Appendix for "Inference in models with partially identified control functions"

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Abstract

This document includes the step-by-step proof of Theorem 1 in the paper, along with the description of our estimator for the variance of our test-statistic and its asymptotic properties. Every section in this document has the format **AX.X** and every equation has the format **(A-XX)**. Any section or equation that we reference here which does not have this format refers to a section or an equation in the paper.

A1 Proof of Theorem 1

We will focus for brevity on proving part (A) of Theorem 1. The proof of part (B) follows parallel and analogous steps, so we will just summarize it towards the end. Part (C) follows immediately from (A) and (B). We begin by presenting a maximal inequality result that will be useful throughout various steps of our proofs.

A1.1 A useful maximal inequality result

Let us begin by presenting once again the definition of Euclidean classes of functions. Our definition is taken from Nolan and Pollard (1987, Definition 8), Pakes and Pollard (1989, Definition 2.7), and Sherman (1994, Definition 3).

Definition: Euclidean classes of functions

Let $\mathcal{T}$ be a space and d be a pseudometric defined on $\mathcal{T}$. For each $\varepsilon > 0$, define the *packing number* $D(\varepsilon, d, \mathcal{T})$ to be the largest number D for which there exist points $m_1, \ldots, m_D$ in $\mathcal{T}$ such that $d(m_i, m_j) > \varepsilon$ for each $i \neq j$. Packing numbers are a measure of how big $\mathcal{T}$ is with respect to d. Let $\mathscr{G}$ be a class of functions defined on a set $\mathcal{S}_Z^k$. We say that G is an *envelope* for $\mathscr{G}$ is $\sup_{\mathscr{G}} |g(\cdot)| \leq G(\cdot)$. Let μ be a measure on $\mathcal{S}_Z^k$ and denote $\mu h \equiv \int h(z_1, \ldots, z_k) d\mu(z_1, \ldots, z_k)$. We say that the class of functions $\mathscr{G}$ is Euclidean (A, V) for the envelope G if, for any measure μ such that $\mu G^2 < \infty$, we have $D(\varepsilon, d_\mu, \mathscr{G}) \leq A\varepsilon^{-V}$ $\forall\ 0 < \varepsilon \leq 1$, where, for $g_1, g_2 \in \mathscr{G}$, $d_\mu(g_1, g_2) = \left(\mu|g_1 - g_2|^2/\mu G^2\right)^{1/2}$. The constants A and V must not depend on μ. ■

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A1.1.1 A maximal inequality for degenerate U-processes

The following result is taken from Sherman (1994), who obtained maximal inequalities for degenerate U-Processes. Let $Z_1,\ldots,Z_n$ be i.i.d observations from a distribution F on a set $\mathcal{S}_Z$. Let k be a positive integer and $\mathscr{G}$ a class of real-valued functions on $\mathcal{S}_Z^k = \mathcal{S}_Z \otimes \cdots \otimes \mathcal{S}_Z$ (k factors). For each $g \in \mathscr{G}$, define

$$U_n^k g = (n)_k^{-1} \sum_{\boldsymbol{i}_k} g(Z_{i_1},\ldots,Z_{i_k}),$$

where $(n)_k = n(n-1)\cdots(n-k+1)$ and $\sum_{\boldsymbol{i}_k}$ denotes the sum over the $(n)_k$ distinct integers $\{i_1,\ldots,i_k\}$ from the set $\{1,\ldots,n\}$. $U_n^k g$ is a U-statistic of order k and the collection $\{U_n^k g\colon g \in \mathscr{G}\}$ is called a U-process of order k, indexed by $\mathscr{G}$. If every $g \in \mathscr{G}$ is such that

$$\underbrace{E_F\left[g(s_1,\ldots,s_{i-1},Z,s_{i+1},\ldots,s_k)\right] \equiv 0}_{E_F[g(Z_1,\ldots,Z_k)|Z_1=s_1,\ldots,Z_{i-1}=s_{i-1},Z_{i+1}=s_{i+1},\ldots,Z_k=s_k]\equiv 0} \quad , \quad i=1,\ldots,k,$$

then $\mathscr{G}$ is called an $F-$*degenerate* class of functions on $\mathcal{S}_Z^k$ and $\{U_n^k g\colon g \in \mathscr{G}\}$ is a *degenerate U-process* of order k.

Result A1 ***(Sherman (1994, Corollary 4A))*** *Let $\mathscr{G}$ be a class of $F-$degenerate functions on $\mathcal{S}_Z^k$, $k \geq 1$. Suppose $\mathscr{G}$ is Euclidean (A,V) for an envelope G such that $E_F\left[G(Z_1,\ldots,Z_k)^{4p}\right] < \infty$ for a positive integer p. Then,*

$$E_F\left[\left(\sup_{\mathscr{G}} \left|n^{k/2} U_n^k g\right|\right)^p\right] \leq \Upsilon \cdot \left(E_F\left[G(Z_1,\ldots,Z_k)^{4p}\right]\right)^{1/2} \equiv \overline{M},$$

where Υ is a constant that depends only on p, A, V and $E_F\left[G(Z_1,\ldots,Z_k)^2\right]$. By a Chebyshev inequality, this implies that for each $\varepsilon > 0$,

$$P_F\left(\sup_{\mathscr{G}} \left|n^{k/2} U_n^k g\right| > \varepsilon\right) \leq \frac{\overline{M}}{\varepsilon^p} \quad \textit{and therefore} \quad P_F\left(\sup_{\mathscr{G}} \left|U_n^k g\right| > \varepsilon\right) \leq \frac{\overline{M}}{\left(n^{k/2}\cdot\varepsilon\right)^p}.$$

From the last result, we also have

$$\sup_{\mathscr{G}} \left|U_n^k g\right| = O_p\left(\frac{1}{n^{k/2}}\right). \quad \blacksquare$$

We will invoke Result A1 at various points throughout our proofs.

A1.2 Asymptotic properties of $\widehat{Q}_2$ and $\widehat{R}_2$

Note: In all the results that follow, $\epsilon > 0$ denotes the constant described in Assumption 4 of the paper.

Recall that, as described in equation (16) in the paper, for a given $v \equiv (v^c, v^d)$, we defined,

$$\mathcal{K}\left(\frac{V_i^c - v^c}{h_n}\right) \equiv \prod_{m=1}^{r} \kappa\left(\frac{V_{mi}^c - v_m^c}{h_n}\right), \quad \Gamma(V_i, v, h_n) \equiv \mathcal{K}\left(\frac{V_i^c - v^c}{h_n}\right) \cdot \mathbb{1}\left\{V_i^d = v^d\right\},$$

and, from here,

$$\widehat{R}_2(v) \equiv \frac{1}{n \cdot h_n^r} \sum_{i=1}^{n} Y_{2i} Y_{1i} \phi_2(V_i) \Gamma(V_i, v, h_n), \quad \widehat{Q}_2(v) \equiv \frac{1}{n \cdot h_n^r} \sum_{i=1}^{n} Y_{1i} \phi_2(V_i) \Gamma(V_i, v, h_n).$$

We proceed next to characterize the asymptotic properties of $\widehat{R}_2(v)$ and $\widehat{Q}_2(v)$ under our assumptions. Let $\lambda(\cdot)$ be a real-valued function of bounded variation on $\mathbb{R}$. By Nolan and Pollard (1987, Lemma 22) (or Pakes and Pollard (1989, Example 10)), the class $\mathscr{G}$ of all functions on $\mathbb{R}^d$ of the form $x \to \lambda(\alpha' x + \beta)$, with α ranging over $\mathbb{R}^d$ and β ranging over $\mathbb{R}$ is Euclidean for the constant envelope $G \equiv \sup|\lambda|$. Therefore, since our kernel is a function of bounded variation, the class of functions $\left\{m(v) = k\left(\frac{v-u}{h}\right) \text{ for some } u \in \mathbb{R}, h > 0\right\}$ is Euclidean (A_k, V_k) for the constant envelope $\overline{k}$ (neither (A_k, V_k), nor $\overline{k}$ depend on F). From here and Sherman (1994, Lemma 5), the following empirical processes $\nu_n^{Q_2}(\cdot)$ and $\nu_n^{R_2}(\cdot)$ defined as follows, satisfy the conditions of Result A1,

$$\begin{aligned} &\left\{\nu_n^{Q_2}(v,h) = \frac{1}{n}\sum_{i=1}^{n}\left(Y_{1i}\phi_2(V_i)\Gamma(V_i,v,h) - E_F\left[Y_1\phi_2(V)\Gamma(V,v,h)\right]\right) : v \in \mathbb{R}^{L_V}, h > 0\right\}, \\ &\left\{\nu_n^{R_2}(v,h) = \frac{1}{n}\sum_{i=1}^{n}\left(Y_{2i}Y_{1i}\phi_2(V_i)\Gamma(V_i,v,h) - E_F\left[Y_2Y_1\phi_2(V)\Gamma(V,v,h)\right]\right) : v \in \mathbb{R}^{L_V}, h > 0\right\} \end{aligned} \tag{A-1}$$

for the constant envelope $\overline{\phi} \cdot \overline{K}$, and the envelope $|Y_2| \cdot \overline{\phi} \cdot \overline{K}$, respectively. From here, Result A1 and the condition that $E_F[|Y_2|^4] \leq \overline{D}_2$ for all $F \in \mathcal{F}$ (Assumption 3) imply that there exists a finite $\overline{M}$ such that, for each $\varepsilon > 0$,

$$P_F\left(\sup_{\substack{v\in\mathbb{R}^{L_v} \\ h>0}} \left|\nu_n^{Q_2}(v,h)\right| > \varepsilon\right) \leq \frac{\overline{M}}{n^{1/2} \cdot \varepsilon}, \quad \text{and} \quad P_F\left(\sup_{\substack{v\in\mathbb{R}^{L_v} \\ h>0}} \left|\nu_n^{R_2}(v,h)\right| > \varepsilon\right) \leq \frac{\overline{M}}{n^{1/2} \cdot \varepsilon} \quad \forall\, F \in \mathcal{F} \tag{A-2}$$

and therefore,

$$\sup_{\substack{v\in\mathbb{R}^{L_v} \\ h>0}} \left|\nu_n^{Q_2}(v,h)\right| = O_p\left(\frac{1}{n^{1/2}}\right), \quad \text{and} \quad \sup_{\substack{v\in\mathbb{R}^{L_v} \\ h>0}} \left|\nu_n^{R_2}(v,h)\right| = O_p\left(\frac{1}{n^{1/2}}\right), \quad \text{uniformly over } \mathcal{F} \tag{A-3}$$

We have,

$$\widehat{Q}_2(v) - Q_{2F}(v) = \frac{1}{h_n^{r_v}} \cdot \nu_n^{Q_2}(v,h_n) + B_{n,F}^{Q_2}(v), \quad \text{where} \quad B_{n,F}^{Q_2}(v) \equiv \frac{1}{h_n^{r_v}} \cdot \Big(Q_{2F}(v) - E_F\left[Y_1 \phi_2(V)\Gamma(V,v,h_n)\right]\Big),$$
$$\widehat{R}_2(v) - R_{2F}(v) = \frac{1}{h_n^{r_v}} \cdot \nu_n^{R_2}(v,h_n) + B_{n,F}^{R_2}(v), \quad \text{where} \quad B_{n,F}^{R_2}(v) \equiv \frac{1}{h_n^{r_v}} \cdot \Big(R_{2F}(v) - E_F\left[Y_2 Y_1 \phi_2(V)\Gamma(V,v,h_n)\right]\Big), \tag{A-4}$$

The smoothness conditions in Assumption 2 and the kernel properties in Assumption 4 an M^{th}–order approximation implies that there exists a finite $\overline{B}$ such that

$$\sup_{v\in\mathcal{V}} \left|B_{n,F}^{Q_2}(v)\right| \leq \overline{B} \cdot h_n^M, \quad \text{and} \quad \sup_{v\in\mathcal{V}} \left|B_{n,F}^{R_2}(v)\right| \leq \overline{B} \cdot h_n^M \quad \forall\ F \in \mathcal{F} \tag{A-5}$$

From (A-4) and (A-5) we have,

$$\left.\begin{aligned} \sup_{v\in\mathcal{V}} \left|\widehat{Q}_2(v) - Q_{2F}(v)\right| &\leq \tfrac{1}{h_n^{r_v}} \cdot \sup_{v\in\mathcal{V}} \left|\nu_n^{Q_2}(v,h_n)\right| + \overline{B} \cdot h_n^M \\ \sup_{v\in\mathcal{V}} \left|\widehat{R}_2(v) - R_{2F}(v)\right| &\leq \tfrac{1}{h_n^{r_v}} \cdot \sup_{v\in\mathcal{V}} \left|\nu_n^{R_2}(v,h_n)\right| + \overline{B} \cdot h_n^M \end{aligned}\right\} \forall\ F \in \mathcal{F} \tag{A-6}$$

From (A-3) and (A-6), and the bandwidth convergence restrictions in Assumption 4, we have

$$\left.\begin{aligned} \sup_{v\in\mathcal{V}} \left|\widehat{Q}_2(v) - Q_{2F}(v)\right| &= O_p\left(\tfrac{1}{h_n^{r_v} \cdot n^{1/2}}\right) + \overline{B} \cdot h_n^M = o_p\left(\tfrac{1}{n^{1/4+\epsilon/2}}\right) \\ \sup_{v\in\mathcal{V}} \left|\widehat{R}_2(v) - R_{2F}(v)\right| &= O_p\left(\tfrac{1}{h_n^{r_v} \cdot n^{1/2}}\right) + \overline{B} \cdot h_n^M = o_p\left(\tfrac{1}{n^{1/4+\epsilon/2}}\right) \end{aligned}\right\} \text{uniformly over } \mathcal{F} \tag{A-7}$$

Where $\epsilon > 0$ denotes the constant described in Assumption 4. Take any sequence $\varepsilon_n > 0$ such that $n^{1/2} \cdot h_n^{r_v} \cdot \varepsilon_n \longrightarrow \infty$. Given the bandwidth convergence restrictions in Assumption 4, there exists $n_0 > 0$ such that $n^{1/2} \cdot h_n^{r_v} \cdot \varepsilon - \overline{B} \cdot n^{1/2} \cdot h_n^{r_v+M} > 0$, for all $n > n_0$, and from the results in (A-2) and (A-6),

$$\left.\begin{aligned} P_F\left(\sup_{v\in\mathcal{V}} \left|\widehat{Q}_2(v) - Q_{2F}(v)\right| > \varepsilon_n\right) &\leq \tfrac{\overline{M}}{n^{1/2} \cdot h_n^{r_v} \cdot \varepsilon_n - \overline{B} \cdot n^{1/2} \cdot h_n^{r_v+M}} \\ P_F\left(\sup_{v\in\mathcal{V}} \left|\widehat{R}_2(v) - R_{2F}(v)\right| > \varepsilon_n\right) &\leq \tfrac{\overline{M}}{n^{1/2} \cdot h_n^{r_v} \cdot \varepsilon_n - \overline{B} \cdot n^{1/2} \cdot h_n^{r_v+M}} \end{aligned}\right\} \forall\ F \in \mathcal{F},\ \forall\ n > n_0 \tag{A-8}$$

Therefore, under the conditions in Assumptions 2 and 4, we have

$$\left.\begin{aligned} \sup_{F\in\mathcal{F}} P_F\left(\sup_{v\in\mathcal{V}} \left|\widehat{Q}_2(v) - Q_{2F}(v)\right| > \varepsilon_n\right) &\longrightarrow 0 \\ \sup_{F\in\mathcal{F}} P_F\left(\sup_{v\in\mathcal{V}} \left|\widehat{R}_2(v) - R_{2F}(v)\right| > \varepsilon_n\right) &\longrightarrow 0 \end{aligned}\right\} \forall\ \varepsilon_n > 0 : n^{1/2} \cdot h_n^{r_v} \cdot \varepsilon_n \longrightarrow \infty$$

A1.3 Asymptotic properties of $\widehat{\tau}_2(v,v',\theta)$

We have,

$$\widehat{\tau}_2(v,v',\theta) =$$
$$\left(\left(\widehat{R}_2(v)\widehat{Q}_2(v') - \widehat{R}_2(v')\widehat{Q}_2(v)\right) - (g_2(x_2,\theta_2) - g_2(x_2',\theta_2))\,\widehat{Q}_2(v)\widehat{Q}_2(v')\right) \cdot \mathbb{1}\{g_{1U}(w_1',\theta_1) \leq g_{1L}(w_1,\theta_1)\}$$
$$\cdot\, \phi_2(v)\phi_2(v'),$$
$$\tau_{2F}(v,v',\theta) =$$
$$\left((R_{2F}(v)Q_{2F}(v') - R_{2F}(v')Q_{2F}(v)) - (g_2(x_2,\theta_2) - g_2(x_2',\theta_2))\,Q_{2F}(v)Q_{2F}(v')\right) \cdot \mathbb{1}\{g_{1U}(w_1',\theta_1) \leq g_{1L}(w_1,\theta_1)\}$$
$$\cdot\, \phi_2(v)\phi_2(v'),$$

Therefore,

$$\widehat{\tau}_2(v,v',\theta) - \tau_{2F}(v,v',\theta) =$$
$$\Bigg(\left(R_{2F}(v) - (g_2(x_2,\theta_2) - g_2(x_2',\theta_2))\,Q_{2F}(v)\right) \cdot \left(\widehat{Q}_2(v') - Q_{2F}(v')\right) - \left(R_{2F}(v') + (g_2(x_2,\theta_2) - g_2(x_2',\theta_2))\,Q_{2F}(v')\right)$$
$$\cdot \left(\widehat{Q}_2(v) - Q_{2F}(v)\right)$$
$$+\, Q_{2F}(v') \cdot \left(\widehat{R}_2(v) - R_{2F}(v)\right) - Q_{2F}(v) \cdot \left(\widehat{R}_2(v') - R_{2F}(v')\right)\Bigg) \cdot \mathbb{1}\{g_{1U}(w_1',\theta_1) \leq g_{1L}(w_1,\theta_1)\}\,\phi_2(v)\phi_2(v')$$
$$+\, \xi^{\tau_2}_{a,n}(v,v',\theta), \tag{A-9}$$

where

$$\xi^{\tau_2}_{a,n}(v,v',\theta) \equiv \Bigg(\left(\widehat{R}_2(v) - R_{2F}(v)\right) \cdot \left(\widehat{Q}_2(v') - Q_{2F}(v')\right) - \left(\widehat{R}_2(v') - R_{2F}(v')\right) \cdot \left(\widehat{Q}_2(v) - Q_{2F}(v)\right)$$
$$-\,(g_2(x_2,\theta_2) - g_2(x_2',\theta_2)) \cdot \left(\widehat{Q}_2(v) - Q_{2F}(v)\right) \cdot \left(\widehat{Q}_2(v') - Q_{2F}(v')\right)\Bigg) \cdot \mathbb{1}\{g_{1U}(w_1',\theta_1) \leq g_{1L}(w_1,\theta_1)\}\,\phi_2(v)\phi_2(v')$$

From the conditions in Assumption 2, there exists a finite constant $\overline{D}$ such that

$$\sup_{\substack{v,v' \in \mathbb{R}^{L_V} \times \mathbb{R}^{L_V} \\ \theta \in \Theta}} \max\left\{|R_{2F}(v)|,\ |Q_{2F}(v)|,\ \left|g_2(x_2,\theta_2) - g_2(x_2',\theta_2)\right|\right\} \leq 2\overline{D} \quad \forall\ F \in \mathcal{F}.$$

Therefore, there exists $\overline{D}_2$ such that, for each $F \in \mathcal{F}$,

$$\sup_{\substack{v,v' \in \mathbb{R}^{L_V} \times \mathbb{R}^{L_V} \\ \theta \in \Theta}} \left|\widehat{\tau}_2(v,v',\theta) - \tau_{2F}(v,v',\theta)\right| \leq \overline{D}_2 \cdot \Bigg(\sup_{v \in \mathcal{V}} \left|\widehat{Q}_2(v) - Q_{2F}(v)\right| + \sup_{v \in \mathcal{V}} \left|\widehat{R}_2(v) - R_{2F}(v)\right|$$
$$+ \sup_{v \in \mathcal{V}} \left|\widehat{Q}_2(v) - Q_{2F}(v)\right| \times \sup_{v \in \mathcal{V}} \left|\widehat{R}_2(v) - R_{2F}(v)\right| + \left(\sup_{v \in \mathcal{V}} \left|\widehat{Q}_2(v) - Q_{2F}(v)\right|\right)^2\Bigg)$$

