Econometrics and Time Series Homework3 N10528700 Yun-Feng(Kyle) Lee

2.7

Verify the calculations for the autocorrelation functions of an ARMA(1,1) process given in example 2.11. compare the form with that of the ACF for the ARMA(1,0) and the ARMA(0,1). Plot the ACFs of the 3 series on the same graph. Use Φ = 0.6 and Θ= 0.9.

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2.7

ARMA(1,1)

$$X_t - \phi X_{t-1} = W_t + \theta W_{t-1}$$

$$\gamma(h) - \phi\gamma(h-1) = 0 \qquad h=2,3,\cdots$$

$$\gamma(h) = \phi\gamma(h-1) = c\phi^h \qquad h=1,2,\cdots$$

$$\gamma(0) = \phi\gamma(1) + \sigma^2 + \theta^2\sigma^2 + \theta\phi\sigma^2$$

$$= \phi\gamma(1) + \sigma^2(1+\theta\phi+\theta^2)$$

$$\gamma(1) = \phi\gamma(0) + \theta\sigma^2$$

$$\gamma(0) = \frac{(1+2\theta\phi+\theta^2)}{1-\phi^2}\sigma^2$$

$$\gamma(1) = \frac{\sigma^2(1+\theta\phi)(\phi+\theta)}{1-\phi^2}$$

since $\gamma(1) = c\phi$, $c = \frac{\gamma(1)}{\phi}$

$$\gamma(h) = \frac{\sigma^2(1+\theta\phi)(\phi+\theta)}{1-\phi^2}\cdot\phi^{h-1}$$

$$\rho(h) = \frac{(1+\theta\phi)(\phi+\theta)}{1+2\theta\phi+\theta^2}\phi^{h-1}$$

ARMA(1, 0)

$$X_t - \phi X_{t-1} = W_t$$

$$X_t = \phi X_{t-1} + W_t$$

$$\gamma(h) - \phi\gamma(h-1) = 0 \qquad h=2,3,\cdots$$

$$\gamma(h) = \phi\gamma(h-1) = c\phi^h \qquad h=1,2,\cdots$$

$$\gamma(0) = \phi\gamma(1) + \sigma^2$$

$$\gamma(1) = \phi\gamma(0)$$

$$\gamma(1) = \frac{\phi\sigma^2}{1-\phi^2}$$

$$\gamma(0) = \frac{\sigma^2}{1-\phi^2}$$

$$\gamma(1) = c\phi \qquad c = \frac{\gamma(1)}{\phi}$$

$$\gamma(h) = \frac{\phi\sigma^2}{1-\phi^2}\phi^{h-1}$$

$$\rho(h) = \phi^h$$

ARMA(0, 1)

