(g) \equiv \frac{1}{nh_n^D}\sum_{i=1}^{n}\frac{\partial g_d(X_i,\widehat{\theta})}{\partial\theta_\ell}K\left(\frac{g(X_i,\widehat{\theta})-g}{h_n}\right),$$

$$(2)\quad \widehat{\Omega}_{R_p,1}^{d,\ell}(g,t) = \frac{\widehat{A}_{R_p,1}^{d,\ell}(g,t)}{\widehat{f}_g(g)}, \quad \widehat{A}_{R_p,1}^{d,\ell}(g,t) \equiv \frac{1}{n\cdot h_n^D}\sum_{i=1}^{n}S_p(Y_i,t)\omega_p(g(X_i,\widehat{\theta}))\frac{\partial g_d(X_i,\widehat{\theta})}{\partial\theta_\ell}K\left(\frac{g(X_i,\widehat{\theta})-g}{h_n}\right),$$

$$(3)\quad \widehat{\Omega}_{R_p,2}(g,t) = \frac{\widehat{A}_{R_p,2}(g,t)}{\widehat{f}_g(g)}, \quad \widehat{A}_{R_p,2}(g,t) \equiv \frac{1}{n\cdot h_n^D}\sum_{i=1}^{n}S_p(Y_i,t)\omega_p(g(X_i,\widehat{\theta}))K\left(\frac{g(X_i,\widehat{\theta})-g}{h_n}\right),$$

$$(4)\quad \widehat{\Omega}_{R_p,3}^{\ell}(g,t) = \frac{\widehat{A}_{R_p,3}^{\ell}(g,t)}{\widehat{f}_g(g)}, \quad \widehat{A}_{R_p,3}^{\ell}(g,t) \equiv \frac{1}{n\cdot h_n^D}\sum_{i=1}^{n}S_p(Y_i,t)\frac{\partial\omega_p(g(X_i,\widehat{\theta}))}{\partial\theta_\ell}K\left(\frac{g(X_i,\widehat{\theta})-g}{h_n}\right),$$

From here, we estimate $\Xi_{f_g}(x,\theta_F^*)$ and $\Xi_{R_p}(x,t,\theta_F^*)$, where these functionals are described in (A-1) and $\Xi_{Q_p}(x,t,\theta_F^*)$, where this functional is described in equation (A-3). Our estimators are,

$$\widehat{\Xi}_{\ell,f_g}(x,\widehat{\theta}) \equiv \sum_{d=1}^{D}\left(\frac{\partial g_d(x,\widehat{\theta})}{\partial\theta_\ell}\cdot\frac{\partial\widehat{f}_g(g(x,\widehat{\theta}))}{\partial g_d} - \frac{\partial\left[\widehat{\Omega}_{f_g}^{d,\ell}(g(x,\widehat{\theta}))\widehat{f}_g(g(x,\widehat{\theta}))\right]}{\partial g_d}\right),$$

$$\widehat{\Xi}_{f_g}(x,\widehat{\theta}) \equiv \left(\widehat{\Xi}_{1,f_g}(x,\widehat{\theta}),\ldots,\widehat{\Xi}_{k,f_g}(x,\widehat{\theta})\right),$$

$$\widehat{\Xi}_{\ell,R_p}(x,t,\widehat{\theta}) \equiv \sum_{d=1}^{D}\left(\frac{\partial\left[\widehat{\Omega}_{R_p,2}(g(x,\widehat{\theta}),t)\widehat{f}_g(g(x,\widehat{\theta}))\right]}{\partial g_d}\cdot\frac{\partial g_d(x,\widehat{\theta})}{\partial\theta_\ell} - \frac{\partial\left[\widehat{\Omega}_{R_p,1}^{d,\ell}(g(x,\widehat{\theta}),t)\widehat{f}_g(g(x,\widehat{\theta}))\right]}{\partial g_d}\right.$$
$$\left. + \widehat{\Omega}_{R_p,3}^{\ell}(g(x,\widehat{\theta}),t)\cdot\widehat{f}_g(g(x,\widehat{\theta}))\right),$$

$$\widehat{\Xi}_{R_p}(x,t,\widehat{\theta}) \equiv \left(\widehat{\Xi}_{1,R_p}(x,t,\widehat{\theta}),\ldots,\widehat{\Xi}_{k,R_p}(x,t,\widehat{\theta})\right),$$

$$\widehat{\Xi}_{Q_p}(x,t,\widehat{\theta}) \equiv \frac{\widehat{\Xi}_{R_p}(x,t,\widehat{\theta}) - \widehat{Q}_p(x,t,\widehat{\theta})\cdot\widehat{\Xi}_{f_g}(x,\widehat{\theta})}{\widehat{f}_g(g(x,\widehat{\theta}))},$$

From here we are ready to estimate the functional $\Xi_{T_1,F}$ described in Proposition 1A, equation (A-12)

$$\widehat{\Xi}_{T_1}^{p} \equiv \frac{1}{n}\sum_{i=1}^{n}\frac{\partial\mathcal{B}\left(\widehat{Q}(X_i,t_i,\widehat{\theta})\right)}{\partial\widehat{Q}_p}\mathbb{1}\left\{\mathcal{B}\left(\widehat{Q}(X_i,t_i,\widehat{\theta})\right)\geq -b_n\right\}\phi(X_i,t_i)\widehat{\Xi}_{Q_p}(X_i,t_i,\widehat{\theta}), \quad \text{and} \quad \widehat{\Xi}_{T_1} \equiv \sum_{p=1}^{P}\widehat{\Xi}_{T_1}^{p}.$$

And for a given $t$ we estimate the functional $\Xi_{T_0,F}(t)$ described in Proposition 1B, equation (A-16) with

$$\widehat{\Xi}^p_{T_0}(t) \equiv \frac{1}{n}\sum_{i=1}^{n} \frac{\partial \mathcal{B}\left(\widehat{Q}(X_i,t,\widehat{\theta})\right)}{\partial \widehat{Q}_p} \mathbb{1}\left\{\mathcal{B}\left(\widehat{Q}(X_i,t,\widehat{\theta})\right) \geq -b_n\right\} \phi(X_i,t) \widehat{\Xi}_{Q_p}(X_i,t,\widehat{\theta}), \quad \text{and} \quad \widehat{\Xi}_{T_0}(t) \equiv \sum_{p=1}^{P} \widehat{\Xi}^p_{T_0}(t).$$