Therefore, there exists a finite constant $\overline{C}_2$ such that, for any $b > 0$

$$P_F\left(\sup_{\substack{v,v'\in\mathbb{R}^{L_V}\times\mathbb{R}^{L_V}\\ \theta\in\Theta}}\left|\widehat{\tau}_2(v,v',\theta)-\tau_{2F}(v,v',\theta)\right|>b\right)\leq P_F\left(\sup_{v\in\mathcal{V}}\left|\widehat{Q}_2(v)-Q_{2F}(v)\right|>\overline{C}_2\cdot\left(b\wedge b^{1/2}\right)\right)$$
$$+P_F\left(\sup_{v\in\mathcal{V}}\left|\widehat{R}_2(v)-R_{2F}(v)\right|>\overline{C}_2\cdot\left(b\wedge b^{1/2}\right)\right)\quad\forall\ F\in\mathcal{F}$$

Fix $b > 0$. From the previous result, equation (A-8) implies that, under Assumptions 2, 3 and 4, there exist constants $\overline{M}$, $\overline{B}$ and $\overline{C}_2$ and n_0 such that, for $n > n_0$,

$$P_F\left(\sup_{\substack{v,v'\in\mathbb{R}^{L_V}\times\mathbb{R}^{L_V}\\ \theta\in\Theta}}\left|\widehat{\tau}_2(v,v',\theta)-\tau_{2F}(v,v',\theta)\right|>b\right)\leq\frac{2\overline{M}}{n^{1/2}\cdot h_n^{r_v}\cdot\overline{C}_2\cdot\left(b\wedge b^{1/2}\right)-\overline{B}\cdot n^{1/2}\cdot h_n^{r_v+M}}\quad\forall\ F\in\mathcal{F}$$

In particular, take any sequence $b_n > 0$ such that $b_n \longrightarrow 0$ and $n^{1/2}\cdot h_n^r\cdot b_n\longrightarrow\infty$. The previous result implies that, under Assumptions 2, 3 and 4, for any such sequence, we have,

$$\sup_{F\in\mathcal{F}}P_F\left(\sup_{\substack{v,v'\in\mathbb{R}^{L_V}\times\mathbb{R}^{L_V}\\ \theta\in\Theta}}\left|\widehat{\tau}_2(v,v',\theta)-\tau_{2F}(v,v',\theta)\right|>b_n\right)\longrightarrow 0. \tag{A-10}$$

Note that (A-10) immediately implies,

$$\sup_{\substack{v,v'\in\mathbb{R}^{L_V}\times\mathbb{R}^{L_V}\\ \theta\in\Theta}}\left|\widehat{\tau}_2(v,v',\theta)-\tau_{2F}(v,v',\theta)\right|=o_p(1),\quad\text{uniformly over }\mathcal{F}, \tag{A-11}$$

and

$$\sup_{\substack{v,v'\in\mathbb{R}^{L_V}\times\mathbb{R}^{L_V}\\ \theta\in\Theta}}\mathbb{1}\left\{\left|\widehat{\tau}_2(v,v',\theta)-\tau_{2F}(v,v',\theta)\right|>b_n\right\}=o_p(1),\quad\text{uniformly over }\mathcal{F}. \tag{A-12}$$

Next, note that

$$\begin{aligned}
&\left|\mathbb{1}\left\{\widehat{\tau}_2(v,v',\theta)\geq -b_n\right\}-\mathbb{1}\left\{\tau_{2F}(v,v',\theta)\geq 0\right\}\right|\\
=&\mathbb{1}\left\{\widehat{\tau}_2(v,v',\theta)\geq -b_n\ ,\ -2b_n\leq\tau_{2F}(v,v',\theta)<0\right\}+\mathbb{1}\left\{\widehat{\tau}_2(v,v',\theta)\geq -b_n\ ,\ \tau_{2F}(v,v',\theta)<-2b_n\right\}\\
+&\mathbb{1}\left\{\widehat{\tau}_2(v,v',\theta)<-b_n\ ,\ \tau_{2F}(v,v',\theta)\geq 0\right\}\\
\leq&\mathbb{1}\left\{-2b_n\leq\tau_{2F}(v,v',\theta)<0\right\}+\mathbb{1}\left\{\left|\widehat{\tau}_2(v,v',\theta)-\tau_{2F}(v,v',\theta)\right|\geq b_n\right\}
\end{aligned} \tag{A-13}$$

And, by the conditions in Assumption 2, there exists a finite constant $\overline{\tau}_2$ such that

$$\sup_{\substack{v,v' \in \mathbb{R}^{L_V} \times \mathbb{R}^{L_V} \\ \theta \in \Theta}} \left| \tau_{2F}(v,v',\theta) \right| \leq \overline{\tau}_2 \quad \forall\, F \in \mathcal{F}. \tag{A-14}$$

We have

$$\begin{aligned}\widehat{\tau}_2(v,v',\theta) \cdot \mathbb{1}\{\widehat{\tau}_2(v,v',\theta) \geq -b_n\} = & \left(\tau_{2F}(v,v',\theta)\right)_+ \\ & + \tau_{2F}(v,v',\theta) \cdot \Big(\mathbb{1}\{\widehat{\tau}_2(v,v',\theta) \geq -b_n\} - \mathbb{1}\{\tau_{2F}(v,v',\theta) \geq 0\} \Big) \\ & + \left(\widehat{\tau}_2(v,v',\theta) - \tau_{2F}(v,v',\theta)\right) \cdot \mathbb{1}\{\widehat{\tau}_2(v,v',\theta) \geq -b_n\}.\end{aligned}$$

From here, using the results in (A-13) and (A-14),

$$\begin{aligned}&\sup_{\substack{v,v' \in \mathbb{R}^{L_V} \times \mathbb{R}^{L_V} \\ \theta \in \Theta}} \left| \widehat{\tau}_2(v,v',\theta) \cdot \mathbb{1}\{\widehat{\tau}_2(v,v',\theta) \geq -b_n\} - \left(\tau_{2F}(v,v',\theta)\right)_+ \right| \\ &\leq \overline{\tau}_2 \cdot \underbrace{\sup_{\substack{v,v' \in \mathbb{R}^{L_V} \times \mathbb{R}^{L_V} \\ \theta \in \Theta}} \mathbb{1}\Big\{ \left| \widehat{\tau}_2(v,v',\theta) - \tau_{2F}(v,v',\theta) \right| > b_n \Big\}}_{o_p(1) \text{ uniformly over } \mathcal{F}, \text{ by (A-12)}} \\ &+ \underbrace{\sup_{\substack{v,v' \in \mathbb{R}^{L_V} \times \mathbb{R}^{L_V} \\ \theta \in \Theta}} \Big(|\tau_{2F}(v,v',\theta)| \cdot \mathbb{1}\{-2b_n \leq \tau_{2F}(v,v',\theta) < 0\} \Big)}_{\leq 2b_n \to 0 \text{ for all } F, \text{ by construction}} + \underbrace{\sup_{\substack{v,v' \in \mathbb{R}^{L_V} \times \mathbb{R}^{L_V} \\ \theta \in \Theta}} \left| \widehat{\tau}_2(v,v',\theta) - \tau_{2F}(v,v',\theta) \right|}_{o_p(1) \text{ uniformly over } \mathcal{F}, \text{ by (A-11)}}\end{aligned}$$

Therefore,

$$\sup_{\substack{v,v' \in \mathbb{R}^{L_V} \times \mathbb{R}^{L_V} \\ \theta \in \Theta}} \left| \widehat{\tau}_2(v,v',\theta) \cdot \mathbb{1}\{\widehat{\tau}_2(v,v',\theta) \geq -b_n\} - \left(\tau_{2F}(v,v',\theta)\right)_+ \right| = o_p(1), \quad \text{uniformly over } \mathcal{F}. \tag{A-15}$$

And from the definition of $\widehat{\mathcal{T}_2}(\theta)$ in equation (17), the result in (A-15) immediately implies,

$$\sup_{\theta \in \Theta} \left| \widehat{\mathcal{T}_2}(\theta) - \mathcal{T}_{2F}(\theta) \right| = o_p(1), \quad \text{uniformly over } \mathcal{F}. \tag{A-16}$$

Let us go back to (A-9). Plugging in (A-4) into (A-9), we have,

$$\begin{aligned}
&\widehat{\tau_2}(v,v',\theta)-\tau_{2F}(v,v',\theta)= \\
&\Big[\Big(R_{2F}(v)-(g_2(x_2,\theta_2)-g_2(x_2',\theta_2))\,Q_{2F}(v)\Big)\cdot\frac{1}{h_n^r}\cdot\nu_n^{Q_2}(v',h_n) \\
&-\Big(R_{2F}(v')+(g_2(x_2,\theta_2)-g_2(x_2',\theta_2))\,Q_{2F}(v')\Big)\cdot\frac{1}{h_n^r}\cdot\nu_n^{Q_2}(v,h_n) \\
&+Q_{2F}(v')\cdot\frac{1}{h_n^r}\cdot\nu_n^{R_2}(v,h_n)-Q_{2F}(v)\cdot\frac{1}{h_n^r}\cdot\nu_n^{R_2}(v',h_n)\Big]\cdot\mathbb{1}\{g_{1U}(w_1',\theta_1)\le g_{1L}(w_1,\theta_1)\}\,\phi_2(v)\phi_2(v') \\
&+\xi_{a,n}^{\tau_2}(v,v',\theta)+\xi_{b,n}^{\tau_2}(v,v',\theta),
\end{aligned}\tag{A-17}$$

where

$$\begin{aligned}
\xi_{b,n}^{\tau_2}(v,v',\theta)\equiv&\Big[\Big(R_{2F}(v)-(g_2(x_2,\theta_2)-g_2(x_2',\theta_2))\,Q_{2F}(v)\Big)\cdot B_{n,F}^{Q_2}(v') \\
&-\Big(R_{2F}(v')+(g_2(x_2,\theta_2)-g_2(x_2',\theta_2))\,Q_{2F}(v')\Big)\cdot B_{n,F}^{Q_2}(v) \\
&+Q_{2F}(v')\cdot B_{n,F}^{R_2}(v)-Q_{2F}(v)\cdot B_{n,F}^{R_2}(v')\Big]\cdot\mathbb{1}\{g_{1U}(w_1',\theta_1)\le g_{1L}(w_1,\theta_1)\}\,\phi_2(v)\phi_2(v')
\end{aligned}$$

By the conditions in Assumption 2 and the result in (A-5), there exist finite constants $\overline{D}_2$ and $\overline{B}$ such that,

$$\begin{aligned}
\sup_{\substack{v,v'\in\mathbb{R}^{L_V}\times\mathbb{R}^{L_V}\\ \theta\in\Theta}}\left|\xi_{b,n}^{\tau_2}(v,v',\theta)\right|\le\overline{D}_2\cdot\left(\sup_{v\in\mathcal{V}}\left|B_{n,F}^{Q_2}(v)\right|+\sup_{v\in\mathcal{V}}\left|B_{n,F}^{Q_2}(v)\right|\right)\le 2\cdot\overline{D}_2\cdot\overline{B}\cdot h_n^M \\
\equiv\overline{B}_3\cdot h_n^M=o\left(\frac{1}{n^{1/2+\epsilon}}\right)\quad\forall\,F\in\mathcal{F}
\end{aligned}\tag{A-18}$$

where the last equality follows from Assumption 4, and $\epsilon>0$ is the constant described there. Next we turn our attention to $\xi_{a,n}^{\tau_2}(v,v',\theta)$. By the conditions in Assumption 2, there exists a finite constant $\overline{D}$ such that,

$$\sup_{\substack{v,v'\in\mathbb{R}^{L_V}\times\mathbb{R}^{L_V}\\ \theta\in\Theta}}\left|\xi_{a,n}^{\tau_2}(v,v',\theta)\right|\le 2\cdot\sup_{v\in\mathcal{V}}\left|\widehat{Q}_2(v)-Q_{2F}(v)\right|\times\sup_{v\in\mathcal{V}}\left|\widehat{R}_2(v)-R_{2F}(v)\right|+\overline{D}\cdot\left(\sup_{v\in\mathcal{V}}\left|\widehat{Q}_2(v)-Q_{2F}(v)\right|\right)^2,$$

for all $F\in\mathcal{F}$. And from here, the result in (A-7) yields,

$$\sup_{\substack{v,v'\in\mathbb{R}^{L_V}\times\mathbb{R}^{L_V}\\ \theta\in\Theta}}\left|\xi_{a,n}^{\tau_2}(v,v',\theta)\right|=\left[o_p\left(\frac{1}{n^{1/4+\epsilon/2}}\right)\right]^2=o_p\left(\frac{1}{n^{1/2+\epsilon}}\right),\quad\text{uniformly over }\mathcal{F}.\tag{A-19}$$

Where $\epsilon>0$ denotes the constant described in Assumption 4. For a given $y_2\in\mathbb{R}$, $y_1\in\{0,1\}$ and

$u, u' \in \mathbb{R}^{L_V} \times \mathbb{R}^{L_V}$ and $h > 0$, let

$$\begin{aligned} \varphi_F^{Q_2}(y_1, u, u', h) &\equiv y_1 \cdot \phi_2(u) \cdot \Gamma(u, u', h) - E_F\left[Y_1 \phi_2(V) \cdot \Gamma(V, u', h)\right], \\ \varphi_F^{R_2}(y_2, y_1, u, u', h) &\equiv y_2 \cdot y_1 \cdot \phi_2(u) \cdot \Gamma(u, u', h) - E_F\left[Y_2 Y_1 \phi_2(V) \cdot \Gamma(V, u', h)\right], \end{aligned} \tag{A-20}$$

and for a given $v, v' \in \mathbb{R}^{L_V} \times \mathbb{R}^{L_V}$ and $\theta \in \Theta$, let

$$\begin{aligned} \zeta_{a,F}^{\tau_2}(v, v', \theta) &\equiv \left(R_{2F}(v) - (g_2(x_2, \theta_2) - g_2(x_2', \theta_2))\, Q_{2F}(v)\right) \cdot \mathbb{1}\{g_{1U}(w_1', \theta_1) \le g_{1L}(w_1, \theta_1)\}\, \phi_2(v)\phi_2(v'), \\ \zeta_{b,F}^{\tau_2}(v, v', \theta) &\equiv \left(R_{2F}(v') + (g_2(x_2, \theta_2) - g_2(x_2', \theta_2))\, Q_{2F}(v')\right) \cdot \mathbb{1}\{g_{1U}(w_1', \theta_1) \le g_{1L}(w_1, \theta_1)\}\, \phi_2(v)\phi_2(v'), \\ \zeta_{c,F}^{\tau_2}(v, v', \theta) &\equiv Q_{2F}(v') \cdot \mathbb{1}\{g_{1U}(w_1', \theta_1) \le g_{1L}(w_1, \theta_1)\}\, \phi_2(v)\phi_2(v'), \\ \zeta_{d,F}^{\tau_2}(v, v', \theta) &\equiv Q_{2F}(v) \cdot \mathbb{1}\{g_{1U}(w_1', \theta_1) \le g_{1L}(w_1, \theta_1)\}\, \phi_2(v)\phi_2(v'), \end{aligned}$$

and

$$\begin{aligned} \zeta_F^{\tau_2}(Y_2, Y_1, V, v, v', \theta, h) \equiv\ & \zeta_{a,F}^{\tau_2}(v, v', \theta)\varphi_F^{Q_2}(Y_1, V, v', h) - \zeta_{b,F}^{\tau_2}(v, v', \theta)\varphi_F^{Q_2}(Y_1, V, v, h) \\ & + \zeta_{c,F}^{\tau_2}(v, v', \theta)\varphi_F^{R_2}(Y_2, Y_1, V, v, h) - \zeta_{d,F}(v, v', \theta)^{\tau_2}\varphi_F^{R_2}(Y_2, Y_1, V, v', h) \end{aligned}$$

Note that $E_F[\zeta_F^{\tau_2}(Y_2, Y_1, V, v, v', \theta, h)] = 0$ for all (v, v', θ, h). Plugging in (A-18) and (A-19) into (A-17), and using the definitions of $\nu_n^{Q_2}(\cdot)$ and $\nu_n^{R_2}(\cdot)$ given in (A-1), we have

$$\begin{aligned} \widehat{\tau}_2(v, v', \theta) - \tau_{2F}(v, v', \theta) &= \frac{1}{h_n^{r_v}} \cdot \frac{1}{n}\sum_{k=1}^{n} \zeta_F^{\tau_2}(Y_{2k}, Y_{1k}, V_k, v, v', \theta, h_n) + \xi_n^{\tau_2}(v, v', \theta), \\ & \text{where} \sup_{\substack{v, v' \in \mathbb{R}^{L_V} \times \mathbb{R}^{L_V} \\ \theta \in \Theta}} \left|\xi_n^{\tau_2}(v, v', \theta)\right| = o_p\left(\frac{1}{n^{1/2+\epsilon}}\right), \quad \text{uniformly over } \mathcal{F} \end{aligned} \tag{A-21}$$

Where $\epsilon > 0$ denotes the constant described in Assumption 4. Let

$$\alpha_F^{\tau_2}(v, v', \theta) \equiv \left(\zeta_{a,F}^{\tau_2}(v, v', \theta),\ \zeta_{b,F}^{\tau_2}(v, v', \theta),\ \zeta_{c,F}^{\tau_2}(v, v', \theta),\ \zeta_{d,F}^{\tau_2}(v, v', \theta)\right).$$

By the conditions in Assumption 2, there exists a finite constant $\overline{M}_2$ such that

$$\sup_{\substack{v, v' \in \mathbb{R}^{L_V} \times \mathbb{R}^{L_V} \\ \theta \in \Theta}} \left\|\alpha_F^{\tau_2}(v, v', \theta)\right\| \le \overline{M}_2 \quad \forall\ F \in \mathcal{F}.$$

Consider the class of functions,

$$\begin{aligned} \mathscr{H}_{1,F} \equiv \Big\{ & m(y_2, y_1, v) = \alpha_1\varphi_F^{Q_2}(y_1, v, u', h) + \alpha_2\varphi_F^{Q_2}(y_1, v, u, h) + \alpha_3\varphi_F^{R_2}(y_2, y_1, v, u, h) + \alpha_4\varphi_F^{R_2}(y_2, y_1, v, u', h) : \\ & u, u' \in \mathbb{R}^{L_V} \times \mathbb{R}^{L_V}\ , \theta \in \Theta\ , h > 0\ , \|(\alpha_1, \alpha_2, \alpha_3, \alpha_4)\| \le \overline{M}_2 \Big\} \end{aligned}$$

By Nolan and Pollard (1987, Lemma 22) (or Pakes and Pollard (1989, Example 10)), and Pakes and Pollard (1989, Lemma 2.14) and the bounded-variation properties of the weight function $\phi_2(\cdot)$ and the kernel $K(\cdot)$, there exist constants (A,V) such that $\mathscr{H}_{1,F}$ is Euclidean (A,V) for all $F \in \mathcal{F}$, for an envelope of the form $H_1 = C_1 + \overline{C}_2 \cdot |Y_2|$, where C_1 and C_2 are constant for all F. Now define,

$$\mathscr{G}_{1,F} \equiv \Big\{ m(y_2, y_1, v) = \zeta_F^{\tau_2}(y_2, y_1, v, u, u', \theta, h) : u, u' \in \mathbb{R}^{L_V} \times \mathbb{R}^{L_V} \ , \theta \in \Theta \ , h > 0 \Big\}.$$

Note that $\mathscr{G}_{1,F} \subseteq \mathscr{H}_{1,F}$. Therefore, there exist constants (A,V) such that $\mathscr{G}_{1,F}$ is Euclidean (A,V) for all $F \in \mathcal{F}$, for an envelope of the form $H_1 = C_1 + C_2 \cdot |Y_2|$, where C_1 and C_2 are constant for all F. Define the empirical process $\nu_n^{\tau_2}(\cdot)$ given by,

$$\left\{ \nu_n^{\tau_2}(u, u', \theta, h) = \frac{1}{n} \sum_{i=1}^{n} \zeta_F^{\tau_2}(Y_{2i}, Y_{1i}, V_i, u, u', \theta, h) : u, u' \in \mathbb{R}^{L_V} \times \mathbb{R}^{L_V} \ , \theta \in \Theta \ , h > 0 \right\}.$$

$\nu_n^{\tau_2}(\cdot)$ satisfies the conditions of Result A1. Since there exists a finite constant $\overline{D}_4$ such that $E_F[|Y_2|^4] \leq \overline{D}_4$ for all $F \in \mathcal{F}$ by Assumption 3, Result A1 implies that there exists a constant $\overline{M}$ such that, for each $\varepsilon > 0$,

$$P_F\left(\sup_{\substack{u,u' \in \mathbb{R}^{L_V} \times \mathbb{R}^{L_V} \\ \theta \in \Theta, h > 0}} \left| \nu_n^{\tau_2}(u, u', \theta, h) \right| > \varepsilon \right) \leq \frac{\overline{M}}{n^{1/2} \varepsilon} \quad \forall \ F \in \mathcal{F}.$$

Therefore,

$$\sup_{\substack{u,u' \in \mathbb{R}^{L_V} \times \mathbb{R}^{L_V} \\ \theta \in \Theta, h > 0}} \left| \nu_n^{\tau_2}(u, u', \theta, h) \right| = O_p\left(\frac{1}{n^{1/2}} \right) \ \text{uniformly over } \mathcal{F}.$$

From here, (A-21) yields,

$$\begin{aligned} \sup_{\substack{v,v' \in \mathbb{R}^{L_V} \times \mathbb{R}^{L_V} \\ \theta \in \Theta}} \left| \widehat{\tau}_2(v, v', \theta) - \tau_{2F}(v, v', \theta) \right| &= O_p\left(\frac{1}{h_n^{r_v} \cdot n^{1/2}} \right) + o_p\left(\frac{1}{n^{1/2+\epsilon}} \right) \\ &= o_p\left(\frac{1}{n^{1/4+\epsilon/2}} \right) \quad \text{uniformly over } \mathcal{F}. \end{aligned} \tag{A-22}$$

Where $\epsilon > 0$ is the constant described in Assumption 4. By the conditions in Assumption 2, there exists a finite constant $\overline{\tau}_2$ such that

$$\sup_{\substack{v,v' \in \mathbb{R}^{L_V} \times \mathbb{R}^{L_V} \\ \theta \in \Theta}} \left| \tau_{2F}(v, v', \theta) \right| \leq \overline{\tau}_2 \quad \forall \ F \in \mathcal{F}.$$

From here and (A-22), we obtain,

$$\sup_{\substack{v,v'\in\mathbb{R}^{L_V}\times\mathbb{R}^{L_V}\\ \theta\in\Theta}} \left|\widehat{\tau}_2(v,v',\theta)\right| = O_p(1) \quad \text{uniformly over } \mathcal{F}. \tag{A-23}$$

The results in (A-10), (A-21), (A-22) and (A-23) summarize the relevant asymptotic properties of $\widehat{\tau}_2(v,v',\theta)$ for our problem.

