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SEMESTER-I

PHYSICS

Mathematical Physics–I

[Revised Syllabus
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Attempt five questions in agll.

Q. 1. (a) By calculating the Wronskian of the functions e^x, xe^x and e^{-x}, check whether the functions are linearly dependent or independent. 4

Ans.

$$y_1 = e^x$$
$$y_2 = xe^x$$
$$y_3 = e^{-x}$$
$$y'_1 = e^x$$
$$y'_2 = xe^x + e^x . 1 = xe^x + e^x$$
$$y'_3 = -e^{-x}$$
$$y''_1 = e^x$$
$$y''_2 = x . e^x + e^x + e^x = x . e^x + 2e^x$$
$$y''_3 = e^{-x}$$

$$w = \begin{vmatrix} y_1 & y_2 & y_3 \\ y'_1 & y'_2 & y'_3 \\ y''_1 & y''_2 & y''_3 \end{vmatrix} = \begin{vmatrix} e^x & xe^x & e^{-x} \\ e^x & xe^x + e^x & -e^{-x} \\ e^x & xe^x + 2e^x & e^{-x} \end{vmatrix}$$

$$= \begin{vmatrix} e^x & xe^x & e^{-x} \\ e^x & e^x(x+1) & -e^{-x} \\ e^x & e^x(x+2) & e^{-x} \end{vmatrix} = e^x \cdot e^x \cdot e^{-x} \begin{vmatrix} 1 & x & 1 \\ 1 & x+1 & -1 \\ 1 & x+2 & 1 \end{vmatrix}$$

$$W = e^x \begin{vmatrix} 1 & x & 1 \\ 1 & x+1 & -1 \\ 1 & x+2 & 1 \end{vmatrix}$$

$$C_1 \to C_1 - C_2$$

$$W = e^x \begin{vmatrix} 0 & x & 1 \\ 2 & x+1 & -1 \\ 0 & x+2 & 1 \end{vmatrix}$$

$$= e^x(-2)[x . 1 - 1 . (x+2)]$$
$$= -2e^x(x - x - 2) = 4e^x \neq 0$$

Hence the solutions are linearly independent.

(b) Solve the inexact equation

$$(y^4 + 2y)\,dx + (xy^3 + 2y^4 - 4x)\,dy = 0$$ 5

Ans. $(y^4 + 2y)\,dx + (xy^3 + 2y^4 - 4x)\,dy = 0$ ... (i)

$$M = y^4 + 2y$$

$$\frac{\partial M}{\partial y} = 4y^3 + 2$$

$$N = xy^3 + 2y^4 - 4x$$

$$\frac{\partial N}{\partial x} = y^3 - 4$$

The equation is not exact.

$$\frac{1}{M}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] = \frac{3y^3+6}{y^4+2y} = \frac{3}{y}$$

This is a function of y

$$\text{I.F. } e^{-\int \frac{3}{y}dy} = e^{-3\log y} = y^{-3}$$

Multiplying (i) by y^{-3}, we have

$$\left(y + \frac{2}{y^2}\right)dx + \left[x + 2y - \frac{4x}{y^3}\right]dy = 0 \quad \text{... (ii)}$$

Or $\quad M_1\,dx + N_1\,dy = 0$

This (ii) is an exact equation.

$$\int M_1 dx = \int\left(y + \frac{2}{y^2}\right)dx = \left(y + \frac{2}{y^2}\right)x$$

$$\int N_1 dy = \int 2y\,dy = y^2$$

Therefore, the required solution is

$$\left(y + \frac{2}{y^2}\right)x + y^2 = C$$ Ans.

(c) Solve the differential equation

$$\frac{d^2y}{dx^2} - y = e^x \cos x$$ 6

Ans. $$\frac{d^2y}{dx^2} - y = e^x \cos x$$

The auxilliary equation is $m^2 - 1 = 0$

$$\Rightarrow \quad m^2 = 1$$

$$\Rightarrow \quad m = -1, 1$$

$$\text{C.F.} = C_1 e^{-x} + C_2 e^x$$

Let $\quad y_1 = e^{-x}$ and $y_2 = e^x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-x} & e^x \\ -e^{-x} & e^x \end{vmatrix} = 1 + 1 = 2 \neq 0$$

Let $\quad \text{P.I.} = u_1 y_1 + u_2 y_2$

$$u_1 = -\int \frac{y_2 R}{w}dx = -\int \frac{e^x \cdot e^x \cos x\,dx}{2}$$

$$= -\frac{1}{2}\int e^{2x}\cos x\,dx$$

We have, $\int e^{ax}\cos bx\,dx = \dfrac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$

Here $\quad a = 2,\ b = 1$

$$\therefore \quad u_1 = -\frac{1}{2}\left[\frac{e^{2x}}{2^2+1^2}(2\sin x - \cos x)\right]$$

$$= -\frac{1}{10}e^{2x}(2\sin x - \cos x)$$

$$u_1 y_1 = -\frac{1}{10}e^{2x}(2\sin x - \cos x)e^{-x}$$

$$= -\frac{1}{10}e^{x}(2\sin x - \cos x)$$

$$u_2 = \int \frac{y_1 R}{W} = \int \frac{e^{-x}\cdot e^x \cos x}{2}dx$$

$$= \frac{1}{2}\int \cos x\,dx = \frac{1}{2}\sin x$$

$$u_2 y_2 = \frac{1}{2}\sin x \cdot e^x = \frac{1}{2}e^x \sin x$$

$$\text{P.I.} = u_1 y_1 + u_2 y_2$$

The complete solution is y=C.F. + P.I. Ans.

Q. 2. (a) Solve the differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = e^{2x} + \sin 3x$$ 8

Ans. $$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = e^{2x} + \sin 3x$$

$$(D^2 - 4D + 4) = e^{2x} + \sin 3x$$

The auxiliary equation is :

$$m^2 - 4m + 4 = 0$$

or $$(m-2)^2 = 0$$

$\Rightarrow$ $$m = 2, 2$$

$$\text{C.F.} = (c_1 + c_2 x)e^{2x}$$

$$\text{P.I.} = \frac{1}{(D-2)^2}(e^{2x} + \sin 3x)$$

$$= \frac{1}{(D-2)^2}\cdot e^{2x} + \frac{1}{(D-2)^2}\cdot \sin 3x$$

Now, $$\frac{1}{(D-2)^2}\cdot e^{2x} = \frac{x^2}{2}e^{2x}$$

$$\frac{1}{(D-2)^2}\sin 3x = \frac{1}{(D^2 - 4D + 4)}\sin 3x$$

Here $$a = 3$$

$$a^2 = 9$$

$$D^2 = -a^2 = -9$$

$\therefore$ $$\frac{1}{(D-2)^2}\sin 3x = \frac{1}{(-9-4D+4)}\sin 3x$$

$$= \frac{1}{(-5-4D)}\sin 3x = \frac{-1}{(5+4D)}\sin 3x$$

$$= \frac{(-1)(5-4D)}{(5+4D)(5-4D)}\cdot \sin 3x = \frac{-(5-4D)}{25-16D^2}\cdot \sin 3x$$

But $D^2 = -9 \Rightarrow$

$$\frac{1}{(D-2)^2}\sin 3x = \frac{-(5-4D)}{25-16(-9)}\sin 3x = \frac{-(5-4D)}{169}\sin 3x$$

$$= -\frac{5}{169}\sin 3x + \frac{4}{169}D(\sin 3x)$$

$$= -\frac{5}{169}\sin 3x + \frac{12}{169}\cos 3x$$

The complete solution is

$$y = \text{C.F.} + \text{P. I.}$$

or, $$y = (C_1 + C_2)e^{2x} + \frac{x^2}{2}e^{2x} - \frac{5}{160}\sin 3x + \frac{12}{169}\cos 3x \text{ Ans.}$$

(b) Solve the differental equation using method of undermined coefficients

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} = x^2 + 2x + 4$$ 7

Ans. $$\frac{d^2y}{dx^2} + \frac{dy}{dx} = x^2 + 2x + 4 \quad \ldots(i)$$

$$y'' + y' = x^2 + 2x + 4$$

Aux. Equation : $$(m^2 + m) = 0$$

$\Rightarrow$ $$m(m+1) = 0$$

$\Rightarrow$ $$m = 0, -1$$

$$\text{C.F.} = C_1 + C_2 e^{-x}$$

Let $$y = Ax^3 + Bx^2 + Cx$$

$$y' = 3Ax^2 + 2Bx + C$$

$$y'' = 6Ax + 2B$$

Putting the value of y' and y'' in (i),

$$6Ax + 2B + 3Ax^2 + 2Bx + C = x^2 + 2x + 4$$

Comparing the co-efficients of same powers of x,

$$3A = 1$$

$\Rightarrow$ $$A = \frac{1}{3}$$

$$6A + 2B = 2$$

$\Rightarrow$ $$3A + B = 1$$

$$3\left(\frac{1}{3}\right) + B = 1$$

$\Rightarrow$ $$B = 0$$

$$C = 4$$

$$y = C_1 + C_2 e^{-x} + Ax^3 + Bx^2 + Cx \quad \ldots(ii)$$

$$y' = -C_2 e^{-x} + 3Ax^2 + C$$

$$y'' = C_2 e^{-x} + 6Ax$$

$$y'' + y' = 3Ax^2 + 6Ax + C$$

$$= 3\left(\frac{1}{3}\right)x^2 + 6\left(\frac{1}{3}\right)x + 4 = x^2 + 2x + 4$$

Ans.

Hence the solution is (ii).

Q. 3. (a) Solve the differential equation

$$\frac{d^2y}{dx^2}+3\frac{dy}{dx} = 1-9x^2$$

given $y(0) = 0$ and $y'(0) =$ [illegible]. 8

Ans.
$$\frac{d^2y}{dx^2}+3\frac{dy}{dx} = 1-9x^2 \qquad \ldots (i)$$

$$y''+3y' = 1-9x^2$$

Aux. Equation is $m^2+3m = 0$

$\Rightarrow \quad m(m+3) = 0$

$\Rightarrow \quad m = 0, -3$

$$\text{C.F.} = C_1 + C_2 e^{-3x}$$

Let the trial solution be $y = Ax^3 + Bx^2 + Cx$

$$y' = 3Ax^2 + 2Bx + C$$
$$y'' = 6Ax + 2B$$

Putting these values in (i)

$$6Ax + 2B + 3(3Ax^2 + 2Bx + C) = 1-9x^2$$

or $\quad 6Ax + 2B + 9Ax^2 + 6Bx + 3C = 1 - 9x^2$

Comparing the co-efficients of various powers of x on both sides,

$$9A = -9$$

$\Rightarrow \quad A = -1$

$$6Ax + 6Bx = 0$$

$\Rightarrow \quad A + B = 0$

Since $\quad A = -1,$

$\therefore \quad B = 1$

$\therefore \quad 2B + 3C = 1$

$\Rightarrow \quad 2(1) + 3C = 1$

$\Rightarrow \quad C = -\dfrac{1}{3}$

$$Y = C_1 + C_2 e^{-3x} + Ax^3 + Bx^2 + Cx$$
$$Y' = -3C_2 e^{-3x} + 3Ax^2 + 2Bx + C$$
$$Y'' = 9C_2 e^{-3x} + 6Ax + 2B$$
$$Y(0) = C_1 - 3C_2 = 0 \Rightarrow C_1 = 3C_2$$

But $\quad C_2 = -\dfrac{4}{9}$

$\therefore \quad C_1 = 3\left(-\dfrac{4}{9}\right) = -\dfrac{4}{3}$

$$Y'(0) = -3C_2 + C = 1$$

$$C = 1+3\left(-\frac{4}{9}\right) = 1-\frac{4}{3} = -\frac{1}{3}$$

Hence the solution is : $\quad Y = -\dfrac{1}{3}-\dfrac{4}{9}e^{-3x} - x^3 + x^2 - \dfrac{1}{3}Cx$

Verification $\quad Y''+3Y' = 9C_2e^{-3x}+6Ax+2B+3(-3C_2e^{-3x}+3Ax^2+2Bx+C)$

$$= 9C_2e^{-3x}+6Ax+2B-9C_2e^{-3x}+9Ax^2+6Bx+3C)$$
$$= 6x(A+B)+9Ax^2+2B+3C$$
$$= 6x(-1+1)+9(-1)x^2+2(1)+3(-1/3)$$
$$= 1-9x^2 \qquad = \text{R. H. S}$$

(b) Solve the differential equation using method of variation of parameters

$$\frac{d^2y}{dx^2}+a^2y = \text{cosec } ax \qquad 7$$

Ans. $\quad$ C.F. $= c_1 \cos ax + c_2 \sin ax$

Let $\quad y_1 = \cos ax$ and $y_2 = \sin ax$

$$w = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & a\cos ax \end{vmatrix} = a \neq 0$$

Noq $\quad u_1 = -\displaystyle\int \frac{y_2 R}{w} = -\int \frac{\sin ax \ \text{cosec } ax}{a}dx$

$$= -\frac{1}{a}\int dx = -\frac{x}{a}$$

$$u_2 = \int \frac{y_1 R}{w} = \int \frac{\cos ax \ \text{cosec } ax}{a}dx$$

$$= \frac{1}{a}\int \cot ax dx = \frac{1}{a}\int \frac{\cos ax}{\sin ax}dx = \frac{1}{a^2}\log \sin ax$$

$$\text{P.I.} = u_1y_1 + u_2y_2 = -\frac{x}{a}\cos ax + \frac{1}{a^2}\sin ax \log \sin ax$$

The complete solution is C.F. + P.I.

$$= c_1 \cos ax + c_2 \sin ax - \frac{x}{a}\cos ax + \frac{1}{a^2}\sin ax \log \sin ax$$

Q. 4. (a) Find $\dfrac{d}{dt}\left(\vec{V}\cdot\dfrac{d\vec{V}}{dt}\times\dfrac{d^2\vec{V}}{dt^2}\right)$

where $\vec{V}$ is a function of t.

Ans. $\frac{d}{dt}\left(V\cdot\frac{dV}{dt}\times\frac{d^2V}{dt^2}\right) = V\cdot\frac{dV}{dt}\times\frac{d^3V}{dt^3} + V\cdot\frac{d^2V}{dt^2}\times\frac{d^2V}{dt^2} + \frac{dV}{dt}\cdot\frac{dV}{dt}\times\frac{d^2V}{dt^2}$

$$= V\cdot\frac{dV}{dt}\times\frac{d^3V}{dt^3} + 0 + 0 = V\cdot\frac{dV}{dt}\times\frac{d^3V}{dt^3}$$

(b) Find the Jacobian $J\left(\frac{x,y,z}{u,v,w}\right)$ of the transformation

$u = x^2 + y^2 + z^2$, $v = x^2 - y^2 - z^2$ **and** $w = x^2 + y^2 - z^2$.

Ans. Jacobians. If u and v are functions of two independent variables x and y, then determinant

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$ is called the Jacobian of u, v with respect to x, y and is written as

$\frac{\partial(u,v)}{\partial(x,y)}$ or $J\left(\frac{u,v}{x,y}\right)$.

Similarly the Jacobian of u, v, w with respect to x, y, z is

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} \quad ...(i)$$

Likewise, we can define Jacobians of four or more variables. An important application of Jacobians is in connection with the change of variables in multiple integrals.

Properties of Jacobians. We give below two of the important properties of Jacobians. For simplicity, the properties are stated in terms of two variables only, but these are evidenely true in general.

I. If $J = \frac{\partial(u,v)}{\partial(x,y)}$ and $J' = \frac{\partial(x,y)}{\partial(u,v)}$ then $JJ' = 1$.

Let $u = f(x, y)$ and $v = g(x, y)$,

Suppose, on solving for x and y, we get $x = \phi(u, v)$ and $y = \psi(u, v)$.

Then
$$\frac{\partial u}{\partial u} = 1 = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial u},$$
$$\frac{\partial u}{\partial v} = 0 = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial v},$$
$$\frac{\partial v}{\partial u} = 0 = \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial u},$$
and
$$\frac{\partial v}{\partial v} = 1 = \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial v},$$

$$\therefore JJ' = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} \times \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial y/\partial x & \partial v/\partial y \end{vmatrix} \times \begin{vmatrix} \partial x/\partial u & \partial y/\partial u \\ \partial x/\partial v & \partial y/\partial v \end{vmatrix}$$

(Interchanging rows and columns of the 2nd determinant)

$$= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

II. If u, v are function of r, s and r, s are functions of x, y then

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)}\cdot\frac{\partial(r,s)}{\partial(x,y)}$$

$$\frac{\partial(u,v)}{\partial(r,s)}\cdot\frac{\partial(r,s)}{\partial(x,y)} = \begin{vmatrix} \partial u/\partial r & \partial u/\partial s \\ \partial r/\partial y & \partial v/\partial s \end{vmatrix} \times \begin{vmatrix} \partial r/\partial x & \partial s/\partial x \\ \partial r/\partial y & \partial s/\partial y \end{vmatrix}$$

[Interchanging rows and columns of the 2nd]

$$\begin{vmatrix} \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial r} & \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y} \\ \frac{\partial v}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial v}{\partial x}\cdot\frac{\partial s}{\partial x} & \frac{\partial v}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial v}{\partial s}\cdot\frac{\partial s}{\partial y} \end{vmatrix} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial u/\partial x & \partial u/\partial y \end{vmatrix} = \frac{\partial(u,v)}{\partial(x,y)}$$

$$u = x^2 + y^2 + z^2$$
$$v = x^2 - y^2 - z^2$$
$$w = x^2 + y^2 - z^2$$

We have to find $J\left(\frac{x,y,z}{u,v,w}\right)$

$$J\left(\frac{x,y,z}{u,v,w}\right) = \frac{1}{J\left(\frac{u,v,w}{x,y,z}\right)}$$

$$J = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$u = x^2 + y^2 + z^2$$

$$\frac{\partial u}{\partial x} = 2x; \frac{\partial u}{\partial y} = 2y; \frac{\partial u}{\partial z} = 2z$$

$$v = x^2 - y^2 - z^2$$

$$\frac{\partial v}{\partial x} = 2x; \frac{\partial v}{\partial y} = -2y; \frac{\partial v}{\partial z} = -2z$$

$$w = x^2 + y^2 - z^2$$

$$\frac{\partial w}{\partial x} = 2x; \frac{\partial w}{\partial y} = 2y; \frac{\partial w}{\partial z} = -2z$$

$J\left(\frac{x, y, z}{u, v, w}\right)$ is written as J^1

$$J = \begin{vmatrix} 2x & 2y & 2z \\ 2x & -2y & -2z \\ 2x & 2y & -2z \end{vmatrix}$$

Please evaluate the above determinant and take its reciprocal.

(c) If $\vec{v} = \vec{w} \times \vec{r}$, find the whether $\vec{v}$ is irrotational or not, where $\vec{w}$ is a constant vector and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

Ans. curl v = $\nabla \times v = \nabla \times (\omega \times r) = \nabla \times \begin{vmatrix} i & j & k \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix}$

$$= \nabla \times [(\omega_2 z - \omega_3 y)\, i + (\omega_3 x - \omega_1 z)\, j + (\omega_1 y - \omega_2 x)\, k]$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \\ \omega_2 z - \omega_3 y & \omega_3 x - \omega_1 z & \omega_1 y - \omega_2 x \end{vmatrix} = 2(\omega_1 i + \omega_2 j + \omega_3 k) = 2\omega$$

Since curl $\bar{v}$ is not zero, it is not irrotational

(d) Find $\vec{\nabla} \times \left(f(r)\vec{r}\right)$, where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

Ans. $\vec{\nabla} \times \left[f(r)\vec{r}\right] = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right) \times \{xf(r)i + yf(r)j + zf(r)k\}$

[$\because r = xi + yj + zk$]

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xf(r) & yf(r) & zf(r) \end{vmatrix}$$

$$= i\left(z\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial z}\right) + j\left(y\frac{\partial f}{\partial z} - z\frac{\partial f}{\partial x}\right) + k\left(\frac{y\partial f}{\partial x} - x\frac{\partial f}{\partial y}\right) \quad \ldots (1)$$

Again $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r} \cdot \frac{\partial r}{\partial x} = \frac{\partial f}{\partial r} \cdot \frac{\partial}{\partial x}\sqrt{(x^2 + y^2 + z^2)}$

$$= \frac{f'(r)z}{\sqrt{(x^2 + y^2 + z^2)}} = \frac{xf'}{r}, \text{ Similarly}$$

$\frac{\partial f}{\partial y} = \frac{\partial f'}{r}, \frac{\partial f}{\partial z} = \frac{\partial f'}{r}$ Thus by putting there values in (1)

We get curl $\{rf(r)\}$

$$= i\left(\frac{zyf'}{r} - \frac{yzf'}{r}\right) + j\left(\frac{zxf'}{r} - \frac{zxf'}{r}\right) + k\left(\frac{yxf'}{r} - \frac{xyf'}{r}\right)$$

or curl $\{r\, f(r) = 0$.

(e) Find the directional derivative of a scalar function $\phi = 2xz - y^2$ at the point (1, 3, 2) in the direction of $xz\hat{i} + yz\hat{j} + xy\hat{k}$. (3×5=15)

Ans. Let $\phi = 2xz - y^2$

Directional derivative of ϕ

$$= \nabla\phi = \left[\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right](2xz - y^2) = 2z\hat{i} - 2y\hat{j} + 2x\hat{k}$$

Directional derivative at (1, 3, 2)

$$= 2(2)\hat{i} - 2(3)\hat{j} + 2(1)\hat{k} = 4\hat{i} - 6\hat{j} + 2\hat{k}$$

We have to find the unit vector in the direction of $xz\hat{i} + yz\hat{j} + xy\hat{k}$ at the point (1, 3, 2).