Next, for a given $x$, $t$, we estimate

$$\widehat{\Gamma}_p(x,t,\widehat{\theta}) = \frac{\frac{1}{n \cdot h_n^D} \sum_{i=1}^{n} S_p(Y_i,t) K\left(\frac{\Delta g(X_i,x,\widehat{\theta})}{h_n}\right)}{\widehat{f_g}(g(x,\widehat{\theta}))}.$$

For a given $(y,g)$, we estimate the functional $\Omega^p_{T_1}(y,g)$ described in Assumption 7A, equation (17) with

$$\widehat{\Omega}^p_{T_1}(y,g) = \frac{\frac{1}{n \cdot h_n^D} \sum_{i=1}^{n} \left(S_p(y,X_i,t_i) - \widehat{\Gamma}_p(X_i,t_i,\widehat{\theta})\right) \frac{\partial \mathcal{B}\left(\widehat{Q}(X_i,t_i,\widehat{\theta})\right)}{\partial Q_p} \phi(X_i,t_i) \mathbb{1}\left\{\mathcal{B}\left(\widehat{Q}(X_i,t_i,\widehat{\theta})\right) \geq -b_n\right\} K\left(\frac{g(X_i,\widehat{\theta})-g}{h_n}\right)}{\widehat{f_g}(g)}$$

And for a given $(y,g,t)$, we estimate the functional $\Omega^p_{T_0}(y,t,g)$ described in Assumption 7B, equation (21) with

$$\widehat{\Omega}^p_{T_0}(y,t,g) = \frac{\frac{1}{n \cdot h_n^D} \sum_{i=1}^{n} \left(S_p(y,X_i,t) - \widehat{\Gamma}_p(X_i,t,\widehat{\theta})\right) \frac{\partial \mathcal{B}\left(\widehat{Q}(X_i,t,\widehat{\theta})\right)}{\partial Q_p} \phi(X_i) \mathbb{1}\left\{\mathcal{B}\left(\widehat{Q}(X_i,t,\widehat{\theta})\right) \geq -b_n\right\} K\left(\frac{g(X_i,\widehat{\theta})-g}{h_n}\right)}{\widehat{f_g}(g)}$$

Our estimator for the influence function $\psi_F^{T_1}(V)$ of the statistic $\widehat{T_1}$ described in equation (A-13) is,

$$\begin{aligned}\widehat{\psi}^{T_1}(V_i) \equiv & \left(\mathcal{B}\left(\widehat{Q}(X_i,t_i,\widehat{\theta})\right) \mathbb{1}\left\{\mathcal{B}\left(\widehat{Q}(X_i,t_i,\widehat{\theta})\right) \geq -b_n\right\} \phi(X_i,t_i) - \widehat{T_1}\right) + \sum_{p=1}^{P} \widehat{\Omega}^p_{T_1}(Y_i, g(X_i,\widehat{\theta})) \cdot \omega(g(X_i,\widehat{\theta})) \\ & + \widehat{\Xi}_{T_1} \widehat{\psi}^{\theta}(Z_i)\end{aligned}$$

And our estimator for the influence function $\psi_F^{T_2}(V)$ of the statistic $\widehat{T_2}$ described in equation (A-17) is,

$$\begin{aligned}\widehat{\psi}^{T_2}(V_i) \equiv & \left(\int_t \mathcal{B}\left(\widehat{Q}(X_i,t,\widehat{\theta})\right) \mathbb{1}\left\{\mathcal{B}\left(\widehat{Q}(X_i,t,\widehat{\theta})\right) \geq -b_n\right\} d\mathcal{W}(t) \phi(X_i) - \widehat{T_2}\right) \\ & + \sum_{p=1}^{P} \int_t \widehat{\Omega}^p_{T_0}(Y_i,t,g(X_i,\widehat{\theta})) d\mathcal{W}(t) \cdot \omega(g(X_i,\widehat{\theta})) + \int_t \widehat{\Xi}_{T_0}(t) d\mathcal{W}(t) \widehat{\psi}^{\theta}(Z_i)\end{aligned}$$

### A2.1 Estimators for $\sigma_{1,F}^2$ and $\sigma_{2,F}^2$

Our estimators for $\sigma_{1,F}^2$ and $\sigma_{2,F}^2$ are $\widehat{\sigma}_1^2 \equiv \frac{1}{n}\sum_{i=1}^n \widehat{\psi}^{T_1}(V_i)^2$, and $\widehat{\sigma}_2^2 \equiv \frac{1}{n}\sum_{i=1}^n \widehat{\psi}^{T_2}(V_i)^2$. As we show in the Econometric Supplement (Section S4), under the conditions in Assumptions 1-6, 7A, 7B and 8,

$$\left.\begin{array}{l} \frac{1}{n}\sum_{i=1}^n \left|\widehat{\psi}^{T_1}(V_i) - \psi_F^{T_1}(V_i)\right|^2 = o_p(1) \\ \frac{1}{n}\sum_{i=1}^n \left|\widehat{\psi}^{T_2}(V_i) - \psi_F^{T_2}(V_i)\right|^2 = o_p(1) \end{array}\right\} \text{uniformly over } \mathcal{F}. \tag{A-20}$$

Next, under the conditions of Propositions 1A and 1B, there exists a finite $\overline{\mu}_T > 0$ such that $E_F\left[\psi_F^{T_1}(V)^2\right] \le \overline{\mu}_T$ and $E_F\left[\psi_F^{T_2}(V)^2\right] \le \overline{\mu}_T$ for all $F \in \mathcal{F}$. From here, a Chebyshev inequality yields,

$$\left|\frac{1}{n}\sum_{i=1}^n \psi_F^{T_1}(V_i)^2 - \sigma_{1,F}^2\right| = o_p(1), \quad \text{and} \quad \left|\frac{1}{n}\sum_{i=1}^n \psi_F^{T_2}(V_i)^2 - \sigma_{2,F}^2\right| = o_p(1), \quad \text{uniformly over } \mathcal{F}. \tag{A-21}$$

(A-20) and (A-21) yield,

$$\left|\widehat{\sigma}_1^2 - \sigma_{1,F}^2\right| = o_p(1), \quad \text{and} \quad \left|\widehat{\sigma}_2^2 - \sigma_{2,F}^2\right| = o_p(1), \quad \text{uniformly over } \mathcal{F}. \tag{A-22}$$

This is the result stated in equation (27) in the paper.