A1.4 Asymptotic properties of $\widehat{\mathcal{T}}_2(\theta)$

Recall that,

$$\widehat{\mathcal{T}}_2(\theta) \equiv \frac{1}{n\cdot(n-1)}\sum_{i=1}^{n}\sum_{j\neq i}\widehat{\tau}_2(V_i,V_j,\theta)\cdot\mathbb{1}\left\{\widehat{\tau}_2(V_i,V_j,\theta)\geq -b_n\right\}.$$

Let

$$\widetilde{\mathcal{T}}_2(\theta) \equiv \frac{1}{n\cdot(n-1)}\sum_{i=1}^{n}\sum_{j\neq i}\widehat{\tau}_2(V_i,V_j,\theta)\cdot\mathbb{1}\left\{\tau_{2F}(V_i,V_j,\theta)\geq 0\right\}.$$

Note that $\widetilde{\mathcal{T}}_2(\theta)$ takes $\widehat{\mathcal{T}}_2(\theta)$ and replaces the indicator function $\mathbb{1}\left\{\widehat{\tau}_2(V_i,V_j,\theta)\geq -b_n\right\}$ with the indicator function $\mathbb{1}\left\{\tau_{2F}(V_i,V_j,\theta)\geq 0\right\}$. Our first step is to analyze $\widehat{\mathcal{T}}_2(\theta)-\widetilde{\mathcal{T}}_2(\theta)$. Denote,

$$r_{n,F}^{\mathcal{T}_2}(\theta) \equiv \widehat{\mathcal{T}}_2(\theta)-\widetilde{\mathcal{T}}_2(\theta) = \frac{1}{n\cdot(n-1)}\sum_{i=1}^{n}\sum_{j\neq i}\widehat{\tau}_2(V_i,V_j,\theta)\cdot\left[\mathbb{1}\left\{\widehat{\tau}_2(V_i,V_j,\theta)\geq -b_n\right\}-\mathbb{1}\left\{\tau_{2F}(V_i,V_j,\theta)\geq 0\right\}\right].$$

Thus,

$$\left|r_{n,F}^{\mathcal{T}_2}(\theta)\right| \leq \frac{1}{n\cdot(n-1)}\sum_{i=1}^{n}\sum_{j\neq i}\left|\widehat{\tau}_2(V_i,V_j,\theta)\right|\cdot\left|\mathbb{1}\left\{\widehat{\tau}_2(V_i,V_j,\theta)\geq -b_n\right\}-\mathbb{1}\left\{\tau_{2F}(V_i,V_j,\theta)\geq 0\right\}\right|.$$

As we pointed out in (A-13), we have

$$\begin{aligned}
&\left|\mathbb{1}\left\{\widehat{\tau}_2(V_i,V_j,\theta)\geq -b_n\right\}-\mathbb{1}\left\{\tau_{2F}(V_i,V_j,\theta)\geq 0\right\}\right|\\
=&\mathbb{1}\left\{\widehat{\tau}_2(V_i,V_j,\theta)\geq -b_n\ ,\ -2b_n\leq\tau_{2F}(V_i,V_j,\theta)<0\right\}+\mathbb{1}\left\{\widehat{\tau}_2(V_i,V_j,\theta)\geq -b_n\ ,\ \tau_{2F}(V_i,V_j,\theta)<-2b_n\right\}\\
+&\mathbb{1}\left\{\widehat{\tau}_2(V_i,V_j,\theta)< -b_n\ ,\ \tau_{2F}(V_i,V_j,\theta)\geq 0\right\}\\
\leq&\mathbb{1}\left\{-2b_n\leq\tau_{2F}(V_i,V_j,\theta)<0\right\}+\mathbb{1}\left\{\left|\widehat{\tau}_2(V_i,V_j,\theta)-\tau_{2F}(V_i,V_j,\theta)\right|\geq b_n\right\}
\end{aligned} \tag{A-24}$$

From here, we have,

$$
\begin{aligned}
&\left|r_{n,F}^{\mathcal{T}_2}(\theta)\right| \\
\leq&\frac{1}{n\cdot(n-1)}\sum_{i=1}^{n}\sum_{j\neq i}\left|\widehat{\tau}_2(V_i,V_j,\theta)\right|\cdot\mathbb{1}\left\{-2b_n\leq\tau_{2F}(V_i,V_j,\theta)<0\right\} \\
+&\frac{1}{n\cdot(n-1)}\sum_{i=1}^{n}\sum_{j\neq i}\left|\widehat{\tau}_2(V_i,V_j,\theta)\right|\cdot\mathbb{1}\left\{\left|\widehat{\tau}_2(V_i,V_j,\theta)-\tau_{2F}(V_i,V_j,\theta)\right|\geq b_n\right\} \\
\leq&\frac{1}{n\cdot(n-1)}\sum_{i=1}^{n}\sum_{j\neq i}\Big(\left|\tau_2(V_i,V_j,\theta)\right|+\left|\widehat{\tau}_2(V_i,V_j,\theta)-\tau_{2F}(V_i,V_j,\theta)\right|\Big)\cdot\mathbb{1}\left\{-2b_n\leq\tau_{2F}(V_i,V_j,\theta)<0\right\} \\
+&\frac{1}{n\cdot(n-1)}\sum_{i=1}^{n}\sum_{j\neq i}\left|\widehat{\tau}_2(V_i,V_j,\theta)\right|\cdot\mathbb{1}\left\{\left|\widehat{\tau}_2(V_i,V_j,\theta)-\tau_{2F}(V_i,V_j,\theta)\right|\geq b_n\right\}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\left|r_{n,F}^{\mathcal{T}_2}(\theta)\right| \\
\leq&\left(2b_n+\sup_{\substack{v,v'\in\mathbb{R}^{L_V}\times\mathbb{R}^{L_V}\\ \theta\in\Theta}}\left|\widehat{\tau}_2(v,v',\theta)-\tau_{2F}(v,v',\theta)\right|\right)\times\frac{1}{n\cdot(n-1)}\sum_{i=1}^{n}\sum_{j\neq i}\mathbb{1}\left\{-2b_n\leq\tau_{2F}(V_i,V_j,\theta)<0\right\} \\
+&\sup_{\substack{v,v'\in\mathbb{R}^{L_V}\times\mathbb{R}^{L_V}\\ \theta\in\Theta}}\left|\widehat{\tau}_2(v,v',\theta)\right|\times\frac{1}{n\cdot(n-1)}\sum_{i=1}^{n}\sum_{j\neq i}\mathbb{1}\left\{\left|\widehat{\tau}_2(V_i,V_j,\theta)-\tau_{2F}(V_i,V_j,\theta)\right|\geq b_n\right\}
\end{aligned}
$$

From here and the results in (A-22) and (A-23), uniformly over $\mathcal{F}$, we have

$$
\begin{aligned}
\sup_{\theta\in\Theta}\left|r_{n,F}^{\mathcal{T}_2}(\theta)\right|\leq&\left(2b_n+o_p\left(\frac{1}{n^{1/4+\epsilon/2}}\right)\right)\times\sup_{\theta\in\Theta}\left|\frac{1}{n\cdot(n-1)}\sum_{i=1}^{n}\sum_{j\neq i}\mathbb{1}\left\{-2b_n\leq\tau_{2F}(V_i,V_j,\theta)<0\right\}\right| \\
&+O_p(1)\times\sup_{\theta\in\Theta}\left|\frac{1}{n\cdot(n-1)}\sum_{i=1}^{n}\sum_{j\neq i}\mathbb{1}\left\{\left|\widehat{\tau}_2(V_i,V_j,\theta)-\tau_{2F}(V_i,V_j,\theta)\right|\geq b_n\right\}\right|
\end{aligned}
\tag{A-25}
$$

Where $\epsilon>0$ is the constant described in Assumption 4. Let us analyze each term on the right hand side of (A-25). In what follows, let V_1,V_2 be independent draws from the distribution F. For a given $\theta\in\Theta$ and $b>0$, let

$$
g_{2F}(V_1,V_2,\theta,b)\equiv\frac{1}{2}\Big(\mathbb{1}\left\{-2b\leq\tau_{2F}(V_1,V_2,\theta)<0\right\}+\mathbb{1}\left\{-2b\leq\tau_{2F}(V_2,V_1,\theta)<0\right\}\Big).
$$

$g_{2F}(V_1,V_2,\theta,b)$ is symmetric in V_1,V_2 by construction. Note that

$$\frac{1}{n\cdot(n-1)}\sum_{i=1}^{n}\sum_{j\neq i}\mathbb{1}\left\{-2b\leq\tau_{2F}(V_i,V_j,\theta)<0\right\}=\binom{2}{n}^{-1}\sum_{i<j}g_{2F}(V_i,V_j,\theta,b)\equiv S^{g}_{2,n}(\theta,b).$$

We will focus on the properties of the U-process $\left\{S^{g}_{2,n}(\theta,b):\theta\in\Theta,\,0<b\leq\frac{c_0}{2}\right\}$, where c_0 is the constant described in Assumption 3. We will proceed by analyzing the Hoeffding decomposition (see Serfling (1980, pages 177-178) or Sherman (1994, equations (6)-(7))) of $S^{g}_{2,n}(\theta,b)$. Let

$$\mu^{g}_{2F}(\theta,b)\equiv E_F\left[\mathbb{1}\{-2b\leq\tau_{2F}(V_1,V_2,\theta)<0\}\right],$$

Note that $\mu^{g}_{2F}(\theta,b)=E_F\left[g_{2F}(V_1,V_2,\theta,b)\right]$ by symmetry. Let

$$\begin{aligned}\widetilde{g}_{2F}(V_1,V_2,\theta,b)&\equiv g_{2F}(V_1,V_2,\theta,b)-\mu^{g}_{2F}(\theta,b),\\ \widetilde{m}_{1,F}(V_1,\theta,b)&\equiv E_F\left[\widetilde{g}_{2F}(V_1,V_2,\theta,b)|V_1\right],\\ \widetilde{m}_{2,F}(V_1,V_2,\theta,b)&\equiv\widetilde{g}_{2F}(V_1,V_2,\theta,b)-\widetilde{m}_{1,F}(V_1,\theta,b)-\widetilde{m}_{1,F}(V_2,\theta,b),\end{aligned}$$

The Hoeffding decomposition of $S^{g}_{2,n}(\theta,b)$ (see Serfling (1980, pages 177-178) or Sherman (1994, equations (6)-(7))) is given by,

$$S^{g}_{2,n}(\theta,b)=\mu^{g}_{2F}(\theta,b)+\frac{2}{n}\sum_{i=1}^{n}\widetilde{m}_{1,F}(V_i,\theta,b)+\binom{n}{2}^{-1}\sum_{i<j}\widetilde{m}_{2,F}(V_i,V_j,\theta,b).\tag{A-26}$$

Let us analyze the second and third terms on the right-hand side of (A-26). By the properties of VC classes of sets described, e.g, in Pakes and Pollard (1989, Lemma 2.5), the conditions described in Assumption 3 imply that, for each $F\in\mathcal{F}$, the following class of sets is a VC class, with VC dimension uniformly bounded over $\mathcal{F}$ by a finite constant,

$$\mathscr{D}^{\tau_2}_{2,F}\equiv\left\{(v_1,v_2)\in\mathbb{R}^{L_v}\times\mathbb{R}^{L_v}:\,-c\leq\tau_{2F}(v_1,v_2,\theta)<0\ \text{ for some } 0<c\leq c_0 \text{ and } \theta\in\Theta\right\},$$

where the constant c_0 is as described in Assumption 3. From here, the result in Pakes and Pollard (1989, p. 1033) implies that there exist constants $(\overline{A},\overline{V})$ such that, for each $F\in\mathcal{F}$, the class of indicator functions,

$$\mathscr{M}_F\equiv\left\{m(v_1,v_2)=\mathbb{1}\{-c\leq\tau_{2F}(v_1,v_2,\theta)<0\}\ \text{ for some } 0<c\leq c_0 \text{ and } \theta\in\Theta\right\}$$

is Euclidean $(\overline{A}, \overline{V})$ for the constant envelope 1. From here and Sherman (1994, Lemma 5), the conditions for Result A1 are satisfied and, from there, we obtain,

$$\left.\begin{aligned} &\sup_{\substack{\theta\in\Theta \\ 0<b\leq\frac{c_0}{2}}} \left|\tfrac{1}{n}\textstyle\sum_{i=1}^{n} \widetilde{m}_{1,F}(V_i,\theta,b)\right| = O_p\left(\tfrac{1}{n^{1/2}}\right) \\ &\sup_{\substack{\theta\in\Theta \\ 0<b\leq\frac{c_0}{2}}} \left|\tbinom{n}{2}^{-1}\sum_{i<j} \widetilde{m}_{2,F}(V_i,V_j,\theta,b)\right| = O_p\left(\tfrac{1}{n}\right) \end{aligned}\right\} \text{uniformly over } \mathcal{F}. \tag{A-27}$$

Combining (A-27) and (A-26), we have

$$S^{g}_{2,n}(\theta,b) = \mu^{g}_{2F}(\theta,b) + \xi^{g}_{n}(\theta,b), \quad \text{where} \quad \sup_{\substack{\theta\in\Theta \\ 0<b\leq\frac{c_0}{2}}} \left|\xi^{g}_{n}(\theta,b)\right| = O_p\left(\frac{1}{n^{1/2}}\right), \quad \text{uniformly over } \mathcal{F}.$$

Next, recall that, from Assumption 5, there exists $b_0 > 0$ and $\overline{m} < \infty$ such that,

$$\sup_{\theta\in\Theta} \left|\mu^{g}_{2F}(\theta,b)\right| \leq \overline{m}\cdot b \quad \forall\ 0 < b \leq b_0, \quad \forall\ F \in \mathcal{F}.$$

Next note that there exists n_0 such that $b_n < \left(\frac{c_0}{2}\right) \wedge b_0$ for all $n > n_0$. Therefore, for all $n > n_0$,

$$\begin{aligned} \sup_{\theta\in\Theta}\left|\frac{1}{n\cdot(n-1)}\sum_{i=1}^{n}\sum_{j\neq i}\mathbb{1}\left\{-2b_n \leq \tau_{2F}(V_i,V_j,\theta) < 0\right\}\right| &\leq \overline{m}\cdot b_n + \sup_{\substack{\theta\in\Theta \\ 0<b\leq\frac{c_0}{2}}} \left|\xi^{g}_{n}(\theta,b)\right| = O(b_n) + O_p\left(\frac{1}{n^{1/2}}\right) \\ &= b_n \times \left(O(1) + o_p\left(\frac{1}{b_n\cdot n^{1/2}}\right)\right) \\ &= b_n \times \left(O(1) + o_p(1)\right) \\ &= O_p(b_n), \quad \text{uniformly over } \mathcal{F}. \end{aligned}$$

Thus, uniformly over $\mathcal{F}$, we have

$$\begin{aligned} \left(2b_n + o_p\left(\frac{1}{n^{1/4+\epsilon/2}}\right)\right)\times\sup_{\theta\in\Theta}\left|\frac{1}{n\cdot(n-1)}\sum_{i=1}^{n}\sum_{j\neq i}\mathbb{1}\left\{-2b_n \leq \tau_{2F}(V_i,V_j,\theta) < 0\right\}\right| &= \left(2b_n + o_p\left(\frac{1}{n^{1/4+\epsilon/2}}\right)\right)\times O_p(b_n) \\ &= O_p\left(b_n^2\right) + o_p\left(\frac{b_n}{n^{1/4+\epsilon/2}}\right) \\ &= o_p\left(\frac{1}{n^{1/2+\epsilon}}\right), \end{aligned}$$

where $\epsilon > 0$ is the constant described in Assumption 4 Going back to (A-25), this result implies that, uniformly over $\mathcal{F}$,

$$\begin{aligned}\sup_{\theta\in\Theta}\left|r^{\mathcal{T}_2}_{n,F}(\theta)\right| \leq \left(2b_n + o_p\left(\frac{1}{n^{1/4+\epsilon/2}}\right)\right)\times\sup_{\theta\in\Theta}\left|\frac{1}{n\cdot(n-1)}\sum_{i=1}^{n}\sum_{j\neq i}\mathbb{1}\left\{-2b_n \leq \tau_{2F}(V_i,V_j,\theta) < 0\right\}\right| \\ + O_p(1)\times\sup_{\theta\in\Theta}\left|\frac{1}{n\cdot(n-1)}\sum_{i=1}^{n}\sum_{j\neq i}\mathbb{1}\left\{\left|\widehat{\tau}_2(V_i,V_j,\theta)-\tau_{2F}(V_i,V_j,\theta)\right| \geq b_n\right\}\right| \\ = O_p(1)\times\sup_{\theta\in\Theta}\left|\frac{1}{n\cdot(n-1)}\sum_{i=1}^{n}\sum_{j\neq i}\mathbb{1}\left\{\left|\widehat{\tau}_2(V_i,V_j,\theta)-\tau_{2F}(V_i,V_j,\theta)\right| \geq b_n\right\}\right| + o_p\left(\frac{1}{n^{1/2+\epsilon}}\right)\end{aligned} \tag{A-28}$$

where $\epsilon > 0$ is the constant described in Assumption 4. Take any $C > 0$ and any $\Delta > 0$. We have,

$$\begin{aligned}P_F\left(\sup_{\theta\in\Theta}\left|\frac{1}{n\cdot(n-1)}\sum_{i=1}^{n}\sum_{j\neq i}\mathbb{1}\left\{\left|\widehat{\tau}_2(V_i,V_j,\theta)-\tau_{2F}(V_i,V_j,\theta)\right| \geq b_n\right\}\right| > \frac{C}{n^{\Delta}}\right) \\ \leq P_F\left(\sup_{\substack{v,v'\in\mathbb{R}^{L_V}\times\mathbb{R}^{L_V}\\ \theta\in\Theta}}\left|\widehat{\tau}_2(v,v',\theta)-\tau_{2F}(v,v',\theta)\right| > b_n\right)\end{aligned}$$

Since the bandwidth sequence b_n satisfies $n^{1/2}\cdot h_n^r\cdot b_n \longrightarrow \infty$ by Assumption 4, the result we obtained in equation (A-10) yields,

$$\sup_{F\in\mathcal{F}} P_F\left(\sup_{\theta\in\Theta}\left|\frac{1}{n\cdot(n-1)}\sum_{i=1}^{n}\sum_{j\neq i}\mathbb{1}\left\{\left|\widehat{\tau}_2(V_i,V_j,\theta)-\tau_{2F}(V_i,V_j,\theta)\right| \geq b_n\right\}\right| > \frac{C}{n^{\Delta}}\right) \longrightarrow 0,$$

for any $C > 0$ and $\Delta > 0$. In particular, this holds for $\Delta = 1/2+\epsilon$, with $\epsilon > 0$ being the the constant described in Assumption 4. Therefore, under Assumptions 2, 3 and 4,

$$\sup_{\theta\in\Theta}\left|\frac{1}{n\cdot(n-1)}\sum_{i=1}^{n}\sum_{j\neq i}\mathbb{1}\left\{\left|\widehat{\tau}_2(V_i,V_j,\theta)-\tau_{2F}(V_i,V_j,\theta)\right| \geq b_n\right\}\right| = o_p\left(\frac{1}{n^{1/2+\epsilon}}\right), \quad \text{uniformly over } \mathcal{F}. \tag{A-29}$$

Plugging (A-29) into (A-28), we obtain that, under Assumptions 2, 3, 4 and 5,

$$\sup_{\theta\in\Theta}\left|r^{\mathcal{T}_2}_{n,F}(\theta)\right| = o_p\left(\frac{1}{n^{1/2+\epsilon}}\right), \quad \text{uniformly over } \mathcal{F}.$$

Where $\epsilon > 0$ is the constant described in Assumption 4. Since we defined $r^{\mathcal{T}_2}_{n,F}(\theta) \equiv \widehat{\mathcal{T}}_2(\theta) - \widetilde{\mathcal{T}}_2(\theta)$, with $\widetilde{\mathcal{T}}_2(\theta) \equiv \frac{1}{n\cdot(n-1)}\sum_{i=1}^{n}\sum_{j\neq i}\widehat{\tau}_2(V_i,V_j,\theta)\cdot\mathbb{1}\left\{\tau_{2F}(V_i,V_j,\theta)\geq 0\right\}$, we have that, under Assumptions 2, 3, 4 and 5,

$$\widehat{\mathcal{T}}_2(\theta) = \widetilde{T}_2(\theta) + r^{\mathcal{T}_2}_{n,F}(\theta), \quad \text{where} \quad \sup_{\theta\in\Theta}\left|r^{\mathcal{T}_2}_{n,F}(\theta)\right| = o_p\left(\frac{1}{n^{1/2+\epsilon}}\right), \quad \text{uniformly over } \mathcal{F}.$$

Where $\epsilon > 0$ denotes the constant described in Assumption 4. Our next step is to analyze the asymptotic properties of $\widetilde{T}_2(\theta)$.