Putting the values (1, 3, 2) for x, y and in the above, we get

$$(1)(2)\hat{i} + (3)(2)\hat{j} + (1)(3)\hat{k}$$

or $\quad 2\hat{i} + 6\hat{j} + 3\hat{k}$

Hence required directional derivative is

$$(4\hat{i} - 6\hat{j} + 2\hat{k})\frac{(2\hat{i} + 6\hat{j} + 3\hat{k})}{\sqrt{4+36+9}} = \frac{8-36+6}{7} = -\frac{22}{7} \quad \textbf{Ans.}$$

Q. 5. (a) Prove that :

$$\left(\vec{B}\times\vec{C}\right)\cdot\left(\vec{A}\times\vec{D}\right) + \left(\vec{C}\times\vec{A}\right)\cdot\left(\vec{B}\times\vec{D}\right) + \left(\vec{A}\times\vec{B}\right)\cdot\left(\vec{C}\times\vec{D}\right) = 0 \qquad 5$$

Ans. $\left(\vec{B}\times\vec{C}\right)\cdot\left(\vec{A}\times\vec{D}\right) + \left(\vec{C}\times\vec{A}\right)\cdot\left(\vec{B}\times\vec{D}\right) + \left(\vec{A}\times\vec{B}\right)\cdot\left(\vec{C}\times\vec{D}\right) = 0$

We have, $\left(\vec{A}\times\vec{B}\right)\cdot\left(\vec{C}\times\vec{D}\right) = \left(\vec{A}\cdot\vec{C}\right)\cdot\left(\vec{B}\cdot\vec{D}\right) - \left(\vec{A}\cdot\vec{D}\right)\cdot\left(\vec{B}\cdot\vec{C}\right) \qquad \text{...(i)}$

$$\left(\vec{B}\times\vec{C}\right)\cdot\left(\vec{A}\times\vec{D}\right) = \left(\vec{B}\cdot\vec{A}\right)\cdot\left(\vec{C}\cdot\vec{D}\right) - \left(\vec{B}\cdot\vec{D}\right)\cdot\left(\vec{C}\cdot\vec{A}\right) \qquad \text{...(ii)}$$

$$\left(\vec{C}\times\vec{A}\right)\cdot\left(\vec{B}\times\vec{D}\right) = \left(\vec{C}\cdot\vec{B}\right)\cdot\left(\vec{A}\cdot\vec{D}\right) - \left(\vec{C}\cdot\vec{D}\right)\cdot\left(\vec{A}\cdot\vec{B}\right) \qquad \text{...(iii)}$$

Add (*i*), (*ii*) and (*iii*),

L.H.S. of given expression

$$\left\{\left(\vec{A}\cdot\vec{C}\right)\cdot\left(\vec{B}\cdot\vec{D}\right) - \left(\vec{B}\cdot\vec{D}\right)\cdot\left(\vec{C}\cdot\vec{A}\right)\right\} + \left\{\left(\vec{B}\cdot\vec{A}\right)\cdot\left(\vec{C}\cdot\vec{D}\right) - \left(\vec{C}\cdot\vec{D}\right)\cdot\left(\vec{A}\cdot\vec{B}\right)\right\}$$

$$+\left\{\left(\vec{C}\cdot\vec{B}\right)\cdot\left(\vec{A}\cdot\vec{D}\right) - \left(\vec{A}\cdot\vec{D}\right)\cdot\left(\vec{B}\cdot\vec{C}\right)\right\}$$

$$= 0 + 0 + 0 = 0$$

$$\left[\text{As } \vec{A}\cdot\vec{C} = \vec{C}\cdot\vec{A},\ \vec{B}\cdot\vec{A} = \vec{A}\cdot\vec{B} \text{ and } \vec{C}\cdot\vec{B} = \vec{B}\cdot\vec{C}\right]$$

Hence Proved.

(b) Evaluate $\vec{\nabla}\left[r\vec{\nabla}\left(\frac{1}{r^3}\right)\right]$ $\qquad 5$

where $\qquad r^2 = x^2 + y^2 + z^2.$

Ans. $\qquad \nabla\left(\frac{1}{r^3}\right) = \text{grad } r^{-3}$

$$= \frac{\partial}{\partial x}\left(r^{-3}\right)i + \frac{\partial}{\partial y}\left(r^{-3}\right)J + \frac{\partial}{\partial z}\left(r^{-3}\right)k$$

But $\qquad \frac{\partial}{\partial x}\left(r^{-3}\right) = -3r^{-4}\frac{\partial r}{\partial x}$

Also $\qquad r^2 = x^2 + y^2 + z^2$

$\Rightarrow \qquad 2r\frac{\partial r}{\partial x} = 2x \qquad \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$

$\Rightarrow \qquad \frac{\partial}{\partial x}\left(r^{-3}\right) = -3r^{-4}\frac{x}{r} = -3r^{-5}x$

Similarly, $\qquad \frac{\partial}{\partial y}\left(r^{-3}\right) = -3r^{-5}y$

and $\qquad \frac{\partial}{\partial z}\left(r^{-3}\right) = -3r^{-5}z$

$\therefore \qquad \nabla\left(\frac{1}{r^3}\right) = -3r^{-5}\left(xi + yj + zk\right)$

$\Rightarrow \qquad r\nabla\left(\frac{1}{r^3}\right) = -3r^{-4}\left(xi + yj + zk\right)$

$\Rightarrow \qquad \nabla\bullet\left(r\nabla\frac{1}{r^3}\right) = \frac{\partial}{\partial x}\left(-3r^{-4}x\right) + \frac{\partial}{\partial y}\left(-3r^{-4}y\right) + \frac{\partial}{\partial z}\left(-3r^{-4}z\right)$

Again $\qquad \frac{\partial}{\partial x}\left(-3r^{-4}x\right) = 12r^{-5}\frac{\partial r}{\partial x}x - 3r^{-4}$

$$= 12r^{-6}x^2 - 3r^{-4}$$

Similarly, $\qquad \frac{\partial}{\partial y}\left(-3r^{-4}y\right) = 12r^{-6}y^2 - 3r^{-4}$

And $\qquad \frac{\partial}{\partial z}\left(-3r^{-4}z\right) = 12r^{-6}z^2 - 3r^{-4}$

$\Rightarrow \qquad \left(r\nabla\left(\frac{1}{r^3}\right)\right) = 12r^{-6}\left(x^2 + y^2 + z^2\right) - 9r^{-4}$

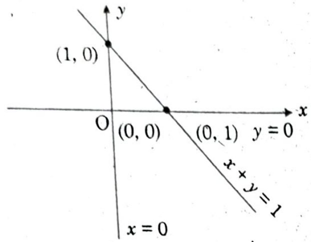
$$= 12r^{-6}\cdot r^2 - 9r^{-4} = 3r^{-4} \textbf{ Ans.}$$

(c) Evaluate $I = \oint_c (3x - 8y^2)dx + (4y - 6xy)dy$

where C is the boundary of the region bounded by $x = 0, y = 0$ and $x + y = 1$. $\qquad 5$

Ans. $I = \oint_c (3x - 8y^2)dx + (4y - 6xy)dy$

$$x = 0$$
$$y = 0$$
$$x + y = 1$$



We have, $M = 3x - 8y^2$

$N = 4y - 6xy$

$$\frac{\partial M}{\partial y} = -16y$$

$$\frac{\partial N}{\partial x} = -6y$$

$$\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = -6y - (-16y) = 10y$$

$$I = \iint 10y\, dx\, dy = \int 5y^2 dx = \int 5(1-x)^2 dx$$

$$= 5\int (1 + x^2 - 2x)dx = 5\int dx + 5\int x^2 dx - 10\int x\, dx$$

or
$$I = 5\left[x + \frac{x^3}{3} - 2\frac{x^2}{2}\right]$$

Here the limits are from $x = 0$ to $x = 1$

$$J = 5\left[x + \frac{x^3}{3} - x^2\right]_0^1 = 5\left[1 + \frac{1}{3} - 1\right] = \frac{5}{3}$$

This is the required value.

Q. 6. (a) Verify Stoke's theorem when $\vec{F} = (2xy - x^2)\hat{i} - (x^2 - y^2)\hat{j}$ where C is the boundary of the region enclosed by $y^2 = x$ and $x^2 = y$ 10

Ans. $\vec{F} = (2xy - x^2)\hat{i} - (x^2 - y^2)\hat{j}$

$$\text{curl } \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy - x^2 & x^2 - y^2 & 0 \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(x^2 - y^2)\right] - j\left[\frac{\partial}{\partial z}(2xy - x^2) - \frac{\partial}{\partial x}(0)\right]$$

$$+k\left[\frac{\partial}{\partial x}(x^2 - y^2) - \frac{\partial}{\partial y}(2xy - x^2)\right] = 0$$

and $\vec{F}\cdot dr = \left[(2xy - x^2)i - (x^2 - y^2)j\right]\cdot\left[idx + jdy + kdz\right]$

$$= dx\,(2xy) - dy\,(x^2 - y^2)$$

Since both(zero), the Stokes theorem is verified.

The boundary values do not come into picture.

(b) Using Gauss Divergence theorem, prove that

$$\iiint_V \vec{\nabla}\times\vec{F}\,dV = \iint_S d\vec{S}\times\vec{F}$$

where V is the volume enclosed by surface S. 5

Ans. By Gauss divergence theorem, we have

$$\iint_S A\bullet dS = \iiint_V \text{div } A\, dV$$

Now let ∴ $A = C\times F$ where C is a constant vector. Then

$$\iint_S (C\times F)\bullet dS = \iiint_V \nabla\bullet(C\times F)\,dV$$

But $(C\times F)\bullet dS = C\bullet F\times dS$

and $\nabla\bullet(C\times F) = C\bullet(\nabla\times F)$

Substituting these values, we get

$$\iint_S C\bullet(F\times dS) = -\iiint_V C\bullet(\nabla\times F)\,dV$$

$$C\bullet\left\{\iiint_V (\nabla\times F)\,dV + \iint_S F\times dS\right\} = 0$$

As C is any arbitrary constant vector, we have

$$\iiint_V (\nabla\times F)\,dV = -\iint_S F\times dS$$

or
$$\iiint_V (\nabla\times F)\,dV = \iint_S dS\times F$$

Q. 7. (a) A class has 15 students whose ages are 14, 17, 15, 14, 21, 17, 19, 20, 16, 18, 20, 17, 16, 19 and 20 years. One student is selected in such a manner that each has the same chance of being choosen and the ages X of the selected is recorded. What is the probability distributed of random variable X? Find mean, variance and standard deviation of X.

Sol. We observe that X takes values 14, 15, 16, 17, 18, 19, 20 and 21 such that

$$P(X=14)=\frac{2}{15},\ P(X=15)=\frac{1}{15},\ P(X=16)=\frac{2}{15},\ P(X=17)=\frac{3}{15}$$

$$P(X=18)=\frac{1}{15},\ P(X=19)=\frac{2}{15},\ P(X=20)=\frac{3}{15},\ P(X=21)\ \frac{1}{15}$$

So, the probability distribution of X is given below:

X :	14	15	16	17	18	19	20	21
P(X) :	$\frac{2}{15}$	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{3}{15}$	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{3}{15}$	$\frac{1}{15}$

Computation of mean of and variance

xi	$pi = P(X = xi)$	$pi\ xi$	$pi\ xi^2$
14	$\frac{2}{15}$	$\frac{28}{15}$	$\frac{392}{15}$
15	$\frac{1}{15}$	$\frac{15}{15}$	$\frac{225}{15}$
16	$\frac{2}{15}$	$\frac{32}{15}$	$\frac{512}{15}$
17	$\frac{3}{15}$	$\frac{51}{15}$	$\frac{867}{15}$
18	$\frac{1}{15}$	$\frac{18}{15}$	$\frac{324}{15}$
19	$\frac{2}{15}$	$\frac{38}{15}$	$\frac{722}{15}$
20	$\frac{3}{15}$	$\frac{60}{15}$	$\frac{1200}{15}$
21	$\frac{1}{15}$	$\frac{21}{15}$	$\frac{441}{15}$
		$\sum pi\,x_i = \frac{263}{15}$	$\sum pi\,x_i^2 = \frac{4683}{15}$

We have,

$$\sum pi\,x_i = \frac{263}{15} \text{ and } \sum pi\,x_i^2 = \frac{4683}{15}$$

$$\therefore \text{Mean} = \sum pi\,x_i = \frac{263}{15} = 17.53$$

$$\text{Variance} = \sum pi\,x_i^2 - \left(\sum pi\,xi\right)^2 = \frac{4683}{15} - \left(\frac{263}{15}\right)^2 = \frac{70245-69169}{225}$$

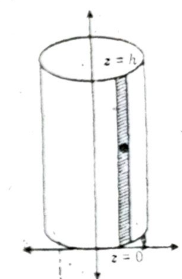
$$= \frac{1076}{225}$$

$$\text{Standard Deviation} = \sqrt{\text{Variance}} = \sqrt{\frac{1076}{15}} = \frac{32.80}{15} = 2.186$$

(b) Evaluate $\iint_V (y^2+z^2)\,dV$, where V is the volume bounded by the cylinder $x^2+y^2=a^2$ and the planes $x=0$ and $z=h$. 6

Ans.

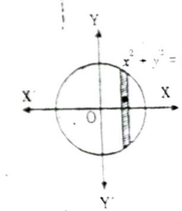
$$I = \int_{-a}^{a} dx \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy \int_0^h (y^2+z^2$$



$$= \int_{-a}^{a} dx \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy \left[y^2 z + \frac{z^3}{3}\right]_0^h$$

$$I = \int_{-a}^{a} dx \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \left(hy^2 + \frac{h^3}{3}\right) dy$$

$$= 2\int_{-a}^{a} dx \left[\frac{hy^3}{3} + \frac{h^3}{3}y\right]_0^{\sqrt{a^2-x^2}}$$



$$= 2\int_{-a}^{a} \left[\frac{h}{3}(a^2-x^2)^{3/2} + \frac{h^3}{3}(a^2-x^2)^{1/2}\right] dx$$

or
$$I = \frac{2\times 2h}{2}\int_0^a \left[(a^2-x^2)^{3/2} + h^2(a^2-x^2)^{1/2}\right] dx$$

Put
$$x = a\cos\theta$$
$$dx = -a\sin\theta\,d\theta$$

$$\therefore I = \frac{4h}{3}\left[\int_0^{\pi/2} (a^2-a^2\cos^2\theta)^{3/2} + h^2(a^2-a^2\cos^2\theta)^{1/2}\right]\left[-a\sin\theta\,d\theta\right]$$

or $I = \frac{4h}{3}\left[\int_0^{\pi/2} a^3\sin^3\theta + ah^2\sin\theta\right](-a\sin\theta d\theta)$

$$= -\frac{4a^2h}{3}\left[\int_0^{\pi/2}(a^2\sin^4\theta \;\; h^2\sin^2\theta)d\theta\right]$$

$$= -\frac{4}{3}a^2h\left[a^2\frac{3\times 1}{4\times 2}\times\frac{\pi}{2}+h^2\frac{1}{2}\times\frac{\pi}{2}\right]$$

or $I = -\frac{a^2h\pi}{3}\left(\frac{3}{4}a^2+h^2\right)$ **Ans.**

(c) **Define the Dirac Delta function and establish**

$$\int_{-\infty}^{+\infty} f(x)\delta'(x)dx = -f'(0)$$ 3

Ans. $\int_{-\infty}^{+\infty} f(x)\delta'(x)dx = -f'(0)$

$$\int_{-\infty}^{+\infty} f(x)\delta'(x)dx = [\delta(x)f(x)]_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty}\delta(x)f'(x)dx$$

If $\lim_{x\to\pm\infty}$ is finite, then the first term in the R.H.S. of above equation is zero, and we get

$$\int_{-\infty}^{+\infty}\delta'(x)f(x)dx = -\int_{-\infty}^{+\infty}\delta(x)f'(x)dv$$

or $$\int_{-\infty}^{+\infty}\delta'(x)f(x)dx = -f'(0)$$

Hence Proved.

Unique Paper Code	:	6671
Name of the Paper	:	Mathematical Physics–I
Name of Course	:	B.Sc. (Hons.) Physics I Year
Semester	:	I
Duration	:	3 Hours
Maximum Marks	:	75

[Nov. / Dec. 2017]

Attempt five questions in all.

Q. 1. Do any five of the following :

(*a*) **Two sides of a triangle are formed by the vectors :**

$$\vec{A} = 3\hat{i} + 6\hat{j} - 2\hat{k} \text{ and } \vec{B} = 4\hat{i} - \hat{j} + 3\hat{k}$$

Determine the angle between these two sides and length of the third side.

Ans. $\vec{A} = 3\hat{i} + 6\hat{j} - 2\hat{k}$

$\vec{B} = 4\hat{i} - \hat{j} + 3\hat{k}$

$\vec{A}.\vec{B} = (3\hat{i} + 6\hat{j} - 2\hat{k}).(4\hat{i} - \hat{j} + 3\hat{k})$

$= 12 - 6 - 6 = 0$

Now $\vec{A}.\vec{B} = |A||B|\cos\theta$

$\therefore$ $\cos\theta = \frac{\vec{A}.\vec{B}}{|A||B|} = 0$

$\therefore$ $\theta = 90°$

Hence, triangle will be a right angled triangle,

$|A| = \sqrt{3^2 + 6^2 + (-2)^2} = \sqrt{49} = 7$

$|B| = \sqrt{4^2 + (-1)^2 + (3)^2} = \sqrt{26}$

Let third side be C

Then $C = \sqrt{49 + 26} = \sqrt{75}$ units.

(*b*) **Show that the area bounded by a simple closed curve C is given by**

$$\frac{1}{2}\oint_C (x\,dy - y\,dx).$$

Ans. In Green's Theorem put

$$M = -y,$$
$$N = x, \quad \text{then,}$$

$$\oint_C (x\,dy - y\,dx) = \iint_R \left[\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y)\right] dx\,dy$$

$$= 2\iint_R dx\,dy = 2A$$

$$\therefore \quad \frac{1}{2}\oint_C (x\,dy - y\,dx) = \frac{1}{2} \times 2A = A$$

Hence, **proved.**

(c) If $\vec{a}$ is a constant vector, then prove that :

$$\vec{\nabla} \times \left(\vec{a} \times \vec{r}\right) = 2\vec{a}.$$

Ans. Let $\quad \vec{a} = \hat{i}\,a_x + \hat{j}a_y + \hat{k}\,a_z$

$$\vec{r} = ix + \hat{j}y + \hat{k}z$$

$$\vec{a} \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ x & y & z \end{vmatrix}$$

$$= \hat{i}\,(2za_y - ya_z) + \hat{j}\,(xa_z - za_x) + \hat{k}\,(ya_x - xa_y)$$

$$\nabla \times (\vec{a} \times \vec{r}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (za_y - ya_z) & (xa_z - za_x) & (ya_x - xa_y) \end{vmatrix}$$

$$= \hat{i}\,(2a_x) + \hat{j}(2a_y) + \hat{k}\,(2a_z) = 2\,\bar{a}$$

Hence, proved.

(d) Solve : $\iint_R \sqrt{x^2 + y^2}\; dx\,dy$.

where, R is the region bounded by the circle, $x^2 + y^2 = 9$.

Sol. Please Refer your Textbook.

(e) Check whether the following functions are linearly independent or not :

$e^x, x\,e^x$.

Sol. Let $\quad y_1 = e^x$

and, $\quad y_2 = xe^x$

The above will be linearly independent if $\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0$

$$y_1' = e^x$$
$$y_2' = xe^x + e^x$$

$$\therefore \quad \Delta = \begin{vmatrix} e^x & xe^x \\ e^x & (xe^x + e^x) \end{vmatrix} = x\,e^{2x} + e^{2x} - xe^{2x} = e^{2x} \neq 0$$

Hence, these are linearly independent.

(f) Solve the differential equation :

$$(b^2 + 2xy + y^2)dx + (x + y)^2 dy = 0.$$

Sol. $(b^2 + 2xy + y^2)dx + (x + y)^2\,dy = 0$

$$M = b^2 + 2xy + y^2$$

$$\frac{\partial M}{\partial y} = 2x + 2y = (2x + y)$$

$$N = (x + y)^2 = x^2 + y^2 + 2xy$$

$$\frac{\partial N}{\partial x} = 2x + 2y = 2(x + y)$$

$$\therefore \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

This is an exact equation.

The solution of the above equation is :

$\int M\,dx + \int (N - x)dy = C$, where E(N − x) denotes the terms of N without x.

∴ Solution is :

$$\int (b^2 + 2xy + y^2)dx + \int y^2\,dy = C$$

or $\quad b^2x + \dfrac{2x^2}{2}y + y^2x + \dfrac{y^3}{3} = C$

or $\quad 3b^2x + 3x^2y + 3xy^2 + y^3 = 3C = K$

This is the required answer.

(g) **Form a differential equation whose solution is given by**

$$\mathbf{y = A\,e^{2x} + B\,e^{3x}}.$$

Sol. $$y = Ae^{2x} + Be^{3x} \qquad ...(i)$$

Differentiate $$y' = 2Ae^{2x} + 3Be^{3x}$$

or $$y' = (2Ae^{2x} + 2Be^{3x}) + Be^{3x}$$

or $$y' = 2y + Be^{3x}$$

or $$y' - 2y = Be^{3x} \qquad ...(ii)$$

Differentiate again,

$$y'' - 2y' = 3Be^{3x} \qquad ...(iii)$$

Divide (iii) By (ii)

$$\frac{y'' - 2y'}{y' - 2y} = \frac{3Be^{3x}}{Be^{3x}} = 3$$

$$y'' - 2y' = 3y' - 6y$$

or $$y'' - 5y' + 6y = 0$$

This is the required differential equation.

(h) **The probabilities that a student pass in Mathematics, Physics and Chemistary are *m*, *p* and *c* respectively of these subjects, The student has a 75% chance of passing in atleast one, a 50% chance of passing in atleast two and a 40% chance of passing in exactly two. what is the value of $p + m + c$?**

Sol. Let A, B, and C be the following events :

A The student passes in Mathematics.

B The student passes in Physics.

C The student passes in Chemistry.