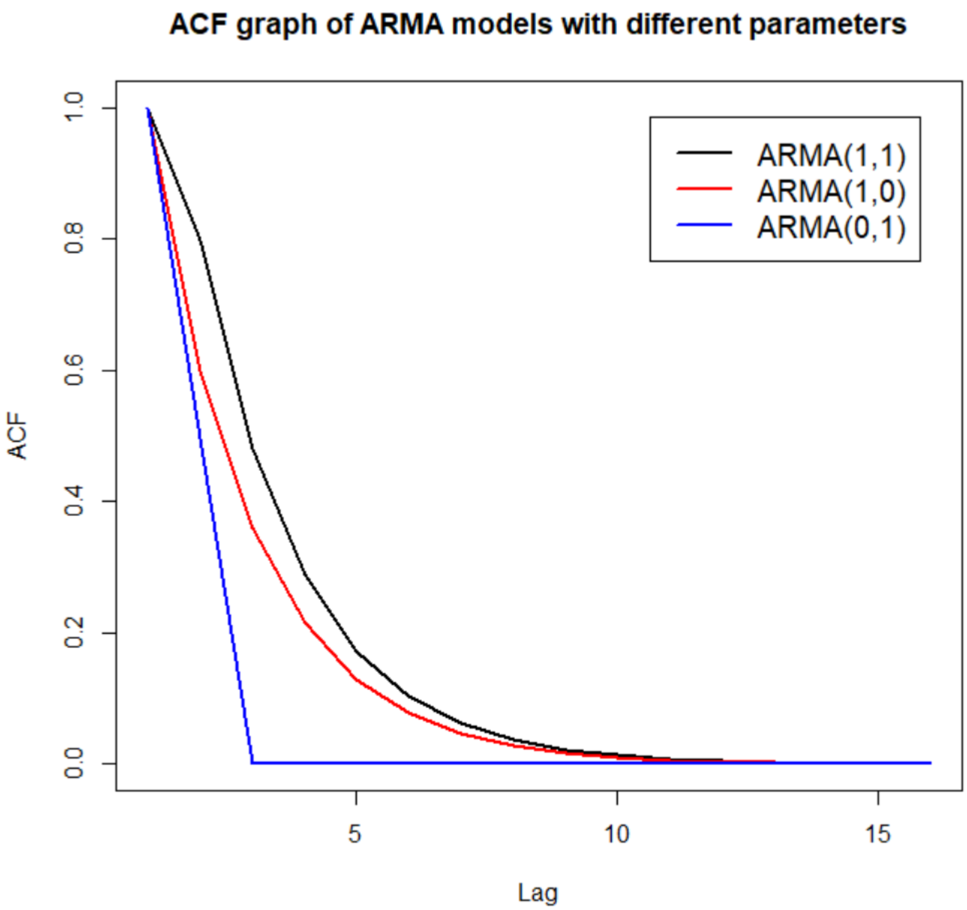
$$X_t = W_t + \theta W_{t-1}$$

$$\gamma(t+h, t) = E[W_{t+h} + \theta W_{t+h-1}, W_t + \theta W_{t-1})$$

$$= \begin{cases} h=0 & \sigma^2 + \theta^2\sigma^2 \\ h=\pm 1 & \theta\sigma^2 \\ \text{else} & 0 \end{cases}$$

$$\rho(1) = \frac{\theta}{1+\theta^2} \qquad \rho(h) = 0 \quad h=2,3,\cdots$$

Plot of the ACFs of the 3 series on the same graph:



2.8

Generate n=100 observations for each of the 3 models discussed in 2.7. Compute the sample ACFs for each model and compare the theoretical values. Compute the same PACF for each of the generated series and compare the same ACFs and PACFs with the general results given in table 2.1.

ARMA(1,1) Theoretical ACF values:

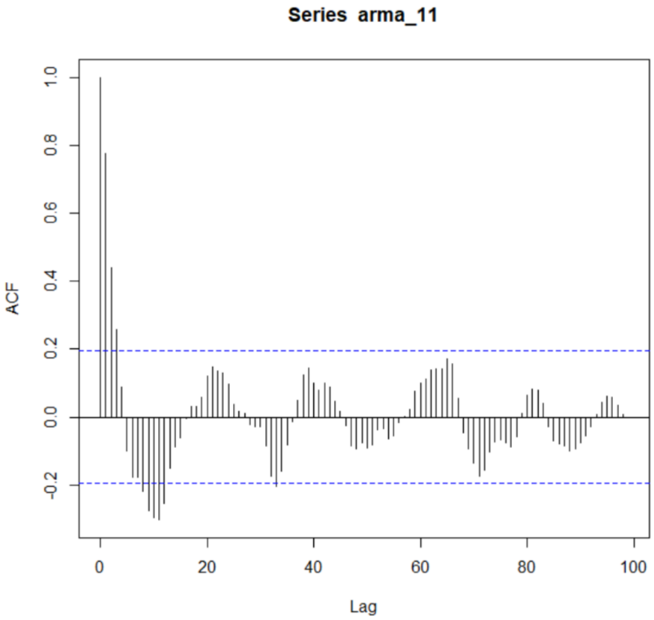
```
> ARMAacf(ar=c(0.6),ma=c(0.9),lag.max=15)
          0           1           2           3           4           5           6           7           8           9          10 
1.000000000 0.799307958 0.479584775 0.287750865 0.172650519 0.103590311 0.062154187 0.037292512 0.022375507 0.013425304 0.008055183 
         11          12          13          14          15 
0.004833110 0.002899866 0.001739919 0.001043952 0.000626371 
```

