Lecture Notes in Mathematics

Arkansas Tech University
Department of Mathematics

The Basics of Linear Algebra

Marcel B. Finan

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Preface

Linear algebra has evolved as a branch of mathematics with wide range of applications to the natural sciences, to engineering, to computer sciences, to management and social sciences, and more.
This book is addressed primarely to students in engineering and mathematics who have already had a course in calculus and discrete mathematics. It is the result of lecture notes given by the author at Arkansas Tech University. I have included as many problems as possible of varying degrees of difficulty. Most of the exercises are computational, others are routine and seek to fix some ideas in the reader's mind; yet others are of theoretical nature and have the intention to enhance the reader's mathematical reasoning.
A solution guide to the book is available by request. Email: mfinan@atu.edu

Marcel B. Finan
Russellville, Arkansas
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Linear Systems of Equations

In this chapter we shall develop the theory of general systems of linear equations. The tool we will use to find the solutions is the row-echelon form of a matrix. In fact, the solutions can be read off from the row-echelon form of the augmented matrix of the system. The solution technique, known as **elimination** method, is developed in Section 4.

1 Systems of Linear Equations

Consider the following problem: At a carry-out pizza restaurant, an order of 3 slices of pizza, 4 breadsticks, and 2 soft drinks cost \$13.35. A second order of 5 slices of pizza, 2 breadsticks , and 3 soft drinks cost \$19.50. If four bread sticks and a can of soda cost \$0.30 more than a slice of pizza, what is the cost of each item?

Let x_1 be the cost of a slice of pizza, x_2 the cost of a breadstick, and x_3 the cost of a soft drink. The assumptions of the problem yield the following three equations:

$$\begin{cases} 3x_1 + 4x_2 + 2x_3 = 13.35 \\ 5x_1 + 2x_2 + 3x_3 = 19.50 \\ \qquad\ \ 4x_2 + \ \ x_3 = 0.30 + x_1 \end{cases}$$

or equivalently

$$\begin{cases} \ \ 3x_1 + 4x_2 + 2x_3 = 13.35 \\ \ \ 5x_1 + 2x_2 + 3x_3 = 19.50 \\ -x_1 + 4x_2 + \ \ x_3 = 0.30. \end{cases}$$

Thus, the problem is to find the values of x_1, x_2, and x_3. A system like the one above is called a linear system.

Many practical problems can be reduced to solving systems of linear equations. The main purpose of linear algebra is to find systematic methods for solving these systems. So it is natural to start our discussion of linear algebra by studying linear equations.

A **linear equation** in n variables is an equation of the form

$$a_1x_1 + a_2x_2 + ... + a_nx_n = b \tag{1.1}$$

where $x_1, x_2, ..., x_n$ are the **unknowns** (i.e., quantities to be found) and $a_1, \cdots, a_n$ are the **coefficients** (i.e., given numbers). We assume that the a_i's are not all zeros. Also given, is the number b known as the **constant term.** In the special case where $b = 0$, Equation (1.1) is called a **homogeneous linear equation.**

Observe that a linear equation does not involve any products, inverses, or roots of variables. All variables occur only to the first power and do not appear as arguments for trigonometric, logarithmic, or exponential functions.

Example 1.1
Determine whether the given equations are linear or not (i.e., non-linear):
(a) $3x_1 - 4x_2 + 5x_3 = 6$.
(b) $4x_1 - 5x_2 = x_1x_2$.
(c) $x_2 = 2\sqrt{x_1} - 6$.
(d) $x_1 + \sin x_2 + x_3 = 1$.
(e) $x_1 - x_2 + x_3 = \sin 3$.

Solution
(a) The given equation is in the form given by (1.1) and therefore is linear.
(b) The equation is non-linear because the term on the right side of the equation involves a product of the variables x_1 and x_2.
(c) A non-linear equation because the term $2\sqrt{x_1}$ involves a square root of the variable x_1.
(d) Since x_2 is an argument of a trigonometric function, the given equation is non-linear.
(e) The equation is linear according to (1.1) $\blacksquare$

In the case of $n = 2$, sometimes we will drop the subscripts and use instead $x_1 = x$ and $x_2 = y$. For example, $ax + by = c$. Geometrically, this is a straight line in the $xy-$coordinate system. Likewise, for $n = 3$, we will use $x_1 = x, x_2 = y$, and $x_3 = z$ and write $ax + by + cz = d$ which is a plane in the $xyz-$coordinate system.
A **solution** of a linear equation (1.1) in n unknowns is a finite ordered collection of numbers $s_1, s_2, ..., s_n$ which make (1.1) a true equality when $x_1 = s_1, x_2 = s_2, \cdots, x_n = s_n$ are substituted in (1.1). The collection of all solutions of a linear equation is called the **solution set** or the **general solution**.

Example 1.2
Show that $(5 + 4s - 7t, s, t)$, where $s, t \in \mathbb{R}$, is a solution to the equation

$$x_1 - 4x_2 + 7x_3 = 5.$$

Solution
$x_1 = 5 + 4s - 7t, x_2 = s$, and $x_3 = t$ is a solution to the given equation because

$$x_1 - 4x_2 + 7x_3 = (5 + 4s - 7t) - 4s + 7t = 5 \blacksquare$$

Many problems in the sciences lead to solving more than one linear equation. The general situation can be described by a linear system.
A **system of linear equations** or simply a **linear system** is any finite collection of linear equations. A linear system of m equations in n variables has the form

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \qquad\qquad\qquad\qquad\quad \vdots = \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m. \end{cases}$$

Note that the coefficients a_{ij} consist of two subscripts. The subscript i indicates the equation in which the coefficient occurs, and the subscript j indicates which unknown it multiplies.
When a linear system has more equations than unknowns, we call the system **overdetermined.** When the system has more unknowns than equations then we call the system **underdetermined.**
A **solution** of a linear system in n unknowns is a finite ordered collection of numbers $s_1, s_2, ..., s_n$ for which the substitution

$$x_1 = s_1, x_2 = s_2, \cdots, x_n = s_n$$

makes each equation a true statement. In compact form, a solution is an ordered $n-$tuple of the form

$$(s_1, s_2, \cdots, s_n).$$

The collection of all solutions of a linear system is called the **solution set** or the **general solution**. To **solve** a linear system is to find its general solution.
A linear system can have infinitely many solutions (**dependent system**), exactly one solution (**independent system**) or no solutions at all. When a linear system has a solution we say that the system is **consistent**. Otherwise, the system is said to be **inconsistent.** Thus, for the case $n = 2$, a linear system is consistent if the two lines either intersect at one point (independent) or they coincide (dependent). In the case the two lines are parallel, the system is inconsistent. For the case, $n = 3$, replace a line by a plane.

Example 1.3
Find the general solution of the linear system

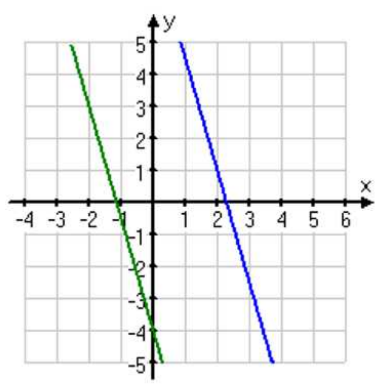
$$\begin{cases} x + \quad y = 7 \\ 2x + 4y = 18. \end{cases}$$

Solution.
Multiply the first equation of the system by -2 and then add the resulting equation to the second equation to find $2y = 4$. Solving for y we find $y = 2$. Plugging this value in one of the equations of the given system and then solving for x one finds $x = 5$ ■

Example 1.4
Solve the system

$$\begin{cases} \quad 7x + 2y = 16 \\ -21x - 6y = 24. \end{cases}$$

Solution.
Graphing the two lines we find

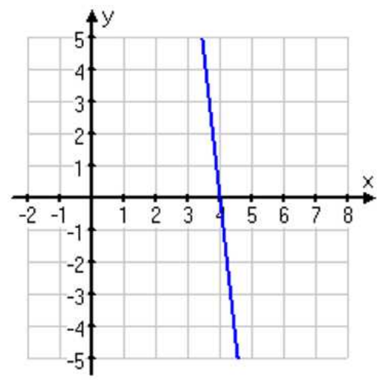


Thus, the system is inconsistent ■

Example 1.5
Solve the system

$$\begin{cases} 9x + \quad y = 36 \\ 3x + \frac{1}{3}y = 12. \end{cases}$$

Solution.
Graphing the two lines we find



Thus, the system is consistent and dependent. Note that the two equations are basically the same: $9x + y = 36$. Letting $y = t$, where t is called a **parameter**, we can solve for x and find $x = \frac{36-t}{9}$. Thus, the general solution is defined by the **parametric equations**

$$x = \frac{36-t}{9}, \; y = t \blacksquare$$

Example 1.6
By letting $x_3 = t$, find the general solution of the linear system

$$\begin{cases} x_1 + \; x_2 + x_3 = 7 \\ 2x_1 + 4x_2 + x_3 = 18. \end{cases}$$

Solution.
By letting $x_3 = t$ the given system can be rewritten in the form

$$\begin{cases} x_1 + \; x_2 = 7 - t \\ 2x_1 + 4x_2 = 18 - t. \end{cases}$$

By multiplying the first equation by -2 and adding to the second equation one finds $x_2 = \frac{4+t}{2}$. Substituting this expression in one of the individual equations of the system and then solving for x_1 one finds $x_1 = \frac{10-3t}{2}$ $\blacksquare$

Practice Problems

Problem 1.1
Define the following terms:
(a) Solution to a system of m linear equations in n unknowns.
(b) Consistent system of linear equations.
(c) Inconsistent system of linear equations.
(d) Dependent system of linear equations.
(e) Independent system of linear equations.

Problem 1.2
Which of the following equations are not linear and why?
(a) $x_1^2 + 3x_2 - 2x_3 = 5$.
(b) $x_1 + x_1x_2 + 2x_3 = 1$.
(c) $x_1 + \frac{2}{x_2} + x_3 = 5$.

Problem 1.3
Show that $(2s + 12t + 13, s, -s - 3t - 3, t)$ is a solution to the system

$$\begin{cases} 2x_1 + 5x_2 + 9x_3 + 3x_4 = -1 \\ x_1 + 2x_2 + 4x_3 \qquad\quad = 1. \end{cases}$$

Problem 1.4
Solve each of the following systems graphically:
(a)

$$\begin{cases} 4x_1 - 3x_2 = 0 \\ 2x_1 + 3x_2 = 18. \end{cases}$$

(b)

$$\begin{cases} 4x_1 - 6x_2 = 10 \\ 6x_1 - 9x_2 = 15. \end{cases}$$

(c)

$$\begin{cases} 2x_1 + x_2 = 3 \\ 2x_1 + x_2 = 1. \end{cases}$$

Which of the above systems is consistent and which is inconsistent?

Problem 1.5
Determine whether the system of equations is linear or non-linear.

(a)
$$\begin{cases} \ln x_1 + x_2 + x_3 = 3 \\ 2x_1 + x_2 - 5x_3 = 1 \\ -x_1 + 5x_2 + 3x_3 = -1. \end{cases}$$

(b)
$$\begin{cases} 3x_1 + 4x_2 + 2x_3 = 13.35 \\ 5x_1 + 2x_2 + 3x_3 = 19.50 \\ -x_1 + 4x_2 + \ \ x_3 = 0.30. \end{cases}$$

Problem 1.6
Find the parametric equations of the solution set to the equation $-x_1 + 5x_2 + 3x_3 - 2x_4 = -1$.

Problem 1.7
Write a system of linear equations consisting of three equations in three unknowns with
(a) no solutions;
(b) exactly one solution;
(c) infinitely many solutions.

Problem 1.8
For what values of h and k the system below has (a) no solution, (b) a unique solution, and (c) many solutions.

$$\begin{cases} x_1 + 3x_2 = 2 \\ 3x_1 + hx_2 = k. \end{cases}$$

Problem 1.9
True/False:
(a) A general solution of a linear system is an explicit description of all the solutions of the system.
(b) A linear system with either one solution or infinitely many solutions is said to be inconsistent.
(c) Finding a parametric description of the solution set of a linear system is the same as solving the system.
(d) A linear system with a unique solution is consistent and dependent.

Problem 1.10
Find a linear equation in the variables x and y that has the general solution $x = 5 + 2t$ and $y = t$.

Problem 1.11
Find a relationship between a, b, c so that the following system is consistent.

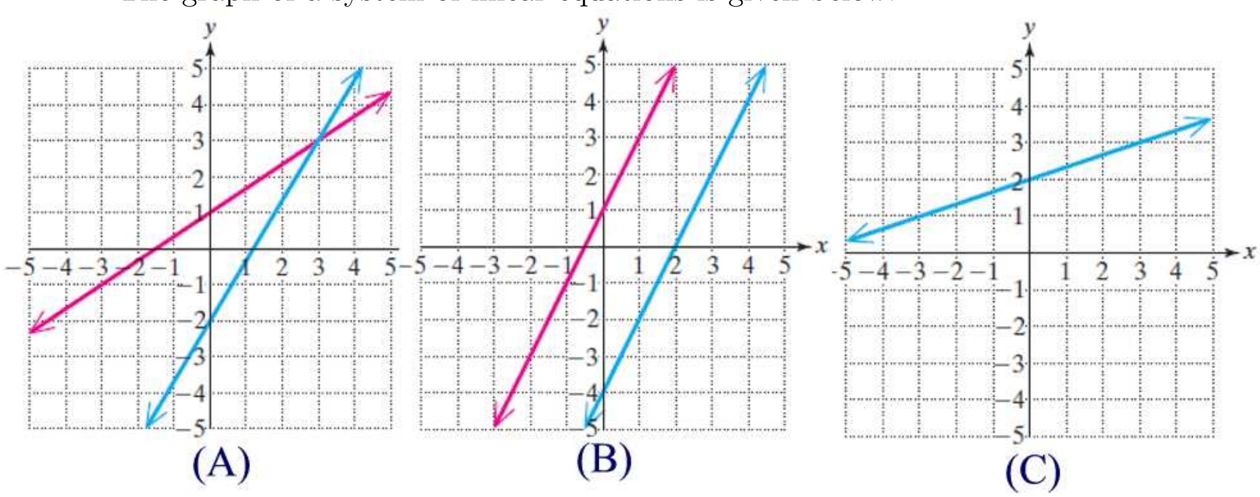
$$\begin{cases} x_1 + x_2 + 2x_3 = a \\ x_1 \qquad + \ \ x_3 = b \\ 2x_1 + x_2 + 3x_3 = c. \end{cases}$$

Problem 1.12
Determine which points are solutions to the system

$$\begin{cases} x_1 - \ \ x_2 = 6 \\ 4x_1 + 3x_2 = -4. \end{cases}$$

(a) $(4, -2)$ (b) $(6, 0)$ (c) $(2, -4)$.

Problem 1.13
The graph of a system of linear equations is given below.



(a) Identify whether the system is consistent or inconsistent.
(b) Identify whether the system is dependent or independent.
(c) Identify the number of solutions to the system.

Problem 1.14
You invest a total of \$5,800 in two investments earning 3.5% and 5.5% simple interest. Your goal is to have a total annual interest income of \$283. Write a system of linear equations that represents this situation where x_1 represents the amount invested in the 3.5% fund and x_2 represents the amount invested in the 5.5% fund.

Problem 1.15
Consider the system

$$\begin{cases} mx_1 + & x_2 = & m^2 \\ x_1 + & mx_2 = & 1. \end{cases}$$

(a) For which value(s) of m does the system have no solutions?
(b) What about infinitely many solutions?

Problem 1.16
A little boy opened his piggy bank which has coins consisting of pennies and nickels. The total amount in the jar is \$8.80. If there are twice as many nickels as pennies, how many pennies does the boy have? How many nickels?

Problem 1.17
Two angles are complementary. One angle is 81° less than twice the other. Find the two angles.

Problem 1.18
The sum of three numbers is 14. The largest is 4 times the smallest, while the sum of the smallest and twice the largest is 18. Find the numbers.

Problem 1.19
Show that the system

$$\begin{cases} x_1 - 2x_2 = -1 \\ 4x_1 + 3x_2 = 7 \\ 3x_1 + 5x_2 = 8. \end{cases}$$

is consistent and independent.

Problem 1.20
A linear equation in three variables can be visualized as a plane in the three

dimensional space. Explain geometrically the meaning of the following statements:
(a) A system with three equations and three unknowns is inconsistent.
(b) A system with three equations and three unknowns is consistent and independent.
(c) A system with three equations and three unknowns is consistent and dependent.

2 Equivalent Systems and Elementary Row Operations: The Elimination Method

Next, we shift our attention for solving linear systems of equations. In this section we introduce the concept of elementary row operations that will be vital for our algebraic method of solving linear systems.
First, we define what we mean by equivalent systems: Two linear systems are said to be **equivalent** if and only if they have the same set of solutions.

Example 2.1
Show that the system

$$\begin{cases} x_1 - 3x_2 = -7 \\ 2x_1 + \quad x_2 = 7 \end{cases}$$

is equivalent to the system

$$\begin{cases} 8x_1 - 3x_2 = 7 \\ 3x_1 - 2x_2 = 0 \\ 10x_1 - 2x_2 = 14. \end{cases}$$

Solution.
Solving the first system one finds the solution $x_1 = 2, x_2 = 3$. Similarly, solving the second system one finds the solution $x_1 = 2$ and $x_2 = 3$. Hence, the two systems are equivalent ■

Example 2.2
Show that if $x_1 + kx_2 = c$ and $x_1 + \ell x_2 = d$ are equivalent then $k = l$ and $c = d$.

Solution.
The general solution to the first equation is the ordered pair $(c - kt, t)$ where t is an arbitrary number. For this solution to be a solution to the second equation as well, we must have $c - kt + \ell t = d$ for all $t \in \mathbb{R}$. In particular, if $t = 0$ we find $c = d$. Thus, $kt = \ell t$ for all $t \in \mathbb{R}$. Letting $t = 1$ we find $k = \ell$ ■

Our basic algebraic method for solving a linear system is known as the **method of elimination**. The method consists of reducing the original system to an equivalent system that is easier to solve. The reduced system has the shape of an upper (resp. lower) triangle. This new system can

be solved by a technique called **backward-substitution** (resp. **forward-substitution**). The unknowns are found starting from the bottom (resp. the top) of the system.
The three basic operations in the above method, known as the **elementary row operations**, are summarized as follows:
(I) Multiply an equation by a non-zero number.
(II) Replace an equation by the sum of this equation and another equation multiplied by a number.
(III) Interchange two equations.

To indicate which operation is being used in the process one can use the following shorthand notation. For example, $r_3 \leftarrow \frac{1}{2}r_3$ represents the row operation of type (I) where each entry of row 3 is being replaced by $\frac{1}{2}$ that entry. Similar interpretations for types (II) and (III) operations.
The following theorem asserts that the system obtained from the original system by means of elementary row operations has the same set of solutions as the original one.

Theorem 2.1
Suppose that an elementary row operation is performed on a linear system. Then the resulting system is equivalent to the original system.

Example 2.3
Use the elimination method described above to solve the system

$$\begin{cases} x_1 + \ x_2 - \ x_3 = 3 \\ x_1 - 3x_2 + 2x_3 = 1 \\ 2x_1 - 2x_2 + \ x_3 = 4. \end{cases}$$

Solution.
Step 1: We eliminate x_1 from the second and third equations by performing two operations $r_2 \leftarrow r_2 - r_1$ and $r_3 \leftarrow r_3 - 2r_1$ obtaining

$$\begin{cases} x_1 + \ x_2 - \ x_3 = 3 \\ \quad - 4x_2 + 3x_3 = -2 \\ \quad - 4x_2 + 3x_3 = -2. \end{cases}$$

Step 2: The operation $r_3 \leftarrow r_3 - r_2$ leads to the system

$$\begin{cases} x_1 + \ x_2 - \ x_3 = 3 \\ \quad - 4x_2 + 3x_3 = -2. \end{cases}$$

By assigning x_3 an arbitrary value t we obtain the general solution $x_1 = \frac{t+10}{4}, x_2 = \frac{2+3t}{4}, x_3 = t$. This means that the linear system has infinitely many solutions (consistent and dependent). Every time we assign a value to t we obtain a different solution ■

Example 2.4
Determine if the following system is consistent or not

$$\begin{cases} 3x_1 + 4x_2 + x_3 = 1 \\ 2x_1 + 3x_2 \qquad = 0 \\ 4x_1 + 3x_2 - x_3 = -2. \end{cases}$$

Solution.
Step 1: To eliminate the variable x_1 from the second and third equations we perform the operations $r_2 \leftarrow 3r_2 - 2r_1$ and $r_3 \leftarrow 3r_3 - 4r_1$ obtaining the system

$$\begin{cases} 3x_1 + 4x_2 + \ \ x_3 = 1 \\ \qquad\quad x_2 - 2x_3 = -2 \\ \quad\ - 7x_2 - 7x_3 = -10. \end{cases}$$

Step 2: Now, to eliminate the variable x_3 from the third equation we apply the operation $r_3 \leftarrow r_3 + 7r_2$ to obtain

$$\begin{cases} 3x_1 + 4x_2 + \quad x_3 = 1 \\ \qquad\quad x_2 - \ \ 2x_3 = -2 \\ \qquad\qquad - 21x_3 = -24. \end{cases}$$

Solving the system by the method of backward substitution we find the unique solution $x_1 = -\frac{3}{7}, x_2 = \frac{2}{7}, x_3 = \frac{8}{7}$. Hence the system is consistent and independent ■

Example 2.5
Determine whether the following system is consistent:

$$\begin{cases} \quad x_1 - 3x_2 = 4 \\ -3x_1 + 9x_2 = 8. \end{cases}$$

Solution.
Multiplying the first equation by 3 and adding the resulting equation to the second equation we find $0 = 20$ which is impossible. Hence, the given system is inconsistent ■

Practice Problems

Problem 2.1
Solve each of the following systems using the method of elimination:
(a)

$$\begin{cases} 4x_1 - 3x_2 = 0 \\ 2x_1 + 3x_2 = 18. \end{cases}$$

(b)

$$\begin{cases} 4x_1 - 6x_2 = 10 \\ 6x_1 - 9x_2 = 15. \end{cases}$$

(c)

$$\begin{cases} 2x_1 + x_2 = 3 \\ 2x_1 + x_2 = 1. \end{cases}$$

Which of the above systems is consistent and which is inconsistent?

Problem 2.2
Find the values of A, B, C in the following partial fraction decomposition

$$\frac{x^2 - x + 3}{(x^2+2)(2x-1)} = \frac{Ax + B}{x^2 + 2} + \frac{C}{2x - 1}.$$

Problem 2.3
Find a quadratic equation of the form $y = ax^2 + bx + c$ that goes through the points $(-2, 20), (1, 5)$, and $(3, 25)$.

Problem 2.4
Solve the following system using the method of elimination.

$$\begin{cases} 5x_1 - 5x_2 - 15x_3 = 40 \\ 4x_1 - 2x_2 - 6x_3 = 19 \\ 3x_1 - 6x_2 - 17x_3 = 41. \end{cases}$$

Problem 2.5
Solve the following system using elimination.

$$\begin{cases} 2x_1 + x_2 + x_3 = -1 \\ x_1 + 2x_2 + x_3 = 0 \\ 3x_1 - 2x_3 = 5. \end{cases}$$

Problem 2.6
Find the general solution of the linear system

$$\begin{cases} x_1 - 2x_2 + 3x_3 + x_4 = -3 \\ 2x_1 - x_2 + 3x_3 - x_4 = 0. \end{cases}$$

Problem 2.7
Find a, b, and c so that the system

$$\begin{cases} x_1 + ax_2 + cx_3 = 0 \\ bx_1 + cx_2 - 3x_3 = 1 \\ ax_1 + 2x_2 + bx_3 = 5 \end{cases}$$

has the solution $x_1 = 3, x_2 = -1, x_3 = 2$.

Problem 2.8
Show that the following systems are equivalent.

$$\begin{cases} 7x_1 + 2x_2 + 2x_3 = 21 \\ - 2x_2 + 3x_3 = 1 \\ 4x_3 = 12 \end{cases}$$

and

$$\begin{cases} 21x_1 + 6x_2 + 6x_3 = 63 \\ - 4x_2 + 6x_3 = 2 \\ x_3 = 3. \end{cases}$$

Problem 2.9
Solve the following system by elimination.

$$\begin{cases} 3x_1 + x_2 + 2x_3 = 13 \\ 2x_1 + 3x_2 + 4x_3 = 19 \\ x_1 + 4x_2 + 3x_3 = 15. \end{cases}$$

Problem 2.10
Solve the following system by elimination.

$$\begin{cases} x_1 - 2x_2 + 3x_3 = 7 \\ 2x_1 + x_2 + x_3 = 4 \\ -3x_1 + 2x_2 - 2x_3 = -10. \end{cases}$$

Problem 2.11
John, Philip, and Grace play a round of golf together. Their combined score is 224. John's score was 8 more than Philip's, and Grace's score was 7 more than John's. What was each person's score?

Problem 2.12
Set up the system of equations to solve the following problem. Do not solve it.
Billy's Restaurant ordered 200 flowers for Mother's Day. They ordered carnations at \$1.50 each, roses at \$5.75 each, and daisies at \$2.60 each. They ordered mostly carnations, and 20 fewer roses than daisies. The total order came to \$589.50. How many of each type of flower was ordered?

Problem 2.13
In each case, tell whether the operation is a valid row operation. If it is, say what it does (in words).
(a) $r_1 \leftrightarrow r_3$.
(b) $r_2 \leftarrow r_2 + 5$.
(c) $r_3 \leftarrow r_3 + 4r_2$.
(d) $r_4 \leftarrow 5r_4$.

Problem 2.14
Find the general solution to the system

$$\left\{\begin{array}{rrrrl} x_1 + & x_2 + & x_3 + & 2x_4 & = 1 \\ & 2x_2 + & 2x_3 + & x_4 & = 0 \\ 2x_1 & & + 2x_3 + & x_4 & = 1. \end{array}\right.$$

Problem 2.15
Determine the value(s) of a so that the following system is inconsistent.

$$\left\{\begin{array}{rrrl} x_1+ & 2x_2+ & x_3 = & a \\ x_1+ & x_2+ & ax_3 = & 1 \\ 3x_1+ & 4x_2+ & (a^2-2)x_3 = & 1. \end{array}\right.$$

Problem 2.16
True or false:
(a) Elementary row operations on an augmented matrix never change the solution set of the associated linear system.
(b) Two matrices are row equivalent if they have the same number of rows.
(c) Two linear systems are equivalent if they have the same solution set.

Problem 2.17
Determine the value(s) of h so that the system

$$\begin{cases} x_1 + & hx_2 = & -3, \\ -2x_1 + & 4x_2 = & 6. \end{cases}$$

(a) is consistent and dependent;
(b) is consistent and indepedent.

Problem 2.18
Suppose the system

$$\begin{cases} x_1 + & 3x_2 = & f, \\ cx_1 + & dx_2 = & g \end{cases}$$

is consistent for all possible values of f and g. What can you say about the coefficients c and d?

Problem 2.19
Show that the following systems are equivalent.

$$\begin{cases} x_1 - & 3x_2 = & -3, \\ 2x_1 + & x_2 = & 8 \end{cases} \qquad \begin{cases} x_1 - & 3x_2 = & -3, \\ & 7x_2 = & 14. \end{cases}$$

Problem 2.20
(a) The elementary row operation $r_i \leftrightarrow r_j$ swaps rows i and j. What elementary row operation will undo this operation?
(b) The elementary row operation $r_i \leftarrow \alpha r_i$ multiplies row i by a non-zero constant α. What elementary row operation will undo this operation?
(c) The elementary row operation $r_i \leftrightarrow r_i + \alpha r_j$ adds α row j to row i. What elementary row operation will undo this operation?

Problem 2.21
Show that $(s_1, s_2, \cdots, s_n)$ is a solution to $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$ if and only if $(s_1, s_2, \cdots, s_n)$ is a solution to $ka_1x_1 + ka_2x_2 + \cdots + a_nx_n = b$ for any non-zero scalar k.

Problem 2.22
Show that $(s_1, s_2, \cdots, s_n)$ is a solution to $a_1x_1 + a_2x_2 + \cdots + a_nx_n = a$ and $b_1x_1 + b_2x_2 + \cdots + b_nx_n = b$ if and only if $(s_1, s_2, \cdots, s_n)$ is a solution to $(ka_1 + b_1)x_1 + (ka_2 + b_2)x_2 + \cdots + (ka_n + b_n)x_n = ka + b$ and $a_1x_1 + a_2x_2 + \cdots + a_nx_n = a$.

Problem 2.23
Using Problems 2.21 and 2.22, prove Theorem 2.1.

3 Solving Linear Systems Using Augmented Matrices

In this section we apply the elimination method described in the previous section to the rectangular array consisting of the coefficients of the unknowns and the right-hand side of a given system rather than to the individual equations. To elaborate, consider the linear system

$$\left\{\begin{aligned} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n &= b_2 \\ \vdots &= \vdots \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n &= b_m. \end{aligned}\right.$$

We define the **augmented matrix** corresponding to the above system to be the rectangular array

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

We then apply elementary row operations on the augmented matrix and reduces it to a triangular matrix. Then the corresponding system is triangular as well and is equivalent to the original system. Next, use either the backward-substitution or the forward-substitution technique to find the unknowns. We illustrate this technique in the following examples.

Example 3.1
Solve the following linear system using elementary row operations on the augmented matrix:

$$\left\{\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9. \end{aligned}\right.$$

Solution.
The augmented matrix for the system is

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

Step 1: The operations $r_2 \leftarrow \frac{1}{2}r_2$ and $r_3 \leftarrow r_3 + 4r_1$ give

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

Step 2: The operation $r_3 \leftarrow r_3 + 3r_2$ gives

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

The corresponding system of equations is

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_2 - 4x_3 = 4 \\ x_3 = 3. \end{cases}$$

Using back-substitution we find the unique solution $x_1 = 29, x_2 = 16, x_3 = 3$ ■

Example 3.2
Solve the following linear system using the method described above.

$$\begin{cases} x_2 + 5x_3 = -4 \\ x_1 + 4x_2 + 3x_3 = -2 \\ 2x_1 + 7x_2 + x_3 = -1. \end{cases}$$

Solution.
The augmented matrix for the system is

$$\begin{bmatrix} 0 & 1 & 5 & -4 \\ 1 & 4 & 3 & -2 \\ 2 & 7 & 1 & -1 \end{bmatrix}$$

Step 1: The operation $r_2 \leftrightarrow r_1$ gives

$$\begin{bmatrix} 1 & 4 & 3 & -2 \\ 0 & 1 & 5 & -4 \\ 2 & 7 & 1 & -1 \end{bmatrix}$$

Step 2: The operation $r_3 \leftarrow r_3 - 2r_1$ gives the system

$$\begin{bmatrix} 1 & 4 & 3 & -2 \\ 0 & 1 & 5 & -4 \\ 0 & -1 & -5 & 3 \end{bmatrix}$$

Step 3: The operation $r_3 \leftarrow r_3 + r_2$ gives

$$\begin{bmatrix} 1 & 4 & 3 & -2 \\ 0 & 1 & 5 & -4 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

The corresponding system of equations is

$$\begin{cases} x_1 + 4x_2 + 3x_3 = -2 \\ \qquad\quad x_2 + 5x_3 = -4 \\ 0x_1 + 0x_2 + 0x_3 = -1. \end{cases}$$

From the last equation we conclude that the system is inconsistent ■

Example 3.3
Determine if the following system is consistent.

$$\begin{cases} \qquad\quad x_2 - 4x_3 = 8 \\ 2x_1 - 3x_2 + 2x_3 = 1 \\ 5x_1 - 8x_2 + 7x_3 = 1. \end{cases}$$

Solution.
The augmented matrix of the given system is

$$\begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{bmatrix}$$

Step 1: The operation $r_3 \leftarrow r_3 - 2r_2$ gives

$$\begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 1 & -2 & 3 & -1 \end{bmatrix}$$

Step 2: The operation $r_3 \leftrightarrow r_1$ leads to

$$\begin{bmatrix} 1 & -2 & 3 & -1 \\ 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \end{bmatrix}$$

Step 3: Applying $r_2 \leftarrow r_2 - 2r_1$ to obtain

$$\begin{bmatrix} 1 & -2 & 3 & -1 \\ 0 & 1 & -4 & 3 \\ 0 & 1 & -4 & 8 \end{bmatrix}$$

Step 4: Finally, the operation $r_3 \leftarrow r_3 - r_2$ gives

$$\begin{bmatrix} 1 & -2 & 3 & -1 \\ 0 & 1 & -4 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

Hence, the equivalent system is

$$\begin{cases} x_1 - 2x_2 + 3x_3 = 0 \\ \qquad\quad x_2 - 4x_3 = 3 \\ 0x_1 + 0x_2 + 0x_3 = 5. \end{cases}$$

This last system has no solution (the last equation requires x_1, x_2, and x_3 to satisfy the equation $0x_1 + 0x_2 + 0x_3 = 5$ and no such x_1, x_2, and x_3 exist). Hence the original system is inconsistent ■

Pay close attention to the last row of the triangular matrix of the previous exercise. This situation is typical of an inconsistent system.

Practice Problems

Problem 3.1
Solve the following linear system using the elimination method of this section.

$$\begin{cases} x_1 + 2x_2 = 0 \\ -x_1 + 3x_2 + 3x_3 = -2 \\ x_2 + x_3 = 0. \end{cases}$$

Problem 3.2
Find an equation involving g, h, and k that makes the following augmented matrix corresponds to a consistent system.

$$\begin{bmatrix} 2 & 5 & -3 & g \\ 4 & 7 & -4 & h \\ -6 & -3 & 1 & k \end{bmatrix}$$

Problem 3.3
Solve the following system using elementary row operations on the augmented matrix:

$$\begin{cases} 5x_1 - 5x_2 - 15x_3 = 40 \\ 4x_1 - 2x_2 - 6x_3 = 19 \\ 3x_1 - 6x_2 - 17x_3 = 41. \end{cases}$$

Problem 3.4
Solve the following system using elementary row operations on the augmented matrix:

$$\begin{cases} 2x_1 + x_2 + x_3 = -1 \\ x_1 + 2x_2 + x_3 = 0 \\ 3x_1 - 2x_3 = 5. \end{cases}$$

Problem 3.5
Solve the following system using elementary row operations on the augmented matrix:

$$\begin{cases} x_1 - x_2 + 2x_3 + x_4 = 0 \\ 2x_1 + 2x_2 - x_4 = 0 \\ 3x_1 + x_2 + 2x_3 + x_4 = 0. \end{cases}$$

Problem 3.6
Find the value(s) of a for which the following system has a nontrivial solution. Find the general solution.

$$\left\{\begin{array}{r} x_1+2x_2+x_3=0\\ x_1+3x_2+6x_3=0\\ 2x_1+3x_2+ax_3=0.\end{array}\right.$$

Problem 3.7
Solve the linear system whose augmented matrix is given by

$$\begin{bmatrix} 1 & 1 & 2 & 8\\ -1 & -2 & 3 & 1\\ 3 & -7 & 4 & 10\end{bmatrix}$$

Problem 3.8
Solve the linear system whose augmented matrix is reduced to the following triangular form

$$\begin{bmatrix} 1 & -3 & 7 & 1\\ 0 & 1 & 4 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

Problem 3.9
Solve the linear system whose augmented matrix is reduced to the following triangular form

$$\begin{bmatrix} 1 & 0 & 0 & -7 & 8\\ 0 & 1 & 0 & 3 & 2\\ 0 & 0 & 1 & 1 & -5\end{bmatrix}$$

Problem 3.10
Reduce the matrix to triangular matrix.

$$\begin{bmatrix} -1 & -1 & 0 & 0\\ 0 & 0 & 2 & 3\\ 4 & 0 & -2 & 1\\ 3 & -1 & 0 & 4\end{bmatrix}$$

Problem 3.11
Solve the following system using elementary row operations on the augmented matrix:

$$\begin{cases} 3x_1 + x_2 + 7x_3 + 2x_4 = 13 \\ 2x_1 - 4x_2 + 14x_3 - x_4 = -10 \\ 5x_1 + 11x_2 - 7x_3 + 8x_4 = 59 \\ 2x_1 + 5x_2 - 4x_3 - 3x_4 = 39. \end{cases}$$

Problem 3.12
Determine the value(s) of a so that the line whose parametric equations are given by

$$\begin{cases} x_1 = -3 + t \\ x_2 = 2 - t \\ x_3 = 1 + at \end{cases}$$

is parallel to the plane $3x_1 - 5x_2 + x_3 = -3$.

Problem 3.13
Find the general solution to the system

$$\begin{cases} x_1 - 2x_2 + 2x_3 - x_4 = 3 \\ 3x_1 + x_2 + 6x_3 + 11x_4 = 16 \\ 2x_1 - x_2 + 4x_3 + x_4 = 9. \end{cases}$$

Problem 3.14
Solve the system

$$\begin{cases} x_1 - 2x_2 - 6x_3 = 12 \\ 2x_1 + 4x_2 + 12x_3 = -17 \\ x_1 - 4x_2 - 12x_3 = 22. \end{cases}$$

Problem 3.15
Solve the system

$$\begin{cases} x_1 - 2x_2 - 3x_3 = 0 \\ -x_1 + x_2 + 2x_3 = 3 \\ \phantom{-x_1 + {}} 2x_2 + x_3 = -8. \end{cases}$$

Problem 3.16
True or false. Justify your answer.
(a) A general solution of a system is an explicit description of all solutions of the system.
(b) If one row in an reduced form of an augmented matrix is $[0 \quad 0 \quad 0 \quad 5 \quad 0]$, then the associated linear system is inconsistent.
(c) If one row in an reduced form of an augmented matrix is $[0 \quad 0 \quad 0 \quad 0 \quad 0]$, then the associated linear system is inconsistent.
(d) The row reduction algorithm applies only to augmented matrices for a linear system.

Problem 3.17
The augmented matrix

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & k & -4 \end{bmatrix}$$

represents a system of equations in the variables x_1, x_2, and x_3.
(a) For what values of k is there no solution?
(b) For what value of k is there exactly one solution? What is that solution?

Problem 3.18
The cubic function $p(x) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$ crosses the points $(20, 106), (30, 123)$, $(40, 132)$ and $(50, 151)$. Find the value of $p(60)$.

Problem 3.19
Solve the system

$$\begin{cases} \quad\quad\quad\quad\; 2x_3 + 2x_4 - \;\; 6x_5 = -2 \\ x_1 + 2x_2 + \;\; x_3 + 4x_4 - 12x_5 = -3 \\ x_1 + 2x_2 + \;\; x_3 + 2x_4 - \;\; 6x_5 = -1. \end{cases}$$

Problem 3.20
Solve the system

$$\begin{cases} x_1 + \;\; x_2 + \;\; x_3 = 11 \\ x_1 - \;\; x_2 + 3x_3 = 5 \\ 2x_1 + 2x_2 + 2x_3 = 15. \end{cases}$$

4 Echelon Form and Reduced Echelon Form: Gaussian Elimination

The elimination method introduced in the previous section reduces the augmented matrix to a "nice" matrix (meaning the corresponding equations are easy to solve). Two of the "nice" matrices discussed in this section are matrices in either row-echelon form or reduced row-echelon form, concepts that we discuss next.
By a **leading entry** of a row in a matrix we mean the leftmost non-zero entry in the row.
A rectangular matrix is said to be in **row-echelon form** if it has the following three characterizations:

(1) All rows consisting entirely of zeros are at the bottom.
(2) The leading entry in each non-zero row is 1 and is located in a column to the right of the leading entry of the row above it.
(3) All entries in a column below a leading entry are zero.

The matrix is said to be in **reduced row-echelon form** if in addition to the above, the matrix has the following additional characterization:

(4) Each leading 1 is the only nonzero entry in its column.

Remark 4.1 From the definition above, note that a matrix in row-echelon form has zeros below each leading 1, whereas a matrix in reduced row-echelon form has zeros both above and below each leading 1.

Example 4.1
Determine which matrices are in row-echelon form (but not in reduced row-echelon form) and which are in reduced row-echelon form
(a)

$$\begin{bmatrix} 1 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Solution.
(a)The given matrix is in row-echelon form but not in reduced row-echelon form since the $(1,2)-$entry is not zero.
(b) The given matrix satisfies the characterization of a reduced row-echelon form ■

The importance of the row-echelon matrices is indicated in the following theorem.

Theorem 4.1
Every nonzero matrix can be brought to (reduced) row-echelon form by a finite number of elementary row operations.

Proof.
The proof consists of the following steps:
Step 1. Find the first column from the left containing a non-zero entry (call it a), and move the row containing that entry to the top position.
Step 2. Multiply the row from Step 1 by $\frac{1}{a}$ to create a leading 1.
Step 3. By subtracting multiples of that row from the rows below it, make each entry below the leading 1 zero.
Step 4. This completes the first row. Now repeat steps 1-3 on the matrix consisting of the remaining rows.
The process stops when either no rows remain in step 4 or the remaining rows consist of zeros. The entire matrix is now in row-echelon form.
To find the reduced row-echelon form we need the following additional step.
Step 5. Beginning with the last nonzero row and working upward, add suitable multiples of each row to the rows above to introduce zeros above the leading 1 ■

The process of reducing a matrix to a row-echelon form is known as **Gaussian elimination**. That of reducing a matrix to a reduced row-echelon form is known as **Gauss-Jordan elimination**.

Example 4.2
Use Gauss-Jordan elimination to transform the following matrix first into

row-echelon form and then into reduced row-echelon form

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

Solution.
The reduction of the given matrix to reduced row-echelon form is as follows.

Step 1: $r_1 \leftrightarrow r_4$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

Step 2: $r_2 \leftarrow r_2 + r_1$ and $r_3 \leftarrow r_3 + 2r_1$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

Step 3: $r_2 \leftarrow \frac{1}{2}r_2$ and $r_3 \leftarrow \frac{1}{5}r_3$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

Step 4: $r_3 \leftarrow r_3 - r_2$ and $r_4 \leftarrow r_4 + 3r_2$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

Step 5: $r_3 \leftrightarrow r_4$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Step 6: $r_5 \leftarrow -\frac{1}{5}r_5$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Step 7: $r_1 \leftarrow r_1 - 4r_2$

$$\begin{bmatrix} 1 & 0 & -3 & 3 & 5 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Step 8: $r_1 \leftarrow r_1 - 3r_3$ and $r_2 \leftarrow r_2 + 3r_3$

$$\begin{bmatrix} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \blacksquare$$

Example 4.3
Use Gauss-Jordan elimination to transform the following matrix first into row-echelon form and then into reduced row-echelon form

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

Solution.
By following the steps in the Gauss-Jordan algorithm we find

Step 1: $r_3 \leftarrow \frac{1}{3}r_3$

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 1 & -3 & 4 & -3 & 2 & 5 \end{bmatrix}$$

Step 2: $r_1 \leftrightarrow r_3$

$$\begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

Step 3: $r_2 \leftarrow r_2 - 3r_1$

$$\begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

Step 4: $r_2 \leftarrow \frac{1}{2}r_2$

$$\begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

Step 5: $r_3 \leftarrow r_3 - 3r_2$

$$\begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Step 6: $r_1 \leftarrow r_1 + 3r_2$

$$\begin{bmatrix} 1 & 0 & -2 & 3 & 5 & -4 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Step 7: $r_1 \leftarrow r_1 - 5r_3$ and $r_2 \leftarrow r_2 - r_3$

$$\begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \blacksquare$$

Remark 4.2
It can be shown that no matter how the elementary row operations are varied, one will always arrive at the same reduced row-echelon form; that is, the reduced row echelon form is unique. On the contrary row-echelon form is **not** unique. However, the number of leading 1's of two different row-echelon forms is the same. That is, two row-echelon matrices have the same number of nonzero rows. This number is known as the **rank** of the matrix and is denoted by rank(A).

Example 4.4
Consider the system

$$\begin{cases} ax + by = k \\ cx + dy = l. \end{cases}$$

Show that if $ad - bc \neq 0$ then the reduced row-echelon form of the coefficient matrix is the matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Solution.
The coefficient matrix is the matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Assume first that $a \neq 0$. Using Gaussian elimination we reduce the above matrix into row-echelon form as follows:

Step 1: $r_2 \leftarrow ar_2 - cr_1$

$$\begin{bmatrix} a & b \\ 0 & ad - bc \end{bmatrix}$$

Step 2: $r_2 \leftarrow \frac{1}{ad-bc} r_2$

$$\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$$

Step 3: $r_1 \leftarrow r_1 - br_2$

$$\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$$

Step 4: $r_1 \leftarrow \frac{1}{a} r_1$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Next, assume that $a = 0$. Then $c \neq 0$ and $b \neq 0$. Following the steps of Gauss-Jordan elimination algorithm we find

Step 1: $r_1 \leftrightarrow r_2$

$$\begin{bmatrix} c & d \\ 0 & b \end{bmatrix}$$

Step 2: $r_1 \leftarrow \frac{1}{c} r_1$ and $r_2 \leftarrow \frac{1}{b} r_2$

$$\begin{bmatrix} 1 & \frac{d}{c} \\ 0 & 1 \end{bmatrix}$$

Step 3: $r_1 \leftarrow r_1 - \frac{d}{c} r_2$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \blacksquare$$

Example 4.5
Find the rank of each of the following matrices
(a)

$$A = \begin{bmatrix} 2 & 1 & 4 \\ 3 & 2 & 5 \\ 0 & -1 & 1 \end{bmatrix}$$

(b)

$$B = \begin{bmatrix} 3 & 1 & 0 & 1 & -9 \\ 0 & -2 & 12 & -8 & -6 \\ 2 & -3 & 22 & -14 & -17 \end{bmatrix}$$

Solution.
(a) We use Gaussian elimination to reduce the given matrix into row-echelon form as follows:

Step 1: $r_2 \leftarrow r_2 - r_1$

$$\begin{bmatrix} 2 & 1 & 4 \\ 1 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

Step 2: $r_1 \leftrightarrow r_2$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 4 \\ 0 & -1 & 1 \end{bmatrix}$$

Step 3: $r_2 \leftarrow -r_2 + 2r_1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & -1 & 1 \end{bmatrix}$$

Step 4: $r_3 \leftarrow -r_3 - r_2$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus, $rank(A) = 3$.
(b) As in (a), we reduce the matrix into row-echelon form as follows:

Step 1: $r_1 \leftarrow r_1 - r_3$

$$\begin{bmatrix} 1 & 4 & -22 & 15 & 8 \\ 0 & -2 & 12 & -8 & -6 \\ 2 & -3 & 22 & -14 & -17 \end{bmatrix}$$

Step 2: $r_3 \leftarrow r_3 - 2r_1$

$$\begin{bmatrix} 1 & 4 & -22 & 15 & 25 \\ 0 & -2 & 12 & -8 & -6 \\ 0 & -11 & -22 & -44 & -33 \end{bmatrix}$$

Step 3: $r_2 \leftarrow -\frac{1}{2}r_2$

$$\begin{bmatrix} 1 & 4 & -22 & 15 & 8 \\ 0 & 1 & -6 & 4 & 3 \\ 0 & -11 & -22 & -44 & -33 \end{bmatrix}$$

Step 4: $r_3 \leftarrow r_3 + 11r_2$

$$\begin{bmatrix} 1 & 4 & -22 & 15 & 8 \\ 0 & 1 & -6 & 4 & 3 \\ 0 & 0 & -88 & 0 & 0 \end{bmatrix}$$

Step 5: $r_3 \leftarrow \frac{1}{8}r_3$

$$\begin{bmatrix} 1 & 4 & -22 & 15 & 8 \\ 0 & 1 & -6 & 4 & 3 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Hence, $rank(B) = 3$ ■

Practice Problems

Problem 4.1
Use Gaussian elimination to reduce the given matrix to row echelon form.

$$\begin{bmatrix} 1 & -2 & 3 & 1 & -3 \\ 2 & -1 & 3 & -1 & 0 \end{bmatrix}.$$

Problem 4.2
Use Gaussian elimination to reduce the given matrix to row echelon form.

$$\begin{bmatrix} -1 & 0 & 2 & -3 \\ 0 & 3 & -1 & 7 \\ 3 & 2 & 0 & 7 \end{bmatrix}.$$

Problem 4.3
Use Gaussian elimination to reduce the given matrix to row echelon form.

$$\begin{bmatrix} 5 & -5 & -15 & 40 \\ 4 & -2 & -6 & 19 \\ 3 & -6 & -17 & 41 \end{bmatrix}.$$

Problem 4.4
Use Gaussian elimination to reduce the given matrix to row echelon form.

$$\begin{bmatrix} 2 & 1 & 1 & -1 \\ 1 & 2 & 1 & 0 \\ 3 & 0 & -2 & 5 \end{bmatrix}.$$

Problem 4.5
Which of the following matrices are not in reduced row-ehelon form and why?
(a)

$$\begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(b)

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & 3 & 0 \end{bmatrix}.$$

(c)

$$\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Problem 4.6
Use Gaussian elimination to convert the following matrix into a row-echelon matrix.

$$\begin{bmatrix} 1 & -3 & 1 & -1 & 0 & -1 \\ -1 & 3 & 0 & 3 & 1 & 3 \\ 2 & -6 & 3 & 0 & -1 & 2 \\ -1 & 3 & 1 & 5 & 1 & 6 \end{bmatrix}.$$

Problem 4.7
Use Gauss-Jordan elimination to convert the following matrix into reduced row-echelon form.

$$\begin{bmatrix} -2 & 1 & 1 & 15 \\ 6 & -1 & -2 & -36 \\ 1 & -1 & -1 & -11 \\ -5 & -5 & -5 & -14 \end{bmatrix}.$$

Problem 4.8
Use Gauss-Jordan elimination to convert the following matrix into reduced row-echelon form.

$$\begin{bmatrix} 3 & 1 & 7 & 2 & 13 \\ 2 & -4 & 14 & -1 & -10 \\ 5 & 11 & -7 & 8 & 59 \\ 2 & 5 & -4 & -3 & 39 \end{bmatrix}.$$

Problem 4.9
Use Gaussian elimination to convert the following matrix into row-echelon form.

$$\begin{bmatrix} -1 & -1 & 0 & 0 \\ 0 & 0 & 2 & 3 \\ 4 & 0 & -2 & 1 \\ 3 & -1 & 0 & 4 \end{bmatrix}.$$

Problem 4.10
Use Gaussian elimination to convert the following matrix into row-echelon

form.

$$\begin{bmatrix} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{bmatrix}.$$

Problem 4.11
Find the rank of each of the following matrices.
(a)

$$\begin{bmatrix} -1 & -1 & 0 & 0 \\ 0 & 0 & 2 & 3 \\ 4 & 0 & -2 & 1 \\ 3 & -1 & 0 & 4 \end{bmatrix}.$$

(b)

$$\begin{bmatrix} 1 & -1 & 3 \\ 2 & 0 & 4 \\ -1 & -3 & 1 \end{bmatrix}.$$

Problem 4.12
Use Gauss-Jordan elimination to reduce the matrix into reduced row echelon form

$$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & 3 & 4 & 11 \\ 4 & 9 & 16 & 41 \end{bmatrix}.$$

Problem 4.13
Use Gauss-Jordan elimination to reduce the matrix into reduced row echelon form

$$\begin{bmatrix} 2 & -2 & 1 & 3 \\ 3 & 1 & -1 & 7 \\ 1 & -3 & 2 & 0 \end{bmatrix}.$$

Problem 4.14
Find the rank of the matrix

$$\begin{bmatrix} 2 & -1 & 3 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \\ 1 & 1 & 4 \end{bmatrix}.$$

Problem 4.15
Find the rank of the following matrix.

