Problem Set #7
Math 467 – Complex Analysis

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Exercise 1.

Calculate

$$\int_{|z|=2} \tan z \, dz.$$

Solution.

Using the argument principle, define $f(z) = \cos z$ and notice that $f(z)$ has two zeros in the set $B_2(0)$ and no poles. Thus, we have

$$\frac{1}{2\pi i} \int_{|z|=2} \tan z \, dz = \frac{-1}{2\pi i} \int_{|z|=2} \frac{g'(z)}{g(z)} \, dz = 2.$$

For the last identity, multiply both sides by $-2\pi i$ and we find

$$\int_{|z|=2} \tan z \, dz = -4\pi i.$$

Q.E.D.

Exercise 2.

Calculate

$$\int_{-\pi}^{\pi} \frac{\sin n\theta}{\sin \theta}\, d\theta.$$

Solution.

We claim

$$\int_{-\pi}^{\pi} \frac{\sin n\theta}{\sin \theta}\, d\theta = \begin{cases} 0 & : \ n \text{ is even} \\ 2\pi & : \ n \text{ is odd} \end{cases}$$

To see how clear this is, recall Euler's formula

$$\sin \pi z = \frac{e^{i\pi z} - e^{-i\pi z}}{2i}.$$

Thus, we can rewrite the integral as

$$\int_{-\pi}^{\pi} \frac{e^{in\theta} - e^{-in\theta}}{e^{i\theta} - e^{-i\theta}}\, d\theta.$$

Setting $z = e^{i\theta}$ so that $\frac{dz}{iz} = d\theta$, and we see the contour arising from this change of variables is precisely S^1 and now we have

$$\begin{aligned} \int_{|z|=1} \frac{z^n - z^{-n}}{z - z^{-1}}\, dz &= \int_{|z|=1} \frac{z^{-n}}{z^{-1}} \cdot \frac{z^{2n} - 1}{z^2 - 1}\, dz \\ &= \int_{|z|=1} \frac{1}{z^{n-1}} \cdot \sum_{k=0}^{n-1} z^{2n}\, dz. \end{aligned}$$

Now we can easily see the order of the residue at $z = 0$ is of order $n - 1$, and an easy calculation shows the result claimed above. During calculations, remember to treat the cases when n is even and when n is odd separately.

Q.E.D.

Exercise 3.

Suppose that f and g are holomorphic on a region Ω and that $fg \equiv 0$ on Ω. Prove that either f or g is identically zero on Ω.

Solution.

We begin with a lemma.

Lemma 1. *Let $f_1, \ldots, f_n$ be holomorphic functions on a bounded region Ω with distinct points $\{z_k\}_{k=1}^{\infty}$ having a limit point in Ω such that*

$$\prod_{i=1}^{n} f_i(z_j) = 0$$

for $j = 1, \ldots$. Then $f_p \equiv 0$ for some $1 \leq p \leq n$.

Proof. Since the points are distinct and have a limit point in Ω, clearly

$$\prod_{i=1}^{n} f_i \equiv 0.$$

To see this, simply define $g(z)$ to be the product and apply a previous theorem.

Moreover, it's clear that at least one f_i will have infinitely many zeros by the pigeonhole principle. Taking a compact subset $\Omega' \subseteq \Omega$ if necessary, we see that these zeros will have a limit point in Ω', hence in Ω, therefore $f_i \equiv 0$, as desired. **Q.E.D.**

Now we see the exercise becomes a trivial corollary to the lemma, since we can define $\Omega_1 \subseteq \Omega$ to be a bounded subset.

Q.E.D.

Exercise 4.

Let f be holomorphic on $|z| \leq 1$ with $|f| < 1$ when $|z| = 1$. Show that $f(z) - z^3 = 0$ has exactly 3 solutions in $|z| < 1$.

Solution.

Clearly $g(z) = z^3$ has three zeros in $\mathbb{D}$. By hypothesis, observe

$$|-g(z)| = |g(z)| = 1 > |f(z)|.$$

By Rouchés theorem, we know $f(z) - g(z) = 0$ will have as many solutions as $g(z)$, and therefore $f(z) = z^3$ has exactly three solutions.

Q.E.D.

Exercise 5.

How many zeros does $p(z) = z^4 + z^2 - 2z + 6$ have in the first quadrant?

Solution.

Using the argument principle, define Z to be the number of zeros of $p(z)$ in the first quadrant, P to be the number of poles in the first quadrant, and Γ to be the semi-circle with radius R such that $\mathrm{Re}(z) > 0$ for $z \in \Gamma$ along with the portion of the imaginary axis between them. Clearly, on the imaginary axis, since all numbers are real except the term of degree 1, no zeros can occur on the imaginary axis except possibly at $z = 0$; however, $p(0) = 6$, thus $z = 0$ is not a root. As there are only finitely many roots, we can be sure to choose R greater than the maximum modulus of the zeros. Since no zeros or poles are contained on Γ, we now we

$$\int_\Gamma \frac{p'(z)}{p(z)}\, dz = Z - P = Z$$

as $p(z)$ is entire, and therefore has no poles. To show that the imaginary axis doesn't contribute any to the argument, letting $z = x + iy$, we find that on the imaginary axis

$$p(z) = y^4 - y^2 + 6 - 2iy.$$

Using elementary algebra and calculus, we find the zeros of the derivative of the real part of $p(z)$ to be $y = -1, 0, 1$, each corresponding to a maximum/minimum of the real part of $p(z)$. Since we can verify all of these are positive, and the function remains positive, we see that for $z = iy$, $\mathrm{Re}(p(z)) > 0$ so the argument is only dependent on the initial and final points. At $z = iR$, we have

$$\theta = \lim_{R\to\infty} \arctan\left(-\frac{2}{R^3}\right) = 0.$$

When $z = -iR$, we have

$$\theta = \lim_{R\to\infty} \arctan\left(\frac{2}{R^3}\right) = 0.$$

Therefore, we see the change in argument on the imaginary axis is 0.