ARMA(1,1) sample observed ACF values

```
Autocorrelations of series ‘arma_11’, by lag

     0      1      2      3      4      5      6      7      8      9     10     11     12     13     14     15 
 1.000  0.777  0.439  0.256  0.089 -0.099 -0.178 -0.176 -0.220 -0.275 -0.297 -0.302 -0.254 -0.149 -0.087 -0.060 
```

ARMA(1,1) sample observed ACF graph



ARMA(1,1) theoretical PACF values

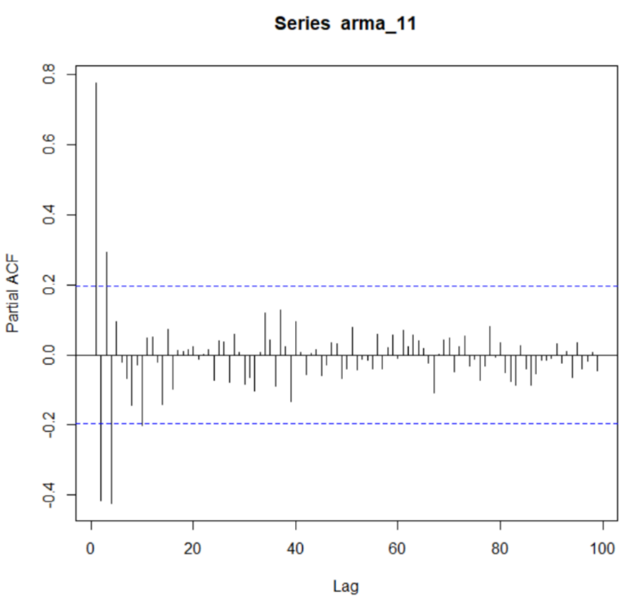
```
> ARMAacf(ar=c(0.6),ma=c(0.9),lag.max=15,pacf=TRUE)
 [1]  0.79930796 -0.44116711  0.30234014 -0.22804489  0.18144231 -0.14927772  0.12561417 -0.10739637  0.09289211 -0.08104604
[11]  0.07117818 -0.06283010  0.05568093 -0.04949870  0.04411096
```

ARMA(1,1) observed PACF values

```
Partial autocorrelations of series 'arma_11', by lag

     1      2      3      4      5      6      7      8      9     10     11     12     13     14     15     16     17     18 
 0.777 -0.415  0.292 -0.424  0.095 -0.019 -0.066 -0.145 -0.028 -0.201  0.047  0.050 -0.020 -0.142  0.073 -0.097  0.011  0.009 
    19     20     21     22     23     24     25     26     27     28     29     30     31     32     33     34     35     36 
 0.016  0.024 -0.013  0.002  0.016 -0.073  0.040  0.036 -0.077  0.060  0.008 -0.084 -0.063 -0.103  0.006  0.119  0.044 -0.090 
```

ARMA(1,1) observed PACF graph



ARMA(1,0) Theoretical ACF values

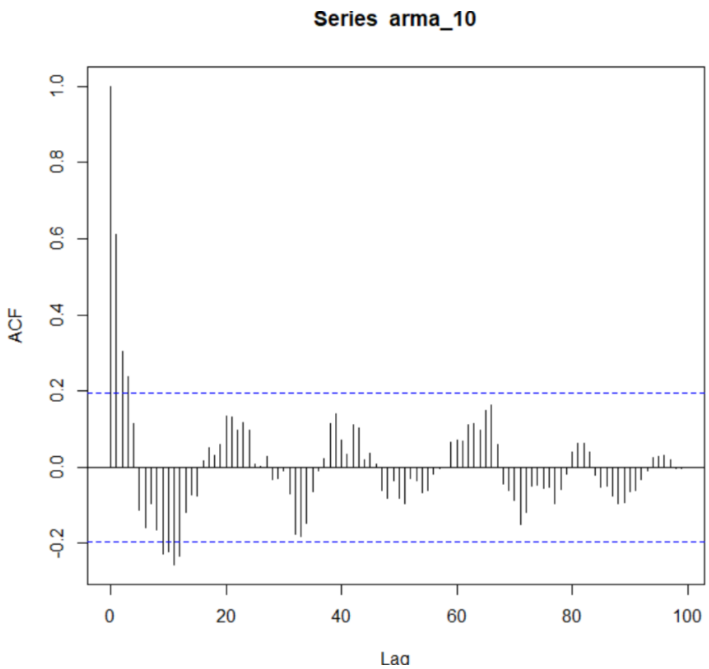
```
> ARMAacf(ar=c(0.6),lag.max=15)
           0            1            2            3            4            5            6            7            8            9
1.0000000000 0.6000000000 0.3600000000 0.2160000000 0.1296000000 0.0777600000 0.0466560000 0.0279936000 0.0167961600 0.0100776960
          10           11           12           13           14           15
0.0060466176 0.0036279706 0.0021767823 0.0013060694 0.0007836416 0.0004701850
```