It is given that

P (A) = *p*, P (B) = *m* and P (C) = *c*

$$P(A \cup B \cup C) = \frac{75}{100} \qquad ...(i)$$

$$P(A \cap B) + P(B \cap C) + P(C \cap A) - 2P(A \cap B \cap C) = \frac{50}{100} \qquad ...(ii)$$

$$P(A \cap B) + P(B \cap C) + P(C \cap A) - 3P(A \cap B \cap C) = \frac{40}{100} \qquad ...(iii)$$

Subtracting (iii) from (ii), we get

$$P(A \cap B \cap C) = \frac{1}{100} \qquad ...(iv)$$

Adding (ii) and (iii), we get

$$2\{P(A \cap B) + P(B \cap C) + P(C \cap A)\} - 5P(A \cap B \cap C) = \frac{9}{10}$$

$$\Rightarrow \quad 2\{P(A \cap B) + P(B \cap C) + P(C \cap A)\} = \frac{9}{10} + \frac{5}{10} \qquad \text{[Using (iv)]}$$

$$\Rightarrow \quad P(A \cap B) + P(B \cap C) + P(C \cap A) = \frac{7}{10} \qquad ...(v)$$

From (i), we have

$$P(A \cup B \cup C) = \frac{75}{100}$$

$$\Rightarrow \quad P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C) = \frac{3}{4}$$

$$\Rightarrow \quad P(A) + P(B) + P(C) - \{P(A \cap B) + P(B \cap C) + P(C \cap A)\} + P(A \cap B \cap C) = \frac{3}{4}$$

$$\Rightarrow \quad p + m + c - \frac{7}{10} + \frac{1}{10} = \frac{3}{4} \qquad \text{[Using (iv) and (v)]}$$

$$\Rightarrow \quad p + m + c = \frac{27}{20}$$

Q. 2. (a) Find the constants '*a*' and '*b*' so that the surface $ax^2 - byz = (a + 2)x$ will be orthogonal to the surface $4x^2y + z^3 = 4$ at the point (1, –1, 2).

Sol.

$$ax^2 - byz = (a + z)x \qquad ...(i)$$

$$4x^2y + z^3 = 4 \qquad ...(ii)$$

Normal to the surface (i)

$$= \nabla\left[ax^2 - byz - (a + 2)x\right]$$

$$= \left[\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right]\left[ax^2 - byz - (a + 2)x\right]$$

$$= \hat{i}\,[2a - a - 2] + \hat{j}\,[-bz] + \hat{k}\,[-by]$$

Normal at (1, –1, 2). $$= \hat{i}\,(2a - a - 2) - \hat{j}\,(-2b) + \hat{k}b$$

$$= \hat{i}\,(a - 2) + \hat{j}\,(2b) + \hat{k}b \qquad ...(iii)$$

Normal at the surface (ii)

$$= \left[\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right](4x^2y + z^3 - 4)$$

$$= \hat{i}\,(8x\,y) + \hat{j}\,(4x^2) + \hat{k}\,(3z^2)$$

Normal at the point (1, –1, 2) $= -8\hat{i} + 4\hat{j} + 12\hat{k}$ ...(iv)

Now (iii) and (iv) are orthogonal,

$$\therefore \quad \left[\hat{i}\,(a - 2) + \hat{j}\,(2b) + \hat{k}b\right]\cdot\left[-8\hat{i} + 4\hat{j} + 12\hat{k}\right] = 0$$

or

$\Rightarrow$ $-8(a-2)+4(2b)+12b=0$

$\Rightarrow$ $-8a+20b+16=0$

$\Rightarrow$ $4(-2a+5b+4)=0$

$-2a+5b+4=0$

Putting (1, –1, 2) will satisfy (*i*)

$\therefore$

$\Rightarrow$ $a(1)^2-b(-1)(2)=(a+2)(1)$

$\Rightarrow$ $a+2=a+2b$

$b=1$

Putting this value in (*ii*),

$$-2a+5+4=0$$

$\therefore$ $$a=\frac{9}{2}$$

Hence $$a=\frac{9}{2},\ b=1$$

(*b*) If $\vec{A}=r^n\vec{r}$, then find the value of *n* for which $\vec{A}$ is solenoidal.

Sol. $$\vec{A}=r^n\vec{r}$$

Let $$\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$$

$$\vec{F}=\vec{\nabla}.\vec{F}=\vec{\nabla}.r^n\vec{r}$$

$$=\nabla.(x^2+y^2+z^2)^{n/2}(x\hat{i}+y\hat{j}+z\hat{k})$$

$$=\left[\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial z}+\hat{k}\frac{\partial}{\partial z}\right].\left[x^2+y^2+z^2\right]^{n/2}n\hat{i}+(x^2+y^2+z^2)^{n/2}y\hat{j}$$

$$+(x^2+y^2+z^2)^{n/2}z\hat{k}$$

$$=\frac{n}{2}(x^2+y^2+z^2)^{n/2-1}(2x^2)+(x^2+y^2+z^2)^{n/2}+\frac{n}{2}(x^2+y^2+z^2)^{n/2-1}(2y^2)+(x^2$$

$$+y^2+z^2)^{n/2}+\frac{n}{2}(x^2+y^2+z^2)^{n/2-1}(2z^2)+(x^2+y^2+z^2)^{n/2}$$

$$=n(x^2+y^2+z^2)^{n/2-1}(x^2+y^2+z^2)+3(x^2+y^2+z^2)^{n/2}$$

$$=n(x^2+y^2+z^2)^{n/2}+3(x^2+y^2+z^2)^{n/2}$$

$$=(n+3)(x^2+y^2+z^2)^{n/2}$$

If $r^n\vec{r}$ is solenoidal, then

$$(x+3)(x^2+y^2+z^2)^{n/2}=0$$

or $n+3=0$ or $n=-3$.

(*c*) Find the probability of drawing a one-rupee coin from a purse with two compartments one of which contains 3 fifty-paise coins and 2 one-rupee coins and other contains 2 fifty-paise coins and 3 one-rupee coins.

Sol. A one rupee coin can be drawn in two mutually exclusive ways.

(I) Selecting compartment I and then drawing a rupee coin from it.

(II) Selecting compartment II and then drawing a rupee coin from it.

Let E_1, E_2 and A be the events defined as follows:

E_1 = the first compartment of the purse is chosen,

E_2 = the second compartment of the purse is chosen,

A = a rupee coin is drawn from the purse.

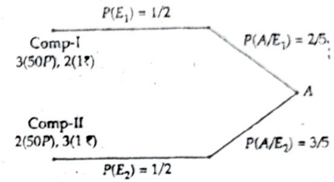


Fig. 31.12

Since one of the two compartments is chosen randomly.

$\therefore$ $$P(E_1)=\frac{1}{2}=P(E_2)$$

Also,

$P(A/E_1)$ = Probability drawing a rupee coin given that the first compartment of the purse is chosen

$=\frac{2}{5}$ [$\because$ First compartment contains 3 fifth paise coins and 2 one rupee coins]

and, $P(A/E_2)$ = Probability drawing a rupee coin given that the second compartment of the purse is chosen

$=\frac{3}{5}$ [$\because$ First compartment c[illegible]ins 2 fifth paise coins and 3 one rupee coins]

By the law of total probability

P (Drawing a one rupee coin) = $P(A)=P(E_1)P(A/E_1)+P(E_2)P(A/E_2)$

$$=\frac{1}{2}\times\frac{2}{5}+\frac{1}{2}\times\frac{3}{5}=\frac{1}{2}.$$

Q. 3. (*a*) Prove that :

$$\vec{A}\times\left(\vec{\nabla}\times\vec{A}\right)=\frac{1}{2}\vec{\nabla}A^2-\left(\vec{A}.\vec{\nabla}\right)\vec{A}$$

Sol. We have,

$$\text{grad}(A.B)=(A.\nabla)B+(B.\nabla)A+A\times(\nabla\times B)+B\times(\nabla\times A)$$

Let $B=A$

$$\text{grad}(A\,.\,A) = (A\,.\,\nabla)A + (A\,.\,\nabla)A + A \times (\nabla \times A) + A \times (\nabla \times A)$$

$$\nabla(A^2) = 2(A\,.\,\nabla)A + 2A \times (\nabla \times A)$$

or

$$2A \times (\nabla \times A) = \nabla(A^2) - 2(A\,.\,\nabla)A$$

$$A \times (\nabla \times A) = \nabla(A^2) - 2(A\,.\,\nabla)A$$

$$A \times (\nabla \times A) = \frac{1}{2}\nabla(A^2) - (A\,.\,\nabla)A$$

Hence, proved.

(b) (i) A, B and C shot to hit a target. If A hits target 4 times in 5 trials; B hits it 3 times in 4 trials and C hits 2 times in 3 trials; what is the probability that the target is hit by a least 2 persons?

Sol. Let E_1, E_2 and E_3 be the events that A hits the target, B hits the target and C hits the target respectively. Then, E_1, E_2, E_3 are independent events such that

$$P(E_1) = \frac{4}{5},\ P(E_2) = \frac{3}{4} \text{ and } P(E_3) = \frac{2}{3}.$$

The target is hit by at least two persons in the following mutually exclusive ways :

(I) A hits, B hits and C does not hit *i.e.* $E_1 \cap E_2 \cap \bar{E}_3$

(II) A hits, B does not hit and C hits *i.e.*, $E_1 \cap \bar{E}_2 \cap E_3$

(III) A does not hit, B hits and C hits *i. e.* $\bar{E}_1 \cap E_2 \cap E_3$

(IV) A hits, B hits and C hits *i.e.* $E_1 \cap E_2 \cap E_3$

$\therefore$ Required probability

$= P(I \cup II \cup III \cup IV)$

$= P[(E_1 \cap E_2 \cap \bar{E}_3) \cup (E_1 \cap \bar{E}_2 \cap E_3) \cup (\bar{E}_1 \cap E_2 \cap E_3) \cup (E_1 \cap E_2 \cap E_3)]$

$= P(E_1 \cap E_2 \cap \bar{E}_3) + P(E_1 \cap \bar{E}_2 \cap E_3) + P(\bar{E}_1 \cap \bar{E}_2 \cap E_3) + P(E_1 \cap E_2 \cap E_3)$

[$\because$ Four events are mutually exclusive]

$= P(E_1)\,P(E_2)\,P(\bar{E}_3) + P(E_1)\,P(\bar{E}_2)\,P(E_3) + P(\bar{E}_1)\,P(E_2)\,P(E_3) + P(E_1)\,P(E_2)\,P(E_3)$

[$\because$ E_1, E_2 and E_3 are independent]

$$= \frac{4}{5}\times\frac{3}{4}\times\frac{1}{3} + \frac{4}{5}\times\frac{1}{4}\times\frac{2}{3} + \frac{1}{5}\times\frac{3}{4}\times\frac{2}{3} + \frac{4}{5}\times\frac{3}{4}\times\frac{2}{3} = \frac{1}{5} + \frac{2}{15} + \frac{1}{10} + \frac{2}{5} = \frac{5}{6}$$

(ii) A bag contains 4 red and 3 black balls. A second bag contains 2 red and 4 black balls. One bag is selected at random! From the selected bag, one ball is drawn. Find the probability that the ball drawn is red.

Sol. A red ball can be drawn in two mutually exclusive ways.

(I) Selecting bag I and then drawing a red ball from it.

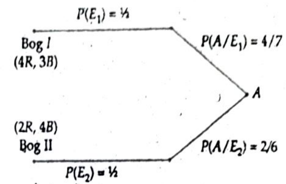


Fig. 34.11

(II) Selecting bag II and then drawing a red ball from it.

Let E_1, E_2 and A denote the events defined as follows :

E_1 = Selecting bag I,

E_2 = Selecting bag II,

A = Drawing a red ball

Since one of the two bags is selected randomly.

$$\therefore\quad P(E_1) = \frac{1}{2} \text{ and } P(E_2) = \frac{1}{2}.$$

Now, P (A/E_1) = Probability of drawing a red ball when the first bad has been chosen.

$$= \frac{4}{7} \qquad [\because \text{First bag contains 4 red and 3 black balls}]$$

and, P (A/E2) = Probability of drawing a red ball when the second bag has been selected

$$= \frac{2}{6} \qquad [\because \text{Second bag contains 2 red and 4 black balls}]$$

Using the law of total probability, we have

Required probability = P (A) = P (E_1) P (A/E_1) + P (E_2) P (A/E_2)

$$= \frac{1}{2}\times\frac{4}{7} + \frac{1}{2}\times\frac{2}{6} = \frac{19}{42}.$$

Q. 4. (a) Prove that :

$$\oint\!\!\oint_S r^5\,\hat{n}\,dS = \iiint_V 5r^3\,\vec{r}\,dV$$

whers, simple closed surface S encloses volume V.

Sol. By Gauss's divergence theorem,

$$\iiint_V \nabla\,.\,\bar{F}dV = \oint\!\!\oint_S \bar{F}\,.\,\hat{n}\,ds$$

Here, $\bar{F} = r^5\,\bar{r}$

Now,

$$\text{grad } r^n = nr^{n-2}\bar{r}$$

$$\nabla.\bar{F} = \nabla.r^5\bar{r} = 5r^3\vec{r}$$

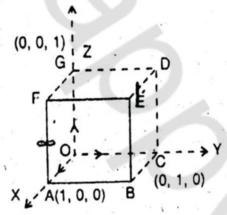
$$\iint_S r^5\hat{n}\,dS = \iiint_V 5r^3\vec{r}\,dV.$$

Hence Proved.

(b) Write the mathematical form of Gauss's Divergence theorem and hence verify it for $\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$, where S is the surface of the cube bounded by $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$.

Sol. $$\nabla.\bar{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).(4\,zx\hat{i} - y^2\hat{j} + yz\hat{k})$$

$$= 4z - 2y + y = 4z - y$$



Volume Integral $= \iiint \nabla.\vec{F}\,dv$

$$= \iiint (4z - y)\,dx\,dy\,dz$$

$$= \int_0^1 dx\int_0^1 dy\int_0^1 (4z - y)\,dz$$

$$= \int_0^1 dx\int_0^1 dy(2z^2 - yz)_0^1 = \int_0^1 dx\int_0^1 dy(2 - y)$$

$$= \int_0^1 dx\left(2y - \frac{y^2}{2}\right)_0^1 = \int_0^1 dx\left(2 - \frac{1}{2}\right)$$

$$= \frac{3}{2}\int_0^1 dx = \frac{3}{2}(x)_0^1 = \frac{3}{2} \qquad \text{...(1)}$$

To evaluate $\iint_S \vec{F}.\hat{n}\,ds$, where S consists of six plane surfaces.

(i) Over the face OABC, $z = 0$, $dz = 0$, $\hat{n} = -\hat{k}$, $ds = dx\,dy$

$$\iint \bar{F}.\hat{n}\,ds = \int_0^1\int_0^1 (-y^2\hat{j}).(-\hat{k})\,dx\,dy = 0$$

(ii) Over the face BCDE, $y = 1$, $dy = 0$

$$\hat{n} = \hat{j},\ ds = dx\,dz$$

$$= \int_0^1\int_0^1 -\,dx\,dz$$

$$\iint \vec{F}.\hat{n}\,ds = \int_0^1\int_0^1 (4xz\hat{i} - \hat{j} + z\hat{k}).(\hat{j})\,dx\,dz$$

S.No.				$\hat{n}$	ds
1	OABC	$z = 0$	$dz = 0$	$-k$	$dx\,dy$
2	BCDE	$y = 1$	$dy = 0$	j	$dx\,dz$
3	DEFG	$z = 1$	$dz = 0$	k	$dx\,dy$
4	OCDG	$x = 0$	$dx = 0$	$-i$	$dy\,dz$
5	AOGF	$x = 0$	$dy = 0$	$-j$	$dx\,dz$
6	ABEF	$x = 1$	$dx = 0$	i	$dy\,dz$

$$= -\int_0^1 dx\int_0^1 dz = -(x)_0^1(z)_0^1 = -(1)(1) = -1$$

(iii) Over the face DEFG, $z = 1$, $dz = 0$, $\hat{n} = \hat{k}$, $ds = dx\,dy$

$$\iint \vec{F}.\hat{n}\,ds = \int_0^1\int_0^1 [4x(1) - y^2\hat{j} + y(1)\hat{k}].(\hat{k})\,dx\,dy$$

$$= \int_0^1\int_0^1 y\,dx\,dy = \int_0^1 dx\int_0^1 y\,dy = (x)_0^1\left(\frac{y^2}{2}\right)_0^1 = \frac{1}{2}$$

(iv) Over the face OCDG, $x = 0$, $dx = 0$, $\hat{n} = -\hat{i}$, $ds = dy\,dz$

$$\iint \vec{F}.\hat{n}\,ds = \int_0^1\int_0^1 (0\hat{i} - y^2\hat{j} + yz\hat{k}).(-\hat{i})\,dy\,dz = 0$$

(v) Over the face AOGF, $y = 0$, $dy = 0$, $\hat{n} = -\hat{j}$, $ds = dx\,dz$

$$\iint \vec{F}.\hat{n}\,ds = \int_0^1\int_0^1 (4xz\hat{i}).(-\hat{j}).(-\hat{j})\,dz\,dz = 0$$

(vi) Over the face ABEF, $x = 1$, $dx = 0$, $\hat{n} = \hat{i}$, $ds = dy\,dz$

$$\iint \vec{F}.\hat{n}\,ds = \int_0^1\int_0^1 [(4z\hat{i} - y^2\hat{j} + yz\hat{k}).(\hat{i})]\,dy\,dz = \int_0^1\int_0^1 4z\,dy\,dz$$

$$= \int_0^1 dy\int_0^1 4z\,dz = \int_0^1 dy\,(2z^2)_0^1 = 2\int_0^1 dy = 2(y)_0^1 = 0$$

On adding we see that over the whole surface

$$\iint \vec{F}\cdot\hat{n}\,ds = \left(0 - 1 + \frac{1}{2} + 0 + 0 + 2\right) = \frac{3}{2} \qquad \text{...(2)}$$

From (1) and (2), we have $\iiint_V \nabla\cdot\vec{F}\,dv = \iint_S \vec{F}\cdot\hat{n}\,ds$

Hence Proved.

Q. 5. (a) Evaluate : $\iiint_V (2x + y)\,dV$,

Where, V is the closed region bounded by the cylinder $z = 4 - x^2$ and the planes, $x = 0, y = 0, y = 2, z = 0$.

Sol. The cylinder $z = 4 - x^2$ meets the x-axis $(y = 0, z = 0)$ at $x^2 = 4$ or $x = 2$, *i.e.*, at the point (2, 0, 0). It meets z-axis $(x = 0, y = 0)$ at $z = 4$, *i.e.*, at (0,0,4). Therefore the intefration limits are :

(*i*) $z = 4$ to $z = 4 - x^2$

(*ii*) $y = 0$ to $y = 2$ and

(*iii*) $x = 0$ to $x = 2$

Further $dv = dxdydz$

$$I = \int_V (2x + y)\,dv = \int_{x=0}^{2}\int_{y=0}^{2}\int_{z=0}^{(4-x^2)} (2x + y)\,dx\,dy\,dz$$

$$= \int_{x=0}^{2}\int_{y=0}^{2} (2x + y)\,[z]_0^{4-x^2}\,dx\,dy = \int_{x=0}^{2}\int_{y=0}^{2} (2x+y)(4-x^2)\,dx\,dy$$

$$= \int_{x=0}^{2}\left[2x(4 - x^2)y + (4 - x^2)\frac{y^2}{2}\right]_{y=0}^{2} dx$$

$$= \int_0^2 [4x(4-x^2) + 2(4-x^2)]\,dx = \int_0^2 2(4-x^2)(1 + 2x)\,dx$$

$$= 2\int_0^2 (4 + 8x - x^2 - 2x^3)\,dx = 2\left[4x + 4x^2 - \frac{x^3}{3} - \frac{x^4}{2}\right]_0^2$$

$$= 2\left[8 + 16 - \frac{8}{3} - 8\right] = \frac{80}{3}$$

(b) (i) In a bolt factory, machines A, B and C manufacture respectively 25%, 35% and 40% of the total bolts. Of their output 5, 4 and 2 percent are respectively defective bolts. A bolt is drawn at randpm from the product. If the bolt drawn is found to be defective, what is the probability that it is manufactured by the machine B?

Sol. Let E_1, E_2, E_3 and A be the events defined as follows:

E_1 = Bolt is manufactured by machine A

E_2 = Bolt is manufactured by machine B

E_3 = Bolt is manufactured by machine C

A = Bolt is defective.

Then $P(E_1)$ = (Probability that the bolt drawn is manufactured by machine A) = 25/100,

$P(E_2)$ = (Probability that the bolt drawn is manufactured by machine B) = 35/100,

$P(E_3)$ = (Probability that the bolt drawn is manufactured by machine C) = 40/100.

$P(A/E_1)$ = Probability that the bolt drawn is defective given that it is manufactured by machine A = 5/100.

Similarly, we have $P(A/E_2) = \frac{4}{100}$ and $P(A/E_3) = \frac{2}{100}$

Required probability = Probability that the bolt is manufactured by machine B given that the bolt drawn is defective

$$= P(E_2/A)$$

$$= \frac{P(E_2)P(A/E_2)}{P(E_1)P(A/E_1) + P(E_2)P(A/E_2) + P(E_3)P(A/E_3)}$$

$$= \frac{\frac{35}{100}\times\frac{4}{100}}{\frac{25}{100}\times\frac{5}{100} + \frac{35}{100}\times\frac{4}{100} + \frac{40}{100}\times\frac{2}{100}} = \frac{140}{125+140+80} = \frac{140}{345} = \frac{28}{69}$$

(ii) Three urns contain 6 red, 4 black ; 4 red, 6 black, and 5 red, 5 black balls respectively. One of the urns is selected at random and a ball is drawn from it. If the ball drawn is red, find the probability that it is drawn from the first urn.

Sol. Let E_1, E_2, E_3 and A be the events defined as follows:

E_1 = First urn is chosen, E_2 = Second urn is chosen,

E_3 = Third urn is chosen, and A = Ball drawn is red.

Since there are three urns and one of the three urns is chosen at randon.

Therefore, $P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}$.

If E_1 has already occurred, then first urn has been chosen which contains 6 red and 4 black balls. The probability of drawing a red ball from it is 6/10.

So, $P(A/E_1) = \frac{6}{10}$.

Similarly, $P(A/E_2) = \frac{4}{10}$ and $P(A/E_3) = \frac{5}{10}$

We have to find $P(E_1/A)$, i.e. given that the ball drawn is red, what is the probability that it is drawn from the first sum urn.