Finally, on the half-circle, for large enough R, $p(z) \sim z^4 = R^4 e^{4i\theta}$ for $-\pi/2 \leq \theta \leq \pi/2$. Calculating this, shows the argument changes by 4π, therefore, after dividing by 2π, we see that there are two zeros with $\mathrm{Re}(z) > 0$. Dividing by 2, since there must be as many in the first quadrant as in the fourth quadrant, we see that there is exactly 1 zero in the first quadrant.

Q.E.D.

Exercise 6.

How many zeros does $f(z) = 3z^{100} - e^z$ have in $|z| < 1$? Are they all distinct?

Solution.

Again, an easy application of Rouché's theorem shows us that there exists exactly 100 solutions in the unit disc. To see that the solutions are all distinct, suppose there is a repeated root, say $f(z) = (z-a)^k g(z)$ with $k \geq 2$ and $g(z)$ holomorphic and non-vanishing at $z = a$. Then we can easily see

$$f'(z) = (z-a)^k g'(z) + k(z-a)^{k-1} g(z).$$

It's clear that $f'(a) = 0$, and therefore a satisfies the following equations:

$$\begin{aligned} 3a^{100} - e^a &= 0 \\ 300a^{99} - e^a &= 0. \end{aligned}$$

Subtracting the equations and factoring, we have

$$3a^{99}(a - 100) = 0.$$

Now we have that $a = 0$ or $a = 100$. Since we've already established the roots to be in the unit disc, $a = 100$ is certainly not a root. Likewise, $a = 0$ is not a root since $f(0) = 1$. This shows there are no repeated roots of $f(z)$.

Q.E.D.

Exercise 7.

Suppose f is entire and there are constants a and b in $\mathbb{R}$ such that

$$|f(z)| \leq a\sqrt{|z|} + b$$

for all $z \in \mathbb{C}$.

Solution.

From a previous problem, we know that if

$$|f(z)| \leq a|z|^s + b,$$

then $f(z)$ is a polynomial of degree at most s. Here, since $s = 1/2$, $f(z)$ must be constant and must be in the disc of radius b.

Q.E.D.

EXERCISE 8.

Find the order of growth of the following entire functions:

(a) $p(z)$ *where* p *is a polynomial.*

(b) e^{bz^n} *for* $b \neq 0$.

(c) e^{e^z}.

SOLUTION.

(a) From class, we know that $\rho = 0$.

(b) From the definition, we know $\rho = n$.

(c) Again, from definition, we know $\rho = \infty$ since e^z does not have polynomial growth.

Q.E.D.

EXERCISE 9.

Establish the following properties of infinite products.

(a) *Show that if $\sum |a_n|^2$ converges, and $a_n \neq -1$, then the product $\prod(1+a_n)$ converges to a non-zero limit if and only if $\sum a_n$ converges.*

(b) *Find an example of a sequence of complex numbers $\{a_n\}$ such that $\sum a_n$ converges but $\prod(1+a_n)$ diverges.*

(c) *Also find an example such that $\prod(1+a_n)$ converges and $\sum a_n$ diverges.*

SOLUTION.

(a) Suppose first that the product converges. The fact that $\sum |a_n|^2$ converges implies that for large enough n, say for $n > N$, we have $|a_n| < \frac{1}{2}$. These together imply that $\sum_{n=N+1}^{\infty} \log(1+a_n)$ converges. From the power series expansion, we know that $0 < a_n - \log(1+a_n) < a_n^2$, which tells us $\sum_{n=N+1}^{\infty} a_n - \log(1+a_n)$ converges absolutely, hence converges. This says the convergence of $\sum_{n=N+1}^{\infty} a_n$ and $\sum_{n=N+1}^{\infty} \log(1+a_n)$ are equivalent. Since we know one converges from above, we know the other must converge.

Conversely, suppose $\sum a_n$ converges. Disregarding finitely many terms if necessary, we can assume $|a_n| < \frac{1}{2}$ for all n. Thus, we know $0 \leq |a_n - \log(1+a_n)| < |a_n|^2$, so by comparison, we know $\sum_{n=1}^{\infty} |a_n - \log(1+a_n)|$ converges. Since it is absolutely convergent, it is convergent, so now we have

$$\sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} (a_n - \log(1+a_n)) = \sum_{n=1}^{\infty} \log(1+a_n),$$

therefore the series is convergent as a sum of two convergent series.

(b) If we take $a_n = \frac{(-1)^n}{n}$, then the sum clearly converges, while the product diverges (according to our definition of convergence, the limit must be nonzero).

(c) Take

$$a_{2n-1} = \frac{-1}{\sqrt{n+1}}, \quad a_{2n} = \frac{1}{\sqrt{n+1}} + \frac{1}{n+1}.$$

It is not complicated to verify this satisfies the conditions.

Q.E.D.