## A3 Bandwidth convergence rates described in Section 3.6

We focus on bandwidth sequences of the form $h_n \propto n^{-\alpha_h}$ and $b_n \propto n^{-\alpha_b}$. Since $b_n$ plays an important role in our problem, we focus on the fastest rates of convergence for it that is compatible with Assumption 4. Let $\alpha_b = \frac{1}{4} + \Delta_b$. To satisfy the restrictions $n^{1/2+\epsilon'} \cdot b_n^2 \longrightarrow 0$ and $n^{1/2} \cdot h_n^{D+1} \cdot b_n \longrightarrow \infty$, we must have $\frac{\epsilon'}{2} < \Delta_b < \frac{1}{4} - (D+1)\cdot\alpha_h$. Thus, we must have $\alpha_h < \frac{1}{4(D+1)} - \frac{\epsilon'}{2(D+1)}$. Accordingly, let $\alpha_h = \frac{1}{4(D+1)} - \Delta_h$, where $\Delta_h > \frac{\epsilon'}{2(D+1)}$. Thus, we can express $\Delta_h = \frac{\epsilon'+\delta'}{2(D+1)}$ with $\delta' > 0$. Set[11] $\epsilon' < \left(\frac{1}{2D(D+1)} \wedge \frac{1}{8}\right)$. Then, having $\alpha_h = \frac{1}{4(D+1)} - \Delta_h$ automatically satisfies the condition $n^{1/2-\epsilon'}\cdot(h_n^{2D} \wedge h_n^{D+2}) \longrightarrow \infty$. The last bandwidth convergence restriction we need to satisfy is $n^{1/2+\epsilon'}\cdot h_n^M \longrightarrow 0$. We are interested in the smallest integer $M$ that can satisfy this condition given the restrictions on $\alpha_h$. The condition will be satisfied if and only if $M\cdot\left(\frac{1-4(D+1)\Delta_h}{4(D+1)}\right) > \frac{1}{2}+\epsilon'$. Since $\Delta_h = \frac{\epsilon'+\delta'}{2(D+1)}$, this becomes $M > \left(\frac{4(D+1)}{1-2(\epsilon'+\delta')}\right)\cdot\left(\frac{1}{2}+\epsilon'\right)$. Note that $\left(\frac{4(D+1)}{1-2(\epsilon'+\delta')}\right)\cdot\left(\frac{1}{2}+\epsilon'\right) > 2(D+1) = 2D+2$, so the smallest integer that can be greater than the right-hand side of the previous expression is $2D+3$. Accordingly, choose $\epsilon' > 0$ and $\delta' > 0$ small enough that $\left(\frac{4(D+1)}{1-2(\epsilon'+\delta')}\right)\cdot\left(\frac{1}{2}+\epsilon'\right) < 2D+3$. Then, $M = 2D+3$ satisfies our restriction and this is the smallest integer that can do so. In summary, the bandwidth convergence rates would be of the form $\alpha_h = \frac{1}{4(D+1)} - \frac{\epsilon'+\delta'}{2(D+1)}$ and $\alpha_b = \frac{1}{4} + \Delta_b$, where $\frac{\epsilon'}{2} < \Delta_b < \frac{\epsilon'+\delta'}{2}$, with $\epsilon' > 0$

[11] We will have $\frac{1}{2D(D+1)} \wedge \frac{1}{8} = \frac{1}{2D(D+1)}$ and $h_n^{2D} \wedge h_n^{D+2} = h_n^{2D}$ for all $D \ge 2$.

and $\delta' > 0$ small enough such that $\epsilon' < \left(\frac{1}{2D(D+1)} \wedge \frac{1}{8}\right)$ and $\left(\frac{4(D+1)}{1-2(\epsilon'+\delta')}\right) \cdot \left(\frac{1}{2} + \epsilon'\right) < 2D + 3$. Then the bandwidth convergence restrictions in Assumption 4 are satisfied with $M \geq 2D + 3$. Thus, we can use a bias-reducing kernel of order $M = 2D + 3$.

## A4 Examples of estimators that satisfy the conditions described in Assumption 1

Here we describe two examples of estimators and the conditions under which they satisfy the restrictions in Assumption 1. The examples we include are OLS and a semiparametric, multiple index estimator. The proofs of the results presented below are included in the online Econometric Supplement, where we present additional examples of estimators, including GMM and density-weighted average derivatives, in addition to OLS and the semiparametric, multiple index estimator we analyze here.

#### A convenient definition

Before proceeding, let us introduce the following definitions for notational convenience. Take a collection of column vectors $(v_\ell)_{\ell=1}^d$ where $v_\ell \in \mathbb{R}^d$ for each $\ell$, and let

$$\underbrace{v \equiv (v_1', v_2', \ldots, v_d')'}_{d^2 \times 1}.$$

For any such $v$ we will define

$$\underbrace{H_d(v)}_{d\times d} \equiv \begin{pmatrix} v_1' \\ v_2' \\ \vdots \\ v_d' \end{pmatrix} \text{ and, when it exists, we will denote } M_d(v) \equiv H_d(v)^{-1}. \tag{A-23}$$

### A4.1 OLS

Consider an iid sample $(Z_{1i}, Z_{2i})_{i=1}^n$ where $Z_i \equiv (Z_{1i}, Z_{2i}) \sim F$, with $Z_{1i} \in \mathbb{R}$ and $Z_{2i} \in \mathbb{R}^k$. Denote the $\ell^{th}$ element in $Z_{2i}$ as $Z_{2i,\ell}$. Define

$$\underbrace{\overline{G}_\ell}_{k\times 1} \equiv \frac{1}{n}\sum_{i=1}^n Z_{2i} Z_{2i,\ell}, \quad \underbrace{\lambda_{\ell,F}}_{k\times 1} \equiv E_F[Z_2 Z_{2,\ell}],$$

$$\overline{G} \equiv \underbrace{\left(\overline{G}_1', \overline{G}_2', \ldots, \overline{G}_k'\right)'}_{k^2\times 1} \quad \text{and} \quad \lambda_F \equiv \underbrace{\left(\lambda_{1,F}', \lambda_{2,F}', \ldots, \lambda_{k,F}'\right)'}_{k^2\times 1}.$$

Let $M_k$ and $H_k$ be as defined in (A-23) and consider the following assumption.