By Baye's theorem, obtain

$$P(E_1/A) = \frac{P(E_1)P(A/E_1)}{P(E_1)P(A/E_1)+P(E_2)P(A/E_2)+P(E_3)P(A/E_3)}$$

$$= \frac{\frac{1}{3}\times\frac{6}{10}}{\frac{1}{3}\times\frac{6}{10}+\frac{1}{3}\times\frac{4}{10}+\frac{1}{3}\times\frac{5}{10}} = \frac{6}{15} = \frac{2}{5}$$

(*iii*) An insurance company insured 2000 scooter drivers, 4000 car drivers and 6000 truck drivers. The probabilities of an accident involving a scooter driver, car driver and a truck driver are 0.01. 0.03 and 0.15 respectively. One of the insured person meets with an accident. What is the probability that he is a scooter driver?

Sol. Let E_1, E_2, E_3 and A be the events defined as follows:

E_1 = Person chosen is a scooter driver,
E_2 = Person chosen is a car driver,
E_3 = Person chosen is a truck driver, and
A = Person meets with and accident.

Since there are 12000 drivers out of which scooter, car and truck drivers are 2000, 4000 and 6000 respectively.

$$\therefore \quad P(E_1) = \frac{2000}{12000} = \frac{1}{6} \text{ and } P(E_2) = \frac{4000}{12000} = \frac{1}{3} \text{ and } P(E_2) = \frac{6000}{12000} = \frac{1}{2}$$

It is given that

$P(A/E_1)$ = Probability that a person meets with an accident given that he is a scooter driver = 0.01.

Similarly, $P(A/E_2) = 0.03$ and $P(A/E_3) = 0.15$.

We are required to find $P(E_2/A)$, i.e. given that the person meets with an accident, what is the probability that he was a scooter driver?

By Baye's rule,

$$P(E_1/A) = \frac{P(E_1)P(A/E_1)}{P(E_1)P(A/E_1)+P(E_2)P(A/E_2)+P(E_3)P(A/E_3)}$$

$$P(E_1/A) = \frac{\frac{1}{6}\times 0.01}{\frac{1}{6}\times 0.01+\frac{1}{3}\times 0.03+\frac{1}{2}\times 0.15} = \frac{1}{1+6+45} = \frac{1}{52}$$

Q. 6. Solve the differential equations :

(*a*) $(x^2y - 2xy^2)dx - (x^3 - 2x^2y)\,dy = 0$

Sol. $\quad (x^2y - 2xy^2)dx - (x^3 - 2x^2y)dy = 0$

$$(x^2y - 2xy^2)dx + (2x^2y - y^3)dy = 0$$

This is of the form $Mdx + Ndy = 0$

$$M = x^2y - 2xy^2$$

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial m}(x^2y - 2xy^2) = x^2 - 4xy$$

$$N = 2x^2y - y^3$$

$$\frac{\partial N}{\partial x} = 4xy - 3y^2$$

$$\therefore \quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

The equation is not exact.

$$(x^2y - 2xy^2)dx + (2x^2y - x^3)dy = 0$$

Multiply by $x^\alpha y^\beta$ both sides....

$$(x^{\alpha+2}y^{\beta+1} - 2x^{\alpha+1}y^{\beta+2})dx + 2x^{\alpha+2}y^{\beta+1} - x^{\alpha+3}y^\beta/dy = 0 \quad ...(i)$$

$$\frac{\partial M}{\partial y} = (\beta+1)x^{\alpha+2}y^\beta - 2(\beta+2)x^{\alpha+1}y^{\beta+1}$$

$$\frac{\partial N}{\partial x} = 2(\alpha+2)x^{\alpha+1}y^{\beta+1} - (\alpha+3)x^{\alpha+2}y^\beta$$

This is exact for values of α and β for which $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Equation the terms of $x^{\alpha+2}$

$$(\beta+1) = -(\alpha+3)$$

Equation the terms of $x^{\alpha+1}$

$$2(\alpha+2) = -2(\beta+2)$$

Equation (*i*) becomes

$$\alpha + 2 = \frac{-1}{3} + 2 = \frac{5}{3} \qquad \beta + 1 = \frac{-13}{3} + 1 = \frac{-10}{3}$$

$$\alpha + 1 = \frac{-1}{3} + 1 = \frac{2}{3} \qquad \beta + 2 = \frac{-13}{3} + 2 = \frac{-7}{3}$$

$$\alpha + 3 = \frac{-1}{3} + 3 = \frac{8}{3} \qquad \beta + 3 = \frac{-13}{3} + 3 = \frac{-4}{3}$$

$$\left(x^{5/3}y^{\frac{-10}{3}} - 2x^{2/3}y^{\frac{-7}{3}}\right)dx + \left(2x^{5/3}y^{\frac{-10}{3}} - x^{8/3}y^{\frac{-13}{3}}\right)dy$$

The solution of this exact equation is :

$$y^{\frac{-10}{3}}\int x^{5/3}\,dx - 2y^{-7/3}\int x^{2/3}dx + 0 = a$$

or $$y^{\frac{-10}{3}}\frac{x^{\frac{5}{3}+1}}{\frac{5}{3}+1} - 2y^{-7/3}\frac{x^{\frac{2}{3}+1}}{\frac{2}{3}+1} = a$$

$$\frac{3}{8}x^{8/3}y^{\frac{-10}{3}} - \frac{2x^3}{5}x^{5/3}y^{-7/3} = a$$

$$\frac{3}{8}x^{8/3}y^{\frac{-10}{3}} - \frac{6}{5}x^{5/3}y^{-7/3} = a$$

or $$15x^{8/3}y^{-10/3} - 48x^{5/3}y^{-7/3} = c$$

Multiplying both sides by $y^{10/3}$,

$$15x^{8/3} - 48x^{5/3}y = cy^{10/3}$$

∴ This is the required answer.

(b) $(D^2+1)y = \text{cosec}\, x \left(D = \frac{d}{dx}\right)$.

Sol. $(D^2+1)\,y = \text{cosec}\, x$

The Auxiliary Equation is : $m^2 + 1 = 0$

$\Rightarrow$ $m = \pm i$

C.F. $= A\cos x + B\sin x$

Here $y_1 = \cos x$

$y_2 = \sin x$

P.I. $= y_1 u + y_2 v$

Here $$u = \int\frac{-y_2\,\text{cosec}\,x\,dx}{y_1y_2' - y_1'\,y_2} = \int\frac{-\sin x\,\text{cosec}\,x\,dx}{\cos x\,(\cos x) - (-\sin x)(\sin x)}$$

$$= \int\frac{-\sin x\,.\frac{1}{\sin x}dx}{\cos^2 x + \sin^2 x} = -\int dx = -x$$

$$v = \int\frac{y_1 \times dx}{y_1\,y_2' - y_1'\,y_2} = \int\frac{\cos x\,\text{cosec}\,x\,dx}{\cos x\,(\cos x) - (-\sin x)\,(\sin x)}$$

$$= \int\frac{\cot x\,dx}{\cos^2 x + \sin^2 x} = \int\cot x\,dx = \log\sin x$$

∴ P.I. $= uy_1 + vy_2 = -x\cos x + \sin x\,(\log\sin x)$

The eeneral solution is ; C.F. + P.I.

$$y = A\cos x + B\sin x - x\cos x + \sin x\log(\sin x)$$

Q. 7. (a) Solve the differential equation :

$$\mathbf{(D^2 - 6D + 8)y = (e^{2x} - 1)^2.}$$

Sol. $(D^2 - 6D + 8)y = (e^{2x} - 1)^2$

The auxiliary equation is $m^2 - 6m + 8 = 0$

$\Rightarrow$ $(m-4)(m-2) = 0$

$\Rightarrow$ $m = 4,\ m = 2.$

∴ C.F. $= C_1e^{4x} + C_2e^{2x}$

$$\text{P.I.} = \frac{1}{(D-4)(D-2)}\left(e^{4x} - 2e^{2x} + 1\right)$$

$$= \frac{1}{(D-4)(D-2)}\left(e^{4x} - 2e^{2x} + e^{0x}\right)$$

$$= \frac{1}{(4-4)(4-2)}xe^{4x} - \frac{2}{(2-2)(2-4)}xe^{2x} + \frac{1}{(0-2)(0-4)}$$

$$= \frac{1}{2}xe^{4x} - \frac{2}{(-2)}xe^{2x} + \frac{1}{8} = \frac{x}{2}e^{4x} + xe^{2x} + \frac{1}{8}$$

The solution is C.F. + P.I.

or $$y = c_1e^{4x} + c_2e^{2x} + \frac{x}{2}e^{4x} + x.e^{2x} + \frac{1}{8}.$$

(b) Using method of variation of parameters, solve the differential equation :

$$\mathbf{(D^2 + 4)y = x\sin 2x.}$$

Sol. $(D^2 + 4)y = x\sin 2x$

The Auxiliary equation is : $m^2 + 4 = 0$

$\Rightarrow$ $m = \pm 2i$

∴ C.F. $= C_1\cos 2x + C_2\sin 2x$

$$\text{P.I.} = \frac{1}{D^2+4}(x\sin 2x) = \left\{x - \frac{1}{D^2+4}.2D\right\}.\frac{1}{D^2+4}\sin 2x$$

$$= \left\{x - \frac{2D}{D^2+4}\right\}\left\{-\frac{x}{4}\cos 2x\right\}$$

$$= -\frac{x^2}{4}\cos 2x + \frac{1}{2}.\frac{1}{D^2+4}(\cos 2x - 2x\sin 2x)$$

$$= +\frac{x^2}{4}\cos 2x + \frac{1}{2}.\frac{1}{D^2+4}\cos 2x - \frac{1}{D^2+4}(x\sin 2x)$$

$$\therefore \quad \text{P.I.} = -\frac{x^2}{8}\cos 2x + \frac{1}{4}\cdot\frac{1}{D^2+4}\cos 2x$$

$$= -\frac{x^2}{8}\cos 2x + \frac{1}{4}\cdot\frac{x}{4}\sin 2x = -\frac{x^2}{8}\cos 2x + \frac{1}{16}x\sin 2x$$

$\therefore$ The solution is C.F. + P.I.

or
$$y = C_1\cos 2x + C_2\sin 2x - \frac{x^2}{8}\cos 2x + \frac{1}{16}x\sin x.$$

Q. 8. (*a*) Solve the differential equation :

$$(D^2 - 4D + 3)y = xe^{2x}.$$

Sol. $(D^2 - 4D + 3)y = xe^{2x}$

The auxiliary equation is :

$$m^2 - 4m + 3 = 0$$
$$(m-3)(m-1) = 0$$
$$\therefore \quad m = 1, 3$$

The C.F. is : $C_1e^x + C_2e^{3x}$

The P.I. is : $\text{P}\,\frac{1}{D^2 - 4D + 3}\cdot xe^{2x}$

or $\quad e^{2x}\frac{1}{(D+2)^2 - 4(D+2) + 3}\cdot x$

or $\quad e^{2x}\frac{1}{D^2 + 4D + 4 - 4D - 8 + 3}\cdot x$

or $\quad e^{2x}\frac{1}{D^2 - 1}x$

or $\quad \left(-e^{2x}\right)\frac{1}{1 - D^2}x$

or $\quad (-e^{2x})(1 - D^2)^{-1}x$

or $\quad (-e^{2x})(1 + D^2)x$

or $\quad (-e^{2x})(1 + 0)x$

or $\quad (-e^{2x})(x)$

or $\quad \text{P.I.} = -xe^{2x}$

$\therefore$ The solution is C.F. + P.I.

or $\quad$ $\boldsymbol{y = C_1e^x + C_2e^{3x} - xe^{2x}}$.

(*b*) Using method of undetermined coefficients, solve the differential equation :

$$(D^2 - 1)y = e^x + 2x.$$

Sol. $(D^2 - 1)y = e^x + 2x$

The auxiliary equation is :

$$m^2 - 1 = 0$$

$$\therefore \quad m = \pm 1$$

$\therefore$ C.F. is : $C_1e^x + C_2e^{-x}$

The P.I. is : $\frac{1}{(D^2-1)}e^x + \frac{1}{(D^2-1)}\cdot 2x$

The term $\quad \frac{1}{(D^2-1)}e^x = \frac{1}{(D+1)(D-1)}e^x$

$$= \frac{1}{(1+1)(D-1)}e^x = \frac{1}{2}x\,e^x$$

The term $\quad \left(\frac{1}{D^2-1}\right)2x = \frac{-1}{(1-D^2)}\cdot 2x$

$$= (-1)(1-D^2)^{-1}\cdot 2x$$
$$= -[1 + D^2]\cdot 2x = -1.2x = -2x$$

$\therefore$ The P.I. is : $\frac{1}{2}xe^x - 2x$

Hence the solution is :

$$y = C_1e^x + C_2e^{-x} + \frac{1}{2}xe^x - 2x$$

Unique Paper Code : 32221101

Name of the Paper : Mathematical Physics-I

Name of Course : B.Sc. (Hons.) Physics I Year

Semester : I

Duration : 3 Hours

Maximum Marks : 75 **[Nov. / Dec. 2018]**

Attempt five questions in all.

Q. 1. Do any five questions :

(a) There cards are drawn from a pack of 52 playing cards. Find the probability distribution of the number of aces.

Ans. Let X denote the number of aces in a sample of 3 cards drawn from a well shuffled pack of 52 playing cards. Since there are four aces in the pack, therefore in the sample of 3 cards drawn either there can be no ace or there can be one ace or two aces or three aces. Thus, X can take value 0, 1, 2 and 3.

Now, P (X = 0) = Probability of getting no ace = Probability of getting 3 other cards

$$= \frac{^{48}C_3}{^{52}C_3} = \frac{4324}{5525}$$

P(X = 1) = Probability of getting one ace and two other cards

$$= \frac{^{4}C_1 \times {^{48}C_2}}{^{52}C_3} = \frac{1128}{5525}$$

P(X = 2) = Probability of getting two aces and one other card

$$= \frac{^{4}C_2 \times {^{48}C_1}}{^{52}C_3} = \frac{72}{5525}$$

$\therefore$ and, P(X = 3) = Probability of getting 3 aces $= \frac{^{4}C_3}{^{52}C_3} = \frac{1}{5525}$.

(b) By calculating the Wronskian of the functions e^x, e^{-x}, and e^{-2x} check whether the functions are linearly dependent or independent.

Ans.

$$y_1 = e^x$$
$$y_2 = e^{-x}$$
$$y_3 = e^{-x}$$
$$y_1' = e^x$$
$$y_2' = -e^{-x}$$
$$y_3' = -2e^{-2x}$$

$$y_1'' = e^x$$
$$y_2'' = e^{-x}$$
$$y_3'' = 4e^{-2x}$$

$$W(y_1, y_2, y_3) = \begin{vmatrix} e^x & e^{-x} & e^{-2x} \\ e^x & -e^{-x} & -2e^{-2x} \\ e^x & e^x & 4e^{-2x} \end{vmatrix} = (e^x)(e^{-x})(e^{-2x})\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & -2 \\ 1 & 1 & 4 \end{vmatrix}$$

$$C_2 \to C_2 - C_1$$
$$C_3 \to C_3 - C_1$$

$$W(y_1, y_2, y_3) = \begin{vmatrix} 1 & 0 & 0 \\ 1 & -2 & -3 \\ 1 & 0 & 3 \end{vmatrix} = 1(-2(3) - 0) = 1(-6) = -6 \neq 0$$

Hence, e^x, e^{-x} and e^{-2x} are linearly independent.

(c) Find the area of the triangle with vertices P(2, 3, 5), Q (4, 2, – 1) and R (3, 6, 4).

Ans. The vertices of the triangle are P (2, 3, 5), Q (4, 2, – 1) and R (3, 6, 4)

$$OQ - OP = 4 - 2, 2 - 3, -1 - 5 = 2, -1, -6$$

$$\therefore \quad \overline{PQ} = 2\hat{i} - \hat{j} - 6\hat{k}$$

$$OR - OP = 3 - 2, 6 - 3, 4 - 5 = 1, 3, -1$$

$$\text{Area of } \Delta PQR = \frac{1}{2}\left|\overline{PQ} \times \overline{PR}\right|$$

$$\overline{PQ} \times \overline{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & -6 \\ 1 & 3 & -1 \end{vmatrix} = \hat{i}\begin{vmatrix} -1 & -6 \\ 3 & -1 \end{vmatrix} \hat{j}\begin{vmatrix} 2 & -6 \\ 1 & -1 \end{vmatrix} + \hat{k}\begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix}$$

$$= \hat{i}(1 + 18) - \hat{j}(-2 + 6) + \hat{k}(6 + 1) = 19\hat{i} - 4\hat{j} + 7\hat{k}$$

$$\text{Area} = \frac{1}{2}\sqrt{(19)^2 + (-4)^2 + (7)^2}$$

$$= \frac{1}{2}\left[\sqrt{361 + 16 + 49}\right] = \frac{1}{2}\sqrt{426} \text{ sq. units}$$

(d) Find the unit vector normal to the surface $x^2 + y^2 + z^2 = 4$ at the point $(1, \sqrt{2}, -1)$.

Ans. Surface : $x^2 + y^2 + z^2 = 4$

$$\nabla (x^2 + y^2 + z^2) = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

Then a unit normal the surface at the point $(1, \sqrt{2}, -1)$

$$= \frac{2(1)\hat{i} + 2(\sqrt{2})\hat{j} + 2(-1)\hat{k}}{\sqrt{(2)^2 + (2\sqrt{2})^2 + (-2)^2}} = \frac{2\hat{i} + 2\sqrt{2}\hat{j} - 2\hat{k}}{4}$$

$$= \frac{1}{2}\hat{i} + \frac{\sqrt{2}}{2}\hat{j} - \frac{1}{2}\hat{k}$$

(e) Show that : $\oiint \left(\vec{\nabla} r^2\right) . \vec{ds} = 6V$ where S is the closed surface enclosing the volume V.

Ans. $\oiint_S (\vec{\nabla} r^2) . \vec{ds}$

$$= \iint_S \nabla (x^2 + y^2 + z^2) . d\bar{S} = \left[\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right] . (x^2 + y^2 + z^2)$$

$$= 2x\hat{i} + 2y\hat{j} + 2z\hat{k} = 2(x\hat{i} + y\hat{j} + 2\hat{k})$$

$$\iint_S \nabla (x^2 + y^2 + z^2) . ds = \iint_S \nabla (x^2 + y^2 + z^2) . \hat{n}\, ds$$

where $\hat{n}$ being outward drawn unit normal vector to S.

$$= \iint_S 2(x\hat{i} + y\hat{j} + z\hat{k}) . \hat{n}\, ds = 2\iiint_V \text{div}\,(x\hat{i} + y\hat{j} + z\hat{k})\, dv \quad ...(i$$

(By divergence theorem

Here V is the volume enclosed By S

Now, we have

$$\text{div} . (x\hat{i} + y\hat{j} + z\hat{k}) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) . (x\hat{i} + y\hat{j} + z\hat{k})$$

$$= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1 + 1 + 1 = 3 \quad ...(ii$$

From (i) and (ii)

$$\iint \nabla (x^2 + y^2 + z^2)\, dS = 2\iiint_V 3\, dv = 6\iiint_V dv = 6V$$

Hence proved.

(f) The probability that a teacher will give an un-announced test during any class meeting is 1/5. If a student is absent twice, what is the probability that he will miss at least one test ?

Ans. Let E_i be event that the student misses ith test ($i = 1, 2$). Then E_1 and E_2 are independent events such that $P(E_1) = \frac{1}{5} = P(E_2)$

$\therefore$ Required probability $= P(E_1 \cup E_2)$

$$= 1 - P(\bar{E}_1)\, P(\bar{E}_2)$$

[$\because$ E_1, E_2 are independent]

$$= 1 - \left(1 - \frac{1}{5}\right)\left(1 - \frac{1}{5}\right) = \frac{9}{25}.$$

(g) If $P(A) = \frac{3}{8}$, $P(B) = \frac{1}{2}$ and $P(A \cap B) = \frac{1}{4}$, find $P(\bar{A}/\bar{B})$ and $P(\bar{B}/\bar{A})$.

Ans. We know that $P(\bar{A}/\bar{B}) = \dfrac{P(\bar{A} \cap \bar{B})}{P(\bar{B})}$ and $P(\bar{B}/\bar{A}) = \dfrac{P(\bar{A} \cap \bar{B})}{P(\bar{A})}$

Therefore, to find $P(\bar{A}/\bar{B})$ and $P(\bar{B}/\bar{A})$, we need the value of $P(\bar{A} \cap \bar{B})$, $P(\bar{A})$ and $P(\bar{B})$. So, let us first, compute these probabilities.

Now, $\quad P(\bar{A} \cap \bar{B}) = P(\overline{A \cup B})$

$$= 1 - P(A \cup B) = 1 - \{P(A) + P(B) - P(A \cap B)\}$$

$$= 1 - \left\{\frac{3}{8} + \frac{1}{2} - \frac{1}{4}\right\} = \frac{3}{8}$$

$$P(\bar{A}) = 1 - P(A) = \frac{5}{8} \quad \text{and} \quad P(\bar{B}) = 1 - P(B) = \frac{1}{2}$$

$$\therefore \quad P(\bar{A}/\bar{B}) = \frac{P(\bar{A} \cap \bar{B})}{P(\bar{B})} = \frac{3/8}{1/2} = \frac{3}{4}$$

and $$P(\bar{B}/\bar{A}) = \frac{P(\bar{A} \cap \bar{B})}{P(\bar{A})} = \frac{3/8}{5/8} = \frac{3}{5}.$$

(h) Form a differential equation whose solutions are e^{2x} and e^{3x}. **(5 × 3 = 15)**

Ans. The two solutions of the differential equation are e^{2x} and e^{3x}

Let, $$y = ae^{2x} + be^{3x} \quad \ldots (1)$$

We have to eliminate two constants an a and b. We have to differntiate (i) twice

Differentiating (i), we get

$$y' = 2ae^{2x} + 3be^{3x}$$

or $$y' = 2y + be^{3x} \quad \ldots (ii)$$

$\therefore$ $$b = \frac{y' - 2y}{e^{3x}} \quad \ldots (iii)$$

Differentiating (ii), we get

$$y'' = 2y' + 3be^{3x}$$

$\therefore$ $$b = \frac{y'' - 2y'}{3e^{3x}} \quad \ldots (iv)$$

Equating (iii) and (iv)

$$\frac{y'' - 2y'}{3e^{3x}} = \frac{y' - 2y}{e^{3x}}$$

$\therefore$ $$y'' - 2y' = 3(y' - 2y)$$

or $$y'' - 2y' = 3y' - 6y$$

or $$y'' - 5y' + 6y = 0$$

This is the required differential equation.