ARMA(1,0) observed ACF values

```
Autocorrelations of series 'arma_10', by lag

     0      1      2      3      4      5      6      7      8      9     10     11     12     13     14     15     16     17
 1.000  0.611  0.303  0.237  0.115 -0.112 -0.160 -0.094 -0.165 -0.229 -0.223 -0.256 -0.234 -0.117 -0.073 -0.075  0.017  0.051
    18     19     20     21     22     23     24     25     26     27     28     29     30     31     32     33     34     35
 0.032  0.059  0.136  0.133  0.096  0.116  0.097  0.009  0.003  0.029 -0.032 -0.030 -0.010 -0.068 -0.176 -0.182 -0.147 -0.065
```

ARMA(1,0) sample observed ACF graph



ARMA(1,0) Theoretical PACF values

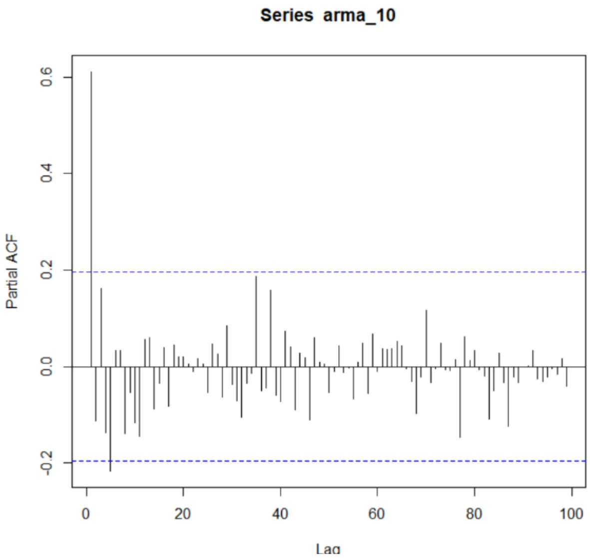
```
 ARMAacf(ar=c(0.6),lag.max=15,pacf=TRUE)
[1] 0.6 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0
```

ARMA(1,0) observed PACF values

```
Partial autocorrelations of series 'arma_10', by lag

     1      2      3      4      5      6      7      8      9     10     11     12     13     14     15     16     17     18
 0.611 -0.113  0.163 -0.138 -0.217  0.033  0.033 -0.138 -0.054 -0.117 -0.145  0.055  0.060 -0.088 -0.036  0.039 -0.082  0.044
    19     20     21     22     23     24     25     26     27     28     29     30     31     32     33     34     35     36
 0.019  0.021  0.005 -0.011  0.016  0.005 -0.054  0.046  0.025 -0.064  0.084 -0.037 -0.071 -0.106 -0.035 -0.014  0.186 -0.051
```