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 4 \\ 0 & 2 & 0 \end{bmatrix}.$$

Problem 4.16
Use Gaussian elimination to reduce the matrix to reduced row echelon form.

$$\begin{bmatrix} 1 & 1 & 1 & 5 \\ 2 & 3 & 5 & 8 \\ 4 & 0 & 5 & 2 \end{bmatrix}$$

Problem 4.17
Use Gaussian elimination to reduce the matrix to reduced row echelon form.

$$\begin{bmatrix} 1 & 2 & -3 & 2 \\ 6 & 3 & -9 & 6 \\ 7 & 14 & -21 & 13 \end{bmatrix}$$

Problem 4.18
Use Gaussian elimination to reduce the matrix to reduced row echelon form.

$$\begin{bmatrix} 0 & 4 & 1 & 2 \\ 2 & 6 & -2 & 3 \\ 4 & 8 & -5 & 4 \end{bmatrix}$$

Problem 4.19
Find the rank of the matrix

$$\begin{bmatrix} -1 & 0 & -1 & 2 \\ 2 & 0 & 2 & 0 \\ 1 & 0 & 1 & -1 \end{bmatrix}$$

Problem 4.20
Find the rank of the matrix

$$\begin{bmatrix} 1 & 2 & -1 & 3 & 0 & 1 \\ -1 & -2 & 2 & -2 & -1 & 1 \\ 1 & 2 & 0 & 4 & 0 & 6 \\ 0 & 0 & 2 & 2 & -1 & 7 \end{bmatrix}$$

5 Echelon Forms and Solutions to Linear Systems

In this section we give a systematic procedure for solving systems of linear equations; it is based on the idea of reducing the augmented matrix to either the row-echelon form or the reduced row-echelon form. The new system is equivalent to the original system.
Unknowns corresponding to leading entries in the echelon augmented matrix are called **dependent** or **leading variables**. If an unknown is not dependent then it is called **free** or **independent** variable.

Example 5.1
Find the dependent and independent variables of the following system

$$\left\{\begin{array}{rrrrrrl} x_1 + 3x_2 & - 2x_3 & & + 2x_5 & & = & 0 \\ 2x_1 + 6x_2 & - 5x_3 & - \ 2x_4 & + 4x_5 & - \ 3x_6 & = & -1 \\ & 5x_3 & + 10x_4 & & + 15x_6 & = & 5 \\ 2x_1 + 6x_2 & & + \ 8x_4 & + 4x_5 & + 18x_6 & = & 6. \end{array}\right.$$

Solution.
The augmented matrix for the system is

$$\left[\begin{array}{rrrrrr|r} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{array}\right]$$

Using the Gaussian algorithm we bring the augmented matrix to row-echelon form as follows:

Step 1: $r_2 \leftarrow r_2 - 2r_1$ and $r_4 \leftarrow r_4 - 2r_1$

$$\left[\begin{array}{rrrrrr|r} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{array}\right]$$

Step 2: $r_2 \leftarrow -r_2$

$$\left[\begin{array}{rrrrrr|r} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{array}\right]$$

Step 3: $r_3 \leftarrow r_3 - 5r_2$ and $r_4 \leftarrow r_4 - 4r_2$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \end{array}\right]$$

Step 4: $r_3 \leftrightarrow r_4$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Step 5: $r_3 \leftarrow \frac{1}{6}r_3$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The leading variables are x_1, x_3, and x_6. The free variables are x_2, x_4, and x_5 ■

One way to solve a linear system is to apply the elementary row operations to reduce the augmented matrix to a (reduced) row-echelon form. If the augmented matrix is in reduced row-echelon form then to obtain the general solution one just has to move all independent variables to the right side of the equations and consider them as parameters. The dependent variables are given in terms of these parameters.

Example 5.2
Solve the following linear system.

$$\begin{cases} x_1 + 2x_2 \quad + \quad x_4 \quad = 6 \\ \qquad\qquad x_3 + 6x_4 \quad = 7 \\ \qquad\qquad\qquad\qquad x_5 = 1. \end{cases}$$

Solution.
The augmented matrix is already in row-echelon form. The free variables are x_2 and x_4. So let $x_2 = s$ and $x_4 = t$. Solving the system starting from the bottom we find $x_1 = -2s - t + 6, x_3 = 7 - 6t$, and $x_5 = 1$ ■

If the augmented matrix does not have the reduced row-echelon form but the row-echelon form then the general solution also can be easily found by using the method of backward substitution.

Example 5.3
Solve the following linear system

$$\begin{cases} x_1 - 3x_2 + \ \ x_3 - x_4 = 2 \\ \qquad\ \ x_2 + 2x_3 - x_4 = 3 \\ \qquad\qquad\quad\ x_3 + x_4 = 1. \end{cases}$$

Solution.
The augmented matrix is in row-echelon form. The free variable is $x_4 = t$. Solving for the leading variables we find, $x_1 = 11t + 4, x_2 = 3t + 1$, and $x_3 = 1 - t$ ■

The questions of existence and uniqueness of solutions are fundamental questions in linear algebra. The following theorem provides some relevant information.

Theorem 5.1
A system of m linear equations in n unknowns can have exactly one solution, infinitely many solutions, or no solutions at all. If the augmented matrix in the (reduced) row echelon form
(1) has a row of the form $[0, 0, \cdots, 0, b]$ where b is a nonzero constant, then the system has no solutions;
(2) has independent variables and no rows of the form $[0, 0, \cdots, 0, b]$ with $b \neq 0$ then the system has infinitely many solutions;
(3) no independent variables and no rows of the form $[0, 0, \cdots, 0, b]$ with $b \neq 0$, then the system has exactly one solution.

Example 5.4
Find the general solution of the system whose augmented matrix is given by

$$\left[\begin{array}{rr|r} 1 & 2 & -7 \\ -1 & -1 & 1 \\ 2 & 1 & 5 \end{array}\right].$$

Solution.
We first reduce the system to row-echelon form as follows.

Step 1: $r_2 \leftarrow r_2 + r_1$ and $r_3 \leftarrow r_3 - 2r_1$

$$\left[\begin{array}{cc|c} 1 & 2 & -7 \\ 0 & 1 & -6 \\ 0 & -3 & 19 \end{array}\right]$$

Step 2: $r_3 \leftarrow r_3 + 3r_2$

$$\left[\begin{array}{cc|c} 1 & 2 & -7 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{array}\right]$$

The corresponding system is given by

$$\begin{cases} x_1 + 2x_2 = -7 \\ \quad\quad\ \ x_2 = -6 \\ 0x_1 + 0x_2 = 1 \end{cases}$$

Because of the last equation the system is inconsistent ■

Example 5.5
Find the general solution of the system whose augmented matrix is given by

$$\left[\begin{array}{ccccc|c} 1 & -2 & 0 & 0 & 7 & -3 \\ 0 & 1 & 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & 1 & 5 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Solution.
By adding two times the second row to the first row we find the reduced row-echelon form of the augmented matrix.

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & 1 & 5 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

It follows that the free variables are $x_3 = s$ and $x_5 = t$. Solving for the leading variables we find $x_1 = -1 - t, x_2 = 1 + 3t$, and $x_4 = -4 - 5t$ ■

Example 5.6
Determine the value(s) of h such that the following matrix is the augmented matrix of a consistent linear system

$$\left[\begin{array}{rr|r} 1 & 4 & 2 \\ -3 & h & -1 \end{array}\right].$$

Solution.
By adding three times the first row to the second row we find

$$\left[\begin{array}{cc|c} 1 & 4 & 2 \\ 0 & 12+h & 5 \end{array}\right]$$

The system is consistent if and only if $12 + h \neq 0$; that is, $h \neq -12$ ■

Example 5.7
Find (if possible) conditions on the numbers a, b, and c such that the following system is consistent

$$\begin{cases} x_1 + 3x_2 + x_3 = a \\ -x_1 - 2x_2 + x_3 = b \\ 3x_1 + 7x_2 - x_3 = c. \end{cases}$$

Solution.
The augmented matrix of the system is

$$\left[\begin{array}{rrr|c} 1 & 3 & 1 & a \\ -1 & -2 & 1 & b \\ 3 & 7 & -1 & c \end{array}\right]$$

Now apply Gaussian elimination as follows.

Step 1: $r_2 \leftarrow r_2 + r_1$ and $r_3 \leftarrow r_3 - 3r_1$

$$\left[\begin{array}{rrr|c} 1 & 3 & 1 & a \\ 0 & 1 & 2 & b+a \\ 0 & -2 & -4 & c-3a \end{array}\right]$$

Step 2: $r_3 \leftarrow r_3 + 2r_2$

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & a \\ 0 & 1 & 2 & b+a \\ 0 & 0 & 0 & c-a+2b \end{array}\right]$$

The system has no solution if $c - a + 2b \neq 0$. The system has infinitely many solutions if $c - a + 2b = 0$. In this case, the solution is given by $x_1 = 5t - (2a + 3b), x_2 = (a + b) - 2t, x_3 = t$ ■

Practice Problems

Problem 5.1
Using Gaussian elimination, solve the linear system whose augmented matrix is given by

$$\left[\begin{array}{rrr|r} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{array}\right].$$

Problem 5.2
Solve the linear system whose augmented matrix is reduced to the following reduced row-echelon form

$$\left[\begin{array}{rrrr|r} 1 & 0 & 0 & -7 & 8 \\ 0 & 1 & 0 & 3 & 2 \\ 0 & 0 & 1 & 1 & -5 \end{array}\right].$$

Problem 5.3
Solve the linear system whose augmented matrix is reduced to the following row-echelon form

$$\left[\begin{array}{rrr|r} 1 & -3 & 7 & 1 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right].$$

Problem 5.4
Solve the following system using Gauss-Jordan elimination.

$$\begin{cases} 3x_1 + x_2 + 7x_3 + 2x_4 = 13 \\ 2x_1 - 4x_2 + 14x_3 - x_4 = -10 \\ 5x_1 + 11x_2 - 7x_3 + 8x_4 = 59 \\ 2x_1 + 5x_2 - 4x_3 - 3x_4 = 39. \end{cases}$$

Problem 5.5
Solve the following system.

$$\begin{cases} 2x_1 + x_2 + x_3 = -1 \\ x_1 + 2x_2 + x_3 = 0 \\ 3x_1 - 2x_3 = 5. \end{cases}$$

Problem 5.6
Solve the following system using elementary row operations on the augmented matrix:

$$\begin{cases} 5x_1 - 5x_2 - 15x_3 = 40 \\ 4x_1 - 2x_2 - 6x_3 = 19 \\ 3x_1 - 6x_2 - 17x_3 = 41. \end{cases}$$

Problem 5.7
Reduce the following system to row echelon form and then find the solution.

$$\begin{cases} 2x_1 + x_2 - x_3 + 2x_4 = 5 \\ 4x_1 + 5x_2 - 3x_3 + 6x_4 = 9 \\ -2x_1 + 5x_2 - 2x_3 + 6x_4 = 4 \\ 4x_1 + 11x_2 - 4x_3 + 8x_4 = 2. \end{cases}$$

Problem 5.8
Reduce the following system to row echelon form and then find the solution.

$$\begin{cases} 2x_1 - 5x_2 + 3x_3 = -4 \\ x_1 - 2x_2 - 3x_3 = 3 \\ -3x_1 + 4x_2 + 2x_3 = -4. \end{cases}$$

Problem 5.9
Reduce the following system to reduced row echelon form and then find the solution.

$$\begin{cases} 2x_1 + 4x_2 + 2x_3 + 4x_4 + 2x_5 = 4 \\ 2x_1 + 4x_2 + 3x_3 + 3x_4 + 3x_5 = 4 \\ 3x_1 + 6x_2 + 6x_3 + 3x_4 + 6x_5 = 6 \\ x_3 - x_4 - x_5 = 4. \end{cases}$$

Problem 5.10
Using the Gauss-Jordan elimination method, solve the following linear system.

$$\begin{cases} x_1 + 2x_2 + 3x_3 + 4x_4 + 3x_5 = 1 \\ 2x_1 + 4x_2 + 6x_3 + 2x_4 + 6x_5 = 2 \\ 3x_1 + 6x_2 + 18x_3 + 9x_4 + 9x_5 = -6 \\ 4x_1 + 8x_2 + 12x_3 + 10x_4 + 12x_5 = 4 \\ 5x_1 + 10x_2 + 24x_3 + 11x_4 + 15x_5 = -4. \end{cases}$$

Problem 5.11
Using the Gauss-Jordan elimination method, solve the following linear system.

$$\begin{cases} x_1 + x_2 + x_3 = 5 \\ 2x_1 + 3x_2 + 5x_3 = 8 \\ 4x_1 + 5x_3 = 2. \end{cases}$$

Problem 5.12
Using the Gauss-Jordan elimination method, solve the following linear system.

$$\begin{cases} x_1 + 2x_2 - 3x_3 = 2 \\ 6x_1 + 3x_2 - 9x_3 = 6 \\ 7x_1 + 14x_2 - 21x_3 = 13. \end{cases}$$

Problem 5.13
Using the Gauss-Jordan elimination method, solve the following linear system.

$$\begin{cases} 4x_1 + 8x_2 - 5x_3 = 4 \\ 2x_1 + 6x_2 - 2x_3 = 3 \\ 4x_2 + x_3 = 2. \end{cases}$$

Problem 5.14
Using the Gauss-Jordan elimination method, solve the following linear system.

$$\begin{cases} x_1 - 2x_2 + x_3 - 3x_4 = 0 \\ 3x_1 - 6x_2 + 2x_3 - 7x_4 = 0. \end{cases}$$

Problem 5.15
Using the Gauss-Jordan elimination method, solve the following linear system.

$$\begin{cases} -x_1 + 3x_2 + x_3 = 16 \\ x_1 - 5x_2 + 3x_3 = 2 \\ -2x_1 + 5x_2 - 3x_3 = 6. \end{cases}$$

Problem 5.16
Using the Gauss-Jordan elimination method, solve the following linear sys-

tem.

$$\begin{cases} x_1 + \ \ x_2 + \ \ x_3 = 5 \\ 2x_1 + 3x_2 + 5x_3 = 8 \\ 4x_1 \qquad\quad + 5x_3 = 2. \end{cases}$$

Problem 5.17
Using the Gauss-Jordan elimination method, solve the following linear system.

$$\begin{cases} x_1 + \ \ 2x_2 - \ \ 3x_3 = 2 \\ 6x_1 + \ \ 3x_2 - \ \ 9x_3 = 6 \\ 7x_1 + 14x_2 - 21x_3 = 13. \end{cases}$$

Problem 5.18
Using the Gauss-Jordan elimination method, solve the following linear system.

$$\begin{cases} \qquad\quad 4x_2 + \ \ x_3 = 2 \\ 2x_1 + 6x_2 - 2x_3 = 3 \\ 4x_1 + 8x_2 - 5x_3 = 4. \end{cases}$$

Problem 5.19
Using the Gaussian elimination method, solve the following linear system.

$$\begin{cases} -x_1 - \ \ x_3 = 2 \\ \ 2x_1 - 2x_3 = 0 \\ \ \ x_1 - \ \ x_3 = -1. \end{cases}$$

Problem 5.20
Using the Gaussian elimination method, solve the following linear system.

$$\begin{cases} \ \ 2x_1 + \ \ 4x_2 + \ \ 4x_3 + \ \ 2x_4 = 16 \\ \ \ 4x_1 + \ \ 8x_2 + \ \ 6x_3 + \ \ 8x_4 = 32 \\ 14x_1 + 29x_2 + 32x_3 + 16x_4 = 112 \\ 10x_1 + 17x_2 + 10x_3 + \ \ 2x_4 = 28. \end{cases}$$

Problem 5.21
Consider a system of m linear equations in n unknowns such that the augmented matrix in the (reduced) row echelon form has a row of the form $[0, 0, \cdots, 0, b]$ where b is a nonzero constant. Show that the system has no solutions.

Problem 5.22
Consider a system of m linear equations in n unknowns such that the augmented matrix in the (reduced) row echelon form has independent variables and no rows of the form $[0, 0, \cdots, 0, b]$ with $b \neq 0$. Show that the system has infinitely many solutions.

Problem 5.23
Consider a system of m linear equations in n unknowns such that the augmented matrix in the (reduced) row echelon form has no independent variables and no rows of the form $[0, 0, \cdots, 0, b]$ with $b \neq 0$. Show that the system has exactly one solution.

6 Homogeneous Systems of Linear Equations

A **homogeneous** linear system is any system of the form

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0 \\ \vdots = \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0. \end{cases}$$

Every homogeneous system is consistent, since $x_1 = 0, x_2 = 0, \cdots, x_n = 0$ is always a solution. This solution is called the **trivial solution**; any other solution is called **non-trivial.**
A homogeneous system has either a unique solution (the trivial solution) or infinitely many solutions. The following theorem provides a criterion where a homogeneous system is assured to have a non-trivial solution (and therefore infinitely many solutions).

Theorem 6.1
A homogeneous system in n unknowns and m equations has infinitely many solutions if either
(1) the rank of the row echelon form of the augmented matrix is less than n; or
(2) the number of unknowns exceeds the number of equations, i.e. $m < n$. That is, the system is underdetermined.

Proof.
Applying the Gauss-Jordan elimination to the augmented matrix $[A|0]$ we obtain the matrix $[B|0]$. The number of nonzero rows of B is equals to $rank(A)$.
(1) Suppose first that $rank(A) = r < n$. In this case, the system $Bx = 0$ has r equations in n unknowns. Thus, the system has $n - r$ independent variables and consequently the system $Bx = 0$ has a nontrivial solution. By Theorem 2.1, the system $Ax = 0$ has a nontrivial solution.
(2) Suppose $m < n$. If the system has only the trivial solution then by (1) we must have $rank(A) \geq n$. This implies that $m < n \leq rank(A) \leq m$, a contradiction ■

Example 6.1
Solve the following homogeneous system using Gauss-Jordan elimination.

$$\begin{cases} 2x_1 + 2x_2 - x_3 + x_5 = 0 \\ -x_1 - x_2 + 2x_3 - 3x_4 + x_5 = 0 \\ x_1 + x_2 - 2x_3 - x_5 = 0 \\ x_3 + x_4 + x_5 = 0. \end{cases}$$

Solution.
The reduction of the augmented matrix to reduced row-echelon form is outlined below.

$$\left[\begin{array}{rrrrr|r} 2 & 2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & -3 & 1 & 0 \\ 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right]$$

Step 1: $r_3 \leftarrow r_3 + r_2$

$$\left[\begin{array}{rrrrr|r} 2 & 2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & -3 & 1 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right]$$

Step 2: $r_3 \leftrightarrow r_4$ and $r_1 \leftrightarrow r_2$

$$\left[\begin{array}{rrrrr|r} -1 & -1 & 2 & -3 & 1 & 0 \\ 2 & 2 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \end{array}\right]$$

Step 3: $r_2 \leftarrow r_2 + 2r_1$ and $r_4 \leftarrow -\frac{1}{3}r_4$

$$\left[\begin{array}{rrrrr|r} -1 & -1 & 2 & -3 & 1 & 0 \\ 0 & 0 & 3 & -6 & 3 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right]$$

Step 4: $r_1 \leftarrow -r_1$ and $r_2 \leftarrow \frac{1}{3}r_2$

$$\left[\begin{array}{rrrrr|r} 1 & 1 & -2 & 3 & -1 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right]$$

Step 5: $r_3 \leftarrow r_3 - r_2$

$$\left[\begin{array}{ccccc|c} 1 & 1 & -2 & 3 & -1 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right]$$

Step 6: $r_4 \leftarrow r_4 - \frac{1}{3}r_3$ and $r_3 \leftarrow \frac{1}{3}r_3$

$$\left[\begin{array}{ccccc|c} 1 & 1 & -2 & 3 & -1 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Step 7: $r_1 \leftarrow r_1 - 3r_3$ and $r_2 \leftarrow r_2 + 2r_3$

$$\left[\begin{array}{ccccc|c} 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Step 8: $r_1 \leftarrow r_1 + 2r_2$

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

The corresponding system is

$$\begin{cases} x_1 + x_2 \qquad + x_5 = 0 \\ \qquad\quad x_3 \quad + x_5 = 0 \\ \qquad\qquad x_4 \qquad = 0. \end{cases}$$

The free variables are $x_2 = s, x_5 = t$ and the general solution is given by the formula: $x_1 = -s - t, x_2 = s, x_3 = -t, x_4 = 0, x_5 = t$ ■

Example 6.2
Solve the following homogeneous system using Gaussian elimination.

$$\begin{cases} x_1 + 3x_2 + 5x_3 + \ \ x_4 = 0 \\ 4x_1 - 7x_2 - 3x_3 - \ \ x_4 = 0 \\ 3x_1 + 2x_2 + 7x_3 + 8x_4 = 0. \end{cases}$$

Solution.
The augmented matrix for the system is

$$\left[\begin{array}{cccc|c} 1 & 3 & 5 & 1 & 0 \\ 4 & -7 & -3 & -1 & 0 \\ 3 & 2 & 7 & 8 & 0 \end{array}\right]$$

We reduce this matrix into a row-echelon form as follows.
Step 1: $r_2 \leftarrow r_2 - r_3$

$$\left[\begin{array}{cccc|c} 1 & 3 & 5 & 1 & 0 \\ 1 & -9 & -10 & -9 & 0 \\ 3 & 2 & 7 & 8 & 0 \end{array}\right]$$

Step 2: $r_2 \leftarrow r_2 - r_1$ and $r_3 \leftarrow r_3 - 3r_1$

$$\left[\begin{array}{cccc|c} 1 & 3 & 5 & 1 & 0 \\ 0 & -12 & -15 & -10 & 0 \\ 0 & -7 & -8 & 5 & 0 \end{array}\right]$$

Step 3: $r_2 \leftarrow -\frac{1}{12}r_2$

$$\left[\begin{array}{cccc|c} 1 & 3 & 5 & 1 & 0 \\ 0 & 1 & \frac{5}{4} & \frac{5}{6} & 0 \\ 0 & -7 & -8 & 5 & 0 \end{array}\right]$$

Step 4: $r_3 \leftarrow r_3 + 7r_2$

$$\left[\begin{array}{cccc|c} 1 & 3 & 5 & 1 & 0 \\ 0 & 1 & \frac{5}{4} & \frac{5}{6} & 0 \\ 0 & 0 & \frac{3}{4} & \frac{65}{6} & 0 \end{array}\right]$$

Step 5: $r_3 \leftarrow \frac{4}{3}r_3$

$$\left[\begin{array}{cccc|c} 1 & 3 & 5 & 1 & 0 \\ 0 & 1 & \frac{5}{4} & \frac{5}{6} & 0 \\ 0 & 0 & 1 & \frac{130}{9} & 0 \end{array}\right]$$

We see that $x_4 = t$ is the only free variable. Solving for the leading variables using back substitution we find $x_1 = \frac{176}{9}t, x_2 = \frac{155}{9}t$, and $x_3 = -\frac{130}{9}t$ ■

Remark 6.1
Part (2) of Theorem 6.1 applies only to homogeneous linear systems. A non-homogeneous system (right-hand side has non-zero entries) with more unknowns than equations need not be consistent as shown in the next example.

Example 6.3
Show that the following system is inconsistent.

$$\begin{cases} x_1 + \ \ x_2 + \ \ x_3 = 0 \\ 2x_1 + 2x_2 + 2x_3 = 4. \end{cases}$$

Solution.
Multiplying the first equation by -2 and adding the resulting equation to the second we obtain $0 = 4$ which is impossible. So the system is inconsistent ■

Practice Problems

Problem 6.1
Justify the following facts:
(a) All homogeneous linear systems are consistent.
(b) A homogeneous linear system with fewer equations than unknowns has infinitely many solutions.

Problem 6.2
Find the value(s) of a for which the following system has a non-trivial solution. Find the general solution.

$$\begin{cases} x_1 + 2x_2 + x_3 = 0 \\ x_1 + 3x_2 + 6x_3 = 0 \\ 2x_1 + 3x_2 + ax_3 = 0. \end{cases}$$

Problem 6.3
Solve the following homogeneous system.

$$\begin{cases} x_1 - x_2 + 2x_3 + x_4 = 0 \\ 2x_1 + 2x_2 - x_4 = 0 \\ 3x_1 + x_2 + 2x_3 + x_4 = 0. \end{cases}$$

Problem 6.4
Solve the homogeneous linear system.

$$\begin{cases} x_1 + x_2 - 2x_3 = 0 \\ 3x_1 + 2x_2 + 4x_3 = 0 \\ 4x_1 + 3x_2 + 3x_3 = 0. \end{cases}$$

Problem 6.5
Solve the homogeneous linear system.

$$\begin{cases} x_1 + x_2 - 2x_3 = 0 \\ 3x_1 + 2x_2 + 4x_3 = 0 \\ 4x_1 + 3x_2 + 2x_3 = 0. \end{cases}$$

Problem 6.6
Solve the homogeneous linear system.

$$\begin{cases} 2x_1 + 4x_2 - \ \ 6x_3 = 0 \\ 4x_1 + 8x_2 - 12x_3 = 0. \end{cases}$$

Problem 6.7
Solve the homogeneous linear system.

$$\begin{cases} x_1 + x_2 \qquad\quad + 3x_4 = 0 \\ 2x_1 + x_2 - x_3 + \ \ x_4 = 0 \\ 3x_1 - x_2 - x_3 + 2x_4 = 0. \end{cases}$$

Problem 6.8
Solve the homogeneous linear system.

$$\begin{cases} x_1 + \ \ x_2 \qquad\qquad - \ \ x_4 = 0 \\ -2x_1 - 3x_2 + 4x_3 + 5x_4 = 0 \\ 2x_1 + 4x_2 \qquad\qquad - 2x_4 = 0. \end{cases}$$

Problem 6.9
Solve the homogeneous system.

$$\begin{cases} 3x_1 + 2x_2 + x_3 = 0 \\ 6x_1 + 4x_2 \qquad\ = 0 \\ \qquad\qquad x_2 + x_3 = 0. \end{cases}$$

Problem 6.10
Use Gaussian elimination to solve the following homogeneous system of equations.

$$\begin{cases} x_1 - \ \ x_2 - \ \ x_3 + 3x_4 = 0 \\ x_1 + \ \ x_2 - 2x_3 + \ \ x_4 = 0 \\ 4x_1 - 2x_2 + 4x_3 + \ \ x_4 = 0. \end{cases}$$

Problem 6.11
Use Gaussian elimination to solve the following homogeneous system of equations.

$$\begin{cases} -x_1 + x_2 - \ \ x_3 = 0 \\ 3x_1 - x_2 - \ \ x_3 = 0 \\ 2x_1 + x_2 - 3x_3 = 0. \end{cases}$$

Problem 6.12
Use Gaussian elimination to solve the following homogeneous system of equations.

$$\begin{cases} x_1 - x_2 - x_3 + x_4 = 0 \\ 2x_1 - 2x_2 + x_3 + x_4 = 0 \\ 5x_1 - 5x_2 - 2x_3 + 4x_4 = 0. \end{cases}$$

Problem 6.13
Without solving the system, justify the reason why the system has infinitely many solutions.

$$\begin{cases} x_1 - x_2 + 3x_3 = 0 \\ 2x_1 + x_2 + 3x_3 = 0. \end{cases}$$

Problem 6.14
By just using the rank of the augmented matrix, show that the following system has infinitely many solutions.

$$\begin{cases} -x_1 - x_2 = 0 \\ 4x_1 - 2x_3 + x_4 = 0 \\ -4x_2 - 2x_3 + x_4 = 0 \\ 3x_1 - x_2 + 4x_4 = 0. \end{cases}$$

Problem 6.15
Show that the following two systems are equivalent:

$$\begin{cases} -x_1 + x_2 + 4x_3 = 0 \\ x_1 + 3x_2 + 8x_3 = 0 \\ x_1 + 2x_2 + 5x_3 = 0 \end{cases}$$

and

$$\begin{cases} x_1 - x_3 = 0 \\ x_2 + 3x_3 = 0. \end{cases}$$

Problem 6.16
True or false. Justify your answer.
(a) All homogeneous linear systems are consistent.
(b) Linear systems with more unknowns than equations are always consistent.
(c) A homogeneous system with fewer equations than unknowns has infinitely many solutions.

Problem 6.17
Solve the system

$$\begin{cases} x_1 + 2x_2 - x_3 + 3x_4 = 0 \\ -x_1 - 2x_2 + 2x_3 - 2x_4 - x_5 = 0 \\ x_1 + 2x_2 + 4x_4 = 0 \\ 2x_3 + 2x_4 - x_5 = 0. \end{cases}$$

Problem 6.18
Show that if $(x_1, x_2, \cdots, x_n)$ and $(y_1, y_2, \cdots, y_n)$ are two solutions to a $m \times n$ homogeneous linear system then $\alpha(x_1, x_2, \cdots, x_n) + \beta(y_1, y_2, \cdots, y_n) = (\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2, \cdots, \alpha x_n + \beta y_n)$ is also a solution.

Problem 6.19
Show with an example that the result of Problem 6.19 fails for linear non-homogeneous systems.

Problem 6.20
If the Gauss-Jordan form of the augmented matrix of a homogeneous system has a row of zeros, are there necessarily any free variables? If there are free variables, is there necessarily a row of zeros?

Matrices

Matrices are essential in the study of linear algebra. The concept of matrices has become a tool in all branches of mathematics, the sciences, and engineering. They arise in many contexts other than as augmented matrices for systems of linear equations. In this chapter we shall consider this concept as objects in their own right and develop their properties for use in our later discussions.

7 Matrices and Matrix Operations

In this section, we discuss the operations on matrices−equality, addition, subtraction, scalar multiplication, trace, and the transpose operation and give their basic properties. Also, we introduce some special types of matrices: symmetric and skew-symmetric matrices.

A **matrix A of size** $m \times n$ is a rectangular array of the form

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

where the a_{ij}'s are the **entries** of the matrix, m is the number of rows, n is the number of columns. The **zero matrix 0** is the matrix whose entries are all 0. The $n \times n$ **identity matrix** I_n is a square matrix whose main diagonal consists of $1's$ and the off diagonal entries are all 0. A matrix A can be represented with the following compact notation $A = [a_{ij}]$. The i^{th} **row** of the matrix A is

$$[a_{i1}, a_{i2}, ..., a_{in}]$$

and the j^{th} **column** is

$$\begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}.$$

In what follows we discuss the basic arithmetic of matrices.

Two matrices are said to be **equal** if they have the same size and their corresponding entries are all equal. If the matrix A is not equal to the matrix B we write $A \neq B$.

Example 7.1
Find x_1, x_2 and x_3 such that

$$\begin{bmatrix} x_1 + x_2 + 2x_3 & 0 & 1 \\ 2 & 3 & 2x_1 + 4x_2 - 3x_3 \\ 4 & 3x_1 + 6x_2 - 5x_3 & 5 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 1 \\ 2 & 3 & 1 \\ 4 & 0 & 5 \end{bmatrix}.$$

Solution.
Because corresponding entries must be equal, this gives the following linear system

$$\begin{cases} x_1 + \ \ x_2 + 2x_3 = 9 \\ 2x_1 + 4x_2 - 3x_3 = 1 \\ 3x_1 + 6x_2 - 5x_3 = 0 \end{cases}$$

The augmented matrix of the system is

$$\left[\begin{array}{ccr|c} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{array}\right].$$

The reduction of this matrix to row-echelon form is

Step 1: $r_2 \leftarrow r_2 - 2r_1$ and $r_3 \leftarrow r_3 - 3r_1$

$$\left[\begin{array}{ccr|r} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{array}\right]$$

Step 2: $r_2 \leftrightarrow r_3$

$$\left[\begin{array}{ccr|r} 1 & 1 & 2 & 9 \\ 0 & 3 & -11 & -27 \\ 0 & 2 & -7 & -17 \end{array}\right]$$

Step 3: $r_2 \leftarrow r_2 - r_3$

$$\left[\begin{array}{ccr|r} 1 & 1 & 2 & 9 \\ 0 & 1 & -4 & -10 \\ 0 & 2 & -7 & -17 \end{array}\right]$$

Step 4: $r_3 \leftarrow r_3 - 2r_2$

$$\left[\begin{array}{ccr|r} 1 & 1 & 2 & 9 \\ 0 & 1 & -4 & -10 \\ 0 & 0 & 1 & 3 \end{array}\right].$$

The corresponding system is

$$\begin{cases} x_1 + x_2 + 2x_3 = 9 \\ \qquad x_2 - 4x_3 = -10 \\ \qquad\qquad x_3 = 3. \end{cases}$$

Using backward substitution we find: $x_1 = 1, x_2 = 2, x_3 = 3$ ■

Example 7.2
Solve the following matrix equation for a, b, c, and d

$$\begin{bmatrix} a-b & b+c \\ 3d+c & 2a-4d \end{bmatrix} = \begin{bmatrix} 8 & 1 \\ 7 & 6 \end{bmatrix}$$

Solution.
Equating corresponding entries we get the system

$$\begin{cases} a-b & & & = 8 \\ & b+c & & = 1 \\ & & c+3d & = 7 \\ 2a & & -4d & = 6. \end{cases}$$

The augmented matrix is

$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 8 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 & 7 \\ 2 & 0 & 0 & -4 & 6 \end{array}\right].$$

We next apply Gaussian elimination as follows.

Step 1: $r_4 \leftarrow r_4 - 2r_1$

$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 8 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 & 7 \\ 0 & 2 & 0 & -4 & -10 \end{array}\right]$$

Step 2: $r_4 \leftarrow r_4 - 2r_2$

$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 8 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 & 7 \\ 0 & 0 & -2 & -4 & -12 \end{array}\right]$$

Step 3: $r_4 \leftarrow r_4 + 2r_3$

$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 8 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 & 7 \\ 0 & 0 & 0 & 2 & 2 \end{array}\right].$$

Step 4: $r_4 \leftarrow \frac{1}{2}r_4$

$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 8 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 & 7 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right].$$

The equivalent system to the given one is

$$\begin{cases} a - b & = 8 \\ \quad b + c & = 1 \\ \qquad c + 3d & = 7 \\ \qquad\quad d & = 1. \end{cases}$$

Using backward substitution to find: $a = 5, b = -3, c = 4, d = 1$ ■

Next, we introduce the operation of addition of two matrices. If A and B are two matrices of the same size, then the **sum** $A + B$ is the matrix obtained by adding together the corresponding entries in the two matrices. Matrices of different sizes cannot be added.

Example 7.3
Consider the matrices

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}, C = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \end{bmatrix}.$$

Compute, if possible, $A + B$, $A + C$ and $B + C$.

Solution.
We have

$$A + B = \begin{bmatrix} 4 & 2 \\ 6 & 9 \end{bmatrix}.$$

$A + C$ and $B + C$ are undefined since A and C are of different sizes as well as B and C ■

From now on, a constant number will be called a **scalar.** The **scalar product** of a real number c and a matrix $A = [a_{ij}]$ is the matrix $cA = [ca_{ij}]$. That is, the product cA is the matrix obtained by multiplying each entry of A by c. Hence, $-A = (-1)A$. We define, $A - B = A + (-B)$.

Example 7.4
Consider the matrices

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 & 2 & 7 \\ 1 & -3 & 5 \end{bmatrix}.$$

Compute $A - 3B$.

Solution.
Using the above definitions we have

$$A - 3B = \begin{bmatrix} 2 & -3 & -17 \\ -2 & 11 & -14 \end{bmatrix} \blacksquare$$

The following theorem list the properties of matrix addition and multiplication of a matrix by a scalar.

Theorem 7.1
Let A, B, and C be $m \times n$ and let c, d be scalars. Then
(i) $A + B = B + A$,
(ii) $(A + B) + C = A + (B + C) = A + B + C$,
(iii) $A + \mathbf{0} = \mathbf{0} + A = A$,
(iv) $A + (-A) = \mathbf{0}$,
(v) $c(A + B) = cA + cB$,
(vi) $(c + d)A = cA + dA$,
(vii) $(cd)A = c(dA)$.

Example 7.5
Solve the following matrix equation.

$$\begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}.$$

Solution.
Adding and then equating corresponding entries we obtain $a = -2, b = -2, c = 0$, and $d = 1$ $\blacksquare$

If A is a square matrix then the sum of the entries on the main diagonal is called the **trace** of A and is denoted by $tr(A)$.

Example 7.6
Find the trace of the coefficient matrix of the system

$$\begin{cases} \quad\quad - \; x_2 + 3x_3 = 1 \\ x_1 \quad\quad\quad + 2x_3 = 2 \\ -3x_1 - 2x_2 \quad\quad = 4. \end{cases}$$

Solution.
If A is the coefficient matrix of the system then

$$A = \begin{bmatrix} 0 & -1 & 3 \\ 1 & 0 & 2 \\ -3 & -2 & 0 \end{bmatrix}.$$

The trace of A is the number $tr(A) = 0 + 0 + 0 = 0$ ■

Two useful properties of the trace of a matrix are given in the following theorem.

Theorem 7.2
Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two $n \times n$ matrices and c be a scalar. Then
(i) $tr(A + B) = tr(A) + tr(B)$,
(ii) $tr(cA) = c\ tr(A)$.

Proof.
(i) $tr(A + B) = \sum_{i=1}^n (a_{ii} + b_{ii}) = \sum_{i=1}^n a_{ii} + \sum_{i=1}^n b_{ii} = tr(A) + tr(B)$.
(ii) $tr(cA) = \sum_{i=1}^n ca_{ii} = c \sum_{i=1}^n a_{ii} = c\ tr(A)$ ■

If A is an $m \times n$ matrix then the **transpose** of A, denoted by A^T, is defined to be the $n \times m$ matrix obtained by interchanging the rows and columns of A, that is the first column of A^T is the first row of A, the second column of A^T is the second row of A, etc. Note that, if $A = [a_{ij}[$ then $A^T = [a_{ji}]$. Also, if A is a square matrix then the diagonal entries on both A and A^T are the same.

Example 7.7
Find the transpose of the matrix

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 1 \end{bmatrix}.$$

Solution.
The transpose of A is the matrix

$$A^T = \begin{bmatrix} 2 & 1 \\ 3 & 2 \\ 4 & 1 \end{bmatrix} \blacksquare$$

The following result lists some of the properties of the transpose of a matrix.

Theorem 7.3
Let $A = [a_{ij}]$, and $B = [b_{ij}]$ be two $m \times n$ matrices, $C = [c_{ij}]$ be an $n \times n$ matrix, and c a scalar. Then
(i) $(A^T)^T = A$,
(ii) $(A + B)^T = A^T + B^T$,
(iii) $(cA)^T = cA^T$,
(iv) $tr(C^T) = tr(C)$.

Proof.
(i) $(A^T)^T = [a_{ji}]^T = [a_{ij}] = A$.
(ii) $(A + B)^T = [a_{ij} + b_{ij}]^T = [a_{ji} + b_{ji}] = [a_{ji}] + [b_{ji}] = A^T + B^T$.
(iii) $(cA)^T = [ca_{ij}]^T = [ca_{ji}] = c[a_{ji}] = cA^T$.
(iv) $tr(C^T) = \sum_{i=1}^{n} c_{ii} = tr(C)$ $\blacksquare$

Example 7.8
A square matrix A is called **symmetric** if $A^T = A$. A square matrix A is called **skew-symmetric** if $A^T = -A$.
(a) Show that the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$$

is a symmetric matrix.
(b) Show that the matrix

$$A = \begin{bmatrix} 0 & 2 & 3 \\ -2 & 0 & -4 \\ -3 & 4 & 0 \end{bmatrix}$$

is a skew-symmetric matrix.
(c) Show that a matrix that is both symmetric and skew-symmetric is the

zero matrix.
(d) Show that for any square matrix A, the matrix $S = \frac{1}{2}(A + A^T)$ is symmetric and the matrix $K = \frac{1}{2}(A - A^T)$ is skew-symmetric.
(e) Show that if A is a square matrix, then $A = S + K$, where S is symmetric and K is skew-symmetric.
(f) Show that the representation in (d) is unique.

Solution.
(a) A is symmetric since

$$A^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix} = A.$$

(b) A is skew-symmetric since

$$A^T = \begin{bmatrix} 0 & -2 & -3 \\ 2 & 0 & 4 \\ 3 & -4 & 0 \end{bmatrix} = -A.$$

(c) Suppose C is a matrix that is both symmetric and skew-symmetric. Then $C^T = C$ and $C^T = -C$. Hence, $C = -C$ or $2C = 0$. Dividing by 2, we find $C = 0$.
(d) Because $S^T = \frac{1}{2}(A + A^T)^T = \frac{1}{2}(A + A^T) = S$ then S is symmetric. Similarly, $K^T = \frac{1}{2}(A - A^T)^T = \frac{1}{2}(A^T - A) = -\frac{1}{2}(A - A^T) = -K$ so that K is skew-symmetric.
(e) $S + K = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T) = A$.
(f) Let S' be a symmetric matrix and K' be skew-symmetric such that $A = S' + K'$. Then $S + K = S' + K'$ and this implies that $S - S' = K' - K$. Since the matrix $S - S'$ is symmetric and the matrix $K' - K$ is skew-symmetric, (c) implies $S - S'$ is the zero matrix. That is $S = S'$. Hence, $K = K'$ ■

Example 7.9
Let A be an $n \times n$ matrix.
(a) Show that if A is symmetric then A and A^T have the same main diagonal.
(b) Show that if A is skew-symmetric then the entries on the main diagonal are 0.
(c) If A and B are symmetric then so is $A + B$.

Solution.
(a) Let $A = [a_{ij}]$ be symmetric. Let $A^T = [b_{ij}]$. Then $b_{ij} = a_{ji}$ for all $1 \leq i, j \leq n$. In particular, when $i = j$ we have $b_{ii} = a_{ii}$. That is, A and A^T have the same main diagonal.
(b) Since A is skew-symmetric, we have $a_{ij} = -a_{ji}$. In particular, $a_{ii} = -a_{ii}$ and this implies that $a_{ii} = 0$.
(c) Suppose A and B are symmetric. Then $(A+B)^T = A^T + B^T = A + B$. That is, $A + B$ is symmetric ■

Example 7.10
Let A be an $m \times n$ matrix and α a real number. Show that if $\alpha A = \mathbf{0}$ then either $\alpha = 0$ or $A = \mathbf{0}$.

Solution.
Let $A = [a_{ij}]$. Then $\alpha A = [\alpha a_{ij}]$. Suppose $\alpha A = \mathbf{0}$. Then $\alpha a_{ij} = 0$ for all $0 \leq i \leq m$ and $0 \leq j \leq n$. If $\alpha \neq 0$ then $a_{ij} = 0$ for all indices i and j. In this case, $A = \mathbf{0}$ ■

Practice Problems

Problem 7.1
Compute the matrix

$$3\begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}^T - 2\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}.$$

Problem 7.2
Find w, x, y, and z.

$$\begin{bmatrix} 1 & 2 & w \\ 2 & x & 4 \\ y & -4 & z \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \\ 2 & -3 & 4 \\ 0 & -4 & 5 \end{bmatrix}.$$

Problem 7.3
Determine two numbers s and t such that the following matrix is symmetric.

$$A = \begin{bmatrix} 2 & s & t \\ 2s & 0 & s+t \\ 3 & 3 & t \end{bmatrix}.$$

Problem 7.4
Let A be the matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Show that

$$A = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Problem 7.5
Let $A = \begin{bmatrix} 1 & 1 & -1 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 & 2 \end{bmatrix}, C = \begin{bmatrix} 3 & 0 & 1 \end{bmatrix}$. Suppose $rA + sB + tC = \mathbf{0}$. Show that $s = r = t = 0$.

Problem 7.6
Compute

$$\begin{bmatrix} 1 & 9 & -2 \\ 3 & 6 & 0 \end{bmatrix} + \begin{bmatrix} 8 & -4 & 3 \\ -7 & 1 & 6 \end{bmatrix}.$$

Problem 7.7
Determine whether the matrix is symmetric or skew-symmetric.

$$A = \begin{bmatrix} 11 & 6 & 1 \\ 6 & 3 & -1 \\ 1 & -1 & -6 \end{bmatrix}.$$

Problem 7.8
Determine whether the matrix is symmetric or skew-symmetric.

$$A = \begin{bmatrix} 0 & 3 & -1 & -5 \\ -3 & 0 & 7 & -2 \\ 1 & -7 & 0 & 0 \\ 5 & 2 & 0 & 0 \end{bmatrix}.$$

Problem 7.9
Consider the matrix

$$A = \begin{bmatrix} 0 & 3 & -1 & -5 \\ -3 & 0 & 7 & -2 \\ 1 & -7 & 0 & 0 \\ 5 & 2 & 0 & 0 \end{bmatrix}.$$

Find $4tr(7A)$.

Problem 7.10
Consider the matrices

$$A = \begin{bmatrix} 11 & 6 & 1 \\ 6 & 3 & -1 \\ 1 & -1 & -6 \end{bmatrix}, \; B = \begin{bmatrix} 0 & 3 & -1 \\ -3 & 0 & 7 \\ 1 & -7 & 0 \end{bmatrix}.$$

Find $tr(A^T - 2B)$.

Problem 7.11
Write the matrix A =

$$\begin{bmatrix} 12 & 7 & 1 \\ -2 & -4 & 0 \\ 0 & -8 & 2 \end{bmatrix}$$

as a sum of a symmetric matrix S and a skew-symmetric matrix K.

Problem 7.12
Show that there is no square matrix A of dimension n such that $A - A^T = I_n$.

Problem 7.13
What value(s) of x satisfies the matrix equation

$$\begin{bmatrix} 0 & -x \\ 3x-2 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 7x+14 \\ x+8 & 5 \end{bmatrix}^T.$$

Problem 7.14
Let $A = \begin{bmatrix} 2 & -3 \\ 0 & 2 \\ 7 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 & 0 \\ 5 & 1 & 0 \end{bmatrix}$. Calculate $3(A^T + 2B)$.

Problem 7.15
Give an example of a 5×5 symmetric matrix with non-zero entries.

Problem 7.16
Solve the system

$$x_1 \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} -5 \\ -2 \\ 4 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \\ -4 \end{bmatrix}$$

Problem 7.17
Let A and B be two square matrices with trace 0. Show that $tr(\alpha A+\beta B) = 0$ where α and β are scalars.

Problem 7.18
Let f be a function defined on the set of all $n \times n$ matrices given by $f(A) = tr(A)$. Show that $f(\alpha A + \beta B) = \alpha f(A) + \beta f(B)$, where α and β are scalars.

Problem 7.19
Let f be a function defined on the set of all $m \times n$ matrices given by $f(A) = A^T$. Show that $f(\alpha A + \beta B) = \alpha f(A) + \beta f(B)$, where α and β are scalars.

Problem 7.20
Suppose that A and B are skew-symmetric. Show that $\alpha A + \beta B$ is also skew-symmetric, where α and β are scalars.

Problem 7.21
Prove Theorem 7.1.

8 Matrix Multiplication

In the previous section we discussed some basic properties associated with matrix addition and scalar multiplication. Here we introduce another important operation involving matrices−the product.
Let $A = [a_{ij}]$ be a matrix of size $m \times n$ and $B = (b_{ij})$ be a matrix of size $n \times p$. Then the **product** matrix is a matrix of size $m \times p$ and entries

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + ... + a_{in}b_{nj},$$

that is, $(AB)_{ij}$ is obtained by multiplying componentwise the entries of the i^{th} row of A by the entries of the j^{th} column of B. It is very important to keep in mind that the number of columns of the first matrix must be equal to the number of rows of the second matrix; otherwise the product is undefined.
An interesting question associated with matrix multiplication is the following: If A and B are square matrices then is it always true that $AB = BA$?
The answer to this question is negative. In general, matrix multiplication is not commutative, as the following example shows.

Example 8.1
Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}, B = \begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}.$$

Show that $AB \neq BA$. Hence, matrix multiplication is not commutative.

Solution.
Using the definition of matrix multiplication we find

$$AB = \begin{bmatrix} -4 & 7 \\ 0 & 5 \end{bmatrix}, BA = \begin{bmatrix} -1 & 2 \\ 9 & 2 \end{bmatrix}.$$

Hence, $AB \neq BA$ ■

Example 8.2
Consider the matrices

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}, C = \begin{bmatrix} -1 & -2 \\ 11 & 4 \end{bmatrix}.$$

(a) Compare $A(BC)$ and $(AB)C$.
(b) Compare $A(B + C)$ and $AB + AC$.
(c) Compute I_2A and AI_2, where I_2 is the 2×2 identity matrix.

Solution.
(a)

$$A(BC) = (AB)C = \begin{bmatrix} 70 & 14 \\ 235 & 56 \end{bmatrix}.$$

(b)

$$A(B+C) = AB + AC = \begin{bmatrix} 16 & 7 \\ 59 & 33 \end{bmatrix}.$$

(c) $AI_2 = I_2A = A$ ■

Example 8.3
Let A be a $m \times n$ and B be a $n \times p$ matrices. Show that if
(a) B has a column of zeros then the same is true for AB.
(b) A has a row of zeros then the same is true for AB.

Solution.
(a) Suppose the the j^{th} column of B is a column of zeros. That is, $b_{1j} = b_{2j} = \cdots = b_{nj} = 0$. Then for $1 \le i \le m$, we have

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj} = 0.$$

Hence, the j^{th} column of AB is a column of zeros.
(b) Suppose that i^{th} row of A is a row of zeros. That is, $a_{i1} = a_{i2} = \cdots = a_{in} = 0$. Then for $1 \le j \le p$, we have

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj} = 0.$$

That is, the i^{th} row of AB is a row of zeros ■

Example 8.4
Let A and B be two $n \times n$ matrices.
(a) Show that $tr(AB) = tr(BA)$.
(b) Show that $AB - BA = I_n$ is impossible.

Solution.
(a) Let $A = [a_{ij}]$ and $B = [b_{ij}]$. Then
$tr(AB) = \sum_{i=1}^{n}(\sum_{k=1}^{n} a_{ik}b_{ki}) = \sum_{k=1}^{n}(\sum_{i=1}^{n} b_{ki}a_{ik}) = tr(BA)$.

(b) If $AB - BA = I_n$ then $0 = tr(AB) - tr(BA) = tr(AB - BA) = tr(I_n) = n \geq 1$, a contradiction ■

Next, consider a system of linear equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots = \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m. \end{cases}$$

Then the matrix of the coefficients of the x_i's is called the **coefficient matrix**:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}.$$

The matrix of the coefficients of the x_i's and the right hand side coefficients is called the **augmented matrix**:

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right].$$

Now, if we let

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

and

$$b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

then the above system can be represented in **matrix notation** as

$$Ax = b.$$

Example 8.5
Consider the linear system

$$\begin{cases} x_1 - 2x_2 + \ \ x_3 = 0 \\ \qquad \ \ 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9. \end{cases}$$

(a) Find the coefficient and augmented matrices of the linear system.
(b) Find the matrix notation.

Solution.
(a) The coefficient matrix of this system is

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix}$$

and the augmented matrix is

$$\left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right].$$

(b) We can write the given system in matrix form as

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ -9 \end{bmatrix} \blacksquare$$

As the reader has noticed so far, most of the basic rules of arithmetic of real numbers also hold for matrices but a few do not. In Example 8.1 we have seen that matrix multiplication is not commutative. The following exercise shows that the cancellation law of numbers does not hold for matrix product.

Example 8.6
(a) Consider the matrices

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Compare AB and AC. Is it true that $B = C$?
(b) Find two square matrices A and B such that $AB = \mathbf{0}$ but $A \neq \mathbf{0}$ and $B \neq \mathbf{0}$.

Solution.
(a) Note that $B \neq C$ even though $AB = AC = \mathbf{0}$.
(b) The given matrices satisfy $AB = \mathbf{0}$ with $A \neq \mathbf{0}$ and $B \neq \mathbf{0}$ ■

Matrix multiplication shares many properties of the product of real numbers which are listed in the following theorem

Theorem 8.1
Let A be a matrix of size $m \times n$. Then
(a) $A(BC) = (AB)C$, where B is of size $n \times p$, C of size $p \times q$.
(b) $A(B + C) = AB + AC$, where B and C are of size $n \times p$.
(c) $(B + C)A = BA + CA$, where B and C are of size $l \times m$.
(d) $c(AB) = (cA)B = A(cB)$, where c denotes a scalar and B is of size $n \times p$.
(e) $I_m A = AI_n = A$.

The next theorem describes a property about the transpose of a matrix.

Theorem 8.2
Let $A = [a_{ij}], B = [b_{ij}]$ be matrices of sizes $m \times n$ and $n \times m$ respectively. Then $(AB)^T = B^T A^T$.

Example 8.7
Let A be any matrix. Show that AA^T and $A^T A$ are symmetric matrices.