Q. 2. (a) Solve : $\frac{d^2x}{dt^2} - 3x - 4y = 0, \frac{d^2y}{dt^2} + x + y = 0.$

Ans. The given equations can be expressed as

$$(D^2 - 3)x - 4y = 0, \quad x + (D^2 + 1)y = 0. \qquad (D = d/bt)$$

Multiply the first equation by $(D^2 + 1)$, second equation by 4 and add,

$$(D^2 + 1)(D^2 - 3)x + 4x = 0 \Rightarrow (D^4 - 2D^2 + 1)x = 0$$

or $$(D2 - 1)2x = 0.$$ Its solution is

$$x = (a + bt)e^t + (c + dt)e^{-t}. \quad \ldots (i)$$

Now, $$\frac{dx}{dt} = (a + bt)e^t + be^t - (c + dt)e^{-t} + de^{-t},$$

$$\frac{d^2x}{dt^2} = (a + bt)e^t + 2be^t + (c + dt)e^{-t} - 2de^{-t}.$$

From the given first equation, we have

$$y = \frac{1}{4}\frac{d^2x}{dt^2} - \frac{3}{4}x$$

Substituting (1) and (2) in the above equation, we get

$$y = -\frac{1}{2}(a + bt)e^t - \frac{1}{2}(c + dt)e^{-t} + \frac{1}{2}be^t - \frac{1}{2}de^{-t}.$$

$\therefore$ $$2y = (b - a - bt)et - (c + d + dt)e^{-1}. \quad \ldots (3)$$

Hence, the relations (1) and (3) constitute the required solution.

(b) Solve the differential equation : $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 4y = e^x \cos x.$ **(4)**

Ans. $$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 4y = e^x \cos x$$

$$y'' - 2y' + 4y = e^x \cos x$$

A. E. $$m^2 - 2m + 4 = 0$$

$$m = \frac{2 \pm \sqrt{4 - 16}}{2} = \frac{2 \pm 2\sqrt{3}i}{2}$$

$$m = 1 \pm \sqrt{3}i$$

$\therefore$ $$\text{C.F.} = e^x\left(C_1 \cos\sqrt{3}\,x + C_2 \sin\sqrt{3}\,x\right)$$

$$\text{P.I.} = \frac{1}{D^2 - 2D + 4}e^x \cos x = e^x \frac{1}{(D+1)^2 - 2(D+1) + 4}\cos x$$

$$= e^x \frac{1}{D^2 + 3}\cos x = e^x \frac{1}{-1^2 + 3}\cos x = \frac{1}{2}e^x \cos x$$

$\therefore$ $$y = e^x\left(C_1 \cos\sqrt{3}\,x + C_2 \sin\sqrt{3}\,x\right) + \frac{1}{2}e^x \cos x$$

This is the required solution.

(c) Solve : $\frac{dx}{dt} + 4x + 3y = t, \frac{dy}{dt} + 2x + 5y = e^t.$ **(6)**

Sol. The given equations can be expressed as

$$(D + 4)x + 3y = t, \quad 2x + (D + 5)y = e^t. \qquad (D = d/dt)$$

Multiply the first equation by 2, second equation by $(D + 4)$ and subtract,

$$(D + 4)(D + 5)y - 6y = (D + 4)et - 2t$$

or $$(D^2 + 9D + 14)y = 5e^t - 2t.$$

Its solution is given by

$$y = ae^{-2t} + be^{-7t} + \frac{5}{(D^2 + 9D + 14)}e^t - \frac{2}{14}\left\{1 + \left(\frac{D^2 + 9D}{14}\right)\right\}^{-1} t$$

$$= ae^{-2t} + be^{-7t} + \frac{5}{24}e^t + \frac{1}{7}\left\{1 - \left(\frac{D^2 + 9D}{14}\right)\right\}t$$

$$\therefore \quad y = ae^{-2t} + be^{-7t} + \frac{5}{24}e^t - \frac{1}{7}t + \frac{9}{98} \qquad \ldots(1)$$

Now, $\frac{dy}{dt} = -2ae^{-2t} - 7be^{-7t} + \frac{5}{24}e^t - \frac{1}{7}$. $\qquad \ldots(2)$

From the given second equation, we obtain

$$x = -\frac{1}{2}\frac{dy}{dt} - \frac{5}{2}y + \frac{1}{2}e^t.$$

Substituting (1) and (2) in the above equation, we get

$$x = -\frac{3}{2}ae^{-2t} + be^{-7t} - \frac{1}{8}e^t + \frac{5}{14}t - \frac{31}{196} \qquad \ldots(3)$$

Hence, the relations (1) and (3) constitute the required solution.

Q. 3. (a) (i) Three coins are tossed. Consider the events : E = three heads or three tails, F = At least two heads and G = At most two heads. Of the pairs (E, F), (E, G) and (F, G) which are independent ? Which are dependent ? (9)

Ans. The sample space S associated with the experiment is given by

S = {HHH, HHT, HTH, THH, TTH, THT, HTT, TTT}

Clearly, E = {HHH, TTT}, F = {HHH, HHT, HTH, THH}

and, G = {HHT, HTH, THH, HTT, THT, TTH, TTT}

Also, $E \cap F$ = {HHH}, $E \cap G$ = {TTT}, $F \cap G$ = {HHT, HTH, THH}

$$\therefore \quad P(E) = \frac{2}{8} = \frac{1}{4},\ P(F) = \frac{4}{8} = \frac{1}{2},\ P(G) = \frac{7}{8},\ P(E \cap F) = \frac{1}{8}$$

$$P(E \cap G) = \frac{1}{8} \quad P(F \cap G) = \frac{3}{8}$$

Clearly, $P(E \cap F) = P(E)\,P(F)$, but $P(E \cap G) \neq P(E)\,P(G)$ and $P(F \cap G) \neq P(F)\,P(G)$

So, E and F are independents, E and G are dependent events and F and G are also dependent events.

(ii) If X is normally distributed with mean 2 and variance 2, express $P[|X - 1| \leq 2]$ in terms of the standard normal cumulative distribution function.

Ans. We are given $X \sim N(2, 2)$.

Let $$Z = \frac{X - \mu}{\sigma} = \frac{X - 2}{\sqrt{2}}$$

We have $P[|X - 1| \leq 2] = P[1 - 2 \leq X \leq 1 + 2]$

$$= P[-1 \leq X \leq 3]$$

$$= P\left[\frac{-1-2}{\sqrt{2}} \leq \frac{X-2}{\sqrt{2}} \leq \frac{3-2}{\sqrt{2}}\right] = P\left[-\frac{3}{\sqrt{2}} \leq Z \leq \frac{1}{\sqrt{2}}\right]$$

$$= \Phi\left(\frac{1}{\sqrt{2}}\right) - \Phi\left(-\frac{3}{\sqrt{2}}\right) = \Phi\left(\frac{1}{\sqrt{2}}\right) - \left[1 - \Phi\left(\frac{3}{\sqrt{2}}\right)\right]$$

$[\because \Phi(x) = 1 - \Phi(-x)]$

$$= \Phi\left(\frac{1}{\sqrt{2}}\right) + \Phi\left(\frac{3}{\sqrt{2}}\right) - 1.$$

(b) Solve the differential equation using method of variation of parameter (6)

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = x^2e^{2x}.$$

Sol. $$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = x^2e^{2x}$$

$$(D^2 - 4D + 4)y = x^2e^{2x}$$

A.E. $$m^2 - 4m + 4 = 0$$

$$(m-2)^2 = 0$$

$$m = 2, 2$$

$\therefore$ Complementary Function : $(C_1 + C_2X)e^{2x}$

Praticuar Integral : $\frac{1}{D^2 - 4D + 4} \cdot x^2 e^{2x}$

$$= e^{2x} \cdot \frac{1}{(D+2)^2 - 4(D+2) + 4} \cdot x^2 = e^{2x} \frac{1}{D^2} x^2$$

$$= e^{2x} \frac{1}{D} \cdot \left(\frac{x^3}{3}\right) = e^{2x} \cdot \frac{x^4}{12}$$

The complete solution is $y = (C_1 + C_2 x)e^{2x} + e^{2x} \cdot \frac{x^4}{12}$

Q. 4. (a) Show that : $$\left[\left(\vec{A} \times \vec{B}\right) \times \vec{C}\right] \times \vec{D} + \left[\left(\vec{B} \times \vec{A}\right) \times \vec{D}\right] \times \vec{C}$$

$$\left[\left(\vec{C} \times \vec{D}\right) \times \vec{A}\right] \times \vec{B} + \left[\left(\vec{D} \times \vec{C}\right) \times \vec{B}\right] \times \vec{A} = 0.$$

Ans. $\left[\left(\vec{A}\times\vec{B}\right)\times\vec{C}\right]\times\vec{D}+\left[\left(\vec{B}\times\vec{A}\right)\times\vec{D}\right]\times\vec{C}$

$$\left[\left(\vec{C}\times\vec{D}\right)\times\vec{A}\right]\times\vec{B}+\left[\left(\vec{D}\times\vec{C}\right)\times\vec{B}\right]\times\vec{A}$$

$$=\left[\left(\vec{A}.\vec{C}\right).\vec{B}-\left(\vec{B}.\vec{C}\right)\vec{A}\right]\times\vec{D}+\left[\left(\vec{B}.\vec{D}\right).\vec{A}-\left(\vec{A}.\vec{D}\right)\vec{B}\right]\times\vec{C}$$

$$+\left[\left(\vec{C}.\vec{A}\right).\vec{D}-\left(\vec{D}.\vec{A}\right)\vec{C}\right]\times\vec{B}+\left[\left(\vec{D}.\vec{B}\right).\vec{C}-\left(\vec{C}.\vec{B}\right)\vec{D}\right]\times\vec{A}$$

$$=\left[\left(\vec{A}.\vec{C}\right)\left(\vec{B}\times\vec{D}\right)-\left(\vec{B}.\vec{C}\right)\left(\vec{A}\times\vec{D}\right)+\left(\vec{B}.\vec{D}\right)\left(\vec{A}\times\vec{C}\right)-\left(\vec{A}.\vec{D}\right)\left(\vec{B}\times\vec{C}\right)\right.$$

$$\left.+\left(\vec{C}.\vec{A}\right)\left(\vec{D}\times\vec{B}\right)-\left(\vec{D}.\vec{A}\right)\left(\vec{C}\times\vec{B}\right)+\left(\vec{D}.\vec{B}\right)\left(\vec{C}\times\vec{A}\right)-\left(\vec{C}.\vec{B}\right)\left(\vec{D}\times\vec{A}\right)\right]$$

$$=\left(\vec{A}.\vec{C}\right)\left[\left(\vec{B}\times\vec{D}\right)+\left(\vec{D}\times\vec{B}\right)\right]-\left(\vec{B}.\vec{C}\right)\left[\left(\vec{A}\times\vec{D}\right)+\left(\vec{D}\times\vec{A}\right)\right]$$

$$+\left(\vec{B}.\vec{D}\right)\left[\left(\vec{A}\times\vec{C}\right)+\left(\vec{C}\times\vec{A}\right)\right]-\left(\vec{A}.\vec{D}\right)\left[\left(\vec{B}\times\vec{D}\right)+\left(\vec{D}\times\vec{B}\right)\right]$$

Now $\left(\vec{A}\times\vec{B}\right)=\left(\vec{B}\times\vec{A}\right)$

$$=\left(\vec{A}.\vec{C}\right)\left[\left(\vec{B}\times\vec{D}\right)+\left(\vec{B}\times\vec{D}\right)\right]-\left(\vec{B}.\vec{C}\right)\left[\left(\vec{A}\times\vec{D}\right)+\left(\vec{A}\times\vec{D}\right)\right]$$

$$+\left(\vec{B}.\vec{D}\right)\left[\left(\vec{A}\times\vec{C}\right)+\left(\vec{A}\times\vec{C}\right)\right]-\left(\vec{A}.\vec{D}\right)\left[\left(\vec{B}\times\vec{D}\right)+\left(\vec{B}\times\vec{D}\right)\right]$$

= 0 Hence Proved.

(b) Show that $\int_S F\times n\, d\,S$, *i.e.*, $\int_S F\times da=-\int_V \text{curl}\, F\, dV$. **(9)**

Ans. Let $f=a\times F$, where a is a constant vector

$$\int_S f.\,nds=\int_V \text{div}\, f dV$$

or $\qquad \int_S a\times F.\,ndS=\int_V \text{div}\,(a\times F)\,dV \qquad$ ...(1)

Now, $\qquad a\times F.\,n=a.\,F\times n \qquad$ ...(2)

because, position of dot and cross can be changed in scalar triple product if cycle order is maintained.

Also, $\qquad \text{div}\,(a\times F)=(\text{curl}\,a).\,F-(\text{curl}\,f).\,a$

$$=-a.\,(\text{curl}\,F) \qquad \text{...(3)}$$

as curl $a=0$

Substituting the above results (2) and (3) in (1), we get

$$\int_S a.\,F\times n\,dS=\int_V a.\,\text{curl}\,F\,dV$$

or, $\qquad a.\left[\int_S (F\times n)\,dS+\int_V \text{curl}F\,dV\right]=0 \qquad$ ...(4)

Since a is any arbitrary vector, we have from (iv)

$$\int_S F\times n\,dS+\int_V \text{curl}\,F\,dV=0$$

$$\therefore \quad \int_S F\times n\,dS=-\int_V \text{curl}\,F\,dV$$

Q. 5. (*a*) If $\vec{a}$ is a constant vector and $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$ then prove that :

$$\text{Curl}\left(\frac{\vec{a}\times\vec{r}}{r^3}\right)=\frac{3(\vec{a}.\vec{r})\vec{r}}{r^5}-\frac{\vec{a}}{r^3}. \qquad (7)$$

Ans. $\qquad \text{curl}\left(\frac{\vec{a}\times\vec{r}}{r^3}\right)=\frac{3\left(\vec{a}.\vec{r}\right)\vec{r}}{r^3}-\frac{\vec{a}}{r^3}$

$$\frac{\vec{a}\times\vec{r}}{r^3}=\frac{1}{r^3}\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\a_1&a_2&a_3\\x&y&z\end{vmatrix}$$

$$=\frac{1}{e^3}\left[(a_2z-a_3y)\hat{i}+(a_3x-a_1z)\hat{j}+(a_1y-a_2x)\hat{k}\right]$$

$$\nabla\times\frac{\left(\vec{a}\times\vec{r}\right)}{r^3}=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\\frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\\frac{a_2z-a_3y}{r^3}&\frac{a_2x-a_1z}{r^3}&\frac{a_1y-a_2x}{r^3}\end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}\left(\frac{a_1 y - a_2 x}{r^3}\right) - \frac{\partial}{\partial z}\left(\frac{a_3 x - a_1 z}{r^3}\right)\right] - \hat{j}\left[\frac{\partial}{\partial x}\left(\frac{a_1 y - a_2 x}{r^3}\right) - \frac{\partial}{\partial z}\left(\frac{a_2 z - a_3 y}{r^3}\right)\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}\left(\frac{a_3 x - a_1 z}{r^3}\right) - \frac{\partial}{\partial y}\left(\frac{a_2 z - a_3 y}{r^3}\right)\right]$$

Now, $r^2 = x^2 + y^2 + z^2$

$$2r\frac{\partial r}{\partial x} = 2x \quad \Rightarrow \quad \frac{\partial r}{\partial x} = \frac{x}{r}$$

Similarly $\quad \dfrac{\partial r}{\partial y} = \dfrac{y}{r}$

and $\quad \dfrac{\partial r}{\partial z} = \dfrac{z}{r}$

$$\nabla \times \left(\frac{\vec{a} \times \vec{r}}{r^3}\right) = \hat{i}\left[\left\{-3r^{-4}\left(\frac{y}{z}\right)(a_1 y - a_2 x) + \frac{1}{r^3}a_1\right\}\right]$$

$$-\left[-3r^{-4}\left(\frac{z}{r}\right)(a_3 x - a_1 z) + \frac{1}{r^3}(-a_1)\right]$$

Two similar term

$$= \hat{i}\left[\frac{-3}{r^5}\left(a_1 y^2 - a_2 xy\right) + \frac{a_1}{r_3} + \frac{3}{r^5}\left(a_3 xz + \frac{a_1}{r_3}\right)\right] \quad + \text{two similar terms}$$

$$= \hat{i}\left[\frac{2a_1}{r^3} - \frac{3}{r^5}a_1(y^2 + z^2) + \frac{3}{r^5}(a_2 xy + a_3 xz)\right] \quad + \text{two similar terms}$$

$$= \hat{i}\left[\frac{2a_1}{r^3} - \frac{3a_1}{r^5}.r^2 + \frac{3}{r^5}x\,(a_1 x + a_2 y + a_3 z)\right] \quad + \text{two similar terms}$$

$$= \hat{i}\left[\frac{2a_1}{r^3} - \frac{3a_1}{r_3} + \frac{3}{r^5}x\,(a_1 x + a_2 y + a_3 z)\right] \quad + \text{two similar terms}$$

$$= \frac{2}{r^3}(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) - \frac{3}{r^3}(a_1\hat{i} + a_2\hat{j} + a_3\hat{k})$$

$$+ \frac{3}{r^5}(a_1 x + a_2 y + a_3 z)(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= \frac{2-3}{r^3}\bar{a} + \frac{3}{r^5}(\bar{a}.\bar{r})\bar{r} = \frac{3}{r^5}(\bar{a}.\bar{r})\bar{r} - \frac{\bar{a}}{r^3}$$

= R.H.S Hence, proved.

(b) Evaluate : $\iint_S \vec{A}.\hat{n}\,dS$ where $\vec{A} = 18z\hat{i} - 12\hat{j} + 3y\hat{k}$ and S is the part of the plane $2x + 3y + 6z = 12$ located in the first octant. (8)

Ans. Here, $\quad \vec{A} = 18z\hat{i} - 12\hat{j} + 3y\hat{k}$

Given surface $\quad f(x, y, z) = 2x + 3y + 6z - 12$

Normal vector $= \nabla f = \left(\hat{i}\dfrac{\partial}{\partial x} + \hat{j}\dfrac{\partial}{\partial y} + \hat{k}\dfrac{\partial}{\partial z}\right)(2x + 3y + 6z - 12) = 2\hat{i} + 3\hat{j} + 6\hat{k}$

$\hat{n}$ = unit normal vector at any point (x, y, z) of $2x + 3y + 6z = 12$

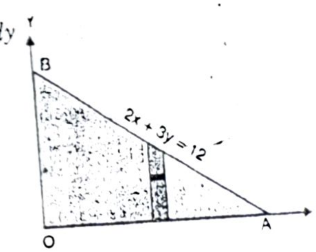
$$= \frac{2\hat{i} + 3\hat{j} + 6\hat{k}}{\sqrt{4+9+36}} = \frac{1}{7}(2\hat{i} + 3\hat{j} + 6\hat{k})$$

$$dS = \frac{dx\,dy}{\hat{n}.\hat{k}} = \frac{dx\,dy}{\frac{1}{7}(2\hat{i} + 3\hat{j} + 6\hat{k}).\hat{k}} = \frac{dx\,dy}{\frac{6}{7}} = \frac{7}{6}$$

Now, $\iint \bar{A}.\hat{n}\,dS = \iint (18z\hat{i} - 12\hat{j} + 3y\hat{k}).\frac{1}{7}(2\hat{i} + 3\hat{j} + 6\hat{k})\frac{1}{7}dx\,dy$

$$= \iint (36z - 36 + 18y)\frac{dx\,dy}{6} = \iint (6z - 6 + 3y)\,dx\,dy$$

Putting the value of $6z = 12 - 2x - 3y$, we get



$$= \int_0^6 \int_0^{\frac{1}{3}(12-2x)} (12 - 2x - 3y - 6 + 3y)\,dx\,dy$$

$$= \int_0^6 \int_0^{\frac{1}{3}(12-2x)} (6 - 2x)\,dx\,dy$$

$$= \int_0^6 (6 - 2x)\,dx \int_0^{\frac{1}{3}(12-2x)} dy$$

$$= \int_0^6 (6 - 2x)\,dx\,(y)_0^{\frac{1}{3}(12-2x)} = \int_0^6 (6 - 2x)\frac{1}{3}(12 - 2x)\,dx$$

$$= \frac{1}{3}\int_0^6 (4x^2 - 36x + 72)\,dx = \frac{1}{3}\left[\frac{4x^3}{3} - 18x^2 + 72x\right]_0^6$$

$= \frac{1}{3}[4\times36\times2-18\times36+72\times6] = \frac{72}{4}[4-9+6] = 24$ **Ans.**

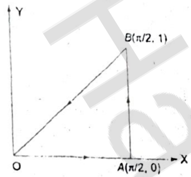
Q. 6. (a) Evaluate : $\oint_C (y - \sin x)dx + \cos x\, dy$

(i) directly

(ii) using Green's theorem in the plane, where C is the boundary of a triangle enclosed by the lines $y = 0, x = \frac{\pi}{2}$, and $y = \frac{2}{\pi}x$. **(10)**

Sol. We have by Green's theorem in the plane,

$$\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dy\, dx$$



Here, $M = y - \sin x$, $N = \cos x$

$$\frac{\partial N}{\partial x} = -\sin x, \quad \frac{\partial M}{\partial y} = 1$$

(i) Try yourself using definite integration.