ARMA(1,0) sample observed PACF graph



ARMA(0,1) Theoretical ACF values

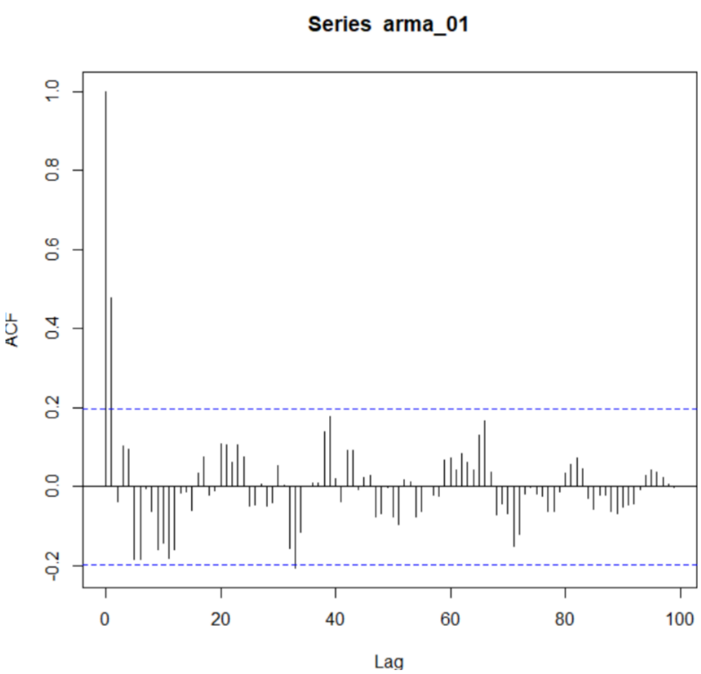
```
> ARMAacf(ma=c(0.9),lag.max=15)
        0         1         2         3         4         5         6         7         8         9        10        11        12
1.0000000 0.4972376 0.0000000 0.0000000 0.0000000 0.0000000 0.0000000 0.0000000 0.0000000 0.0000000 0.0000000 0.0000000 0.0000000
       13        14        15
0.0000000 0.0000000 0.0000000
```

ARMA(0,1) observed ACF values

```
Autocorrelations of series ‘arma_01’, by lag

     0      1      2      3      4      5      6      7      8      9     10     11     12     13     14     15     16     17
 1.000  0.477 -0.036  0.102  0.095 -0.183 -0.183 -0.003 -0.062 -0.159 -0.143 -0.180 -0.158 -0.016 -0.012 -0.059  0.033  0.076
    18     19     20     21     22     23     24     25     26     27     28     29     30     31     32     33     34     35
-0.020 -0.011  0.108  0.105  0.062  0.105  0.075 -0.047 -0.045  0.008 -0.049 -0.041  0.053  0.003 -0.155 -0.206 -0.115  0.002
```

ARMA(0,1) sample observed ACF graph



ARMA(0,1) Theoretical PACF values

```
> ARMAacf(ma=c(0.9),lag.max=15,pacf=TRUE)
 [1]  0.49723757 -0.32845383  0.24319934 -0.19139394  0.15635134 -0.13092530  0.11154614 -0.09623280  0.08379774 -0.07348555
[11]  0.06479226 -0.05736822  0.05096256 -0.04539004  0.04051031
```

ARMA(0,1) observed PACF values

```
Partial autocorrelations of series 'arma_01', by lag

     1      2      3      4      5      6      7      8      9     10     11     12     13     14     15     16     17     18 
 0.477 -0.342  0.422 -0.372  0.034 -0.069  0.035 -0.128  0.029 -0.242 -0.058  0.025  0.003 -0.111  0.061 -0.085  0.009 -0.053 
    19     20     21     22     23     24     25     26     27     28     29     30     31     32     33     34     35     36 
 0.048 -0.020  0.054  0.015  0.037 -0.057 -0.028  0.029 -0.026 -0.007  0.070 -0.011 -0.035 -0.132 -0.121  0.060  0.092 -0.108 
```