Solution.
First note that for any matrix A the matrices AA^T and $A^T A$ are well-defined. Since $(AA^T)^T = (A^T)^T A^T = AA^T$, AA^T is symmetric. Similarly, $(A^T A)^T = A^T (A^T)^T = A^T A$ ■

Finally, we discuss the **powers** of a square matrix. Let A be a square matrix of size $n \times n$. Then the non-negative powers of A are defined as follows: $A^0 = I_n, A^1 = A$, and for a positive integer $k \geq 2$, $A^k = (A^{k-1})A$.

Example 8.8
Suppose that

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

Compute A^3.

Solution.
Multiplying the matrix A by itself three times we obtain

$$A^3 = \begin{bmatrix} 37 & 54 \\ 81 & 118 \end{bmatrix} \blacksquare$$

We end this section with the following result about matrix powers properties.

Theorem 8.3
For any non-negative integers s, t we have
(a) $A^{s+t} = A^s A^t$
(b) $(A^s)^t = A^{st}$.

Proof.
(a) Fix s. Let $t = 1$. Then by definition above we have $A^{s+1} = A^s A$. Now, we prove by induction on t that $A^{s+t} = A^s A^t$. The equality holds for $t = 1$. As the induction hypothesis, suppose that $A^{s+t} = A^s A^t$. Then $A^{s+(t+1)} = A^{(s+t)+1} = A^{s+t}A = (A^s A^t)A = A^s(A^t A) = A^s A^{t+1}$.
(b) Fix s. We prove by induction on t that $(A^s)^t = A^{st}$. The equality holds for $t = 1$. As the induction hypothesis, suppose that $(A^s)^t = A^{st}$. Then $(A^s)^{t+1} = (A^s)^t(A^s) = (A^{st})A^s = A^{st+s} = A^{s(t+1)}$ $\blacksquare$

Practice Problems

Problem 8.1
Write the linear system whose augmented matrix is given by

$$\left[\begin{array}{rrr|r} 2 & -1 & 0 & -1 \\ -3 & 2 & 1 & 0 \\ 0 & 1 & 1 & 3 \end{array}\right].$$

Problem 8.2
Consider the linear system

$$\left\{\begin{array}{rcrcrcrcl} 2x_1 & + & 3x_2 & - & 4x_3 & + & x_4 & = & 5 \\ -2x_1 & & & + & x_3 & & & = & 7 \\ 3x_1 & + & 2x_2 & - & 4x_3 & & & = & 3. \end{array}\right.$$

(a) Find the coefficient and augmented matrices of the linear system.
(b) Find the matrix notation.

Problem 8.3
Let A be an arbitrary matrix. Under what conditions is the product AA^T defined?

Problem 8.4
An $n \times n$ matrix A is said to be **idempotent** if $A^2 = A$.
(a) Show that the matrix

$$A = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

is idempotent.
(b) Show that if A is idempotent then the matrix $(I_n - A)$ is also idempotent.

Problem 8.5
The purpose of this exercise is to show that the rule $(ab)^n = a^n b^n$ does not hold with matrix multiplication. Consider the matrices

$$A = \begin{bmatrix} 2 & -4 \\ 1 & 3 \end{bmatrix}, B = \begin{bmatrix} 3 & 2 \\ -1 & 5 \end{bmatrix}.$$

Show that $(AB)^2 \neq A^2B^2$.

Problem 8.6
Show that $AB = BA$ if and only if $A^TB^T = B^TA^T$.

Problem 8.7
Let A and B be symmetric matrices. Show that AB is symmetric if and only if $AB = BA$.

Problem 8.8
A matrix B is said to be the **square root** of a matrix A if $BB = A$. Find two square roots of the matrix

$$A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}.$$

Problem 8.9
Find k such that

$$\begin{bmatrix} k & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & -3 \end{bmatrix} \begin{bmatrix} k \\ 1 \\ 1 \end{bmatrix} = 0.$$

Problem 8.10
Express the matrix notation as a system of linear equations.

$$\begin{bmatrix} 3 & -1 & 2 \\ 4 & 3 & 7 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}.$$

Problem 8.11
The augmented matrix $\begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & a & -4 \end{bmatrix}$ represents a system of equations in the variables x_1, x_2, and x_3.
(a) For what values of a is there no solution?
(b) For what values of a is there exactly one solution?
(c) Find the solution to this system of equations if $a = -2$.

Problem 8.12
Consider the following system of equations

$$\begin{cases} x_1 + \ \ x_2 + 2x_3 = 1 \\ 2x_1 + 3x_2 + 3x_3 = -2 \\ 3x_1 + 3x_2 + 7x_3 = 3. \end{cases}$$

(a) Write the matrix notation $Ax = b$.
(b) Solve this system if

$$A^{-1} = \begin{bmatrix} 12 & -1 & -3 \\ -5 & 1 & 1 \\ -3 & 0 & 1 \end{bmatrix}.$$

Problem 8.13
Consider the matrices $A = \begin{bmatrix} 12 & -1 & -3 \\ -5 & 1 & 1 \\ -3 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 & -3 & 2 \\ -1 & 3 & 3 & 0 \\ -2 & 4 & -1 & 2 \end{bmatrix}$.
Determine the dimension of AB and find the value of $(2,3)$ entry of AB.

Problem 8.14
Compute $\begin{bmatrix} 1 & -2 & 0 \\ 0 & 3 & 1 \\ -4 & 2 & 1 \end{bmatrix}^2$.

Problem 8.15
Let $A = \begin{bmatrix} 1 & -2 & 3 \\ -1 & 2 & -3 \\ 2 & -3 & 1 \\ -1 & 1 & 2 \end{bmatrix}$. Solve the system $A^T x = 0$.

Problem 8.16
A square matrix A of size n is said to be **orthogonal** if $AA^T = A^T A = I_n$.
Show that the matrix

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

is orthogonal.

Problem 8.17
If

$$A = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}$$

and

$$AB = \begin{bmatrix} -1 & 2 & -1 \\ 6 & -9 & 3 \end{bmatrix},$$

deteremine the first and the second columns of B.

Problem 8.18
Let L and D be two square matrices each of size n with D being a diagonal matrix. Show that the matrix $A = LDL^T$ is symmetric.

Problem 8.19
Let L and D be two square matrices each of size n, where L is an orthogonal matrix, i.e., $LL^T = L^TL = I_n$. Show that if $A = LDL^T$ then $A^n = LD^nL^T$, where n is a positive integer.

Problem 8.20
Let

$$A = \begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 2 & 3 \\ 1 & \frac{3}{2} \end{bmatrix}.$$

Show that $AB = \mathbf{0}$. Thus, the zero product property of real numbers[1] does not hold for matrices.

Problem 8.21
Let A be a matrix of size $m \times n$, B is of size $n \times p$, and C of size $p \times q$. Show that $A(BC) = (AB)C$.

Problem 8.22
Let A be a matrix of size $m \times n$, B and C of size $n \times p$. Show that $A(B+C) = AB + AC$.

Problem 8.23
Let A be a matrix of size $m \times n$, B and C of size $\ell \times m$. Show that $(B+C)A = BA + CA$.

Problem 8.24
Let A be a matrix of size $m \times n$ and B size $n \times p$. Show that $c(AB) = (cA)B = A(cB)$.

Problem 8.25
Prove Theorem 8.2.

[1]If $a, b, \in R$ and $ab = 0$ then $a = 0$ or $b = 0$.

9 The Inverse of a Square Matrix

Most problems in practice reduces to a system with matrix notation $Ax = b$. Thus, in order to get x we must somehow be able to eliminate the coefficient matrix A. One is tempted to try to divide by A. Unfortunately such an operation has not been defined for matrices. In this section we introduce a special type of square matrices and formulate the matrix analogue of numerical division. Recall that the $n \times n$ identity square matrix is the matrix I_n whose main diagonal entries are 1 and off diagonal entries are 0.
A square matrix A of size n is called **invertible** or **non-singular** if there exists a square matrix B of the same size such that $AB = BA = I_n$. In this case B is called the **inverse** of A. A square matrix that is not invertible is called **singular.**

Example 9.1
Show that the matrix

$$B = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

is the inverse of the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

Solution.
Using matrix multiplication one checks that $AB = BA = I_2$ ■

Example 9.2
Show that the matrix

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

is singular.

Solution.
Let $B = (b_{ij})$ be a 2×2 matrix. Then

$$BA = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} b_{11} & 0 \\ b_{21} & 0 \end{bmatrix} \neq I_2.$$

Thus, A is singular ■

It is important to keep in mind that the concept of invertibility is defined only for square matrices. In other words, it is possible to have a matrix A of size $m \times n$ and a matrix B of size $n \times m$ such that $AB = I_m$. It would be wrong to conclude that A is invertible and B is its inverse.

Example 9.3
Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Show that $AB = I_2$.

Solution.
Simple matrix multiplication shows that $AB = I_2$. However, this does not imply that B is the inverse of A since

$$BA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so that the condition $BA = I_2$ fails ■

Example 9.4
Show that the identity matrix is invertible but the zero matrix is not.

Solution.
Since $I_n I_n = I_n$, I_n is nonsingular and its inverse is I_n. Now, for any $n \times n$ matrix B we have $B\mathbf{0} = \mathbf{0} \neq I_n$ so that the zero matrix is not invertible ■

Now, if A is a nonsingular matrix then how many different inverses does it possess? The answer to this question is provided by the following theorem.

Theorem 9.1
The inverse of a matrix is unique.

Proof.
The proof is by contradiction. Suppose A has more than one inverse. Let B and C be two distinct inverses of A. We have, $B = BI_n = B(AC) = (BA)C = I_nC = C$. A contradiction. Hence, A has a unique inverse ■

Since an invertible matrix A has a unique inverse, we will denote it from now on by A^{-1}.

Using the definition, to show that an $n \times n$ matrix A is invertible we find a matrix B of the same size such that $AB = I_n$ and $BA = I_n$. The next theorem shows that one of these equality is enough to assure invertibilty. For the proof, see Problem 11.20.

Theorem 9.2
If A and B are two square matrices of size $n \times n$ such that $AB = I_n$ then $BA = I_n$ and $B = A^{-1}$.

For an invertible matrix A one can now define the negative power of a square matrix as follows: For any positive integer $n \geq 1$, we define $A^{-n} = (A^{-1})^n$. The next theorem lists some of the useful facts about inverse matrices.

Theorem 9.3
Let A and B be two square matrices of the same size $n \times n$.
(a) If A and B are invertible matrices then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.
(b) If A is invertible then A^{-1} is invertible and $(A^{-1})^{-1} = A$.
(c) If A is invertible then A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$

Proof.
(a) If A and B are invertible then $AA^{-1} = A^{-1}A = I_n$ and $BB^{-1} = B^{-1}B = I_n$. In This case, $(AB)(B^{-1}A^{-1}) = A[B(B^{-1}A^{-1})] = A[(BB^{-1})A^{-1}] = A(I_nA^{-1}) = AA^{-1} = I_n$. Similarly, $(B^{-1}A^{-1})(AB) = I_n$. It follows that $B^{-1}A^{-1}$ is the inverse of AB.
(b) Since $A^{-1}A = AA^{-1} = I_n$, A is the inverse of A^{-1}, i.e. $(A^{-1})^{-1} = A$.
(c) Since $AA^{-1} = A^{-1}A = I_n$, by taking the transpose of both sides we get $(A^{-1})^TA^T = A^T(A^{-1})^T = I_n$. This shows that A^T is invertible with inverse $(A^{-1})^T$ ■

Example 9.5
If A is invertible and $k \neq 0$ show that $(kA)^{-1} = \frac{1}{k}A^{-1}$.

Solution.
Suppose that A is invertible and $k \neq 0$. Then $(kA)A^{-1} = k(AA^{-1}) = kI_n$. This implies $(kA)(\frac{1}{k}A^{-1}) = I_n$. Thus, kA is invertible with inverse equals to $\frac{1}{k}A^{-1}$ ■

Example 9.6
(a) Under what conditions a diagonal matrix is invertible?
(b) Is the sum of two invertible matrices necessarily invertible?

Solution.
(a) Let $D = [d_{ii}]$ be a diagonal $n \times n$ matrix. Let $B = [b_{ij}]$ be an $n \times n$ matrix such that $DB = I_n$. Using matrix multiplication, we find $(DB)_{ij} = \sum_{k=1}^{n} d_{ik}b_{kj}$. It follows that $(DB)_{ij} = d_{ii}b_{ij} = 0$ for $i \neq j$ and $(DB)_{ii} = d_{ii}b_{ii} = 1$. If $d_{ii} \neq 0$ for all $1 \leq i \leq n$ then $b_{ij} = 0$ for $i \neq j$ and $b_{ii} = \frac{1}{d_{ii}}$. Thus, if $d_{11}d_{22}\cdots d_{nn} \neq 0$ then D is invertible and its inverse is the diagonal matrix $D^{-1} = \left[\frac{1}{d_{ii}}\right]$.
(b) The following two matrices are invertible but their sum , which is the zero matrix, is not.

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \blacksquare$$

Example 9.7
Consider the 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Show that if $ad - bc \neq 0$ then A^{-1} exists and is given by

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Solution.
Let

$$B = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$$

be a matrix such that $BA = I_2$. Then using matrix multiplication we find

$$\begin{bmatrix} ax + cy & bx + dy \\ az + cw & bz + dw \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Equating corresponding entries we obtain the following systems of linear equations in the unknowns x, y, z and w.

$$\begin{cases} ax + cy = 1 \\ bx + dy = 0 \end{cases}$$

and

$$\begin{cases} az + cw = 0 \\ bz + dw = 0. \end{cases}$$

In the first system, using elimination we find $(ad - bc)y = -b$ and $(ad - bc)x = d$. Similarly, using the second system we find $(ad - bc)z = -c$ and $(ad - bc)w = a$. If $ad - bc \neq 0$ then one can solve for x, y, z, and w and in this case $B = A^{-1}$ as given in the statement of the problem $\blacksquare$

Finally, we mention here that matrix inverses can be used to solve systems of linear equations as suggested by the following theorem.

Theorem 9.4
If A is an $n \times n$ invertible matrix and b is a column matrix then the equation $Ax = b$ has a unique solution $x = A^{-1}b$.

Proof.
Since $A(A^{-1}b) = (AA^{-1})b = I_nb = b$, we find that $A^{-1}b$ is a solution to the equation $Ax = b$. Now, if y is another solution then $y = I_ny = (A^{-1}A)y = A^{-1}(Ay) = A^{-1}b$ $\blacksquare$

Practice Problems

Problem 9.1
(a) Find two 2×2 singular matrices whose sum in nonsingular.
(b) Find two 2×2 nonsingular matrices whose sum is singular.

Problem 9.2
Show that the matrix

$$A = \begin{bmatrix} 1 & 4 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{bmatrix}$$

is singular.

Problem 9.3
Let

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}.$$

Find A^{-3}.

Problem 9.4
Let

$$A^{-1} = \begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix}.$$

Find A.

Problem 9.5
Let A and B be square matrices such that $AB = \mathbf{0}$. Show that if A is invertible then B is the zero matrix.

Problem 9.6
Find the inverse of the matrix

$$A = \begin{bmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{bmatrix}.$$

Problem 9.7
Find the matrix A given that

$$(I_2 + 2A)^{-1} = \begin{bmatrix} -1 & 2 \\ 4 & 5 \end{bmatrix}.$$

Problem 9.8
Find the matrix A given that

$$(5A^T)^{-1} = \begin{bmatrix} -3 & -1 \\ 5 & 2 \end{bmatrix}.$$

Problem 9.9
Show that if a square matrix A satisfies the equation $A^2 - 3A - I_n = 0$ then $A^{-1} = A - 3I_n$.

Problem 9.10
Simplify: $(AB)^{-1}(AC^{-1})(D^{-1}C^{-1})^{-1}D^{-1}$.

Problem 9.11
Let A and B be two square matrices. Show that $(ABA^{-1})^2 = AB^2A^{-1}$.

Problem 9.12
Two square matrices A and B are said to be **similar**, denoted $A \approx B$, if there exists an invertible matrix P such that $B = P^{-1}AP$.
(a) Show that $A \approx A$.
(b) Show that if $A \approx B$ then $B \approx A$.
(c) Show that if $A \approx B$ and $B \approx C$ then $A \approx C$.

Problem 9.13
Let $A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$. Compare A, A^T, and A^{-1}.

Problem 9.14
Suppose A, B, and X are $n \times n$ matrices such that A, X, and $A - AX$ are invertible. Moreover, suppose that

$$(A - AX)^{-1} = X^{-1}B.$$

(a) Show that B is invertible.
(b) Solve for X.

Problem 9.15
Suppose that A is similar to a diagonal matrix D. Show that A^T is also similar to D.

Problem 9.16
Suppose that A is a $n \times n$ orthogonal square matrix. What is its inverse?

Problem 9.17
Find the inverse of the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

Problem 9.18
Solve the system below by finding A^{-1}.

$$\left\{\begin{array}{r} 8x_1 - x_2 = 17 \\ -4x_1 + x_2 = -5. \end{array}\right.$$

Problem 9.19
Solve the system below by finding A^{-1}.

$$\left\{\begin{array}{r} -2x_1 + \quad x_2 = 1 \\ 3x_1 - 3x_2 = 2. \end{array}\right.$$

Problem 9.20
Let A be an $n \times n$ invertible square matrix. Show that the homogeneous system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

10 Elementary Matrices

In this section we introduce a special type of invertible matrices, the so-called elementary matrices, and we discuss some of their properties. As we shall see, elementary matrices will be used in the next section to develop an algorithm for finding the inverse of a square matrix.
An $n \times n$ **elementary matrix** is a matrix obtained from the identity matrix by performing *one* single elementary row operation.

Example 10.1
Show that the following matrices are elementary matrices
(a)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

(b)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

(c)

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Solution.
We list the operations that produce the given elementary matrices.
(a) $r_1 \leftarrow r_1$.
(b) $r_2 \leftrightarrow r_3$.
(c) $r_1 \leftarrow r_1 + 3r_3$ ■

Example 10.2
Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 1 & 4 & 4 & 0 \end{bmatrix}.$$

(a) Find the row equivalent matrix to A obtained by adding 3 times the first row of A to the third row. Call the equivalent matrix B.

(b) Find the elementary matrix E corresponding to the above elementary row operation.
(c) Compare EA and B.

Solution.
(a)

$$B = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 4 & 4 & 10 & 9 \end{bmatrix}.$$

(b)

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}.$$

(c) $EA = B$ ■

The conclusion of the above example holds for any matrix of size $m \times n$.

Theorem 10.1
If the elementary matrix E results from performing a certain row operation on I_m and if A is an $m \times n$ matrix, then the product of EA is the matrix that results when this same row operation is performed on A.

It follows from the above theorem that a matrix A is **row equivalent to a matrix** B, denoted by $A \sim B$, if and only if $B = E_k E_{k-1} \cdots E_1 A$, where $E_1, E_2, \cdots, E_k$ are elementary matrices. See Problem 10.16.
The above theorem is primarily of theoretical interest and will be used for developping some results about matrices and systems of linear equations. From a computational point of view, it is preferred to perform row operations directly rather than multiply on the left by an elementary matrix. Also, this theorem says that an elementary row operation on A can be achieved by premultiplying A by the corresponding elementary matrix E.
Given any elementary row operation, there is another row operation (called its **inverse**) that reverse the effect of the first operation. The inverses are described in the following chart.

Type	*Operation*	*Inverse operation*
I	$r_i \leftarrow cr_i$	$r_i \leftarrow \frac{1}{c} r_i$
II	$r_j \leftarrow cr_i + r_j$	$r_j \leftarrow -cr_i + r_j$
III	$r_i \leftrightarrow r_j$	$r_i \leftrightarrow r_j$

The following theorem gives an important property of elementary matrices.

Theorem 10.2
Every elementary matrix is invertible, and the inverse is an elementary matrix.

Proof.
Let A be any $n \times n$ matrix. Let E be an elementary matrix obtained by applying a row elementary operation ρ on I_n. By Theorem 10.1, applying ρ on A produces a matrix EA. Applying the inverse operation ρ^{-1} to EA gives $F(EA)$ where F is the elementary matrix obtained from I_n by applying the operation ρ^{-1}. Since inverse row operations cancel the effect of each other, it follows that $FEA = A$. Since A was arbitrary, we can choose $A = I_n$. Hence, $FE = I_n$. A similar argument shows that $EF = I_n$. Hence E is invertible and $E^{-1} = F$ ■

Example 10.3
Write down the inverses of the following elementary matrices:

$$(a)E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, (b)E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 9 \end{bmatrix}, (c)E_3 = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Solution.
Using the inverses of elementary row operations introduced above, we find
(a) $E_1^{-1} = E_1$ since $E_1E_1 = I_3$. Note that $E_1 = E_1^T$.
(b)

$$E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{9} \end{bmatrix}.$$

Note that $E_2^T = E_2$. (c)

$$E_3^{-1} = \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \blacksquare$$

Example 10.4
If E is an elementary matrix show that E^T is also an elementary matrix of the same type.

Solution.
(a) Suppose that E is the elementary matrix obtained by swapping rows i and j of I_n with $i < j$. But then E^T is obtained by interchanging rows i and j of I_n and so is an elementary matrix. Note that $E^T = E$.
(b) Suppose E is obtained by multiplying the i^{th} row of I_n by a non-zero constant k. But then E^T is obtained by multiplying the i^{th} row of I_n by k and so is an elementary matrix. Note that $E^T = E$.
(c) Suppose E is obtained by adding k times the i^{th} row of I_n to the j^{th} row $(i < j)$. But then E^T is obtained by adding k times the j^{th} row of I_n to the i^{th} row. Hence, E^T is an elementary matrix ■

Practice Problems

Problem 10.1
Which of the following are elementary matrices?
(a)
$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$
(b)
$$\begin{bmatrix} 1 & 0 \\ -5 & 1 \end{bmatrix}.$$
(c)
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 9 \\ 0 & 0 & 1 \end{bmatrix}.$$
(d)
$$\begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Problem 10.2
Let A be a 4×3 matrix. Find the elementary matrix E, which as a premultiplier of A, that is, as EA, performs the following elementary row operations on A :
(a) Multiplies the second row of A by -2.
(b) Adds 3 times the third row of A to the fourth row of A.
(c) Interchanges the first and third rows of A.

Problem 10.3
For each of the following elementary matrices, describe the corresponding elementary row operation and write the inverse.
(a)
$$E = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

(b)

$$E = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(c)

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$$

Problem 10.4
Consider the matrices

$$A = \begin{bmatrix} 3 & 4 & 1 \\ 2 & -7 & -1 \\ 8 & 1 & 5 \end{bmatrix}, B = \begin{bmatrix} 8 & 1 & 5 \\ 2 & -7 & -1 \\ 3 & 4 & 1 \end{bmatrix}, C = \begin{bmatrix} 3 & 4 & 1 \\ 2 & -7 & -1 \\ 2 & -7 & 3 \end{bmatrix}.$$

Find elementary matrices E_1, E_2, E_3, and E_4 such that
(a) $E_1A = B$, (b) $E_2B = A$, (c) $E_3A = C$, (d) $E_4C = A$.

Problem 10.5
What should we premultiply a 3×3 matrix if we want to interchange rows 1 and 3?

Problem 10.6
Let

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Find the corresponding inverse operations.

Problem 10.7
List all 3×3 elementary matrices corresponding to type I elementary row operations.

Problem 10.8
List all 3×3 elementary matrices corresponding to type II elementary row operations.

Problem 10.9
Write down the inverses of the following elementary matrices:

$$E_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, E_2 = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Problem 10.10
Consider the following elementary matrices:

$$E_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, E_2 = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Find

$$E_1E_2E_3 \begin{bmatrix} 1 & 0 & 2 \\ -2 & 3 & 4 \\ 0 & 5 & -3 \end{bmatrix}.$$

Problem 10.11
Write the following matrix as a product of elementary matrices:

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}.$$

Problem 10.12
Explain why the following matrix cannot be written as a product of elementary matrices.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 3 & 3 & 3 \end{bmatrix}.$$

Problem 10.13
Let E_1, E_2, E_3 be 3×3 elementary matrices that act on an $3 \times m$ matrix A, such that
(i) E_1A is obtained from A by multiplying the third row by -1;
(ii) E_2A is obtained from A by subtracting the third row from the second;
(iii) E_3A is obtained from A by adding the third row to the first.

(a) Find E_1, E_2, E_3.
(b) Find $(E_3E_2E_1)^{-1}$.

Problem 10.14
Let
$$A=\begin{bmatrix}1&0&-2\\0&4&3\\0&0&1\end{bmatrix}$$
(a) Find elementary matrices E_1, E_2, E_3 such that $E_3E_2E_1A = I_3$.
(b) Write A as a product of elementary matrices.

Problem 10.15
Let A be a matrix with 4 rows. For each elementary row operation on A below, find the corresponding elementary matrices and their inverses.
(a) Adding 2 times row 3 to row 4;
(b) Swapping rows 1 and 4;
(c) Multiplying row 2 by 6.

Problem 10.16
We say that a matrix A is **row equivalent** to a matrix B if and only if $B = E_kE_{k-1}\cdots E_1A$, where $E_1, E_2, \cdots, E_k$ are elementary matrices. We write $A \sim B$.
(a) Show that $A \sim A$ for any matrix A.
(b) Show that if $A \sim B$ then $B \sim A$.
(c) Show that is $A \sim B$ and $B \sim C$ then $A \sim C$.

Problem 10.17
Let A be an $n \times n$ matrix. Show that if $A \sim I_n$ then A is the product of elementary matrices.

Problem 10.18
Show that if a square matrix A can be written as the product of elementary matrices then A is invertible.

Problem 10.19
Let A be an $n \times n$ matrix such that $A = E_1E_2\cdots E_k$ where each E_i swaps some two rows of I_n. Show that $AA^T = I_n$.

Problem 10.20
Let
$$A=\begin{bmatrix}3&4&5\\2&-1&0\end{bmatrix}.$$

Let E_1 be the elementary matrix that multiplies the first row of I_2 by $\frac{1}{3}$. Let E_2 be the elementary matrix that adds -2 times row 1 of I_2 to row 2. Let E_3 be the elmentary matrix that multiplies row 2 of I_2 by $-\frac{3}{11}$. Let E_4 be the elementary matrix that adds $-\frac{4}{3}$ times row 2 to row 1 of I_2. Find $E_4E_3E_2E_1A$.

11 Finding A^{-1} Using Elementary Matrices

Before we establish the main results of this section, we recall the reader of the following method of mathematical proofs. To say that statements $p_1, p_2, \cdots, p_n$ are all equivalent means that either they are all true or all false. To prove that they are equivalent, one assumes p_1 to be true and proves that p_2 is true, then assumes p_2 to be true and proves that p_3 is true, continuing in this fashion, one assumes that p_{n-1} is true and proves that p_n is true and finally, assumes that p_n is true and proves that p_1 is true. This is known as the **proof by circular argument**.
Now, back to our discussion of inverses. The following result establishes relationships between square matrices and systems of linear equations. These relationships are very important and will be used many times in later sections.

Theorem 11.1
If A is an $n \times n$ matrix then the following statements are equivalent.
(a) A is invertible.
(b) $Ax = \mathbf{0}$ has only the trivial solution.
(c) A is row equivalent to I_n.
(d) $rank(A) = n$.

Proof.
$(a) \Rightarrow (b)$: Suppose that A is invertible and x_0 is a solution to $Ax = \mathbf{0}$. Then $Ax_0 = \mathbf{0}$. Multiply both sides of this equation by A^{-1} to obtain $A^{-1}Ax_0 = A^{-1}\mathbf{0}$, that is, $x_0 = \mathbf{0}$. Hence, the trivial solution is the only solution.
$(b) \Rightarrow (c)$: Suppose that $Ax = \mathbf{0}$ has only the trivial solution. Then the reduced row-echelon form of the augmented matrix has no rows of zeros or free variables. Hence it must look like

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{array}\right].$$

If we disgard the last column of the previous matrix we can conclude that A can be reduced to I_n by a sequence of elementary row operations, i.e., A is row equivalent to I_n.
$(c) \Rightarrow (d)$: Suppose that A is row equivalent to I_n. Then $rank(A) =$

$rank(I_n) = n.$

$(d) \Rightarrow (a)$: Suppose that $rank(A) = n$. Then A is row equivalent to I_n. That is I_n is obtained by a finite sequence of elementary row operations performed on A. Then by Theorem 10.1, each of these operations can be accomplished by premultiplying on the left by an appropriate elementary matrix. Hence, obtaining

$$E_k E_{k-1} \dots E_2 E_1 A = I_n,$$

where k is the necessary number of elementary row operations needed to reduce A to I_n. Now, by Theorem 10.2, each E_i is invertible. Hence, $E_k E_{k-1} \dots E_2 E_1$ is invertible. By Theorem 9.2, A is invertible and $A^{-1} = (E_k E_{k-1} \dots E_2 E_1)$ ■

As an application of Theorem 11.1, we describe an algorithm for finding A^{-1}. We perform elementary row operations on A until we get I_n; say that the product of the elementary matrices is $E_k E_{k-1} \dots E_2 E_1$. Then we have

$$\begin{aligned} (E_k E_{k-1} \dots E_2 E_1)[A|I_n] &= [(E_k E_{k-1} \dots E_2 E_1)A|(E_k E_{k-1} \dots E_2 E_1)I_n] \\ &= [I_n|A^{-1}]. \end{aligned}$$

We illustrate this algorithm next.

Example 11.1
Find the inverse of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}.$$

Solution.
We first construct the matrix

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right].$$

Applying the above algorithm to obtain

Step 1: $r_2 \leftarrow r_2 - 2r_1$ and $r_3 \leftarrow r_3 - r_1$

$$\left[\begin{array}{rrr|rrr} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right]$$

Step 2: $r_3 \leftarrow r_3 + 2r_2$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right]$$

Step 3: $r_1 \leftarrow r_1 - 2r_2$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 9 & 5 & -2 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right]$$

Step 4: $r_2 \leftarrow r_2 - 3r_3$ and $r_1 \leftarrow r_1 + 9r_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right]$$

Step 5: $r_3 \leftarrow -r_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

It follows that

$$A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix} \blacksquare$$

Example 11.2
Show that the following homogeneous system has only the trivial solution.

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 0 \\ 2x_1 + 5x_2 + 3x_3 = 0 \\ x_1 \qquad\quad + 8x_3 = 0. \end{cases}$$

Solution.
The coefficient matrix of the given system is invertible by the previous example. Thus, by Theorem 11.1 the system has only the trivial solution $\blacksquare$

How can we tell when a square matrix A is singular? i.e., when does the algorithm of finding A^{-1} fail? The answer is provided by the following theorem

Theorem 11.2
An $n \times n$ matrix A is singular if and only if A is row equivalent to a matrix B that has a row of zeros.

Proof.
Suppose first that A is singular. Then by Theorem 11.1, A is not row equivalent to I_n. Thus, A is row equivalent to a matrix $B \neq I_n$ which is in reduced row echelon form. By Theorem 11.1, B must have a row of zeros.
Conversely, suppose that A is row equivalent to matrix B with a row consisting entirely of zeros. Then $rank(B) < n$ so that B is singular by Theorem 11.1. Now, $B = E_kE_{k-1}\dots E_2E_1A$. If A is nonsingular then B is nonsingular, a contradiction. Thus, A must be singular ■

Corollary 11.1
If A is a square matrix with a row consisting entirely of zeros then A is singular.

When a square linear system $Ax = b$ has an invertible coefficient matrix then the unique solution is given by $x = A^{-1}b$.

Example 11.3
Solve the following system.

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 5 \\ 2x_1 + 5x_2 + 3x_3 = 3 \\ x_1 \qquad\quad + 8x_3 = 17. \end{cases}$$

Solution.
The coefficient matrix is invertible by Example 11.1. Hence, the unique solution is given by $x = A^{-1}b$. That is,

$$\begin{aligned} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}. \end{aligned}$$

Hence, $x_1 = 1$, $x_2 = -1$, and $x_3 = 2$ ■

Practice Problems

Problem 11.1
Determine if the following matrix is invertible.

$$\begin{bmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 5 \end{bmatrix}.$$

Problem 11.2
For what values of a does the following homogeneous system have a nontrivial solution?

$$\begin{cases} (a-1)x_1 + & 2x_2 = 0 \\ 2x_1 + & (a-1)x_2 = 0. \end{cases}$$

Problem 11.3
Find the inverse of the matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 5 & 5 & 1 \end{bmatrix}.$$

Problem 11.4
Prove that if A is symmetric and non-singular then A^{-1} is symmetric.

Problem 11.5
If

$$D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix},$$

find D^{-1}.

Problem 11.6
Prove that a square matrix A is non-singular if and only if A is a product of elementary matrices.

Problem 11.7
Prove that two $m \times n$ matrices A and B are row equivalent if and only if there exists a non-singular matrix P such that $B = PA$.

Problem 11.8
Let A and B be two $n\times n$ matrices. Suppose A is row equivalent to B. Prove that A is non-singular if and only if B is non-singular.

Problem 11.9
Show that a 2×2 lower triangular matrix is invertible if and only if the product of the diagonal entries is not zero and in this case the inverse is also lower triangular.

Problem 11.10
Let A be an $n \times n$ matrix and suppose that the system $Ax = \mathbf{0}$ has only the trivial solution. Show that $A^k x = \mathbf{0}$ has only the trivial solution for any positive integer k.

Problem 11.11
Show that if A and B are two $n \times n$ invertible matrices then A is row equivalent to B.

Problem 11.12
(a) Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix}.$$

(b) Solve the following system.

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ \phantom{x_1 - {}} 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9. \end{cases}$$

Problem 11.13
(a) Find the inverse of the matrix

$$A = \begin{bmatrix} 5 & -5 & -15 \\ 4 & -2 & -6 \\ 3 & -6 & -17 \end{bmatrix}.$$

(b) Solve the following system.

$$\begin{cases} 5x_1 - 5x_2 - 15x_3 = 40 \\ 4x_1 - 2x_2 - 6x_3 = 19 \\ 3x_1 - 6x_2 - 17x_3 = 41. \end{cases}$$

Problem 11.14
Let A and B be two square matrices. Show that AB is non-singular if and only if both A and B are non-singular.

Problem 11.15
If P is an $n \times n$ matrix such that $P^TP = I_n$ then the matrix $H = I_n - 2PP^T$ is called the **Householder matrix**. Show that H is symmetric and $H^TH = I_n$.

Problem 11.16
Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 0 & 4 \\ -1 & 1 & -1 \\ -1 & 0 & -3 \end{bmatrix}.$$

Problem 11.17
Find all 2×2 matrices

$$\begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix}$$

such that $A^{-1} = A$.

Problem 11.18
Find the inverse of the matrix

$$A = \begin{bmatrix} 3 & 5 & 7 \\ 1 & 2 & 3 \\ 2 & 3 & 5 \end{bmatrix}.$$

Problem 11.19
Find a 3×3 matrix A such that

$$(A^T + 5I_3)^{-1} = \begin{bmatrix} 1 & 3 & 0 \\ 0 & -1 & 0 \\ 2 & 2 & 1 \end{bmatrix}.$$

Problem 11.20
Show that if A and B are two $n \times n$ matrices such that $AB = I_n$ then $B^{-1} = A$ and $BA = I_n$.

Determinants

With each square matrix we can associate a real number called the determinant of the matrix. Determinants have important applications to the theory of systems of linear equations. More specifically, determinants give us a method (called Cramer's method) for solving linear systems. Also, determinant tells us whether or not a matrix is invertible.
Throughout this chapter we use only square matrices.

12 Determinants by Cofactor Expansion

The determinant of a 2×2 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

is the number

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}.$$

The determinant of a 3×3 matrix can be found using the determinants of 2×2 matrices and a cofactor expansion which we discuss next.
If A is a square matrix of order n then the **minor** of the entry a_{ij}, denoted by M_{ij}, is the determinant of the submatrix obtained from A by deleting the i^{th} row and the j^{th} column. The **cofactor** of the entry a_{ij} is the number $C_{ij} = (-1)^{i+j}M_{ij}$.

Example 12.1
Let

$$A = \begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix}.$$

Find the minor and the cofactor of the entry $a_{32} = 4$.

Solution.
The minor of the entry a_{32} is

$$M_{32} = \begin{vmatrix} 3 & -4 \\ 2 & 6 \end{vmatrix} = 26$$

and the cofactor is $C_{32} = (-1)^{3+2}M_{32} = -26$ ■

Example 12.2
Find the cofactors C_{11}, C_{12}, and C_{13} of the matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

Solution.
We have

$$C_{11} = (-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{32}a_{23}$$

$$C_{12} = (-1)^{1+2}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = -(a_{21}a_{33} - a_{31}a_{23})$$

$$C_{13} = (-1)^{1+3}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21}a_{32} - a_{31}a_{22} \blacksquare$$

The **determinant of a matrix** A of order n can obtained by multiplying the entries of a row (or a column) by the corresponding cofactors and adding the resulting products. More precisely, the **expansion along row** i is

$$|A| = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}.$$

The **expansion along column** j is given by

$$|A| = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}.$$

Any row or column chosen will result in the same answer.

Example 12.3
Find the determinant of the matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

Solution.
Using the previous example, we can find the determinant using the cofactor along the first row to obtain

$$\begin{aligned} |A| =& a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\ =& a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ =& a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22}) \blacksquare \end{aligned}$$

Remark 12.1
In general, the best strategy for evaluating a determinant by cofactor expansion is to expand along a row or a column having the largest number of zeros.

Example 12.4
If a square matrix has a row of zeros then its determinant is zero.

Solution.
Finding the determinant of the matrix by cofactor expansion along the row of zeros, we find that the determinant is 0 ■

A square matrix is called **lower triangular** if all the entries above the *main diagonal* are zero. A square matrix is called **upper triangular** if all the entries below the *main diagonal* are zero. A **triangular matrix** is one that is either lower triangular or upper triangular. A matrix that is both upper and lower triangular is called a **diagonal matrix**.

Example 12.5
Find the determinant of each of the following matrices.
(a)

$$A = \begin{bmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

(b)

$$B = \begin{bmatrix} 0 & 0 & 0 & a_{14} \\ 0 & 0 & a_{23} & a_{24} \\ 0 & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}.$$

(c)

$$C = \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}.$$

Solution.
(a) Expanding along the first column we find

$$|A| = a_{31}C_{31} = -a_{31}a_{22}a_{13}.$$

(b) Again, by expanding along the first column we obtain

$$|B| = a_{41}C_{41} = a_{41}a_{32}a_{23}a_{14}.$$

(c) Expanding along the last column we find

$$|C| = a_{44}C_{44} = a_{11}a_{22}a_{33}a_{44} \blacksquare$$

Remark 12.2
Be aware that the matrices in (a) and (b) do not fall into the category of triangular matrices whereas (c) does.

Example 12.6
Evaluate the determinant of the following matrix.

$$\begin{vmatrix} 2 & 7 & -3 & 8 & 3 \\ 0 & -3 & 7 & 5 & 1 \\ 0 & 0 & 6 & 7 & 6 \\ 0 & 0 & 0 & 9 & 8 \\ 0 & 0 & 0 & 0 & 4 \end{vmatrix}$$

Solution.
The given matrix is upper triangular so that the determinant is the product of entries on the main diagonal, i.e., equals to -1296 $\blacksquare$

Example 12.7
Use cofactor expansion along the first column to find $|A|$ where

$$A = \begin{bmatrix} 3 & 5 & -2 & 6 \\ 1 & 2 & -1 & 1 \\ 2 & 4 & 1 & 5 \\ 3 & 7 & 5 & 3 \end{bmatrix}.$$

Solution.
Expanding along the first column we find

$$\begin{aligned} |A| &= 3C_{11} + C_{21} + 2C_{31} + 3C_{41} \\ &= 3M_{11} - M_{21} + 2M_{31} - 3M_{41} \\ &= 3(-54) + 78 + 2(60) - 3(18) = -18 \blacksquare \end{aligned}$$

Practice Problems

Problem 12.1
Evaluate the determinant of each of the following matrices
(a)

$$A = \begin{bmatrix} 3 & 5 \\ -2 & 4 \end{bmatrix}.$$

(b)

$$A = \begin{bmatrix} -2 & 7 & 6 \\ 5 & 1 & -2 \\ 3 & 8 & 4 \end{bmatrix}.$$

Problem 12.2
Find all values of t for which the determinant of the following matrix is zero.

$$A = \begin{bmatrix} t-4 & 0 & 0 \\ 0 & t & 0 \\ 0 & 3 & t-1 \end{bmatrix}.$$

Problem 12.3
Solve for x

$$\begin{vmatrix} x & 1 \\ 1 & 1-x \end{vmatrix} = \begin{vmatrix} 1 & 0 & -3 \\ 2 & x & -6 \\ 1 & 3 & x-5 \end{vmatrix}.$$

Problem 12.4
Evaluate the determinant of the following matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 0 \end{bmatrix}.$$

Problem 12.5
Let

$$A = \begin{bmatrix} 4 & -1 & 1 & 6 \\ 0 & 0 & -3 & 3 \\ 4 & 1 & 0 & 14 \\ 4 & 1 & 3 & 2 \end{bmatrix}.$$

Find M_{23} and C_{23}.

Problem 12.6
Find all values of λ for which $|A| = 0$, where

$$A = \begin{bmatrix} \lambda - 1 & 0 \\ 2 & \lambda + 1 \end{bmatrix}.$$

Problem 12.7
Evaluate the determinant of the matrix

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 2 & -1 & 5 \\ 1 & 9 & -4 \end{bmatrix},$$

(a) along the first column;
(b) along the third row.

Problem 12.8
Evaluate the determinant of the matrix by a cofactor expansion along a row or column of your choice.

$$A = \begin{bmatrix} 3 & 3 & 0 & 5 \\ 2 & 2 & 0 & -2 \\ 4 & 1 & -3 & 0 \\ 2 & 10 & 3 & 2 \end{bmatrix}.$$

Problem 12.9
Evaluate the determinant of the following matrix by inspection.

$$A = \begin{bmatrix} 1 & 2 & 7 & -3 \\ 0 & 1 & -4 & 1 \\ 0 & 0 & 2 & 7 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

Problem 12.10
Evaluate the determinant of the following matrix.

$$A = \begin{bmatrix} \sin\theta & \cos\theta & 0 \\ -\cos\theta & \sin\theta & 0 \\ \sin\theta - \cos\theta & \sin\theta + \cos\theta & 1 \end{bmatrix}.$$

Problem 12.11
Find all values of λ such that $|A| = 0$.
(a)

$$A = \begin{bmatrix} \lambda - 1 & -2 \\ 1 & \lambda - 4 \end{bmatrix},$$

(b)

$$A = \begin{bmatrix} \lambda - 6 & 0 & 0 \\ 0 & \lambda & -1 \\ 0 & 4 & \lambda - 4 \end{bmatrix}.$$

Problem 12.12
Find

$$\begin{vmatrix} 3 & 0 & 0 & -2 & 4 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 5 & -3 \\ -4 & 0 & 1 & 0 & 6 \\ 0 & -1 & 0 & 3 & 2 \end{vmatrix}.$$

Problem 12.13
Solve the equation

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & t \\ 1 & 4 & t^2 \end{vmatrix} = 0.$$

Problem 12.14
Find

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -1 & 4 \\ 0 & 0 & \pi \end{vmatrix}.$$

Problem 12.15
Find

$$\begin{vmatrix} 3 & 0 & 2 & -1 \\ 1 & 2 & 0 & -2 \\ 4 & 0 & 6 & -3 \\ 5 & 0 & 2 & 0 \end{vmatrix}.$$

Problem 12.16
Find

$$|A| = \begin{vmatrix} a & b & c \\ ka & kb & kc \\ e & f & g \end{vmatrix}.$$

Problem 12.17
Solve for λ.

$$\begin{vmatrix} \lambda - 5 & -8 & -16 \\ -4 & \lambda - 1 & -8 \\ 4 & 4 & \lambda + 11 \end{vmatrix} = 0.$$

Problem 12.18
Solve for λ.

$$\begin{vmatrix} \lambda - 2 & -1 & -1 \\ -2 & \lambda - 1 & 2 \\ 1 & 0 & \lambda + 2 \end{vmatrix} = 0.$$

Problem 12.19
Solve for λ.

$$\begin{vmatrix} \lambda + 2 & 0 & -1 \\ 6 & \lambda + 2 & 0 \\ -19 & -5 & \lambda + 4 \end{vmatrix} = 0.$$

Problem 12.20
Solve for λ.

$$A = \begin{vmatrix} \lambda - 2 & -1 & 0 \\ 0 & \lambda - 2 & 0 \\ -2 & -3 & \lambda - 1 \end{vmatrix} = 0.$$

13 Evaluating Determinants by Row Reduction

In this section we provide a simple procedure for finding the determinant of a matrix. The idea is to reduce the matrix into row-echelon form which in this case is a triangular matrix. The following theorem provides a formula for finding the determinant of a triangular matrix.

Theorem 13.1
If A is an $n \times n$ triangular matrix then $|A| = a_{11}a_{22}\dots a_{nn}$.

Example 13.1
Compute $|A|$.
(a)

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}.$$

(b)

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix}.$$

Solution.
(a) Since A is upper triangular and hence triangular, $|A| = (1)(4)(6) = 24$.
(b) $|B| = (1)(3)(6) = 18$ ■

Example 13.2
Compute the determinant of the identity matrix I_n.

Solution.
Since the identity matrix is triangular with entries equal to 1 on the main diagonal, $|I_n| = 1$ ■

Theorem 13.2
If a square matrix has two identical rows then its determinant is zero.

Proof.
We will prove this result by induction of the dimension of the matrix.
(i) Basis of induction: Suppose $A = [a_{ij}]$ is a 2×2 matrix with two indentical rows. That is,

$$\begin{bmatrix} a & b \\ a & b \end{bmatrix}.$$

Then $|A| = ab - ab = 0$.
(ii) Induction hypothesis: Suppose the result holds for all numbers $1, 2, \cdots, n$.
(iii) Induction step: Let A be an $(n+1) \times (n+1)$ matrix with two identical rows i and k with $i < k$. Choose a row m that is different from i and k. Finding the determinant of A by cofactor expansion along the m^{th} row, we find

$$|A| = \sum_{\ell=1}^{n} a_{m\ell}(-1)^{m+\ell} M_{m\ell}.$$

But $M_{m\ell}$ is the determinant of an $n \times n$ matrix with two identical rows so that by the induction hypothesis, $M_{m\ell} = 0$. Hence, $|A| = 0$ ■

The following theorem is of practical use. It provides a technique for evaluating determinants by greatly reducing the labor involved. We shall show that the determinant can be evaluated by reducing the matrix to row-echelon form.

Theorem 13.3
Let A be an $n \times n$ matrix.
(a) Let B be the matrix obtained from A by multiplying a row of A by a scalar c. Then $|B| = c|A|$ or $|A| = \frac{1}{c}|B|$.
(b) Let B be the matrix obtained from A by adding c times a row of A to another row. Then $|B| = |A|$.
(c) Let B be the matrix obtained from A by interchanging two rows of A. Then $|B| = -|A|$.

Example 13.3
Use Theorem 13.3 to evaluate the determinant of the following matrix

$$A = \begin{bmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{bmatrix}.$$

Solution.
We use Gaussian elimination as follows.

Step 1: $r_1 \leftrightarrow r_2$

$$|A| = -\begin{vmatrix} 3 & -6 & 9 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix}.$$

Step 2: $r_1 \leftarrow r_1 - r_3$

$$|A| = -\begin{vmatrix} 1 & -12 & 8 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix}.$$

Step 3: $r_3 \leftarrow r_3 - 2r_1$

$$|A| = -\begin{vmatrix} 1 & -12 & 8 \\ 0 & 1 & 5 \\ 0 & 30 & -15 \end{vmatrix}.$$

Step 4: $r_3 \leftarrow r_3 - 30r_2$

$$|A| = -\begin{vmatrix} 1 & -12 & 8 \\ 0 & 1 & 5 \\ 0 & 0 & -165 \end{vmatrix} = -(1)(1)(-165) = 165 \blacksquare$$

Example 13.4
Show that if a square matrix has two proportional rows then its determinant is zero.

Solution.
Suppose that A is a square matrix such that row j is k times row i with $k \neq 0$. By adding $-\frac{1}{k}r_j$ to r_i then the i^{th} row will consist of zeros. By Thereom 13.2 and Theorem 13.3(b), $|A| = 0$ $\blacksquare$

Example 13.5
Find, by inspection, the determinant of the following matrix.

$$A = \begin{bmatrix} 3 & -1 & 4 & -2 \\ 6 & -2 & 5 & 2 \\ 5 & 8 & 1 & 4 \\ -9 & 3 & -12 & 6 \end{bmatrix}.$$

Solution.
Since the first and the fourth rows are proportional, the determinant is zero by Example 13.4 ■

Example 13.6
Show that if A is an $n \times n$ matrix and c is a scalar then $|cA| = c^n|A|$.

Solution.
The matrix cA is obtained from the matrix A by multiplying the rows of A by $c \neq 0$. By mutliplying the first row of A by c we obtain a matrix B_1 such that $\frac{1}{c}|B_1| = |A|$ (Theorem 13.3(a)). Now, by multiplying the second row of B_1 by c we obtain a matrix B_2 such that $|B_1| = \frac{1}{c}|B_2|$. Thus, $|A| = \frac{1}{c^2}|B_2|$. Repeating this process, we find $|A| = \frac{1}{c^n}B_n = \frac{1}{c^n}|cA|$ or $|cA| = c^n|A|$ ■

Example 13.7
(a) Let E_1 be the elementary matrix corresponding to type I elementary row operation, i.e., multiplying a row by a scalar. Find $|E_1|$.
(b) Let E_2 be the elementary matrix corresponding to type II elementary row operation, i.e., adding a multiple of a row to another row. Find $|E_2|$.
(c) Let E_3 be the elementary matrix corresponding to type III elementary row operation, i.e., swapping two rows. Find $|E_3|$.

Solution.
(a) The matrix E_1 is obtained from the identity matrix by multiplying a row of I_n by a nonzero scalar c. By Theorem 13.3(a), we have $|E_1| = c|I_n| = c$.
(b) E_2 is obtained from I_n by adding a multiple of a row to another row. By Theorem 13.3(b), we have $|E_2| = |I_n| = 1$.
(c) The matrix E_3 is obtained from the matrix I_n by interchanging two rows. By Theorem 13.3(c), we have $|E_3| = -|I_n| = -1$ ■

Practice Problems

Problem 13.1
Use the row reduction technique to find the determinant of the following matrix.

$$A = \begin{bmatrix} 2 & 5 & -3 & -2 \\ -2 & -3 & 2 & -5 \\ 1 & 3 & -2 & 2 \\ -1 & -6 & 4 & 3 \end{bmatrix}.$$

Problem 13.2
Given that

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -6,$$

find
(a)

$$\begin{vmatrix} d & e & f \\ g & h & i \\ a & b & c \end{vmatrix},$$

(b)

$$\begin{vmatrix} 3a & 3b & 3c \\ -d & -e & -f \\ 4g & 4h & 4i \end{vmatrix},$$

(c)

$$\begin{vmatrix} a+g & b+h & c+i \\ d & e & f \\ g & h & i \end{vmatrix},$$

(d)

$$\begin{vmatrix} -3a & -3b & -3c \\ d & e & f \\ g-4d & h-4e & i-4f \end{vmatrix}.$$

Problem 13.3
Determine by inspection the determinant of the following matrix.

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \\ 16 & 17 & 18 & 19 & 20 \\ 2 & 4 & 6 & 8 & 10 \end{bmatrix}.$$

Problem 13.4
Let A be a 3×3 matrix such that $|2A| = 6$. Find $|A|$.

Problem 13.5
Find the determinant of the following elementary matrix by inspection.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Problem 13.6
Find the determinant of the following elementary matrix by inspection.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Problem 13.7
Find the determinant of the following elementary matrix by inspection.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -9 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Problem 13.8
Use the row reduction technique to find the determinant of the following matrix.

$$A = \begin{bmatrix} 2 & 1 & 3 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix}.$$

Problem 13.9
Use row reduction to find the determinant of the following **Vandermonde** matrix.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix}.$$

Problem 13.10
Let a, b, c be three numbers such that $a + b + c = 0$. Find the determinant of the following matrix.

$$A = \begin{bmatrix} b+c & a+c & a+b \\ a & b & c \\ 1 & 1 & 1 \end{bmatrix}.$$

Problem 13.11
Compute the following determinant using elementary row operations.

$$\begin{vmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{vmatrix}.$$

Problem 13.12
Compute the following determinant using elementary row operations.

$$\begin{vmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{vmatrix}.$$

Problem 13.13
If

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

and $c \neq 0$, then calculate $|cA|$.