(ii) Now, $\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dy\, dx$

$$= \iint_R (-\sin x - 1)\, dy\, dx = \int_{x=0}^{\pi/2}\left[\int_{y=0}^{\frac{2x}{\pi}} (-\sin x - 1)\, dy\, dx\right]$$

$$= \int_0^{\pi/2} [-y\sin x - y]_0^{\frac{2x}{\pi}}\, dx = \int_0^{\pi/2}\left(-\frac{2x}{\pi}\sin x - \frac{2x}{\pi}\right) dx$$

$$= -\frac{2}{\pi}\left[(-x\cos x + \sin x) - \frac{x^2}{\pi}\right]_0^{\pi/2} = -\frac{2}{\pi} - \frac{\pi}{4}$$

(b) Very that : $\nabla^2 r^n = n(n+1)r^{n-2}$.

Sol. $\nabla^2 r^2 = n(n+1)r^{n-2}$

Let $\phi = r^n$, Then

$$\frac{\partial^2\phi}{\partial x^2} = n\left[1.r^{n-2} + x(n-2)r^{n-3}\frac{x}{r}\right] = n r^{n-2}\left[1 + \frac{(n-2)}{r^2}r^2\right]$$

Similarly,

$$\frac{\partial^2\phi}{\partial y^2} = n r^{n-2}\left[1 + \frac{(n-2)}{r^2}y^2\right]$$

And $$\frac{\partial^2\phi}{\partial z^2} = n r^{n-2}\left[1 + \frac{(n-2)}{r^2}z^2\right]$$

Now, $$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

$$= n r^{n-2}\left[3 + \frac{(n-2)}{r^2}(x^2 + y^2 + z^2)\right] = n r^{n-2}\left[3 + \frac{(n-2)}{r^2}r^2\right]$$

$= nr^{n-2}[3 + n - 2] = n(n+1)r^{n-2} =$ R.H.S. Hence Proved.

Q. 7. (a) Verify divergence theorem for

$\vec{F} = (x^2 - yz)\hat{i} + (y^2 - xz)\hat{j} + (z^2 - xy)\hat{k}$ **taken over the rectangular parallelopiped $0 \le x \le a, 0 \le y \le b, 0 \le z \le c$.** **(10)**

Sol. We have

$$\text{div } \vec{F}\, \nabla.\vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).[(x^2 - yz)\hat{i} + (y^2 - zx)\hat{j} + (z^2 - xy)\hat{k}]$$

$$= \frac{\partial}{\partial x}(x^2 - yz) + \frac{\partial}{\partial y}(y^2 - yz) + \frac{\partial}{\partial z}(z^2 - yz) = 2x + 2y + 2z$$

$\therefore$ Volume integral $= \iiint_V \nabla.\vec{F}\, dV = \iiint_V 2(x + y + z)\, dV$

$$= 2\int_{x=0}^{a}\int_{y=0}^{b}\int_{z=0}^{c}(x + y + z)\, dx\, dy\, dz = 2\int_0^a\int_0^b\int_0^c (x + y + z)\, dz$$

$$= 2\int_0^a dx\int_0^b dy\left(xz + yz + \frac{z^2}{2}\right)_0^c = 2\int_0^a dx\int_0^b dy\left(cx + cy + \frac{c^2}{2}\right)$$

$$= 2\int_0^a dx\left(cxy + c + \frac{y^2}{2} + \frac{c^2 y}{2}\right)_0^b = 2\int_0^a dx\left(bcx + c + \frac{b^2 c}{2} + \frac{bc^2}{2}\right)$$

$$= 2\left[\frac{bcx^2}{2} + \frac{b^2 cx}{2} + \frac{bc^2 x}{2}\right]_0^a = [a^2bc + ab^2c + abc^2]$$

$= abc(a + b + c)$ ...(A)

To evaluate $\iint_S \vec{F}.\hat{n}\,ds$, where S consists of six plane surfaces.

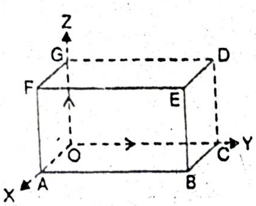
$$\iint_S \vec{F}.\hat{n}\,ds = \iint_{OABC}\vec{F}.\hat{n}\,ds+\iint_{DEFC}\vec{F}.\hat{n}\,ds+\iint_{OAFG}\vec{F}.\hat{n}\,ds$$

$$+\iint_{BCDE}\vec{F}.\hat{n}\,ds+\iint_{ABEF}\vec{F}.\hat{n}\,ds+\iint_{OCDG}\vec{F}.\hat{n}\,ds$$

$$\iint_{OABC}\vec{F}.\hat{n}\,ds = \iint_{OABC}\{(x^2-yz)\hat{i}+(y^2-xz)\hat{j}+(z^2-xy)\hat{k}\}(-\hat{k})\,dx\,dy$$

$$= -\int_0^a\int_0^b(z^2-xy)\,dx\,dy = -\int_0^a\int_0^b(0-xy)\,dx\,dy = \frac{a^2b^2}{4} \quad ...(1)$$

$$\iint_{DEFG}\vec{F}.\hat{n}\,ds = \iint_{DEFG}\{(x^2-yz)\hat{i}+(y^2-xz)\hat{j}+(z^2-xy)\hat{k}\}(\hat{k})\,dx\,dy$$



$$= \int_0^a\int_0^b(z^2-xy)\,dx\,dy = \int_0^a\int_0^b(c^2-xy)\,dx\,dy$$

$$= \int_0^a\left[c^2y-\frac{xy^2}{2}\right]_0^b dx = \int_0^a\left(c^2b-\frac{xb^2}{2}\right)dx = \left[c^2bx-\frac{x^2b^2}{4}\right]_0^a = abc^2-\frac{a^2b^2}{4} \quad ...(2)$$

S. No.	Surface	Outward normal	ds	
1.	OABC	$-k$	$dx\,dy$	$z = 0$
2.	DEFG	k	$dx\,dy$	$z = c$
3.	OAFG	$-j$	$dx\,dz$	$y = 0$
4.	BCDE	j	$dx\,dz$	$y = b$
5.	ABEF	i	$dy\,dz$	$x = a$
6.	OCDG	$-i$	$dy\,dz$	$x = 0$

$$\iint_{OAFG}\vec{F}.\hat{n}\,ds = \iint_{OAFG}\{(x^2-yz)\hat{i}+(y^2-zx)\hat{j}+(z^2-xy)\hat{k}\}(-\hat{j})\,dx\,dz$$

$$= -\iint_{OAFG}(y^2-zx)\,dx\,dz$$

$$= -\int_0^a dx\int_0^a(0-zx)\,dz = \int_0^a dx\left(\frac{xz^2}{2}\right)_0^c = \int_0^a\frac{xc^2}{2}dx = \left[\frac{x^2c^2}{4}\right]_0^a = \frac{a^2c^2}{4} \quad ...(3)$$

$$\iint_{BCDE}\vec{F}.\hat{n}\,ds = \iint\{(x^2-yz)\hat{i}+(y^2-zx)\hat{j}+(z^2-xy)\hat{k}\}.\hat{j}\,dx\,dz$$

$$= \iint_{BCDE}(y^2-xz)\,dx\,dz$$

$$= -\int_0^a dx\int_0^a(b^2-xz)\,dz = \int_0^a\left(b^2z-\frac{xz^2}{2}\right)_0^c dx = \int_0^a\left(b^2c-\frac{xc^2}{2}\right)dx$$

$$= \left[b^2cx-\frac{x^2c^2}{4}\right]_0^a = ab^2c-\frac{a^2c^2}{4} \quad ...(4)$$

$$\iint_{ABEF}\vec{F}.\hat{n}\,ds = \iint_{ABEF}\{(x^2-yz)\hat{i}+(y^2-xz)\hat{j}+(z^2-xy)\hat{k}\}\hat{i}\,dy\,dz$$

$$= \iint_{ABEF}(x^2-yz)\,dy\,dz = \int_0^b dy\int_0^c(a^2-yz)\,dz = \int_0^b dy\left(a^2z-\frac{yz^2}{2}\right)_0^c$$

$$= \int_0^b\left(a^2c-\frac{yc^2}{2}\right)dy = \left[a^2cy-\frac{y^2c^2}{4}\right]_0^b = a^2bc-\frac{b^2c^2}{4} \quad ...(5)$$

$$\iint_{OCDG}\vec{F}.\hat{n}\,ds = \iint_{OCDG}\{(x^2-yz)\hat{i}+(y^2-zx)\hat{j}+(z^2-xy)\hat{k}\}(-\hat{i})\,dy\,dz$$

$$= \int_0^b\int_0^c(x^2-yz)\,dy\,dz = -\int_0^b dy\int_0^c(-yz)\,dz = -\int_0^b dy\left[\frac{-yz^2}{2}\right]_0^c$$

$$= \int_0^b\left[\frac{yc^2}{2}dy\right] = \left[\frac{y^2c^2}{4}\right]_0^b = \frac{b^2c^2}{4} \quad ...(6)$$

Adding (1), (2), (3), (4), (5) and (6), we get

$$\iint\vec{F}.\hat{n}\,ds = \left(\frac{a^2b^2}{4}\right)+\left(abc^2-\frac{a^2b^2}{4}\right)+\left(\frac{a^2c^2}{4}\right)+\left(ab^2c-\frac{a^2c^2}{4}\right)$$

$$+\left(\frac{b^2c^2}{4}\right)+\left(a^2bc-\frac{b^2c^2}{4}\right) \quad ...(B)$$

$$= abc^2+ab^2c+a^2bc = abc\,(a+b+c)$$

From (A) and (B), Gauss diveragence Theorem is verified.

(b) Express the position and velocity of a particle in cylindrical coordinates. (5)

Sol. Please Refer I Q 3, Unit - 2D

Q. 8. (a) Derive an expression for the divergence of a vector field in orthogonal curvilinear coordinate system. (10)

Sol. Please Refer I Q 13, Unit - 2D

(b) Evaluate Jacobin $J\left(\dfrac{x, y, z}{u_1, u_2, u_3}\right)$ **for the transformation from rectangular coordinate system to spherical coordinate system.** (5)

Sol. Please Refer I Q 27, Unit - 2C.

Sr. No. of Question Paper	:	8599
Unique Paper Code	:	32221101
Name of the Paper	:	Mathematical Physics – I
Name of the Course	:	B.Sc. (Hons.) Physics
Semester	:	I

Nov./Dec. 2019

Duration : 3 Hours **Maximum Marks : 75**

Instructions for Candidates

1. Question **No. 1** is compulsory.
2. Attempt **four** more questions out of the rest.

Q. 1. (a) Determine the linear independence/linear dependence of e^x, xe^x, $x^2 e^x$.

Ans. Let $y_1(x) = e^x$; $y_2(x) = xe^x$. $Y_3(x) = x^2 e^x$

Then $y_1^1(x) = e^x$; $y_2^1(x) = e^x(x+1)$; $y_3^1(x) = e^x(x^2+2x)$

and

$y_1''(x) e^x$; $y_2''(x) = e^x(x+2)$; $y_3''(x) = e^x(x^2+4x+2)$

Now wronskian is given By

$$w = \begin{vmatrix} e^x & e^x(x) & e^x x^2 \\ e^x & e^x(x+1) & e^x(x^2+2x) \\ e^x & e^x(x+2) & e^x(x^2+4x+2) \end{vmatrix}$$

Or

$$W = e^x \begin{vmatrix} e^x(x+1) & e^x(x^2+2x) \\ e^x(x+2) & e^x(x^2+4x+2) \end{vmatrix}$$

$$-e^x \begin{vmatrix} e^x & e^x(x^2+2x) \\ e^x & e^x(x^2+4x+4) \end{vmatrix} + e^x x^2 \begin{vmatrix} e^x & e^x(x+1) \\ e^x & e^x(x+2) \end{vmatrix}$$

$$= e^{3x}\left[(x+1)(x^2+4x+2) - (x+2)(x^2+2x)\right]$$

$$- e^{3x}\left[x(x^2+4x+2-x^2-2x)\right] + e^{3x}\left[x^2(x+2-x-1)\right]$$

$$= e^{3x}\left[6x+2-4x+1(2x+2)x\right]$$

$$= e^{3x}\left(4x^2+4x+2\right)$$

$\therefore\ w \neq 0$

So the functions are linearly independent.

(b) Determine the order, degree and linearity of the following differential equation.

$$\frac{d^3 y}{dx^3}+x^2\left(\frac{d^2 y}{dx^2}\right)^2=0$$

Ans. Order of a differential equation. The order of a differential equation is the order of the highest order derivative appearing in the equation Degree of a differential equation degree of a differential equation

The degree of a differential equation is the degree of the highest order derivative when differential coeficients are made free from radicals and fractions.

Linear and non-linear differential equations A diffrential equation when expressed in the form of a polynomial involves the derivatues and dependent variable in the first power and there are in product of these, and also the co-efficient of the various terms are either constanto or functions of the independent variable is linear.

$$\frac{d^3 y}{dx^3}+x^2\left(\frac{d^2 y}{dx2}\right)^2=0$$

Order .... 3, Degree 1

Linearity Non-linear $\left[\because\left(\frac{d_y^2}{dx\,2}\right)^2\right]$

(c) Find the area of the triangle having vertices at P (1, 3, 2), Q (2, – 1, 1) and (–1, 2, 3).

Ans. The three vertices of ΔPQR are :

P (1, 3, 2) ; Q (2, – 1,1) ; R (–1, 2, 3)

$$\overline{PQ}=\overline{OQ}-\overline{OP}=\left(2\hat{i}-\hat{j}+k\right)-\left(\hat{i}+3\hat{j}+2k\right)=\left(i-4j-k\right)$$

$$\overline{QR}=\overline{OQ}=\left(-i+2j+3k\right)-\left(2i-j+k\right)=(-3i+3j+2k)$$

$$\text{Now area of } \Delta\, PQR=\frac{1}{2}\left(\overline{PQ}\times\overline{QR}\right)$$

$$=\frac{1}{2}\begin{vmatrix} i & j & k\\ 1 & -4 & -1\\ -3 & +3 & +2\end{vmatrix}=i(-8+3)-j(2-3)+k(3-12)$$

$$= -5i+j-9k$$

$$\therefore \left|\overline{PQ}\times\overline{QR}\right|=\sqrt{(-5)^2+(1)^2+(-9)^2}=\sqrt{(25)+1+81}=\sqrt{107}\ \text{sqmtrs}$$

$\therefore$ area of Δ PQR is $\frac{1}{2}\sqrt{107}$ sq. units.

(d) Let $\overline{A}$ be a constant vector. Prove that $\vec{\nabla}\left(\bar{r}.\overline{A}\right)=\overline{A}$

Ans. $\bar{\Delta}\left(\bar{r}.\overline{A}\right)=\overline{A}$

Since $\overline{A}.\overline{B}=\overline{B}.A$ as $\cos\theta=\cos(-\theta)$

$$\therefore\quad \bar{\Delta}\left(\bar{r}.\overline{A}\right)=\bar{\Delta}\left(\overline{A}.\bar{r}\right)$$

Let $\quad \overline{A}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k},\ \bar{r}=x\hat{i}+y\hat{j}+z\hat{k}$

$$\therefore\quad \overline{A}.\bar{r}=\left(a_1\hat{i}+a_2\hat{j}+a_3k\right).\left(x\hat{i}+y\hat{j}+z\hat{k}\right)$$

$$=a_1x+a_2y+a_3z$$

$$\text{Therefore}\quad \bar{\nabla}\left(\overline{A}.\bar{r}\right)=\left(i\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\left(a_1x+a_2y+a_3z\right)$$

$$=\hat{i}a_1+\hat{j}a_2+\hat{k}a_3=\overline{A}.\ \text{Hence Proved}$$

(e) Find the acute angle between the surfaces $xy^2z-3x-z^2=0$ and $3x^2-y^2+2z=1$ at the point (1, –2, 1)

Ans. The angle between the surfaces at the point is the angle between the normals to the surfaces at the point.

A normal to $xy^2z-3x-z^2=0$

At (1, – 2,1) is : $\nabla\phi_1$

$$\nabla=\phi_1=\nabla\left(xy^2z-3x-z^2\right)$$

$$=\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\left(xy^2z-3x-z^2\right)$$

$$= \left(y^2 z\, i - 3i\right) + \left(2xyz\,\hat{j}\right) + \left(xy^2\,\hat{k} - 2z\hat{k}\right)$$

$$= \left(y^2 z - 3\right)\hat{i} + 2x\,yz\,\hat{j} + \left(xy^2 - 2z\right)\hat{k}$$

Normal at the point (1, – 2, 1) is :

$$\left((-2)^2\,(1) - 3\right)\hat{i} + 2(1)(-2)(1)\hat{j} + \left\{1\,(-2)^2 - 2(1)\right\}\hat{k}$$

$$= \hat{i} - 4\hat{j} + 2\hat{k} \qquad \ldots(i)$$

Normal to the surface $(3x^2 - y^2 + 2z - 1)$

At (1, –z, –1) :

$$\nabla\,\phi_2 = \left(\hat{i}\,\frac{\partial}{\partial x} + \hat{j}\,\frac{\partial}{\partial z} + \hat{k}\,\frac{\partial}{\partial 2}\right)\left(3x^2 - y^2 + 2z - 1\right)$$

$$\frac{\left(x^2+y^2+z^2\right)^{n/2}.\,1 - y\frac{n}{2}\left(x^2+y^2+2^2\right)(2y)}{\left(x^2+y^2+z^2\right)n} = 6(1)\,\hat{i} - 2\,(-2)\,\hat{j} + 2\hat{k}$$

$$= 6\hat{i} + 4\hat{j} + 2\hat{k} \qquad (ii)$$

Let θ be the angle between the normals (i) and (ii)

$$\left(\hat{i} - 4\hat{j} + 2\hat{k}\right).\left(6\hat{i} + 4\hat{j} + 2\hat{k}\right)$$

$$(P_1) = \left(\frac{5}{8}\right) \times \left(\frac{3}{5}\right) = \frac{3}{8}$$

$$= \sqrt{21}\,\sqrt{56}\,\cos\theta = \sqrt{7}\,\sqrt{3}\,\;\sqrt{7}\,\sqrt{4}\,\sqrt{2}\,\cos\theta = 14\sqrt{6}\,\cos\theta$$

, Q

$$\therefore\ 6 - 16 + 4 = 14\sqrt{6}\,\cos\theta$$

$$\cos\theta = \frac{-6}{14\sqrt{6}} = \frac{-\sqrt{6}}{14} \Rightarrow \theta\,\cos^{-1}\left(-\frac{\sqrt{6}}{14}\right)$$

(f) A random variable X has probability density function

$$f(x) = \frac{1}{\sqrt{2\pi}}\,e - (x-5)^{2/2}$$

$$-\infty < X < \infty$$

Ans. Please Refer I Q. 34, Unit – 3 p – 162 to 164 Also

We explain the concept of probability density function.

When a varibale X takes every value in an interval, it gives rise to continous distribution of X. The distributions defined by the variables like heights, weights

are continous distributions. The probability distributions of a continous variable x is defined by the function $f(x)$ such that the probability of the variable x in the small interval $x - \frac{1}{2}dx$ To $x + \frac{1}{2}dx$. Symbiolically it cam be expressed by

$$P\left(x - \frac{1}{2}dx \le x \le x + \frac{1}{2}dx\right) = f(x)\,dx$$

The function $f(x)$ is called the probability density function and the continous curve $y = f(x)$ is called the probability curve.

The range of variable may be finite or infinite, it is convenient to consider it as infinite by supporting the density function to be zero outside the given range.

Let $f(x) = \phi\,(x)$ be density function for x in the the interval (a, b), then in can be written as

$$f(x) = 0,\ x < a$$
$$= \phi,\ a \le x \le b$$
$$= 0,\ x > b$$

The density function $f(x)$ is always positive and $\int_{-\infty}^{\infty} f(x)\,dx = 1$, i.e, the total area under the probability curve and the x-axis is unity.

This means that total probability of happening of an event is unity.

The mean value (u) of the probability distribution of a variabnce X commonly known as its expectation is denoted by E (x). Let $f(x)$ be the probability density function of the variate X, then

We have, $E(x) = \int_{-\infty}^{\infty} x\,f(x)\,x$, for continous distribution.

In general, expection of any function

$\phi(x)$ is $E\left[\phi(x)\right] = \sum_i \phi(x_i)\,f(x_i)$ for a discrete distribution.

Or $E\,[\phi\,(x)] = \int_{-\infty}^{\infty} \phi(x)\,f(x)\,dx$ for a continous distribution.

Please calculate the mean or expectation value based on above.

Q. 2. (a) Solve the simultaneous differential equations given below.

$$\frac{dy}{dt} = y \qquad \frac{dx}{dt} = 2y + x$$

Ans. $\frac{dy}{dt} = y\ ;\ \frac{dx}{dt} = 2y + x$

$$\frac{dy}{dt} = y$$

Or $\log y = t + \log a$

Or $y = t + \log a$

Or $y = ae^t$ ...(i)

Now we have, $\frac{dx}{dt} - x = zae^t$ $\left[\because y = ae^t\right]$

I. F. $= e^{-t}$

The required solution is

$$xe^{-t} = \int 2ae^t e^{-t}\, dt + b$$

Or $xe^{-t} = 2at + b$ ...(ii)

Hence, (i) and (ii) Constitute the required solution.

(b) Two independent random variable X and Y have probability density functions $f(x) = e^{-x}$ and $g(y) = 2e^{-2y}$ respectively. What is the probability that X and Y lie in the intervals $1 < x \le 2$ and $0 < y \le 1$

The time rate of change of the temperature of a body at an instant t is proportional to the temperature difference between the body and its surrounding medium that instant.

Ans. $f(x) \ge 0$ For all x in (1, 2) and

$$\int_{-\infty}^{\infty} f(x)\, dx = \int_{-\infty}^{0} 0.x + \int_{0}^{\infty} e^{-x}\, dx = 1$$

Hence the function is a density function.

Probability $= P(1 \le x \le 2)$

$$= \int_{1}^{2} e^{-x}\, dx = e^{-1} - e^{-2} = 0.368 - 0.135 = 0.233$$

Please complete the question on similar lines.