**Assumption LS** *$\exists\ \overline{M}_{z_2z_2}$ such that $\left\|(E_F[Z_2Z_2'])^{-1}\right\| \leq \overline{M}_{z_2z_2}$ $\forall\ F \in \mathcal{F}$. For some $q \geq 2$, there exist $\overline{\mu}_{z_2z_2}$ and $\overline{\mu}_{z_2\nu}$ such that, for each $\ell, m$,*

$$E_F\left[\left|Z_{2,\ell}Z_{2,m} - E_F[Z_{2,\ell}Z_{2,m}]\right|^q\right] \leq \overline{\mu}_{z_2z_2} \quad \text{and} \quad E_F\left[\left|Z_{2,\ell}\nu\right|^q\right] \leq \overline{\mu}_{z_2\nu} \quad \forall\ F \in \mathcal{F}.$$

*There exists $\overline{M}_\lambda$ such that $\|M_k(\lambda_F)\| \leq \overline{M}_\lambda$ for all $F \in \mathcal{F}$ and there exist $K_1 > 0$, $K_2 > 0$ and $\alpha > 0$ such that, for any $F \in \mathcal{F}$ and $v \in \mathbb{R}^{k^2}$,*

$$\|v - \lambda_F\| \leq K_1 \quad \Longrightarrow \quad \|M_k(v) - M_k(\lambda_F)\| \leq K_2 \cdot \|v - \lambda_F\|^\alpha,$$

*and there exists $K_3 < \infty$ such that*

$$\sup_{v:\|v-\lambda_F\|\leq K_1}\left\{\|M_k(v) - M_k(\lambda_F)\|\right\} \leq K_3 \quad \forall\ F \in \mathcal{F}$$

Consider the OLS estimator

$$\widehat{\theta} = \left(\frac{1}{n}\sum_{i=1}^n Z_{2i}Z_{2i}'\right)^{-1}\frac{1}{n}\sum_{i=1}^n Z_{2i}Z_{1i}, \tag{A-24}$$

and let

$$\theta_F^* \equiv (E_F[Z_2Z_2'])^{-1} \cdot E_F[Z_2Z_1],$$

and let us express $Z_1 = Z_2'\theta_F^* + \left(Z_1 - Z_2'\theta_F^*\right) \equiv Z_2'\theta_F^* + \nu$, where $\nu \equiv \left(Z_1 - Z_2'\theta_F^*\right)$. Note that $E_F[Z_2\nu] = 0$ by the definition of $\theta_F^*$. In the usual linear regression model where we assume a structural relationship given by $Z_1 = Z_2'\beta_0 + \varepsilon$ with $E_F[Z_2\varepsilon] = 0\ \forall\ F \in \mathcal{F}$, we would have $\nu = \varepsilon$ and $\theta_F^* = \beta_0$ for all $F \in \mathcal{F}$.

**Result OLS** *Let $\theta_F^* \equiv \left(E_F[Z_2Z_2']\right)^{-1} \cdot E_F[Z_2Z_1]$, $\nu_i \equiv \left(Z_{1i} - Z_{2i}'\theta_F^*\right)$, and $\psi_F^\theta(Z_i) \equiv \left(E_F[Z_2Z_2']\right)^{-1} \cdot Z_{2i}\nu_i$. Note that $E_F[\psi_F^\theta(Z_i)] = 0$. Under Assumption LS, the OLS estimator in* (A-24) *satisfies*

$$\widehat{\theta} = \theta_F^* + \frac{1}{n}\sum_{i=1}^n \psi_F^\theta(Z_i) + \varepsilon_n^\theta,$$

*and the conditions in Assumption 1 are satisfied, with $\psi_F^\theta(Z_i) = \left(E_F[Z_2Z_2']\right)^{-1} \cdot Z_{2i}\nu_i$, $r_n = n^{1/2}$, and for any $\tau$ and $\overline{\delta}$ such that $0 < \tau < \alpha/2$, and $0 < \overline{\delta} < (q-1)/2$.*

**Proof:** A step-by-step proof is included in the online Econometric Supplement. ■

### A4.2 A semiparametric, multiple index estimator that satisfies the conditions in Assumption 1

Consider a collection of $d$ single-valued indices $(m_\ell(W_\ell,\theta_\ell))_{\ell=1}^d$ where $\theta_\ell \in \mathbb{R}^{k_\ell}$. Each $m_\ell$ has a known parametric functional form (e.g, $m_\ell(W_\ell,\theta_\ell) = W_\ell'\theta_\ell$). Group $\cup_{\ell=1}^d W_\ell \equiv Z_2$ and let $\theta \equiv \left(\theta_1',\theta_2',\ldots,\theta_d'\right)' \in \mathbb{R}^k$ and denote

$$m(Z_2,\theta) \equiv (m_1(W_1,\theta_1), m_2(W_2,\theta_2),\ldots,m_d(W_d,\theta_d))' \in \mathbb{R}^d.$$

For simplicity let us focus on the case where $Z_2$ is a vector of jointly continuously distributed random variables. Let $Z_1$ be a scalar random variable and group $Z \equiv (Z_1,Z_2) \sim F \in \mathcal{F}$. Let $\Theta$ denote the parameter space for $\theta$, assume $\Theta$ to be bounded and consider a model where there exists a $\theta^* \in \Theta$ such that

$$E_F[Z_1|Z_2] = E_F\left[Z_1\middle|m(Z_2,\theta^*)\right] \quad \forall\ F \in \mathcal{F}$$

For a given $\theta \in \Theta$ let $\mu_F(m(Z_2,\theta)) \equiv E_F[Z_1|m(Z_2,\theta)]$. Our model therefore assumes $E_F[Z_1|Z_2] = \mu_F(m(Z_2,\theta^*))$. Let $\phi \in \mathbb{R}^k$ denote a vector of pre-specified instrument functions and consider an estimator based on the moment conditions

$$E_F\left[\phi(Z_2)\cdot\left(Z_1 - \mu_F(m(Z_2,\theta^*))\right)\right] = 0$$

Suppose we have a random sample $(Z_{1i},Z_{2i})_{i=1}^n$ where $Z_i \equiv (Z_{1i},Z_{2i}) \sim F \in \mathcal{F}$. Let $\mathcal{S}_\xi$ denote the support of the r.v $\xi$ and for simplicity assume throughout that $\mathcal{S}_Z$ is the same for all $F \in \mathcal{F}$. Suppose that the instrument functions are designed such that $\phi(z_2) = 0\ \forall\ z_2 \notin \mathcal{Z}_2$, where $\mathcal{Z}_2 \subset \mathcal{S}_{Z_2}$ is a pre-specified set belonging in the interior of $\mathcal{S}_{Z_2}$ for all $F \in \mathcal{F}$. We refer to $\mathcal{Z}_2$ as our *inference range*. Thus, the instrument functions also serve as trimming functions to keep inference confined to the set $\mathcal{Z}_2$. Finally, suppose $\|\phi(z_2)\| \leq \overline{\phi}\ \forall\ z_2$. Let

$$\mathcal{M} \equiv \left\{m \in \mathbb{R}^d\colon\ m = m(z_2,\theta)\ \text{ for some } (z_2,\theta) \in \mathcal{Z}_2 \times \Theta\right\}.$$