A1.4.1 Asymptotic properties of $\widetilde{T}_2(\theta)$

Denote $(A)_+ \equiv \max\{A, 0\}$. We have,

$$\begin{aligned}
\widetilde{T}_2(\theta) &\equiv \frac{1}{n\cdot(n-1)}\sum_{i=1}^{n}\sum_{j\neq i}\widehat{\tau}_2(V_i,V_j,\theta)\cdot\mathbb{1}\left\{\tau_{2F}(V_i,V_j,\theta)\geq 0\right\}\\
&= \frac{1}{n\cdot(n-1)}\sum_{i=1}^{n}\sum_{j\neq i}\Big(\tau_{2F}(V_i,V_j,\theta)\Big)_+\\
&+ \frac{1}{n\cdot(n-1)}\sum_{i=1}^{n}\sum_{j\neq i}\Big(\widehat{\tau}_2(V_i,V_j,\theta)-\tau_{2F}(V_i,V_j,\theta)\Big)\cdot\mathbb{1}\left\{\tau_{2F}(V_i,V_j,\theta)\geq 0\right\}
\end{aligned}$$

For a pair $v \equiv (x_2, w_1, w_1)$, $v' \equiv (x_2', w_1', w_1')$, denote,

$$\mathbb{I}_{2F}(v,v',\theta) \equiv \mathbb{1}\left\{g_{1U}(w_1',\theta_1)\leq g_{1L}(w_1,\theta_1)\right\}\cdot\mathbb{1}\left\{\tau_{2F}(v,v',\theta)\geq 0\right\}\cdot\phi_2(v)\cdot\phi_2(v'). \tag{A-30}$$

And, for a given $v, v' \in \mathbb{R}^{L_V}\times\mathbb{R}^{L_V}$ and $\theta\in\Theta$, let

$$\begin{aligned}
\delta^{\tau_2}_{a,F}(v,v',\theta) &\equiv \left(R_{2F}(v) - (g_2(x_2,\theta_2)-g_2(x_2',\theta_2))\,Q_{2F}(v)\right)\cdot\mathbb{I}_{2F}(v,v',\theta),\\
\delta^{\tau_2}_{b,F}(v,v',\theta) &\equiv \left(R_{2F}(v') - (g_2(x_2',\theta_2)-g_2(x_2,\theta_2))\,Q_{2F}(v')\right)\cdot\mathbb{I}_{2F}(v,v',\theta),\\
\delta^{\tau_2}_{c,F}(v,v',\theta) &\equiv Q_{2F}(v')\cdot\mathbb{I}_{2F}(v,v',\theta),\\
\delta^{\tau_2}_{d,F}(v,v',\theta) &\equiv Q_{2F}(v)\cdot\mathbb{I}_{2F}(v,v',\theta).
\end{aligned}$$

And let $\varphi^{Q_2}_F(u,u',h)$ and $\varphi^{R_2}_F(y_2,u,u',h)$ be as defined in (A-20). As we defined previously, let us group all the observable covariates in the model as $Z \equiv (Y_1, Y_2, V)$. For given $(z,z',z'')\in\mathbb{R}^{L_v+2}\times\mathbb{R}^{L_v+2}\times\mathbb{R}^{L_v+2}$, $\theta\in\Theta$ and $h>0$, let

$$\begin{aligned}
\varphi^{\tau_2}_F(z,z',z'',\theta,h) \equiv\; &\delta^{\tau_2}_{a,F}(v,v',\theta)\varphi^{Q_2}_F(y_1'',v'',v',h) - \delta^{\tau_2}_{b,F}(v,v',\theta)\varphi^{Q_2}_F(y_1'',v'',v,h)\\
&+ \delta^{\tau_2}_{c,F}(v,v',\theta)\varphi^{R_2}_F(y_2'',y_1'',v'',v,h) - \delta^{\tau_2}_{d,F}(v,v',\theta)\varphi^{R_2}_F(y_2'',y_1'',v'',v',h)
\end{aligned} \tag{A-31}$$

Note by inspection of the definitions in (A-20) that,

$$E_F\left[\varphi_F^{Q_2}(Y_1,V,v,h)\right]=0,\quad \text{and}\quad E_F\left[\varphi_F^{R_2}(Y_2,Y_1,V,v,h)\right]=0\quad \forall\ v\in\mathbb{R}^{L_v},\ h>0. \tag{A-32}$$

Recall that we have defined $\mu_{2F}(v)\equiv E_F[Y_2|V=v,Y_1=1]$. By the smoothness conditions in Assumption 2 and the kernel properties in Assumption 4, an M^{th}–order approximation implies that there exists a finite $\overline{B}$ such that

$$\begin{aligned}\varphi_F^{Q_2}(Y_1,V,v,h_n)=&Y_1\phi_2(V)\Gamma(V,v,h_n)-h_n^{r_v}\cdot\phi_2(v)f_{V,1}(v)+B_{n,F}^{Q_2}(v),\\ \varphi_F^{R_2}(Y_2,Y_1,V,v,h_n)=&Y_2Y_1\phi_2(V)\Gamma(V,v,h_n)-h_n^{r_v}\cdot\mu_{2F}(v)\cdot\phi_2(v)f_{V,1}(v)+B_{n,F}^{R_2}(v),\\ \text{where}\ \sup_{v\in\mathcal{V}}&\left|B_{n,F}^{Q_2}(v)\right|\le\overline{B}\cdot h_n^{r_v+M},\quad \sup_{v\in\mathcal{V}}\left|B_{n,F}^{R_2}(v)\right|\le\overline{B}\cdot h_n^{r_v+M}\quad\forall\ F\in\mathcal{F}.\end{aligned} \tag{A-33}$$

From (A-21), we have

$$\begin{aligned}\widetilde{\mathcal{T}}_2(\theta)=&\frac{1}{n\cdot(n-1)}\sum_{i=1}^n\sum_{j\neq i}\Big(\tau_{2F}(V_i,V_j,\theta)\Big)_+ +\frac{1}{h_n^{r_v}}\cdot\frac{1}{n^2\cdot(n-1)}\sum_{i=1}^n\sum_{j\neq i}\sum_{k=1}^n\varphi_F^{\tau_2}(Z_i,Z_j,Z_k,\theta,h_n)+\xi_{a,n}^{\widetilde{\mathcal{T}}_2}(\theta),\\ &\text{where}\ \sup_{\theta\in\Theta}\left|\xi_{a,n}^{\widetilde{\mathcal{T}}_2}(\theta)\right|=o_p\left(\frac{1}{n^{1/2+\epsilon}}\right),\ \text{uniformly over}\ \mathcal{F}.\end{aligned} \tag{A-34}$$

Where $\epsilon>0$ denotes the constant described in Assumption 4. Let

$$\begin{aligned}U_{a,n}(\theta,h)&\equiv\frac{1}{n\cdot(n-1)\cdot(n-2)}\sum_{i=1}^n\sum_{j\neq i}\sum_{k\neq i,j}\varphi_F^{\tau_2}(Z_i,Z_j,Z_k,\theta,h),\\ U_{b,n}(\theta,h)&\equiv\frac{1}{n\cdot(n-1)}\sum_{i=1}^n\sum_{j\neq i}\Big(\varphi_F^{\tau_2}(Z_i,Z_j,Z_i,\theta,h)+\varphi_F^{\tau_2}(Z_i,Z_j,Z_j,\theta,h)\Big)\end{aligned}$$

Then, (A-34) can be re-expressed as,

$$\begin{aligned}\widetilde{\mathcal{T}}_2(\theta)=&\frac{1}{n\cdot(n-1)}\sum_{i=1}^n\sum_{j\neq i}\Big(\tau_{2F}(V_i,V_j,\theta)\Big)_+ +\frac{(n-2)}{n}\cdot\frac{1}{h_n^{r_v}}\cdot U_{a,n}(\theta,h_n)+\frac{1}{n\cdot h_n^{r_v}}\cdot U_{b,n}(\theta,h_n)+\xi_{a,n}^{\widetilde{\mathcal{T}}_2}(\theta),\\ &\text{where}\ \sup_{\theta\in\Theta}\left|\xi_{a,n}^{\widetilde{\mathcal{T}}_2}(\theta)\right|=o_p\left(\frac{1}{n^{1/2+\epsilon}}\right),\ \text{uniformly over}\ \mathcal{F}.\end{aligned} \tag{A-35}$$

Where $\epsilon>0$ is the constant described in Assumption 4. Recall from Assumption 3 that the class of sets

$$\mathscr{C}\equiv\Big\{(w_1,w_1)\in\mathbb{R}^{d_U}\times\mathbb{R}^{d_L}:g_{1U}(w_1,\theta_1)\le g_{1L}(w_1,\theta_1)\ \text{for some}\ \theta_1\in\Theta\Big\}$$

is a VC class with VC dimension $\overline{V}_C$, and that the following is a VC class of sets for each F, with VC dimension uniformly bounded over $\mathcal{F}$ by a finite constant $\overline{V}_D$,

$$\mathscr{D}^{\tau_2}_{1,F} \equiv \Big\{(v_1, v_2) \in \mathbb{R}^{L_v} \times \mathbb{R}^{L_v} : \tau_{2F}(v_1, v_2, \theta) \geq 0 \text{ for some } \theta \in \Theta\Big\}$$

Going back to the definition of $\mathbb{I}_{2F}$ in equation (A-30), these VC properties imply, by the results in Pakes and Pollard (1989, p. 1033) (the result that classes of indicator functions over VC classes of sets are Euclidean (A, V), with (A, V) depending only on the VC-dimension of the underlying class of sets), and Pakes and Pollard (1989, Lemma 2.14) (the product of Euclidean classes of functions is also a Euclidean class) that there exist constants $(\overline{A}, \overline{V})$ such that, for each $F \in \mathcal{F}$, the class of indicator functions

$$\mathscr{I}_{2,F} \equiv \{m(v, v') = \mathbb{I}_{2F}(v, v', \theta) : \theta \in \Theta\}, \tag{A-36}$$

is Euclidean $(\overline{A}, \overline{V})$ for the constant envelope 1. From here, let $\varphi^{\tau_2}_F$ be as defined in (A-31) and consider the class of functions,

$$\mathscr{H}_{2,F} \equiv \Big\{m(z_1, z_2, z_3) = \varphi^{\tau_2}_F(z_1, z_2, z_3, \theta, h) : \theta \in \Theta,\, h > 0\Big\}. \tag{A-37}$$

By the conditions in Assumptions 2, 3 and 4 (the bounded properties of the functionals involved, the bounded-variation properties of the weight function $\phi_2(\cdot)$ and the kernel $K(\cdot)$, and the VC property of the classes of sets involved, which led to the Euclidean property of the class of functions described in equation (A-36)), by Nolan and Pollard (1987, Lemma 22) and Pakes and Pollard (1989, Lemma 2.14), there exist constants $(\overline{A}, \overline{V})$ such that $\mathscr{H}_{1,F}$ is Euclidean $(\overline{A}, \overline{V})$ for all $F \in \mathcal{F}$, for an envelope of the form $H_1 = D_1 + D_2 \cdot |Y_2|$, where D_1 and D_2 are constant for all F. Since there exists a finite constant $\overline{D}_4$ such that $E_F[|Y_2|^4] \leq \overline{D}_4$ for all $F \in \mathcal{F}$ by Assumption 3, Result A1 can be used to show that,

$$\sup_{\substack{\theta \in \Theta \\ h > 0}} \Big|U_{b,n}(\theta, h)\Big| = O_p(1), \quad \text{uniformly over } \mathcal{F}.$$

Therefore, using the bandwidth convergence conditions described in Assumption 4, equation (A-35) becomes,

$$\begin{aligned}
&\widetilde{\mathcal{T}}_2(\theta) = \frac{1}{n \cdot (n-1)} \sum_{i=1}^{n} \sum_{j \neq i} \Big(\tau_{2F}(V_i, V_j, \theta)\Big)_+ + \frac{(n-2)}{n} \cdot \frac{1}{h_n^{r_v}} \cdot U_{a,n}(\theta, h_n) + \xi^{\widetilde{\mathcal{T}}_2}_{b,n}(\theta), \\
&\text{where } \sup_{\theta \in \Theta} \Big|\xi^{\widetilde{\mathcal{T}}_2}_{b,n}(\theta)\Big| = O_p\left(\frac{1}{n \cdot h_n^{r_v}}\right) + o_p\left(\frac{1}{n^{1/2+\epsilon}}\right) = o_p\left(\frac{1}{n^{1/2+\epsilon}}\right), \text{ uniformly over } \mathcal{F}.
\end{aligned} \tag{A-38}$$

Where $\epsilon > 0$ is the constant described in Assumption 4. Next we focus on the Hoeffding decomposition (see Serfling (1980, pages 177-178) or Sherman (1994, equations (6)-(7))) of $U_{a,n}(\theta, h_n)$. In what follows, let Z_1, Z_2, Z_3 be iid draws from the distribution F. Let

$$\overline{\varphi}_F^{\tau_2}(Z_1, Z_2, Z_3, \theta, h) \equiv \frac{1}{3!} \sum_p \varphi_F^{\tau_2}(Z_{m_1}, Z_{m_2}, Z_{m_3}, \theta, h), \tag{A-39}$$

where $\sum_p$ denotes the sum over the 3! permutations (m_1, m_2, m_3) of $(1,2,3)$. By construction, $\overline{\varphi}_F^{\tau_2}(Z_1, Z_2, Z_3, \theta, h)$ is symmetric in (Z_1, Z_2, Z_3), and $U_{a,n}(\theta, h)$ can be expressed as,

$$U_{a,n}(\theta, h) = \binom{n}{3}^{-1} \sum_{i<j<k} \overline{\varphi}_F^{\tau_2}(Z_1, Z_2, Z_3, \theta, h).$$

Note from (A-32) that $E_F\left[\overline{\varphi}_F^{\tau_2}(Z_1, Z_2, Z_3, \theta, h)\right] = E_F\left[\varphi_F^{\tau_2}(Z_1, Z_2, Z_3, \theta, h)\right] = 0$. For a given (z, z', z''), let

$$\begin{aligned}
m_{1F}^{\tau_2}(z, \theta, h) &\equiv E_F\left[\overline{\varphi}_F^{\tau_2}(z, Z_2, Z_3, \theta, h)\right], \\
m_{2F}^{\tau_2}(z, z', \theta, h) &\equiv E_F\left[\overline{\varphi}_F^{\tau_2}(z, z', Z_3, \theta, h)\right] - m_{1F}^{\tau_2}(z, \theta, h) - m_{1F}^{\tau_2}(z', \theta, h), \\
m_{3F}(z, z', z'', \theta, h) &\equiv \overline{\varphi}_F^{\tau_2}(z, z', z'', \theta, h) - m_{2F}^{\tau_2}(z, z', \theta, h) - m_{2F}^{\tau_2}(z, z'', \theta, h) - m_{2F}^{\tau_2}(z', z'', \theta, h) \\
&\quad - m_{1F}^{\tau_2}(z, \theta, h) - m_{1F}^{\tau_2}(z', \theta, h) - m_{1F}^{\tau_2}(z'', \theta, h)
\end{aligned}$$

Let,

$$S_{2,n}^{\tau_2}(\theta, h) \equiv \binom{n}{2}^{-1} \sum_{i<j} m_{2F}^{\tau_2}(Z_i, Z_j, \theta, h), \quad S_{3,n}^{\tau_2}(\theta, h) \equiv \binom{n}{3}^{-1} \sum_{i<j<k} m_{3F}^{\tau_2}(Z_i, Z_j, Z_k, \theta, h)$$

The Hoeffding decomposition of $U_{a,n}(\theta, h_n)$ (see Serfling (1980, pages 177-178) or Sherman (1994, equations (6)-(7))) is given by,

$$U_{a,n}(\theta, h_n) = \frac{3}{n} \sum_{i=1}^{n} m_{1F}^{\tau_2}(Z_i, \theta, h_n) + 3 \cdot S_{2,n}^{\tau_2}(\theta, h_n) + S_{3,n}^{\tau_2}(\theta, h_n) \tag{A-40}$$

$\left\{S_{2,n}^{\tau_2}(\theta, h) : \theta \in \Theta, h > 0\right\}$ is a degenerate U-process of order 2 and $\left\{S_{3,n}^{\tau_2}(\theta, h) : \theta \in \Theta, h > 0\right\}$ is a degenerate U-process of order 3. The Euclidean properties of the class of functions $\mathscr{H}_{2,F}$ defined in (A-37) and described above yield, via Result A1,

$$\sup_{\substack{\theta \in \Theta \\ h > 0}} \left|S_{2,n}^{\tau_2}(\theta, h)\right| = O_p\left(\frac{1}{n}\right), \quad \text{and} \quad \left|S_{3,n}^{\tau_2}(\theta, h)\right| = O_p\left(\frac{1}{n^{3/2}}\right), \quad \text{uniformly over } \mathcal{F}.$$

Therefore, combining (A-40) and (A-38), we have,

$$\widetilde{\mathcal{T}}_2(\theta) = \frac{1}{n\cdot(n-1)}\sum_{i=1}^{n}\sum_{j\neq i}\Big(\tau_{2F}(V_i,V_j,\theta)\Big)_+ + \frac{(n-2)}{n}\cdot\frac{3}{n}\sum_{i=1}^{n}\frac{m_{1F}^{\tau_2}(Z_i,\theta,h_n)}{h_n^{r_v}} + \xi_{c,n}^{\widetilde{\mathcal{T}}_2}(\theta),$$
$$\text{where } \sup_{\theta\in\Theta}\Big|\xi_{c,n}^{\widetilde{\mathcal{T}}_2}(\theta)\Big| = O_p\left(\frac{1}{n\cdot h_n^{r_v}}\right) + o_p\left(\frac{1}{n^{1/2+\epsilon}}\right) = o_p\left(\frac{1}{n^{1/2+\epsilon}}\right),\ \text{uniformly over } \mathcal{F}. \tag{A-41}$$

Where $\epsilon > 0$ denotes the constant described in Assumption 4. Let us turn our attention to $m_{1F}^{\tau_2}(Z_i,\theta,h_n)$. Recall from (A-31) that,

$$\begin{aligned}\varphi_F^{\tau_2}(Z_i,Z_j,Z_k,\theta,h) \equiv\ & \delta_{a,F}^{\tau_2}(V_i,V_j,\theta)\varphi_F^{Q_2}(Y_{1k},V_k,V_j,h) - \delta_{b,F}^{\tau_2}(V_i,V_j,\theta)\varphi_F^{Q_2}(Y_{1k},V_k,V_i,h)\\ &+ \delta_{c,F}^{\tau_2}(V_i,V_j,\theta)\varphi_F^{R_2}(Y_{2k},Y_{1k},V_k,V_i,h) - \delta_{d,F}^{\tau_2}(V_i,V_j,\theta)\varphi_F^{R_2}(Y_{2k},Y_{1k},V_k,V_j,h)\end{aligned}$$

where $\varphi_F^{Q_2}$ and $\varphi_F^{R_2}$ are as described in (A-20) and $\delta_{a,F}^{\tau_2}$, $\delta_{b,F}^{\tau_2}$, $\delta_{c,F}^{\tau_2}$ and $\delta_{d,F}^{\tau_2}$ are as described in (A1.4.1). Note from (A-20) that,

$$\begin{aligned}E_F\left[\varphi_F^{Q_2}(Y_{1k},V_k,V_j,h)|Z_i,Z_j\right] &= E_F\left[\varphi_F^{Q_2}(Y_{1k},V_k,V_i,h)|Z_i,Z_j\right] = 0,\\ E_F\left[\varphi_F^{R_2}(Y_{2k},Y_{1k},V_k,V_i,h)|Z_i,Z_j\right] &= E_F\left[\varphi_F^{R_2}(Y_{2k},Y_{1k},V_k,V_j,h)|Z_i,Z_j\right] = 0.\end{aligned}$$