(c) Box A contains 8 items out of which 3 are defective. Box B contains 5 items out of which 2 are defective. An item is drawn randomly from each box. **5 + 5 + 5**

Ans. Total number of items in box A = 8 Defective items in box A = 3

Non defective items in box A = 5

$$P_{A\text{-Def}} = \frac{3}{8}$$

$$P_{A\text{-N-Def}} = \frac{5}{8}$$

Total No. of items in Box B = 5

defective items in Box B = 2

Non-defective items in Box B = 3

$$P_{B\text{-Def}} = \frac{2}{5}$$

$$P_{B\text{-N-Def}} = \frac{3}{5}$$

(i) What is the probability that both the items are non-defective ?

Ans. (i) Both items are non-defective (P_1)

$$(P_1) = \left(\frac{5}{8}\right) \times \left(\frac{3}{5}\right) = \frac{3}{8}$$

(ii) What is the probability that only one item is defective ?

Ans. Only one item is defective (p_2)

This defective itme can be the first item or the second item.

$$\therefore p_2 = \left(\frac{3}{8}\right)\left(\frac{3}{5}\right) + \left(\frac{5}{8}\right)\left(\frac{2}{5}\right) = \frac{9}{40} + \frac{10}{40} = \frac{19}{40}$$

(iii) What is the probability that the defective item came from box A ?

Ans. Please try on the above lines.

Q. 3. Solve the following differential equations. 8

(a) $y^{11} + y = \sec x$

Ans. $y^{11} + y = \sec x$

$$\frac{d^2y}{dx^2} + y = \sec x$$ (I)

Auxiliary equation is : $m^2 + 1 = 0$

$\Rightarrow m^2 = -1$

$\Rightarrow m = \pm i$

Complementary function of (I) is :

A cos x + B sin x

Here, $y_1 = \cos x$

$y_2 = \sin x$ (II)

$$\text{P. I.} = uy_1 + vy_2$$

Where $u = \int \frac{-y_2 \sec x}{y_1 \cdot y_2^1 - y_1^1 y_2} dx$

$$u = \int \frac{-y_2 \sec x}{y_1 \cdot y_2^1 - y_1^1 y_2} dx$$

$[y_1 y_2^1 - y_1^1 y_2 = \cos x \cos x - \sin x(-\sin x) = \cos^2 x + \sin^2 x = 1]$

On putting the values of y_2 and $y_1 y_2^1 - y_1^1 y_2$, we get

$$u = \int \frac{-\sin x \sec x}{1} dx = -\int \tan x dx = \log \cos x$$

And $v = \int -\frac{y_1 \sec x}{y_1 y_2^1 - y_1^1 y_2} dx$

putting the values of y_1 and $(y_1 y_2^1 - y_1^1 y_2)$

we get $v = \int \frac{\cos x . \sec x}{1} = \int dx = x$

putting the values of u and v in (II),

P.I = cos x. log cos x + x sin x

complete solution is

y = C. F. + P. I

Or $y = A \cos x + B \sin x + \cos x \log \cos x + x \sin x$

(b) $(z + ye^{xy})dx + (xe^{xy} - 2y)\, dy = 0$ 7

Ans. Please refer your text as the question is misleading due to presence of Z in it.

Q. 4. (a) Solve the initial value problem. 8

(i) y" + 4y' + 8y = sin x

Ans. Auxiliary equation :

$m^2 + 4m + 8 = 0$

$$m = \frac{-4 \pm \sqrt{16-32}}{2} = \frac{-4+4i}{2} = -2 \pm 2i$$

$\therefore \alpha = -2 + 2i$ And $\beta = -2 - 2i$

Complementary function is :

$$= Ae^{(-2+2i)x} + Be^{(-2-2i)x} = e^{-2x}\left[Ae^{2ix} + Be^{-2ix}\right]$$

$$= e^{-2x}\left[A\{\cos 2x + i \sin 2x\} + B\{\cos 2x - i \sin 2x\}\right]$$

$$= e^{-2x}\left[\cos 2x + iD \sin 2x\right]$$

P. I. = $(D^2 + 4D + 8) \sin x$

Please refer Q. 2 (a), 2016 p - 190 and calculate P. I. on similar lines and compute the complete solution.

The value of the constants can be calculated using the initial values given.

(b) A metal bar at a temperature 100° F is placed in a room at a constant temperature of 0° F. After 20 minutes the temperature of the bar is 50° F. Find : 7

Ans. Initial temperature of bar = 100° F

Room Temperature = 0° F

temperature of bar after

Zominutes = 50° F

Let T be the temperature of water at any time t. Then by Newtons law of cooling.

$$\frac{dT}{dt} \propto (T-0)$$

Or $\frac{dT}{dt} \propto T$

$\therefore \frac{dT}{T} = -\lambda\, dt$

Or $\int \frac{dT}{T} = -\lambda \int dt$

$\therefore$ $|\log|T|| = -\lambda t + C$

At $t = 0, T = 100°$ F

Substituting above values in

$\log 100° = C$

$\therefore$ $\log|T| = -\lambda t + \log 100$

Or $\log\left(\frac{T}{100}\right) = \lambda t$

Temperature of bar after 20 minutes is 50° F

$\therefore \log\left(\frac{50}{100}\right) = -20\lambda$

Or $-20l = \log \frac{1}{2} = \log 1 - 1\,]^2$

$\therefore$ $-20\lambda = -\log 2$

$\therefore \lambda = \frac{1}{20} \log^2$

$$\therefore \quad \log\left|\frac{T}{100}\right| = \frac{1}{20}\log 2$$

(i) The time it will take the bar to reach a temperature of 25° F

Ans. We have to find the time the bar takes to reach 25° F

$$\log\left(\frac{T}{100}\right) = -\frac{t}{20}\log 2$$

$$\log\frac{25}{100} = -\frac{t}{20}\log 2$$

$$\log\frac{1}{4} = -\frac{t}{20}\log 2$$

$$\log 1 - \log = -\frac{t}{20}\log 2$$

$$\frac{-\log 4}{-\log 4} = \frac{t}{20}$$

$$\therefore \frac{t}{20} = 2$$

$\therefore t = 40$ minutes

(ii) Temperature of the bar after 10 minutes

Ans. Bar temperature after 10 minutes

$$\log\left(\frac{T_0}{100}\right) = -\frac{10}{20}\log 2$$

Or $$\log\left|\frac{T}{100}\right| = -\frac{1}{2}\log^2$$

$$\log 2 = -2\log\frac{T}{100}$$

Or $$\log 2 = 2\log\frac{100}{1}$$

Or $$\log 2 = \log\left(\frac{100}{T}\right)^2$$

Taking anti Cogarithms

$$Z = \left(\frac{100}{T}\right)^2$$

$$\therefore \quad 2\,T^2\;100^2$$

$$T^2 = \left(\frac{100}{\sqrt{2}}\right)^2 = (70.7)^2$$

$$\therefore \quad T = 70.7° \;\; F$$

Q. 5. If v denotes the region inside the semicircular cylinder

$$0 \le x \le \sqrt{a^2 - y^2} \quad 0 \le z \le 2a$$

Evaluate $\iiint_v x\,dv$

Ans. Please refer your text

(b) 17

Ans. The above can also be written as

$$\int_{x_1}^{x_2} x\,dx \int_{y_1}^{y_2} dy \int_{z_1}^{z_2} dz = \int_{x_1}^{x_2} x\,dx \int_{y_1}^{y_2} dy\,(z_2 - z_1)$$

$$= \int_{x_1}^{x_2} x\,dx \int_{y_1}^{y_2} (z_2 - z_1)\,dy = \int_{x_1}^{x_2} x\,dx\left[y\,(z_2 - z_1)\right]_{y=y_1}^{y=y_2}$$

$$= \int_{x_1}^{x_2} x\,dx\,(z_2 - z_1)(y_2 - y_1) = (y_2 - y_1)(z_2 - z_1)\left(\frac{x^2}{2}\right)_{x=x_1}^{x=x_2}$$

$$= \frac{(x_2^2 - x_1^2)(y_2 - y_1)(z_2 - z_1)}{2} = \frac{1}{2}(x_2^2 - x_1^2)(y_2 - y_1)(z_2 - z_1)$$

The boundary conditions will specify the values z_1, z_2, y_1, y_2 and x_1, x_2

Q. 6. (a) Find the directional derivative of $\varphi = 4xz^3 - 3x^2y^2z$ at (2, – 1, 2) in the direction $2\hat{i} - 3\hat{j} + 6\hat{k}$ **5**

Ans. $\phi = 4\,x\,z^3 - 3\,x^2\,y^2\,z$

$$\nabla\phi = \left[\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right](4\,x\,z^3 - 3x^2\,y^2\,z)$$

$$= (4z^3 - 6\,x\,y^2\,z)\,\hat{i} + (-6\,x^2\,y\,z)\,\hat{j} + (12\,x\,z^2 - 3x^2)\hat{k} \quad \nabla\phi \text{ AT } (2, -1, 2)$$

$$= \left[4(2)^3 - 6(2)(-1)^2(2)\right]\hat{i} + \left[-6(2)^2(-1)(2)\right]\hat{j} + \left[12(2)(2)^2 - 3(2)^2(-1)^2\right]\hat{k}$$

$$= [32 - 24]\hat{i} + [48]\hat{j} + [96 - 12]\hat{k} = 8\hat{i} + 48\hat{j} + 84\hat{k}$$

$\hat{a}$ = unit vector

$$= \frac{2\hat{i} - 3\hat{j} + 6\hat{k}}{\sqrt{4+9+36}} = \frac{1}{7}\left(2\hat{i} - 3\hat{j} + 6\hat{k}\right)$$

So the required derivative at (2, –1, 2)

$$= \nabla\phi\,\hat{a}\ (8\hat{i} + 48\hat{j} + 84\hat{k})\,\frac{1}{7}\,(2\hat{i} - 3\hat{j} + 6\hat{k})$$

$$= \frac{1}{7}(16 - 144 + 504) = \frac{376}{7}$$

(b) **Find the value of $\nabla^2(\ln r)$** 5

Ans. $\nabla^2\left(\ln \bar{r}\right)$

we have $\nabla^2\phi = \nabla.\nabla\phi$

$\Rightarrow \nabla^2(\ln \bar{r}) = \nabla.\nabla(\ln \bar{r})$

$\Delta \ln |r|$

Let $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Then $|\bar{r}| = \sqrt{x^2 + y^2 + z^2}$

And $\ln|\bar{r}| = \frac{1}{2}\ln\left(x^2 + y^2 + z^2\right)$

$$= \frac{1}{2}\left\{\hat{i}\frac{\partial}{\partial 2}\ln\left(x^2 + y^2 + z^2\right) + \hat{j}\frac{\partial}{\partial y}\ln\left(x^2 \times y^2 + z^2\right) + \hat{k}\frac{\partial}{\partial 2}\ln\left(x^2 + y^2 + z^2\right)\right\}$$

$$= \frac{1}{2}\left[\hat{i}\frac{2x}{x^2 + y^2 + 2^2} + \hat{j}\frac{2y}{x^2 + y^2 + z^2} + \hat{k}\frac{2z}{x^2 + y^2\ z^2}\right] = \frac{\bar{r}}{r^2}$$

Let us evaluate $\bar{\Delta}\left(\frac{\bar{r}}{r^n}\right), n > 0$

And $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\bar{\nabla}\left(\frac{\bar{r}}{r^n}\right) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left[\frac{x\hat{i} + y\hat{j} + z\hat{k}}{\left(x^2 + y^2 + z^2\right)^{n/2}}\right]$$

$$= \left[\hat{i}\frac{\partial}{\partial x}\frac{x\hat{i}}{\left(x^2 + y^2 + z^2\right)^{n/2}} + \hat{j}\frac{\partial}{\partial y}\frac{y\hat{j}}{\left(x^2 + y^2 + z^2\right)^{n/2}} + \hat{k}\frac{\partial}{\partial z}\frac{z\hat{k}}{\left(x^2\ y^2\ 2^2\right)^{n/2}}\right]$$

$$= \frac{\partial}{\partial x}\frac{x}{\left(x^2 + y^2 + z^2\right)^{n/2}} + \frac{\partial}{\partial y}\frac{y}{\left(x^2 + y^2 + z^2\right)^{n/2}} + \frac{\partial}{\partial z}\frac{z}{\left(x^2 + y^2 + z^2\right)^{n/2}}$$

$$\frac{\left(x^2 + y^2 + z^2\right)^{n/2}1 - x\frac{n}{2}\left(x^2 + y^2 + z^2\right)^{\frac{n}{2}-1}(2x)}{\left(x^2 + y^2 + z^2\right)n}$$

$$+ \frac{\left(x^2 + y^2 + z^2\right)^{n/2}.1 - y\frac{n}{2}\left(x^2 + y^2 + z^2\right)(2y)}{\left(x^2 + y^2 + z^2\right)n}$$

$$+ \frac{\left(x^2 + y^2 + z^2\right)^{n/2} - y\frac{n}{2}\left(x^2 + y^2 + z^2\right)^{-1}(z)}{\left(x^2 + y^2 + z^2\right)n}$$

$$= \frac{\left(x^2 + y^2 + z^2\right) - nx^2}{\left(x^2 + y^2 + z^2\right)\left(x^2 + y^2 + z^2\right)^{\frac{n}{2}+1}} + \frac{\left(x^2 + y^2 + z^2\right) - ny^2}{\left(x^2 + y^2 + z^2\right)^{\frac{n}{2}+1}} + \frac{\left(x^2 + y^2 + z^2\right) - nz^2}{\left(x^2 + y^2 + z^2\right)^{\frac{n}{2}+1}}$$

$$= \frac{x^2 + nx^2 + y^2 + z^2}{\left(x^2 + y^2 + z^2\right)^{\frac{n}{2}+1}} + \frac{\left(x^2 + y^2 - ny^2 + z^2\right) - ny^2}{\left(x^2 + y^2 + z^2\right)^{\frac{n}{2}+1}} + \frac{x^2 + y^2 + z^2 - nz^2}{\left(x^2 + y^2 + z^2\right)^{\frac{n}{2}+1}}$$

$$= \frac{\left\{(1-n)x^2 + y^2 + z^2\right\} + \left\{(1-n)y^2 + x^2 + z^2\right\} + \left\{(1-n)z^2 + x^2 + y^2\right\}}{\left(x^2 + y^2 + z^2\right)^{\frac{n}{2}+1}}$$

$$= \frac{(3-n)\left(x^2 + y^2 + z^2\right)}{\left(x^2 + y^2 + z^2\right)^{\frac{n}{2}+1}} = \frac{3-n}{\left(x^2 + y^2 + z^2\right)^{\frac{n}{2}}} = \frac{3-n}{r^n}$$

Giving the value $n = 2$

$$\bar{\nabla}\left(\frac{\bar{r}}{r^2}\right) = \frac{3-2}{r^2} = \frac{1}{r^2}$$

Hence the required answer is $\frac{1}{r^2}$

(c) Prove that : $\iiint \frac{dv}{r^2} = \oiint \frac{\bar{r}.\hat{n}}{r^2}\, ds$ 5

Where v is the volume of region enclosed by surface

Ans. Volume integral is a triple integration whereas a surface integral is a double integral.

The above can be written as $\int_V \frac{dv}{r^2} = \int_s \frac{\bar{r}.\bar{n}}{r^2}\, ds$

By divergence theorem, we have

$$\int_S \frac{\bar{r}.\bar{n}}{r^2} = \int_S \left(\frac{\bar{r}}{r^2}\right).\bar{n}\, ds = \int_v div\left(\frac{\bar{r}}{r^2}\right) dv \quad ...(i)$$

Now $div\left(\frac{\bar{r}}{r^2}\right) = \nabla\left(\frac{\bar{r}}{r^2}\right) = \frac{1}{r^2}(\nabla.\bar{r}) + \bar{r}.\nabla\left(\frac{1}{r^2}\right)$

$$= \frac{1}{r^2}(3) + r.\left(-\frac{2}{r}\Delta\bar{r}\right) \qquad [\because div. r = 3; \text{etc.}]$$

$$= \frac{3}{r^2} - \frac{2}{r^2}\left(\frac{\bar{r}.\bar{r}}{r}\right)$$

Q. 7. (a) Suppose $\vec{A} = (2y+3)\hat{i} + xz\hat{j} + (yz-x)\hat{k}$ Evaluate $\int_c \vec{A}.\, d\bar{r}$ along the following paths : 9

Ans. $\vec{A} = (2y+3)\hat{i} + xz\hat{j} + (yz-x)\hat{k}$

We have to evaluate $\int_C \vec{A}.d\bar{r}$ in the following three cases :

$$\int_C \vec{A}.d\bar{r} = \int_C \left[(2y+3)\,\hat{i} + xz\hat{j} + (yz-x)\hat{k}\right]:\left(dx\,\hat{i} + dx\,\hat{j} + dz\,\hat{k}\right)$$

$x = 2t^2$

$y = t$

$z = t^3 \quad = \int_C (2y+3)\,dx + xz\,dy + (yz-x)dz$

(i) $x = 2t^2$, $y = t$, $z = t^3$ from $t = 0$ to $t = 1$

Ans. $x = 2t^2$, $y = t$, $z = t^3$ from $t = 0$ to $t = 1$

$$= \int_C \vec{A}\, dx = \int_{t=0}^{1} (2t+3)(2t^2) + (2t^2) + (2t^2)(t^3)\,dt + (t.t^3 - 2t^2)\,d(t^3)$$

$$= \int_{t=0}^{1} (2t+3)\,4t\,dt + 2t^5\,dt + (t^4 - 2t^2)\,3t^2\,dt(2t+3)d(2t^2)$$

$$= \int_{t=0}^{1} (8t^2 + 12t)\,dt + 2t^5\,dt - 3t^6\,dt - 6t^4\,dt$$

$$= \left[\frac{8.t3}{3} + \frac{12t^2}{2} + \frac{2t^6}{6} - 3\frac{t^7}{7} - 6\frac{t^5}{5}\right]_{t=0}^{1}$$

$$= \left[\frac{8}{3}t^3 + 6t^2 + \frac{1}{3}t^6 - \frac{3}{7}t^7\right]_{t=0}^{1} = \left[\frac{8}{3}t^3 + 6t^2 + \frac{1}{3}t^6 - \frac{3}{7}t^7\right]_{t=0}^{1}$$

$$= \frac{8}{3} + \frac{6}{1} + \frac{1}{3} - \frac{3}{7} = \frac{56 + 126 + 7 - 9}{21} = \frac{180}{21} = \frac{60}{7}$$

(ii) The straight line from (0, 0, 0) to (0, 0, 1) then to (0, 1, 1) and then to (2, 1, 1)

Ans. Please try yourself on similar lines.

(iii) The straight line joining (0, 0, 0) and (2, 1, 1)

Ans. Please try yourself on similar lines.

(b) Evaluate $\iint \vec{A}.\hat{n}\, dS$

where $\vec{A} = z\hat{i} + x\hat{j} + x\hat{j} - 3\hat{y}^2 z\hat{k}$ and S is the Surface of the cylinder $x^2 + y^2 = 16$ included in the first octant between $z = 0$ to $z = 5$ 6

Ans. The vector normal to the surfaces is given by

$$\nabla(x^2 + y^2) = 2x\hat{i} + 2y\hat{j}$$

$\therefore\ \hat{x}$ = unit normal At (x, y) of S

$$= \frac{2x\,\hat{i} + 2y\,\hat{j}}{\sqrt{4x^2 + 4y^2}} = \frac{2x\hat{i} + 2y\hat{j}}{\sqrt{16}} \quad [\because x^2 + y^2 = 16]$$

Now, $\int\limits_S F.\hat{n}\, ds = \iint\limits_R F.n \frac{dx\,dz}{|n.\hat{j}|}$

Where R is the projection of S on xz-place

$$= \iint\limits_R (z\hat{i} + x\hat{j} - 3y^2 z\hat{k}) \left(x\frac{\hat{i} + y\hat{j}}{4}\right) \frac{dx\,dz}{\frac{1}{4}[(x\hat{i} + y\hat{j}).\hat{j}]}$$

$$= \iint\limits_R \frac{zx + xy}{4} \cdot \frac{dx\,dz}{y/4} = \int\limits_R \frac{zx + xy}{y}\, dx\,dz$$

$$= \iint\limits_R \left(\frac{zx}{y} + x\right) dx\,dz = \iint\limits_R \left(\frac{zx}{\sqrt{16 - x^2}} + x\right) dx\,dz$$

$$= \int_{z=0}^{5}\int_{x=0}^{4} \left(\frac{zx}{\sqrt{16 - x^2}} + x\right) dx\,dz = \left[\frac{4z^2}{2}\right]_0^5 + [8z]_0^5 = 2(25 - 0) + 8(5 - 0)$$

$= 50 + 40 = 90$

AGS

Paper Code	:	32221101 (CBCS)
Name of the Paper	:	Mathematical Physics–I
Name of Course	:	B.Sc. (Hons.) PHYSICS I Year
Semester	:	I

2020

Duration : 3 Hours **Maximum Marks : 75**

Attempt any four questions out of six. Each question carries equal marks.