$\mathcal{M}$ is the range of all possible values of the index $m(z_2,\theta)$ over our inference range and the parameter space. Let $\sigma_n \to 0$ denote a bandwidth sequence and let $K$ denote a kernel function. For a given $\theta \in \Theta$ and $z_2 \in \mathcal{Z}_2$, let $f_m(m(Z_2,\theta))$ denote the density of $m(Z_2,\theta)$. Consider a kernel-based estimator of $\mu_F(m(z_2,\theta))$ of the form

$$
\begin{aligned}
\widehat{\mu}(m(z_2,\theta)) &= \frac{\widehat{R}(m(z_2,\theta))}{\widehat{f}_m(m(z_2,\theta))}, \quad \text{where} \\
\widehat{R}(m(z_2,\theta)) &= \frac{1}{n\cdot\sigma_n^d}\sum_{i=1}^{n} Z_{1i} K\left(\frac{m(Z_{2i},\theta)-m(z_2,\theta)}{\sigma_n}\right), \\
\widehat{f}_m(m(z_2,\theta)) &= \frac{1}{n\cdot\sigma_n^d}\sum_{i=1}^{n} K\left(\frac{m(Z_{2i},\theta)-m(z_2,\theta)}{\sigma_n}\right).
\end{aligned}
$$

Consider an estimator $\widehat{\theta}$ defined by the sample analog moment conditions,

$$
\frac{1}{n}\sum_{i=1}^{n}\phi(Z_{2i})\cdot\left(Z_{1i}-\widehat{\mu}(m(Z_{2i},\widehat{\theta}))\right)=0. \tag{A-25}
$$

**Assumption SMIM1** *For some $q\geq 2$, we have $E_F[Z_1^{4q}]\leq\overline{\mu}_{4q}<\infty$ for all $F\in\mathcal{F}$. Also, there exist constants $\underline{f}_m>0$, $\overline{f}_m<\infty$ and $\overline{\mu}<\infty$ such that $\overline{f}_m\geq f_m(m)\geq\underline{f}_m$ and $|\mu_F(m)|\leq\overline{\mu}$ $\forall$ $m\in\mathcal{M}$ and all $F\in\mathcal{F}$. Also assume that both $f_m(m)$ and $\mu_F(m)$ are $L$-times continuously differentiable with respect to $m$ for $F-$a.e $m\in\mathcal{M}$, with derivatives that are uniformly bounded over $\mathcal{M}$ for all $F\in\mathcal{F}$. The kernel $K:\mathbb{R}^d\to\mathbb{R}$ is a multiplicative kernel of the form $K(\psi)=\prod_{\ell=1}^{d}\kappa(\psi_\ell)$ (with $\psi\equiv(\psi_1,\ldots,\psi_d)$), where $\kappa(\cdot)$ is a function of bounded-variation, a bias-reducing kernel of order $L$ with support of the form $[-S,S]$ (i.e, $\int_{-S}^{S}v^j\kappa(v)dv=0$ for $j=1,\ldots,L-1$ and $\int_{-S}^{S}|v|^L\kappa(v)dv<\infty$) and symmetric around zero. We have $\sup_{\psi\in\mathbb{R}^d}\left|K(\psi)\right|\leq\overline{K}$. The bandwidth sequence $\sigma_n>0$ satisfies $\sigma_n\longrightarrow 0$, with $n^{1/2+\Delta}\cdot\sigma_n^L\longrightarrow 0$ and $n^{1/2-\Delta}\cdot\sigma_n^d\longrightarrow\infty$ for some $0<\Delta<1/2$. $q$ and $\Delta$ are such that $q\Delta>\frac{1}{2}$.*

**Assumption SMIM2** *Consider the following class of functions defined on $\mathcal{S}_{Z_2}$*

$$
\mathcal{G}_1=\left\{g:\mathcal{S}_{Z_2}\to\mathbb{R}: g(z_2)=K\left(\alpha\cdot m(z_2,\theta)+\beta\cdot m(v,\theta)\right)\right) \text{ for some } v\in\mathcal{S}_{Z_2},\,\theta\in\Theta,\,\alpha,\beta\in\mathbb{R}\right\}
$$

*Then, $\mathcal{G}_1$ is Euclidean for the constant envelope $\overline{K}$.*

For indices of the form $m(z_2,\theta)=z_2'\theta$, the condition in Assumption SMIM2 follows immediately from Lemma 22 in Nolan and Pollard (1987), who showed that if $\lambda(\cdot)$ is a real-valued function of bounded variation on $\mathbb{R}$, the class of all functions of the form $x\to\lambda(\gamma'x+\tau)$ with $\gamma$ ranging over $\mathbb{R}^d$ and $\tau$ ranging over $\mathbb{R}$ is Euclidean for a constant envelope. Let

$$
\eta_F(m(Z_2,\theta))\equiv E_F[\phi(Z_2)\mid m(Z_2,\theta)],
$$

and assume that, like the other functionals analyzed before, $\eta_F(m)$ is also $L$-times continuously differentiable with respect to $m$ for $F-$a.e $m\in\mathcal{M}$, with derivatives that are uniformly bounded over $\mathcal{M}$ for all $F \in \mathcal{F}$.