Problem 13.14
Use row reduction to compute the determinant of

$$A = \begin{bmatrix} 2 & 3 & 3 & 1 \\ 0 & 4 & 3 & -3 \\ 2 & -1 & -1 & -3 \\ 0 & -4 & -3 & 2 \end{bmatrix}.$$

Problem 13.15
Solve for x,

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 4 & 5 & x & 0 \\ 6 & 7 & 8 & 9 \end{vmatrix} = 54.$$

Problem 13.16
Let A be an $n \times n$ matrix and E an elementary matrix. Show that $|EA| = |E||A|$.

Problem 13.17
By induction on k, one can show that the previous result is true when A is premultiplied by any number of elementary matrice. Recall that if a square matrix A is invertible then A can be written as the product of elementary matrices (Theorem 11.1). Use this theorem and the previous exercise to show that if A is invertible then $|A| \neq 0$. Taking the contrapositive, if $|A| = 0$ then A is singular.

Problem 13.18
Using determinants, show that the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ -1 & 5 & -7 & 4 \\ -3 & 5 & -9 & 2 \\ 4 & 4 & 6 & -1 \end{bmatrix}$$

is singular.

Problem 13.19
Find the determinant of

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}.$$

Problem 13.20
Let $A = [a_{ij}]$ be a triangular matrix of size $n \times n$. Use induction on $n \geq 2$, to show that $|A| = a_{11}a_{22}\cdots a_{nn}$.

Problem 13.21
Let A be an $n \times n$ matrix. Let B be the matrix obtained from A by multiplying a row of A by a scalar c. Show that $|B| = c|A|$ or $|A| = \frac{1}{c}|B|$.

Problem 13.22
Let $A = [a_{ij}]$, $B = [b_{ij}]$, and $C = [c_{ij}]$ be $n \times n$ matrices such that $a_{kj} = b_{kj} + c_{kj}$ for all $1 \leq j \leq n$ and $b_{ij} = c_{ij} = a_{ij}$ for all $i \neq k$. That is, the k^{th} row of A is the sum of the k^{th} row of B and the k^{th} row of C and all the rows of B and C different from row k are the same as the rows of A. Show that $|A| = |B| + |C|$.

Problem 13.23
Let A be an $n \times n$ matrix. Let B be the matrix obtained from A by adding c times a row of A to another row of A. Show that $|B| = |A|$.

Problem 13.24
Let A be an $n \times n$ matrix. Let B be the matrix obtained from A by interchanging rows i and k of A with $i < k$. Show that $|B| = -|A|$.

14 Additional Properties of the Determinant

In this section we shall exhibit some additional properties of the determinant. One of the immediate consequences of these properties will be an important determinant test for the invertibility of a square matrix.
The first result relates the invertibility of a square matrix to its determinant.

Theorem 14.1
If A is an $n \times n$ matrix then A is non-singular if and only if $|A| \neq 0$.

Proof.
Suppose that A is non-singular. Then there are elementary matrices $E_1, E_2, \cdots, E_k$ such that $A = E_1E_2 \cdots E_k$. By Problem 13.16, we have $|A| = |E_1||E_2| \cdots |E_k|$. For $1 \leq i \leq k$, $E)i$ is an elementary matrix and therefore $|E_i| \neq 0$ (Example 13.7). Hence, $|A| \neq 0$.
Conversely, suppose that $|A| \neq 0$. If A is singular then by Theorem 11.1, A is row equivalent to a matrix B that has a row of zeros. Hence, $0 = |B|$. Write $B = E_k \cdots E_2E_1A$. Then $0 \neq |E_k| \cdots |E_2||E_1||A| = |E_k \cdots E_2E_1A| = |B| = 0$, a contradiction. Hence, A must be non-singular $\blacksquare$

Combining Theorem 11.1 with Theorem 14.1, we have

Theorem 14.2
The following statements are all equivalent:
(i) A is non-singular.
(ii) $|A| \neq 0$.
(iii) A is row equivalent to I_n.
(iv) The homogeneous systen $Ax = \mathbf{0}$ has only the trivial solution.
(v) $rank(A) = n$.

Example 14.1
Prove that $|A| = 0$ if and only if $Ax = \mathbf{0}$ has a non-trivial solution.

Solution.
If $|A| = 0$ then according to Theorem 14.2 the homogeneous system $Ax = \mathbf{0}$ must have a non-trivial solution. Conversely, if the homogeneous system $Ax = \mathbf{0}$ has a non-trivial solution then A must be singular by Theorem 14.2. By Theorem 14.1, $|A| = 0$ $\blacksquare$

Next, we discuss results concerning the determinant of a transpose.

Theorem 14.3
If A is an $n \times n$ matrix then $|A^T| = |A|$.

Proof.
(i) Suppose first that A is singular. Then $|A| = 0$. By Theorem 9.3(c), A^T is singular and by Theorem 14.1, $|A^T| = 0$.
(ii) Suppose that A is non-singular. Then A can be written as the product of elementary matrices $A = E_1E_2 \cdots E_k$. By Example 10.4 and Theorem 13.3, $|E_j^T| = |E_j$ for $1 \leq j \leq k$. Hence, $|A^T| = |E_k^T \cdot E_2^T E_1^T| = |E_k^T| \cdots |E_2^T||E_1^T| = |E_k| \cdots |E_2||E_1| = |E_1E_2 \cdots E_k| = |A|$ ■

It is worth noting here that the above Theorem says that every property about determinants that contains the word "row" in its statement is also true when the word 'column" is substituted for 'row".

Example 14.2
(a) Show that a square matrix with two identical columns has a zero determinant.
(b) Show that a square matrix with a column of zeros has a zero determinant.
(c) Show that if a square matrix has two proportional columns then its determinant is zero.

Solution.
(a) If A has two identical columns then A^T has two identical rows so that by Theorem 13.2, we have $|A^T| = 0$. By Theorem 14.3, $|A| = 0$.
(b) If A has acolmun of zeros then A^T has a row of zeros so that by Example 12.4, we have $|A^T| = 0$. By Theorem 14.3, $|A| = 0$.
(c) If A has two proportional columns then A^T has two proportional rows so that by Example 13.4, we have $|A^T| = 0$. By Theorem 14.3, $|A| = 0$ ■

Our next major result in this section concerns the determinant of a product of matrices.

Theorem 14.4
If A and B are $n \times n$ matrices then $|AB| = |A||B|$.

Example 14.3
Is it true that $|A + B| = |A| + |B|$?

Solution.
No. Consider the following matrices.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then $|A + B| = |\mathbf{0}| = 0$ and $|A| + |B| = -2$ ■

Example 14.4
Show that if A is invertible then $|A^{-1}| = \frac{1}{|A|}$.

Solution.
If A is invertible then $A^{-1}A = I_n$. Taking the determinant of both sides we find $|A^{-1}||A| = 1$. That is, $|A^{-1}| = \frac{1}{|A|}$. Note that, since A is invertible, $|A| \neq 0$ ■

Example 14.5
Let A and B be two **similar** square matrices , i.e. there exists a nonsingular matrix P such that $A = P^{-1}BP$. Show that $|A| = |B|$.

Solution.
Using Theorem 14.4 and Example 14.4 we have, $|A| = |P^{-1}BP| = |P^{-1}||B||P| = \frac{1}{|P|}|B||P| = |B|$. Note that since P is nonsingular, $|P| \neq 0$ ■

Example 14.6
Find all values of x such that the matrix

$$A = \begin{bmatrix} 2 & 0 & 10 \\ 0 & 7+x & -3 \\ 0 & 4 & x \end{bmatrix}$$

is invertible.

Solution.
Expanding along the first column, we find $|A| = 2[x(7 + x) + 12] = 2(x^2 + 7x + 12)$. Now, A is invertible if and only if $2(x^2 + 7x + 12) \neq 0$. This is equivalent to $x^2 + 7x + 12 \neq 0$. Factoring the quadratic expression, we find $x^2 + 7x + 12 = (x + 3)(x + 4)$. Hence, A is invertible if and only if $x \neq -3$ and $x \neq -4$ ■

Practice Problems

Problem 14.1
Show that if n is any positive integer then $|A^n| = |A|^n$.

Problem 14.2
Show that if A is an $n \times n$ skew-symmetric and n is odd then $|A| = 0$.

Problem 14.3
Show that if A is **orthogonal**, i.e. $A^T A = AA^T = I_n$ then $|A| = \pm 1$. Note that $A^{-1} = A^T$.

Problem 14.4
If A is a non-singular matrix such that $A^2 = A$, what is $|A|$?

Problem 14.5
Find out, without solving the system, whether the following system has a non-trivial solution

$$\begin{cases} x_1 - 2x_2 + \ \ x_3 = 0 \\ 2x_1 + 3x_2 + \ \ x_3 = 0 \\ 3x_1 + \ \ x_2 + 2x_3 = 0. \end{cases}$$

Problem 14.6
For which values of c does the matrix

$$A = \begin{bmatrix} 1 & 0 & -c \\ -1 & 3 & 1 \\ 0 & 2c & -4 \end{bmatrix}$$

have an inverse?

Problem 14.7
If $|A| = 2$ and $|B| = 5$, calculate $|A^3 B^{-1} A^T B^2|$.

Problem 14.8
Show that $|AB| = |BA|$.

Problem 14.9
Show that $|A + B^T| = |A^T + B|$ for any $n \times n$ matrices A and B.

Problem 14.10
Let $A = [a_{ij}]$ be a triangular matrix. Show that $|A| \neq 0$ if and only if $a_{ii} \neq 0$, for $1 \leq i \leq n$.

Problem 14.11
Using determinant, show that the rank of the matrix

$$A = \begin{bmatrix} 3 & 0 & 2 & -1 \\ 1 & 2 & 0 & -2 \\ 4 & 0 & 6 & -3 \\ 5 & 0 & 2 & 0 \end{bmatrix}$$

is 4.

Problem 14.12
Show, by inspection, that the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 8 & 9 \\ 10 & 20 & 21 \end{bmatrix}$$

is singular.

Problem 14.13
Let A be a 3×3 matrix with $|A| = 5$. Find each of the following
(a) $|A^4|$ (b) $|A^T|$ (c) $|5A|$ (d) $|A^{-1}|$.

Problem 14.14
Let A be a 3×3 matrix such that $|A| = 3$. Find $|(3A)^{-1}|$.

Problem 14.15
Show that the matrix

$$\begin{bmatrix} 2x^2+3 & x^2 & 1 \\ 4x & 2x & 0 \\ 4 & 2 & 0 \end{bmatrix}$$

is singular for all values of x.

Problem 14.16
Using determinants, show that the rank of the following matrix is less than 4.

$$A = \begin{bmatrix} 1 & 4 & 2 & 3 \\ 0 & 1 & 4 & 4 \\ -1 & 0 & 1 & 0 \\ 2 & 0 & 4 & 1 \end{bmatrix}$$

Problem 14.17
Let A be a 3×3 invertible matrix such that $|-8A^2| = |A^{-1}|$. Find the value of $|A|$.

Problem 14.18
Show that the matrix

$$\begin{bmatrix} 3 & 2 & -2 & 0 \\ 5 & -6 & -1 & 0 \\ -6 & 0 & 3 & 0 \\ 4 & 7 & 0 & -3 \end{bmatrix}$$

is singular.

Problem 14.19
Let

$$A = \begin{bmatrix} 2 & 4 & 2 & 1 \\ 4 & 3 & 0 & -1 \\ -6 & 0 & 2 & 0 \\ 0 & 1 & 1 & 2 \end{bmatrix}.$$

Find $|A^T|$.

Problem 14.20
Show that $|AB| = |A||B|$ if A is singular.

Problem 14.21
Suppose that A and B are two $n \times n$ matrices such that A is invertible. Using this fact and Problem 13.16, show that $|AB| = |A||B|$.

15 Finding A^{-1} Using Cofactors

In Section 11, we discussed the row reduction method for finding the inverse of a matrix. In this section, we introduce a method for finding the inverse of a matrix that uses cofactors.
If A is an $n \times n$ square matrix and C_{ij} is the cofactor of the entry a_{ij} then we define the **adjoint** of A to be the matrix

$$\mathrm{adj}(A) = \begin{bmatrix} C_{11} & C_{12} & \dots & C_{1n} \\ C_{21} & C_{22} & \dots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \dots & C_{nn} \end{bmatrix}^T.$$

Example 15.1
Let

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix}.$$

Find $\mathrm{adj}(A)$.

Solution.
We first find the matrix of cofactors of A.

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} = \begin{bmatrix} 12 & 6 & -16 \\ 4 & 2 & 16 \\ 12 & -10 & 16 \end{bmatrix}.$$

The adjoint of A is the transpose of this cofactor matrix.

$$\mathrm{adj}(A) = \begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix} \blacksquare$$

Our next goal is to find another method for finding the inverse of a nonsingular square matrix based on the adjoint. To this end, we need the following result.

Theorem 15.1
For $i \neq j$ we have

$$a_{i1}C_{j1} + a_{i2}C_{j2} + \cdots + a_{in}C_{jn} = 0.$$

Proof.
Let B be the matrix obtained by replacing the j^{th} row of A by the i^{th} row of A. Then B has two identical rows and therefore $|B| = 0$(See Theorem 13.2 (a)). Expand $|B|$ along the j^{th} row. The elements of the j^{th} row of B are $a_{i1}, a_{i2}, \ldots, a_{in}$. The cofactors are $C_{j1}, C_{j2}, \ldots, C_{jn}$. Thus

$$0 = |B| = a_{i1}C_{j1} + a_{i2}C_{j2} + \cdots + a_{in}C_{jn}.$$

This concludes a proof of the theorem ■

The following theorem states that the product $A \cdot \text{adj}(A)$ is a scalar multiple of the identity matrix.

Theorem 15.2
If A is an $n \times n$ matrix then $A \cdot \text{adj}(A) = |A|I_n$.

Proof.
The (i, j) entry of the matrix

$$A \cdot \text{adj}(A) = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix} \begin{bmatrix} C_{11} & C_{21} & \ldots & C_{n1} \\ C_{12} & C_{22} & \ldots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \ldots & C_{nn} \end{bmatrix}$$

is given by the sum

$$a_{i1}C_{j1} + a_{i2}C_{j2} + \cdots + a_{in}C_{jn} = |A|$$

if $i = j$ and 0 if $i \neq j$. Hence,

$$A \cdot \text{adj}(A) = \begin{bmatrix} |A| & 0 & \ldots & 0 \\ 0 & |A| & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & |A| \end{bmatrix} = |A|I_n.$$

This ends a proof of the theorem ■

The following theorem provides a way for finding the inverse of a matrix using the notion of the adjoint.

Theorem 15.3
If $|A| \neq 0$ then A is invertible and $A^{-1} = \frac{\text{adj}(A)}{|A|}$. Hence, $\text{adj}(A) = A^{-1}|A|$.

Proof.
By the previous theorem we have that $A(\text{adj}(A)) = |A|I_n$. If $|A| \neq 0$ then $A\left(\frac{adj(A)}{|A|}\right) = I_n$. By Theorem 9.2, A is invertible with inverse $A^{-1} = \frac{adj(A)}{|A|}$ ■

Example 15.2
Let

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix}.$$

Use Theorem 15.3 to find A^{-1}.

Solution.
First we find the determinant of A given by $|A| = 64$. By Theorem 15.3 and Example 15.1, we find

$$A^{-1} = \frac{1}{|A|}\text{adj}(A) = \begin{bmatrix} \frac{3}{16} & \frac{1}{16} & \frac{3}{16} \\ \frac{3}{32} & \frac{1}{32} & -\frac{5}{32} \\ -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix} \blacksquare$$

In the next theorem we discuss three properties of the adjoint matrix.

Theorem 15.4
Let A and B denote invertible $n \times n$ matrices. Then,
(a) $\text{adj}(A^{-1}) = (\text{adj}(A))^{-1}$.
(b) $\text{adj}(A^T) = (\text{adj}(A))^T$.
(c) $\text{adj}(AB) = \text{adj}(B)\text{adj}(A)$.

Proof.
(a) Since $A(\text{adj}(A)) = |A|I_n$, $\text{adj}(A)$ is invertible (Theorem 9.2) and $(\text{adj}(A))^{-1} = \frac{A}{|A|} = (A^{-1})^{-1}|A^{-1}| = \text{adj}(A^{-1})$.

(b) $\text{adj}(A^T) = (A^T)^{-1}|A^T| = (A^{-1})^T|A| = [A^{-1}|A|]^T = (\text{adj}(A))^T$.

(c) We have $\text{adj}(AB) = (AB)^{-1}|AB| = B^{-1}A^{-1}|A||B| = (B^{-1}|B|)(A^{-1}|A|) = \text{adj}(B)\text{adj}(A)$ ■

Example 15.3
Show that if A is singular then $A \cdot \text{adj}(A) = \mathbf{0}$, the zero matrix.

Solution.
If A is singular then $|A| = 0$. But then $A \cdot \text{adj}(A) = |A|I_n = \mathbf{0}$ ■

Practice Problems

Problem 15.1
Let
$$A = \begin{bmatrix} 3 & -2 & 1 \\ 5 & 6 & 2 \\ 1 & 0 & -3 \end{bmatrix}.$$
(a) Find $\mathrm{adj}(A)$.
(b) Compute $|A|$.

Problem 15.2
Let A be an $n \times n$ matrix. Show that $|\mathrm{adj}(A)| = |A|^{n-1}$.

Problem 15.3
If
$$A^{-1} = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & 3 \\ 3 & 1 & -1 \end{bmatrix},$$
find $\mathrm{adj}(A)$.

Problem 15.4
If $|A| = 2$, find $|A^{-1} + \mathrm{adj}(A)|$.

Problem 15.5
Show that $\mathrm{adj}(\alpha A) = \alpha^{n-1}\mathrm{adj}(A)$.

Problem 15.6
Consider the matrix
$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 1 & 5 & 7 \end{bmatrix}.$$
(a) Find $|A|$.
(b) Find $\mathrm{adj}(A)$.
(c) Find A^{-1}.

Problem 15.7
Prove that if A is symmetric then $\mathrm{adj}(A)$ is also symmetric.

Problem 15.8
Prove that if A is a non-singular triangular matrix then $\text{adj}(A)$ is a lower triangular matrix.

Problem 15.9
Prove that if A is a non-singular triangular matrix then A^{-1} is also triangular.

Problem 15.10
Let A be an $n \times n$ matrix.
(a) Show that if A has integer entries and $|A| = 1$ then A^{-1} has integer entries as well.
(b) Let $Ax = b$. Show that if the entries of A and b are integers and $|A| = 1$ then the entries of x are also integers.

Problem 15.11
Let

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 4 & 0 & 6 \\ 0 & 1 & -1 \end{bmatrix}.$$

(a) Find $|A|$.
(b) Find $\text{adj}(A)$.
(c) Find A^{-1}.

Problem 15.12
Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 2 & 1 \\ 4 & 5 & 1 \end{bmatrix}.$$

(a) Find $|A|$.
(b) Find the $(3,2)-$entry of A^{-1} without finding A^{-1}.

Problem 15.13
Let

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}.$$

(a) Find the cofactor matrix of A.
(b) Find $\text{adj}(A)$.
(c) Find $|A|$.
(d) Find A^{-1}.

Problem 15.14
Find the inverse of

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 2 \\ 2 & 1 & 0 \end{bmatrix}$$

using the adjoint method.

Problem 15.15
Let A be an $n \times n$ non-singular matrix. Show that $\mathrm{adj}(\mathrm{adj}(A)) = |A|^{n-2}A$.

Problem 15.16
Let $D = [d_{ii}]$ be an invertible $n \times n$ diagonal matrix. Find the (i, i) entry of $\mathrm{adj}(D)$.

Problem 15.17
Decide whether the given matrix is invertible, and if so, use the adjoint method to find its inverse.

$$A = \begin{bmatrix} 2 & -3 & 5 \\ 0 & 1 & -3 \\ 0 & 0 & 2 \end{bmatrix}.$$

Problem 15.18
Let A be a 3×3 matrix such that $|A| = 5$. Find $|\mathrm{adj}(A)|$.

Problem 15.19
Let A be a 3×3 matrix such that $|A| = 5$. Find $|A \cdot \mathrm{adj}(A)|$.

Problem 15.20
Show that $\mathrm{adj}(A^n) = [\mathrm{adj}(A)]^n$, where n is a positive integer.

16 Application of Determinants to Systems: Cramer's Rule

Cramer's rule is another method for solving a linear system of n equations in n unknowns. This method is reasonable for inverting, for example, a 3×3 matrix by hand; however, the inversion method discussed before is more efficient for larger matrices.

Theorem 16.1
Let $Ax = b$ be a matrix equation with $A = [a_{ij}], x = [x_i], b = [b_i]$. We have

$$|A|x_i = |A_i|, \quad i = 1, 2, \cdots, n$$

where A_i is the matrix obtained from A by replacing its i^{th} column by b. It follows that

(1) If $|A| \neq 0$ then the system $Ax = b$ has a unique solution given by

$$x_i = \frac{|A_i|}{|A|},$$

where $1 \leq i \leq n$.
(2) If $|A| = 0$ and $|A_i| \neq 0$ for some i then the system $Ax = b$ has no solution.
(3) If $|A| = |A_1| = \cdots = |A_n| = 0$ then the system $Ax = b$ has an infinite number of solutions.

Proof.
First of all, from the equation $Ax = b$, we have,

$$b_i = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n, \quad i = 1, 2, \cdots, n.$$

Now, let $I_i(x)$ be the matrix obtained by replacing the i^{th} column of I_n by the vector x. Expanding along the i^{th} row of $I_i(x)$, we find $|I_i(x)| = x_i$ for $i = 1, 2, \cdots, n$.
Next, we have

$$AI_i(x) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1,i-1} & (a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n) & a_{1,i+1} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2,i-1} & (a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n) & a_{2,i+1} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n,i-1} & (a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n) & a_{n,i+1} & \cdots & a_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1,i-1} & b_1 & a_{1,i+1} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2,i-1} & b_2 & a_{2,i+1} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n,i-1} & b_n & a_{n,i+1} & \cdots & a_{nn} \end{bmatrix} = A_i.$$

Hence,

$$|A_i| = |AI_i(x)| = |A||I_i(x)| = |A|x_i, \quad i = 1, 2, \cdots, n.$$

Now, (1), (2), and (3) follow easily. This ends a proof of the theorem ■

Example 16.1
Use Cramer's rule to solve

$$\begin{cases} -2x_1 + 3x_2 - x_3 = 1 \\ \quad x_1 + 2x_2 - x_3 = 4 \\ -2x_1 - \quad x_2 + x_3 = -3. \end{cases}$$

Solution.
By Cramer's rule we have

$$A = \begin{bmatrix} -2 & 3 & -1 \\ 1 & 2 & -1 \\ -2 & -1 & 1 \end{bmatrix}, |A| = -2.$$

$$A_1 = \begin{bmatrix} 1 & 3 & -1 \\ 4 & 2 & -1 \\ -3 & -1 & 1 \end{bmatrix}, |A_1| = -4.$$

$$A_2 = \begin{bmatrix} -2 & 1 & -1 \\ 1 & 4 & -1 \\ -2 & -3 & 1 \end{bmatrix}, |A_2| = -6.$$

$$A_3 = \begin{bmatrix} -2 & 3 & 1 \\ 1 & 2 & 4 \\ -2 & -1 & -3 \end{bmatrix}, |A_3| = -8.$$

Thus, $x_1 = \frac{|A_1|}{|A|} = 2, x_2 = \frac{|A_2|}{|A|} = 3, x_3 = \frac{|A_3|}{|A|} = 4$ ■

Example 16.2
Use Cramer's rule to solve

$$\begin{cases} 5x_1 - 3x_2 - 10x_3 = -9 \\ 2x_1 + 2x_2 - \ \ 3x_3 = 4 \\ -3x_1 - \ \ x_2 + \ \ 5x_3 = 1. \end{cases}$$

Solution.
By Cramer's rule we have

$$A = \begin{bmatrix} 5 & -3 & -10 \\ 2 & 2 & -3 \\ -3 & -1 & 5 \end{bmatrix}, |A| = -2.$$

$$A_1 = \begin{bmatrix} -9 & -3 & -10 \\ 4 & 2 & -3 \\ 1 & -1 & 5 \end{bmatrix}, |A_1| = 66.$$

$$A_2 = \begin{bmatrix} 5 & -9 & -10 \\ 2 & 4 & -3 \\ -3 & 1 & 5 \end{bmatrix}, |A_2| = -16.$$

$$A_3 = \begin{bmatrix} 5 & -3 & -9 \\ 2 & 2 & 4 \\ -3 & -1 & 1 \end{bmatrix}, |A_3| = 36.$$

Thus, $x_1 = \frac{|A_1|}{|A|} = -33, x_2 = \frac{|A_2|}{|A|} = 8, x_3 = \frac{|A_3|}{|A|} = -18$ ■

Practice Problems

Problem 16.1
Use Cramer's Rule to solve

$$\begin{cases} x_1 + 2x_3 = 6 \\ -3x_1 + 4x_2 + 6x_3 = 30 \\ -x_1 - 2x_2 + 3x_3 = 8. \end{cases}$$

Problem 16.2
Use Cramer's Rule to solve

$$\begin{cases} 5x_1 + x_2 - x_3 = 4 \\ 9x_1 + x_2 - x_3 = 1 \\ x_1 - x_2 + 5x_3 = 2. \end{cases}$$

Problem 16.3
Use Cramer's Rule to solve

$$\begin{cases} 4x_1 - x_2 + x_3 = -5 \\ 2x_1 + 2x_2 + 3x_3 = 10 \\ 5x_1 - 2x_2 + 6x_3 = 1. \end{cases}$$

Problem 16.4
Use Cramer's Rule to solve

$$\begin{cases} 3x_1 - x_2 + 5x_3 = -2 \\ -4x_1 + x_2 + 7x_3 = 10 \\ 2x_1 + 4x_2 - x_3 = 3. \end{cases}$$

Problem 16.5
Use Cramer's Rule to solve

$$\begin{cases} -x_1 + 2x_2 + 3x_3 = -7 \\ -4x_1 - 5x_2 + 6x_3 = -13 \\ 7x_1 - 8x_2 - 9x_3 = 39. \end{cases}$$

Problem 16.6
Use Cramer's Rule to solve

$$\begin{cases} 3x_1 - 4x_2 + 2x_3 = 18 \\ 4x_1 + \ \ x_2 - 5x_3 = -13 \\ 2x_1 - 3x_2 + \ \ x_3 = 11. \end{cases}$$

Problem 16.7
Use Cramer's Rule to solve

$$\begin{cases} 5x_1 - 4x_2 + \ \ x_3 = 17 \\ 6x_1 + 2x_2 - 3x_3 = 1 \\ \ \ x_1 - 4x_2 + 3x_3 = 15. \end{cases}$$

Problem 16.8
Use Cramer's Rule to solve

$$\begin{cases} 2x_1 - 3x_2 + 2x_3 = 1 \\ 3x_1 + 2x_2 - \ \ x_3 = 16 \\ \ \ x_1 - 5x_2 + 3x_3 = -7. \end{cases}$$

Problem 16.9
Use Cramer's Rule to solve

$$\begin{cases} \ \ x_1 - 2x_2 + 2x_3 = 5 \\ 3x_1 + 2x_2 - 3x_3 = 13 \\ 2x_1 - 5x_2 + \ \ x_3 = 2. \end{cases}$$

Problem 16.10
Use Cramer's Rule to solve

$$\begin{cases} 5x_1 - \ \ x_2 + 3x_3 = 10 \\ 6x_1 + 4x_2 - \ \ x_3 = 19 \\ \ \ x_1 - 7x_2 + 4x_3 = -15. \end{cases}$$

Problem 16.11
Using Cramer's Rule, determine whether the system is consistent or inconsistent. If the system is consistent, determine whether it is independent or dependent.

$$\begin{cases} \ \ x_1 + 2x_2 + \ \ x_3 = 3 \\ \ \ x_1 + \ \ x_2 + 3x_3 = 1 \\ 3x_1 + 4x_2 + 7x_3 = 1. \end{cases}$$

Problem 16.12
Using Cramer's Rule, determine whether the system is consistent or inconsistent. If the system is consistent, determine whether it is independent or dependent.

$$\begin{cases} 3x_1 + x_2 + 7x_3 + 2x_4 = 13 \\ 2x_1 - 4x_2 + 14x_3 - x_4 = -10 \\ 5x_1 + 11x_2 - 7x_3 + 8x_4 = 59 \\ 2x_1 + 5x_2 - 4x_3 - 3x_4 = 39. \end{cases}$$

Problem 16.13
Using Cramer's Rule, determine whether the system is consistent or inconsistent. If the system is consistent, determine whether it is independent or dependent.

$$\begin{cases} x_1 - 2x_2 - 6x_3 = 12 \\ 2x_1 + 4x_2 + 12x_3 = -17 \\ x_1 - 4x_2 - 12x_3 = 22. \end{cases}$$

Problem 16.14
Use Cramer's rule to solve for x_3 only.

$$\begin{cases} x_1 + x_2 + 2x_3 = 8 \\ -x_1 - 2x_2 + 3x_3 = 1 \\ 3x_1 - 7x_2 + 4x_3 = 10. \end{cases}$$

Problem 16.15
Use Cramer's rule to express $\cos\alpha$ in terms of a, b, and c.

$$\begin{array}{lllllll} b\cos\alpha & + & a\cos\beta & & & = & c, \\ c\cos\alpha & & & + & a\cos\gamma & = & b, \\ & & c\cos\beta & + & b\cos\gamma & = & a. \end{array}$$

Problem 16.16
Solve the following linear system by using Cramer's rule.

$$\begin{bmatrix} 11 & 13 & 17 \\ 5 & 3 & 2 \\ 7 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Problem 16.17
True or false: One can find the value of a variable in a square linear system without using the values of the remaining variables.

Problem 16.18
True or false: Cramer's Rule can be applied to overdetermined and underdetermined linear systems.

Problem 16.19
Consider the system

$$\left\{\begin{array}{ccccc} a_{11}x_1 & + & a_{12}x_2 & = & b_1, \\ a_{21}x_1 & + & a_{22}x_2 & = & b_2. \end{array}\right.$$

Assume that $|A| = a_{11}a_{22} - a_{21}a_{12} \neq 0$.
(a) Show that $A^{-1} = |A|^{-1}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$.
(b) Show that

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = |A|^{-1}\begin{bmatrix} b_1a_{22} - b_2a_{12} \\ b_2a_{11} - b_1a_{21} \end{bmatrix}.$$

(c) Using (b), derive Cramer's Rule.

Problem 16.20
List three techniques that you have learned so far for solving square linear systems.

The Theory of Vector Spaces

In Chapter 2, we saw (Theorem 7.1) that the operations of addition and scalar multiplication on the set M_{mn} of $m \times n$ matrices possess many of the same algebraic properties as addition and scalar multiplication on the set $\mathbb{R}$ of real numbers. In fact, there are many other sets with operations that share these same properties. Instead of studying these sets individually, we study them as a class.

In this chapter, we define vector spaces to be sets with algebraic operations having the properties similar to those of addition and scalar multiplication on $\mathbb{R}$ and M_{mn}. We then establish many important results that apply to all vector spaces, not just $\mathbb{R}$ and M_{mn}.

17 Vector Spaces and Subspaces

Why study vector spaces? Simply because they are applied throughout mathematics, science and engineering. For example, they provide an environment that can be used for solution techniques for ordinary linear differential equations. To give an example, when solving a second order homogeneous linear ordinary differential equation, one looks for two solutions y_1 and y_2 that are linearly independent. In this case, the general solution (i.e. the formula that generates all the solutions to the differential equation) is given by

$$y(t) = c_1y_1(t) + c_2y_2(t).$$

The concept of linear independence applies to elements of algebraic systems known as vector spaces.
In this section, we define vector spaces to be sets with algebraic operations having the properties similar to those of addition and scalar multiplication on $\mathbb{R}^n$ and M_{mn}.

Let's start with an example. Let n be a positive integer. Let $\mathbb{R}^n$ be the collection of elements of the form $(x_1, x_2, \ldots, x_n)$, where the $x_i's$ are real numbers. Define the following operations on $\mathbb{R}^n$:

(a) Addition: $(x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n)$.
(b) Multiplication of a vector by a scalar:

$$\alpha(x_1, x_2, \ldots, x_n) = (\alpha x_1, \alpha x_2, \ldots, \alpha x_n).$$

The basic properties of addition and scalar multiplication of vectors in $\mathbb{R}^n$ are listed in the following theorem.

Theorem 17.1
The following properties hold, for u, v, w in $\mathbb{R}^n$ and α, β scalars:
(a) $u + v = v + u$
(b) $u + (v + w) = (u + v) + w$
(c) $u + 0 = 0 + u = u$ where $0 = (0, 0, \ldots, 0)$
(d) $u + (-u) = 0$
(e) $\alpha(u + v) = \alpha u + \alpha v$
(f) $(\alpha + \beta)u = \alpha u + \beta u$
(g) $\alpha(\beta u) = (\alpha\beta)u$
(h) $1u = u$.

The set $\mathbb{R}^n$ with the above operations and properties is called the **Euclidean space.**

Many concepts concerning vectors in $\mathbb{R}^n$ can be extended to other mathematical systems. We can think of a vector space in general, as a collection of objects that behave as vectors do in $\mathbb{R}^n$. The objects of such a set are called **vectors.**
A **vector space** is a set V together with the following operations:
(i) Addition: If $u, v \in V$ then $u + v \in V$. We say that V is **closed under addition**.
(ii) Multiplication of an element by a scalar: If $\alpha \in \mathbb{R}$ and $u \in V$ then $\alpha u \in V$. That is, V is **closed under scalar multiplication.**
(iii) These operations satisfy the properties (a) - (h) of Theorem 17.1.

Example 17.1
Let M_{mn} be the collection of all $m \times n$ matrices. Show that M_{mn} is a vector space using matrix addition and scalar multiplication.

Solution.
See Problem 17.17 ■

Example 17.2
Let $V = \{(x, y) : x \geq 0, y \geq 0\}$. Show that the set V fails to be a vector space under the standard operations on $\mathbb{R}^2$.

Solution.
For any $(x, y) \in V$ with $x, y > 0$, we have $-(x, y) = (-x, -y) \notin V$. Thus, V is not a vector space ■

The following theorem exhibits some properties which follow directly from the axioms of the definition of a vector space and therefore hold for every vector space.

Theorem 17.2
Let V be a vector space, u a vector in V and α is a scalar. Then the following properties hold:
(a) $(-1)u = -u$
(b) $0u = 0$.

(c) $\alpha 0 = 0$
(d) If $\alpha u = 0$ then $\alpha = 0$ or $u = 0$.

Proof.
See Problem 17.18 ■

Vector spaces may be formed from subsets of other vectors spaces. These are called subspaces. A non-empty subset W of a vector space V is called a **subspace** of V if the following two properties are satisfied:

(i) If u, v are in W then $u + v$ is also in W.
(ii) If α is a scalar and u is in W then αu is also in W.

Every vector space V has at least two subspaces: V itself and the subspace consisting of the zero vector of V. These are called the **trivial** subspaces of V.

Example 17.3
Let S be the subset of M_{nn} consisting of symmetric matrices. Show that S is a subspace of M_{nn}.

Solution.
Since $I_n^T = I_n$, $I_n \in S$ and S is non-empty. Now, if A and B belong to S then $(A+B)^T = A^T + B^T = A + B$ and $(\alpha A)^T = \alpha A^T = \alpha A$. That is, $A + B \in S$ and $\alpha A \in S$. By the previous theorem, S is a subspace of M_{nn} ■

Example 17.4
Show that a subspace of a vector space is itself a vector space.

Solution.
The elements of the subspace inherits all the axioms of a vector space ■

The following theorem provides a criterion for deciding whether a subset S of a vector space V is a subspace of V.

Theorem 17.3
Show that $W \neq \emptyset$ is a subspace of V if and only if $\alpha u + v \in W$ for all $u, v \in W$ and $\alpha \in \mathbb{R}$.

Proof.
Suppose that W is a non-empty subspace of V. If $u, v \in W$ and $\alpha \in \mathbb{R}$ then $\alpha u \in W$ and therefore $\alpha u + v \in W$. Conversely, suppose that for all $u, v \in W$ and $\alpha \in \mathbb{R}$ we have $\alpha u + v \in \mathbb{R}$. In particular, if $\alpha = 1$ then $u + v \in W$. If $v = 0$ then $\alpha u + v = \alpha u \in W$. Hence, W is a subspace ■

Example 17.5
Let M_{22} be the collection of 2×2 matrices. Show that the set W of all 2×2 matrices having zeros on the main diagonal is a subspace of M_{22}.

Solution.
The set W is the set

$$W = \left\{ \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix} : a, b \in \mathbb{R} \right\}.$$

Clearly, the 2×2 zero matrix belongs to W so that $W \neq \emptyset$. Also,

$$\alpha \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix} + \begin{bmatrix} 0 & a' \\ b' & 0 \end{bmatrix} = \begin{bmatrix} 0 & \alpha a + a' \\ \alpha b + b' & 0 \end{bmatrix} \in W$$

Thus, W is a subspace of M_{22} ■

Practice Problems

Problem 17.1
Let $D([a,b])$ be the collection of all differentiable functions on $[a,b]$. Show that $D([a,b])$ is a subspace of the vector space of all functions defined on $[a,b]$.

Problem 17.2
Let A be an $m \times n$ matrix. Show that the set $S = \{x \in \mathbb{R}^n : Ax = \mathbf{0}\}$ is a subspace of $\mathbb{R}^n$.

Problem 17.3
Let $\mathbf{P}$ be the collection of polynomials in the indeterminate x. Let $p(x) = a_0 + a_1x + a_2x^2 + \cdots$ and $q(x) = b_0 + b_1x + b_2x^2 + \cdots$ be two polynomials in $\mathbf{P}$. Define the operations:
(a) Addition: $p(x) + q(x) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \cdots$
(b) Multiplication by a scalar: $\alpha p(x) = \alpha a_0 + (\alpha a_1)x + (\alpha a_2)x^2 + \cdots$.
Show that $\mathbf{P}$ is a vector space.

Problem 17.4
Let $F(\mathbb{R})$ be the set of functions $f : \mathbb{R} \to \mathbb{R}$. Define the operations

$$(f + g)(x) = f(x) + g(x)$$

and

$$(\alpha f)(x) = \alpha f(x).$$

Show that $F(\mathbb{R})$ is a vector space under these operations.

Problem 17.5
Define on $\mathbb{R}^2$ the following operations:
(i) $(x, y) + (x', y') = (x + x', y + y')$;
(ii) $\alpha(x, y) = (\alpha y, \alpha x)$.
Show that $\mathbb{R}^2$ with the above operations is not a vector space.

Problem 17.6
Let $U = \{p(x) \in \mathbf{P} : p(3) = 0\}$. Show that U is a subspace of **P.**

Problem 17.7
Let P_n denote the collection of all polynomials of degree n. Show that P_n is a subspace of $\mathbf{P}$.

Problem 17.8
Show that the set $S = \{(x, y) : x \leq 0\}$ is not a vector space of $\mathbb{R}^2$ under the usual operations of $\mathbb{R}^2$.

Problem 17.9
Show that the collection $C([a, b])$ of all continuous functions on $[a, b]$ with the operations:

$$(f + g)(x) = f(x) + g(x)$$
$$(\alpha f)(x) = \alpha f(x)$$

is a vector space.

Problem 17.10
Let $S = \{(a, b, a + b) : a, b \in \mathbb{R}\}$. Show that S is a subspace of $\mathbb{R}^3$ under the usual operations.

Problem 17.11
Let V be a vector space. Show that if $u, v, w \in V$ are such that $u+v = u+w$ then $v = w$.

Problem 17.12
Let H and K be subspaces of a vector space V.
(a) The **intersection** of H and K, denoted by $H \cap K$, is the subset of V that consists of elements that belong to both H and K. Show that $H \cap K$ is a subspace of V.
(b) The **union** of H and K, denoted by $H \cup K$, is the susbet of V that consists of all elements that belong to either H or K. Give, an example of two subspaces of V such that $H \cup K$ is not a subspace.
(c) Show that if $H \subset K$ or $K \subset H$ then $H \cup K$ is a subspace of V.

Problem 17.13
Show that $S = \left\{ \begin{bmatrix} a & a-2 \\ b & c \end{bmatrix} \right\}$ is not a subspace of M_{22}, the vector space of 2×2 matrices.

Problem 17.14
Show that $V = \{(a, b, c) : a + b = 2c\}$ with the usual operations on $\mathbb{R}^3$ is a vector space.

Problem 17.15
Show that $S = \{(a, b) \in \mathbb{R}^2 : a^2 - b^2 = 0\}$ is not a subspace of $\mathbb{R}^2$.

Problem 17.16
Prove Theorem 17.1.

Problem 17.17
Let M_{mn} be the collection of all $m \times n$ matrices. Show that M_{mn} is a vector space using matrix addition and scalar multiplication.

Problem 17.18
Prove Theorem 17.2

Problem 17.19
Let V be a vector space over $\mathbb{R}$ and $v_1, v_2, \cdots, v_n$ be elements of V. Let

$$S = \{c_1 v_1 + c_2 v_2 + \cdots + c_n v_n : \text{where } c_1, c_2, \cdots, c_n \in \mathbb{R}\}.$$

Show that S is a subspace of V.

Problem 17.20
Show that the set S all $n \times n$ diagonal matrices is a subspace of M_{nn}.

18 Basis and Dimension

Consider the question of finding the general solution to a second order linear homogeneous ODE with constant coefficient:

$$ay'' + by' + cy = 0 \quad a < t < b.$$

Let S be the solution set of this equation. By the principle of superposition, if u and v are in S and $\alpha \in \mathbb{R}$ then $\alpha u + v \in S$. Hence, S is a subspace of the vector space $F[(a,b)]$ of all real valued functions defined on the interval (a,b). Let y_1 and y_2 be two elements of S with the property

$$W(y_1(t), y_2(t)) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2 \neq 0, \text{ for some } a < t < b. \tag{18.1}$$

We call W the **Wronskian** of y_1 and y_2 and the set $\{y_1, y_2\}$ a **fundamental set**.

Now, let T be the set of all linear combinations of y_1 and y_2. Then T is a subspace of $F[(a,b)]$ and $T \subseteq S$. It is shown in the theory of differential equations that if y is a solution to the ODE then condition (18.1) asserts the existence of constants c_1 and c_2 such that $y = c_1 y_1 + c_2 y_2$. That is, $S \subseteq T$. Hence, $S = T$ and all the solutions to the ODE are generated from y_1 and y_2.

The concepts of linear combination, spanning set, and basis for a vector space play a major role in the investigation of the structure of any vector space. In this section, we introduce and discuss these concepts.

The concept of linear combination will allow us to generate vector spaces from a given set of vectors in a vector space .

Let V be a vector space and $v_1, v_2, \cdots, v_n$ be vectors in V. A vector $w \in V$ is called a **linear combination** of the vectors $v_1, v_2, \ldots, v_n$ if it can be written in the form

$$w = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$$

for some scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$.

Example 18.1

Show that the vector $\vec{w} = (9, 2, 7)$ is a linear combination of the vectors $\vec{u} = (1, 2, -1)$ and $\vec{v} = (6, 4, 2)$ whereas the vector $\vec{w'} = (4, -1, 8)$ is not.

Solution.
We must find numbers s and t such that

$$(9,2,7) = s(1,2,-1) + t(6,4,2).$$

This leads to the system

$$\begin{cases} s + 6t = 9 \\ 2s + 4t = 2 \\ -s + 2t = 7. \end{cases}$$

Solving the first two equations one finds $s = -3$ and $t = 2$ both values satisfy the third equation.
Turning to $(4,-1,8)$, the question is whether s and t can be found such that $(4,-1,8) = s(1,2,-1) + t(6,4,2)$. Equating components gives

$$\begin{cases} s + 6t = 4 \\ 2s + 4t = -1 \\ -s + 2t = 8. \end{cases}$$

Solving the first two equations one finds $s = -\frac{11}{4}$ and $t = \frac{9}{8}$ and these values do not satisfy the third equation. That is, the system is inconsistent ■

The process of forming linear combinations leads to a method of constructing subspaces, as follows.

Theorem 18.1
Let $W = \{v_1, v_2, \dots, v_n\}$ be a subset of a vector space V. Let $span(W)$ be the collection of all linear combinations of elements of W. Then $span(W)$ is a subspace of V.

Proof.
Let $u = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$ and $v = \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_n v_n$. Then for any scalar α, we have

$$\begin{aligned} \alpha u + v =& \alpha(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n) + (\beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_n v_n) \\ =& (\alpha\alpha_1 + \beta_1)v_1 + (\alpha\alpha_2 + \beta_2)v_2 + \cdots + (\alpha\alpha_n + \beta_n)v_n \in W. \end{aligned}$$

Hence, W is a subspace of V ■

Example 18.2
Show that $P_n = span\{1, x, x^2, \cdots, x^n\}$.

Solution.
Clearly, $span\{1, x, x^2, \cdots, x^n\} \subseteq P_n$. If $p(x) \in P_n$ then there are scalars $a_0, a_1 \cdots, a_n$ such that $p(x) = a_0 + a_1 x + \cdots + a_n x^n \in span\{1, x, \cdots, x^n\}$. That is, $P_n \subseteq span\{1, x, x^2, \cdots, x^n\}$ ■

Example 18.3
Show that $\mathbb{R}^n = span\{e_1, e_2, \cdots, e_n\}$ where e_i is the vector with 1 in the i^{th} component and 0 otherwise.

Solution.
Clearly, $span\{e_1, e_2, \cdots, e_n\} \subseteq \mathbb{R}^n$. We must show that if $u \in \mathbb{R}^n$ then u is a linear combination of the $e_i's$. Indeed, if $u = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$ then

$$u = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n$$

Hence u lies in $span\{e_1, e_2, \cdots, e_n\}$. That is, $\mathbb{R}^n \subseteq span\{e_1, e_2, \cdots, e_n\}$ ■

If every element of V can be written as a linear combination of elements of W then we have $V = span(W)$ and in this case we say that W is a **span** of V or W **generates** V.

Example 18.4
(a)Determine whether $\vec{v_1} = (1, 1, 2), \vec{v_2} = (1, 0, 1)$ and $\vec{v_3} = (2, 1, 3)$ span $\mathbb{R}^3$.
(b) Show that the vectors $\vec{i} = (1, 0, 0), \vec{j} = (0, 1, 0)$, and $\vec{k} = (0, 0, 1)$ span $\mathbb{R}^3$.

Solution.
(a) We must show that an arbitrary vector $\vec{v} = (a, b, c)$ in $\mathbb{R}^3$ is a linear combination of the vectors $\vec{v_1}, \vec{v_2}$, and $\vec{v_3}$. That is $\vec{v} = s\vec{v_1} + t\vec{v_2} + w\vec{v_3}$. Expressing this equation in terms of components gives

$$\begin{cases} s + t + 2w = a \\ s \quad\ + \ w = b \\ 2s + t + 3w = c. \end{cases}$$

The problem is reduced of showing that the above system is consistent. This system will be consistent if and only if the coefficient matrix A

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix}$$

is invertible. Since $|A| = 0$, the system is inconsistent and therefore $\mathbb{R}^3 \neq span\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$.
(b) This is Example 18.3 with $n = 3$ ■

Next, we introduce a concept which guarantees that any vector in the span of a set $S \subseteq V$ has only one representation as a linear combination of vectors in S. Spanning sets with this property play a fundamental role in the study of vector spaces as we shall see later in this section.
If $v_1, v_2, \ldots, v_n$ are vectors in a vector space with the property that

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$$

holds only for $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$ then the vectors are said to be **linearly independent.** If there are scalars not all 0 such that the above equation holds then the vectors are called **linearly dependent.**

Example 18.5
Show that the set $S = \{1, x, x^2, \cdots, x^n\}$ is a linearly independent set in P_n.

Solution.
Suppose that $a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = 0$ for all $x \in \mathbb{R}$. By the Fundamental Theorem of Algebra, a polynomial of degree n has at most n roots. But by the above equation, every real number is a root of the equation. This forces the numbers $a_0, a_1, \cdots, a_n$ to be 0 ■

Example 18.6
Let u be a nonzero vector. Show that $\{u\}$ is linearly independent.

Solution.
Suppose that $\alpha u = 0$. If $\alpha \neq 0$ then we can multiply both sides by α^{-1} and obtain $u = 0$. But this contradicts the fact that u is a nonzero vector. Hence, we must have $\alpha = 0$ ■

Example 18.7
(a) Show that the vectors $\vec{v_1} = (1, 0, 1, 2), \vec{v_2} = (0, 1, 1, 2)$, and $\vec{v_3} = (1, 1, 1, 3)$ are linearly independent.
(b) Show that the vectors $\vec{v_1} = (1, 2, -1), \vec{v_2} = (1, 2, -1)$, and $\vec{v_3} = (1, -2, 1)$ are linearly dependent.

Solution.
(a) Suppose that s, t, and w are real numbers such that $s\vec{v_1} + t\vec{v_2} + w\vec{v_3} = \mathbf{0}$. Then equating components gives

$$\begin{cases} s \qquad + \; w = 0 \\ \qquad t + \; w = 0 \\ s + \; t + \; w = 0 \\ 2s + 2t + 3w = 0. \end{cases}$$

The second and third equation leads to $s = 0$. The first equation gives $w = 0$ and the second equation gives $t = 0$. Thus, the given vectors are linearly independent.
(b) These vectors are linearly dependent since $\vec{v_1} + \vec{v_2} - 2\vec{v_3} = \mathbf{0}$. Alternatively, the coefficient matrix of homogeneous system

$$\begin{cases} s + 2t - w = 0 \\ s + 2t - w = 0 \\ s - 2t + w = 0. \end{cases}$$

is singular ■

Example 18.8
Show that the unit vectors $e_1, e_2, \cdots, e_n$ in $\mathbb{R}^n$ are linearly independent.

Solution.
Suppose that $x_1e_1 + x_2e_2 + \cdots + x_ne_n = (0, 0, \cdots, 0)$. Then $(x_1, x_2, \cdots, x_n) = (0, 0, \cdots, 0)$ and this leads to $x_1 = x_2 = \cdots = x_n = 0$. Hence the vectors $e_1, e_2, \cdots, e_n$ are linearly independent ■

Let $S = \{v_1, v_2, \dots, v_n\}$ be a subset of a vector space V. We say that S is a **basis** for V if and only if
(i) S is linearly independent set.
(ii) $V = span(S)$.

Example 18.9
Let e_i be the vector of $\mathbb{R}^n$ whose i^{th} component is 1 and zero otherwise. Show that the set $S = \{e_1, e_2, \dots, e_n\}$ is a basis for $\mathbb{R}^n$. This is called the **standard basis** of $\mathbb{R}^n$.

Solution.
By Example 18.3, we have $\mathbb{R}^n = span\{e_1, e_2, \cdots, e_n\}$. By Example 18.8, the vectors $e_1, e_2, \cdots, e_n$ are linearly independent. Thus $\{e_1, e_2, \cdots, e_n\}$ is a basis of $\mathbb{R}^3$ ■

Example 18.10
Show that $\{1, x, x^2, \cdots, x^n\}$ is a basis of P_n.

Solution.
By Example 18.2, $P_n = span\{1, x, x^2, \cdots, x^n\}$ and by Example 18.5, the set $S = \{1, x, x^2, \cdots, x^n\}$ is linearly independent. Thus, S is a basis of P_n ■

If $S = \{v_1, v_2, \ldots, v_n\}$ is a basis for V then we say that V is a **finite dimensional space of dimension** n. We write $dim(V) = n$. A vector space which is not finite dimensional is said to be **infinite dimensional** vector space. We define the zero vector space to have dimension zero. The vector spaces $M_{mn}, \mathbb{R}^n$, and P_n are finite-dimensional spaces whereas the space P of all polynomials and the vector space of all real-valued functions defined on $\mathbb{R}$ are inifinite dimensional vector spaces.
Unless otherwise specified, the term vector space shall always mean a finite-dimensional vector space.