ARMA(0,1) sample observed PACF graph

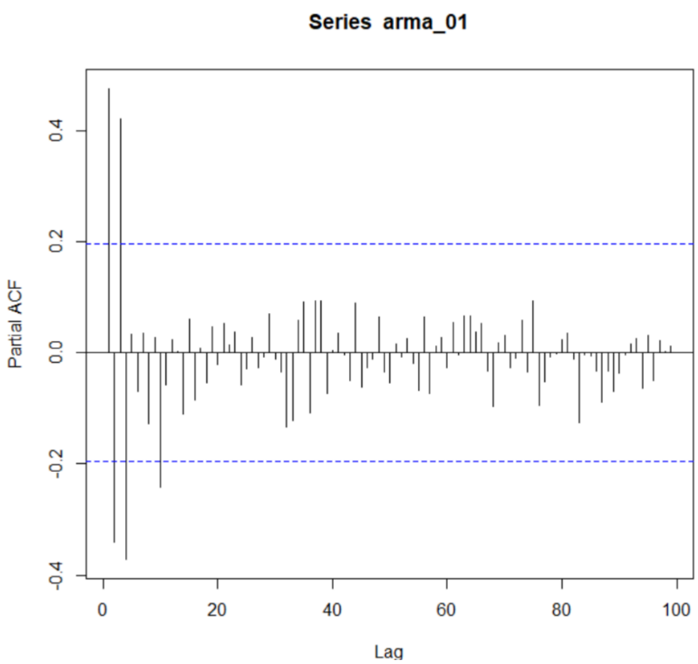


Table 2.1 Behavior of the ACF and PACF for Causal and Invertible ARMA Models

	AR(p)	MA(q)	ARMA(p,q)
ACF	Tails off	Cuts off after lag q	Tails off
PACF	Cuts off after lag p	Tails off	Tails off

We can see that our data is like what is on table 2.1.

For ARMA(1,1), both ACF and PACF tails off as shown in the plots above.

For ARMA(1,0), which represents AR(1), we can see from our theoretical values that the ACF tails off while the PACF cuts off to 0 right after lag 1.

Last, for ARMA(0,1), which in other words means MA(1), we can also see from the theoretical values that the ACF of it drops to 0 starting from lag 2, but the PACF tails off.

Codes used(written in R):

```
#Problem 2.7 in Time Series Analysis and its applications(Shumway and Stoffer)

#1.Create ACF of ARMA(1,1),(1,0),(0,1), and plot them
plot(as.ts(ARMAacf(ar=c(0.6),ma=c(0.9),lag.max=15)),col="black",lwd=2,
    main="ACF graph of ARMA models with different parameters",ylab="ACF",xlab="Lag")
lines(as.ts(ARMAacf(ar=c(0.6),lag.max=15)),col="red",lwd=2)
lines(as.ts(ARMAacf(ma=c(0.9),lag.max=15)),col="blue",lwd=2)
legend(x="topright", legend=c("ARMA(1,1)","ARMA(1,0)","ARMA(0,1)"),
      inset=0.05, cex=1.25, lwd=2, lty=c(1, 1, 1), col=c("black", "red", "blue"))

#2.Generate 100 observations from above, compute sample ACF and PACF
#ARMA(1,1)
rand_om <- rnorm(100,0,1)
arma_11<- c(1:100)
arma_11[1] <- (1+0.6*0.9)*(0.6+0.9)/(1+2*0.6*0.9+0.9^2)
for(i in c(2:100)){
  arma_11[i] <- 0.6*arma_11[i-1]+rand_om[i]+0.9*rand_om[i-1]
}

arma11_acf <- acf(arma_11,type="correlation",plot=T,lag.max=100)
arma11_acf
arma11_pacf <- acf(arma_11,type="partial",lag.max=100)

#ARMA(1,0)
arma_10<- c(1:100)
arma_10[1] <- 0.6
for(i in c(2:100)){
  arma_10[i] <- 0.6*arma_10[i-1]+rand_om[i]
}

arma10_acf <- acf(arma_10,type="correlation",plot=T,lag.max=100)
arma10_acf
arma10_pacf <- acf(arma_10,type="partial",lag.max=100)

#ARMA(0,1)
arma_01<- c(1:100)
arma_01[1] <- 0.9/(1+0.9^2)
for(i in c(2:100)){
  arma_01[i] <- rand_om[i]+0.9*rand_om[i-1]
}

arma01_acf <- acf(arma_01,type="correlation",plot=T,lag.max=100)
arma01_acf
arma01_pacf <- acf(arma_01,type="partial",lag.max=100)

##Making sure data is as wanted
#plot.ts(arma_11)
#lines(as.ts(arma_10),col="red")
#lines(as.ts(arma_01),col="blue")
```