**Assumption SMIM3** *The index $m(z_2,\theta)$ is smooth with respect to $\theta$ and, for every $F \in \mathcal{F}$, the following Jacobians are well-defined for $F-$a.e $z_2 \in \mathcal{S}_{Z_2}$ and for all $\theta \in \Theta$,*

$$\underbrace{\nabla_\theta \mu_F(m(z_2;\theta))}_{1\times k} \equiv \begin{pmatrix} \frac{\partial \mu_F(m(z_2;\theta)'}{\partial \theta_1} & \frac{\partial \mu_F(m(z_2;\theta)'}{\partial \theta_2} & \cdots & \frac{\partial \mu_F(m(z_2;\theta)'}{\partial \theta_d} \end{pmatrix},$$
$$\underbrace{\nabla_\theta f_m(m(z_2;\theta))}_{1\times k} \equiv \begin{pmatrix} \frac{\partial f_m(m(z_2;\theta)'}{\partial \theta_1} & \frac{\partial f_m(m(z_2;\theta)'}{\partial \theta_2} & \cdots & \frac{\partial f_m(m(z_2;\theta)'}{\partial \theta_d} \end{pmatrix}$$

*There exists a nonnegative function $\overline{H}_1(\cdot)$ such that, for each $F \in \mathcal{F}$,*

$$\sup_{\theta\in\Theta} \left\|\nabla_\theta \mu_F(m(z_2;\theta))\right\| \le \overline{H}_1(z_2) \quad \forall\ z_2 \in \mathcal{Z}_2,$$
$$\sup_{\theta\in\Theta} \|\nabla_\theta f_m(m(z_2;\theta))\| \le \overline{H}_1(z_2) \quad \forall\ z_2 \in \mathcal{Z}_2,$$

*and there exists $\overline{\mu}_{\overline{H}_1} < \infty$ such that $E_F[\overline{H}_1(Z_2)^{4q}] \le \overline{\mu}_{\overline{H}_1}$ $\forall\ F \in \mathcal{F}$, where $q$ is the integer described in Assumption SMIM1.*

Let

$$\underbrace{\eta_F(m(Z_2,\theta))}_{k\times 1} \equiv E_F\left[\phi(Z_2)|m(Z_2,\theta)\right] = \begin{pmatrix} E_F\left[\phi_1(Z_2)|m(Z_2,\theta)\right] \\ E_F\left[\phi_2(Z_2)|m(Z_2,\theta)\right] \\ \vdots \\ E_F\left[\phi_k(Z_2)|m(Z_2,\theta)\right] \end{pmatrix} \equiv \begin{pmatrix} \eta_{1,F}(m(Z_2,\theta)) \\ \eta_{2,F}(m(Z_2,\theta)) \\ \vdots \\ \eta_{k,F}(m(Z_2,\theta)) \end{pmatrix}$$

and assume that, like the other functionals analyzed before, $\eta_F(m)$ is also $L$-times continuously differentiable with respect to $m$ for $F-$a.e $m \in \mathcal{M}$, with derivatives that are uniformly bounded over $\mathcal{M}$ for all $F \in \mathcal{F}$. We add the following smoothness conditions to those described in Assumption SMIM3.

**Assumption SMIM4** *For every $F \in \mathcal{F}$, the following Jacobians are well-defined for $F-$a.e $z_2 \in \mathcal{S}_{Z_2}$ and everywhere on $\Theta$,*

$$\underbrace{\nabla_\theta \eta_{\ell,F}(m(z_2,\theta))}_{1\times k} \equiv \begin{pmatrix} \frac{\partial \eta_{\ell,F}(m(z_2,\theta)'}{\partial \theta_1} & \frac{\partial \eta_{\ell,F}(m(z_2,\theta)'}{\partial \theta_2} & \cdots & \frac{\partial \eta_{\ell,F}(m(z_2,\theta)'}{\partial \theta_d} \end{pmatrix}, \quad \ell = 1,\ldots,k$$

*Let*

$$\underbrace{\nabla_\theta \eta_F(m(z_2,\theta)}_{k\times k} \equiv \begin{pmatrix} \nabla_\theta \eta_{1,F}(m(z_2,\theta)) \\ \nabla_\theta \eta_{2,F}(m(z_2,\theta)) \\ \vdots \\ \nabla_\theta \eta_{k,F}(m(z_2,\theta)) \end{pmatrix}$$

*and express* $\phi(z_2) \equiv (\phi_1(z_2), \phi_2(z_2), \ldots, \phi_k(z_2))' \in \mathbb{R}^k$. *For* $\ell = 1, \ldots, k$, *define*

$$\begin{aligned}
\underbrace{T_{\ell,F}(Z,\theta)}_{1\times k} &\equiv \Big(\phi_\ell(Z_2) - \eta_{\ell,F}(m(Z_2,\theta))\Big) \cdot \nabla_\theta \mu_F(m(Z_2,\theta)) + \nabla_\theta \eta_{\ell,F}(m(Z_2,\theta)) \cdot \Big(Z_1 - \mu_F(m(Z_2,\theta))\Big), \\
\underbrace{T_F(Z,\theta)}_{k^2\times 1} &\equiv \Big(T_{1,F}(Z,\theta) \quad T_{2,F}(Z,\theta) \quad \cdots \quad T_{k,F}(Z,\theta)\Big)', \\
\underbrace{\lambda_{\ell,F}(\theta)}_{1\times k} &\equiv E_F\left[T_{\ell,F}(Z,\theta)\right], \\
\underbrace{\lambda_F(\theta)}_{k^2\times 1} &\equiv E\left[T_F(Z,\theta)\right] = \Big(\lambda_{1,F}(\theta) \quad \lambda_{2,F}(\theta) \quad \cdots \quad \lambda_{k,F}(\theta)\Big)'
\end{aligned}$$

*(i) There exists a nonnegative function* $\overline{H}_2(\cdot)$ *such that, for each* $F \in \mathcal{F}$,

$$\sup_{\theta\in\Theta} \left\|\nabla_\theta \eta_F(m(z_2,\theta)\right\| \leq \overline{H}_2(z_2) \quad \forall\, z_2 \in \mathcal{Z}_2$$

*and there exists* $\overline{\mu}_{\overline{H}_6} < \infty$ *such that* $E_F\left[\overline{H}_2(Z_2)^{4q}\right] \leq \overline{\mu}_{\overline{H}_6}$ *for all* $F \in \mathcal{F}$, *where* $q$ *is the integer described in Assumption SMIM1. Note that this condition, combined with Assumptions SMIM1 and SMIM3 imply that there exists a nonnegative function* $\overline{G}_6(\cdot)$ *such that, for all* $F \in \mathcal{F}$,

$$\left\|T_F(z,\theta) - T_F(z,\theta')\right\| \leq \overline{G}_6(z) \cdot \left\|\theta - \theta'\right\| \quad \forall\, z \in \mathcal{S}_Z \quad \text{and} \quad \theta, \theta' \in \Theta,$$