Example 18.11
Determine a basis and the dimension for the solution space of the homogeneous system

$$\begin{cases} 2x_1 + 2x_2 - \ \ x_3 \qquad\quad + x_5 + \ = 0 \\ -x_1 - \ \ x_2 + 2x_3 - 3x_4 + x_5 \quad = 0 \\ x_1 + \ \ x_2 - 2x_3 \qquad\quad - x_5 \quad = 0 \\ \qquad\qquad\quad x_3 + \ \ x_4 + x_5 \quad = 0. \end{cases}$$

Solution.
By Example 6.1, we found that $x_1 = -s - t, x_2 = s, x_3 = -t, x_4 = 0, x_5 = t$. So if S is the vector space of the solutions to the given system then $S = \{(-s-t, s, -t, 0, t) : s, t \in \mathbb{R}\} = \{s(-1, 1, 0, 0, 0) + t(-1, 0, -1, 0, 1) : s, t \in \mathbb{R}\} = span\{(-1, 1, 0, 0, 0), (-1, 0, -1, 0, 1)\}$. Moreover, if $s(-1, 1, 0, 0, 0) + t(-1, 0, -1, 0, 1) = (0, 0, 0, 0, 0)$ then $s = t = 0$. Thus, the set

$$\{(-1, 1, 0, 0, 0), (-1, 0, -1, 0, 1)\}$$

is a basis for the solution space of the homogeneous system ■

The following theorem will indicate the importance of the concept of a basis in investigating the structure of vector spaces. In fact, a basis for a vector space V determines the representation of each vector in V in terms of the vectors in that basis.

Theorem 18.2
If $S = \{v_1, v_2, \dots, v_n\}$ is a basis for V then any element of V can be written in one and only one way as a linear combination of the vectors in S.

Proof.
Suppose $v \in V$ has the following two representations $v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$ and $v = \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_n v_n$. We want to show that $\alpha_i = \beta_i$ for $1 \le i \le n$. But this follows from $(\alpha_1 - \beta_1)v_1 + (\alpha_2 - \beta_2)v_2 + \cdots + (\alpha_n - \beta_n)v_n = 0$ and the fact that S is a linearly independent set ■

Remark 18.1
A vector space can have different bases; however, all of them have the same number of elements.

The following theorem tells us that one of the conditions of the definition of a basis is sufficient in the case the vector space is finite dimensional.

Theorem 18.3
Let V be a finite dimensional vector space with $\dim(V) = n$.
(i) Any set of n linearly independent vectors of V is a basis of V.
(ii) Any span of V that consists of n vectors is a basis of V.

Proof.
See Problem 18.22 ■

Practice Problems

Problem 18.1
Let $W = span\{v_1, v_2, \cdots, v_n\}$, where $v_1, v_2, \cdots, v_n$ are vectors in V. Show that any subspace U of V containing the vectors $v_1, v_2, \cdots, v_n$ must contain W, i.e. $W \subset U$. That is, W is the smallest subspace of V containing $v_1, v_2, \cdots, v_n$.

Problem 18.2
Show that the polynomials $p_1(x) = 1 - x, p_2(x) = 5 + 3x - 2x^2$, and $p_3(x) = 1 + 3x - x^2$ are linearly dependent vectors in P_2.

Problem 18.3
Express the vector $\vec{u} = (-9, -7, -15)$ as a linear combination of the vectors $\vec{v_1} = (2, 1, 4), \vec{v_2} = (1, -1, 3), \vec{v_3} = (3, 2, 5)$.

Problem 18.4
(a) Show that the vectors $\vec{v_1} = (2, 2, 2), \vec{v_2} = (0, 0, 3)$, and $\vec{v_3} = (0, 1, 1)$ span $\mathbb{R}^3$.
(b) Show that the vectors $\vec{v_1} = (2, -1, 3), \vec{v_2} = (4, 1, 2)$, and $\vec{v_3} = (8, -1, 8)$ do not span $\mathbb{R}^3$.

Problem 18.5
Show that

$$M_{22} = span\left\{\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right\}.$$

Problem 18.6
Show that the vectors $\vec{v_1} = (2, -1, 0, 3), \vec{v_2} = (1, 2, 5, -1)$, and $\vec{v_3} = (7, -1, 5, 8)$ are linearly dependent.

Problem 18.7
Show that the vectors $\vec{v_1} = (4, -1, 2)$ and $\vec{v_2} = (-4, 10, 2)$ are linearly independent.

Problem 18.8
Show that the $\{u, v\}$ is linearly dependent if and only if one is a scalar multiple of the other.

Problem 18.9
Find a basis for the vector space M_{22} of 2×2 matrices.

Problem 18.10
Show that $\mathcal{B} = \{1 + 2x, x - x^2, x + x^2\}$ is a basis for P_2.

Problem 18.11
Determine the value(s) of a so that $\{p_1(x), p_2(x), p_3(x)\}$ is linearly independent in P_2 where

$$p_1(x) = a,\; p_2(x) = -2 + (a-4)x,\; p_3(x) = 1 + 2x + (a-1)x^2.$$

Problem 18.12
How many different representations can the vector $\vec{b} = (2, -1, 3)$ be written as a linear combination of the vectors $\vec{v}_1 = (1, -2, 0)$, $\vec{v}_2 = (0, 1, 1)$, and $\vec{v}_3 = (5, -6, 4)$?

Problem 18.13
Based on the previous problem, explain why $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ can not be a basis of $\mathbb{R}^3$.

Problem 18.14
Let W be the subspace of M_{22} :

$$W = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : a, b \in \mathbb{R} \right\}.$$

Find a basis for W and state its dimension.

Problem 18.15
Suppose that $V = \text{span}\{v_1, v_2, \cdots, v_n\}$. Show that if v_k is a linear combination of the remaining $v's$ then $V = \text{span}\{v_1, \cdots, v_{k-1}, v_{k+1}, \cdots, v_n\}$.

Problem 18.16
Let $\{v_1, v_2, \cdots, v_n\}$ be a basis for a vector space V and $S = \{w_1, w_2, \cdots, w_m\}$ be a subset of V with $m > n$. I want to show that S is linearly dependent.
(a) Show that there is a matrix $A = [a_{ij}]$ of dimension $m \times n$ such that

$Av = w$, where $v = [v_i]$ is $n \times 1$ and $w = [w_i]$ is $m \times 1$.
(b) Suppose that $\alpha_1 w_1 + \alpha_2 w_2 + \cdots + \alpha_m w_m = 0$. Show that

$$\begin{bmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \dots & \dots & \dots & \dots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

(c) Show that the homogeneous system in (b) has a non-trivial solution. Hence, concluding that S is linearly dependent.

Problem 18.17
Let $\{v_1, v_2, \cdots, v_n\}$ be a basis for a vector space V and $S = \{w_1, w_2, \cdots, w_m\}$ be a subset of V with $m < n$. I want to show that S can not span V.
(a) Suppose that $V = \text{span}(S)$. Show that there is a matrix $A = [a_{ij}]$ of dimension $n \times m$ such that $Aw = v$, where $v = [v_i]$ is $n \times 1$ and $w = [w_i]$ is $m \times 1$.
(b) Show that the homogeneous system $A^T \alpha = 0$, where

$$\begin{bmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ a_{12} & a_{22} & \dots & a_{n2} \\ \dots & \dots & \dots & \dots \\ a_{1m} & a_{2m} & \dots & a_{nm} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

has a non-trivial solution.
(c) Show that $\alpha^T v = 0$. That is, $\alpha_1 v_1 + \alpha_2 v_2 + \cdots \alpha_n v_n = 0$.
(d) Conclude from (c) that $\{v_1, v_2, \cdots, v_n\}$ is linearly dependent which contradicts the definition of a basis. Hence, we must have $V \neq \text{span}(S)$.

Problem 18.18
Show that if $\{v_1, v_2, \cdots, v_n\}$ is a basis of a vector space V then every basis of V has exactly n elements.

Problem 18.19
Let V be a vector space with $\dim(V) = n$. I want to show that any subset of independent vectors of V can be extended to a basis of V.
(a) Let $S = \{v_1, v_2, \cdots, v_m\}$ be a linearly independent subset of V with $V \neq \text{span}(S)$. Show that there is a vector $v \in V$ and $v \notin \text{span}(S)$ such that

$\{v, v_1, v_2, \cdots, v_m\}$ is linearly independent.
(b) Show that if $V \neq \text{span}\{v, v_1, v_2, \cdots, v_m\}$ then there is a vector w such that $\{w, v, v_1, v_2, \cdots, v_m\}$ is linearly independent.
(c) Show that eventually, S can be extended to a basis of V.

Problem 18.20
Let V be a vector space with $\dim(V) = n$. I want to show that any spanning subset of V can be extended to a basis of V.
(a) Suppose that $V = \{\text{span}\}\{\text{v}_1, \text{v}_2, \cdots, \text{v}_\text{m}\}$. Show that if $\{v_1, v_2, \cdots, v_m\}$ is linearly dependent then there is a subset $S_1 \subseteq \{v_1, v_2, \cdots, v_m\}$ such that $V = \{\text{span}\}(\text{S}_1)$.
(b) Continuing the process in (a), show that there is a subset S of $\{v_1, v_2, \cdots, v_m\}$ that is a basis of V.

Problem 18.21
Let V be a vector space of dimension n and W a subspace of V. Show that if $\dim(W) = \dim(V)$ then $W = V$.

Problem 18.22
Use Problem 18.21 to prove Theorem 18.3.

Eigenvalues and Eigenvectors

Eigenvalues and eigenvectors arise in many physical applications such as the study of vibrations, electrical systems, genetics, chemical reactions, quantum mechanics, economics, etc. In this chapter we introduce these two concepts and we show how to find them.

19 The Eigenvalues of a Square Matrix

Consider the following linear system

$$\begin{cases} \frac{dx_1}{dt} =& x_1 - 2x_2 \\ \frac{dx_2}{dt} =& 3x_1 - 4x_2. \end{cases}$$

In matrix form, this sysem can be written as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

A solution to this system has the form $x = e^{\lambda t}y$ where

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \text{ and } y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

That is, x is known once we know λ and y. Substituting, we have

$$\lambda e^{\lambda t} y = e^{\lambda t} A y$$

where

$$A = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}$$

or

$$Ay = \lambda y.$$

Thus, we need to find λ and y from this matrix equation.

If A is an $n \times n$ matrix and x is a nonzero vector in $\mathbb{R}^n$ such that $Ax = \lambda x$ for some complex number λ then we call x an **eigenvector** or **proper vector** corresponding to the **eigenvalue** (or **proper value**) λ.

Example 19.1
Show that $x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector of the matrix

$$A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$$

corresponding to the eigenvalue $\lambda = 3$.

Solution.
The value $\lambda = 3$ is an eigenvalue of A with eigenvector x since

$$Ax = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} = 3x \blacksquare$$

Eigenvalues can be either real numbers or complex numbers. To find the eigenvalues of a square matrix A we rewrite the equation $Ax = \lambda x$ as

$$Ax = \lambda I_n x$$

or equivalently

$$(\lambda I_n - A)x = 0.$$

For λ to be an eigenvalue, there must be a nonzero solution to the above homogeneous system. But, the above system has a nontrivial solution if and only if the coefficient matrix $(\lambda I_n - A)$ is singular, that is , if and only if

$$|\lambda I_n - A| = 0.$$

This equation is called the **characteristic equation** of A.

Example 19.2
Find the characteristic equation of the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix}.$$

Solution.
The characteristic equation of A is the equation

$$\begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -4 & 17 & \lambda - 8 \end{vmatrix} = 0.$$

That is, the equation: $\lambda^3 - 8\lambda^2 + 17\lambda - 4 = 0 \blacksquare$

It can be shown (See Problem 19.18) that

$$\begin{aligned} p(\lambda) =& |\lambda I_n - A| \\ =& \lambda^n - (a_{11} + a_{22} + \cdots + a_{nn})\lambda^{n-1} + \text{ terms of lower degree} \end{aligned} \qquad (19.1)$$

That is, $p(\lambda)$ is is a polynomial function in λ of degree n and leading coefficient 1. This is called the **characteristic polynomial** of A.

Example 19.3
Show that the constant term in the characteristic polynomial of a matrix A is $(-1)^n|A|$.

Solution.
The constant term of the polynomial $p(\lambda)$ corresponds to $p(0)$. It follows that $p(0) = \text{constant term} = |-A| = (-1)^n|A|$ ■

Example 19.4
Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}.$$

Solution.
The characteristic equation of A is given by

$$\begin{vmatrix} \lambda - 3 & -2 \\ 1 & \lambda \end{vmatrix} = 0.$$

Expanding the determinant and simplifying, we obtain

$$\lambda^2 - 3\lambda + 2 = 0$$

or

$$(\lambda - 1)(\lambda - 2) = 0.$$

Thus, the eigenvalues of A are $\lambda = 2$ and $\lambda = 1$ ■

Example 19.5
Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix}.$$

Solution.
According to Example 19.2, the characteristic equation of A is $\lambda^3 - 8\lambda^2 + 17\lambda - 4 = 0$. Using the rational root test we find that $\lambda = 4$ is a solution to this equation. Using synthetic division of polynomials we find

$$(\lambda - 4)(\lambda^2 - 4\lambda + 1) = 0.$$

The eigenvalues of the matrix A are the solutions to this equation, namely, $\lambda = 4, \lambda = 2 + \sqrt{3}$, and $\lambda = 2 - \sqrt{3}$ ■

Example 19.6
(a) Show that the eigenvalues of a triangular matrix are the entries on the main diagonal.
(b) Find the eigenvalues of the matrix

$$A = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -1 & \frac{2}{3} & 0 \\ 5 & -8 & -\frac{1}{4} \end{bmatrix}.$$

Solution.
(a) Suppose that A is an upper triangular $n \times n$ matrix. Then the matrix $\lambda I_n - A$ is also upper triangular with entries on the main diagonal are $\lambda - a_{11}, \lambda - a_{22}, \cdots, \lambda - a_{nn}$. Since the determinant of a triangular matrix is just the product of the entries of the main diagonal, the characteristic equation of A is

$$(\lambda - a_{11})(\lambda - a_{22}) \cdots (\lambda - a_{nn}) = 0.$$

Hence, the eigenvalues of A are $a_{11}, a_{22}, \cdots, a_{nn}$.
(b) Using (a), the eigenvalues of A are $\lambda = \frac{1}{2}, \lambda = \frac{2}{3}$, and $\lambda = -\frac{1}{4}$ ■

The **algebraic multiplicity** of an eigenvalue λ of a matrix A is the multiplicity of λ as a root of the characteristic polynomial.

Example 19.7
Find the algebraic multiplicity of the eigenvalues of the matrix

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 2 & 3 & 1 \end{bmatrix}.$$

Solution.
The characteristic equation of the matrix A is

$$\begin{vmatrix} \lambda - 2 & -1 & 0 \\ 0 & \lambda - 2 & 0 \\ -2 & -3 & \lambda - 1 \end{vmatrix} = 0.$$

Expanding the determinant and simplifying we obtain

$$(\lambda - 2)^2(\lambda - 1) = 0.$$

The eigenvalues of A are $\lambda = 2$ (of algebraic multiplicity 2) or $\lambda = 1$ (of algebraic multiplicity 1) ■

There are many matrices with real entries but with no real eigenvalues. An example is given next.

Example 19.8
Show that the following matrix has no real eigenvalues.

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Solution.
The characteristic equation of the matrix A is

$$\begin{vmatrix} \lambda & -1 \\ 1 & \lambda \end{vmatrix} = 0.$$

Expanding the determinant we obtain

$$\lambda^2 + 1 = 0.$$

The solutions to this equation are the imaginary complex numbers $\lambda = i$ and $\lambda = -i$ ■

We next introduce a concept for square matrices that will be fundamental in the next section. We say that two $n \times n$ matrices A and B are **similar** if there exists a non-singular matrix P such that $B = P^{-1}AP$. We write $A \sim B$. The matrix P is not unique. For example, if $A = B = I_n$ then any invertible matrix P will satisfy the definition.

Example 19.9
Let A and B be similar matrices. Show the following:
(a) $|A| = |B|$.
(b) $tr(A) = tr(B)$.
(c) $|\lambda I_n - A| = |\lambda I_n - B|$.

Solution.
Since $A \sim B$, there exists an invertible matrix P such that $B = P^{-1}AP$.
(a) $|B| = |P^{-1}AP| = |P^{-1}||A||P| = |A|$ since $|P^{-1}| = |P|^{-1}$.
(b) $tr(B) = tr(P^{-1}(AP)) = tr((AP)P^{-1}) = tr(A)$ (See Example 8.4(a)).
(c) Indeed, $|\lambda I_n - B| = |\lambda I_n - P^{-1}AP| = |P^{-1}(\lambda I_n - A)P| = |\lambda I_n - A|$. It follows that two similar matrices have the same eigenvalues ■

Example 19.10
Show that the following matrices are not similar.

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Solution.
The eigenvalues of A are $\lambda = 3$ and $\lambda = -1$. The eigenvalues of B are $\lambda = 0$ and $\lambda = 2$. According to Example 19.9 (c), these two matrices cannot be similar ■

Example 19.11
Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ including repetitions. Show the following.
(a) $tr(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$.
(b) $|A| = \lambda_1 \lambda_2 \cdots \lambda_n$.

Solution.
Factoring the characteristic polynomial of A we find

$$\begin{aligned} p(\lambda) =&(\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n) \\ =&\lambda^n - (\lambda_1 + \lambda_2 + \cdots + \lambda_n)\lambda^{n-1} + \cdots + (-1)^n \lambda_1 \lambda_2 \cdots \lambda_n. \end{aligned}$$

(a) By Equation 19.1, $tr(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$.
(b) $|-A| = p(0) = (-1)^n \lambda_1 \lambda_2 \cdots \lambda_n$. But $|-A| = (-1)^n |A|$. Hence, $|A| = \lambda_1 \lambda_2 \cdots \lambda_n$ ■

Example 19.12
(a) Find the characteristic polynomial of

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

(b) Find the matrix $A^2 - 5A - 2I_2$.
(c) Compare the result of (b) with (a).

Solution.
(a) $p(\lambda) = \begin{vmatrix} \lambda - 1 & -2 \\ -3 & \lambda - 4 \end{vmatrix} = \lambda^2 - 5\lambda - 2$.
(b) Simple algebra shows $A^2 - 5A - 2I_2 = \mathbf{0}$.
(c) A satisfies $p(A) = \mathbf{0}$. That is, A satisfies its own characteristic equation ■

More generally, we have

Theorem 19.1 *(Cayley-Hamilton)*
Every square matrix is the zero of its characteristic polynomial.

Proof.
Let $p(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$ be the characteristic polynomial corresponding to A. We will show that $p(A) = 0$. The cofactors of A are polynomials in λ of degree at most $n-1$. Thus,

$$adj(\lambda I_n - A) = B_{n-1}\lambda^{n-1} + \cdots + B_1\lambda + B_0$$

where the B_i are $n \times n$ matrices with entries independent of λ. Hence,

$$(\lambda I_n - A)adj(\lambda I_n - A) = |\lambda I_n - A|I_n.$$

That is

$$\begin{aligned} &B_{n-1}\lambda^n + (B_{n-2} - AB_{n-1})\lambda^{n-1} + (B_{n-3} - AB_{n-2})\lambda^{n-2} + \cdots + (B_0 - AB_1)\lambda - AB_0 \\ =&I_n\lambda^{n-1} + a_{n-1}I_n\lambda^{n-1} + a_{n-2}I_n\lambda^{n-2} + \cdots a_1I_n\lambda + a_0I_n. \end{aligned}$$

Equating coefficients of corresponding powers of λ,

$$\begin{aligned} B_{n-1} =&I_n \\ B_{n-2} - AB_{n-1} =&a_{n-1}I_n \\ B_{n-3} - AB_{n-2} =&a_{n-2}I_n \\ &\vdots \\ B_0 - AB_1 =&a_1I_n \\ -AB_0 =&a_0I_n \end{aligned}$$

Multiplying the above matrix equations by $A^n, A^{n-1}, \cdots, A, I_n$ respectively and adding the resulting matrix equations to obtain

$$0 = A^n + a_{n-1}A^{n-1} + \cdots + a_1A + a_0I_n.$$

In other words, $p(A) = 0$ ■

Example 19.13
Use the Cayley-Hamilton theorem to find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

Solution.
Since $|A| = 4 - 6 = -2 \neq 0$, A^{-1} exists. By Cayley-Hamilton Theorem we have

$$
\begin{aligned}
A^2 - 5A - 2I_2 =& \mathbf{0} \\
2I_2 =& A^2 - 5A \\
2A^{-1} =& A - 5I_2 \\
2A^{-1} =& \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} \\
=& \begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix} \\
A^{-1} =& \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} \blacksquare
\end{aligned}
$$

Practice Problems

Problem 19.1
Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 5 & 8 & 16 \\ 4 & 1 & 8 \\ -4 & -4 & -11 \end{bmatrix}.$$

Problem 19.2
Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}.$$

Problem 19.3
Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 1 & -2 \\ -1 & 0 & -2 \end{bmatrix}.$$

Problem 19.4
Find the eigenvalues of the matrix

$$A = \begin{bmatrix} -2 & 0 & 1 \\ -6 & -2 & 0 \\ 19 & 5 & -4 \end{bmatrix}.$$

Problem 19.5
Show that if λ is a nonzero eigenvalue of an invertible matrix A then $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}.

Problem 19.6
Show that if λ is an eigenvalue of a matrix A then λ^m is an eigenvalue of A^m for any positive integer m.

Problem 19.7
Show that if A is similar to a diagonal matrix D then A^k is similar to D^k.

Problem 19.8
Show that the identity matrix I_n has exactly one eigenvalue.

Problem 19.9
Let A be an $n \times n$ **nilpotent** matrix, i.e. $A^k = \mathbf{0}$ for some positive integer k.
(a) Show that $\lambda = 0$ is the only eigenvalue of A.
(b) Show that $p(\lambda) = \lambda^n$.

Problem 19.10
Suppose that A and B are $n \times n$ similar matrices and $B = P^{-1}AP$. Show that if λ is an eigenvalue of A with corresponding eigenvector x then λ is an eigenvalue of B with corresponding eigenvector $P^{-1}x$.

Problem 19.11
Let A be an $n \times n$ matrix with n odd. Show that A has at least one real eigenvalue.

Problem 19.12
Consider the following $n \times n$ matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 \end{bmatrix}.$$

Show that the characterisitc polynomial of A is given by $p(\lambda) = \lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0$. Hence, every monic polynomial (i.e. the coefficient of the highest power of λ is 1) is the characteristic polynomial of some matrix. A is called the **companion matrix** of $p(\lambda)$.

Problem 19.13
Show that if D is a diagonal matrix then D^k, where k is a positive integer, is a diagonal matrix whose entries are the entries of D raised to the power k.

Problem 19.14
Show that A and A^T have the same characteristic polynomial and hence the same eigenvalues.

Problem 19.15
Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}.$$

Problem 19.16
Find the eigenvalues of the matrix

$$B = \begin{bmatrix} -2 & -1 \\ -5 & 2 \end{bmatrix}.$$

Problem 19.17
Show that $\lambda = 0$ is an eigenvalue of a matrix A if and only if A is singular.

Problem 19.18
Show that

$$|\lambda I_n - A| = \lambda^n - (a_{11} + a_{22} + \cdots + a_{nn})\lambda^{n-1} + \text{ terms of lower degree.}$$

Problem 19.19
Consider the matrix

$$\begin{bmatrix} 6 & -2 \\ 6 & -1 \end{bmatrix}.$$

Using Cayley-Hamilton theorem, show that $A^6 = 665A - 1266I_2$.

Problem 19.20
Use the Cayley-Hamilton theorem to find the inverse of the matrix

$$A = \begin{bmatrix} 2 & 0 & 1 \\ -2 & 3 & 4 \\ -5 & 5 & 6 \end{bmatrix}.$$

20 Finding Eigenvectors and Eigenspaces

In this section, we turn to the problem of finding the eigenvectors of a square matrix. Recall that an eigenvector is a nontrivial solution to the matrix equation $(\lambda I_n - A)\mathbf{x} = 0$.
For a square matrix of size $n \times n$, the set of all eigenvectors together with the zero vector is a vector space as shown in the next result.

Theorem 20.1
Let V_λ denote the set of eigenvectors of a matrix corresponding to an eigenvalue λ. The set $V^\lambda = V_\lambda \cup \{\mathbf{0}\}$ is a subspace of $\mathbb{R}^n$. This subspace is called the **eigenspace** of A corresponding to λ.

Proof.
Let $V_\lambda = \{\mathbf{x} \in \mathbb{R}^n \text{ and } \mathbf{x} \neq \{\mathbf{0}\} : A\mathbf{x} = \lambda\mathbf{x}\}$. We will show that $V^\lambda = V_\lambda \cup \{\mathbf{0}\}$ is a subspace of $\mathbb{R}^n$.
(i) Let $\mathbf{u}, \mathbf{v} \in V^\lambda$. If $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$ then the sum is either $\mathbf{u}, \mathbf{v}$, or $\mathbf{0}$ which belongs to V^λ. So assume that both $\mathbf{u}, \mathbf{v} \in V_\lambda$. We have $A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v} = \lambda\mathbf{u} + \lambda\mathbf{v} = \lambda(\mathbf{u}+\mathbf{v})$. That is $\mathbf{u}+\mathbf{v} \in V^\lambda$.
(ii) Let $\mathbf{u} \in V^\lambda$ and $\alpha \in \mathbb{R}$. Then $A(\alpha\mathbf{u}) = \alpha A\mathbf{u} = \lambda(\alpha\mathbf{u})$ so $\alpha\mathbf{u} \in V^\lambda$. Hence, V^λ is a subspace of $\mathbb{R}^n$ $\blacksquare$

By the above theorem, determining the eigenspaces of a square matrix is reduced to two problems: First find the eigenvalues of the matrix, and then find the corresponding eigenvectors which are solutions to linear homogeneous systems.

Example 20.1
Find the eigenspaces of the matrix

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 2 & 3 & 1 \end{bmatrix}.$$

Solution.
The characteristic equation of the matrix A is

$$\begin{bmatrix} \lambda - 2 & -1 & 0 \\ 0 & \lambda - 2 & 0 \\ -2 & -3 & \lambda - 1 \end{bmatrix}.$$

Expanding the determinant and simplifying we obtain

$$(\lambda-2)^2(\lambda-1)=0.$$

The eigenvalues of A are $\lambda=2$ and $\lambda=1$.
A vector $\mathbf{x}=[x_1,x_2,x_3]^T$ is an eigenvector corresponding to an eigenvalue λ if and only if $\mathbf{x}$ is a solution to the homogeneous system

$$\begin{cases} (\lambda-2)x_1- & x_2 & & =0 \\ & (\lambda-2)x_2 & & =0 \\ -2x_1- & 3x_2+ & (\lambda-1)x_3 & =0. \end{cases} \tag{20.1}$$

If $\lambda=1$, then (20.1) becomes

$$\begin{cases} -x_1- \ \ x_2=0 \\ \qquad - \ \ x_2=0 \\ -2x_1-3x_2=0. \end{cases}$$

Solving this system yields

$$x_1=0, x_2=0, x_3=s.$$

The eigenspace corresponding to $\lambda=1$ is

$$V^1=\left\{\begin{bmatrix}0\\0\\s\end{bmatrix} : s\in\mathbb{R}\right\}=\text{span}\left\{\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$$

and $[0,0,1]^T$ is a basis for V^1 and $\dim(V^1)=1$.
If $\lambda=2$, then (20.1) becomes

$$\begin{cases} \qquad - \ \ x_2 \qquad =0 \\ -2x_1-3x_2+x_3=0. \end{cases}$$

Solving this system yields

$$x_1=\frac{1}{2}s, x_2=0, x_3=s.$$

The eigenspace corresponding to $\lambda = 2$ is

$$\begin{aligned} V^2 &= \left\{ \begin{bmatrix} \frac{1}{2}s \\ 0 \\ s \end{bmatrix} : s \in \mathbb{R} \right\} \\ &= \text{span}\left\{ \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix} \right\} \end{aligned}$$

and the vector $[\frac{1}{2}, 0, 1]^T$ is a basis for V^2 and $\dim(V^2) = 1$ ■.

The **algebraic multiplicity** of an eigenvalue λ of a matrix A is the multiplicity of λ as a root of the characteristic polynomial, and the dimension of the eigenspace corresponding to λ is called the **geometric multiplicity** of λ.

Example 20.2
Find the algebraic and geometric multiplicities of the eigenvalues of the matrix

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 2 & 3 & 1 \end{bmatrix}.$$

Solution.
From the previous example, the eigenvalue $\lambda = 2$ has algebraic multiplicity 2 and geometric multiplicity 1. The eigenvalue $\lambda = 1$ has algebraic and geometric multiplicity equal to 1 ■

Example 20.3
Solve the homogeneous linear system

$$\begin{cases} \dfrac{dx_1}{dt} = x_1 - 2x_2 \\ \dfrac{dx_2}{dt} = 3x_1 - 4x_2 \end{cases}$$

using eigenvalues and eigenvectors.

Solution.
In matrix form, this sysem can be written as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

A solution to this system has the form $\mathbf{x} = e^{\lambda t}\mathbf{y}$ where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \text{ and } \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

That is, $\mathbf{x}$ is known once we know λ and $\mathbf{y}$. Substituting, we have

$$\lambda e^{\lambda t}\mathbf{y} = e^{\lambda t} A\mathbf{y}$$

where

$$A = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}$$

or

$$A\mathbf{y} = \lambda\mathbf{y}.$$

To find λ, we solve the characteristic equation

$$|\lambda I_2 - A| = \lambda^2 + 3\lambda + 2 = 0.$$

The eigenvalues are $\lambda = -1$ and $\lambda = -2$. Next, we find the eigenspaces of A. A vector $\mathbf{x} = [x_1, x_2, x_3]^T$ is an eigenvector corresponding to an eigenvalue λ if and only if $\mathbf{x}$ is a solution to the homogeneous system

$$\begin{cases} (\lambda - 1)x_1 + & 2x_2 = 0 \\ -3x_1 + & (\lambda + 4)x_2 = 0. \end{cases} \tag{20.2}$$

If $\lambda = -1$, then (20.2) becomes

$$\begin{cases} -2x_1 + 2x_2 = 0 \\ -3x_1 + 3x_2 = 0. \end{cases}$$

Solving this system yields

$$x_1 = s, x_2 = s.$$

The eigenspace corresponding to $\lambda = -1$ is

$$V^{-1} = \left\{ \begin{bmatrix} s \\ s \end{bmatrix} : s \in \mathbb{R} \right\} = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}.$$

If $\lambda = -2$, then (20.2) becomes

$$\begin{cases} -3x_1 + 2x_2 = 0 \\ -3x_1 + 2x_2 = 0. \end{cases}$$

Solving this system yields

$$x_1 = \frac{2}{3}s, x_2 = s.$$

The eigenspace corresponding to $\lambda = -2$ is

$$\begin{aligned} V^{-2} &= \left\{ \begin{bmatrix} \frac{3}{2}s \\ s \end{bmatrix} : s \in \mathbb{R} \right\} \\ &= \text{span} \left\{ \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix} \right\}. \end{aligned}$$

The general solution to the system is

$$\mathbf{x} = c_1 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 e^{-t} + \frac{3}{2} c_2 e^{-2t} \\ c_1 e^{-t} + c_2 e^{-2t} \end{bmatrix} \blacksquare$$

Practice Problems

Problem 20.1
Show that $\lambda = -3$ is an eigenvalue of the matrix

$$A = \begin{bmatrix} 5 & 8 & 16 \\ 4 & 1 & 8 \\ -4 & -4 & -11 \end{bmatrix}$$

and then find the corresponding eigenspace V^{-3}.

Problem 20.2
Find the eigenspaces of the matrix

$$A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}.$$

Problem 20.3
Find the eigenspaces of the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 1 & -2 \\ -1 & 0 & -2 \end{bmatrix}.$$

Problem 20.4
Find the bases of the eigenspaces of the matrix

$$A = \begin{bmatrix} -2 & 0 & 1 \\ -6 & -2 & 0 \\ 19 & 5 & -4 \end{bmatrix}.$$

Problem 20.5
Find the eigenvectors and the eigenspaces of the matrix

$$A = \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}.$$

Problem 20.6
Find the eigenvectors and the eigenspaces of the matrix

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}.$$

Problem 20.7
Find the eigenvectors and the eigenspaces of the matrix

$$A = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}.$$

Problem 20.8
Find the eigenvectors and the eigenspaces of the matrix

$$A = \begin{bmatrix} -1 & 1 & 1 & -2 \\ -1 & 1 & 3 & 2 \\ 1 & 1 & -1 & -2 \\ 0 & -1 & -1 & 1 \end{bmatrix}.$$

Problem 20.9
Find the geometric and algebraic multiplicities of the eigenvalues of the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Problem 20.10
When an $n \times n$ matrix has a eigenvalue whose geometric multiplicity is less than the algebraic multiplicity, then it is called a **defective** matrix. Is A defective?

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Problem 20.11
Find bases for the eigenspaces of the matrix

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}.$$

Problem 20.12
Find the eigenspaces of the matrix

$$A = \begin{bmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$$

Problem 20.13
Find the eigenspaces of the matrix

$$A = \begin{bmatrix} 2 & -1 \\ 5 & -2 \end{bmatrix}.$$

Problem 20.14
Suppose that λ_1 and λ_2 are two distinct eigenvalues of a square matrix A with corresponding eigenvectors v_1 and v_2. Show that $\{v_1, v_2\}$ is linearly independent.

Problem 20.15
Show that if λ is an eigenvalue of A with corresponding eigenvector v then for any scalar r, $\lambda - r$ is an eigenvalue of $A - rI_n$ with corresponding eigenvector v.

Problem 20.16
Let

$$A = \begin{bmatrix} 2 & 0 \\ 2 & 3 \end{bmatrix}.$$

Show that A and A^T do not have the same eigenspaces.

Problem 20.17
Let A be a 2×2 matrix such that $A^2 = I_2$. Suppose that the eigenvectors of A are $x + Ax$ and $x - Ax$. Find the corresponding eigenvalue.

Problem 20.18
Show that if A is invertible then it can not have 0 as an eigenvalue.

Problem 20.19
Consider the following matrix

$$A = \begin{bmatrix} -9 & 15 & 3 \\ -12 & 18 & 3 \\ 24 & -30 & -3 \end{bmatrix}.$$

(a) Find the characteristic polynomial.
(b) Find the eigenvalues and their corresponding algebraic multiplicities.
(c) Find the basis for each eigenvalue of A.

Problem 20.20
Show that if A^2 is the zero matrix, then the only eigenvalue of A is zero.

Problem 20.21
Suppose A is 3, and v is an eigenvector of A corresponding to an eigenvalue of 7. Is v an eigenvector of $2I_3 - A$? If so, find the corresponding eigenvalue.

Problem 20.22
If each row of A sums to the same number s, what is one eigenvalue and eigenvector?

21 Diagonalization of a Matrix

In this section, we shall discuss a method for finding a basis of $\mathbb{R}^n$ consisting of the eigenvectors of a given $n \times n$ matrix A. It turns out that this is equivalent to finding an invertible matrix P such that $P^{-1}AP = D$ is a diagonal matrix. That is, A is similar to D. In this case, the columns of P are the eigenvectors of A and the diagonal entries of D are the eigenvalues of A. Also, the columns of P form a basis of the Euclidean space $\mathbb{R}^n$. We say that A is a **diagonalizable** matrix.
The computations of diagonalizable matrices is one of the most frequently applied numerical processes in applications such as quantum mechanics or differential equations.
The next theorem gives a characterization of diagonalizable matrices. In fact, this theorem and Theorem 18.3 support our statement mentioned at the beginning of this section that the problem of finding a basis of $\mathbb{R}^n$ consisting of eigenvectors of A is equivalent to diagonalizing A.

Theorem 21.1
If A is an $n \times n$ square matrix, then the following statements are equivalent.
(a) A is diagonalizable.
(b) A has n linearly independent eigenvectors.

Proof.
$(a) \Rightarrow (b)$: Suppose A is diagonalizable. Then there are an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$. That is

$$AP = PD = \left[\begin{array}{cccc} p_{11} & p_{12} & \dots & p_{1n} \\ p_{21} & p_{22} & \dots & p_{2n} \\ \vdots & & & \vdots \\ p_{n1} & p_{n2} & \dots & p_{nn} \end{array}\right] \left[\begin{array}{ccccc} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \dots & \lambda_n \end{array}\right].$$

For $1 \le i \le n$, let

$$p_i = \left[\begin{array}{c} p_{1i} \\ p_{2i} \\ \vdots \\ p_{ni} \end{array}\right].$$

Then the columns of AP are $\lambda_1 p_1, \lambda_2 p_2, \dots, \lambda_n p_n$. But $AP = [Ap_1, Ap_2, \dots, Ap_n]$. Hence, $Ap_i = \lambda_i p_i$, for $1 \le i \le n$. Thus $\lambda_1, \lambda_2, \dots, \lambda_n$ are eigenvalues of

A and $p_1, p_2, \ldots, p_n$ are the corresponding eigenvectors. The eigenvectors $p_1, p_2, \cdots, p_n$ are linearly independent (See Problem 21.16).

$(b) \Rightarrow (a)$: Suppose that A has n linearly independent eigenvectors $p_1, p_2, \ldots, p_n$ with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Let

$$P = \begin{bmatrix} p_{11} & p_{12} & \ldots & p_{1n} \\ p_{21} & p_{22} & \ldots & p_{2n} \\ \vdots & & & \vdots \\ p_{n1} & p_{n2} & \ldots & p_{nn} \end{bmatrix}.$$

Then the columns of AP are $Ap_1, Ap_2, \ldots, Ap_n$. But $Ap_1 = \lambda_1 p_1, Ap_2 = \lambda_2 p_2, \ldots, Ap_n = \lambda_n p_n$. Hence

$$\begin{aligned} AP &= \begin{bmatrix} \lambda_1 p_{11} & \lambda_2 p_{12} & \ldots & \lambda_n p_{1n} \\ \lambda_1 p_{21} & \lambda_2 p_{22} & \ldots & \lambda_n p_{2n} \\ \vdots & & & \vdots \\ \lambda_1 p_{n1} & \lambda_2 p_{n2} & \ldots & \lambda_n p_{nn} \end{bmatrix} \\ &= \begin{bmatrix} p_{11} & p_{12} & \ldots & p_{1n} \\ p_{21} & p_{22} & \ldots & p_{2n} \\ \vdots & & & \vdots \\ p_{n1} & p_{n2} & \ldots & p_{nn} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 & \ldots & 0 \\ 0 & \lambda_2 & 0 & \ldots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \ldots & \lambda_n \end{bmatrix} = PD. \end{aligned}$$

Since the column vectors of P ar linearly independent, the homogeneous system $Px = 0$ has only the trivial solution so that by Theorem 11.1, P is invertible. Hence $A = PDP^{-1}$, that is A is similar to a diagonal matrix ■

How do we find P and D? From the above proof we obtain the following procedure for diagonalizing a diagonalizable matrix.
Step 1. Find n linearly independent eigenvectors of A, say $p_1, p_2, \cdots, p_n$.
Step 2. Form the matrix P having $p_1, p_2, \cdots, p_n$ as its column vectors.
Step 3. The matrix $P^{-1}AP$ will then be diagonal with $\lambda_1, \lambda_2, \ldots, \lambda_n$ as its diagonal entries, where λ_i is the eigenvalue corresponding to p_i, $1 \le i \le n$.

Example 21.1
Find a matrix P that diagonalizes

$$A = \begin{bmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$$

Solution.
By Problem 20.12, the eigenspaces corresponding to the eigenvalues $\lambda = 1$ and $\lambda = 5$ are

$$V^1 = \left\{ \begin{bmatrix} s \\ s \\ 0 \end{bmatrix} : s \in \mathbb{R} \right\} = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

and

$$\begin{aligned} V^5 &= \left\{ \begin{bmatrix} -t \\ t \\ s \end{bmatrix} : s, t \in \mathbb{R} \right\} \\ &= \left\{ t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} : s, t \in \mathbb{R} \right\} \\ &= \text{span} \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}. \end{aligned}$$

Let $v_1 = [1, 1, 0]^T$, $v_2 = [-1, 1, 0]^T$, and $v_3 = [0, 0, 1]^T$. It is easy to verify that these vectors are linearly independent. The matrices

$$P = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

satisfy $AP = PD$ or $D = P^{-1}AP$ ■

Example 21.2
Show that the matrix

$$A = \begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix}$$

is not diagonalizable.

Solution.
The characteristic equation of the matrix A is

$$\begin{vmatrix} \lambda+3 & -2 \\ 2 & \lambda-1 \end{vmatrix} = 0.$$

Expanding the determinant and simplifying we obtain

$$(\lambda+1)^2 = 0.$$

The only eigenvalue of A is $\lambda = -1$.
An eigenvector $x = [x_1, x_2]^T$ corresponding to the eigenvalue $\lambda = -1$ is a solution to the homogeneous system

$$\begin{cases} 2x_1 - 2x_2 = 0 \\ 2x_1 - 2x_2 = 0. \end{cases}$$

Solving this system yields $x_1 = s, x_2 = s$. Hence the eigenspace corresponding to $\lambda = -1$ is

$$V^{-1} = \left\{ \begin{bmatrix} s \\ s \end{bmatrix} : s \in \mathbb{R} \right\} = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

Since $dim(V^{-1}) = 1$, A does not have two linearly independent eigenvectors and is therefore not diagonalizable ■

In many applications one is concerned only with knowing whether a matrix is diagonalizable without the need of finding the matrix P. The answer is provided with the following theorem.

Theorem 21.2
If A is an $n \times n$ matrix with n distinct eigenvalues then A is diagonalizable.

This theorem follows from the following result.

Theorem 21.3
If $v_1, v_2, \ldots, v_n$ are nonzero eigenvectors that correspond to distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ then the set $\{v_1, v_2, \ldots, v_n\}$ is linearly indepedent.

Proof.
The proof is by induction on n. If $n = 1$ then $\{v_1\}$ is linearly independent (Example 18.6). So assume that the vectors $\{v_1, v_2, \cdots, v_{n-1}\}$ are linearly independent. Suppose that

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots \alpha_n v_n = 0. \tag{21.1}$$

Apply A to both sides of (21.1) and using the fact that $Av_i = \lambda_i v_i$ to obtain

$$\alpha_1 \lambda_1 v_1 + \alpha_2 \lambda_2 v_2 + \cdots \alpha_n \lambda_n v_n = 0. \tag{21.2}$$

Now, multiplying (21.1) by λ_n and subtracting the resulting equation from (21.2) we obtain

$$\alpha_1(\lambda_1 - \lambda_n)v_1 + \alpha_2(\lambda_2 - \lambda_n)v_2 + \cdots \alpha_{n-1}(\lambda_{n-1} - \lambda_n)v_{n-1} = 0. \tag{21.3}$$

By the induction hypothesis, all the coefficients must be zero. Since the λ_i are distinct, i.e. $\lambda_i - \lambda_n \neq 0$ for $i \neq n$, we arrive at $\alpha_1 = \alpha_2 = \cdots = \alpha_{n-1} = 0$. Substituting this into (21.1) to obtain $\alpha_n v_n = 0$, and hence $\alpha_n = 0$. This shows that $\{v_1, v_2, \cdots, v_n\}$ is linearly indepedent ■

Example 21.3
Show that the following matrix is diagonalizable.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & -3 \\ 1 & -1 & 0 \end{bmatrix}$$

Solution.
The characteristic equation of the matrix A is

$$\begin{vmatrix} \lambda - 1 & 0 & 0 \\ -1 & \lambda - 2 & 3 \\ -1 & 1 & \lambda \end{vmatrix} = 0.$$

Expanding the determinant along the first row and simplifying we obtain

$$(\lambda - 1)(\lambda - 3)(\lambda + 1) = 0.$$

The eigenvalues are $1, 3$ and -1, so A is diagonalizable by Theorem 21.1 ■
The converse of Theorem 21.2 is false as illustrated in the next example.

Example 21.4
(a) Find a matrix P that diagonalizes

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

(b) Show that the eigenvectors form a basis of $\mathbb{R}^3$.

Solution.
(a) By Problem 20.11, the eigenspaces corresponding to the eigenvalues $\lambda = 1$ and $\lambda = 2$ are

$$V^1 = \text{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\}$$

and

$$V^2 = \text{span}\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Let $v_1 = [-2, 1, 1]^T, v_2 = [-1, 0, 1]^T$, and $v_3 = [0, 1, 0]^T$. It is easy to verify that these vectors are linearly independent. The matrices

$$P = \begin{bmatrix} -2 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

and

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

satisfy $AP = PD$ or $D = P^{-1}AP$.
(b) Since $\dim(\mathbb{R}^3) = 3$ and the eigenvectors v_1, v_2, and v_3 are linearly independent, Theorem 18.3(i), $\{v_1, v_2, v_3\}$ is a basis of $\mathbb{R}^3$ ■

Practice Problems

Problem 21.1
Recall that a matrix A is similar to a matrix B if and only if there is an invertible matrix P such that $P^{-1}AP = B$. In symbol, we write $A \sim B$. Show that if $A \sim B$ then
(a) $A^T \sim B^T$.
(b) $A^{-1} \sim B^{-1}$.

Problem 21.2
If A is invertible show that $AB \sim BA$ for all B.

Problem 21.3
Show that the matrix A is diagonalizable.

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix}$$

Problem 21.4
Show that the matrix A is not diagonalizable.

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 1 & -2 \\ -1 & 0 & -2 \end{bmatrix}$$

Problem 21.5
Show that the matrix

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

is diagonalizable with only one eigenvalue.

Problem 21.6
Show that A is diagonalizable if and only if A^T is diagonalizable.

Problem 21.7
Show that if A and B are similar then A is diagonalizable if and only if B is diagonalizable.

Problem 21.8
Give an example of two diagonalizable matrices A and B such that $A+B$ is not diagonalizable.

Problem 21.9
Find P and D such that $P^{-1}AP = D$ where

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Problem 21.10
Find P and D such that $P^{-1}AP = D$ where

$$A = \begin{bmatrix} -1 & 1 & 1 & -2 \\ -1 & 1 & 3 & 2 \\ 1 & 1 & -1 & -2 \\ 0 & -1 & -1 & 1 \end{bmatrix}$$

Problem 21.11
Is the matrix

$$A = \begin{bmatrix} 6 & 3 & -8 \\ 0 & -2 & 0 \\ 1 & 0 & -3 \end{bmatrix}$$

diagonalizable?

Problem 21.12
Is the matrix

$$A = \begin{bmatrix} 2 & -2 & 1 \\ -1 & 3 & 1 \\ 2 & -4 & 3 \end{bmatrix}$$

diagonalizable?

Problem 21.13
The matrix A is a 3×3 matrix with eigenvalues $1, 2$, and -1, and eigenvectors $[1,1,0]^T, [1,2,1]^T$, and $[0,1,2]^T$. Find A.

Problem 21.14
Let A be a diagonalizable 2×2 matrix with $P = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ and $D = \begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix}$.
Find A^5.

Problem 21.15
Is the matrix

$$A = \begin{bmatrix} 2 & 4 & 6 \\ 0 & 2 & 2 \\ 0 & 0 & 4 \end{bmatrix}$$

diagonalizable?

Problem 21.16
Show that the eigenvectors $p_1, p_2, \cdots, p_n$ of Theorem 21.1 are linearly independent.

Problem 21.17
Given the diagonalized matrix

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

Find A^{10}.

Problem 21.18
Why is

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 6 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

diagonalizable?

Problem 21.19
An $n \times n$ matrix A is said to be **orthogonally diagonalizable** if and only if there exist an orthogonal matrix P (i.e., $P^T = P^{-1}$) and a diagonal matrix D such that $A = PDP^T$. Show that an orthogonally diagonalizable is symmetric.

Problem 21.20
Show that the matrix

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

is diagonalizable.

Linear Transformations

In this chapter we shall discuss a special class of functions whose domains and ranges are vector spaces. Such functions are referred to as linear transformations, a concept to be defined in Section 22.

22 Linear Transformation: Definition and Elementary Properties

A **linear transformation** T from a vector space V to a vector space W is a function $T : V \to W$ that satisfies the following two conditions
(i) $T(u + v) = T(u) + T(v)$, for all u, v in V.
(ii) $T(\alpha u) = \alpha T(u)$ for all u in V and scalar α.
If $W = \mathbb{R}$ then we call T a **linear functional** on V.
It is important to keep in mind that the addition in $u+v$ refers to the addition operation in V whereas that in $T(u) + T(v)$ refers to the addition operation in W. Similar remark for the scalar multiplication.

Example 22.1
Show that $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ x + y \\ x - y \end{bmatrix}$$

is a linear transformation.

Solution.
We verify the two conditions of the definition. Given $[x_1, y_1]^T$ and $[x_2, y_2]^T$ in $\mathbb{R}^2$, compute

$$\begin{aligned} T\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) =& T\left(\begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ x_1 + x_2 + y_1 + y_2 \\ x_1 + x_2 - y_1 - y_2 \end{bmatrix} \\ =& \begin{bmatrix} x_1 \\ x_1 + y_1 \\ x_1 - y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ x_2 + y_2 \\ x_2 - y_2 \end{bmatrix} = T\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right) + T\left(\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right). \end{aligned}$$

This proves the first condition. For the second condition, we let $\alpha \in \mathbb{R}$ and compute

$$\begin{aligned} T\left(\alpha \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right) =& T\left(\begin{bmatrix} \alpha x_1 \\ \alpha y_1 \end{bmatrix}\right) = \begin{bmatrix} \alpha x_1 \\ \alpha x_1 + \alpha y_1 \\ \alpha x_1 - \alpha y_1 \end{bmatrix} \\ =& \alpha \begin{bmatrix} x_1 \\ x_1 + y_1 \\ x_1 - y_1 \end{bmatrix} = \alpha T\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right). \end{aligned}$$

Hence, T is a linear transformation ■

Example 22.2
Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}.$$

Show that T is not linear.

Solution.
We show that the first condition of the definition is violated. Indeed, for any two vectors $[x_1, y_1]^T$ and $[x_2, y_2]^T$ we have

$$\begin{aligned} T\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) =& T\left(\begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ 1 \end{bmatrix} \\ \neq & \begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ 1 \end{bmatrix} = T\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right) + T\left(\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right). \end{aligned}$$

Hence, the given transformation is not linear ■

The next theorem collects four useful properties of all linear transformations.

Theorem 22.1
If $T : V \to W$ is a linear transformation then
(a) $T(u - w) = T(u) - T(w)$
(b) $T(0) = 0$
(c) $T(-u) = -T(u)$
(d) $T(\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n) = \alpha_1 T(u_1) + \alpha_2 T(u_2) + \cdots + \alpha_n T(u_n)$.

Proof.
See Problem 22.16 ■

The following theorem provides a criterion for showing that a transformation is linear.

Theorem 22.2
A function $T : V \to W$ is linear if and only if $T(\alpha u + v) = \alpha T(u) + T(v)$ for all $u, v \in V$ and $\alpha \in \mathbb{R}$.

Proof.
See Problem 22.17 ■

Example 22.3
Let M_{mn} denote the vector space of all $m \times n$ matrices.
(a) Show that $T : M_{mn} \to M_{nm}$ defined by $T(A) = A^T$ is a linear transformation.
(b) Show that $T : M_{nn} \to \mathbb{R}$ defined by $T(A) = tr(A)$ is a linear functional.