Q. 1. Solve the following first order differential equations.

(a) $x\,dx + y\,dy = \dfrac{x\,dy}{x^2 + y^2} - \dfrac{y\,dx}{x^2 + y^2}$

Ans. Integrate both sides of (i)

$$\int x\,dx + \int y\,dy = \int \frac{x}{x^2 + y^2}\,dy - \int \frac{y}{x^2 + y^2}\,dx$$

or $$\frac{x^2}{2} + \frac{y^2}{2} = x\tan^{-1}\left(\frac{x}{y}\right) - y\tan^{-1}\left(\frac{y}{x}\right) + C$$

(b) $(2x^2y^2 + xy)\,y\,dx - (x^2y^2 - xy)\,x\,dy = 0$

Ans. Divide (ii) by xy, we get

$$\frac{1}{xy}(2x^2y^2 + xy)\,y\,dx - \frac{1}{xy}(x^2y^2 - xy) \times dy = 0$$

or $$\frac{xy}{xy}(2xy + 1)\,y\,dx - \frac{xy}{xy}(xy - 1) \times dy = 0$$

or $$y(2xy + 1)\,dx - x(xy - 1)\,dy = 0$$

This can be written as,

$$y(1 + 2xy)dx + x(1 - xy)\,dy = 0 \qquad \ldots(iii)$$

Let $M = y f_1(xy)$

and $N = x f_2(xy)$

$$\Rightarrow \qquad \text{I.F.} = \frac{1}{Mx - Ny}$$

$$= \frac{1}{xy(1 + 2xy) - xy(1 - xy)}$$

$$= \frac{1}{3x^2y^2}$$

Multiplying (iii) by $\dfrac{1}{3x^2y^2}$, we get

$$\left(\frac{1}{3x^2 y}+\frac{2}{3x}\right)dx+\left(\frac{1}{3x^2 y^2}-\frac{1}{3y}\right)dy=0$$

This is an exact differential eqn.

$$\therefore \quad \left[\frac{1}{3x^2 y}+\frac{2}{3x}\right]dx+\left[\frac{1}{3x^2 y^2}-\frac{1}{3y}\right]dy=0$$

or $$\int\left(\frac{1}{3x^2 y}+\frac{2}{3x}\right)dx+\int-\frac{1}{3y}\,dy=C$$

or $$-\frac{1}{3xy}+\frac{2}{3}\log x-\frac{1}{3}\log y=C$$

or $$-\frac{1}{xy}+2\log x-\log y=C$$

(c) Show that standard deviation of a Poisson distribution is equal to square root of its mean. The number of admissions in a hospital follows a Poisson distribution with an average of 4 per day. What will be the probability that there is no admission on a given day ?

Ans. If x is a Poisson distributed random variabls, then

$$E[x]=\sum_{x=0}^{\infty} x\,f_x(x)=\sum_{x=0}^{\infty} x\,\frac{e^{-\lambda}\lambda^x}{x!}$$

$$=e^{-\lambda}\left\{1\cdot\lambda+\frac{\lambda^2}{2!}+\frac{\lambda^3}{3!}+\ldots\right\}$$

$$=\lambda\,.\,e^{-\lambda}\,.\,e^{\lambda}$$

$$=\lambda$$

$$\therefore \quad E[x]=\lambda$$

Now $$E[x^2]=\sum_{x=0}^{\infty} x^2 f_x(x)$$

$$=\sum_{x=0}^{\infty}[x(x-1)+x]\,f_x(x)$$

$$=e^{-\lambda}\left\{2\cdot1\cdot\frac{\lambda^2}{2!}+3\cdot2\cdot\frac{\lambda^3}{3!}+\ldots\right\}+\lambda$$

$$=e^{-\lambda}\lambda^2\left\{1+\lambda+\frac{\lambda^2}{2!}+\ldots\right\}+\lambda$$

$$=e^{-\lambda}\,.\,\lambda^2\,.\,e^{\lambda}+\lambda$$

$$=\lambda^2+\lambda$$

Now, $$\text{Var}[x]=E(x^2)-(E(x))^2$$

$$=\lambda^2+\lambda-\lambda^2$$

$$\therefore \quad \text{Var}[x]=\lambda$$

The m.g.f. of x is given by

$$m_x(t)=E(r^{tx})$$

$$=\sum_{l=0}^{\infty} e^{tx}\,\frac{e^{-\lambda}\lambda^{\lambda}}{x!}$$

$$=e^{-\lambda}\sum_{x=0}^{\infty}\frac{(\lambda e^t)^x}{x!}$$

$$=e^{-\lambda}e^{\lambda e^t}$$

$$=e^{\lambda}(e^t-1)$$

Hence, $m_x(t)=e^x(e^t-1)$

Hence, for a Poisson distribution,

Mean = Variance = λ

But S.D = $\sqrt{Variation}=\sqrt{\lambda}$

Hence Proved

Please try the numerical yourself.

Q. 2. Solve the following second order differential equations :

(a) $(D^2-2D+4)\,y=e^x\cos x$

Ans. $(D^2-2D+4)\,y=e^x\cos x$

Auxiliary equation will be,

$$m^2-2m+y=0$$

or $$(m-2)^2=0$$

$$\Rightarrow \quad m=2,2$$

$$\therefore \quad CF=e^x(A\cos x+B\sin x)$$

$$P.F.=\frac{1}{D^2-2D+y}-e^x\cos x$$

$$=e^x\frac{1}{(D-2)^2}\cos x$$

$$= e^x \frac{1}{(2-D)^2}\cos x$$

$$= e^x \frac{1}{4\left(1-\frac{D}{2}\right)^2}\cos x$$

$$= \frac{e^x}{4}\left(1-\frac{D}{2}\right)^{-2}\cos x$$

$$= \frac{e^x}{4}\left[1+2\left(\frac{D}{2}\right)+\frac{(-2)(-3)}{1.2}\cdot\frac{D^2}{4}+\ldots\right]\cos x$$

$$= \frac{e^x}{4}(1+D)\cos x$$

$$= \frac{e^x}{2}\cdot(\cos x+\sin x)$$

$\therefore$ Complete solution will be,

$$y = e^x(A\cos x+B\sin x)+\frac{e^x}{2}(\cos x+\sin x)$$

(b) $(D^2+n^2)\,y = \cot nx$

Ans. $(D^2+n^2)\,y = \cot nx$

Auxiliary equation will be,

$$m^2+n^2 = 0$$

$\Rightarrow$ $m^2 = -n^2$

or $m = \pm\, i\, n$

$\therefore$ C.F. $= C_1\cos nx + C_2\sin nx$

and P.I $= \dfrac{1}{D^2+n^2}\cot nx$

$$= \frac{1}{(D+in)(D-in)}\cot nx$$

$$= \frac{1}{2in}\left[\frac{1}{D-in}-\frac{1}{D+in}\right]\cot nx$$

$$= \frac{1}{2in}\cdot\frac{1}{(D-in)}\cot nx-\frac{1}{2ni}\frac{1}{(D+in)}\cdot\cot nx$$

But $\dfrac{1}{(D-in)}\cot nx = e^{ni}\int e^{-ni}\cot nx\,dx$

$$= e^{ni}\int(\cos nx-i\sin nx)\cot nx\,dx$$

$$= e^{ni}\int\left(\frac{\cos^2 nx}{\sin nx}-i\cot nx\right)dx$$

$$= e^{ni}\int\left[\frac{1-\sin^2 nx}{\sin nx}-i\cos nx\right]dx$$

$$= e^{ni}\int(\operatorname{cosec} nx-\sin nx-i\cos nx)\,dx$$

$$= e^{ni}\left[\log\frac{(\operatorname{cosec} nx+\cot nx)}{n}+\frac{\cos nx}{n}-\frac{i\sin nx}{n}\right]$$

Similarly,

$$\frac{1}{D+ni}\cot nx = e^{-ni}\left[\frac{1}{n}\cdot\log(\operatorname{cosec} nx+\cot nx)+\frac{\cos nx}{n}+\frac{i\sin nx}{n}\right]$$

$$\therefore\ \text{P.I.} = \left[\begin{array}{l}\frac{1}{n}\cdot\log(\operatorname{cosec} nx+\cot nx)(e^{ni}-e^{-ni})+\frac{\cos nx}{n}(e^{ni}-e^{-ni})\\ \qquad -\frac{i\sin nx}{n}(e^{ni}+e^{-ni})\end{array}\right]$$

$$= \frac{1}{2n}[\sin nx+\log(\operatorname{cosec} nx+\cot nx)+\sin nx\cos nx-i\sin nx\cos nx]$$

$\therefore$ Complete solution will be :

$$y = C_1\cos nx+C_2\sin nx+\frac{1}{2n}\sin nx[\log(\operatorname{cosec} nx+\cot nx)+(1-i)\cos nx]$$

(c) $(D^2+1)\,y = 2\cos x$ (Use the method of undetermined coefficients)

Ans. $(D^2+1)\,y = 2\,x$

Its Auxiliary equation will be,

$$m^2+1 = 0$$

This gives, $m^2 = -1$

or $m = \pm\, i$

$\therefore$ C.F. $= C_1\cos x + C_2\sin x$

P.F. $= \dfrac{1}{(D^2+1)}2\cos x$

$$= \frac{1}{(D^2+i)(D-i)} \cdot 2\cos x$$

$$= \frac{1}{2i}\left[\frac{1}{D-i} - \frac{1}{D+i}\right] 2\cos x$$

$$= \frac{1}{i}\left[\frac{1}{D-i}\cos x - \frac{1}{D+i}\cos x\right]$$

But
$$\frac{1}{D-i}\cos x = e^{ix}\int e^{-i}\cos x\,dx$$

$$= e^{ix}\int(\cos x - i\sin x)\cos x\,dx$$

$$= e^{ix}\int(\cos^2 x - i\sin x\cos x)\,dx$$

$$= e^{ix}\int\left[\frac{\cos 2x - 1}{2} - \frac{1}{2}\sin 2x\right]dx$$

$$= e^{ix}\left[\frac{\sin 2x}{4} - \frac{x}{2} + \frac{i}{4}\cos 2x\right]$$

Similarly,

$$\frac{1}{D+i}\cos x = e^{-ix}\left[\frac{\sin 2x}{4} - \frac{x}{2} - \frac{i}{4}\cos 2x\right]$$

$\therefore$
$$\text{P.I.} = \frac{1}{i}\left[\frac{\sin 2x}{4}\left(e^{ix} - e^{-ix}\right) - \frac{x}{2}\left(e^{ix} - e^{ix}\right) + \frac{i}{4}\cos 2x\left(e^{ix} + e^{ix}\right)\right]$$

$$= \frac{\sin 2x \cdot \sin x}{2} - x\sin x + \frac{i}{2}\cos 2x \cdot \cos x$$

Hence, the complete solution is,

$$y = C_1\cos x + C_2\sin x + \sin x\left(\frac{\sin 2x}{2} - x\right) + \frac{1}{2}\cos x\cos 2x.$$

Q. 3. Find the projection of a vector $\vec{A} = 4\hat{j} - 3\hat{j} + \hat{k}$ on the line passing through the points (2, 3, – 1) and (– 2, – 4, 3).

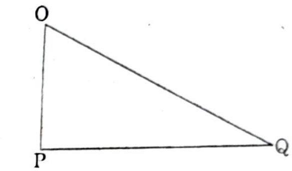
Find the volume of parallelopiped whose edges are represented by

$$\vec{A} = 2\hat{j} - 3\hat{j} + 4\hat{k}$$

$$\vec{B} = \hat{j} + 2\hat{j} - \hat{k}$$

$$\vec{C} = 3\hat{j} - \hat{j} + 2\hat{k}$$

Ans.



Point $P = (2, 3, -1)$

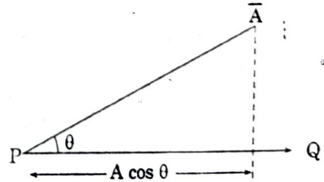
$\therefore$ $\bar{O}P = 2i + 3j - k$

Point $Q = -2, -4, 3$

$\therefore$ $\bar{O}Q = -2i - 4j + 3k$

$\therefore$ $\bar{P}Q = \bar{O}Q - \bar{O}P$

$= -4i - 7j + 4k$



Now,
$$\cos\theta = \frac{\bar{A}\cdot\overline{PQ}}{A\cdot PQ}$$

$$= \frac{(4i - 3j + k)\cdot(-4i - 7j + 4k)}{\sqrt{4^2 + (-3)^2 + 1^2}\cdot\sqrt{(-4)^2 + (-7)^2 + 4^2}}$$

$$= \frac{-16 + 21 + 4}{\sqrt{26}\,\sqrt{81}}$$

$$= \frac{9}{9\cdot\sqrt{26}}$$

$$= \frac{1}{\sqrt{26}}$$

Now, protection of $\bar{A}$ on $\overline{PQ} = A\cos\theta$

$$= \sqrt{26}\cdot\frac{1}{\sqrt{26}}$$

$$= 1$$

Now $\bar{A} = 2i - 3j + 4k$

$\bar{B} = i + 2j - k$

$\bar{C} = 3i - j + 2k$

Volume of parallelopiped $= \bar{A}\cdot(\bar{B}\times\bar{C})$

$$= \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

$$= \begin{vmatrix} 2 & -3 & 4 \\ 1 & 2 & -1 \\ 3 & -1 & 2 \end{vmatrix}$$

$$= 2(4-1) - 3(2+3) + 4(-1-6)$$

$$= 2\times 3 - 3\times 5 + 4(-7) = 37 \text{ units}$$ Ans.

Q. 4. Prove that, $\nabla^2(\phi\,\psi) = \phi\nabla^2\psi + 2\vec{\nabla}\phi\cdot\vec{\nabla}\psi + \psi\nabla^2\phi$

Show that $\vec{A}\ (6xy + z^3)\,\hat{i} + (3x^2 - z)\,\hat{j} + (3xz^2 - y)\,\hat{k}$ is irrotatinal.

Find ϕ such that $\vec{A} = \vec{\nabla}\phi$.

Ans. (i) $\nabla^2(\phi\,\psi) = \bar{\nabla}^2(\phi\,\psi)$

$$= \bar{\nabla}\cdot[\bar{\nabla}\cdot(\phi\,\psi)]$$

$$= \bar{\nabla}[\phi\,\bar{\nabla}\,\psi + \bar{\nabla}\,\phi\cdot\psi]$$

$$= \bar{\nabla}\,\phi\cdot\bar{\nabla}\cdot\psi + \phi\,\bar{\nabla}^2\,\psi + \bar{\nabla}^2\,\phi\cdot\psi + \bar{\nabla}\cdot\phi\,\bar{\nabla}\cdot\psi$$

$$= \phi\,\bar{\nabla}^2\,\psi + 2\bar{\nabla}\,\phi\cdot\bar{\nabla}\psi + \psi\cdot\bar{\nabla}^2\,\phi$$

$[\because \bar{A}\cdot\bar{B} = \bar{B}\cdot\bar{A} \text{ and } \bar{A}^2 = [\bar{A}]^2 = A^2]$

(ii) $\bar{A} = (6xy + Z^3)\,i + (3x^2 - z)\,j + (3xz^2 - y)\,k$

$$\text{Curl } \bar{A} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 6xy + z^3 & 3x^2 - z & 3xz^2 - y \end{vmatrix}$$

$$= i[-1+1] + j[3z^2 - 3z^2] + k[6x - 6x]$$

$$= 0$$

Hence, $\bar{A}$ is irrotational

Now, $\bar{\nabla} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)$

Now, $\bar{A} = \bar{\nabla}\phi$

$\therefore\ (6xy + z^3)\,i + (3x^2 - z)\,j + (3xz^2 - y)\,k$

$$= i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

Comparing, $\frac{\partial\phi}{\partial x} = 6xy + z^3$

On integratin, we get,

$$\phi = \frac{6x^2y}{2} + z^3x$$

or $\phi = 3x^2y + z^3x$

Q. 5. Prove that $\iiint_V \frac{dV}{r^2} = \iint_S \frac{\vec{r}\cdot\hat{n}}{r^2}\,dS$

Verify Stoke's theorem for the function $\vec{F} = x(\hat{i}\,x + \hat{j}\,y)$, integrated round the square in the plane $z = 0$, with sides are along the lines $x = 0$, $y = 0$, $x = a$, $y = a$.

Ans. $\iint \bar{A}\cdot d\bar{S} = \iiint \text{div}\,\bar{A}\,\overline{dx}$

$$\therefore \iint \frac{\bar{r}}{r^2}\,d\bar{S} = \iiint \text{div}\left(\frac{\bar{r}}{r^2}\right)dV$$

Now, $\text{div}\left(\frac{\bar{r}}{r^2}\right)$

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\cdot\left(\frac{x}{x^2+y^2+z^2}\,i + \frac{y}{x^2+y^2+z^2}\,j + \frac{z}{x^2+y^2+z^2}\cdot k\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{z}{x^2+y^2+z^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2+z^2}\right) + \frac{\partial}{\partial z}\left(\frac{z}{x^2+y^2+z^2}\right)$$

$$= \frac{1}{x^2+y^2+z^2}$$

$$= \frac{1}{r^2}$$

$$\therefore \quad \iint \frac{\bar{r}}{r^2}\cdot \hat{n}\, dS = \iint \frac{\bar{r}\cdot d\bar{S}}{r^2} = \iiint \frac{1}{r^2}\,\partial V$$

or
$$\iiint \frac{\partial V}{r^2} = \iint \frac{\bar{r}\cdot\hat{n}}{r^2}\, dS$$

(ii) For square *OABC*,

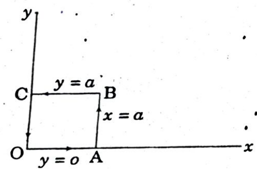
$$\oint_C \bar{F}\cdot d\bar{r} = \int_{OA} \bar{F}\cdot d\bar{r} + \int_{AB} \bar{F}\cdot d\bar{r} + \int_{BC} \bar{F}\cdot dr + \int_{CO} \bar{F}\cdot dr$$

But
$$\int_{OA} \bar{F}\cdot dr = \int_0^a x\,(ix+jy)-i\,dx$$

$$= \int_0^a x^2\, dx$$

$$= \left.\frac{x^3}{3}\right|_0^a$$

$$= \frac{a^3}{3}$$



$$\int_{AB} \bar{F}\cdot d\bar{r} = \int_0^a x\,(ix+jy)-j\,dy$$

$$= \int_0^a xy\, dy = \int_0^a ay\, dy$$

$$= \frac{a^3}{2}$$

$$\int_{BC} \bar{F}\cdot d\bar{r} = \int_a^0 x\,(ix+jy)-i\,dx$$

$$= \int_a^0 x^2\, dx = -\frac{1}{3}a^3$$

$$\int_{CO} \bar{F}\cdot d\bar{r} = \int_a^0 x\,(ix+jy)\cdot j\, dy$$

$$= \int_a^0 xy\, dy = 0$$

$$\oint_C \bar{F}\cdot d\bar{r} = \frac{1}{3}a^3 + \frac{1}{2}a^3 - \frac{1}{3}a^3 + 0$$

$$= \frac{a^3}{2}$$

Using stoke's theorem,

$$\oint_C \bar{F}\cdot d\bar{r} = \iint_S \text{curl}\, \bar{F}\cdot d\bar{S}$$

But
$$\text{curl}\, \bar{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 & xy & 0 \end{vmatrix}$$

and
$$d\bar{S} = k\, dx\, dy$$

$$\therefore \quad \iint_S \text{curl}\, \bar{F}\cdot d\bar{S} = \int_0^a\int_0^a k\, y\cdot k\, dx\, dy$$

$$= \int_0^a\int_0^a y\, dx\, dy$$

$$= \frac{a^4}{2}$$

This verifies Stoke's theorem.

Q. 6. Prove that the cylindrical coordinate system is orthogonal.

A covariant tensor has components **xy, $2y - z^2$, xz in rectangular co-ordinates. Find its covariant components in spherical co-ordinates.**

Find an expression for curl of a vector field V in terms of curvilinear coordinates.

Ans. (i) Pl. Ref. IQ. 13, Unit 2D, p 122–123

(ii) Let A_j denote the covariant component in rectangular co-ordinates

$$x^1 = x; \quad x^2 = y; \quad x^3 = z$$

Then,
$$A_1 = xy = x^1 x^2$$
$$A_2 = 2y - z^2 = 2x^2 - (x^3)^2$$
$$A_3 = xz = x^1 x^3$$

Let $\bar{A}_k$ denote the covariant component in spherical co-ordinates

Then, $$\bar{A}_k = \frac{\partial \bar{x}^j}{\partial \bar{x}^k} A_j \qquad \text{...(i)}$$

In spherical co-ordinates,
$$x = r \sin\theta \cos\theta$$
$$x^1 = \bar{x}^1 \sin\bar{x}^2 \cos\bar{x}^3 \qquad \text{...(ii)}$$
$$y = r \sin\theta \sin\phi$$
$$x^2 = \bar{x}^1 \sin\bar{x}^2 \sin\bar{x}^3 \qquad \text{...(iii)}$$
$$z = r \cos q$$
$$x^3 = \bar{x}^1 \cos\bar{x}^2 \qquad \text{...(iv)}$$

Therefore, Eq. (i) yields the covariant components
$$\bar{A}_1 = \frac{\partial x^1}{\partial \bar{x}^1} A_1 + \frac{\partial x^2}{\partial \bar{x}^1} A_2 + \frac{\partial x^3}{\partial \bar{x}^1} A_3$$
$$= (\sin\bar{x}^2 \cos\bar{x}^3)(x^1 x^2) + (\sin\bar{x}^2 \sin\bar{x}^3) \times [(2x^2 - x^3)] + (\cos\bar{x}^2)(x^1 x^3)$$
$$= (\sin\theta\cos\phi)(r^2 \sin^2\phi\cos\phi) + (\sin\theta\sin\phi)(2r\sin\theta\sin\phi - r^2\cos^2\theta)$$
$$+ (\cos\theta)(r^2 \sin\theta\cos\theta\cos\phi)$$
$$\bar{A}_2 = \frac{\partial x^1}{\partial \bar{x}^2} A_1 + \frac{\partial x^2}{\partial \bar{x}^2} A_2 + \frac{\partial x^3}{\partial \bar{x}^2} A_3$$
$$= (\bar{x}\cos\bar{x}^2 \cos\bar{x}^3)(x^1 x^2) + (\bar{x}^1 \cos\bar{x}^2 \sin\bar{x}^3)$$
$$[(2x^2 - (-x^3)^2] + \bar{x}^1(-\sin\bar{x}^2)(x^1 x^2)$$

or $$\bar{A}_2 = (r\cos\theta\cos\phi)(r^2\sin^2\theta\sin\phi\cos\phi) + (r\cos\theta\sin\phi)$$
$$(2r\sin\theta\sin\phi - r^2\cos\theta)$$
$$+ (-r\sin\theta)(r^2\sin\theta\cos\theta\cos\phi)\ \bar{A}_3 + \frac{\partial x^2}{\partial \bar{x}^3} A_2 + \frac{\partial x^3}{\partial \bar{x}^3} A_3$$
$$= (-r\sin\theta\sin\phi)(r^2\sin^2\theta\sin\phi\cos\phi) + (r\sin\theta\cos\phi)(2r\sin\theta\sin\phi - r^2\cos^2\theta)$$

(iii) Pl. Ref. IQ. 13, Unit 2D, p 124–125.

OOO