*and there exists* $\overline{\mu}_{\overline{G}_6} < \infty$ *such that* $E_F\left[\overline{G}_6(Z)^{4q}\right] \leq \overline{\mu}_{\overline{G}_6}$ $\forall\, F \in \mathcal{F}$, *where* $q$ *is the integer described in Assumption SMIM1.*

*(ii) Let* $H_k$ *and* $M_k$ *be as defined in* (A-23). *Assume that* $\exists\, \underline{d} > 0$, $\overline{M}_\lambda$, $K_5 > 0$, $K_6 > 0$ *and* $\alpha_1 > 0$ *such that, for every* $F \in \mathcal{F}$,

$$\inf_{\theta\in\Theta}\left|det(H_k(\lambda_F(\theta)))\right| \geq \underline{d} \quad \sup_{\theta\in\Theta}\left\|M_k(\lambda_F(\theta))\right\| \leq \overline{M}_\lambda$$
$$\left\|M_k(\lambda_F(\theta)) - M_k(v)\right\| \leq K_6 \cdot \|\lambda_F(\theta) - v\|^{\alpha_1} \quad \forall\, v, \theta : \|v - \lambda_F(\theta)\| \leq K_5,\ \theta \in \Theta.$$

*And,*

$$\sup_{\substack{v:\|v-\lambda_F(\theta)\|\le K_5\\ \theta\in\Theta}}\left\{\left\|M_k(\lambda_F(\theta))-M_k(v)\right\|\right\}\le K_7<\infty$$

*(iii)* $\exists\ K_8>0$, $K_9>0$ *and* $\alpha_2>0$ *such that, for every* $F\in\mathcal{F}$,

$$\left\|\lambda_F(\theta)-\lambda_F(\theta^*)\right\|\le K_9\cdot\|\theta-\theta^*\|^{\alpha_2}\quad\forall\ \theta:\|\theta-\theta^*\|\le K_8$$

**Result SMIM** *Define*

$$\begin{aligned}\zeta_F(Z_i)&\equiv\Big(\phi(Z_{2i})-\eta_F\Big(m(Z_{2i},\theta^*)\Big)\Big)\cdot\Big(Z_{1i}-\mu_F\Big(m(Z_{2i},\theta^*)\Big)\Big),\\ \psi_F^{\theta}(Z_i)&\equiv M_k\left(\lambda_F(\theta^*)\right)\cdot\zeta_F(Z_i).\end{aligned}$$

*Note that* $E_F[\zeta_F(Z)]=E_F[\psi_F^{\theta}(Z)]=0$. *Under Assumptions SMIM1-SMIM4, the estimator defined by* (A-25) *satisfies,*

$$\widehat{\theta}=\theta^*+\frac{1}{n}\sum_{i=1}^{n}\psi_F^{\theta}(Z_i)+\varepsilon_n^{\theta},$$

*and the conditions in Assumption 1 are satisfied, with* $\psi_F^{\theta}(Z_i)=M_k\left(\lambda_F(\theta^*)\right)\cdot\zeta_F(Z_i)$, $r_n=n^{1/2}\cdot\sigma_n^d$, *and for any* $\tau$ *and* $\overline{\delta}$ *such that* $0<\tau<\min\left\{\left(\frac{\alpha_1}{2}\right),\ (\alpha_1\cdot\alpha_2\cdot\Delta),\ \Delta\right\}$ *and* $0<\overline{\delta}<q\Delta-\frac{1}{2}$.

**Proof:** A step-by-step proof is included in the online Econometric Supplement. ∎

## A5 The examples in Section 2.2.4 satisfy Assumption 5

**Examples 1, 2 and 5 (FOSD, SOSD and CMI):** Recall that in these two examples we have $\mathcal{B}(Q)=Q$. Therefore, $|\mathcal{B}(\widehat{Q})-\mathcal{B}(Q)|=|\widehat{Q}-Q|$ and part (i) of Assumption 5 is immediately satisfied for any $M_2$, with $M_1=1$. Parts (ii) and (iii) are trivially satisfied since $\nabla_Q\mathcal{B}(Q)=1$.

**Example 3 (Covariance restrictions):** Here, $P=3$ and $\mathcal{B}(Q)=Q_1-Q_2\cdot Q_3$. We have, $\mathcal{B}(\widehat{Q})=\widehat{Q}_1-\widehat{Q}_2\cdot\widehat{Q}_3=Q_1-Q_2\cdot Q_3+(\widehat{Q}_1-Q_1)-Q_3\cdot(\widehat{Q}_2-Q_2)-\Big(Q_2+(\widehat{Q}_2-Q_2)\Big)\cdot(\widehat{Q}_3-Q_3)$. From here we obtain, $\left|\mathcal{B}(\widehat{Q})-\mathcal{B}(Q)\right|\le\Big(1+\|Q\|_\infty+\left\|\widehat{Q}-Q\right\|_\infty\Big)\cdot\left\|\widehat{Q}-Q\right\|_1$. Take any finite $M_2>0$. Then, for any $(x,t)\in\mathcal{X}\times\mathcal{T}$ and any $F\in\mathcal{F}$, if $\left\|\widehat{Q}(x,t,\widehat{\theta}))-Q_F(x,t,\theta_F^*)\right\|\le M_2$, then

$$\begin{aligned}\left|\mathcal{B}(\widehat{Q}(x,t,\widehat{\theta}))-\mathcal{B}(Q_F(x,t,\theta_F^*))\right|&\le\Big(1+\overline{Q}+c_a\cdot M_2\Big)\cdot c_b\left\|\widehat{Q}(x,t,\widehat{\theta}))-Q_F(x,t,\theta_F^*)\right\|\\&\equiv M_1\cdot\left\|\widehat{Q}(x,t,\widehat{\theta}))-Q_F(x,t,\theta_F^*)\right\|,\end{aligned}$$

where $M_1 \equiv \left(1+\overline{Q}+c_a\cdot M_2\right)\cdot c_b$. Thus, the first part of Assumption 5 is satisfied. Next,

$$\nabla_Q \mathcal{B}(Q) = \begin{pmatrix} 1 & -Q_3 & -Q_2 \end{pmatrix}, \quad \nabla_{QQ'}\mathcal{B}(Q) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