Solution.
(a) For any $A, B \in M_{mn}$ and $\alpha \in \mathbb{R}$ we find $T(\alpha A + B) = (\alpha A + B)^T = \alpha A^T + B^T = \alpha T(A) + T(B)$. Hence, T is a linear transformation.
(b) For any $A, B \in M_{nn}$ and $\alpha \in \mathbb{R}$ we have $T(\alpha A + b) = tr(\alpha A + B) = \alpha tr(A) + tr(B) = \alpha T(A) + T(B)$ so T is a linear functional ■

Example 22.4
Let $S : V \to W$ and $T : V \to W$ be two linear transformations. Show the following:
(a) $S + T$ and $S - T$ are linear transformations.
(b) αT is a linear transformation where α denotes a scalar.

Solution.
(a) Let $u, v \in V$ and $\alpha \in \mathbb{R}$ then

$$\begin{aligned}(S \pm T)(\alpha u + v) =& S(\alpha u + v) \pm T(\alpha u + v)\\ =& \alpha S(u) + S(v) \pm (\alpha T(u) + T(v))\\ =& \alpha(S(u) \pm T(u)) + (S(v) \pm T(v))\\ =& \alpha(S \pm T)(u) + (S \pm T)(v).\end{aligned}$$

(b) Let $u, v \in V$ and $\beta \in \mathbb{R}$ then

$$\begin{aligned}(\alpha T)(\beta u + v) =& (\alpha T)(\beta u) + (\alpha T)(v)\\ =& \alpha\beta T(u) + \alpha T(v)\\ =& \beta(\alpha T(u)) + \alpha T(v)\\ =& \beta(\alpha T)(u) + (\alpha T)(v).\end{aligned}$$

Hence, αT is a linear transformation ■

The following theorem shows that two linear transformations defined on V are equal whenever they have the same effect on any span of the vector space V.

Theorem 22.3
Let $V = span\{v_1, v_2, \cdots, v_n\}$. If T and S are two linear transformations from V into a vector space W such that $T(v_i) = S(v_i)$ for each i then $T = S$.

Proof.
See Problem 22.18 ■

The following very useful theorem tells us that once we say what a linear transformation does to a basis for V, then we have completely specified T.

Theorem 22.4
Let V be an $n-$dimensional vector space with basis $\{v_1, v_2, \cdots, v_n\}$. If $T : V \longrightarrow W$ is a linear transformation then for any $v \in V$, Tv is completely determined by $\{Tv_1, Tv_2, \cdots, Tv_n\}$.

Proof.
Indeed, if $v \in V$ then $v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$. Thus, $Tv = \alpha_1 T(v_1) + \alpha_2 T(v_2) + \cdots + \alpha_n T(v_n)$. Hence, if the vectors $T(v_1), T(v_2), \cdots, T(v_n)$ are known then $T(v)$ is known ■

Example 22.5
Let V and W be $n-$dimensional vector spaces and $\{v_1, v_2, \cdots, v_n\}$ be a basis of V. Given any vectors $w_1, w_2, \cdots, w_n$ in W, there exists a unique linear transformation $T : V \longrightarrow W$ such that $T(v_i) = w_i$ for each i.

Solution.
Let $v \in V$. Then $v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$. Hence, we can define the transformation $T : V \to W$ by

$$T(v) = \alpha_1 T(v_1) + \alpha_2 T(v_2) + \cdots + \alpha_n T(v_n) = \alpha_1 w_1 + \alpha_2 w_2 + \cdots + \alpha_n w_n.$$

One can easily show that T is linear. If S is another linear transformation with the property that $S(v_i) = w_i$ for $i = 1, 2, \cdots, n$ then $S = T$ by Theorem (22.3) ■

Example 22.6
Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Show that there exists an $m \times n$ matrix A such that $T(x) = Ax$ for all $x \in \mathbb{R}^n$. The matrix A is called the **standard matrix** of T.

Solution.
Consider the standard basis of $\mathbb{R}^n$, $\{e_1, e_2, \cdots, e_n\}$ where e_i is the vector with 1 at the i^{th} component and 0 otherwise. Let $\mathbf{x} = [x_1, x_2, \cdots, x_n]^T \in \mathbb{R}^n$. Then $\mathbf{x} = x_1e_1 + x_2e_2 + \cdots + x_ne_n$. Thus,

$$T(\mathbf{x}) = x_1T(e_1) + x_2T(e_2) + \cdots + x_nT(e_n) = A\mathbf{x}$$

where $A = \left[\begin{array}{cccc} T(e_1) & T(e_2) & \cdots & T(e_n) \end{array}\right]$ ■

Example 22.7
Find the standard matrix of $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$T\left(\left[\begin{array}{c} x \\ y \\ z \end{array}\right]\right) = \left[\begin{array}{c} x - 2y + z \\ x - z \end{array}\right].$$

Solution.
We have

$$T(e_1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ T(e_2) = \begin{bmatrix} -2 \\ 0 \end{bmatrix},\ T(e_3) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

and

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$ ■

Practice Problems

Problem 22.1
Let A be an $m \times n$ and let $T : \mathbb{R}^n \to \mathbb{R}^m$ be defined by $T(x) = Ax$ where $x \in \mathbb{R}^n$. Show that T is a linear transformation.

Problem 22.2
(a) Show that the identity transformation defined by $I(v) = v$ for all $v \in V$ is a linear transformation.
(b) Show that the zero transformation is linear.

Problem 22.3
Let $\{v_1, v_2, \ldots, v_n\}$ be a basis for a vector space V and let $T : V \to W$ be a linear transformation. Show that if $T(v_1) = T(v_2) = \cdots = T(v_n) = 0$ then $T(v) = 0$ for any vector v in V.

Problem 22.4
Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation such that $T(\mathbf{e_1}) = [1, 1, 1]^T$, $T(\mathbf{e_2}) = [0, 1, -1]^T$, where $\{\mathbf{e_1}, \mathbf{e_2}\}$ is the standard basis of $\mathbb{R}^2$. Find $T(\mathbf{x})$ for any $\mathbf{x} \in \mathbb{R}^2$.

Problem 22.5
Consider the matrix

$$E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Show that the transformation

$$T_E\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} y \\ x \end{bmatrix}$$

is linear. This transformation is a **reflection** about the line $y = x$.

Problem 22.6
Consider the matrix

$$F = \begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix}.$$

Show that

$$T_F\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \alpha x \\ y \end{bmatrix}$$

is linear. Such a transformation is called an **expansion** if $\alpha > 1$ and a **compression** if $\alpha < 1$.

Problem 22.7
Consider the matrix

$$G = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Show that

$$T_G\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \alpha x + y \\ y \end{bmatrix}$$

is linear. This transformation is called a **shear**.

Problem 22.8
Show that the function $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + y \\ x - 2y \\ 3x \end{bmatrix}$$

is a linear transformation.

Problem 22.9
(a) Show that $D : P_n \longrightarrow P_{n-1}$ given by $D(p) = p'$ is a linear transformation.
(b) Show that $I : P_n \longrightarrow P_{n+1}$ given by $I(p) = \int_0^x p(t)dt$ is a linear transformation.

Problem 22.10
If $T : \mathbb{R}^3 \longrightarrow \mathbb{R}$ is a linear transformation with $T\left(\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}\right) = 5$ and $T\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right) = 2$. Find $T\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right)$.

Problem 22.11
Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be the transformation

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix}.$$

Show that T is linear. This transformation is called a **projection.**

Problem 22.12
Show that the following transformation is not linear: $T : M_{nn} \longrightarrow \mathbb{R}$ where $T(A) = |A|$.

Problem 22.13
If $T_1 : U \longrightarrow V$ and $T_2 : V \longrightarrow W$ are linear transformations, then the composition $T_2 \circ T_1 : U \longrightarrow W$ is also a linear transformation.

Problem 22.14
Let T be a linear transformation on a vector space V such that $T(v-3v_1) = w$ and $T(2v - v_1) = w_1$. Find $T(v)$ and $T(v_1)$ in terms of w and w_1.

Problem 22.15
Let $T : V \longrightarrow W$ be a linear transformation. Show that if the vectors

$$T(v_1), T(v_2), \cdots, T(v_n)$$

are linearly independent then the vectors $v_1, v_2, \cdots, v_n$ are also linearly independent.

Problem 22.16
Prove Theorem 22.1.

Problem 22.17
Prove Theorem 22.2.

Problem 22.18
Prove Theorem 22.3.

Problem 22.19
Define $T : P_2 \to \mathbb{R}^3$, where P_2 is the vector space of all polynomials of degree at most 2, by

$$T(p) = \begin{bmatrix} p(-1) \\ p(0) \\ p(1) \end{bmatrix}.$$

(a) Find the image of $p(t) = 5 + 3t$.
(b) Show that T is a linear transformation.

Problem 22.20
Let $T : V \to W$ be a linear transformation.
(a) Show that the set $\mathrm{Ker}(T) = \{v \in V : T(v) = 0\}$ is a subspace of V.
(b) Show that the set $\mathrm{Im}(T) = \{w \in W : T(v) = w \text{ for some } v \in V\}$ is a subspace of W.

23 Kernel and Range of a Linear Transformation

In this section we discuss two important subspaces associated with a linear transformation T, namely the kernel of T and the range of T.
Let $T : V \to W$ be a linear transformation. The **kernel** of T (denoted by $\mathrm{Ker}(T)$) and the **range** of T (denoted by $\mathrm{Im}(T)$) are defined by

$$\begin{aligned}\mathrm{Ker}(T) =&\{v \in V : T(v) = 0\} \subseteq V.\\ \mathrm{Im}(T) =&\{w \in W : T(v) = w, v \in V\} \subseteq W.\end{aligned}$$

The following theorem asserts that $\mathrm{Ker}(T)$ and $\mathrm{Im}(T)$ are subspaces.

Theorem 23.1
Let $T : V \to W$ be a linear transformation. Then
(a) $\mathrm{Ker}(T)$ is a subspace of V.
(b) $\mathrm{Im}(T)$ is a subspace of W.

Proof.
(a) Let $v_1, v_2 \in \mathrm{Ker}(T)$ and $\alpha \in \mathbb{R}$. Then $T(\alpha v_1 + v_2) = \alpha T(v_1) + T(v_2) = 0$. That is, $\alpha v_1 + v_2 \in \mathrm{Ker}(T)$. This proves that $\mathrm{Ker}(T)$ is a subspace of V.
(b) Let $w_1, w_2 \in \mathrm{Im}(T)$. Then there exist $v_1, v_2 \in V$ such that $T(v_1) = w_1$ and $T(v_2) = w_2$. Let $\alpha \in \mathbb{R}$. Then $T(\alpha v_1 + v_2) = \alpha T(v_1) + T(v_2) = \alpha w_1 + w_2$. Hence, $\alpha w_1 + w_2 \in \mathrm{Im}(T)$. This shows that $\mathrm{Im}(T)$ is a subspace of W ■

Example 23.1
If $T : \mathbb{R}^3 \to \mathbb{R}^3$ is defined by

$$T\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right) = \begin{bmatrix}x_1 - x_2\\x_3\\x_2 - x_1\end{bmatrix},$$

find $\mathrm{Ker}(T)$ and $\mathrm{Im}(T)$.

Solution.
If $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} \in \mathrm{Ker}(T)$ then

$$\begin{bmatrix}0\\0\\0\end{bmatrix} = T\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right) = \begin{bmatrix}x_1 - x_2\\x_3\\x_2 - x_1\end{bmatrix}.$$

This leads to the system

$$\begin{cases} x_1 - x_2 & = 0 \\ -x_1 + x_2 & = 0 \\ \quad\quad x_3 & = 0. \end{cases}$$

The general solution is given by $\begin{bmatrix} s \\ s \\ 0 \end{bmatrix}$ and therefore

$$\mathrm{Ker}(T) = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Now, for the range of T, we have

$$\begin{aligned} \mathrm{Im}(T) &= \left\{ T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) : x_1, x_2, x_3 \in \mathbb{R} \right\} \\ &= \left\{ \begin{bmatrix} x_1 - x_2 \\ x_3 \\ -(x_1 - x_2) \end{bmatrix} : x_1, x_2, x_3 \in \mathbb{R} \right\} \\ &= \left\{ \begin{bmatrix} u \\ v \\ -u \end{bmatrix} : u, v \in \mathbb{R} \right\} \\ &= \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} \blacksquare \end{aligned}$$

Example 23.2
Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be given by $Tx = Ax$. Find $\mathrm{Ker}(T)$ and $\mathrm{Im}(T)$.

Solution.
We have

$$\mathrm{Ker}(T) = \{x \in \mathbb{R}^n : Ax = \mathbf{0}\}$$

and

$$\mathrm{Im}(T) = \{Ax : x \in \mathbb{R}^n\} \blacksquare$$

Example 23.3
Let V be any vector space and α be a scalar. Let $T : V \to V$ be the transformation defined by $T(v) = \alpha v$.
(a) Show that T is linear.
(b) What is the kernel of T?
(c) What is the range of T?

Solution.
(a) Let $u, v \in V$ and $\beta \in \mathbb{R}$. Then $T(\beta u + v) = \alpha(\beta u + v) = \alpha\beta u + \alpha v = \beta T(u) + T(v)$. Hence, T is linear
(b) If $v \in \text{Ker}(T)$ then $0 = T(v) = \alpha v$. If $\alpha = 0$ then T is the zero transformation and $\text{Ker}(T) = V$. If $\alpha \neq 0$ then $\text{Ker}(T) = \{0\}$.
(c) If $\alpha = 0$ then $\text{Im}(T) = \{0\}$. If $\alpha \neq 0$ then $\text{Im}(T) = V$ since $T(\frac{1}{\alpha}v) = v$ for all $v \in V$ ■

Since the kernel and the range of a linear transformation are subspaces of given vector spaces, we may speak of their dimensions. The dimension of the kernel is called the **nullity** of T (denoted $nullity(T)$) and the dimension of the range of T is called the **rank** of T (denoted $rank(T)$).
The following important result is called the **dimension theorem**.

Theorem 23.2
If $T : V \to W$ is a linear transformation with $dim(V) = n$, then

$$nullity(T) + rank(T) = n.$$

Proof.
See Problem 23.17 ■

Example 23.4
Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_1 + x_2 \\ x_2 \end{bmatrix}.$$

(a) Show that T is linear.
(b) Find $nullity(T)$ and $rank(T)$.

Solution.
(a) Let $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ be two vectors in $\mathbb{R}^2$. Then for any $\alpha \in \mathbb{R}$ we have

$$\begin{aligned}
T\left(\alpha \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right) =& T\left(\begin{bmatrix} \alpha x_1 + y_1 \\ \alpha x_2 + y_2 \end{bmatrix}\right) \\
=& \begin{bmatrix} \alpha x_1 + y_1 \\ \alpha x_1 + y_1 + \alpha x_2 + y_2 \\ \alpha x_2 + y_2 \end{bmatrix} \\
=& \begin{bmatrix} \alpha x_1 \\ \alpha(x_1 + x_2) \\ \alpha x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_1 + y_2 \\ y_2 \end{bmatrix} \\
=& \alpha T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) + T\left(\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right).
\end{aligned}$$

(b) Let $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \text{Ker}(T)$. Then $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_1 + x_2 \\ x_2 \end{bmatrix}$ and this leads to $\text{Ker}(T) = \left\{\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\}$. Hence, $nullity(T) = \dim(\text{Ker}(T)) = 0$.
Now,

$$\begin{aligned}
\text{Im}(T) =& \left\{T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) : x_1, x_2 \in \mathbb{R}\right\} \\
=& \left\{\begin{bmatrix} x_1 \\ x_1 + x_2 \\ x_2 \end{bmatrix} : x_1, x_2 \in \mathbb{R}\right\} \\
=& \text{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right\}.
\end{aligned}$$

Thus, $rank(T) = \dim(\text{Im}(T)) = 2$ ■

Practice Problems

Problem 23.1
Let M_{nn} be the vector space of all $n \times n$ matrices. Let $T : M_{nn} \to M_{nn}$ be given by $T(A) = A - A^T$.
(a) Show that T is linear.
(b) Find $ker(T)$ and $\mathrm{Im}(T)$.

Problem 23.2
Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Find a basis for $\mathrm{Ker}(T)$.

Problem 23.3
Consider the linear transformation $T : M_{22} \to M_{22}$ defined by $T(X) = AX - XA$, where $A = \begin{bmatrix} 0 & 2 \\ 3 & 3 \end{bmatrix}$. Find the rank and nullity of T.

Problem 23.4
Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ x_1 \end{bmatrix}.$$

Find $\mathrm{Ker}(T)$ and $\mathrm{Im}(T)$.

Problem 23.5
Let $T : \mathbb{R}^3 \to \mathbb{R}^5$ be a linear transformation such that its kernel is of dimension 2. How many vectors needed to span the range of T?

Problem 23.6
The **nullity** of a $m \times n$ matrix A is defined to be the nullity of the linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ with $T(x) = Ax$. Find the value(s) of a so that the nullity of the matrix

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & a & 1 \\ 0 & 2 & a \end{bmatrix}$$

is zero.

Problem 23.7
Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}.$$

Find a basis for $\mathrm{Ker}(T)$ and $\mathrm{Im}(T)$.

Problem 23.8
Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_1 + x_2 \\ x_1 + 2x_2 \end{bmatrix}.$$

Find a basis for $\mathrm{Ker}(T)$ and $\mathrm{Im}(T)$.

Problem 23.9
Consider the linear transformation $T : P_2 \to P_3$ defined by

$$T(at^2 + bt + c) = ct^2 + (a + b)t.$$

Find a basis for $\mathrm{Ker}(T)$ and $\mathrm{Im}(T)$.

Problem 23.10
Suppose that $S : P_{12} \to P_{15}$ and $T : P_{15} \to P_{12}$ are linear transformations.
(a) If T has rank 6, what is the nullity of T?
(b) If S has nullity 6, what is the rank of S?

Problem 23.11
Consider the linear transformation $T : P_3 \to \mathbb{R}^2$ defined by $T(p) = [p(0), p(1)]^T$. Describe the kernel and range of T.

Problem 23.12
Consider the linear transformation $T : P_n \to P_{n-1}$ defined by $T(p) = p'(t)$. Find the nullity and rank of T.

Problem 23.13
Let V be the vector space of twice differentiable function over an interval I and $\mathcal{F}(I)$ the vector space of functions defined on I. Define $T : V \to \mathcal{F}(I)$ by $T(y) = y'' + y$. Show that T is linear and find its kernel.

Problem 23.14
Let V be a finite dimensional vector space and $T : V \to W$ be a linear transformation. Show that any basis of $\mathrm{Ker}(T)$ can be extended to a basis of V.

Problem 23.15
Let $T : V \to W$ be a linear transformation and $\dim(V) = n$. Suppose that $\{v_1, v_2, \cdots, v_k\}$ is a basis of $\mathrm{Ker}(T)$. By the previous problem, we can extend this basis to a basis $\{v_1, v_2, \cdots, v_k, v_{k+1}, \cdots, v_n\}$ of V. Show that $\mathrm{Im}(T) = \mathrm{span}\{T(v_{k+1}), \cdots, T(v_n)\}$.

Problem 23.16
Show that the set $\{T(v_{k+1}), \cdots, T(v_n)\}$ in Problem 23.15 is linearly independent.

Problem 23.17
Using Problems 23.15 - 23.16, prove Theorem 23.2.

Problem 23.18
Let $T : V \to W$ and $S : W \to X$ be two linear transformations. Prove that (a) $\mathrm{Ker}(T) \subseteq \mathrm{Ket}(S \circ T)$ and (b) $\mathrm{Im}(S \circ T) \subseteq \mathrm{Im}(S)$.

Problem 23.19
Find the kernel and range of the linear transformation $T : P_2 \to P_2$ defined by $T(p) = xp'(x)$.

Problem 23.20
Let $T : V \to W$ be a linear transformation with the property that no two distinct vectors of V have the same image in W. Show that $\mathrm{Ker}(T) = \{0\}$.

24 Isomorphisms

Since linear transformations are functions, it makes sense to talk about one-to-one and onto functions. We say that a linear transformation $T: V \to W$ is **one-to-one** if $T(v) = T(w)$ implies $v = w$. We say that T is **onto** if $\mathrm{Im}(T) = W$. If T is both one-to-one and onto we say that T is an **isomorphism** and the vector spaces V and W are said to be **isomorphic** and we write $V \cong W$. The identity transformation is an isomorphism of any vector space onto itself. That is, if V is a vector space then $V \cong V$.
The following theorem is used as a criterion for proving that a linear transformation is one-to-one.

Theorem 24.1
Let $T: V \to W$ be a linear transformation. Then T is one-to-one if and only if $\mathrm{Ker}(T) = \{0\}$.

Example 24.1
Consider the transformation $T: \mathbb{R}^3 \to \mathbb{R}^2$ given by $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ x_1 - x_2 \end{bmatrix}$.
(a) Show that T is linear.
(b) Show that T is onto but not one-to-one.

Solution.
(a) Let $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ be two vectors in $\mathbb{R}^3$ and $\alpha \in \mathbb{R}$. Then

$$\begin{aligned}
T\left(\alpha \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}\right) =& T\left(\begin{bmatrix} \alpha x_1 + y_1 \\ \alpha x_2 + y_2 \\ \alpha x_3 + y_3 \end{bmatrix}\right) \\
=& \begin{bmatrix} \alpha x_1 + y_1 + \alpha x_2 + y_2 \\ \alpha x_1 + y_1 - \alpha x_2 - y_2 \end{bmatrix} \\
=& \begin{bmatrix} \alpha(x_1 + x_2) \\ \alpha(x_1 - x_2) \end{bmatrix} + \begin{bmatrix} y_1 + y_2 \\ y_1 - y_2 \end{bmatrix} \\
=& \alpha T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) + T\left(\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}\right).
\end{aligned}$$

(b) Since $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \in \text{Ker}(T)$, by Theorem 24.1 T is not one-to-one. Now, let $\begin{bmatrix} u \\ v \\ w \end{bmatrix} \in \mathbb{R}^3$ be such that $T\left(\begin{bmatrix} u \\ v \\ w \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. In this case, $x_1 = \frac{1}{2}(u+v)$ and $x_2 = \frac{1}{2}(u-v)$. Hence, $\text{Im}(T) = \mathbb{R}^2$ so that T is onto ■

Example 24.2
Let $T : V \to W$ be a one-to-one linear transformation. Show that if $\{v_1, v_2, \ldots, v_n\}$ is a basis for V then $\{T(v_1), T(v_2), \ldots, T(v_n)\}$ is a basis for $\text{Im}(T)$.

Solution.
The fact that $\{T(v_1), T(v_2), \cdots, T(v_n)\}$ is linearly independent follows from Problem 24.6. It remains to show that $\text{Im}(T) = \text{span}\{T(v_1), T(v_2), \cdots, T(v_n)\}$. Indeed, let $w \in \text{Im}(T)$. Then there exists $v \in V$ such that $T(v) = w$. Since $\{v_1, v_2, \cdots, v_n\}$ is a basis of V, v can be written uniquely in the form $v = \alpha v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$. Hence, $w = T(v) = \alpha_1 T(v_1) + \alpha_2 T(v_2) + \cdots + \alpha_n T(v_n)$. That is, $w \in span\{T(v_1), T(v_2), \cdots, T(v_n)\}$. We conclude that $\{T(v_1), T(v_2), \cdots, T(v_n)\}$ is a basis of $\text{Im}(T)$ ■

Theorem 24.2
If W is a subspace of a finite-dimensional vector space V and $dim(W) = dim(V)$ then $V = W$.

Proof.
See Problem 18.21 ■

We have seen that a linear transformation $T : V \to W$ can be one-to-one and onto, one-to-one but not onto, and onto but not one-to-one. The foregoing theorem shows that each of these properties implies the other if the vector spaces V and W have the same dimension.

Theorem 24.3
Let $T : V \to W$ be a linear transformation such that $dim(V) = dim(W) = n$. Then
(a) if T is one-to-one, then T is onto;
(b) if T is onto, then T is one-to-one.

Proof.
(a) If T is one-to-one then $\text{Ker}(T) = \{0\}$. Thus, $dim(\text{Ker}(T)) = 0$. By Theorem 23.2 we have $dim(\text{Im}(T)) = n$. Hence, by Theoren 24.2, we have $\text{Im}(T) = W$. That is, T is onto.
(b) If T is onto then $dim(\text{Im}(T)) = dim(W) = n$. By Theorem 23.2, $dim(\text{Ker}(T)) = 0$. Hence, $\text{Ker}(T) = \{0\}$, i.e., T is one-to-one ■

A linear transformation $T : V \to W$ is said to be **invertible** if and only if there exists a unique function $T^{-1} : W \to V$ such that $T \circ T^{-1} = id_W$ and $T^{-1} \circ T = id_V$ where id_V and id_W are the identity functions on V and W respectively.

Theorem 24.4
Let $T : V \to W$ be an invertible linear transformation. Then
(a) T^{-1} is linear.
(b) $(T^{-1})^{-1} = T$.

Proof.
(a) Suppose $T^{-1}(w_1) = v_1, T^{-1}(w_2) = v_2$ and $\alpha \in \mathbb{R}$. Then $\alpha w_1 + w_2 = \alpha T(v_1) + T(v_2) = T(\alpha v_1 + v_2)$. That is, $T^{-1}(\alpha w_1 + w_2) = \alpha v_1 + v_2 = \alpha T^{-1}(w_1) + T^{-1}(w_2)$.
(b) Since $T \circ T^{-1} = id_W$ and $T^{-1} \circ T = id_V$, T is the inverse of T^{-1}. That is, $(T^{-1})^{-1} = T$ ■

What types of linear transformations are invertible?

Theorem 24.5
A linear transformation $T : V \to W$ is invertible if and only if T is one-to-one and onto.

Example 24.3
Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by $T(x) = Ax$ where A is the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 2 \end{bmatrix}$$

(a) Prove that T is invertible.
(b) What is $T^{-1}(x)$?

Solution.

(a) We must show that T is one-to-one and onto. Let $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \text{Ker}(T)$. Then $Tx = Ax = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Since $|A| = -1 \neq 0$, A is invertible and therefore $x = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Hence, $\text{Ker}(T) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$. Now, since A is invertible the system $Ax = b$ is always solvable. This shows that $\text{Im}(T) = \mathbb{R}^3$. Hence, by the above theorem, T is invertible.

(b) $T^{-1}x = A^{-1}x$ ■

Practice Problems

Problem 24.1
Let $T : M_{mn} \to M_{mn}$ be given by $T(X) = AX$ for all $X \in M_{mn}$, where A is an $m \times m$ invertible matrix. Show that T is both one-one and onto.

Problem 24.2
Show that the projection transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is not one-one.

Problem 24.3
Let $T : V \to W$. Prove that T is one-one if and only if $dim(\mathrm{Im}(T)) = dim(V)$.

Problem 24.4
Show that the linear transformation $T : M_{nn} \to M_{nn}$ given by $T(A) = A^T$ is an isomorphism.

Problem 24.5
Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ x_1 + 2x_2 \\ x_2 \end{bmatrix}.$$

Show that T is one-to-one.

Problem 24.6
Let $T : V \to W$ be a linear transformation. Suppose that T is one-to-one. Show that if S is a finite linearly independent set of vectors then $T(S)$ is also linearly independent.

Problem 24.7
Let $T : V \to W$ be a linear transformation. Suppose that $T(S)$ is linearly independent set of vectors whenever S is a set of linearly independent vectors. Show that T is one-to-one.

Problem 24.8

Consider the transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ given by $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ x_1 - x_2 \\ x_1 \end{bmatrix}$.

(a) Show that T is linear.
(b) Show that T is one-to-one but not onto.

Problem 24.9
Prove Theorem 24.1.

Problem 24.10
Prove Theorem 24.5.

Problem 24.11
Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 5x_1 - 3x_2 \\ -7x_1 + 8x_2 \\ 2x_1 \end{bmatrix}$.
(a) Find the standard matrix of T.
(b) Find $\mathrm{Ker}(T)$. Is T one-to-one?

Problem 24.12
Consider the linear transformation $T : P_2 \to P_1$ defined by $T(p) = p'$. Show that T is onto but not one-to-one.

Problem 24.13
Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be the linear transformation defined by $T(x) = Ax$ with A singular. Show that T is neither one-to-one nor onto.

Problem 24.14

Consider the linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^4$ defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ x_1 + x_2 \\ x_2 + x_3 \\ x_3 + x_4 \end{bmatrix}$. Determine whether T is one-to-one and onto.

Problem 24.15
Find the inverse of the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 3x_2 \\ x_1 + 4x_2 \end{bmatrix}$.

Problem 24.16
Let $T : V \to W$ and $S : W \to X$ be two one-to-one linear transformations. Show that $S \circ T$ is also one-to-one.

Problem 24.17
Let $T : R^4 \to M_{22}$ be the linear transfomation defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}.$$

Find T^{-1}.

Problem 24.18
Let $T : V \to W$ and $S : W \to X$ be two onto linear transformations. Show that $S \circ T$ is also onto.

Problem 24.19
Consider the linear transformation $T : M_{22} \to M_{22}$ defined by

$$T(A) = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} A - A \begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix}.$$

Show that T is one-to-one and onto.

Problem 24.20
Suppose that $T : V \to W$ is an isomorphism where V and W are finite dimensional. Show that $\dim(V) = \dim(W)$.

Answer Key

Section 1

1.1 (a) $(a_1, a_2, \cdots, a_n)$ is a solution to a system of m linear equations in n unknowns if and only if the a_i's make each equation a true equality.
(b) A consistent system of linear equations is a system that has a solution.
(c) An inconsistent system of linear equations is a system with no solution.
(d) A dependent system of linear equations is a system with infinitely many solutions.
(e) An independent system of linear equations is a system with exactly one solution.

1.2 (a) Non-inear (b) Non-linear (c) Non-linear.

1.3 Substituting these values for x_1, x_2, x_3, and x_4 in each equation.

$$\begin{aligned} 2x_1 + 5x_2 + 9x_3 + 3x_4 &= 2(2s + 12t + 13) + 5s + 9(-s - 3t - 3) + 3t = -1 \\ x_1 + 2x_2 + 4x_3 &= (2s + 12t + 13) + 2s + 4(-s - 3t - 3) = 1. \end{aligned}$$

Since both equations are satisfied, it is a solution for all s and t.

1.4 (a) The two lines intersect at the point $(3, 4)$ so the system is consistent.
(b) The two equations represent the same line. Hence, $x_2 = s$ is a parameter. Solving for x_1 we find $x_1 = \frac{5+3t}{2}$.The system is consistent.
(c) The two lines are parallel. So the given system is inconsistent.

1.5 (a) Non-linear because of the term $\ln x_1$. (b) Linear.

1.6 $x_1 = 1 + 5w - 3t - 2s,\ x_2 = w,\ x_3 = t,\ x_4 = s.$

1.7 (a)

$$\begin{cases} x_1 + \ \ x_2 + 2x_3 = 9 \\ x_1 + \ \ x_2 + 2x_3 = 2 \\ 3x_1 + 6x_2 - 5x_3 = 0. \end{cases}$$

Note that the first two equations imply $2 = 9$ which is impossible.
(b)

$$\begin{cases} x_1 + \ \ x_2 + 2x_3 = 9 \\ 2x_1 + 4x_2 - 3x_3 = 1 \\ 3x_1 + 6x_2 - 5x_3 = 0. \end{cases}$$

Solving for x_3 in the third equation, we find $x_3 = \frac{3}{5}x_1 + \frac{6}{5}x_2$. Substituting this into the first two equations we find the system

$$\begin{cases} 11x_1 + 17x_2 = 45 \\ x_1 + \ \ 2x_2 = 5. \end{cases}$$

Solving this system by elimination, we find $x_1 = 1$ and $x_2 = 2$. Finally, $x_3 = 3$.
(c)

$$\begin{cases} x_1 + \ \ x_2 + 2x_3 = 1 \\ 2x_1 + 2x_2 + 4x_3 = 2 \\ -3x_1 - 3x_2 - 6x_3 = -3. \end{cases}$$

The three equations reduce to the single equation $x_1 + x_2 + 2x_3 = 1$. Letting $x_3 = t,\ x_2 = s$, we find $x_1 = 1 - s - 2t$.

1.8 (a) The system has no solutions if $k \neq 6$ and $h = 9$.
(b) The system has a unique solution if $h \neq 9$ and any k. In this case, $x_2 = \frac{k-6}{h-9}$ and $x_1 = 2 - \frac{3(k-6)}{h-9}$.
(c) The system has infinitely many solutions if $h = 9$ and $k = 6$ since in this case the two equations reduces to the single equation $x_1 + 3x_2 = 2$. All solutions to this equation are given by the parametric equations $x_1 = 2 - 3t,\ x_2 = t$.

1.9 (a) True (b) False (c) True (d) False.

1.10 $x - 2y = 5$.

1.11 $c = a + b$.

1.12 (a) No (b) No (c) Yes.

1.13 (A) (a) Consistent (b) Independent (c) One solution. (B) (a) Inconsistent (b) Does not apply (c) No solution.
(C) (a) Consistent (b) Dependent (c) Infinitely many solutions.

1.14

$$\begin{cases} x_1 + \quad x_2 = 5800 \\ 0.035x_1 + 0.055x_2 = 283. \end{cases}$$

1.15 (a) $m = \pm 1$ (b) $m = 1$.

1.16 80 pennies and 160 nickels.

1.17 $57°$ and $33°$.

1.18 The three numbers are 2, 4, and 8.

1.19 $x_1 = x_2 = 1$.

1.20 (a) There is no point that lies on all three of the planes.
(b) The three planes have a single point in common.
(c) The three planes have an infinite number of points in common.

Section 2

2.1 (a) The unique solution is $x_1 = 3,\ x_2 = 4$.
(b) The system is consistent. The general solution is given by the parametric equations: $x_1 = \frac{5+3t}{2},\ x_2 = t$.
(c) System is inconsistent.

2.2 $A = -\frac{1}{9}, B = -\frac{5}{9}$, and $C = \frac{11}{9}$.

2.3 $a = 3, b = -2$, and $c = 4$.

2.4 $x_1 = \frac{3}{2}\ , x_2 = 1,\ x_3 = -\frac{5}{2}$.

2.5 $x_1 = \frac{1}{9}, x_2 = \frac{10}{9}, x_3 = -\frac{7}{3}$.

2.6 $x_3 = s$ and $x_4 = t$ are parameters. Solving one finds $x_1 = 1 - s + t$ and $x_2 = 2 + s + t,\ x_3 = s,\ x_4 = t$.

2.7 $a = 1, b = 2, c = -1$.

2.8 Solving both systems using backward-substitution technique, we find that both systems have the same solution $x_1 = 1,\ x_2 = 4,\ x_3 = 3$.

2.9 $x_1 = 2,\ x_2 = 1,\ x_3$.

2.10 $x_1 = 2,\ x_2 = -1,\ x_3 = 1$.

2.11 John = 75 , Philip = 67, Grace = 82.

2.12

$$\begin{cases} 1.50x_1 + 5.75x_2 + 2.60x_3 = 589.50 \\ \quad\ \ x_1 + \qquad x_2 + \qquad x_3 = 200 \\ \qquad\qquad\quad\ \ x_2 - \qquad x_3 = -20. \end{cases}$$

2.13 (a) Swap rows 1 and 3. (b) This is not a valid row operation. You can't add or subtract a number from the elements in a row. (c) Add four times row 2 to row 3. (d) Multiply row 4 by 5.

2.14 $x_1 = -\frac{3}{2}t + 1,\ x_2 = -\frac{3}{2}t + \frac{1}{2},\ x_3 = t - \frac{1}{2},\ x_4 = t.$

2.15 $a = 3.$

2.16 (a) True. The row reduced augmented matrix leads to a linear system that is equivalent to the original system.
(b) False. They are row equivalent if you can get from one to the other using elementary row operations.
(c) True by the definition of equivalent systems.

2.17 (a) The system is consistent and dependent if $h = -2$. In this case, the parametric equations are: $x_1 = 2t - 3,\ x_2 = t.$
(b) The system is consistent and independent if $h \neq -2$. The unique solution is $x_1 = -3,\ x_2 = 0.$

2.18 $d - 3c \neq 0.$

2.19 Both systems have the solution $(3, 2)$.

2.20 (a) $r_i \leftrightarrow r_j$ (b) $r_i \leftarrow \frac{1}{\alpha} r_i$ (c) $r_i \leftarrow r_i - \alpha r_j$.

2.21 If $(s_1, s_2, \cdots, s_n)$ is a solution to $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$ then $a_1s_1 + a_2s_2 + \cdots + a_ns_n = b$. Multiplying this by a non-zero scalar k, we obtain $ka_1s_1 + ka_2s_2 + \cdots + ka_ns_n = kb$. That is, $(s_1, s_2, \cdots, s_n)$ is a solution to $ka_1x_1 + ka_2x_2 + \cdots + ka_nx_n = kb$. Conversely, suppose that $(s_1, s_2, \cdots, s_n)$ is a solution to $ka_1x_1 + ka_2x_2 + \cdots + ka_nx_n = kb,\ k \neq 0$. Dividing this equation by k, we obtain $a_1s_1 + a_2s_2 + \cdots + a_ns_n = b$. Hence, $(s_1, s_2, \cdots, s_n)$ is a solution to $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$.

2.22 $(s_1, s_2, \cdots, s_n)$ is a solution to $a_1x_1 + a_2x_2 + \cdots + a_nx_n = a$ and $b_1x_1 + b_2x_2 + \cdots + b_nx_n = b$ then $a_1s_1 + a_2s_2 + \cdots + a_ns_n = a$ and $b_1s_1 + b_2s_2 + \cdots + b_ns_n = b$. Thus, $(ka_1 + b_1)s_1 + (ka_2 + b_2)s_2 + \cdots + (ka_n + b_n)s_n = ka + b$. That is, $(s_1, s_2, \cdots, s_n)$ is a solution to $(ka_1 + b_1)x_1 + (ka_2 + b_2)x_2 + \cdots + (ka_n + b_n)x_n = ka + b$. Conversely, if $(s_1, s_2, \cdots, s_n)$ is a solution to $(ka_1 + b_1)x_1 + (ka_2 + b_2)x_2 + \cdots + (ka_n + b_n)x_n = ka + b$ and $a_1x_1 + a_2x_2 + \cdots + a_nx_n = a$ then

$$b_1s_1 + b_2s_2 + \cdots + b_ns_n = ka + b - k(a_1s_1 + a_2s_2 + \cdots + a_ns_n) = ka + b - ka = b.$$

That is, $(s_1, s_2, \cdots, s_n)$ is a solution to $b_1x_1 + b_2x_2 + \cdots + b_nx_n = b$.

2.23 If two equations in a system are swapped then the new system is trivially equivalent to the original system since it is just a rearrangement of the equations in the old system.
If an equation in the original system is multiplied by a non-zero scalar then the new system is equivalent to the original system by Problem 2.21.
If a multiple of an equation of a system is added to another equation then the new system is equivalent to the original system by Problem 2.22.

Section 3

3.1 $x_1 = 2, x_2 = -1, x_3 = 1.$

3.2 $-5g + 4h + k = 0.$

3.3 $x_1 = -\frac{11}{2}.\ x_2 = -6,\ x_3 = -\frac{5}{2}.$

3.4 $x_1 = \frac{1}{9}, x_2 = \frac{10}{9}, x_3 = -\frac{7}{3}.$

3.5 $x_1 = -s, x_2 = s, x_3 = s$, and $x_4 = 0$.

3.6 $x_1 = 9s$ and $x_2 = -5s,\ x_3 = s.$

3.7 $x_1 = 3, x_2 = 1, x_3 = 2.$

3.8 Because of the last row the system is inconsistent.

3.9 $x_1 = 8 + 7s,\ x_2 = 2 - 3s,\ x_3 = -5 - s,\ x_4 = s.$

3.10

$$\begin{bmatrix} -1 - 1 & 0 & 0 & \\ 0 & -4 & -2 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

3.11 $x_1 = 4 - 3t, x_2 = 5 + 2t, x_3 = t, x_4 = -2.$

3.12 $a = -8.$

3.13 $x_1 = 5 - t - 2s,\ x_2 = 1 - t,\ x_3 = s,\ x_4 = t.$

3.14 The system is inconsistent.

3.15 $x_1 = -4, x_2 = -5, x_3 = 2.$

3.16 (a) True.
(b) False. This row shows only that $x_4 = 0$.

(c) True. This rows shows that $ox_1 + 0x_2 + 0x_3 + 0x_4 = 0$ and this equation has infinite number of solutions.
(d) False. Any matrix can be reduced. The matrix does not know where it came from.

3.17 (a) If $k = 0$ then the last row gives $0x_1 + 0x_2 + 0x_3 = -4$ which does not have a solution. Hence, the system is inconsistent.
(b) If $k \neq 0$, the system has the unique solution $x_1 = \frac{24-3k}{k}, x_2 = -\frac{12}{k}, x_3 = -\frac{4}{k}$.

3.18 198.

3.19 $x_5 = t,\ x_4 = 3t - 1,\ x_3 = 0,\ x_2 = s,\ x_1 = 1 - 2s$.

3.20 Inconsistent.

Section 4

4.1

$$\begin{bmatrix} 1 & -2 & 3 & 1 & -3 \\ 0 & 1 & -1 & -1 & 2 \end{bmatrix}.$$

4.2

$$\begin{bmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & -7 & 9 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

4.3

$$\begin{bmatrix} 1 & -1 & -3 & 8 \\ 0 & 1 & 2 & -4 \\ 0 & 0 & 1 & -\frac{5}{2} \end{bmatrix}.$$

4.4

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{7}{3} \end{bmatrix}.$$

4.5 (a) No, because the matrix fails condition 1 of the definition. Rows of zeros must be at the bottom of the matrix.
(b) No, because the matrix fails condition 2 of the definition. Leading entry in row 2 must be 1 and not 2.
(c) Yes. The given matrix satisfies conditions 1 - 4.

4.6

$$\begin{bmatrix} 1 & -3 & 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

4.7

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

4.8

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 4 \\ 0 & 1 & -2 & 0 & 5 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

4.9

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & .5 & -.25 \\ 0 & 0 & 1 & 1.5 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

4.10

$$\begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

4.11 (a) 3 (b) 2.

4.12

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

4.13

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

4.14 3.

4.15 3.

4.16

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

4.17

$$\left[\begin{array}{ccr|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

4.18

$$\begin{bmatrix} 1 & 0 & -\frac{5}{4} & 0 \\ 0 & 1 & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

4.19 The rank of the matrix is 2.

4.20 The rank of the matrix is 3.

Section 5

5.1 $x_1 = 3, x_2 = 1, x_3 = 2.$

5.2 $x_1 = 8 + 7s, x_2 = 2 - 3s$, and $x_3 = -5 - s.$

5.3 The system is inconsistent.

5.4 $x_1 = 4 - 3t, x_2 = 5 + 2t, x_3 = t, x_4 = -2.$

5.5 $x_1 = \frac{1}{9}, x_2 = \frac{10}{9}, x_3 = -\frac{7}{3}.$

5.6 $x_1 = -\frac{11}{2},\ x_2 = -6,\ x_3 = -\frac{5}{2}.$

5.7 $x_1 = 1,\ x_2 = -2,\ x_3 = 1,\ x_4 = 3.$

5.8 $x_1 = 2,\ x_2 = 1,\ x_3 = -1.$

5.9 $x_1 = 2 - 2t - 3s,\ x_2 = t,\ x_3 = 2 + s,\ x_4 = s,\ x_5 = -2.$

5.10 $x_1 = 4 - 2s - 3t,\ x_2 = s,\ x_3 = -1,\ x_4 = 0,\ x_5 = t.$

5.11 $x_1 = 3,\ x_2 = 4,\ x_3 = -2.$

5.12 Inconsistent.

5.13 $x_1 = \frac{7}{4}t,\ x_2 = \frac{1}{2} - \frac{1}{4}t,\ x_3 = t.$

5.14 $x_1 = 2s + t,\ x_2 = s,\ x_3 = 2t,\ x_4 = t.$

5.15 $x_1 = -8,\ x_2 = 1,\ x_3 = 5.$

5.16 $x_1 = 3,\ x_2 = 4,\ x_3 = -2.$

5.17 The system of equations is inconsistent.

5.18 $x_1 = \frac{7}{4}t,\ x_2 = \frac{1}{2} - \frac{1}{4}t,\ x_3 = t.$

5.19 $x_1 = -2 - s,\ x_2 = s,\ x_3 = t.$

5.20 $x_1 = -902,\ x_2 = 520,\ x_3 = -52,\ x_4 = -26.$

5.21 Suppose first that the reduced augmented matrix has a row of the form $(0, \cdots, 0, b)$ with $b \neq 0$. That is, $0x_1 + 0x_2 + ... + 0x_m = b$. Then the left side is 0 whereas the right side is not. This cannot happen. Hence, the system has no solutions.

5.22 If the reduced augmented matrix has independent variables and no rows of the form $(0, 0, \cdots, 0, b)$ for some $b \neq 0$ then these variables are treated as parameters and hence the system has infinitely many solutions.

5.23 If the reduced augmented matrix has no row of the form $(0, 0, \cdots, 0, b)$ where $b \neq 0$ and no indepedent variables then the system looks like

$$
\begin{aligned}
x_1 &= c_1\\
x_2 &= c_2\\
x_3 &= c_3\\
\vdots &= \vdots\\
x_m &= c_m
\end{aligned}
$$

i.e., the system has a unique solution.

Section 6

6.1 (a) This is true since the trivial solution is always a solution.
(b) The system will have free variables, so it either has infinitely many solutions or no solutions. However, by part (a), it always has at least one solution. Therefore, such a system will always have infinitely many solutions.

6.2 $x_1 = 9s,\ x_2 = -5s,\ x_3 = s.$

6.3 $x_1 = -s,\ x_2 = s,\ x_3 = s,\ x_4 = 0.$

6.4 $x_1 = x_2 = x_3 = 0.$

6.5 Infinitely many solutions: $x_1 = -8t,\ x_2 = 10t,\ x_3 = t.$

6.6 $x_1 = -s + 3t,\ x_2 = s,\ x_3 = t.$

6.7 $x_1 = -\frac{7}{3}t,\ x_2 = -\frac{2}{3}t,\ x_3 = -\frac{13}{3}t,\ x_4 = t.$

6.8 $x_1 = 8s + 7t,\ x_2 = -4s - 3t,\ x_3 = s,\ x_4 = t.$

6.9 $x_1 = x_2 = x_3 = 0.$

6.10 $x_1 = -\frac{1}{2}t,\ x_2 = \frac{3}{2}t,\ x_3 = x_4 = t.$

6.11 The trivial solution is the only solution.

6.12 $x_1 = t - \frac{2}{3}s,\ x_2 = t,\ x_3 = \frac{1}{3}s,\ x_4 = s.$

6.13 There are more unknowns than equations.

6.14 The rank of the augmented matrix is 3 which is less than the number of unknowns. Hence, by Theorem 6.1(a), the system has infinitely many solutions.

6.15 The augmented matrix of the first system is row equivalent to the augmented matrix of the second system.

6.16 (a) True. The trivial solution is a common solution to all homogeneous linear systems.
(b) False. See Example 6.3.
(c) The system will have free variables, so it either has infinitely many solutions or no solutions. However, by part (a), it always has at least one solution. Therefore, such a system will always have infinitely many solutions.

6.17 $x_1 = -2s - 4t,\ x_2 = s,\ x_3 = -t,\ x_4 = t,\ x_5 = 0.$

6.18 For $1 \le i \le m$ we have

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = 0$$

and

$$a_{i1}y_1 + a_{i2}y_2 + \cdots + a_{in}y_n = 0.$$

Thus,

$$\begin{aligned} a_{i1}(\alpha x_1 + \beta y_1) + a_{i2}(\alpha x_2 + \beta y_2) + \cdots + a_{in}(\alpha x_n + \beta y_n) &= \alpha(a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n) \\ &\quad + \beta(a_{i1}y_1 + a_{i2}y_2 + \cdots + a_{in}y_n) \\ &= 0 + 0 = 0. \end{aligned}$$

Hence, $\alpha(x_1, x_2, \cdots, x_n) + \beta(y_1, y_2, \cdots, y_n)$ is also a solution.

6.19 Consider the equation $x_1 - 4x_2 + 7x_3 = 5$ (see Example 1.2). Two solutions to this equation are $(5, 0, 0)$ and $(2, 1, 1)$. However, $(5, 0, 0) + (2, 1, 1) = (7, 1, 1)$ is not a solution.

6.20 It is possible to have the reduced row echelon form of the augmented matrix with a row of zeros and no free variables. For example,

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

It is possible to have free variables and yet the reduced row echelon form of the augmented matrix has no rows of zeros. See Example 6.2.

Section 7

7.1 $\begin{bmatrix} 4 & -1 \\ -1 & -6 \end{bmatrix}$

7.2 $w = -1, x = -3, y = 0$, and $z = 5$.

7.3 $s = 0$ and $t = 3$.

7.4 We have

$$\begin{aligned} a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} &= \\ \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & d \end{bmatrix} &= \\ \begin{bmatrix} a & b \\ c & d \end{bmatrix} &= A. \end{aligned}$$

7.5 A simple arithmetic yields the matrix

$$rA + sB + tC = \begin{bmatrix} r + 3t & r + s & -r + 2s + t \end{bmatrix}.$$

The condition $rA + sB + tC = \mathbf{0}$ yields the system

$$\begin{cases} r + 3t = 0 \\ r + s = 0 \\ -r + 2s + t = 0. \end{cases}$$

The augmented matrix is row equivalent to the row echelon matrix

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 10 & 0 \end{bmatrix}$$

Solving the corresponding system we find $r = s = t = 0$.

7.6

$$\begin{bmatrix} 9 & 5 & 1 \\ -4 & 7 & 6 \end{bmatrix}$$

7.7 The transpose of A is equal to A.

7.8 $A^T = -A$ so the matrix is skew-symmetric.

7.9 $4tr(7A) = 0$.

7.10 $tr(A^T - 2B) = 8$.

7.11

$$S = \frac{1}{2}(A + A^T) = \begin{bmatrix} 12 & 2.5 & 0.5 \\ 2.5 & -4 & -4 \\ 0.5 & -4 & 2 \end{bmatrix}$$

and

$$K = \frac{1}{2}(A - A^T) = \begin{bmatrix} 0 & 4.5 & 0.5 \\ -4.5 & 0 & 4 \\ -0.5 & -4 & 0 \end{bmatrix}.$$

7.12 If such a matrix exists then we shall have $tr(A - A^T) = tr(I_n) = n$. But $tr(A - A^T) = tr(A) - tr(A^T) = 0$. Thus, $n = 0$, which contradicts the definition of $n \geq 1$.

7.13 $x = -4$.

7.14

$$\begin{bmatrix} 12 & -12 & 21 \\ 21 & 12 & -6 \end{bmatrix}.$$

7.15

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \\ 3 & 4 & 5 & 1 & 2 \\ 4 & 5 & 1 & 2 & 3 \\ 5 & 1 & 2 & 3 & 4 \end{bmatrix}.$$

7.16 $x_1 == 2,\ x_2 = 1,\ x_3 = -1$.

7.17 We have $tr(\alpha A + \beta B) = tr(\alpha A) + tr(\beta B) = \alpha tr(A) + \beta tr(B) =$

$\alpha \cdot 0 + \beta \cdot 0 = 0$.

7.18 We have $f(\alpha A + \beta B) = tr(\alpha A + \beta B) = tr(\alpha A) + tr(\beta B) = \alpha tr(A) + \beta tr(B) = \alpha f(A) + \beta f(B)$.

7.19 We have $f(\alpha A+\beta B) = (\alpha A+\beta B)^T = (\alpha A)^T + (\beta B)^T = \alpha A^T + \beta B^T = \alpha f(A) + \beta f(B)$.

7.20 We have $(\alpha A + \beta B)^T = \alpha A^T + \beta B^T = -\alpha A - \beta B = -(\alpha A + \beta B)$. Hence, $\alpha A + \beta B$ is skew-symmetric.