Thus, from the definition of $\overline{\varphi}_F^{\tau_2}$ in (A-39), we have

$$m_{1F}^{\tau_2}(Z_i,\theta,h) \equiv E_F\left[\overline{\varphi}_F^{\tau_2}(Z_i,Z_j,Z_k,\theta,h)\right] = \frac{1}{3!}\Big(E_F\left[\varphi_F^{\tau_2}(Z_j,Z_k,Z_i,\theta,h)\Big|Z_i\right] + E_F\left[\varphi_F^{\tau_2}(Z_k,Z_j,Z_i,\theta,h)\Big|Z_i\right]\Big) \tag{A-42}$$

As we defined in equation (18) prior to Assumption 2, for a given $v \equiv (x_2, w_1, w_1)$, let

$$\begin{aligned}\eta_{a,F}^{\tau_2}(v,\theta) &\equiv E_F\left[(R_{2F}(V) - (g_2(X_2,\theta_2) - g_2(x_2,\theta_2))\,Q_{2F}(V))\,\mathbb{1}\{g_{1U}(w_1,\theta_1) \le g_{1L}(W_1,\theta_1)\}\,\mathbb{1}\{\tau_{2F}(V,v,\theta)\ge 0\}\,\phi_2(V)\right],\\ \eta_{b,F}^{\tau_2}(v,\theta) &\equiv E_F\left[(R_{2F}(V) - (g_2(X_2,\theta_2) - g_2(x_2,\theta_2))\,Q_{2F}(V))\,\mathbb{1}\{g_{1U}(W_1,\theta_1) \le g_{1L}(w_1,\theta_1)\}\,\mathbb{1}\{\tau_{2F}(v,V,\theta)\ge 0\}\,\phi_2(V)\right],\\ \eta_{c,F}^{\tau_2}(v,\theta) &\equiv E_F\left[Q_{2F}(V)\mathbb{1}\{g_{1U}(W_1,\theta_1) \le g_{1L}(w_1,\theta_1)\}\,\mathbb{1}\{\tau_{2F}(v,V,\theta)\ge 0\}\,\phi_2(V)\right],\\ \eta_{d,F}^{\tau_2}(v,\theta) &\equiv E_F\left[Q_{2F}(V)\mathbb{1}\{g_{1U}(w_1,\theta_1) \le g_{1L}(W_1,\theta_1)\}\,\mathbb{1}\{\tau_{2F}(V,v,\theta)\ge 0\}\,\phi_2(V)\right]\end{aligned}$$

Using iterated expectations, we have

$$\begin{aligned}&E_F\left[\varphi_F^{\tau_2}(Z_1,Z_2,Z_3,\theta,h_n)\Big|Z_3\right] =\\ &E_F\left[\eta_{a,F}^{\tau_2}(V_2,\theta)\phi_2(V_2)\varphi_F^{Q_2}(Y_{13},V_3,V_2,h)\Big|Z_3\right] - E_F\left[\eta_{b,F}^{\tau_2}(V_1,\theta)\phi_2(V_1)\varphi_F^{Q_2}(Y_{13},V_3,V_1,h)\Big|Z_3\right]\\ &+E_F\left[\eta_{c,F}^{\tau_2}(V_1,\theta)\phi_2(V_1)\varphi_F^{R_2}(Y_{23},Y_{13},V_3,V_1,h)\Big|Z_3\right] - E_F\left[\eta_{d,F}^{\tau_2}(V_2,\theta)\phi_2(V_2)\varphi_F^{R_2}(Y_{23},Y_{13},V_3,V_2,h)\Big|Z_3\right]\end{aligned} \tag{A-43}$$

We will analyze each of the terms on the right-hand side of (A-43). Using the result in (A-33), we have

$$
\begin{aligned}
&E_F\left[\eta^{\tau_2}_{a,F}(V_2,\theta)\phi_2(V_2)\varphi^{Q_2}_F(Y_{13},V_3,V_2,h_n)\Big|Z_3\right]=\\
&E_F\left[\eta^{\tau_2}_{a,F}(V_2,\theta)\phi_2(V_2)\Gamma(V_3,V_2,h_n)\Big|V_3\right]Y_{13}\phi_2(V_3)-h_n^{r_v}\cdot E_F\left[\eta^{\tau_2}_{a,F}(V,\theta)\phi_2(V)^2 f_{V,1}(V)\right]\\
&+E_F\left[\eta^{\tau_2}_{a,F}(V,\theta)\phi_2(V)B^{Q_2}_{n,F}(V)\right].
\end{aligned}
$$

By the result shown in (A-33) and the boundedness conditions described in Assumption 2, there exists a finite constant $\overline{D}_a$ such that

$$
\sup_{\substack{v\in\mathbb{R}^{L_v}\\ \theta\in\Theta}}\left|\eta^{\tau_2}_{a,F}(v,\theta)\phi_2(v)B^{Q_2}_{n,F}(v)\right|\leq\overline{D}_a\cdot h_n^{r_v+M}\quad\forall\ F\in\mathcal{F}.
$$

Next, by the smoothness conditions in Assumption 2 and the kernel properties in Assumption 4, an M^{th}–order approximation implies that there exists a finite $\overline{B}_a$ such that,

$$
\begin{aligned}
&E_F\left[\eta^{\tau_2}_{a,F}(V_2,\theta)\phi_2(V_2)\Gamma(V_3,V_2,h_n)\Big|V_3\right]Y_{13}\phi_2(V_3)=h_n^{r_v}\cdot\eta^{\tau_2}_{a,F}(V_3,\theta)\phi_2(V_3)^2Y_{13}f_V(V_3)+B^a_{n,F}(V_3,\theta)Y_{13}\phi_2(V_3),\\
&\text{where}\quad\sup_{\substack{v\in\mathbb{R}^{L_v}\\ \theta\in\Theta}}\left|B^a_{n,F}(v,\theta)\phi_2(v)\right|\leq\overline{B}_a\cdot h_n^{r_V+M}\quad\forall\ F\in\mathcal{F}.
\end{aligned}
$$

Combining these results, we obtain that, under Assumptions 2, 3 and 4, there exists a finite constant $\overline{C}$ such that,

$$
\begin{aligned}
&E_F\left[\eta^{\tau_2}_{a,F}(V_2,\theta)\phi_2(V_2)\varphi^{Q_2}_F(Y_{13},V_3,V_2,h_n)\Big|Z_3\right]=\\
&h_n^{r_v}\cdot\left(\eta^{\tau_2}_{a,F}(V_3,\theta)Y_{13}f_V(V_3)\phi_2(V_3)^2-E_F\left[\eta^{\tau_2}_{a,F}(V,\theta)f_{V,1}(V)\phi_2(V)^2\right]\right)+\xi_{a,n}(Y_{13},V_3,\theta),\\
&\text{where}\quad\sup_{\substack{v\in\mathbb{R}^{L_v}\\ \theta\in\Theta}}\left|\xi_{a,n}(Y_{13},v,\theta)\right|\leq\overline{C}\cdot h_n^{r_v+M}\quad\forall\ F\in\mathcal{F}.
\end{aligned}
$$

Note by iterated expectations that $E_F\left[\eta^{\tau_2}_{a,F}(V,\theta)f_{V,1}(V)\phi_2(V)^2\right]=E_F\left[\eta^{\tau_2}_{a,F}(V,\theta)Y_1f_V(V)\phi_2(V)^2\right]$. Therefore, the previous result becomes,

$$
\begin{aligned}
&E_F\left[\eta^{\tau_2}_{a,F}(V_2,\theta)\phi_2(V_2)\varphi^{Q_2}_F(Y_{13},V_3,V_2,h_n)\Big|Z_3\right]=\\
&h_n^{r_v}\cdot\left(\eta^{\tau_2}_{a,F}(V_3,\theta)Y_{13}f_V(V_3)\phi_2(V_3)^2-E_F\left[\eta^{\tau_2}_{a,F}(V,\theta)Y_1f_V(V)\phi_2(V)^2\right]\right)+\xi_{a,n}(Y_{13},V_3,\theta),\\
&\text{where}\quad\sup_{\substack{v\in\mathbb{R}^{L_v}\\ \theta\in\Theta}}\left|\xi_{a,n}(Y_{13},v,\theta)\right|\leq\overline{C}\cdot h_n^{r_v+M}\quad\forall\ F\in\mathcal{F}.
\end{aligned}
\tag{A-44}
$$

Analogous steps can be used to show that, under our assumptions,

$$
\begin{aligned}
& E_F\left[\eta_{b,F}^{\tau_2}(V_1,\theta)\phi_2(V_1)\varphi_F^{Q_2}(Y_{13},V_3,V_1,h_n)\Big|Z_3\right] = \\
& h_n^{r_v}\cdot\left(\eta_{b,F}^{\tau_2}(V_3,\theta)Y_{13}f_V(V_3)\phi_2(V_3)^2 - E_F\left[\eta_{b,F}^{\tau_2}(V,\theta)Y_1 f_V(V)\phi_2(V)^2\right]\right) + \xi_{b,n}(Y_{13},V_3,\theta), \\
& \text{where} \quad \sup_{\substack{v\in\mathbb{R}^{L_v}\\ \theta\in\Theta}} \left|\xi_{b,n}(Y_{13},v,\theta)\right| \leq \overline{C}\cdot h_n^{r_v+M} \quad \forall\ F\in\mathcal{F}.
\end{aligned}
$$

Next, using again the result in (A-33), we have

$$
\begin{aligned}
& E_F\left[\eta_{c,F}^{\tau_2}(V_1,\theta)\phi_2(V_1)\varphi_F^{R_2}(Y_{23},Y_{13},V_3,V_1,h_n)\Big|Z_3\right] = \\
& E_F\left[\eta_{c,F}^{\tau_2}(V_1,\theta)\phi_2(V_1)\Gamma(V_3,V_1,h_n)\Big|V_3\right]Y_{23}Y_{13}\phi_2(V_3) - h_n^{r_v}\cdot E_F\left[\eta_{c,F}^{\tau_2}(V,\theta)\phi_2(V)^2\mu_{2F}(V)f_{V,1}(V)\right] \\
& + E_F\left[\eta_{c,F}^{\tau_2}(V,\theta)\phi_2(V)B_{n,F}^{R_2}(V)\right].
\end{aligned}
$$

By the result shown in (A-33) and the boundedness conditions described in Assumption 2, there exists a finite constant $\overline{D}_c$ such that

$$
\sup_{\substack{v\in\mathbb{R}^{L_v}\\ \theta\in\Theta}} \left|\eta_{c,F}^{\tau_2}(v,\theta)\phi_2(v)B_{n,F}^{R_2}(v)\right| \leq \overline{D}_c\cdot h_n^{r_v+M} \quad \forall\ F\in\mathcal{F}.
$$

Next, by the smoothness conditions in Assumption 2 and the kernel properties in Assumption 4, an M^{th}–order approximation implies that there exists a finite $\overline{B}_c$ such that,

$$
\begin{aligned}
& E_F\left[\eta_{c,F}^{\tau_2}(V_1,\theta)\phi_2(V_1)\Gamma(V_3,V_1,h_n)\Big|V_3\right]Y_{23}Y_{13}\phi_2(V_3) = h_n^{r_v}\cdot\eta_{c,F}^{\tau_2}(V_3,\theta)Y_{23}Y_{13}f_V(V_3)\phi_2(V_3)^2 \\
& + Y_{23}Y_{13}B_{n,F}^{c}(V_3,\theta)\phi_2(V_3), \quad \text{where} \quad \sup_{\substack{v\in\mathbb{R}^{L_v}\\ \theta\in\Theta}} \left|B_{n,F}^{c}(v,\theta)\phi_2(v)\right| \leq \overline{B}_c\cdot h_n^{r_V+M} \quad \forall\ F\in\mathcal{F}.
\end{aligned}
$$

By iterated expectations, $E_F\left[\eta_{c,F}^{\tau_2}(V,\theta)\phi_2(V)^2\mu_{2F}(V)f_{V,1}(V)\right] = E_F\left[\eta_{c,F}^{\tau_2}(V,\theta)\phi_2(V)^2 Y_2 Y_1 f_V(V)\right]$. Combining the previous results, we obtain that, under Assumptions 2, 3 and 4, there exists a finite constant $\overline{C}$ such that,

$$
\begin{aligned}
& E_F\left[\eta_{c,F}^{\tau_2}(V_1,\theta)\phi_2(V_1)\varphi_F^{R_2}(Y_{23},Y_{13},V_3,V_1,h_n)\Big|Z_3\right] = \\
& h_n^{r_v}\cdot\left(\eta_{c,F}^{\tau_2}(V_3,\theta)Y_{23}Y_{13}f_V(V_3)\phi_2(V_3)^2 - E_F\left[\eta_{c,F}^{\tau_2}(V,\theta)Y_2Y_1 f_V(V)\phi_2(V)^2\right]\right) + \xi_{c,n}(Y_{23},Y_{13},V_3,\theta), \\
& \text{where} \quad \sup_{\substack{v\in\mathbb{R}^{L_v}\\ \theta\in\Theta}} \left|\xi_{c,n}(Y_{23},Y_{13},v,\theta)\right| \leq \overline{C}\cdot h_n^{r_v+M}\cdot|Y_{23}| \quad \forall\ F\in\mathcal{F}
\end{aligned}
$$

Analogous steps can be used to show that, under our assumptions,

$$
\begin{aligned}
&E_F\left[\eta^{\tau_2}_{d,F}(V_2,\theta)\phi_2(V_2)\varphi^{R_2}_F(Y_{23},Y_{13},V_3,V_2,h)\Big|Z_3\right]=\\
&h_n^{r_v}\cdot\left(\eta^{\tau_2}_{d,F}(V_3,\theta)Y_{23}Y_{13}f_V(V_3)\phi_2(V_3)^2-E_F\left[\eta^{\tau_2}_{d,F}(V,\theta)Y_2Y_1f_V(V)\phi_2(V)^2\right]\right)+\xi_{d,n}(Y_{23},Y_{13},V_3,\theta),\\
&\text{where}\quad \sup_{\substack{v\in\mathbb{R}^{L_v}\\ \theta\in\Theta}}\left|\xi_{d,n}(Y_{23},Y_{13},v,\theta)\right|\leq\overline{C}\cdot h_n^{r_v+M}\cdot|Y_{23}|\quad\forall\ F\in\mathcal{F}
\end{aligned}
\tag{A-45}
$$

Let

$$
\begin{aligned}
H^{\mathcal{T}_2}_{2F}(Z_i,\theta)\equiv&\left(\left(\eta^{\tau_2}_{a,F}(V_i,\theta)-\eta^{\tau_2}_{b,F}(V_i,\theta)\right)\cdot Y_{1i}+\left(\eta^{\tau_2}_{c,F}(V_i,\theta)-\eta^{\tau_2}_{d,F}(V_i,\theta)\right)\cdot Y_{2i}Y_{1i}\right)\cdot f_V(V_i)\cdot\phi_2(V_i)^2\\
&-E_F\left[\left(\left(\eta^{\tau_2}_{a,F}(V,\theta)-\eta^{\tau_2}_{b,F}(V,\theta)\right)\cdot Y_1+\left(\eta^{\tau_2}_{c,F}(V,\theta)-\eta^{\tau_2}_{d,F}(V,\theta)\right)\cdot Y_2Y_1\right)\cdot f_V(V)\cdot\phi_2(V)^2\right].
\end{aligned}
\tag{A-46}
$$

Note that $E_F\left[H^{\mathcal{T}_2}_{2F}(Z,\theta)\right]=0$. Combining the results in (A-44)-(A-45), we have that, under Assumptions 2, 3 and 4, there exists a finite constant $\overline{C}$ such that,

$$
\begin{aligned}
&E_F\left[\varphi^{\tau_2}_F(Z_j,Z_k,Z_i,\theta,h)\Big|Z_i\right]=E_F\left[\varphi^{\tau_2}_F(Z_k,Z_j,Z_i,\theta,h)\Big|Z_i\right]=h_n^{r_v+M}\cdot H^{\mathcal{T}_2}_{2F}(Z_i,\theta)+\xi_{e,n}(Z_i,\theta),\\
&\text{where}\quad\sup_{\theta\in\Theta}\left|\xi_{e,n}(Z_i,\theta)\right|\leq\overline{C}\cdot h_n^{r_v+M}\cdot|Y_{2i}|\quad\forall\ F\in\mathcal{F}
\end{aligned}
$$

Plugging this result in to (A-42), we obtain,

$$
\begin{aligned}
\frac{1}{h_n^{r_v}}\cdot m^{\tau_2}_{1F}(Z_i,\theta,h)&=\frac{1}{3!}\left(E_F\left[\varphi^{\tau_2}_F(Z_j,Z_k,Z_i,\theta,h)\Big|Z_i\right]+E_F\left[\varphi^{\tau_2}_F(Z_k,Z_j,Z_i,\theta,h)\Big|Z_i\right]\right)\\
&=\frac{2}{3!}H^{\mathcal{T}_2}_{2F}(Z_i,\theta)+\xi_{f,n}(Z_i,\theta)\\
&=\frac{1}{3}H^{\mathcal{T}_2}_{2F}(Z_i,\theta)+\xi_{f,n}(Z_i,\theta),\\
&\text{where}\quad\sup_{\theta\in\Theta}\left|\xi_{f,n}(Z_i,\theta)\right|\leq\overline{C}\cdot h_n^{M}\cdot|Y_{2i}|\quad\forall\ F\in\mathcal{F}.
\end{aligned}
\tag{A-47}
$$

By Assumption 3, there exists a finite constant $\overline{D}_4$ such that $E_F[|Y_2|^4]\leq\overline{D}_4$ for all $F\in\mathcal{F}$. Therefore, using a Chebyshev inequality argument we have $\frac{1}{n}\sum_{i=1}^n|Y_{2i}|=O_p(1)$, uniformly over $\mathcal{F}$, and from the above results, we have

$$
\sup_{\theta\in\Theta}\frac{1}{n}\sum_{i=1}^n\left|\xi_{e,n}(Z_i,\theta)\right|=O_p\left(h_n^{r_v+M}\right),\quad\text{uniformly over }\mathcal{F}.
$$

From here, plugging (A-47) into (A-41), we obtain

$$\widetilde{\mathcal{T}}_2(\theta) = \frac{1}{n\cdot(n-1)}\sum_{i=1}^{n}\sum_{j\neq i}\Big(\tau_{2F}(V_i,V_j,\theta)\Big)_+ + \frac{(n-2)}{n}\cdot\frac{1}{n}\sum_{i=1}^{n}H_{2F}^{\mathcal{T}_2}(Z_i,\theta) + \xi_{d,n}^{\widetilde{\mathcal{T}}_2}(\theta),$$
$$\text{where } \sup_{\theta\in\Theta}\left|\xi_{d,n}^{\widetilde{\mathcal{T}}_2}(\theta)\right| = O_p\left(h_n^{r_v+M}\right) + o_p\left(\frac{1}{n^{1/2+\epsilon}}\right) = o_p\left(\frac{1}{n^{1/2+\epsilon}}\right), \text{ uniformly over } \mathcal{F}. \quad \text{(A-48)}$$

Where $\epsilon > 0$ is the constant described in Assumption 4. Consider the class of functions,

$$\mathscr{H}_{3,F} \equiv \left\{m(z) = H_{2F}^{\mathcal{T}_2}(z,\theta) : \theta\in\Theta\right\}.$$

By Assumptions 2 and 3, there exist finite constants $\overline{A}_4$ and $\overline{B}_4$ such that, for all $\theta,\theta'\in\Theta$,

$$\left|H_{2F}^{\mathcal{T}_2}(z,\theta) - H_{2F}^{\mathcal{T}_2}(z,\theta')\right| \leq \left(\overline{A}_4 + \overline{B}_4\cdot|y_2|\right)\cdot\left\|\theta-\theta'\right\| \;\forall\, y_2, v, \quad \forall\, F\in\mathcal{F}.$$