By inspection, $\|\nabla_{QQ'}\mathcal{B}(Q)\|$ is trivially bounded above by a constant $c$ for all $Q$ and we have $\|\nabla_Q\mathcal{B}(Q)\| \leq \|Q\|+1$. Therefore, $\|Q\| \leq \overline{Q}+C_Q$ immediately implies $\|\nabla_Q\mathcal{B}(Q)\| \leq \overline{Q}+C_Q+1$. Therefore, the second part of Assumption 5 is satisfied trivially, for example, if we take $\overline{H}_Q \equiv c \vee (\overline{Q}+C_Q+1)$. ■

**Example 4 (Affiliation):** In this case, $P=4$ and $\mathcal{B}(Q) = Q_1\cdot Q_2 - Q_3\cdot Q_4$. We have,

$$\begin{aligned}
\mathcal{B}(\widehat{Q}) &= \widehat{Q}_1\cdot\widehat{Q}_2 - \widehat{Q}_3\cdot\widehat{Q}_4 \\
&= \left(Q_1+(\widehat{Q}_1-Q_1)\right)\cdot\left(Q_2+(\widehat{Q}_2-Q_2)\right) - \left(Q_3+(\widehat{Q}_3-Q_3)\right)\cdot\left(Q_4+(\widehat{Q}_4-Q_4)\right) \\
&= Q_1\cdot Q_2 - Q_3\cdot Q_4 \\
&+\left(Q_2+(\widehat{Q}_2-Q_2)\right)\cdot\left(\widehat{Q}_1-Q_1\right)+Q_1\cdot\left(\widehat{Q}_2-Q_2\right)+\left(Q_4+(\widehat{Q}_4-Q_4)\right)\cdot\left(\widehat{Q}_3-Q_3\right)+Q_3\cdot\left(\widehat{Q}_4-Q_4\right).
\end{aligned}$$

As usual, for $b\in\mathbb{R}^n$ we define $\|b\|_\infty = \max_{i=1,\ldots,n}|b_i|$ and $\|b\|_1 = \sum_{i=1}^n |b_i|$. From the above expression we have

$$\begin{aligned}
\left|\mathcal{B}(\widehat{Q})-\mathcal{B}(Q)\right| &\leq \left(\|Q\|_\infty+\left\|\widehat{Q}-Q\right\|_\infty\right)\cdot\left(\left|\widehat{Q}_1-Q_1\right|+\left|\widehat{Q}_3-Q_3\right|\right)+\|Q\|_\infty\cdot\left(\left|\widehat{Q}_2-Q_2\right|+\left|\widehat{Q}_4-Q_4\right|\right) \\
&\leq \left(\|Q\|_\infty+\left\|\widehat{Q}-Q\right\|_\infty\right)\cdot\left(\sum_{p=1}^4\left|\widehat{Q}_p-Q_p\right|\right) = \left(\|Q\|_\infty+\left\|\widehat{Q}-Q\right\|_\infty\right)\cdot\left\|\widehat{Q}-Q\right\|_1
\end{aligned}$$

Therefore, for any $(x,t)\in\mathcal{X}\times\mathcal{T}$ and any $F\in\mathcal{F}$,

$$\left|\mathcal{B}(\widehat{Q}(x,t,\widehat{\theta}))-\mathcal{B}(Q_F(x,t,\theta_F^*))\right| \leq \left(\overline{Q}+\left\|\widehat{Q}(x,t,\widehat{\theta}))-Q_F(x,t,\theta_F^*)\right\|_\infty\right)\cdot\left\|\widehat{Q}(x,t,\widehat{\theta}))-Q_F(x,t,\theta_F^*)\right\|_1$$

By the equivalence of norms in $\mathbb{R}^n$, for any norm of choice $\|\cdot\|$ there exist constants $c_a$ and $c_b$ depending only on $n$ such that $\|\cdot\|_\infty \leq c_a\|\cdot\|$ and $\|\cdot\|_1 \leq c_b\|\cdot\|$. Take any finite $M_2>0$, then for any $(x,t)\in\mathcal{X}\times\mathcal{T}$ and any $F\in\mathcal{F}$, if $\left\|\widehat{Q}(x,t,\widehat{\theta}))-Q_F(x,t,\theta_F^*)\right\| \leq M_2$, then

$$\begin{aligned}
\left|\mathcal{B}(\widehat{Q}(x,t,\widehat{\theta}))-\mathcal{B}(Q_F(x,t,\theta_F^*))\right| &\leq \left(\overline{Q}+c_a\cdot M_2\right)\cdot\left\|\widehat{Q}(x,t,\widehat{\theta}))-Q_F(x,t,\theta_F^*)\right\|_1 \\
&\leq \left(\overline{Q}+c_a\cdot M_2\right)\cdot c_b\cdot\left\|\widehat{Q}(x,t,\widehat{\theta}))-Q_F(x,t,\theta_F^*)\right\| \\
&\equiv M_1\cdot\left\|\widehat{Q}(x,t,\widehat{\theta}))-Q_F(x,t,\theta_F^*)\right\|,
\end{aligned}$$

where $M_1 \equiv \left(\overline{Q} + c_a \cdot M_2\right) \cdot c_b$. Thus, the first part of Assumption 5 is satisfied. Next,

$$\nabla_Q \mathcal{B}(Q) = \begin{pmatrix} Q_2 & Q_1 & -Q_4 & -Q_3 \end{pmatrix}, \quad \nabla_{QQ'} \mathcal{B}(Q) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

By inspection, $\|\nabla_{QQ'} \mathcal{B}(Q)\|$ is bounded above by a constant $c$ for all $Q$ and, since $\|\nabla_Q \mathcal{B}(Q)\| = \|Q\|$, having $\|Q\| \leq \overline{Q} + C_Q$ immediately implies $\|\nabla_Q \mathcal{B}(Q)\| \leq \overline{Q} + C_Q$. Thus, the second part of Assumption 5 is satisfied trivially, e.g, by taking $\overline{H}_Q \equiv c \vee (\overline{Q} + C_Q)$. ■

## References


Nolan, D. and D. Pollard (1987). U-processes: Rates of convergence. *Annals of Statistics 15*, 780–799.

Serfling, R. J. (1980). *Approximation Theorems of Mathematical Statistics*. Wiley. New York, NY.

Sherman, R. (1994). Maximal inequalities for degenerate u-processes with applications to optimization estimators. *Annals of Statistics 22*, 439–459.