7.21 (i) $A + B = [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}] = [b_{ij} + a_{ij}] = [b_{ij}] + [a_{ij}] = B + A$, since addition of scalars is commutative.
(ii) $(A+B)+C = ([a_{ij}]+[b_{ij}])+[c_{ij}] = [a_{ij}+b_{ij}]+[c_{ij}] = [(a_{ij}+b_{ij})+c_{ij}] = [a_{ij}+(b_{ij}+c_{ij})] = [a_{ij}]+[b_{ij}+c_{ij}] = [a_{ij}]+([b_{ij}]+[c_{ij}]) = A+(B+C)$ since addition of scalars is associative.
(iii) $A + \mathbf{0} = [a_{ij}] + [0] = [a_{ij} + 0] = [a_{ij}] = A$ since 0 is the 0 element for scalars.
(iv) $A + (-A) = [a_{ij}] + [-a_{ij}] = [a_{ij} + (-a_{ij}] = [0] = \mathbf{0}$.
(v) $c(A + B) = c([a_{ij} + b_{ij}]) = [c(a_{ij} + b_{ij})] = [ca_{ij} + cb_{ij}] = [ca_{ij}] + [cb_{ij}] = c[a_{ij}] + c[b_{ij}] = cA + cB$ since multiplication of scalars is distributive with respect to addition.
(vii) $(c + d)A = (c + d)[a_{ij}] = [(c + d)a_{ij}] = [ca_{ij} + da_{ij}] = [ca_{ij}] + [da_{ij}] = c[a_{ij}] + d[a_{ij}] = cA + dA$.
(viii) $(cd)A = (cd)[a_{ij}] = [(cd)a_{ij}] = [c(da_{ij})] = c[da_{ij}] = c(dA)$.

Section 8

8.1

$$\begin{cases} 2x_1 - x_2 = -1 \\ -3x_1 + 2x_2 + x_3 = 0 \\ x_2 + x_3 = 3. \end{cases}$$

8.2 (a) If A is the coefficient matrix and B is the augmented matrix then

$$A = \begin{bmatrix} 2 & 3 & -4 & 1 \\ -2 & 0 & 1 & 0 \\ 3 & 2 & -4 & 0 \end{bmatrix}, B = \begin{bmatrix} 2 & 3 & -4 & 1 & 5 \\ -2 & 0 & 1 & 0 & 7 \\ 3 & 2 & -4 & 0 & 3 \end{bmatrix}.$$

(b) The given system can be written in matrix form as follows

$$\begin{bmatrix} 2 & 3 & -4 & 1 \\ -2 & 0 & 1 & 0 \\ 3 & 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ -9 \end{bmatrix}.$$

8.3 AA^T is always defined.

8.4 (a) Easy calculation shows that $A^2 = A$.
(b) Suppose that $A^2 = A$ then $(I_n - A)^2 = I_n - 2A + A^2 = I_n - 2A + A = I_n - A$.

8.5 We have

$$(AB)^2 = \begin{bmatrix} 100 & -432 \\ 0 & 289 \end{bmatrix}$$

and

$$A^2B^2 = \begin{bmatrix} 160 & -460 \\ -5 & 195 \end{bmatrix}.$$

8.6 $AB = BA$ if and only if $(AB)^T = (BA)^T$ if and only if $B^TA^T = A^TB^T$.

8.7 AB is symmetric if and only if $(AB)^T = AB$ if and only if $B^TA^T = AB$ if and only if $AB = BA$.

8.8

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}.$$

8.9 $k = -1.$

8.10

$$\left\{\begin{array}{rl} 3x_1 - x_2 + 2x_3 &= 2\\ 4x_1 + 3x_2 + 7x_3 &= -1\\ -2x_1 + x_2 + 5x_3 &= 4. \end{array}\right.$$

8.11 (a) $a = 0$ (b) The system has exactly one solution if $a \neq 0$ (c) $x_1 = -15,\ x_2 = 6$, and $x_3 = 2$.

8.12 (a) We have

$$\begin{bmatrix} 1 & 1 & 2\\ 2 & 3 & 3\\ 3 & 3 & 7 \end{bmatrix}\begin{bmatrix} x_1\\ x_2\\ x_3 \end{bmatrix} = \begin{bmatrix} 1\\ -2\\ 3 \end{bmatrix}.$$

(b) $x_1 = 5,\ x_2 = -4,\ x_3 = 0.$

8.13 AB is of dimension 2×4 and $c_{23} = 17$.

8.14

$$\begin{bmatrix} 1 & -8 & -2\\ -4 & 11 & 4\\ -8 & 16 & 3 \end{bmatrix}.$$

8.15 $x_1 = t - s,\ x_2 = t,\ x_3 = x_4 = s.$

8.16 $AA^T = A^TA = I_2.$

8.17

$$\begin{bmatrix} 7\\ 4 \end{bmatrix}, \begin{bmatrix} -8\\ -5 \end{bmatrix}.$$

8.18 First, notice that $D^T = D$ since D is diagonal. Hence, $A^T = (LDL^T)^T = (L^T)^TD^TL^T = LDL^T = A$. That is, A is symmetric.

8.19 We have $A^n = (LDL^T)(LDL^T)\cdots(LDL^T) = LD^nL^T.$

8.20 Indeed, we have

$$\begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & -\frac{3}{2} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

8.21 Let $A = [a_{ij}], B = [b_{ij}], C = [c_{ij}], AB = [d_{ij}], BC = [e_{ij}], A(BC) = [f_{ij}]$, and $(AB)C = [g_{ij}]$. Then from the definition of matrix multiplication we have

$$\begin{aligned}
f_{ij} &= (\text{ith row of } A)(\text{jth column of } BC) \\
&= a_{i1}e_{1j} + a_{i2}e_{2j} + \cdots + a_{in}e_{nj} \\
&= a_{i1}(b_{11}c_{1j} + b_{12}c_{2j} + \cdots + b_{1p}c_{pj}) \\
&+ a_{i2}(b_{21}c_{1j} + b_{22}c_{2j} + \cdots + b_{2p}c_{pj}) \\
&+ \cdots + a_{in}(b_{n1}c_{1j} + b_{n2}c_{2j} + \cdots + b_{np}c_{pj}) \\
&= (a_{i1}b_{11} + a_{i2}b_{21} + \cdots + a_{in}b_{n1})c_{1j} \\
&+ (a_{i1}b_{12} + a_{i2}b_{22} + \cdots + a_{in}b_{n2})c_{2j} \\
&+ \cdots + (a_{i1}b_{1p} + a_{i2}b_{2p} + \cdots + a_{in}b_{np})c_{pj} \\
&= d_{i1}c_{1j} + d_{i2}c_{2j} + \cdots + d_{ip}c_{pj} \\
&= (\text{ith row of } AB)(\text{jth column of } C) \\
&= g_{ij}.
\end{aligned}$$

8.22 Let $A = [a_{ij}], B = [b_{ij}], C = [c_{ij}], A(B + C) = [k_{ij}], AB = [d_{ij}]$, and $AC = [h_{ij}]$. Then

$$\begin{aligned}
k_{ij} &= (\text{ith row of } A)(\text{jth column of } B + C) \\
&= a_{i1}(b_{1j} + c_{1j}) + a_{i2}(b_{2j} + c_{2j}) + \cdots + a_{in}(b_{nj} + c_{nj}) \\
&= (a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}) \\
&+ (a_{i1}c_{1j} + a_{i2}c_{2j} + \cdots + a_{in}c_{nj}) \\
&= d_{ij} + h_{ij}.
\end{aligned}$$

8.23 Let $A = [a_{ij}], B = [b_{ij}], C = [c_{ij}], (B + C)A = [k_{ij}], BA = [d_{ij}$, and

$CA = [h_{ij}]$. Then

$$\begin{aligned} k_{ij} &= (\text{ith row of } B+C)(\text{jth column of } A) \\ &= (b_{i1}+c_{i1})a_{1j} + (b_{i2}+c_{i2})a_{2j} + \cdots + (b_{in}+c_{in})a_{nj} \\ &= (b_{i1}a_{1j} + b_{i2}a_{2j} + \cdots + b_{in}a_{nj}) \\ &+ (c_{i1}a_{1j} + c_{i2}a_{2j} + \cdots + c_{in}a_{nj}) \\ &= d_{ij} + h_{ij}. \end{aligned}$$

8.24 Let $A = [a_{ij}], B = [b_{ij}]$, and $AB = [d_{ij}]$. Then

$$\begin{aligned} cd_{ij} &= c(a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}) \\ &= (ca_{i1})b_{1j} + (ca_{i2})b_{2j} + \cdots + (ca_{in})b_{nj} \\ &= (\text{ith row of } cA)(\text{jth column of } B. \end{aligned}$$

8.25 Let $A = [a_{ij}]$, $B = [b_{ij}$, and $AB = [c_{ij}]$. Then $(AB)^T = [c_{ji}]$. Let $B^TA^T = [d_{ij}]$.Then

$$\begin{aligned} d_{ij} &= (\text{ith row of } B^T)(\text{jth column of } A^T) \\ &= a_{j1}b_{1i} + a_{j2}b_{2i} + \cdots + a_{jn}b_{ni} \\ &= c_{ji}. \end{aligned}$$

Section 9

9.1 (a)

$$A=\begin{bmatrix}1&0\\0&0\end{bmatrix}, B=\begin{bmatrix}0&0\\0&1\end{bmatrix}.$$

(b)

$$A=\begin{bmatrix}1&0\\0&1\end{bmatrix}, B=\begin{bmatrix}-1&0\\0&-1\end{bmatrix}.$$

9.2 If B is a 3×3 matrix such that $BA=I_3$ then

$$b_{31}(0)+b_{32}(0)+b_{33}(0)=0.$$

But this is equal to the $(3,3)-$entry of I_3 which is 1. A contradiction.

9.3

$$\begin{bmatrix}41&-30\\-15&11\end{bmatrix}.$$

9.4

$$\begin{bmatrix}\frac{5}{13}&\frac{1}{13}\\-\frac{3}{13}&\frac{2}{13}\end{bmatrix}$$

9.5 If A is invertible then $B=I_nB=(A^{-1}A)B=A^{-1}(AB)=A^{-1}\mathbf{0}=\mathbf{0}$.

9.6

$$\begin{bmatrix}\sin\theta&-\cos\theta\\\cos\theta&\sin\theta\end{bmatrix}.$$

9.7 $A=\begin{bmatrix}-\frac{9}{13}&\frac{1}{13}\\\frac{2}{13}&-\frac{6}{13}\end{bmatrix}.$

9.8 $A=\begin{bmatrix}-\frac{2}{5}&1\\-\frac{1}{5}&\frac{3}{5}\end{bmatrix}.$

9.9 We have $A(A-3I_n)=I_n$ and $(A-3I_n)A=I_n$. Hence, A is invertible with $A^{-1}=A-3I_n$.

9.10 B^{-1}.

9.11 $(ABA^{-1})^2 = (ABA^{-1})(ABA^{-1}) = AB(A^{-1}A)BA^{-1} = AB^2A^{-1}$.

9.12 (a) $A = I_n^{-1}AI_n$ so that $A \sim A$.
(b) Suppose $A \sim B$. Then $B = P^{-1}AP$. Thus, $A = PBP^{-1} = (P^{-1})^{-1}BP^{-1} = Q^{-1}BQ$, where $Q = P^{-1}$. Hence, $B \sim A$.
(c) Suppose that $A \sim B$ and $B \sim C$. Then $B = P^{-1}AP$ and $C = Q^{-1}BQ$. Thus, $C = Q^{-1}P^{-1}APQ = R^{-1}AR$ where $R = PQ$. Hence, $A \sim C$.

9.13 $A^T = A^{-1} = A$.

9.14 (a) Since $B = X(A - AX)^{-1}$ = product of two invertible matrice, B is also invertible.
(b) We have $A - AX = B^{-1}X \Longrightarrow (A + B^{-1})X = A \Longrightarrow A + B^{-1} = AX^{-1}$ = a product of two invertible matrices. Hence, $A + B^{-1}$ is invertible. Thus, $X = (A + B^{-1})^{-1}A$.

9.15 Since $A \sim D$, we have $D = P^{-1}AP$. Thus, $D = D^T = P^TA^T(P^{-1})^T = [(P^T)^{-1}]^{-1}A^T(P^T)^{-1}$. Hence, $A^T \sim D$.

9.16 $A^{-1} = A^T$.

9.17

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}$$

9.18 $x_1 = 3$ and $x_2 = 7$.

9.19 $x_1 = -\frac{5}{3}$ and $x_2 = -\frac{7}{3}$.

9.20 $\mathbf{x} = I_n\mathbf{x} = (A^{-1}A)\mathbf{x} = A^{-1}(A\mathbf{x}) = A^{-1}\mathbf{0} = \mathbf{0}$.

Section 10

10.1 (a) No. This matrix is obtained by performing two operations: $r_2 \leftrightarrow r_3$ and $r_1 \leftarrow r_1 + r_3$.
(b) Yes: $r_2 \leftarrow r_2 - 5r_1$.
(c) Yes: $r_2 \leftarrow r_2 + 9r_3$.
(d) No: $r_1 \leftarrow 2r_1$ and $r_1 \leftarrow r_1 + 2r_4$.

10.2 (a)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(b)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 1 \end{bmatrix}.$$

(c)

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

10.3.(a) $r_1 \leftrightarrow r_3$, $E^{-1} = E$.
(b) $r_2 \leftarrow r_2 - 2r_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(c) $r_3 \leftarrow 5r_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{5} \end{bmatrix}.$$

10.4 (a)

$$E_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

(b) $E_2 = E_1$.
(c)

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}.$$

(d)

$$E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}.$$

10.5

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

10.6
$r_2 \leftarrow \frac{1}{2}r_2,\ r_1 \leftarrow -r_2 + r_1,\ r_2 \leftrightarrow r_3.$

10.7

$$\begin{bmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & a \end{bmatrix}.$$

10.8

$$\begin{bmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & a \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & a & 1 \end{bmatrix}.$$

10.9 (a) $E_1^{-1} = E_1$.
(b)

$$E_2^{-1} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(c)

$$E_3^{-1} = \begin{bmatrix} 1 & 0 & -0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

10.10

$$\begin{bmatrix} 0 & 5 & -3 \\ -4 & 3 & 0 \\ 3 & 0 & 6 \end{bmatrix}.$$

10.11

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

10.12 The third row is a multiple of the first row. If we perform $r_3 \leftarrow r_3 - 3r_1$ on A, we will get a row of all zeroes in the third row. Thus, there is no way we can do elementary row operations on A to get I_3, because the third row will never have a leading one. So, the reduced row elementary operations of A is not I_3. So, A cannot be written as a product of elementary matrices.

10.13 (a) We have

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) We have

$$(E_3E_2E_1)^{-1} = E_1^{-1}E_2^{-1}E_3^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}.$$

10.14 (a)

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{3}{4} \\ 0 & 0 & 1 \end{bmatrix}.$$

(b)

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{3}{4} \\ 0 & 0 & 1 \end{bmatrix}$$

10.15 (a)

$$E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix}, E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{bmatrix}$$

(b)

$$E_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, E_2^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

(c)

$$E_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{6} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

10.16 (a) is obvious, since I_n can row reduce a matrix to itself by performing the identity row operation.
(b) Supose that $B = E_kE_{k-1}\cdots E_1A$. Then $A = E_1^{-1}E_2^{-1}\cdots E_k^{-1}B$. Since the inverse of an elementary matrix is an elementary matrix, it follows that B row reduces to A. That is, $B \sim A$.
(c) Supose that $B = E_kE_{k-1}\cdots E_1A$ and $C = F_nF_{n-1}\cdots F_1B$. Then $C = F_nF_{n-1}\cdots F_1E_kE_{k-1}\cdots E_1A$. Hence, $A \sim C$.

10.17 Since $A \sim I_n$, we can find square elementary matrices $E_1, E_2, \cdots, E_k$ such that $I_n = E_kE_{k-1}\cdots E_1A$. Hence, $A = E_1^{-1}E_2^{-1}\cdots E_k^{-1}$. Since the inverse of an elementary matrix is an elementary matrix, A is a product of elementary matrices.

10.18 Suppose that $A = E_1E_2\cdots E_k$ where $E_1, E_2, \cdots, E_k$ are elementary matrices. But an elementary matrix is always invertible so A is the product of invertible matrices. Hence, A must be invertible by Theorem 9.3(a).

10.19 Note that $E_i^T = E_i$ and $E_i^2 = I_n$ for each i. Thus, $AA^T = E_1E_2\cdots E_kE_k^TE_{k-1}^T\cdots E_1^T = E_1E_2\cdots E_{k-1}E_k^2E_{k-1}^T\cdots E_1^T = E_1E_2\cdots E_{k-1}E_{k-1}^T\cdots E_1^T = \cdots = I_n$.

10.20

$$\begin{bmatrix} 1 & 0 & \frac{5}{11} \\ 0 & 1 & \frac{10}{11} \end{bmatrix}.$$

Section 11

11.1 The matrix is singular.

11.2 $a = -1$ or $a = 3$.

11.3

$$A^{-1} = \begin{bmatrix} \frac{13}{8} & -\frac{1}{2} & -\frac{1}{8} \\ -\frac{15}{8} & \frac{1}{2} & \frac{3}{8} \\ \frac{5}{4} & 0 & -\frac{1}{4} \end{bmatrix}.$$

11.4 Let A be an invertible and symmetric $n \times n$ matrix. Then $(A^{-1})^T = (A^T)^{-1} = A^{-1}$. That is, A^{-1} is symmetric.

11.5 According to Example 9.6(a), we have

$$D^{-1} = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}.$$

11.6 Suppose first that A is nonsingular. Then by Theorem 11.1, A is row equivalent to I_n. That is, there exist elementary matrices $E_1, E_2, \cdots, E_k$ such that $I_n = E_kE_{k-1}\cdots E_1A$. Then $A = E_1^{-1}E_2^{-1}\cdots E_k^{-1}$. But each E_i^{-1} is an elementary matrix by Theorem 10.2. Conversely, suppose that $A = E_1E_2\cdots E_k$. Then $(E_1E_2\cdots E_k)^{-1}A = I_n$. That is, A is nonsingular.

11.7 Suppose that A is row equivalent to B. Then there exist elementary matrices $E_1, E_2, \cdots, E_k$ such that $B = E_kE_{k-1}\cdots E_1A$. Let $P = E_kE_{k-1}\cdots E_1$. Then by Theorem 10.2 and Theorem 9.3(a), P is nonsingular.
Conversely, suppose that $B = PA$, for some nonsingular matrix P. By Theorem 11.1, P is row equivalent to I_n. That is, $I_n = E_kE_{k-1}\cdots E_1P$. Thus, $B = E_1^{-1}E_2^{-1}\cdots E_k^{-1}A$ and this implies that A is row equivalent to B.

11.8 Suppose that A is row equivalent to B. Then by the previous exercise, $B = PA$, with P nonsingular. If A is nonsingular then by Theorem 9.3(a), B is nonsingular. Conversely, if B is nonsingular then $A = P^{-1}B$ is nonsingular.

11.9 $A^{-1} = \left[\begin{array}{cc} \frac{1}{a_{11}} & 0 \\ -\frac{a_{21}}{a_{11}a_{22}} & \frac{1}{a_{22}} \end{array}\right].$

11.10 Since $Ax = \mathbf{0}$ has only the trivial solution, A is invertible. By induction on k and Theorem 9.3(a), A^k is invertible and consequently the system $A^k x = \mathbf{0}$ has only the trivial solution by Theorem 11.1.

11.11 Since A is invertible, by Theorem 11.1, A is row equivalent to I_n. That is, there exist elementary matrices $E_1, E_2, \cdots, E_k$ such that $I_n = E_k E_{k-1} \cdots E_1 A$. Similarly, there exist elementary matrices $F_1, F_2, \cdots, F_\ell$ such that $I_n = F_\ell F_{\ell-1} \cdots F_1 B$. Hence, $A = E_1^{-1} E_2^{-1} \cdots E_k^{-1} F_\ell F_{\ell-1} \cdots F_1 B$. That is, A is row equivalent to B.

11.12 (a)

$$A^{-1} = \begin{bmatrix} 29 & \frac{23}{2} & 7 \\ 16 & \frac{13}{2} & 4 \\ 4 & \frac{3}{2} & 1 \end{bmatrix}$$

(b) $x_1 = 29,\ x_2 = 16,\ x_3 = 3$

11.13 (a) We have

$$A^{-1} = \begin{bmatrix} -\frac{1}{5} & \frac{1}{2} & 0 \\ 5 & -4 & -3 \\ -\frac{9}{5} & \frac{3}{2} & 1 \end{bmatrix}$$

(b) $x_1 = \frac{3}{2},\ x_2 = 1,\ x_3 = -\frac{5}{2}.$

11.14 Suppose that AB is non-singular. Suppose that A is singular. Then $C = E_k E_{k-1} \cdots E_1 A$ with C having a row consisting entirely of zeros. But then $CB = E_k E_{k-1} \cdots E_1(AB)$ and CB has a row consisting entirely of zeros (Example 8.3(b)). This implies that AB is singular, a contradiction.
The converse is just Theorem 9.3(a).

11.15 Taking the transpose of H we have $H^T = I_n^T - 2(P^T)^T P^T = H$. That is, H is symmetric. On the other hand, $H^T H = H^2 = (I_n - 2PP^T)^2 = I_n - 4PP^T + 4(PP^T)^2 = I_n - 4PP^T + 4P(P^T P)P^T = I_n - 4PP^T + 4PP^T = I_n$.

11.16

$$A^{-1} = \begin{bmatrix} -3 & 0 & -4 \\ -2 & 1 & -3 \\ 1 & 0 & 1 \end{bmatrix}.$$

11.17 Since $A^{-1} = A$, using Example 9.7, we find

$$\frac{1}{ab-1}\begin{bmatrix} b & -1 \\ -1 & a \end{bmatrix} = \begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix}.$$

Thus, $ab - 1 = -1 \Longrightarrow ab = 0$ by the zero product property, $a = 0$ or $b = 0$. Since $\frac{b}{ab-1} = a$ and $\frac{1}{ab-1} = -1$, we conclude that $a = b = 0$.

11.18 Using elementary row operations on $[A|I_3]$, we find

$$A^{-1} = \begin{bmatrix} 1 & -4 & 1 \\ 1 & 1 & -2 \\ -1 & 1 & 1 \end{bmatrix}.$$

11.19

$$A = \begin{bmatrix} -4 & 0 & -2 \\ 3 & -6 & -4 \\ 0 & 0 & -4 \end{bmatrix}.$$

11.20 We will show that $B\mathbf{x} = \mathbf{0}$ has only the trivial solution. Indeed, if $\mathbf{x}$ is any solution of $B\mathbf{x} = \mathbf{0}$ then $\mathbf{x} = I_n\mathbf{x} = (AB)\mathbf{x} = A(B\mathbf{x}) = A\mathbf{0} = \mathbf{0}$. It follows from Theorem 11.1 that B is invertible and $B^{-1} = A$. Also, $BA = BB^{-1} = I_n$.

Section 12

12.1 (a) $|A| = 22$ (b) $|A| = 0$.

12.2 $t = 0, t = 1$, or $t = 4$.

12.3 $x = 1$ or $x_2 = \frac{1}{2}$.

12.4 $|A| = 0$.

12.5 $M_{23} = -96,\ C_{23} = 96$.

12.6 $\lambda = \pm 1$.

12.7 (a) -123 (b) -123.

12.8 -240

12.9 $|A| = 6$.

12.10 $|A| = 1$.

12.11 (a) $\lambda = 3$ or $\lambda = 2$ (b) $\lambda = 2$ or $\lambda = 6$.

12.12 -114.

12.13 $t = 1$ or $t = 2$.

12.14 $|A| = -\pi$.

12.15 20.

12.16 $|A| = 0$. Note that the second row is k times the first row.

12.17 $\lambda = -3$ or $\lambda = 1$.

12.18 $\lambda = 3$ or $\lambda = -1$.

12.19 $\lambda = -8$.

12.20 $\lambda = 2$ or $\lambda = 1$.

Section 13

13.1 $|A| = -4$.

13.2 (a)

$$\begin{vmatrix} d & e & f \\ g & h & i \\ a & b & c \end{vmatrix} = -6.$$

(b)

$$\begin{vmatrix} 3a & 3b & 3c \\ -d & -e & -f \\ 4g & 4h & 4i \end{vmatrix} = 72.$$

(c)

$$\begin{vmatrix} a+g & b+h & c+i \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -6$$

(d)

$$\begin{vmatrix} -3a & -3b & -3c \\ d & e & f \\ g-4d & h-4e & i-4f \end{vmatrix} = 18.$$

13.3 The determinant is 0 since the first and the fifth rows are proportional.

13.4 $|A| = \frac{3}{4}$.

13.5 The determinant is -5.

13.6 The determinant is -1.

13.7 The determinant is 1.

13.8 The determinant is 6.

13.9 $(b-c)(a-c)(b-a)$.

13.10 The determinant is 0.

13.11 15.

13.12 -36.

13.13 $|cA| = c^3$.

13.14 8.

13.15 $x = 2$.

13.16 (i) Let E_1 be the elementary matrix corresponding to type I elementary row operation, i.e., multiplying row i by a scalar c.. Then $|E_1| = c$ (Example 13.7(a)). Now, EA is the matrix obtained from A by multiplying row i of A by c. Then $|EA| = c|A| = |E_1||A|$ (Theorem 13.3 (a)).
(b) Let E_2 be the elementary matrix corresponding to type II elementary row operation, i.e., adding cr_i to r_j. Then $|E_2| = 1$ (Example 13.7(b)). Now, EA is the matrix obtained from A by adding c times row i of A by row j. Then $|EA| = |A| = |E_2||A|$ (Theorem 13.3 (b)).
(c) Let E_3 be the elementary matrix corresponding to type III elementary row operation, i.e., swapping rows i and j. Then $|E_3| = -1$ (Example 13.7(c)). Now, EA is the matrix obtained from A by swapping rows i and j. Then $|EA| = -|A| = |E_3||A|$ (Theorem 13.3 (c)).

13.17 Since A is invertible, $A = E_1E_2\cdots E_k$, where the E_i's are elementary matrices. By the previous exercise, we have $|A| = |E_1||E_2|\cdots|E_k|$. Since $|E_i| \neq 0$ for all $1 \leq i \leq k$ (Example 13.7), we conclude that $|A| \neq 0$.

13.18 By applying the operations $r_2 \leftarrow r_2 + r_1$ and $r_3 \leftarrow r_3 + 3r_1$, we find

$$|A| = \begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 5 & -6 & 5 \\ 0 & 5 & -6 & 5 \\ 4 & 4 & 6 & -1 \end{vmatrix} = 0.$$

Hence, A is singular.

13.19 48.

13.20 We will prove the result for upper triangular matrices.

(i) Basis of induction:

$$\begin{vmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{vmatrix} = a_{11}a_{22}$$

where we expand along the first column.
(ii) Induction hypothesis: Suppose that if $A = [a_{ij}]$ is an $n \times n$ upper triangular matrix then $|A| = a_{11}a_{22}\cdots a_{nn}$.
(iii) Induction step: Let $A = [a_{ij}]$ be an $(n+1)\times(n+1)$ upper triangular matrix. Let B be the $n \times n$ matrix obtained from A by deleting the first row and the first column of A. Expanding along the first column of A, we find $|A| = a_{11}|B| = a_{11}a_{22}\cdots a_{nn}a_{(n+1)(n+1)}$.

13.21 Multiply row k of $A = [a_{ij}]$ by a scalar c. Let $B = [b_{ij}]$ be the resulting matrix. Then $b_{ij} = a_{ij}$ for $i \neq k$ and $b_{kj} = ca_{kj}$ for $1 \leq j \leq n$. Finding the determinant of A by cofactor expansion along the k^{th} row, we find

$$|A| = \sum_{j=1}^{n} (-1)^{k+j} a_{kj} M_{kj}.$$

Now, finding $|B|$ by cofactor expandion along the k^{th} row, we find

$$|B| = \sum_{j=1}^{n} (-1)^{k+j}(ca_{kj})M_{kj} = c\left[(-1)^{k+j}a_{kj}M_{kj}\right] = c|A|.$$

13.22 Finding the determinant of A along the k^{th} row, we find

$$|A| = \sum_{m=1}^{n}(-1)^{k+m}(b_{km}+c_{km})M_{km} = \sum_{m=1}^{n}(-1)^{k+m}b_{km}M_{km}+\sum_{m=1}^{n}(-1)^{k+m}c_{km}M_{km}.$$

The first sum is the determinant of B using cofactor expansion along the k^{th} row and the second sum is the determinant of C using cofactor expansion along the k^{th} row. Hence, $|A| = |B| + |C|$.

13.23 Let $B = [b_{ij}]$ be obtained from $A = [a_{ij}]$ by adding to each element of the k^{th} row of A c times the corresponding element of the i^{th} row of A with $i < k$. That is, $b_{mj} = a_{mj}$ if $m \neq k$ and $b_{kj} = ca_{ij} + a_{kj}$. Define $C = [c_{ij}]$ where $c_{ij} = a_{ij}$ if $i \neq k$ and $c_{kj} = ca_{ij}$. Then the k^{th} row of B is the sum of the k^{th} row of A and the k^{th} row of the matrix C. By the previous problem, we have $|B| = |A| + |C| = |A| + c|D|$, where D is the matrix A with two identical rows i and k. By Theorem 13.2, $|D| = 0$. Hence, $|B| = |A|$.

13.24 Let $r_1, r_2, \cdots, r_n$ denote the rows of A. We have

$$
\begin{aligned}
0 &= \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_i + r_k \\ \vdots \\ r_i + r_k \\ \vdots \\ r_n \end{bmatrix} \\
&= \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_i + r_k \\ \vdots \\ r_i \\ \vdots \\ r_n \end{bmatrix} + \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_i + r_k \\ \vdots \\ r_k \\ \vdots \\ r_n \end{bmatrix} \\
&= \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_i \\ \vdots \\ r_i \\ \vdots \\ r_n \end{bmatrix} + \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_k \\ \vdots \\ r_i \\ \vdots \\ r_n \end{bmatrix} + \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_i \\ \vdots \\ r_k \\ \vdots \\ r_n \end{bmatrix} + \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_k \\ \vdots \\ r_k \\ \vdots \\ r_n \end{bmatrix}
\end{aligned}
$$

$$= \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_k \\ \vdots \\ r_i \\ \vdots \\ r_n \end{bmatrix} + \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_i \\ \vdots \\ r_k \\ \vdots \\ r_n \end{bmatrix}.$$

Hence,

$$\begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_k \\ \vdots \\ r_i \\ \vdots \\ r_n \end{bmatrix} = - \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_i \\ \vdots \\ r_k \\ \vdots \\ r_n \end{bmatrix}.$$

Section 14

14.1 The proof is by induction on $n \geq 1$. The equality is valid for $n = 1$. Suppose that it is valid up to n. Then $|A^{n+1}| = |A^n A| = |A^n||A| = |A|^n|A| = |A|^{n+1}$.

14.2 Since A is skew-symmetric, $A^T = -A$. Taking the determinant of both sides we find $|A| = |A^T| = |-A| = (-1)^n|A| = -|A|$ since n is odd. Thus, $2|A| = 0$ and therefore $|A| = 0$.

14.3 Taking the determinant of both sides of the equality $A^T A = I_n$ to obtain $|A^T||A| = 1$ or $|A|^2 = 1$ since $|A^T| = |A|$. It follows that $|A| = \pm 1$.

14.4 Taking the determinant of both sides to obtain $|A^2| = |A|$ or $|A|(|A| - 1) = 0$. Hence, either A is singular or $|A| = 1$.

14.5 The coefficient matrix

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$$

has determinant $|A| = 0$. By Theorem 14.2, the system has a nontrivial solution.

14.6 Finding the determinant we get $|A| = 2(c+2)(c-3)$. The determinant is 0 if $c = -2$ or $c = 3$.

14.7 $|A^3 B^{-1} A^T B^2| = |A|^3 |B|^{-1} |A||B|^2 = |A|^4 |B| = 80$.

14.8 We have $|AB| = |A||B| = |B||A| = |BA|$.

14.9 We have $|A + B^T| = |(A + B^T)^T| = |A^T + B|$.

14.10 Let $A = (a_{ij})$ be a triangular matrix. By Theorem 14.2, A is nonsingular if and only if $|A| \neq 0$ and this is equivalent to $a_{11}a_{22}\cdots a_{nn} \neq 0$.

14.11 Since $|A| = 200 \neq 0$, the rank of A is 4 by Theorem 14.2.

14.12 The second column of A is twice the first column so that by Example 13.4, $|A| = 0$. Hence, by Theorem 14.1, A is singular.

14.13 (a) $|A^4| = |A|^4 = 5^4 = 625$ (b) $|A^T| = |A| = 5$ (c) $|5A| = 5^3|A| = 5^4 = 625$ (d) $|A^{-1}| = \frac{1}{5}$.

14.14 $\frac{1}{81}$.

14.15 Expanding along the third column, we find $|A| = 8x - 8x = 0$ for all x. Hence, A is singular for all $x \in \mathbb{R}$.

14.16 $|A| = 0$ so by Theorem 14.1, $\text{rank}(A) < 4$.

14.17 $|A| = -\frac{1}{8}$.

14.18 $|A| = 0$.

14.19 14.

14.20 If A is singular then A is row equivalent to a matrix C with rows of zeros. Write $C = E_k \cdots E_2 E_1 A$ or $A = E_1^{-1} E_2^{-1} \cdots E_k^{-1} C$. Post multiply by B to obtain $AB = E_1^{-1} E_2^{-1} \cdots E_k^{-1} CB$. But the matrix on the right has a row of zeros so that $|AB| = 0 = |A||B|$.

14.21 Since A is invertible, there are elementary matrices $E_1, E_2, \cdots, E_k$ such that $A = E_1 E_2 \cdots E_k$. Hence, $AB = E_1 E_2 \cdots E_k B$. By Problem 13.16, we have $|AB| = |E_1||E_2| \cdots |E_k||B| = |A||B|$.

Section 15

15.1 (a)

$$\operatorname{adj}(A) = \begin{bmatrix} -18 & -6 & -10 \\ 17 & -10 & -1 \\ -6 & -2 & 28 \end{bmatrix}.$$

(b) $|A| = -94$.

15.2 Suppose first that A is invertible. Then $\operatorname{adj}(A) = A^{-1}|A|$ so that $|\operatorname{adj}(A)| = ||A|A^{-1}| = |A|^n|A^{-1}| = \frac{|A|^n}{|A|} = |A|^{n-1}$. If A is singular then $\operatorname{adj}(A)$ is singular. Suppose the contrary, then there exists a square matrix B such that $B \cdot \operatorname{adj}(A) = \operatorname{adj}(A) \cdot B = I_n$. Then $A = AI_n = A(\operatorname{adj}(A)B) = (A \cdot \operatorname{adj}(A))B = |A|B = 0$ and this leads to $\operatorname{adj}(A) = 0$, a contradiction to the fact that $\operatorname{adj}(A)$ is non-singular. Thus, $\operatorname{adj}(A)$ is singular and consequently $|\operatorname{adj}(A)| = 0 = |A|^{n-1}$.

15.3

$$\operatorname{adj}(A) = |A|A^{-1} = \begin{bmatrix} -\frac{1}{7} & 0 & -\frac{1}{21} \\ 0 & -\frac{2}{21} & -\frac{1}{7} \\ -\frac{1}{7} & -\frac{1}{21} & \frac{1}{21} \end{bmatrix}.$$

15.4 $|A^{-1} + \operatorname{adj}(A)| = \frac{3^n}{2}$.

15.5 The equality is valid for $\alpha = 0$. So suppose that $\alpha \neq 0$. Then $\operatorname{adj}(\alpha A) = |\alpha A|(\alpha A)^{-1} = (\alpha)^n|A|\frac{1}{\alpha}A^{-1} = (\alpha)^{n-1}|A|A^{-1} = (\alpha)^{n-1}adj(A)$.

15.6 (a) $|A| = 1(21 - 20) - 2(14 - 4) + 3(10 - 3) = 2$.
(b) The matrix of cofactors of A is

$$\begin{bmatrix} 1 & -10 & 7 \\ 1 & 4 & -3 \\ -1 & 2 & -1 \end{bmatrix}.$$

The adjoint is the transpose of this cofactors matrix

$$\operatorname{adj}(A) = \begin{bmatrix} 1 & 1 & -1 \\ -10 & 4 & 2 \\ 7 & -3 & -1 \end{bmatrix}.$$

(c)

$$A^{-1} = \frac{\text{adj}(A)}{|A|} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -5 & 2 & 1 \\ \frac{7}{2} & -\frac{3}{2} & -\frac{1}{2} \end{bmatrix}.$$

15.7 Suppose that $A^T = A$. Then $(\text{adj}(A))^T = (|A|A^{-1})^T = |A|(A^{-1})^T = |A|(A^T)^{-1} = |A|A^{-1} = \text{adj}(A)$.

15.8 Suppose that $A = [a_{ij}]$ is a lower triangular invertible matrix. Then $a_{ij} = 0$ if $i < j$. Thus, $C_{ij} = 0$ if $i > j$ since in this case C_{ij} is the determinant of a matrix with at least one zero on the diagonal. Hence, adj(A) is lower triangular.

15.9 Suppose that A is a lower triangular invertible matrix. Then adj(A) is also a lower triangular matrix. Hence, $A^{-1} = \frac{\text{adj}(A)}{|A|}$ is a lower triangular matrix.

15.10 (a) If A has integer entries then adj(A) has integer entries. If $|A| = 1$ then $A^{-1} = \text{adj}(A)$ has integer entries.
(b) Since $|A| = 1$, A is invertible and $x = A^{-1}b$. By (a), A^{-1} has integer entries. Since b has integer entries, $A^{-1}b$ has integer entries.

15.11 (a) $|A| = -2$ (b)

$$\text{adj}(A) = \begin{bmatrix} -6 & 1 & -6 \\ 4 & -1 & 2 \\ 4 & -1 & 4 \end{bmatrix}.$$

(c)

$$A^{-1} = -\frac{1}{2}\text{adj}(A) = \begin{bmatrix} 3 & -\frac{1}{2} & 3 \\ -2 & \frac{1}{2} & -1 \\ -2 & \frac{1}{2} & -2 \end{bmatrix}.$$

15.12 (a) $|A| = 5$ (b) $-\frac{1}{5}$.

15.13 (a) The cofactor matrix of A is

$$\begin{bmatrix} 0 & 3 & 3 \\ 1 & -3 & -2 \\ 1 & 3 & 1 \end{bmatrix}.$$

(b) We have

$$\operatorname{adj}(A) = \begin{bmatrix} 0 & 1 & 1 \\ 3 & -3 & 3 \\ 3 & -2 & 1 \end{bmatrix}.$$

(c) Expanding along the third row, we find

$$|A| = (-1+2) - (1-4) - (-1+2) = 3.$$

(d)

$$A^{-1} = \begin{bmatrix} 0 & \frac{1}{3} & \frac{1}{3} \\ 1 & -1 & 1 \\ 1 & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}.$$

15.14

$$A^{-1} = \frac{\operatorname{adj}(A)}{|A|} = \begin{bmatrix} -1 & \frac{1}{2} & \frac{1}{2} \\ 2 & -1 & 0 \\ 2 & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}.$$

15.15 Using Problem 15.2, we have

$$\begin{aligned} \operatorname{adj}(\operatorname{adj}(A)) =& [\operatorname{adj}(A)]^{-1}|\operatorname{adj}(A)| \\ =& \operatorname{adj}(A^{-1})|\operatorname{adj}(A)| \\ =& (A^{-1})^{-1}|A^{-1}||\operatorname{adj}(A)| \\ =& |A|^{-1}|\operatorname{adj}(A)|A \\ =& |A|^{-1}|A|^{n-1}A \\ =& |A|^{n-2}A. \end{aligned}$$

15.16 $d_{11}\cdots d_{i-1,i-1}d_{i+1,i+1}\cdots d_{nn}$.

15.17

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{3}{2} & 1 \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & \frac{1}{2} \end{bmatrix}.$$

15.18 25.

15.19 125.

15.20 The prove is by induction on $n \geq 1$.
(i) Basis of indunction: $\mathrm{adj}(A^1) = [\mathrm{adj}(A)]^1$.
(ii) Induction hypothesis: Suppose that $\mathrm{adj}(A^k) = [\mathrm{adj}(A)]^k$, for $k = 1, 2, \cdots, n$.
(iii) Induction step: $\mathrm{adj}(A^{n+1}) = \mathrm{adj}(A^n \cdot A) = \mathrm{adj}(A)\mathrm{adj}(A^n) = \mathrm{adj}(A)[\mathrm{adj}(A)]^n = [\mathrm{adj}(A)]^{n+1}$.

Section 16

16.1 $x_1 = \frac{|A_1|}{|A|} = -\frac{10}{11}, x_2 = \frac{|A_2|}{|A|} = \frac{18}{11}, x_3 = \frac{|A_3|}{|A|} = \frac{38}{11}$.

16.2 $x_1 = \frac{|A_1|}{|A|} = -\frac{3}{4}, x_2 = \frac{|A_2|}{|A|} = \frac{83}{8}, x_3 = \frac{|A_3|}{|A|} = \frac{21}{8}$.

16.3 $x_1 = \frac{|A_1|}{|A|} = -1, x_2 = \frac{|A_2|}{|A|} = 3, x_3 = \frac{|A_3|}{|A|} = 2$.

16.4 $x_1 = \frac{|A_1|}{|A|} = \frac{212}{187}, x_2 = \frac{|A_2|}{|A|} = \frac{273}{187}, x_3 = \frac{|A_3|}{|A|} = \frac{107}{187}$.

16.5 $x_1 = \frac{|A_1|}{|A|} = 4, x_2 = \frac{|A_2|}{|A|} = -1, x_3 = \frac{|A_3|}{|A|} = -\frac{1}{3}$.

16.6 $x_1 = \frac{|A_1|}{|A|} = 2, x_2 = \frac{|A_2|}{|A|} = -1, x_3 = \frac{|A_3|}{|A|} = 4$.

16.7 $x_1 = \frac{|A_1|}{|A|} = 2, x_2 = \frac{|A_2|}{|A|} = -1, x_3 = \frac{|A_3|}{|A|} = 3$.

16.8 $x_1 = \frac{|A_1|}{|A|} = 4, x_2 = \frac{|A_2|}{|A|} = 1, x_3 = \frac{|A_3|}{|A|} = -2$.

16.9 $x_1 = \frac{|A_1|}{|A|} = 5, x_2 = \frac{|A_2|}{|A|} = 2, x_3 = \frac{|A_3|}{|A|} = 2$.

16.10 $x_1 = \frac{|A_1|}{|A|} = 1, x_2 = \frac{|A_2|}{|A|} = 4, x_3 = \frac{|A_3|}{|A|} = 3$.

16.11 Inconsistent.

16.12 Consistent and dependent.

16.13 Inconsistent.

16.14 $x_3 = 2$.

16.15 $\cos\alpha = \frac{b^2+c^2-a^2}{2bc}$.

16.16 $x_1 = \frac{|A_1|}{|A|} = -\frac{7}{27},\ x_2 = \frac{|A_2|}{|A|} = \frac{4}{3},\ x_3 = \frac{|A_3|}{|A|} = -\frac{23}{27}$.

16.17 True. For example, Problem 16.14.

16.18 False. Cramer's Rule uses the concept of determinant, a concept that is valid only for square matrices.

16.19 (a) We have

$$A^{-1} = |A|^{-1}\text{adj}(A) = |A|^{-1}\begin{bmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{bmatrix}^T = |A|^{-1}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}.$$

(b) We have

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A^{-1}\mathbf{b} = |A|^{-1}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = |A|^{-1}\begin{bmatrix} b_1a_{22} - b_2a_{12} \\ b_2a_{11} - b_1a_{21} \end{bmatrix}.$$

(c) We have

$$x_1 = \frac{b_1a_{22} - b_2a_{12}}{|A|} = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{|A|}$$

and

$$x_2 = \frac{b_2a_{11} - b_1a_{21}}{|A|} = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{|A|}.$$

16.20 (i) Elimination (ii) The method of inverse matrix $x = A^{-1}b$ and (iii) Cramer's Rule.

Section 17

17.1 We know from calculus that if f, g are differentiable functions on $[a, b]$ and $\alpha \in \mathbb{R}$ then $\alpha f + g$ is also differentiable on $[a, b]$. Hence, $D([a, b])$ is a subspace of $F([a, b]$.

17.2 Let $x, y \in S$ and $\alpha \in \mathbb{R}$. Then $A(\alpha x + y) = \alpha Ax + Ay = \alpha \times \mathbf{0} + \mathbf{0} = \mathbf{0}$. Thus, $\alpha x + y \in S$ so that S is a subspace of $\mathbb{R}^n$.

17.3 Since $\mathbf{P}$ is a subset of the vector space of all functions defined on $\mathbb{R}$, it suffices to show that $\mathbf{P}$ is a subspace. Indeed, the sum of two polynomials is again a polynomial and the scalar multiplication by a polynomial is also a polynomial.

17.4 The proof is based on the properties of the vector space $\mathbb{R}$.
(a) $(f + g)(x) = f(x) + g(x) = g(x) + f(x) = (g + f)(x)$ where we have used the fact that the addition of real numbers is commutative.
(b) $[(f + g) + h](x) = (f + g)(x) + h(x) = (f(x) + g(x)) + h(x) = f(x) + (g(x) + h(x)) = f(x) + (g + h)(x) = [f + (g + h)](x)$.
(c) Let $\mathbf{0}$ be the zero function. Then for any $f \in F(\mathbb{R})$ we have $(f + 0)(x) = f(x) + \mathbf{0}(x) = f(x) = (\mathbf{0} + f)(x)$.
(d) $[f + (-f)](x) = f(x) + (-f(x)) = f(x) - f(x) = 0 = \mathbf{0}(x)$.
(e) $[\alpha(f + g)](x) = \alpha(f + g)(x) = \alpha f(x) + \alpha g(x) = (\alpha f + \alpha g)(x)$.
(f) $[(\alpha + \beta)f](x) = (\alpha + \beta)f(x) = \alpha f(x) + \beta f(x) = (\alpha f + \beta f)(x)$.
(g) $[\alpha(\beta f)](x) = \alpha(\beta f)(x) = (\alpha\beta)f(x) = [(\alpha\beta)f](x)$
$(h)(1f)(x) = 1f(x) = f(x)$.
Thus, $F(\mathbb{R})$ is a vector space.

17.5 Let $x \neq y$. Then $\alpha(\beta(x, y)) = \alpha(\beta y, \beta x) = (\alpha\beta x, \alpha\beta y) \neq (\alpha\beta)(x, y)$. Thus, $\mathbb{R}^2$ with the above operations is not a vector space.

17.6 Let $p, q \in U$ and $\alpha \in \mathbb{R}$. Then $\alpha p + q$ is a polynomial such that $(\alpha p + q)(3) = \alpha p(3) + q(3) = 0$. That is, $\alpha p + q \in U$. This says that U is a subspace of $\mathbf{P}$.

17.7 Let $p(x) = a_0 + a_1x + \cdots + a_nx^n, q(x) = b_0 + b_1x + \cdots + b_nx^n$, and $\alpha \in R$. Then $(\alpha p + q)(x) = (\alpha a_0 + b_0) + (\alpha a_1 + b_1)x + \cdots + (\alpha a_n + b_n)x^n \in P_n$. Thus, P_n is a subspace of $\mathbf{P}$.

17.8 $(-1,0) \in S$ but $-2(-1,0) = (2,0) \notin S$ so S is not a vector space.

17.9 Since for any continuous functions f and g and any scalar α the function $\alpha f + g$ is continuous, $C([a,b])$ is a subspace of $F([a,b])$ and hence a vector space.

17.10 Indeed, $\alpha(a,b,a+b)+(a',b',a'+b') = (\alpha a+a', \alpha b+b', \alpha a+\alpha b+a'+b')$.

17.11 Using the properties of vector spaces we have $v = v + 0 = v + (u + (-u)) = (v+u)+(-u) = (w+u)+(-u) = w+(u+(-u)) = w+0 = w$.

17.12 (a) Let $u, v \in H \cap K$ and $\alpha \in R$. Then $u, v \in H$ and $u, v \in K$. Since H and K are subspaces, $\alpha u + v \in H$ and $\alpha u + v \in K$ that is $\alpha u + v \in H \cap K$. This shows that $H \cap K$ is a subspace.
(b) One can easily check that $H = \{(x,0) : x \in \mathbb{R}\}$ and $K = \{(0,y) : y \in \mathbb{R}\}$ are subspaces of $\mathbb{R}^2$. The vector $(1,0)$ belongs to H and the vector $(0,1)$ belongs to K. But $(1,0)+(0,1) = (1,1) \notin H \cup K$. It follows that $H \cup K$ is not a subspace of $\mathbb{R}^2$.
(c) If $H \subset K$ then $H \cup K = K$, a subspace of V. Similarly, if $K \subset H$ then $H \cup K = H$, again a subspace of V.

17.13 We have $\begin{bmatrix} 0 & -2 \\ 1 & 2 \end{bmatrix} \in S$ and $\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} \in S$ but

$$\begin{bmatrix} 0 & -2 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 2 & 4 \end{bmatrix} \notin S.$$

Hence, S is not a subspace of M_{22}.

17.14 Let (a,b,c) and (a',b',c') be two elements of S and $\alpha \in \mathbb{R}$. Then

$$\alpha(a,b,c) + (a',b',c') = (\alpha a + a', \alpha b + b', \alpha c + c') \in V$$

since

$$\alpha a + a' + \alpha b + b' = \alpha(a+b) + a' + b' = \alpha(2c) + 2c' = 2(\alpha c + c').$$

Thus, V is a subspace of $\mathbb{R}^3$ and hence a vector space.

17.15 We have $(1,1) \in S$ and $(1,-1) \in S$ but $(1,1)+(1,-1) = (2,0) \notin S$. Hence, S is not a vector space of $\mathbb{R}^2$.