From here, Pakes and Pollard (1989, Lemma 2.13) yields that there exist constants $(\overline{A},\overline{V})$ such that, for each $F\in\mathcal{F}$, the class of functions $\mathscr{H}_{3,F}$ is Euclidean $(\overline{A},\overline{V})$ for the envelope $\overline{H}_3(z) = \left|H_{2F}^{\mathcal{T}_2}(z,\theta_0)\right| + \overline{M}_3\cdot\left(\overline{A}_4+\overline{B}_4\cdot|y_2|\right)$, where θ_0 is an arbitrary point of Θ and $\overline{M}_3 \equiv 2\sqrt{k}\sup_\theta\|\theta-\theta_0\|$ (recall that $k\equiv\dim(\theta)$). By Assumptions 2 and 3, there exists a finite constant $\overline{D}_3$ such that $E_F\left[\overline{H}_3(Z)^4\right] \leq \overline{D}_3$ for all $F\in\mathcal{F}$. Thus, the conditions in Result A1 are satisfied and from there we obtain,

$$\sup_{\theta\in\Theta}\left|\frac{1}{n}\sum_{i=1}^{n}H_{2F}^{\mathcal{T}_2}(Z_i,\theta)\right| = O_p\left(\frac{1}{n^{1/2}}\right), \quad \text{uniformly over } \mathcal{F}.$$

Plugging this result into (A-48), we obtain,

$$\widetilde{\mathcal{T}}_2(\theta) = \frac{1}{n\cdot(n-1)}\sum_{i=1}^{n}\sum_{j\neq i}\Big(\tau_{2F}(V_i,V_j,\theta)\Big)_+ + \frac{1}{n}\sum_{i=1}^{n}H_{2F}^{\mathcal{T}_2}(Z_i,\theta) + \xi_{e,n}^{\widetilde{\mathcal{T}}_2}(\theta),$$
$$\text{where} \quad \xi_{e,n}^{\widetilde{\mathcal{T}}_2}(\theta) \equiv -\left(\frac{2}{n}\right)\cdot\frac{1}{n}\sum_{i=1}^{n}H_{2F}^{\mathcal{T}_2}(Z_i,\theta) + \xi_{d,n}^{\widetilde{\mathcal{T}}_2}(\theta), \quad \text{and} \quad \text{(A-49)}$$
$$\sup_{\theta\in\Theta}\left|\xi_{e,n}^{\widetilde{\mathcal{T}}_2}(\theta)\right| = O_p\left(\frac{1}{n^{3/2}}\right) + o_p\left(\frac{1}{n^{1/2+\epsilon}}\right) = o_p\left(\frac{1}{n^{1/2+\epsilon}}\right), \text{ uniformly over } \mathcal{F},$$

where $\epsilon > 0$ is the constant described in Assumption 4. We move on to the last step and focus on $\frac{1}{n\cdot(n-1)}\sum_{i=1}^{n}\sum_{j\neq i}\left(\tau_{2F}(V_i,V_j,\theta)\right)_+$ and its Hoeffding decomposition. Let V_1, V_2 be iid draws from F and recall that we defined

$$\mathcal{T}_{2F}(\theta) \equiv E_F\left[(\tau_{2F}(V_1,V_2,\theta))_+\right].$$

Let

$$H_{1F}^{\mathcal{T}_2}(V_1,\theta) \equiv \frac{1}{2}\cdot\Big(E_F\left[(\tau_{2F}(V_1,V_2,\theta))_+\,\middle|\,V_1\right] + E_F\left[(\tau_{2F}(V_2,V_1,\theta))_+\,\middle|\,V_1\right]\Big) - \mathcal{T}_{2F}(\theta) \tag{A-50}$$

and note that $E_F\left[H_{1F}^{\mathcal{T}_2}(V,\theta)\right] = 0$. Let

$$\begin{aligned}\widetilde{g}_F^{\mathcal{T}_2}(V_1,V_2,\theta) &\equiv \left(\frac{1}{2}\cdot\Big((\tau_{2F}(V_1,V_2,\theta))_+ + (\tau_{2F}(V_2,V_1,\theta))_+\Big) - \mathcal{T}_{2F}(\theta)\right) - H_{1F}^{\mathcal{T}_2}(V_1,\theta) - H_{1F}^{\mathcal{T}_2}(V_2,\theta),\\ S_{2,n}^{\mathcal{T}_2}(\theta) &\equiv \binom{n}{2}^{-1}\sum_{i<j}\widetilde{g}_F^{\mathcal{T}_2}(V_i,V_j,\theta).\end{aligned} \tag{A-51}$$

The Hoeffding decomposition of $\frac{1}{n\cdot(n-1)}\sum_{i=1}^n\sum_{j\neq i}\Big(\tau_{2F}(V_i,V_j,\theta)\Big)_+$ yields,

$$\frac{1}{n\cdot(n-1)}\sum_{i=1}^n\sum_{j\neq i}\Big(\tau_{2F}(V_i,V_j,\theta)\Big)_+ = \mathcal{T}_{2F}(\theta) + \frac{2}{n}\sum_{i=1}^n H_{1F}^{\mathcal{T}_2}(V_i,\theta) + S_{2,n}^{\mathcal{T}_2}(\theta) \equiv \binom{n}{2}^{-1}\sum_{i<j}\widetilde{g}_F^{\mathcal{T}_2}(V_i,V_j,\theta). \tag{A-52}$$

We proceed by focusing on the degenerate U-process $\left\{S_{2,n}^{\mathcal{T}_2}(\theta) : \theta\in\Theta\right\}$. Fix any finite $\overline{M}$ and consider the class of functions,

$$\mathscr{H}_4^{\overline{M}} \equiv \left\{m(x_2,x_2') = \alpha_1 + (x_2-x_2')'\alpha_2 : \|(\alpha_1,\alpha_2')'\| \leq \overline{M}\right\}.$$

By Pakes and Pollard (1989, Example 2.9), there exist $(\overline{A},\overline{V})$ such that $\mathscr{H}_4$ is a Euclidean $(\overline{A},\overline{V})$ class of functions for envelope $\overline{H}(x_2,x_2') \equiv \overline{M}\cdot(1\vee\|x_2-x_2'\|)$. Now let

$$\begin{aligned}&\mathscr{H}_{4,F} \equiv\\ &\left\{m(v,v') = \Big(R_{2F}(v)Q_{2F}(v') - R_{2F}(v')Q_{2F}(v) - (g_2(x_2,\theta_2) - g_2(x_2',\theta_2))\,Q_{2F}(v)Q_{2F}(v')\Big)\cdot\phi_2(v)\phi_2(v') : \beta_2\in\Theta\right\}.\end{aligned}$$

Assumptions 2 and 3 imply that there exists $\overline{M}<\infty$ such that $\mathscr{H}_{4,F}\subseteq\mathscr{H}_4^{\overline{M}}$ for all $F\in\mathcal{F}$. Therefore, there exist constants $(\overline{A},\overline{V})$ such that $\mathscr{H}_{4,F}$ is Euclidean $(\overline{A},\overline{V})$ for all $F\in\mathcal{F}$. Next, recall from Assumption 3 that the class of sets

$$\mathscr{C} \equiv \left\{(w_1,w_1)\in\mathbb{R}^{d_U}\times\mathbb{R}^{d_L} : g_{1U}(w_1,\theta_1)\leq g_{1L}(w_1,\theta_1)\ \text{ for some } \theta_1\in\Theta\right\}$$

is a VC class with VC dimension $\overline{V}_C$, and that the following is a VC class of sets for each F, with VC dimension uniformly bounded over $\mathcal{F}$ by a finite constant $\overline{V}_D$,

$$\mathscr{D}_{1,F}^{\tau_2} \equiv \left\{(v_1,v_2)\in\mathbb{R}^{L_v}\times\mathbb{R}^{L_v} : \tau_{2F}(v_1,v_2,\theta)\geq 0\ \text{ for some } \theta\in\Theta\right\}$$

These VC properties imply, by the results in Pakes and Pollard (1989, p. 1033) (the result that classes of indicator functions over VC classes of sets are Euclidean (A,V), with (A,V) depending only on the VC-dimension of the underlying class of sets), and Pakes and Pollard (1989, Lemma 2.14) (the product of Euclidean classes of functions is also a Euclidean class) that there exist constants $(\overline{A}', \overline{V}')$ such that, for each $F \in \mathcal{F}$, the class of indicator functions

$$\mathscr{I}_{4,F} \equiv \Big\{ m(v,v') = \mathbb{1}\{g_{1U}(w_1', \theta_1) \leq g_{1L}(w_1,\theta_1)\} \cdot \mathbb{1}\{\tau_{2F}(v,v',\theta) \geq 0\} \Big\},$$

is Euclidean $(\overline{A}', \overline{V}')$ for the constant envelope 1. Recall that

$$\begin{aligned}
&\tau_{2F}(v,v',\theta) = \\
&\Big((R_{2F}(v)Q_{2F}(v') - R_{2F}(v')Q_{2F}(v)) - (g_2(x_2,\theta_2) - g_2(x_2',\theta_2))\, Q_{2F}(v)Q_{2F}(v') \Big) \cdot \mathbb{1}\{g_{1U}(w_1',\theta_1) \leq g_{1L}(w_1,\theta_1)\} \\
&\cdot \phi_2(v)\phi_2(v').
\end{aligned}$$

and $(\tau_{2F}(v,v',\theta))_+ \equiv \tau_{2F}(v,v',\theta) \cdot \mathbb{1}\{\tau_{2F}(v,v',\theta) \geq 0\}$. Using the Euclidean properties of the classes of functions $\mathscr{D}^{\tau_2}_{1,F}$ and $\mathscr{H}_{4,F}$ described above, applying Pakes and Pollard (1989, Lemma 2.14), there exist constants $(\overline{A}_2, \overline{V}_2)$ such that, for each $F \in \mathcal{F}$, the class of functions

$$\mathscr{G}^{\tau_2}_F \equiv \{ m(v,v') = (\tau_{2F}(v,v',\theta))_+ \, : \, \theta \in \Theta \}$$

is Euclidean $(\overline{A}_2, \overline{V}_2)$ for an envelope of the form $G(v_1,v_2) = \overline{C}_1 + \overline{C}_2 \cdot \left\| x_2 - x_2' \right\| \cdot \phi(v)\phi(v')$, where $\overline{C}_1$ and $\overline{C}_2$ are finite constants. From the conditions in Assumption 2, there exists a finite constant $\overline{D}$ such that,

$$\sup_{\substack{x_2, x_2' \in \mathcal{V}\times\mathcal{V} \\ \theta_2 \in \Theta}} \left| g_2(x_2,\theta_2) - g_2(x_2',\theta_2) \right| \leq \overline{D}$$

Therefore, trivially there exists a constant $\overline{\mu}_4$ such that $E_F\left[G(V_1,V_2)^4 \right] \leq \overline{\mu}_4 \ \forall\, F \in \mathcal{F}$, and the conditions for Result A1 are satisfied, and from there we have that the degenerate U-process $S^{\mathcal{T}_2}_{2,n}(\cdot)$ defined in (A-51) satisfies,

$$\sup_{\theta\in\Theta} \left| S^{\mathcal{T}_2}_{2,n}(\theta) \right| = O_p\left(\frac{1}{n}\right) = o_p\left(\frac{1}{n^{1/2+\epsilon}}\right), \quad \text{uniformly over } \mathcal{F}, \tag{A-53}$$

where $\epsilon > 0$ is the constant described in Assumption 4. Let $H^{\mathcal{T}_2}_{2F}(Z_i,\theta)$ be as defined in (A-46), and denote

$$\psi^{\mathcal{T}_2}_F(Z_i,\theta) \equiv 2 \cdot H^{\mathcal{T}_2}_{1F}(V_i,\theta) + H^{\mathcal{T}_2}_{2F}(Z_i,\theta). \tag{A-54}$$

Note that $E_F\left[\psi_F^{\mathcal{T}_2}(Z,\theta)\right]=0$. Plugging the result in (A-53) into (A-52) and (A-49), we obtain the linear representation result for $\widehat{T}_2(\theta)$ given in part (A) of Theorem 1,

$$\widehat{\mathcal{T}}_2(\theta)=\mathcal{T}_{2F}(\theta)+\frac{1}{n}\sum_{i=1}^{n}\psi_F^{\mathcal{T}_2}(Z_i,\theta)+\xi_n^{\mathcal{T}_2}(\theta),\quad \text{where}\quad \sup_{\theta\in\Theta}\left|\xi_n^{\mathcal{T}_2}(\theta)\right|=o_p\left(\frac{1}{n^{1/2+\epsilon}}\right),\quad \text{uniformly over } \mathcal{F},$$

where $\epsilon>0$ is the constant described in Assumption 4. **This concludes the proof of part (A) of Theorem 1.** Part (B) is proved following analogous steps. Let

$$\begin{aligned}
\eta_{a,F}^{\tau_1}(w_1,\theta_1)&\equiv E_F\left[R_{1F}(W_1)\mathbb{1}\{g_{1U}(w_1,\theta_1)\le g_{1L}(W_1,\theta_1)\}\mathbb{1}\{\tau_{1F}(W_1,w_1,\theta)\ge 0\}\phi_1(W_1)\right],\\
\eta_{b,F}^{\tau_1}(w_1,\theta_1)&\equiv E_F\left[R_{1F}(W_1)\mathbb{1}\{g_{1U}(W_1,\theta_1)\le g_{1L}(w_1,\theta_1)\}\mathbb{1}\{\tau_{1F}(w_1,W_1,\theta)\ge 0\}\phi_1(W_1)\right],\\
\eta_{c,F}^{\tau_1}(w_1,\theta_1)&\equiv E_F\left[Q_{1F}(W_1)\mathbb{1}\{g_{1U}(W_1,\theta_1)\le g_{1L}(w_1,\theta_1)\}\mathbb{1}\{\tau_{1F}(w_1,W_1,\theta)\ge 0\}\phi_1(W_1)\right],\\
\eta_{d,F}^{\tau_1}(w_1,\theta_1)&\equiv E_F\left[Q_{1F}(W_1)\mathbb{1}\{g_{1U}(w_1,\theta_1)\le g_{1L}(W_1,\theta_1)\}\mathbb{1}\{\tau_{1F}(W_1,w_1,\theta)\ge 0\}\phi_1(W_1)\right]
\end{aligned}$$

and,

$$\begin{aligned}
H_{1F}^{\mathcal{T}_1}(W_1,\theta_1)&\equiv\frac{1}{2}\cdot\left(E_F\left[(\tau_{1F}(W_1,W_2,\theta_1))_+\,\middle|\,W_1\right]+E_F\left[(\tau_{1F}(W_2,W_1,\theta_1))_+\,\middle|\,W_1\right]\right)-\mathcal{T}_{1F}(\theta_1),\\
H_{2F}^{\mathcal{T}_1}(Z_i,\theta_1)&\equiv\left(\left(\eta_{a,F}^{\tau_1}(W_{1i},\theta_1)-\eta_{b,F}^{\tau_1}(W_{1i},\theta_1)\right)+\left(\eta_{c,F}^{\tau_1}(W_{1i},\theta_1)-\eta_{d,F}^{\tau_1}(W_{1i},\theta_1)\right)\cdot Y_{1i}\right)\cdot f_{W_1}(W_{1i})\cdot\phi_1(W_{1i})^2\\
&\quad-E_F\left[\left(\left(\eta_{a,F}^{\tau_1}(W_1,\theta_1)-\eta_{b,F}^{\tau_1}(W_1,\theta_1)\right)+\left(\eta_{c,F}^{\tau_1}(W_1,\theta_1)-\eta_{d,F}^{\tau_1}(W_1,\theta_1)\right)\cdot Y_1\right)\cdot f_{W_1}(W_1)\cdot\phi_1(W_1)^2\right],\\
\psi_F^{\mathcal{T}_1}(Z_i,\theta_1)&\equiv 2\cdot H_{1F}^{\mathcal{T}_1}(W_{1i},\theta_1)+H_{2F}^{\mathcal{T}_1}(Z_i,\theta_1).
\end{aligned}\tag{A-55}$$

Using parallel steps to the proof of part (A), we can show that,

$$\begin{aligned}
&\widehat{\mathcal{T}}_1(\theta_1)=\mathcal{T}_{1F}(\theta_1)+\frac{1}{n}\sum_{i=1}^{n}\psi_F^{\mathcal{T}_1}(Z_i,\theta_1)+\xi_n^{\mathcal{T}_1}(\theta_1),\quad\text{where}\\
&\sup_{\theta_1\in\Theta}\left|\xi_n^{\mathcal{T}_1}(\theta_1)\right|=o_p\left(\frac{1}{n^{1/2+\epsilon}}\right),\quad\text{uniformly over }\mathcal{F}.
\end{aligned}$$

where $\epsilon>0$ is the constant described in Assumption 4. **This is the result in part (B) of Theorem 1.** Part (C) of the theorem follows immediately from (A) and (B). **This completes the proof of Theorem 1.** ■

A2 Estimation of $\sigma_F^2(\theta)$

In this section we study the asymptotic properties of the estimator for $\sigma_F^2(\theta)\equiv E_F[\psi_F^{\mathcal{T}}(Z,\theta)^2]$ we described in Section 3.6.1 of the paper. Our construction uses the structure of the influence function $\psi_F^{\mathcal{T}}(z,\theta)$ in Theorem 1.

A2.1 Estimation of the influence function $\psi_F^{\mathcal{T}}(z,\theta)$

We use sample analog estimators of the components described in the structure of the influence function $\psi_F^{\mathcal{T}}(z,\theta)$ in Theorem 1. We will describe separately how we estimated $\psi_F^{\mathcal{T}_2}(z,\theta)$ and $\psi_F^{\mathcal{T}_1}(z,\theta_1)$.

A2.1.1 Estimation of $\psi_F^{\mathcal{T}_2}(z,\theta)$

We construct our estimators using sample analogs. Based on the structure described in (A-50), for a given (v,θ), we estimate $H_{1F}^{\mathcal{T}_2}(v,\theta)$ as,

$$\widehat{H}_1^{\mathcal{T}_2}(v,\theta) \equiv \frac{1}{2}\cdot\frac{1}{n}\sum_{j=1}^{n}\Big[\widehat{\tau}_2(v,V_j,\theta)\mathbb{1}\Big\{\widehat{\tau}_2(v,V_j,\theta)\geq -b_n\Big\}+\widehat{\tau}_2(V_j,v,\theta)\mathbb{1}\Big\{\widehat{\tau}_2(V_j,v,\theta)\geq -b_n\Big\}\Big]-\widehat{\mathcal{T}}_2(\theta).$$

And, based on the structure described in (A-46), for a given $z\equiv(y_1,y_2,v)$, we estimate $H_{2F}^{\mathcal{T}_2}(z,\theta)$ as,

$$\begin{aligned}\widehat{H}_2^{\mathcal{T}_2}(z,\theta) \equiv &\Big(\Big(\widehat{\eta}_a^{\mathcal{T}_2}(v,\theta)-\widehat{\eta}_b^{\mathcal{T}_2}(v,\theta)\Big)\cdot y_1+\Big(\widehat{\eta}_c^{\mathcal{T}_2}(v,\theta)-\widehat{\eta}_d^{\mathcal{T}_2}(v,\theta)\Big)\cdot y_2 y_1\Big)\cdot\widehat{f}_V(v)\cdot\phi_2(v)^2\\ &-\frac{1}{n}\sum_{j=1}^{n}\Big[\Big(\Big(\widehat{\eta}_a^{\mathcal{T}_2}(V_j,\theta)-\widehat{\eta}_b^{\mathcal{T}_2}(V_j,\theta)\Big)\cdot Y_{1j}+\Big(\widehat{\eta}_c^{\mathcal{T}_2}(V_j,\theta)-\widehat{\eta}_d^{\mathcal{T}_2}(V_j,\theta)\Big)\cdot Y_{2j}Y_{1j}\Big)\cdot\widehat{f}_V(V_j)\cdot\phi_2(V_j)^2\Big].\end{aligned} \tag{A-56}$$

From here, using the definition in (A-54), for a given $z\equiv(y_1,y_2,v)$, we estimate $\psi_F^{\mathcal{T}_2}(z,\theta)$ as

$$\widehat{\psi}^{\mathcal{T}_2}(z,\theta)\equiv 2\cdot\widehat{H}_1^{\mathcal{T}_2}(v,\theta)+\widehat{H}_2^{\mathcal{T}_2}(z,\theta) \tag{A-57}$$

Let us analyze $\widehat{H}_1^{\mathcal{T}_2}(v,\theta)$ first. First, by the results in (A-15) and (A-16), we have

$$\sup_{\substack{v\in\mathbb{R}^{L_V}\\ \theta\in\Theta}}\left|\widehat{H}_1^{\mathcal{T}_2}(v,\theta)-\left(\frac{1}{2}\cdot\frac{1}{n}\sum_{j=1}^{n}\Big[\Big(\tau_{2F}(v,V_j,\theta)\Big)_+ +\Big(\tau_{2F}(V_j,v,\theta)\Big)_+\Big]-\mathcal{T}_{2F}(\theta)\right)\right| = o_p(1),$$

uniformly over $\mathcal{F}$.

As we have pointed out previously (see equation A-14), by the conditions in Assumption 2, there exists a finite constant $\overline{\tau}_2$ such that $\sup_{\substack{v,v'\in\mathbb{R}^{L_V}\times\mathbb{R}^{L_V}\\ \theta\in\Theta}}|\tau_{2F}(v,v',\theta)|\leq\overline{\tau}_2\ \forall\ F\in\mathcal{F}$. By a Chebyshev inequality argument, this implies

$$\sup_{\substack{v\in\mathbb{R}^{L_V}\\ \theta\in\Theta}}\left|\frac{1}{n}\sum_{j=1}^{n}\Big[\Big(\tau_{2F}(v,V_j,\theta)\Big)_+ +\Big(\tau_{2F}(V_j,v,\theta)\Big)_+\Big]-E_F\Big[(\tau_{2F}(v,V,\theta))_+ +(\tau_{2F}(V,v,\theta))_+\Big]\right| = o_p(1),$$

uniformly over $\mathcal{F}$.

Combining both previous results, we obtain

$$\sup_{\substack{v\in\mathbb{R}^{L_V}\\ \theta\in\Theta}} \left|\widehat{H}_1^{\mathcal{T}_2}(v,\theta) - H_{1F}^{\mathcal{T}_2}(v,\theta)\right| = o_p(1), \quad \text{uniformly over } \mathcal{F}. \tag{A-58}$$

Next, we analyze $\widehat{H}_2^{\mathcal{T}_2}(z,\theta)$. We begin by analyzing the estimators used in (A-56). Using the definitions in (18), we construct the estimators in on the right hand side of (A-56) as,

$$\begin{aligned}
\widehat{\eta_a^{\tau_2}}(v,\theta) &\equiv \frac{1}{n}\sum_{j=1}^{n}\left(\widehat{R}_2(V_j) - \left(g_2(X_{2j},\theta_2) - g_2(x_2,\theta_2)\right)\widehat{Q}_2(V_j)\right)\mathbb{1}\left\{g_{1U}(w_1,\theta_1) \le g_{1L}(W_{1j},\theta_1)\right\}\phi_2(V_j)\\
&\quad\cdot\mathbb{1}\left\{\widehat{\tau}_2(V_j,v,\theta) \ge -b_n\right\},\\
\widehat{\eta_b^{\tau_2}}(v,\theta) &\equiv \frac{1}{n}\sum_{j=1}^{n}\left(\widehat{R}_2(V_j) - \left(g_2(X_{2j},\theta_2) - g_2(x_2,\theta_2)\right)\widehat{Q}_2(V_j)\right)\mathbb{1}\left\{g_{1U}(W_{1j},\theta_1) \le g_{1L}(w_1,\theta_1)\right\}\phi_2(V_j)\\
&\quad\cdot\mathbb{1}\left\{\widehat{\tau}_2(v,V_j,\theta) \ge -b_n\right\},\\
\widehat{\eta_c^{\tau_2}}(v,\theta) &\equiv \frac{1}{n}\sum_{j=1}^{n}\widehat{Q}_2(V_j)\mathbb{1}\left\{g_{1U}(W_{1j},\theta_1) \le g_{1L}(w_1,\theta_1)\right\}\phi_2(V_j)\mathbb{1}\left\{\widehat{\tau}_2(v,V_j,\theta) \ge -b_n\right\},\\
\widehat{\eta_d^{\tau_2}}(v,\theta) &\equiv \frac{1}{n}\sum_{j=1}^{n}\widehat{Q}_2(V_j)\mathbb{1}\left\{g_{1U}(w_1,\theta_1) \le g_{1L}(W_{1j},\theta_1)\right\}\phi_2(V_j)\mathbb{1}\left\{\widehat{\tau}_2(V_j,v,\theta) \ge -b_n\right\}.
\end{aligned} \tag{A-59}$$

Let

$$\begin{aligned}
\varphi^{\eta_a^{\tau_2}}(Z_i,Z_j,v,\theta,h) &\equiv \left(Y_{2i} - \left(g_2(X_{2j},\theta_2) - g_2(x_2,\theta_2)\right)\right)Y_{1i}\Gamma(V_i,V_j,h)\phi_2(V_i)\phi_2(V_j)\\
&\quad\cdot\mathbb{1}\left\{g_{1U}(w_1,\theta_1) \le g_{1L}(W_{1j},\theta_1)\right\},\\
\varphi^{\eta_b^{\tau_2}}(Z_i,Z_j,v,\theta,h) &\equiv \left(Y_{2i} - \left(g_2(X_{2j},\theta_2) - g_2(x_2,\theta_2)\right)\right)Y_{1i}\Gamma(V_i,V_j,h)\phi_2(V_i)\phi_2(V_j)\\
&\quad\cdot\mathbb{1}\left\{g_{1U}(W_{1j},\theta_1) \le g_{1L}(w_1,\theta_1)\right\},\\
\varphi^{\eta_c^{\tau_2}}(Z_i,Z_j,w_1,\theta_1,h) &\equiv \mathbb{1}\left\{g_{1U}(W_{1j},\theta_1) \le g_{1L}(w_1,\theta_1)\right\}Y_{1i}\Gamma(V_i,V_j,h)\phi_2(V_i)\phi_2(V_j),\\
\varphi^{\eta_d^{\tau_2}}(Z_i,Z_j,w_1,\theta_1,h) &\equiv \mathbb{1}\left\{g_{1U}(w_1,\theta_1) \le g_{1L}(W_{1j},\theta_1)\right\}Y_{1i}\Gamma(V_i,V_j,h)\phi_2(V_i)\phi_2(V_j),
\end{aligned}$$

From the constructions of $\widehat{R}_2$ and $\widehat{Q}_2$ (see (16)), our estimators in (A-59) are,

$$\begin{aligned}
\widehat{\eta_a^{\tau_2}}(v,\theta) &= \frac{1}{h_n^r}\cdot\frac{1}{n(n-1)}\sum_{j=1}^{n}\sum_{i\neq j}\varphi^{\eta_a^{\tau_2}}(Z_i,Z_j,v,\theta,h_n)\mathbb{1}\left\{\widehat{\tau_2}(V_j,v,\theta)\geq -b_n\right\},\\
\widehat{\eta_b^{\tau_2}}(v,\theta) &= \frac{1}{h_n^r}\cdot\frac{1}{n(n-1)}\sum_{j=1}^{n}\sum_{i\neq j}\varphi^{\eta_b^{\tau_2}}(Z_i,Z_j,v,\theta,h_n)\mathbb{1}\left\{\widehat{\tau_2}(v,V_j,\theta)\geq -b_n\right\},\\
\widehat{\eta_c^{\tau_2}}(v,\theta) &= \frac{1}{h_n^r}\cdot\frac{1}{n(n-1)}\sum_{j=1}^{n}\sum_{i\neq j}\varphi^{\eta_c^{\tau_2}}(Z_i,Z_j,w_1,\theta_1,h_n)\mathbb{1}\left\{\widehat{\tau_2}(v,V_j,\theta)\geq -b_n\right\},\\
\widehat{\eta_d^{\tau_2}}(v,\theta) &= \frac{1}{h_n^r}\cdot\frac{1}{n(n-1)}\sum_{j=1}^{n}\sum_{i\neq j}\varphi^{\eta_d^{\tau_2}}(Z_i,Z_j,w_1,\theta_1,h_n)\mathbb{1}\left\{\widehat{\tau_2}(V_j,v,\theta)\geq -b_n\right\}.
\end{aligned}\tag{A-60}$$

If Assumptions 1-5 hold, we have

$$\left.\begin{array}{ll}
\sup\limits_{\substack{v\in\mathcal{V}\\ \theta\in\Theta}}\left|\widehat{\eta_a^{\tau_2}}(v,\theta)-\eta_{a,F}^{\tau_2}(v,\theta)\right| = o_p(1) & \sup\limits_{\substack{v\in\mathcal{V}\\ \theta\in\Theta}}\left|\widehat{\eta_b^{\tau_2}}(v,\theta)-\eta_{a,F}^{\tau_2}(v,\theta)\right| = o_p(1)\\
\sup\limits_{\substack{v\in\mathcal{V}\\ \theta\in\Theta}}\left|\widehat{\eta_c^{\tau_2}}(v,\theta)-\eta_{a,F}^{\tau_2}(v,\theta)\right| = o_p(1) & \sup\limits_{\substack{v\in\mathcal{V}\\ \theta\in\Theta}}\left|\widehat{\eta_d^{\tau_2}}(v,\theta)-\eta_{a,F}^{\tau_2}(v,\theta)\right| = o_p(1)
\end{array}\right\}\text{ uniformly over } \mathcal{F}.\tag{A-61}$$

We will show the above result for $\widehat{\eta_a^{\tau_2}}(v,\theta)$. The proof for the remaining estimators in (A-61) follows analogous steps. Our first step is to express,

$$\begin{aligned}
\widehat{\eta_a^{\tau_2}}(v,\theta) &= \frac{1}{h_n^r}\cdot\frac{1}{n(n-1)}\sum_{j=1}^{n}\sum_{i\neq j}\varphi^{\eta_a^{\tau_2}}(Z_i,Z_j,v,\theta,h_n)\mathbb{1}\left\{\tau_{2F}(V_j,v,\theta)\geq 0\right\}+\xi_n^{\eta_a^{\tau_2}}(v,\theta),\quad\text{where}\\
\xi_n^{\eta_a^{\tau_2}}(v,\theta) &\equiv \frac{1}{h_n^r}\cdot\frac{1}{n(n-1)}\sum_{j=1}^{n}\sum_{i\neq j}\varphi^{\eta_a^{\tau_2}}(Z_i,Z_j,v,\theta,h_n)\left(\mathbb{1}\left\{\widehat{\tau_2}(V_j,v,\theta)\geq -b_n\right\}-\mathbb{1}\left\{\tau_{2F}(V_j,v,\theta)\geq 0\right\}\right)
\end{aligned}\tag{A-62}$$

We will first show that $\sup\limits_{\substack{v\in\mathcal{V}\\ \theta\in\Theta}}|\xi_n^{\eta_a^{\tau_2}}(v,\theta)| = o_p(1)$, uniformly over $\mathcal{F}$. Note first that, as we pointed out in equations (A-13) and (A-24), we have

$$\begin{aligned}
&\left|\mathbb{1}\left\{\widehat{\tau_2}(V_j,v,\theta)\geq -b_n\right\}-\mathbb{1}\left\{\tau_{2F}(V_j,v,\theta)\geq 0\right\}\right|\\
&\qquad\qquad\leq \mathbb{1}\left\{\left|\widehat{\tau_2}(V_j,v,\theta)-\tau_{2F}(V_j,v,\theta)\right|\geq b_n\right\}+\mathbb{1}\left\{-2b_n\leq \tau_{2F}(V_j,v,\theta)<0\right\}.
\end{aligned}$$

Next, recall from Assumption 2 that, there exists a finite constant $\overline{D}$ such that, $|g_2(x_2,\theta_2)|\leq\overline{D}$ $\forall$ $(x_2,\theta_2)\in\mathcal{V}\times\Theta$. Combined with the bounded properties of the weight function $\phi_2(\cdot)$ and the kernel $K(\cdot)$, Assumption 2 implies,

$$\begin{aligned}\left|\xi_n^{\eta_a^{\tau_2}}(v,\theta)\right| \leq & \left(\frac{1}{h_n^r}\cdot\frac{1}{n}\sum_{j=1}^{n}\left(\mathbb{1}\left\{\left|\widehat{\tau}_2(V_j,v,\theta)-\tau_{2F}(V_j,v,\theta)\right|\geq b_n\right\}+\mathbb{1}\left\{-2b_n\leq\tau_{2F}(V_j,v,\theta)<0\right\}\right)\right) \\ & \times\overline{\phi}^2\overline{K}\left(\frac{1}{n-1}\sum_{i\neq j}|Y_{2i}|+2\overline{D}\right)\end{aligned} \tag{A-63}$$

By Assumption 3, there exists $\overline{D}_4<\infty$ such that $E_F[|Y_2|^4]\leq\overline{D}_4$ for all $F\in\mathcal{F}$. Therefore, a Chebyshev inequality argument yields,

$$\frac{1}{n-1}\sum_{i\neq j}|Y_{2i}|=O_p(1),\quad \text{uniformly over } \mathcal{F}. \tag{A-64}$$

Take any $\delta>0$, note that

$$P_F\left(\sup_{\substack{v\in\mathcal{V}\\ \theta\in\Theta}}\left|\frac{1}{h_n^r}\cdot\frac{1}{n}\sum_{j=1}^{n}\mathbb{1}\left\{\left|\widehat{\tau}_2(V_j,v,\theta)-\tau_{2F}(V_j,v,\theta)\right|\geq b_n\right\}\right|>\delta\right)\leq P_F\left(\sup_{\substack{v,v'\in\mathbb{R}^{L_V}\times\mathbb{R}^{L_V}\\ \theta\in\Theta}}\left|\widehat{\tau}_2(v,v',\theta)-\tau_{2F}(v,v',\theta)\right|>b_n\right)$$

From equation (A-10),

$$\sup_{F\in\mathcal{F}}P_F\left(\sup_{\substack{v,v'\in\mathbb{R}^{L_V}\times\mathbb{R}^{L_V}\\ \theta\in\Theta}}\left|\widehat{\tau}_2(v,v',\theta)-\tau_{2F}(v,v',\theta)\right|>b_n\right)\longrightarrow 0.$$

Therefore,

$$\frac{1}{h_n^r}\cdot\frac{1}{n}\sum_{j=1}^{n}\mathbb{1}\left\{\left|\widehat{\tau}_2(V_j,v,\theta)-\tau_{2F}(V_j,v,\theta)\right|\geq b_n\right\}=o_p(1),\quad \text{uniformly over } \mathcal{F}. \tag{A-65}$$

Next, for a given (v,θ) and $c>0$, let

$$\overline{m}_F^{\eta_a^{\tau_2}}(v,\theta,c)\equiv\frac{1}{n}\sum_{j=1}^{n}\left(\mathbb{1}\left\{-c\leq\tau_{2F}(V_j,v,\theta)<0\right\}-E_F\left[\mathbb{1}\left\{-c\leq\tau_{2F}(V_j,v,\theta)<0\right\}\right]\right).$$

Note that,

$$\frac{1}{h_n^r}\cdot\frac{1}{n}\sum_{j=1}^{n}\mathbb{1}\left\{-2b_n\leq\tau_{2F}(V_j,v,\theta)<0\right\}=\frac{1}{h_n^r}\cdot\overline{m}_F^{\eta_a^{\tau_2}}(v,\theta,2b_n)+\frac{1}{h_n^r}E_F\left[\mathbb{1}\left\{-2b_n\leq\tau_{2F}(V,v,\theta)<0\right\}\right]. \tag{A-66}$$

By the properties of VC classes of sets described, e.g, in Pakes and Pollard (1989, Lemma 2.5), the conditions described in Assumption 3 imply that, for each $F \in \mathcal{F}$, the following class of sets is a VC class, with VC dimension uniformly bounded over $\mathcal{F}$ by a finite constant,

$$\mathscr{C}_{2,F}^{\tau_2} \equiv \Big\{ v \in \mathbb{R}^{L_v} : -c \leq \tau_{2F}(v,u,\theta) < 0 \text{ for some } 0 < c \leq c_0,\, u \in \mathcal{V}, \text{ and } \theta \in \Theta \Big\},$$

where the constant c_0 is as described in Assumption 3. From here, the result in Pakes and Pollard (1989, p. 1033) implies that there exist constants $(\overline{A},\overline{V})$ such that, for each $F \in \mathcal{F}$, the class of indicator functions,

$$\mathscr{H}_F \equiv \Big\{ m(u) = \mathbb{1}\{-c \leq \tau_{2F}(v,u,\theta) < 0\} \text{ for some } 0 < c \leq c_0,\, u \in \mathcal{V} \text{ and } \theta \in \Theta \Big\}$$

is Euclidean $(\overline{A},\overline{V})$ for the constant envelope 1. From here and Sherman (1994, Lemma 5), the conditions for Result A1 are satisfied and, from there, we obtain,

$$\sup_{\substack{\theta\in\Theta \\ v\in\mathcal{V} \\ 0<c\leq c_0}} \left| \frac{1}{n}\sum_{i=1}^{n} \overline{m}_F^{\eta_a^{\tau_2}}(v,\theta,c) \right| = O_p\left(\frac{1}{n^{1/2}}\right), \quad \text{uniformly over } \mathcal{F}.$$

For n large enough, we have $2b_n \leq c_0$. Therefore, by the above result and the condition in part (ii) of Assumption 5, equation (A-66) yields,

$$\frac{1}{h_n^r}\cdot \sup_{\substack{v\in\mathcal{V} \\ \theta\in\Theta}} \frac{1}{n}\sum_{j=1}^{n} \mathbb{1}\Big\{-2b_n \leq \tau_{2F}(V_j,v,\theta) < 0\Big\} \leq O_p\left(\frac{1}{h_n^r \cdot n^{1/2}}\right) + 2\overline{m}\cdot\frac{b_n}{h_n^r} = o_p(1), \quad \text{uniformly over } \mathcal{F} \tag{A-67}$$

where $\overline{m}$ is the constant described in Assumption 5. The last line follows from the bandwidth convergence conditions in Assumption 4, which require $h_n^r \cdot n^{1/2} \to \infty$ and $\frac{b_n}{h_n^r} \to 0$. Combining (A-63), (A-64), (A-65), and (A-67), we have

$$\sup_{\substack{v\in\mathcal{V} \\ \theta\in\Theta}} \left| \xi_n^{\eta_a^{\tau_2}}(v,\theta) \right| = o_p(1), \quad \text{uniformly over } \mathcal{F}.$$

Plugging this into (A-62), we obtain,

$$\begin{aligned} &\widehat{\eta_a^{\tau_2}}(v,\theta) = \frac{1}{h_n^r}\cdot\frac{1}{n(n-1)}\sum_{j=1}^{n}\sum_{i\neq j} \varphi^{\eta_a^{\tau_2}}(Z_i,Z_j,v,\theta,h_n)\mathbb{1}\Big\{\tau_{2F}(V_j,v,\theta) \geq 0\Big\} + \xi_n^{\eta_a^{\tau_2}}(v,\theta), \\ &\text{where } \sup_{\substack{v\in\mathcal{V} \\ \theta\in\Theta}} \left| \xi_n^{\eta_a^{\tau_2}}(v,\theta) \right| = o_p(1), \quad \text{uniformly over } \mathcal{F}. \end{aligned} \tag{A-68}$$

Next, let

$$U_{n,F}^{\eta_a^{\tau_2}}(v,\theta,h) \equiv$$
$$\frac{1}{n(n-1)}\sum_{j=1}^{n}\sum_{i\neq j}\Big(\varphi^{\eta_a^{\tau_2}}(Z_i,Z_j,v,\theta,h)\mathbb{1}\Big\{\tau_{2F}(V_j,v,\theta)\geq 0\Big\} - E_F\Big[\varphi^{\eta_a^{\tau_2}}(Z_i,Z_j,v,\theta,h)\mathbb{1}\Big\{\tau_{2F}(V_j,v,\theta)\geq 0\Big\}\Big]\Big).$$

We can rewrite (A-68) as,

$$\widehat{\eta}_a^{\tau_2}(v,\theta) = \frac{1}{h_n^r}\cdot U_{n,F}^{\eta_a^{\tau_2}}(v,\theta,h_n) + \frac{1}{h_n^r}\cdot E_F\Big[\varphi^{\eta_a^{\tau_2}}(Z_i,Z_j,v,\theta,h_n)\mathbb{1}\Big\{\tau_{2F}(V_j,v,\theta)\geq 0\Big\}\Big] + \xi_n^{\eta_a^{\tau_2}}(v,\theta) \qquad \text{(A-69)}$$

where, in the above expectation, Z_i, Z_j are two independent draws from F. We will analyze $U_{n,F}^{\eta_a^{\tau_2}}(v,\theta,h_n)$ first. Define the class of functions,

$$\mathscr{H}_F^{\eta_a^{\tau_2}} \equiv \Big\{m(z_1,z_2) = \varphi^{\eta_a^{\tau_2}}(z_1,z_2,u,\theta,h)\mathbb{1}\{\tau_{2F}(v_2,u,\theta)\geq 0\} \ \text{ for some } u\in\mathcal{V},\ \theta\in\Theta,\ h>0\Big\}$$

Invoking arguments and results from empirical process theory we have used previously, the smoothness, regularity and manageability conditions in Assumptions 2 and 3, and the bounded-variation properties of the kernel described in Assumption 2 imply, by Pakes and Pollard (1989, Lemma 2.14), that there exist constants $(\overline{A}_2,\overline{V}_2)$ such that, for each $F\in\mathcal{F}$, the class of functions $\mathscr{H}_F^{\eta_a^{\tau_2}}$ is Euclidean $(\overline{A}_2,\overline{V}_2)$ for an envelope $\overline{G}_2(z_1,z_2)$ such that there exists a constant $\overline{C}_2<\infty$ for which $E_F[\overline{G}_2(Z_1,Z_2)^4]\leq\overline{C}_2$ for all $F\in\mathcal{F}$. Thus, the conditions in Result A1 are satisfied and from there we obtain,

$$\sup_{\substack{\theta\in\Theta\\ v\in\mathcal{V}}}\Big|U_{n,F}^{\eta_a^{\tau_2}}(v,\theta,h_n)\Big| = O_p\Big(\frac{1}{n^{1/2}}\Big), \quad \text{uniformly over } \mathcal{F} \qquad \text{(A-70)}$$

Next, using an M^{th}–order approximation, the smoothness conditions in Assumption 2, and the bias-reducing properties of the kernel described in Assumption 4 imply that there exists a constant $\overline{B}^{\eta_a^{\tau_2}}<\infty$ such that,

$$\begin{aligned}&\frac{1}{h_n^r}\cdot E_F\Big[\varphi^{\eta_a^{\tau_2}}(Z_i,Z_j,v,\theta,h_n)\mathbb{1}\Big\{\tau_{2F}(V_j,v,\theta)\geq 0\Big\}\Big] = \eta_{a,F}^{\tau_2}(v,\theta) + \underbrace{B_n^{\eta_a^{\tau_2}}(v,\theta)}_{\text{bias}},\\ &\text{where} \quad \sup_{\substack{\theta\in\Theta\\ v\in\mathcal{V}}}\Big|B_n^{\eta_a^{\tau_2}}(v,\theta)\Big| \leq \overline{B}^{\eta_a^{\tau_2}}\cdot h_n^M \quad \forall\, F\in\mathcal{F}\end{aligned} \qquad \text{(A-71)}$$

Plugging (A-70) and (A-71) into (A-69), we obtain

$$\begin{aligned}\sup_{\substack{\theta\in\Theta\\ v\in\mathcal{V}}}\left|\widehat{\eta_a}^{\tau_2}(v,\theta)-\eta^{\tau_2}_{a,F}(v,\theta)\right| &\leq \frac{1}{h_n^r}\cdot\sup_{\substack{\theta\in\Theta\\ v\in\mathcal{V}}}\left|U_{n,F}^{\eta_a^{\tau_2}}(v,\theta,h_n)\right|+\sup_{\substack{\theta\in\Theta\\ v\in\mathcal{V}}}\left|B_n^{\eta_a^{\tau_2}}(v,\theta)\right|\\ &= O_p\left(\frac{1}{h_n^r\cdot n^{1/2}}\right)+O\left(h_n^M\right)=o_p(1),\quad \text{uniformly over } \mathcal{F}.\end{aligned}$$

which proves the claim in (A-61) for $\widehat{\eta_a}^{\tau_2}(v,\theta)$. Using our assumptions, proving the claim in (A-61) for $\widehat{\eta_b}^{\tau_2}(v,\theta)$, $\widehat{\eta_c}^{\tau_2}(v,\theta)$ and $\widehat{\eta_d}^{\tau_2}(v,\theta)$ follows analogous steps.

Let us continue with $\widehat{f_V}(v)$, which is also used in (A-56). As we have detailed before, for a given v, we have

$$\widehat{f_V}(v)\equiv\frac{1}{h_n^r}\cdot\frac{1}{n}\sum_{i=1}^n\Gamma(V_i,v,h_n).$$

A result we have used previously is that, by Nolan and Pollard (1987, Lemma 22) (or Pakes and Pollard (1989, Example 10)), the bounded variation nature of our kernel implies that the class of functions $\left\{m(v)=k\left(\frac{v-u}{h}\right) \text{ for some } u\in\mathbb{R},\, h>0\right\}$ is Euclidean (A_k,V_k) for the constant envelope $\overline{k}$ (neither (A_k,V_k), nor $\overline{k}$ depend on F). From here and Sherman (1994, Lemma 5), the following empirical process satisfies the conditions of Result A1,

$$\nu_n^{f_V}(v)\equiv\frac{1}{n}\sum_{i=1}^n\left(\Gamma(V_i,v,h_n)-E_F\left[\Gamma(V_i,v,h_n)\right]\right),$$

and we have, $\sup_{v\in\mathbb{R}^{L_V}}\left|\nu_n^{f_V}(v)\right|=O_p\left(\frac{1}{n^{1/2}}\right)$, uniformly over $\mathcal{F}$. Next, using an M^{th}–order approximation, the smoothness conditions in Assumption 2, and the bias-reducing properties of the kernel described in Assumption 4 imply that there exists a constant $\overline{B}^{f_v}<\infty$ such that,

$$\frac{1}{h_n^r}\cdot E_F\left[\Gamma(V_i,v,h_n)\right]=f_V(v)+\underbrace{B_n^{f_V}(v)}_{\text{bias}},$$

$$\text{where}\quad \sup_{v\in\mathcal{V}}\left|B_n^{f_V}(v)\right|\leq\overline{B}^{f_v}\cdot h_n^M\quad\forall\, F\in\mathcal{F}$$

Combining these results, we have

$$\begin{aligned}\sup_{v\in\mathcal{V}}\left|\widehat{f_V}(v)-f_V(v)\right| &\leq \frac{1}{h_n^r}\cdot\sup_{v\in\mathcal{V}}\left|\nu_n^{f_V}(v)\right|+\sup_{v\in\mathcal{V}}\left|B_n^{f_V}(v)\right|\\ &= O_p\left(\frac{1}{h_n^r\cdot n^{1/2}}\right)+O\left(h_n^M\right)=o_p(1),\quad\text{uniformly over }\mathcal{F}.\end{aligned}\tag{A-72}$$

Plugging in the results in (A-61) and (A-72) into (A-56), for any y_1, y_2, we have[14]

$$\sup_{\substack{\theta\in\Theta\\ v\in\mathbb{R}^{L_V}}} \Bigg|\widehat{H}_2^{\mathcal{T}_2}(z,\theta) - \Bigg\{\Big(\Big(\eta_{a,F}^{\tau_2}(v,\theta)-\eta_{b,F}^{\tau_2}(v,\theta)\Big)\cdot y_1 + \Big(\eta_{c,F}^{\tau_2}(v,\theta)-\eta_{d,F}^{\tau_2}(v,\theta)\Big)\cdot y_2 y_1\Big)\cdot f_V(v)\cdot \phi_2(v)^2$$
$$-\frac{1}{n}\sum_{j=1}^{n}\Big[\Big(\Big(\eta_{a,F}^{\tau_2}(V_j,\theta)-\eta_{b,F}^{\tau_2}(V_j,\theta)\Big)\cdot Y_{1j} + \Big(\eta_{c,F}^{\tau_2}(V_j,\theta)-\eta_{d,F}^{\tau_2}(V_j,\theta)\Big)\cdot Y_{2j}Y_{1j}\Big)\cdot f_V(V_j)\cdot\phi_2(V_j)^2\Big]\Bigg\}\Bigg| = o_p(1),$$

uniformly over $\mathcal{F}$. (A-73)

By the conditions of Assumption 2, there exists a $\overline{\mu}_4^{\tau_2}$ such that,

$$\sup_{\theta\in\Theta} E_F\left[\left|\Big(\Big(\eta_{a,F}^{\tau_2}(V,\theta)-\eta_{b,F}^{\tau_2}(V,\theta)\Big)\cdot Y_1 + \Big(\eta_{c,F}^{\tau_2}(V,\theta)-\eta_{d,F}^{\tau_2}(V,\theta)\Big)\cdot Y_2Y_1\Big)\cdot f_V(V)\cdot\phi_2(V)^2\right|^4\right] \leq \overline{\mu}_4^{\tau_2}$$

$\forall\ F\in\mathcal{F}$.

From here, a Chebyshev inequality argument yields,

$$\sup_{\theta\in\Theta}\Bigg|\frac{1}{n}\sum_{j=1}^{n}\Big[\Big(\Big(\eta_{a,F}^{\tau_2}(V_j,\theta)-\eta_{b,F}^{\tau_2}(V_j,\theta)\Big)\cdot Y_{1j} + \Big(\eta_{c,F}^{\tau_2}(V_j,\theta)-\eta_{d,F}^{\tau_2}(V_j,\theta)\Big)\cdot Y_{2j}Y_{1j}\Big)\cdot f_V(V_j)\cdot\phi_2(V_j)^2\Big]$$
$$- E_F\Big[\Big(\Big(\eta_{a,F}^{\tau_2}(V,\theta)-\eta_{b,F}^{\tau_2}(V,\theta)\Big)\cdot Y_1 + \Big(\eta_{c,F}^{\tau_2}(V,\theta)-\eta_{d,F}^{\tau_2}(V,\theta)\Big)\cdot Y_2Y_1\Big)\cdot f_V(V)\cdot\phi_2(V)^2\Big] = o_p(1),$$

uniformly over $\mathcal{F}$.

Plugging in this result into (A-73), we have that for any y_1, y_2,

$$\sup_{\substack{\theta\in\Theta\\ v\in\mathbb{R}^{L_V}}}\left|\widehat{H}_2^{\mathcal{T}_2}(z,\theta) - H_{2F}^{\mathcal{T}_2}(z,\theta)\right| = o_p(1), \quad \text{uniformly over } \mathcal{F}. \tag{A-74}$$

Combining (A-58) and (A-74) with the definition of $\widehat{\psi}^{\mathcal{T}_2}(z,\theta)$ in (A-57), for any y_1, y_2, under Assumptions 1-5, we have

$$\sup_{\substack{\theta\in\Theta\\ v\in\mathbb{R}^{L_V}}}\left|\widehat{\psi}^{\mathcal{T}_2}(z,\theta) - \psi_F^{\mathcal{T}_2}(z,\theta)\right| = o_p(1), \quad \text{uniformly over } \mathcal{F} \tag{A-75}$$

[14] Note that the presence of the weight function $\phi_2(v)$, which is zero for all $v\notin\mathcal{V}$, implies that the results in (A-61) and (A-72), which hold uniformly over $v\in\mathcal{V}$, immediately produce the result in (A-73), which holds uniformly over $v\in\mathbb{R}^{L_V}$ (since any $v\notin\mathcal{V}$ is trimmed away by $\phi_2(\cdot)$).

A2.1.2 Estimation of $\psi_F^{\mathcal{T}_1}(z,\theta_1)$

As in our estimation of $\psi_F^{\mathcal{T}_2}(z,\theta)$, we proceed using sample analogs based on the definition of $\psi_F^{\mathcal{T}_1}(z,\theta_1)$. Based on the structure described in (A-55), for a given (w_1,θ_1), we estimate $H_{1F}^{\mathcal{T}_1}(w_1,\theta_1)$ as,

$$\widehat{H}_1^{\mathcal{T}_1}(w_1,\theta_1) \equiv \frac{1}{2}\cdot\frac{1}{n}\sum_{j=1}^{n}\Big[\widehat{\tau}_1(w_1,W_{1j},\theta_1)\mathbb{1}\left\{\widehat{\tau}_1(w_1,W_{1j},\theta_1)\geq -b_n\right\}+\widehat{\tau}_1(W_{1j},w_1,\theta_1)\mathbb{1}\left\{\widehat{\tau}_1(W_{1j},w_1,\theta_1)\geq -b_n\right\}\Big]$$
$$-\widehat{\mathcal{T}}_1(\theta_1).$$

And, for a given $z\equiv(y_1,y_2,w_1)$, we estimate $H_{2F}^{\mathcal{T}_1}(z,\theta_1)$ as,

$$\begin{aligned}\widehat{H}_2^{\mathcal{T}_1}(z,\theta_1) &\equiv \left(\left(\widehat{\eta}_a^{\tau_1}(w_1,\theta_1)-\widehat{\eta}_b^{\tau_1}(w_1,\theta_1)\right)+\left(\widehat{\eta}_c^{\tau_1}(w_1,\theta_1)-\widehat{\eta}_d^{\tau_1}(w_1,\theta_1)\right)\cdot y_1\right)\cdot\widehat{f}_{W_1}(w_1)\cdot\phi_1(w_1)^2\\ &-\frac{1}{n}\sum_{j=1}^{n}\left[\left(\left(\widehat{\eta}_a^{\tau_1}(W_{1j},\theta_1)-\widehat{\eta}_b^{\tau_1}(W_{1j},\theta_1)\right)+\left(\widehat{\eta}_c^{\tau_1}(W_{1j},\theta_1)-\widehat{\eta}_d^{\tau_1}(W_{1j},\theta_1)\right)\cdot Y_{1j}\right)\cdot\widehat{f}_{W_1}(W_{1j})\cdot\phi_1(W_{1j})^2\right]\end{aligned} \tag{A-76}$$

From here, using the definition in (A-55), for a given z, we estimate $\psi_F^{\mathcal{T}_1}(z,\theta_1)$ as

$$\widehat{\psi}^{\mathcal{T}_1}(z,\theta_1)\equiv 2\cdot\widehat{H}_1^{\mathcal{T}_1}(w_1,\theta_1)+\widehat{H}_2^{\mathcal{T}_1}(z,\theta_1)$$

Using the definitions in (19), we construct the estimators on the right hand side of (A-76) as,

$$\begin{aligned}\widehat{\eta}_a^{\tau_1}(w_1,\theta_1) &\equiv \frac{1}{n}\sum_{j=1}^{n}\widehat{R}_1(W_{1j})\mathbb{1}\left\{g_{1U}(w_1,\theta_1)\leq g_{1L}(W_{1j},\theta_1)\right\}\mathbb{1}\left\{\widehat{\tau}_1(W_{1j},w_1,\theta)\geq -b_n\right\}\phi_1(W_{1j}),\\ \widehat{\eta}_b^{\tau_1}(w_1,\theta_1) &\equiv \frac{1}{n}\sum_{j=1}^{n}\widehat{R}_1(W_{1j})\mathbb{1}\left\{g_{1U}(W_{1j},\theta_1)\leq g_{1L}(w_1,\theta_1)\right\}\mathbb{1}\left\{\widehat{\tau}_1(w_1,W_{1j},\theta)\geq -b_n\right\}\phi_1(W_{1j}),\\ \eta_{c,F}^{\tau_1}(w_1,\theta_1) &\equiv \frac{1}{n}\sum_{j=1}^{n}\widehat{Q}_1(W_{1j})\mathbb{1}\left\{g_{1U}(W_{1j},\theta_1)\leq g_{1L}(w_1,\theta_1)\right\}\mathbb{1}\left\{\widehat{\tau}_1(w_1,W_{1j},\theta)\geq -b_n\right\}\phi_1(W_{1j}),\\ \eta_{d,F}^{\tau_1}(w_1,\theta_1) &\equiv \frac{1}{n}\sum_{j=1}^{n}\widehat{Q}_1(W_{1j})\mathbb{1}\left\{g_{1U}(w_1,\theta_1)\leq g_{1L}(W_{1j},\theta_1)\right\}\mathbb{1}\left\{\widehat{\tau}_1(W_{1j},w_1,\theta)\geq -b_n\right\}\phi_1(W_{1j})\end{aligned} \tag{A-77}$$

Let

$$\begin{aligned}\varphi^{\eta_a^{\tau_1}}(Z_i,Z_j,w_1,\theta_1,h) &\equiv \mathbb{1}\left\{g_{1U}(w_1,\theta_1)\leq g_{1L}(W_{1j},\theta_1)\right\}Y_{1i}\Gamma(W_{1i},W_{1j},h)\phi_1(W_{1i})\phi_1(W_{1j}),\\ \varphi^{\eta_b^{\tau_1}}(Z_i,Z_j,w_1,\theta_1,h) &\equiv \mathbb{1}\left\{g_{1U}(W_{1j},\theta_1)\leq g_{1L}(w_1,\theta_1)\right\}Y_{1i}\Gamma(W_{1i},W_{1j},h)\phi_1(W_{1i})\phi_1(W_{1j}),\\ \varphi^{\eta_c^{\tau_1}}(Z_i,Z_j,w_1,\theta_1,h) &\equiv \mathbb{1}\left\{g_{1U}(W_{1j},\theta_1)\leq g_{1L}(w_1,\theta_1)\right\}\Gamma(W_{1i},W_{1j},h)\phi_1(W_{1i})\phi_1(W_{1j}),\\ \varphi^{\eta_d^{\tau_1}}(Z_i,Z_j,w_1,\theta_1,h) &\equiv \mathbb{1}\left\{g_{1U}(w_1,\theta_1)\leq g_{1L}(W_{1j},\theta_1)\right\}\Gamma(W_{1i},W_{1j},h)\phi_1(W_{1i})\phi_1(W_{1j}).\end{aligned}$$

From the constructions of $\widehat{R}_1$ and $\widehat{Q}_1$ (see (16)), our estimators in (A-77) are,

$$\widehat{\eta}_a^{\tau_1}(w_1,\theta_1) = \frac{1}{h_n^{\ell}} \cdot \frac{1}{n(n-1)} \sum_{j=1}^{n} \sum_{i \neq j} \varphi^{\eta_a^{\tau_1}}(Z_i, Z_j, w_1, \theta_1, h_n) \mathbb{1}\left\{\widehat{\tau}_1(W_{1j}, w_1, \theta_1) \geq -b_n\right\},$$
$$\widehat{\eta}_b^{\tau_1}(w_1,\theta_1) = \frac{1}{h_n^{\ell}} \cdot \frac{1}{n(n-1)} \sum_{j=1}^{n} \sum_{i \neq j} \varphi^{\eta_b^{\tau_1}}(Z_i, Z_j, w_1, \theta_1, h_n) \mathbb{1}\left\{\widehat{\tau}_1(w_1, W_{1j}, \theta_1) \geq -b_n\right\},$$
$$\widehat{\eta}_c^{\tau_1}(w_1,\theta_1) = \frac{1}{h_n^{\ell}} \cdot \frac{1}{n(n-1)} \sum_{j=1}^{n} \sum_{i \neq j} \varphi^{\eta_c^{\tau_1}}(Z_i, Z_j, w_1, \theta_1, h_n) \mathbb{1}\left\{\widehat{\tau}_1(w_1, W_{1j}, \theta_1) \geq -b_n\right\},$$
$$\widehat{\eta}_d^{\tau_1}(w_1,\theta_1) = \frac{1}{h_n^{\ell}} \cdot \frac{1}{n(n-1)} \sum_{j=1}^{n} \sum_{i \neq j} \varphi^{\eta_d^{\tau_1}}(Z_i, Z_j, w_1, \theta_1, h_n) \mathbb{1}\left\{\widehat{\tau}_1(W_{1j}, w_1, \theta_1) \geq -b_n\right\}.$$

The above expressions are equivalent to those in (A-60). From here, using analogous arguments to those we used in the steps from equation (A-58) to the final result in equation (A-75), we can show that, for any y_1, under Assumptions 1-5, we have

$$\sup_{\substack{\theta_1 \in \Theta \\ v \in \mathbb{R}^{L_V}}} \left| \widehat{\psi}^{\mathcal{T}_1}(z,\theta_1) - \psi_F^{\mathcal{T}_1}(z,\theta_1) \right| = o_p(1), \quad \text{uniformly over } \mathcal{F} \tag{A-78}$$

A2.2 Estimation of $\psi_F^{\mathcal{T}}(z,\theta)$

The influence function $\psi_F^{\mathcal{T}}(z,\theta)$ is defined in Theorem 1 as $\psi_F^{\mathcal{T}}(z,\theta) \equiv \psi_F^{\mathcal{T}_2}(z,\theta) + \psi_F^{\mathcal{T}_1}(z,\theta_1))$. Accordingly, we estimate it as $\widehat{\psi}^{\mathcal{T}}(z,\theta) \equiv \widehat{\psi}^{\mathcal{T}_2}(z,\theta) + \widehat{\psi}^{\mathcal{T}_1}(z,\theta_1))$. From the results in (A-75) and (A-78), for any y_1, y_2, we have

$$\sup_{\substack{\theta \in \Theta \\ v \in \mathbb{R}^{L_V}}} \left| \widehat{\psi}^{\mathcal{T}}(z,\theta) - \psi_F^{\mathcal{T}}(z,\theta) \right| = o_p(1), \quad \text{uniformly over } \mathcal{F} \tag{A-79}$$

A2.3 Our estimator for $\sigma_F^2(\theta)$

We estimate $\sigma_F^2(\theta) \equiv E_F[\psi_F^{\mathcal{T}}(Z,\theta)^2]$ as

$$\widehat{\sigma}^2(\theta) \equiv \frac{1}{n} \sum_{i=1}^{n} \widehat{\psi}^{\mathcal{T}}(Z_i,\theta)^2$$

Recall that $Y_{1i} \in \{0,1\}$ and also recall that, by Assumption 3, there exists a finite constant $\overline{D}_4$ such that $E_F[|Y_2|^4] \leq \overline{D}_4$ for all $F \in \mathcal{F}$. Combining this with the result in (A-79), we obtain that, under Assumptions 1-5,

$$\sup_{\theta \in \Theta} \left| \widehat{\sigma}^2(\theta) - \sigma_F^2(\theta) \right| = o_p(1), \quad \text{uniformly over } \mathcal{F}.$$

This proves the claim in equation (29) in the paper. ■

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