17.16 The prove of this theorem uses the properties of addition and scalar multiplication of real numbers.
Let $u = (u_1, u_2, \cdots, u_n),\ v = (v_1, v_2, \cdots, v_n)$, and $w = (w_1, w_2, \cdots, w_n)$.
(a) We have

$$\begin{aligned} u + v =&(u_1, u_2, \cdots, u_n) + (v_1, v_2, \cdots, v_n) = (u_1 + v_1, u_2 + v_2, \cdots, u_n + v_n)\\ =&(v_1 + u_1, v_2 + u_2, \cdots, v_n + u_n)\\ =&(v_1, v_2, \cdots, v_n) + (u_1, u_2, \cdots, u_n) = v + u. \end{aligned}$$

(b) We have

$$\begin{aligned} u + (v + w) =&(u_1, u_2, \cdots, u_n) + [(v_1, v_2, \cdots, v_n) + (w_1, w_2, \cdots, w_n)]\\ =&(u_1, u_2, \cdots, u_n) + (v_1 + w_1, v_2 + w_2, \cdots, v_n + w_n)\\ =&(u_1 + (v_1 + w_1), u_2 + (v_2 + w_2), \cdots, u_n + (v_n + w_n))\\ =&((u_1 + v_1) + w_1, (u_2 + v_2) + w_2, \cdots, (u_n + v_n) + w_n)\\ =&(u_1 + v_1, u_2 + v_2, \cdots, u_n + v_n) + (w_1, w_2, \cdots, w_n)\\ =&[(u_1, u_2, \cdots, u_n) + (v_1, v_2, \cdots, v_n)] + (w_1, w_2, \cdots, w_n)\\ =&(u + v) + w. \end{aligned}$$

(c) We have

$$\begin{aligned} u + \mathbf{0} =&(u_1, u_2, \cdots, u_n) + (0, 0, \cdots, 0) = (u_1 + 0, u_2 + 0, \cdots, u_n + 0)\\ =&(u_1, u_2, \cdots, u_n) = u. \end{aligned}$$

(d) We have

$$\begin{aligned} u + (-u) =&(u_1, u_2, \cdots, u_n) + (-u_1, -u_2, \cdots, -u_n)\\ =&(u_1 + (-u_1), u_2 + (-u_2), \cdots, u_n + (-u_n))\\ =&(0, 0, \cdots, 0) = \mathbf{0}. \end{aligned}$$

(e) We have

$$\begin{aligned}\alpha(u+v) =&\alpha[(u_1,u_2,\cdots,u_n)+(v_1,v_2,\cdots,v_n)]\\ =&\alpha(u_1+v_1,u_2+v_2,\cdots,u_n+v_n)\\ =&(\alpha(u_1+v_1),\alpha(u_2+v_2),\cdots,\alpha(u_n+v_n))\\ =&(\alpha u_1+\alpha v_1,\alpha u_2+\alpha v_2,\cdots,\alpha u_n+\alpha v_n)\\ =&(\alpha u_1,\alpha u_2,\cdots,\alpha u_n)+(\alpha v_1,\alpha v_2,\cdots,\alpha v_n)\\ =&\alpha(u_1,u_2,\cdots,u_n)+\alpha(v_1,v_2,\cdots,v_n)=\alpha u+\alpha v.\end{aligned}$$

(f) We have

$$\begin{aligned}(\alpha+\beta)u =&(\alpha+\beta)(u_1,u_2,\cdots,u_n)\\ =&((\alpha+\beta)u_1,(\alpha+\beta)u_2,\cdots,(\alpha+\beta)u_n)\\ =&(\alpha u_1+\beta u_1,\alpha u_2+\beta u_2,\cdots,\alpha u_n+\beta u_n)\\ =&(\alpha u_1,\alpha u_2,\cdots,\alpha u_n)+(\beta u_1,\beta u_2,\cdots,\beta u_n)\\ =&\alpha(u_1,u_2,\cdots,u_n)+\beta(u_1,u_2,\cdots,u_n)\\ =&\alpha u+\beta u.\end{aligned}$$

(g) We have

$$\begin{aligned}\alpha(\beta u) =&\alpha(\beta u_1,\beta u_2,\cdots,\beta u_n)\\ =&(\alpha(\beta u_1),\alpha(\beta u_2),\cdots,\alpha(\beta u_n))\\ =&((\alpha\beta)u_1,(\alpha\beta)u_2,\cdots,(\alpha\beta)u_n)\\ =&\alpha\beta(u_1,u_2,\cdots,u_n)\\ =&(\alpha\beta)u.\end{aligned}$$

(h) We have

$$1\cdot u=1\cdot(u_1,u_2,\cdots,u_n)=(1\cdot u_1,1\cdot u_2,\cdots,1\cdot u_n)=(u_1,u_2,\cdots,u_n).$$

17.17 Let $A=[a_{ij}],\ B=[b_{ij}]$, and $C=[c_{ij}]$.
(a) We have

$$\begin{aligned}A+B =&[a_{ij}]+[b_{ij}]=[a_{ij}+b_{ij}]=[b_{ij}+a_{ij}]\\ =&[b_{ij}]+[a_{ij}]=B+A.\end{aligned}$$

(b) We have

$$\begin{aligned} A + (B + C) =& [a_{ij}] + ([b_{ij}] + [c_{ij}]) \\ =& [a_{ij}] + [b_{ij} + c_{ij}] \\ =& [a_{ij} + (b_{ij} + c_{ij})] \\ =& [(a_{ij} + b_{ij}) + c_{ij}] \\ =& [a_{ij} + b_{ij}] + [c_{ij}] \\ =& ([a_{ij}] + [b_{ij}]) + [c_{ij}] \\ =& (A + B) + C. \end{aligned}$$

(c) We have

$$\begin{aligned} A + \mathbf{0} =& [a_{ij}] + [0] = [a_{ij} + 0] \\ =& [a_{ij}] = A. \end{aligned}$$

(d) We have

$$\begin{aligned} A + (-A) =& [a_{ij}] + [-a_{ij}] = [a_{ij} + (-a_{ij})] \\ =& [0] = \mathbf{0}. \end{aligned}$$

(e) We have

$$\begin{aligned} \alpha(A + B) =& \alpha([a_{ij}] + [b_{ij}]) = \alpha[a_{ij} + b_{ij}] \\ =& [\alpha(a_{ij} + b_{ij})] = [\alpha a_{ij} + \alpha b_{ij}] \\ =& [\alpha a_{ij}] + [\alpha b_{ij}] \\ =& \alpha[a_{ij}] + \alpha[b_{ij}] = \alpha A + \alpha B. \end{aligned}$$

(f) We have

$$\begin{aligned} (\alpha + \beta)A =& (\alpha + \beta)[a_{ij}] = [(\alpha + \beta)a_{ij}] \\ =& [\alpha a_{ij} + \beta a_{ij}] = [\alpha a_{ij}] + [\beta a_{ij}] \\ =& \alpha[a_{ij}] + \beta[a_{ij}] \\ =& \alpha A + \beta B. \end{aligned}$$

(g) We have

$$\begin{aligned} \alpha(\beta A) =& \alpha(\beta[a_{ij}]) = \alpha[\beta a_{ij}] \\ =& [\alpha(\beta a_{ij})] = [(\alpha\beta)a_{ij}] = (\alpha\beta)[a_{ij}] \\ =& (\alpha\beta)A. \end{aligned}$$

(h) We have
$$1 \cdot A = 1 \cdot [a_{ij}] = [1 \cdot a_{ij}] = [a_{ij}].$$

17.18 (a) We have $u + (-u) = 0$ and $u + (-1)u = (1 + (-1))u = 0u = 0$. By Problem 17.11, $-u = (-1)u$.
(b) For any scalar $\alpha \in \mathbb{R}$ we have $0u = (\alpha + (-\alpha))u = \alpha u + (-\alpha)u = \alpha u + (-(\alpha u)) = 0$.
(c) Let $u \in V$. Then $\alpha 0 = \alpha(u + (-u)) = \alpha u + \alpha(-u) = \alpha u + (\alpha(-1))u = \alpha u + [-(\alpha u)] = 0$.
(d) Suppose $\alpha u = 0$. If $\alpha \neq 0$ then α^{-1} exists and $u = 1u = (\alpha^{-1}\alpha)u = \alpha^{-1}(\alpha u) = \alpha^{-1}0 = 0$.

17.19 Since $0 = 0v_1 + 0v_2 + \cdots 0v_n$, $0 \in S$ and so $S \neq \emptyset$. Let $c_1v_1 + c_2v_2 + \cdots + c_nv_n \in S$, $d_1v_1 + d_2v_2 + \cdots + d_nv_n \in S$, and $\alpha \in \mathbb{R}$. Then

$$\alpha(c_1v_1+c_2v_2+\cdots+c_nv_n)+d_1v_1+d_2v_2+\cdots+d_nv_n = (\alpha c_1+d_1)v_1+(\alpha c_2+d_2)v_2+\cdots+(\alpha c_n+d_n)v_n \in S.$$

Hence, S is a subspace of V.

17.20 Since the zero matrix is in S, we have $S \neq \emptyset$. Now, if $D = [d_{ii}]$ and $D' = [d'_{ii}]$ are two members of S and $\alpha \in \mathbb{R}$ then

$$\alpha D + D' = [\alpha d_{ii} + d'_{ii}] \in S.$$

Hence, S is a subspace of M_{nn}.

Section 18

18.1 Let U be a subspace of V containing the vectors $v_1, v_2, \cdots, v_n$. Let $x \in W$. Then $x = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$ for some scalars $\alpha_1, \alpha_2, \cdots, \alpha_n$. Since U is a subspace, $x \in U$. This gives $x \in U$ and consequently $W \subset U$.

18.2 Indeed, $3p_1(x) - p_2(x) + 2p_3(x) = 0$.

18.3 The equation $\vec{u} = x_1\vec{v_1} + x_2\vec{v_2} + x_3\vec{v_3}$ gives the system

$$\begin{cases} 2x_1 + x_2 + 3x_3 = -9 \\ x_1 - x_2 + 2x_3 = -7 \\ 4x_1 + 3x_2 + 5x_3 = -15. \end{cases}$$

Solving this system (details omitted) we find $x_1 = -2, x_2 = 1$ and $x_3 = -2$.

18.4 (a) Indeed, this follows because the coefficient matrix

$$A = \begin{bmatrix} 2 & 2 & 2 \\ 0 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix}$$

of the system $Ax = b$ is invertible for all $b \in \mathbb{R}^3$ ($|A| = -6$).
(b) This follows from the fact that the coefficient matrix with rows the vectors $\vec{v_1}, \vec{v_2}$, and $\vec{v_3}$ is singular.

18.5 every $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ can be written as

$$a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

18.6 $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ is linearly dependent.

18.7 Suppose that $\alpha(4,-1,2) + \beta(-4,10,2) = (0,0,0)$ this leads to the system

$$\begin{cases} 4\alpha_1 & - & 4\alpha_2 & = & 0 \\ -\alpha_1 & + & 10\alpha_2 & = & 0 \\ 2\alpha_2 & + & 2\alpha_2 & = & 0 \end{cases}$$

This system has only the trivial solution so that the given vectors are linearly independent.

18.8 Suppose that $\{u, v\}$ is linearly dependent. Then there exist scalars α and β not both zero such that $\alpha u + \beta v = 0$. If $\alpha \neq 0$ then $u = -\frac{\beta}{\alpha} v$, i.e. u is a scalar multiple of v. A similar argument if $\beta \neq 0$.
Conversely, suppose that $u = \lambda v$ then $1u + (-\lambda)v = 0$. This shows that $\{u, v\}$ is linearly dependent.

18.9 We have already shown that

$$M_{22} = \text{span}\left\{ M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, M_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, M_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, M_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

Now, if $\alpha_1 M_1 + \alpha_2 M_2 + \alpha_3 M_3 + \alpha_4 M_4 = \mathbf{0}$ then

$$\begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

and this shows that $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$. Hence, $\{M_1, M_2, M_3, M_4\}$ is a basis for M_{22}.

18.10 Suppose that $\alpha(1 + 2x) + \beta(x - x^2) + \gamma(x + x^2) = 0$ for all $x \in \mathbb{R}$. This is equivalent to $(-\beta + \gamma)x^2 + (2\alpha + \beta + \gamma)x + \alpha = 0$ for all $x \in \mathbb{R}$. This means that the quadratic equation has infinitely many solutions. By the Fundamental Theorem of Algebra, this true only when the coefficients are all 0. This leads to $\alpha = \beta = \gamma = 0$. Thus, the vectors in $\mathcal{B}$ are linearly independent. Now, clearly, $\mathcal{B} \subseteq P_2$. If $p(x) = ax^2 + bx + c \in P_2$ then $p(x) = \alpha(1 + 2x) + \beta(x - x^2) + \gamma(x + x^2)$ then $\alpha = c$, $\beta = \frac{-a+b-2c}{2}$, and $\gamma = \frac{a+b-2c}{2}$. Hence, $p(x) \in \mathcal{B}$. This shows that $P_2 = \text{span}\{1+2x, x-x^2, x+x^2\}$ and $\mathcal{B}$ is a basis of P_2.

18.11 Suppose that $\alpha p_1(x) + \beta p_2(x) + \gamma p_3(x) = 0$ for all $x \in \mathbb{R}$. This leads to the system

$$\begin{cases} a\alpha - 2\beta + \gamma = 0 \\ (a-4)\beta + 2\gamma = 0 \\ (a-1)\gamma = 0. \end{cases}$$

The set $\{p_1(x), p_2(x), p_3(x)\}$ is linearly independent if the above homogeneous system has only the trivial solution. This is the case, if the coefficient matrix is invertible. But the determinant of the coefficient matrix is

$$\begin{vmatrix} a & -2 & 1 \\ 0 & a-4 & 2 \\ 0 & 0 & a-1 \end{vmatrix} = a(a-4)(a-1).$$

Hence, the coefficient matrix is invertible if $a \neq 0, 1, 4$.

18.12 Infinitely many representations.

18.13 By Theorem 18.2, if $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis of $\mathbb{R}^3$ then every vector in $\mathbb{R}^3$ has a unique representation as a linear combination of $\vec{v}_1, \vec{v}_2$ and $\vec{v}_3$. By the previous problem, this is not true for the vector $\vec{b}$. Hence, $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ can not be a basis for $\mathbb{R}^3$.

18.14 First note that

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus,

$$W = \text{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

Next, we prove linear independence. Suppose that

$$\alpha \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \beta \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Then $\alpha = \beta = 0$. Hence,

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

is a basis of W and $\dim(W) = 2$.

18.15 Suppose that $v_k = \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_{k-1} v_{k-1} + \beta_{k+1} v_{k+1} + \cdots + \beta_n v_n$. We want to show

$$\text{span}\{v_1, v_2, \cdots, v_n\} = \text{span}\{v_1, \cdots, v_{k-1}, v_{k+1}, \cdots, v_n\}.$$

Let $x \in \text{span}\{v_1, v_2, \cdots, v_n\}$. Then

$$
\begin{aligned}
x =& \alpha_1 v_1 + \cdots + \alpha_{k-1}v_{k-1} + \alpha_k v_k + \alpha_{k+1}v_{k+1} + \cdots + v_n \\
=& \alpha_1 v_1 + \cdots + \alpha_{k-1}v_{k-1} + \alpha_k(\beta_1 v_1 + \cdots + \beta_{k-1}v_{k-1} + \beta_{k+1}v_{k+1} + \cdots + \beta_n v_n) \\
&+ \alpha_{k+1}v_{k+1} + \cdots + v_n \\
=& (\alpha_1 + \alpha_k\beta_1)v_1 + \cdots + (\alpha_{k-1} + \alpha_k\beta_{k-1})v_{k-1} + (\alpha_{k+1} + \alpha_k\beta_{k+1})v_{k+1} + \cdots + (\alpha_n + \alpha_k\beta_n)v_n.
\end{aligned}
$$

Hence, $x \in \text{span}\{v_1, \cdots, v_{k-1}, v_{k+1}, \cdots, v_n\}$. Now, let $y \in \text{span}\{v_1, \cdots, v_{k-1}, v_{k+1}, \cdots, v_n\}$. Then

$$
\begin{aligned}
y =& \alpha_1 v_1 + \cdots + \alpha_{k-1}v_{k-1} + \alpha_{k+1}v_{k+1} + \cdots + \alpha_n v_n \\
=& (\alpha_1 - \beta_1)v_1 + \cdots + (\alpha_{k-1} - \beta_{k-1})v_{k-1} + v_k + (\alpha_{k+1} - \beta_{k+1})v_{k+1} + \cdots + (\alpha_n - \beta_n)v_n.
\end{aligned}
$$

Hence, $x \in \text{span}\{v_1, v_2, \cdots, v_n\}$.

18.16 (a) Since $\{v_1, v_2, \cdots, v_n\}$ is a basis of V, every element in S is a linear combination of $\{v_1, v_2, \cdots, v_n\}$. That is,

$$
\begin{aligned}
w_1 =& a_{11}v_1 + a_{12}v_2 + \cdots + a_{1n}v_n \\
w_2 =& a_{21}v_1 + a_{22}v_2 + \cdots + a_{2n}v_n \\
\vdots =& \vdots \\
w_m =& a_{m1}v_1 + a_{m2}v_2 + \cdots + a_{mn}v_n
\end{aligned}
$$

or in matrix notation, $Av = w$.
(b) The condition $\alpha_1 w_1 + \alpha_2 w_2 + \cdots + \alpha_m w_m = 0$ leads to the homogeneous system

$$
\left\{
\begin{aligned}
a_{11}\alpha_1 + a_{21}\alpha_2 + \cdots + a_{m1}\alpha_m &= 0 \\
a_{12}\alpha_1 + a_{22}\alpha_2 + \cdots + a_{m2}\alpha_m &= 0 \\
\vdots &= \vdots \\
a_{1n}\alpha_1 + a_{2n}\alpha_2 + \cdots + a_{mn}\alpha_m &= 0
\end{aligned}
\right.
$$

or in matrix notation

$$
\begin{bmatrix}
a_{11} & a_{21} & \dots & a_{m1} \\
a_{12} & a_{22} & \dots & a_{m2} \\
\dots & \dots & \dots & \dots \\
a_{1n} & a_{2n} & \dots & a_{mn}
\end{bmatrix}
\begin{bmatrix}
\alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m
\end{bmatrix}
=
\begin{bmatrix}
0 \\ 0 \\ \vdots \\ 0
\end{bmatrix}.
$$

(c) The homogeneous system in (b) has more unknowns than equations so that by Theorem 6.1(2), the system has a non-trivial solution. Thus, S is linearly dependent.

18.17 (a) Since $V = \text{span}(S)$, every element in $\{v_1, v_2, \cdots, v_n\}$ is a linear combination of $\{w_1, w_2, \cdots, w_m\}$. That is,

$$\begin{aligned} v_1 =& a_{11}w_1 + a_{12}w_2 + \cdots + a_{1m}w_m \\ v_2 =& a_{21}w_1 + a_{22}w_2 + \cdots + a_{2m}m_m \\ \vdots =& \vdots \\ v_n =& a_{n1}w_1 + a_{n2}w_2 + \cdots + a_{nm}w_m \end{aligned}$$

or in matrix notation, $Aw = v$.
(b) Since the homogeneous system $A^T\alpha = 0$ has more unknowns than equations, the system has a non-trivial solution by Theorem 6.1(2).
(c) We have $\alpha^T v = \alpha^T Aw = (\alpha^T A)w = 0 \cdot w = 0$.
(d) From (c), we have $\alpha_1 v_1 + \alpha_2 v_2 + \cdots \alpha_n v_n = 0$ with $\alpha_i \neq 0$ for some $1 \leq i \leq n$. But this says that $\{v_1, v_2, \cdots, v_n\}$ is linearly dependent contradicting the independence of the v_i's. Hence, we must have $V \neq \text{span}(S)$.

18.18 Suppose that $S = \{w_1, w_2, \cdots, w_m\}$ is another basis of V. Since $V = \text{span}(S)$, by Problem 18.17, we must have $m \geq n$. Since S is linearly independent, by Problem 18.16, we must have $m \leq n$. Hence, we must have $m = n$.

18.19 (a) Since $V \neq \text{span}(S)$, there is $v \in V$ and $v \notin \text{span}(S)$. Suppose that $av + a_1v_1 + a_2v_2 + \cdots + a_mv_m = 0$. If $a \neq 0$ then $v = -\frac{a_1}{a}v_1 - \frac{a_2}{a}v_2 - \cdots - \frac{a_m}{a}v_m$. But this says that $v \in \text{span}(S)$. A contradiction. Hence, we must have $a = 0$. But in this case, $a_1v_1 + a_2v_2 + \cdots + a_mv_m = 0$ and the linear independence of the v_i's implies that $a_1 = a_2 = \cdots = a_m = 0$. Hence, $a = a_1 = a_2 = \cdots = a_m = 0$ and this shows that $\{v, v_1, v_2, \cdots, v_m\}$ is linearly independent.
(b) Since $V \neq \text{span}\{v, v_1, v_2, \cdots, v_m\}$, there is $w \in V$ and $w \notin \text{span}\{v, v_1, v_2, \cdots, v_m\}$. Suppose that $bw + av + a_1v_1 + a_2v_2 + \cdots + a_mv_m = 0$. If $b \neq 0$ then $w = -\frac{a}{b}v - \frac{a_1}{b}v_1 - \frac{a_2}{b}v_2 - \cdots - \frac{a_m}{b}v_m$. But this says that $w \in \text{span}\{v, v_1, v_2, \cdots, v_m\}$. A contradiction. Hence, we must have $b = 0$. But in this case, $av + a_1v_1 + a_2v_2 + \cdots + a_mv_m = 0$ and the linear independence of the v's implies that $a = a_1 = a_2 = \cdots = a_m = 0$. Hence, $b = a = a_1 = a_2 = \cdots = a_m = 0$ and

this shows that $\{w, v, v_1, v_2, \cdots, v_m\}$ is linearly independent.
(c) By repeating the process in (a) and (b), we can continue adding vectors in this way until we get a set which is independent and spans V. The process must terminate, since no independent set in V can have more than n elements. (See Problem 18.16).

18.20 (a) Since $\{v_1, v_2, \cdots, v_m\}$ is linearly dependent, there is $v_i \in \{v_1, v_2, \cdots, v_m\}$ such that v_i is a linear combination of the remaining v's. Let S_1 be the set $\{v_1, v_2, \cdots, v_m\}$ with v_i removed. In this case, $V = \text{span}(S_1)$.
(b) Continuing throwing out vectors as in (a) until one reaches a linearly independent set S which spans V, i.e., a basis of V. The process must terminate, because no set containing fewer than n vectors can span V. See Problem 18.17.

18.21 By the definition of a subspace, we have $W \subseteq V$. Now, suppose that $W \neq V$. Let $\{w_1, w_2, \cdots w_n\}$ be a basis of W. If $\{w_1, w_2, \cdots w_n\}$ is not a basis of V, then by the linear independence pf $\{w_1, w_2, \cdots w_n\}$ and Problem 18.19, we can extend $\{w_1, w_2, \cdots w_n\}$ to a basis of V. But a basis of V cannot have more than n elements. Hence, $\{w_1, w_2, \cdots w_n\}$ must be a basis of V. In this case, for any $v \in V$ we can write $v = a_1w_1 + a_2w_2 + \cdots a_nw_n \in W$. Hence, $V \subseteq W$.

18.22 (a) Suppose $S \subseteq V$ has n linearly independent vectors. Let $W = \text{span}(S) \subseteq V$. Then S is independent and spans W, so S is a basis for W. Since S has n elements, $\dim(W) = n$. But $W \subseteq V$ and $\dim(V) = n$. By the preceding problem, $V = W$. Hence, S spans V, and S is a basis for V.
(b) Suppose S spans V. Suppose S is not independent. By Problem 18.20, we can remove some elements of S to get a set T which is a basis for V. But now we have a basis T for V with fewer than n elements (since we removed elements from S, which had n elements). But this contradicts Problem 18.18, and hence S must be independent.

Section 19

19.1 $\lambda = -3$ and $\lambda = 1$.

19.2 $\lambda = -3$ and $\lambda = -1$.

19.3 $\lambda = 3$ and $\lambda = -1$.

19.4 $\lambda = -8$, $\lambda = -i$, and $\lambda = i$.

19.5 Let $\mathbf{x}$ be an eigenvector of A corresponding to the nonzero eigenvalue λ. Then $A\mathbf{x} = \lambda\mathbf{x}$. Multiplying both sides of this equality by A^{-1} and then dividing the resulting equality by λ to obtain $A^{-1}\mathbf{x} = \frac{1}{\lambda}\mathbf{x}$. That is, $\mathbf{x}$ is an eigenvector of A^{-1} corresponding to the eigenvalue $\frac{1}{\lambda}$.

19.6 Let $\mathbf{x}$ be an eigenvector of A corresponding to the eigenvalue λ. Then $A\mathbf{x} = \lambda\mathbf{x}$. Multiplying both sides by A to obtain $A^2\mathbf{x} = \lambda A\mathbf{x} = \lambda^2\mathbf{x}$. Now, multiplying this equality by A to obtain $A^3\mathbf{x} = \lambda^3\mathbf{x}$. Continuing in this manner, we find $A^m\mathbf{x} = \lambda^m\mathbf{x}$.

19.7 Suppose that $D = P^{-1}AP$. Then $D^2 = (P^{-1}AP)(P^{-1}AP) = P^{-1}AP^2$. Thus, by induction on k one finds that $D^k = P^{-1}A^kP$.

19.8 The characteristic equation of I_n is $(\lambda - 1)^n = 0$. Hence, $\lambda = 1$ is the only eigenvalue of I_n.

19.9 (a) If λ is an eigenvalue of A then there is a nonzero vector x such that $Ax = \lambda x$. By Problem 19.6, λ^k is an eigenvalue of A^k and $A^kx = \lambda^kx$. But $A^k = \mathbf{0}$ so $\lambda^kx = \mathbf{0}$ and since $x \neq \mathbf{0}$ we must have $\lambda = 0$.
(b) Since $p(\lambda)$ is of degree n and 0 is the only eigenvalue of A, $p(\lambda) = \lambda^n$.

19.10 Since λ is an eigenvalue of A with corresponding eigenvector x, we have $Ax = \lambda x$. Postmultiply B by P^{-1} to obtain $BP^{-1} = P^{-1}A$. Hence, $BP^{-1}x = P^{-1}Ax = \lambda P^{-1}x$. This says that λ is an eigenvalue of B with corresponding eigenvector $P^{-1}x$.

19.11 The characteristic polynomial is of degree n. The Fundamental Theorem of Algebra asserts that such a polynomial has exactly n roots. A root in this case can be either a complex number or a real number. But if a root is complex then its conjugate is also a root. Since n is odd,there must be at least one real root.

19.12 The characterisitc polynomial of A is

$$p(\lambda) = \lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0.$$

19.13 We will show by induction on k that if

$$D = \begin{bmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_{nn} \end{bmatrix}$$

then

$$D^k = \begin{bmatrix} d_{11}^k & 0 & \cdots & 0 \\ 0 & d_{22}^k & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_{nn}^k \end{bmatrix}$$

Indeed, the result is true for $k = 1$. Suppose true up to $k - 1$ then

$$\begin{aligned} D^k =& D^{k-1}D = \begin{bmatrix} d_{11}^{k-1} & 0 & \cdots & 0 \\ 0 & d_{22}^{k-1} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_{nn}^{k-1} \end{bmatrix} \begin{bmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_{nn} \end{bmatrix} \\ =& \begin{bmatrix} d_{11}^k & 0 & \cdots & 0 \\ 0 & d_{22}^k & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_{nn}^k \end{bmatrix}. \end{aligned}$$

19.14 We use the fact that a matrix and its transpose have the same determinant. Hence,

$$|\lambda I_n - A^T| = |(\lambda I_n - A)^T| = |\lambda I_n - A|.$$

Thus, A and A^T have the same characteristic equation and therefore the same eigenvalues.

19.15 $\lambda = 4$ and $\lambda = -2$.

19.16 The characteristic equation of the matrix B is

$$\begin{vmatrix} \lambda + 2 & 1 \\ -5 & \lambda - 2 \end{vmatrix} = 0.$$

Expanding the determinant and simplifying, we obtain $\lambda^2 - 9 = 0$ and the eigenvalues are $\lambda = \pm 3$.

19.17 If $\lambda = 0$ is an eigenvalue of A then it must satisfy $|0I_n - A| = |-A| = 0$. That is $|A| = 0$ and this implies that A is singular. Conversely, if A is singular then $0 = |A| = |0I_n - A|$ and therefore 0 is an eigenvalue of A.

19.18 Using cofactor expansion along the first row, we find

$$\begin{aligned}
|\lambda I_n - A| =& (\lambda - a_{11}) \begin{vmatrix} \lambda - a_{22} & -a_{23} & \cdots & -a_{2n} \\ -a_{32} & \lambda - a_{33} & \cdots & -a_{3n} \\ \vdots & \vdots & \vdots & \vdots \\ -a_{n2} & -a_{n3} & \cdots & \lambda - a_{nn} \end{vmatrix} \\
& + \text{ polynomial of degree } \leq \text{ (n - 2)} \\
=& (\lambda - a_{11})(\lambda - a_{22}) \begin{vmatrix} \lambda - a_{33} & -a_{34} & \cdots & -a_{3n} \\ -a_{43} & \lambda - a_{44} & \cdots & -a_{4n} \\ \vdots & \vdots & \vdots & \vdots \\ -a_{n3} & -a_{n4} & \cdots & \lambda - a_{nn} \end{vmatrix} \\
& + \text{ polynomial of degree } \leq \text{ (n - 2)} \\
& \vdots \\
=& (\lambda - a_{11})(\lambda - a_{22}) \cdots (\lambda - a_{nn}) + \text{ polynomial of degree } \leq \text{ (n - 2)} \\
=& \lambda^n - (a_{11} + a_{22} + \cdots + a_{nn})\lambda^{n-1} + \text{ terms of lower degree.}
\end{aligned}$$

19.19 The characteristic equation is $\lambda^2 - 5\lambda + 6 = 0$. By Cayley-Hamilton

Theorem, we have $A^2 - 5A + 6I_2 = 0$. Using this, we have

$$\begin{aligned}
A^2 =& 5A - 6I_2\\
A^3 =& 5A^2 - 6A = 25A - 6A - 30I_2\\
=& 19A - 30I_2\\
A^4 =& 19A^2 - 30I_2 = 19(5A - 6I_2) - 30I_2\\
=& 65A - 114I_2\\
A^5 =& 65A^2 - 114A = 65(5A - 6I_2) - 114A\\
=& 211A - 190I_2\\
A^6 =& 211(5A - 6I_2) - 190I_2\\
=& 665A - 1266I_2.
\end{aligned}$$

19.20 $A^{-1} = \begin{bmatrix} -2 & 5 & -3 \\ -8 & 17 & -10 \\ 5 & -10 & 6 \end{bmatrix}.$

Section 20

20.1

$$V^{-3} = \left\{ \begin{bmatrix} -2t-s \\ s \\ t \end{bmatrix} : s,t \in \mathbb{R} \right\} = \text{span} \left\{ \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

20.2

$$V^{3} = \left\{ \begin{bmatrix} \frac{1}{2}s \\ s \end{bmatrix} : s \in \mathbb{R} \right\} = \text{span} \left\{ \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} \right\}$$

and

$$V^{-1} = \left\{ \begin{bmatrix} 0 \\ s \end{bmatrix} : s \in \mathbb{R} \right\} = \text{span} \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}.$$

20.3

$$V^{3} = \left\{ \begin{bmatrix} -5s \\ -6s \\ s \end{bmatrix} : s \in \mathbb{R} \right\} = \text{span} \left\{ \begin{bmatrix} -5 \\ -6 \\ 1 \end{bmatrix} \right\}$$

and

$$V^{-1} = \left\{ \begin{bmatrix} -s \\ 2s \\ s \end{bmatrix} : s \in \mathbb{R} \right\} = \text{span} \left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right\}.$$

20.4

$$V^{-8} = \left\{ \begin{bmatrix} -\frac{1}{6}s \\ -\frac{1}{6}s \\ s \end{bmatrix} : s \in \mathbb{R} \right\} = \text{span} \left\{ \begin{bmatrix} -\frac{1}{6} \\ -\frac{1}{6} \\ 1 \end{bmatrix} \right\}.$$

$$V^{-i} = \left\{ \begin{bmatrix} (10+5i)s \\ (-18-24i)s \\ 25s \end{bmatrix} : s \in \mathbb{R} \right\} = \text{span} \left\{ \begin{bmatrix} \frac{2}{5}+\frac{i}{5} \\ -\frac{18}{25}-\frac{24}{25}i \\ 1 \end{bmatrix} \right\}.$$

$$V^{i} = \left\{ \begin{bmatrix} (10-5i)s \\ (-18+24i)s \\ 25s \end{bmatrix} : s \in \mathbb{R} \right\} = \text{span} \left\{ \begin{bmatrix} \frac{2}{5}-\frac{i}{5} \\ -\frac{18}{25}+\frac{24}{25}i \\ 1 \end{bmatrix} \right\}.$$

20.5

$$V^{1} = \left\{ \begin{bmatrix} s \\ s \\ s \end{bmatrix} : s \in \mathbb{R} \right\} = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\},$$

$$V^2 = \left\{ \begin{bmatrix} \frac{2}{3}s \\ s \\ s \end{bmatrix} : s \in \mathbb{R} \right\} = \text{span} \left\{ \begin{bmatrix} \frac{2}{3} \\ 1 \\ 1 \end{bmatrix} \right\},$$

and

$$V^3 = \left\{ \begin{bmatrix} \frac{1}{4}s \\ \frac{3}{4}s \\ s \end{bmatrix} : s \in \mathbb{R} \right\} = \text{span} \left\{ \begin{bmatrix} \frac{1}{4} \\ \frac{3}{4} \\ 1 \end{bmatrix} \right\}.$$

20.6

$$V^1 = \left\{ \begin{bmatrix} -2s \\ s \\ s \end{bmatrix} : s \in \mathbb{R} \right\} = \text{span} \left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\}$$

and

$$V^2 = \left\{ \begin{bmatrix} -s \\ t \\ s \end{bmatrix} : s, t \in \mathbb{R} \right\} = \text{span} \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

20.7

$$V^1 = \left\{ \begin{bmatrix} s \\ -s \\ s \end{bmatrix} : s \in \mathbb{R} \right\} = \text{span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

and

$$V^{-2} = \left\{ \begin{bmatrix} -s \\ s \\ 0 \end{bmatrix} : s \in \mathbb{R} \right\} = \text{span} \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

20.8

$$V^1 = \left\{ \begin{bmatrix} -s \\ s \\ -s \\ s \end{bmatrix} : s \in \mathbb{R} \right\} = \text{span} \left\{ \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\},$$

$$V^{-1} = \left\{ \begin{bmatrix} s \\ -s \\ s \\ 0 \end{bmatrix} : s \in \mathbb{R} \right\} = \text{span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\},$$

$$V^2 = \left\{ \begin{bmatrix} -s \\ 0 \\ -s \\ s \end{bmatrix} : s \in \mathbb{R} \right\} = \text{span} \left\{ \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\},$$

and

$$V^{-2} = \left\{ \begin{bmatrix} 0 \\ -s \\ s \\ 0 \end{bmatrix} : s \in \mathbb{R} \right\} = \text{span} \left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

20.9 Algebraic multiplicity of $\lambda = 1$ is equal to the geometric multiplicity of 1.

20.10 The matrix is non-defective.

20.11

$$V^1 = \left\{ \begin{bmatrix} -2s \\ s \\ s \end{bmatrix} : s \in \mathbb{R} \right\} = \text{span} \left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\}$$

and $[-2, 1, 1]^T$ is a basis for V^1. Hence, $\dim(V^1) = 1$.

$$V^2 \text{span} \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

The set $\{[-1, 0, 1]^T, [0, 1, 0]^T\}$ is a basis of V^2 and $\dim(V^2) = 2$.

20.12

$$V^1 = \left\{ \begin{bmatrix} s \\ s \\ 0 \end{bmatrix} : s \in \mathbb{R} \right\} = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Note that the vector $[1, 1, 0]^T$ is a basis of V^1 and $\dim V^1 = 1$.

$$V^5 = \text{span} \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \right\}$$

Note that the vectors $[-1, 1, 0]^T$ and $[0, 0, 1]^T$ form a basis of V^5 and $\dim(V^5) = 2$.

20.13

$$V^i = \left\{ \begin{bmatrix} \left(\frac{2}{5} + \frac{i}{5}\right) s \\ s \end{bmatrix} : s \in \mathbb{R} \right\} = \text{span} \left\{ \begin{bmatrix} 2 + i \\ 5 \end{bmatrix} \right\}.$$

and

$$V^{-i} = \left\{ \begin{bmatrix} \left(\frac{2}{5} - \frac{i}{5}\right) s \\ s \end{bmatrix} : s \in \mathbb{R} \right\} = \text{span} \left\{ \begin{bmatrix} 2 - i \\ 5 \end{bmatrix} \right\}.$$

20.14 Suppose the contrary. That is, $\{v_1, v_2\}$ is linearly dependent. Then there is a non-zero scalar α such that $v_1 = \alpha v_2$. Applying A to both sides to obtain $\lambda_1 v_1 = \alpha\lambda_2 v_2$. Multiplying $v_1 = \alpha v_2$ by λ_1 to obtain $\lambda_1 v_1 = \alpha\lambda_1 v_2$. Substracting this equation from $\lambda_1 v_1 = \alpha\lambda_2 v_2$ to obtain $\alpha(\lambda_2 - \lambda)v_2 = 0$. Since $\{v_2\}$ is linearly independent and $\lambda_1 \neq \lambda_2$, we arrive at $\alpha = 0$. This contradicts that $\alpha \neq 0$. Hence, the statement $\{v_1, v_2\}$ is dependent is false and therefore $\{v_1, v_2\}$ is independent.

20.15 Since λ is an eigenvalue of A with corresponding eigenvector v, we have $Av = \lambda v$. Thus, $(A - rI_n)v = Av - rv = \lambda v - rv = (\lambda - r)v$ with $v \neq 0$. That is, $\lambda - r$ is an eigenvalue of $A - rI_n$ with corresponding eigenvector v.

20.16 The eigenvalues of A are $\lambda = 2$ and $\lambda = 3$, since the matrix is lower triangular. One can easily show that

$$V_A^2 = \text{span} \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

and

$$V_A^3 = \text{span} \left\{ \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\}.$$

The eigenvalues of A^T are $\lambda = 2$ and $\lambda = 3$, since the matrix is upper triangular. One can easily show that

$$V_{A^T}^2 = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$$

and

$$V_{A^T}^3 = \text{span} \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}.$$

20.17 $\lambda = 1$.

20.18 Since A is invertible, the homogeneous system has only the trivial solution. That is, 0 cannot be an eigenvalue of A.

20.19 (a) The characterisitic equation is $-\lambda^3 + 6\lambda^2 - 9\lambda = 0$.
(b) The eigenvalues are $\lambda = 0$ (with algebraic multiplicity 1) and $\lambda = 3$ with algebraic multiplicity 2.
(c) The eigenspace corresponding to $\lambda = 0$ is

$$V^0 = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}\right\}.$$

Thus, $\lambda = 0$ has geometric multiplicity 1.
The eigenspace corresponding to $\lambda = 3$ is

$$V^0 = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -5 \end{bmatrix}\right\}.$$

Thus, $\lambda = 3$ has geometric multiplicity 2.

20.20 Let λ be an eigenvalue of A with corresponding eigenvector $v \neq 0$. Then $Av = \lambda v$. Applying A to both sides, we find $A^2 v = \lambda v$ or $\lambda v = 0$. Since $v \neq= 0$, we must have $\lambda = 0$.

20.21 We have

$$(2I_3 - A)v = 2v - Av = 2v - 7v = -5v.$$

Thus, v is an eigenvector of $2I_3 - A$ with corresponding eigenvalue -5.

20.22 If the entries of each row of A sums to s then

$$A\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = s\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}.$$

Hence, s is an eigenvalue of A with corresponding eigenvector

$$\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}.$$

Section 21

21.1 (a) Suppose that $A \sim B$ and let P be an invertible matrix such that $B = P^{-1}AP$. Taking the transpose of both sides we obtain $B^T = (P^T)^{-1}A^TP^T$; that is, $A^T \sim B^T$.
(b) Suppose that A and B are invertible and $B = P^{-1}AP$. Taking the inverse of both sides we obtain $B^{-1} = P^{-1}A^{-1}P$. Hence $A^{-1} \sim B^{-1}$.

21.2 Suppose that A is an $n \times n$ invertible matrix. Then $BA = A^{-1}(AB)A$. That is $AB \sim BA$.

21.3 The eigenvalues of A are $\lambda = 4, \lambda = 2 + \sqrt{3}$ and $\lambda = 2 - \sqrt{3}$. Hence, by Theorem 21.2, A is diagonalizable.

21.4 By Problem 20.3, the eigenspaces of A are

$$V^{-1} = \left\{ \begin{bmatrix} -s \\ 2s \\ s \end{bmatrix} : s \in \mathbb{R} \right\} = \text{span}\left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right\}$$

and

$$V^3 = \left\{ \begin{bmatrix} -5s \\ -6s \\ s \end{bmatrix} : s \in \mathbb{R} \right\} = \text{span}\left\{ \begin{bmatrix} -5 \\ -6 \\ 1 \end{bmatrix} \right\}.$$

Since there are only two eigenvectors, A is not diagonalizable.

21.5 The only eigenvalue of A is $\lambda = 3$. By letting $P = I_n$ and $D = A$, we see that $D = P^{-1}AP$, i.e., A is diagonalizable.

21.6 Suppose that A is diagonalizable. Then there exist matrices P and D such that $D = P^{-1}AP$, with D diagonal. Taking the transpose of both sides to obtain $D = D^T = P^TA^T(P^{-1})^T = Q^{-1}A^TQ$ with $Q = (P^{-1})^T = (P^T)^{-1}$. Hence, A^T is diagonalizable. Similar argument for the converse.

21.7 Suppose that $A \sim B$. Then there exists an invertible matrix P such that $B = P^{-1}AP$. Suppose first that A is diagonalizable. Then there exist an invertible matrix Q and a diagonal matrix D such that $D = Q^{-1}AQ$. Hence, $B = P^{-1}QDQ^{-1}$ and this implies $D = (P^{-1}Q)^{-1}B(P^{-1}Q)$. That is, B is diagonalizable. For the converse, repeat the same argument using $A = (P^{-1})^{-1}BP^{-1}$.

21.8 Consider the matrices

$$A = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}, B = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}.$$

The matrix A has the eigenvalues $\lambda = 2$ and $\lambda = -1$ so by Theorem 21.2, A is diagonalizable. Similar argument for the matrix B. Let $C = A + B$ then

$$C = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

This matrix has only one eigenvalue $\lambda = 1$ with corresponding eigenspace (details omitted)

$$V^1 = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}.$$

Hence, there is only one eigenvector of C and by Theorem 21.1, C is not diagonalizable.

21.9

$$P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

21.10

$$P = \begin{bmatrix} -1 & 1 & -1 & 0 \\ 1 & -1 & 0 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}.$$

21.11 Not diagonalizable.

21.12 $P = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 6 \end{bmatrix}.$

21.13

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 6 & 7 & -4 \\ -6 & 6 & -4 \end{bmatrix}.$$

21.14 $A^5 = \begin{bmatrix} 5226 & 4202 \\ -2101 & -1077 \end{bmatrix}.$

21.15 Not diagonalizable.

21.16 The equation $\alpha_1 p_1 + \alpha_2 p_2 + \cdots + \alpha_n p_n = 0$ leads to the homogeneous system $P\alpha = 0$ where $\alpha = [\alpha_1, \alpha_2, \cdots, \alpha_n]^T$. Since P is invertible, $\alpha = 0$. That is, $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$. This shows that $p_1, p_2, \cdots, p_n$ are linearly independent.

21.17

$$\frac{1}{2}\begin{bmatrix} 3^{10}+1 & 3^{10}-1 \\ 3^{10}-1 & 3^{10}+1 \end{bmatrix}.$$

21.18 The characteristic polynomial is of degree 3 with distinct eigenvalues $\lambda = 2,\ \lambda = 6$, and $\lambda = 1$. By Theorem 21.2, A is diagonalizable.

21.19 Since $A = PDP^T$, by taking transpose of both sides, we find $A^T = (PDP^T)^T = (P^T)^T D^T P^T = PDP^T = A$. That is, A is symmetric.

Section 22

22.1 Given $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$ and $\alpha \in \mathbb{R}$, matrix arithmetic yields $T(\mathbf{x}+\mathbf{y}) = A(\mathbf{x}+\mathbf{y}) = A\mathbf{x}+A\mathbf{y} = T\mathbf{x}+T\mathbf{y}$ and $T(\alpha\mathbf{x}) = A(\alpha\mathbf{x}) = \alpha A\mathbf{x} = \alpha T\mathbf{x}$. Thus, T is linear.

22.2 (a) For all $u, v \in V$ and $\alpha \in \mathbb{R}$ we have $I(u+v) = u+v = Iu+Iv$ and $I(\alpha u) = \alpha u = \alpha I u$. So I is linear.
(b) For all $u, v \in V$ and $\alpha \in \mathbb{R}$ we have $\mathbf{0}(u+v) = 0 = \mathbf{0}u + \mathbf{0}v$ and $\mathbf{0}(\alpha u) = 0 = \alpha\mathbf{0}u$. So $\mathbf{0}$ is linear.

22.3 Let $v \in V$. Then there exist scalars $\alpha_1, \alpha_2, \cdots, \alpha_n$ such that $v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$. Since T is linear, we have $T(v) = \alpha T v_1 + \alpha T v_2 + \cdots + \alpha_n T v_n = 0$.

22.4 If $\mathbf{x} = (x_1, x_2)^T$ then $\mathbf{x} = x_1\mathbf{e_1} + x_2\mathbf{e_2}$. Hence,

$$T(\mathbf{x}) = x_1 T(\mathbf{e_1}) + x_2 T(\mathbf{e_2}) = (x_1, x_1, x_1)^T + (0, x_2, -x_2)^T = \begin{bmatrix} x_1 \\ x_1 + x_2 \\ x_1 - x_2 \end{bmatrix}.$$

22.5 Given $[x_1, y_1]^T$ and $[x_2, y_2]^T$ is $\mathbb{R}^2$ and $\alpha \in \mathbb{R}$ we find

$$\begin{aligned} T_E\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) =& T_E\left(\begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix}\right) = \begin{bmatrix} y_1 + y_2 \\ x_1 + x_2 \end{bmatrix} \\ =& \begin{bmatrix} y_1 \\ x_1 \end{bmatrix} + \begin{bmatrix} y_2 \\ x_2 \end{bmatrix} = T_E\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right) + T_E\left(\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) \end{aligned}$$

and

$$\begin{aligned} T_E\left(\alpha\begin{bmatrix} x \\ y \end{bmatrix}\right) =& T_E\left(\begin{bmatrix} \alpha x \\ \alpha y \end{bmatrix}\right) = \begin{bmatrix} \alpha y \\ \alpha x \end{bmatrix} \\ =& \alpha\begin{bmatrix} y \\ x \end{bmatrix} = \alpha T_E\left(\begin{bmatrix} x \\ y \end{bmatrix}\right). \end{aligned}$$

Hence, T_E is linear.

22.6 Given $[x_1, y_1]^T$ and $[x_2, y_2]^T$ is $\mathbb{R}^2$ and $\beta \in \mathbb{R}$ we find

$$\begin{aligned}
T_F\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) =& T_F\left(\begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix}\right) \\
=& \begin{bmatrix} \alpha(x_1 + x_2) \\ y_1 + y_2 \end{bmatrix} = \begin{bmatrix} \alpha x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} \alpha x_2 \\ y_2 \end{bmatrix} \\
=& T_F\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right) + T_F\left(\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right)
\end{aligned}$$

and

$$\begin{aligned}
T_F\left(\beta \begin{bmatrix} x \\ y \end{bmatrix}\right) =& T_F\left(\begin{bmatrix} \beta x \\ \beta y \end{bmatrix}\right) \\
=& \begin{bmatrix} \beta\alpha x \\ \beta y \end{bmatrix} = \beta \begin{bmatrix} \alpha x \\ y \end{bmatrix} = \beta T_F\left(\begin{bmatrix} x \\ y \end{bmatrix}\right).
\end{aligned}$$

Hence, T_F is linear.

22.7 Given $[x_1, y_1]^T$ and $[x_2, y_2]^T$ is $\mathbb{R}^2$ and $\alpha \in \mathbb{R}$ we find

$$\begin{aligned}
T_G\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) =& T_F\left(\begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix}\right) \\
=& \begin{bmatrix} x_1 + x_2 + y_1 + y_2 \\ y_1 + y_2 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 + y_2 \\ y_2 \end{bmatrix} \\
=& T_G\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right) + T_G\left(\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right)
\end{aligned}$$

and

$$\begin{aligned}
T_G\left(\alpha \begin{bmatrix} x \\ y \end{bmatrix}\right) =& T_G\left(\begin{bmatrix} \alpha x \\ \alpha y \end{bmatrix}\right) \\
=& \begin{bmatrix} \alpha(x + y) \\ \alpha y \end{bmatrix} = \alpha \begin{bmatrix} x + y \\ y \end{bmatrix} = \alpha T_G\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)
\end{aligned}$$

Hence, T_G is linear.

22.8 Let $[x_1, y_1]^T, [x_2, y_2]^T \in \mathbb{R}^2$ and $\alpha \in \mathbb{R}$. Then

$$\begin{aligned}
T\left(\alpha \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right) =& T\left(\begin{bmatrix} \alpha x_1 + x_2 \\ \alpha y_1 + y_2 \end{bmatrix}\right) \\
=& \begin{bmatrix} \alpha x_1 + x_2 + \alpha y_1 + y_2 \\ \alpha x_1 + x_2 - 2\alpha y_1 - 2y_2 \\ 3\alpha x_1 + 3x_2 \end{bmatrix} \\
=& \alpha \begin{bmatrix} x_1 + y_1 \\ x_1 - 2y_1 \\ 3x_1 \end{bmatrix} + \begin{bmatrix} x_2 + y_2 \\ x_2 - 2y_2 \\ 3x_2 \end{bmatrix} \\
=& \alpha T\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right) + T\left(\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right)
\end{aligned}$$

Hence, T is a linear transformation.

22.9 (a) Let $p, q \in P_n$ and $\alpha \in \mathbb{R}$ then

$$\begin{aligned}
D[\alpha p(x) + q(x)] =& (\alpha p(x) + q(x))' \\
=& \alpha p'(x) + q'(x) = \alpha D[p(x)] + D[q(x)]
\end{aligned}$$

Thus, D is a linear transformation.
(b) Let $p, q \in P_n$ and $\alpha \in \mathbb{R}$ then

$$\begin{aligned}
I[\alpha p(x) + q(x)] =& \int_0^x (\alpha p(t) + q(t))dt \\
=& \alpha \int_0^x p(t)dt + \int_0^x q(t)dt = \alpha I[p(x)] + I[q(x)]
\end{aligned}$$

Hence, I is a linear transformation.

22.10 Suppose that $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \alpha \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. This leads to a linear system in the unknowns α and β. Solving this system we find $\alpha = -1$ and $\beta = 2$. Since T is linear, we have

$$T\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right) = -T\left(\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}\right) + 2T\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right) = -5 + 4 = -1.$$

22.11 Let $[x_1, y_1, z_1]^T \in \mathbb{R}^3, [x_2, y_2, z_2]^T \in \mathbb{R}^3$ and $\alpha \in \mathbb{R}$. Then

$$\begin{aligned} T\left(\alpha \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right) =& T\left(\begin{bmatrix} \alpha x_1 + x_2 \\ \alpha y_1 + y_2 \\ \alpha z_1 + z_2 \end{bmatrix}\right) \\ =& \begin{bmatrix} \alpha x_1 + x_2 \\ \alpha y_1 + y_2 \end{bmatrix} = \alpha \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \\ =& \alpha T\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}\right) + T\left(\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right). \end{aligned}$$

Hence, T is a linear transformation.

22.12 Since $|A + B| \neq |A| + |B|$ in general, the given transformation is not linear.

22.13 Let $u_1, u_1 \in U$ and $\alpha \in \mathbb{R}$. Then

$$\begin{aligned} (T_2 \circ T_1)(\alpha u_1 + u_2) =& T_2(T_1(\alpha u_1 + u_2)) \\ =& T_2(\alpha T_1(u_1) + T_1(u_2)) \\ =& \alpha T_2(T_1(u_1)) + T_2(T_1(u_2)) \\ =& \alpha (T_2 \circ T_1)(u_1) + (T_2 \circ T_1)(u_2). \end{aligned}$$

22.14 Consider the system in the unknowns $T(v)$ and $T(v_1)$

$$\left\{ \begin{array}{rrl} T(v)- & 3T(v_1) = & w \\ 2T(v)- & T(v_1) = & w_1 \end{array} \right.$$

Solving this system to find $T(v) = \frac{1}{4}(3w_1 - 2w)$ and $T(v_1) = \frac{1}{4}(w_1 - 2w)$.

22.15 Suppose that $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$. Then $\alpha_1 T(v_1) + \alpha_2 T(v_2) + \cdots + \alpha_n T(v_n) = T(0) = 0$. Since the vectors $T(v_1), T(v_2), \cdots, T(v_n)$ are linearly independent, $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$. This shows that the vectors $v_1, v_2, \cdots, v_n$ are linearly independent.

22.16 (a) $T(u-w) = T(u+(-w)) = T(u)+T(-w) = T(u)+T[(-1)w] = T(u)+(-T(w)) = T(u)-T(w)$.
(b) $T(0) = T(0+0) = T(0)+T(0)$. By the uniquenss of the zero vector, we must have $T(0) = 0$.
(c) $T(-u) = T(0-u) = T(0)-T(u) = 0-T(u) = -T(u)$.
(d) We use induction on $n \geq 2$.
(i) Basis of induction: If $n = 2$, then by the definition of a linear transformation, we have $T(\alpha_1 u_1 + \alpha_2 u_2) = T(\alpha_1 u_1) + T(\alpha_2 u_2) = \alpha_1 T(u_1) + \alpha_2 T(u_2)$.
(ii) Induction hypothesis: Suppose that $T(\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_k u_k) = \alpha_1 T(u_1) + \alpha_2 T(u_2) + \cdots + \alpha_k T(u_k)$ for $2 \leq k \leq n$.
(iii) Induction step: We have

$$\begin{aligned}
T(\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n + \alpha_{n+1} u_{n+1}) =& T[(\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n) + \alpha_{n+1} u_{n+1}] \\
=& T(\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n) + T(\alpha_{n+1} u_{n+1}) \\
=& \alpha_1 T(u_1) + \alpha_2 T(u_2) + \cdots + \alpha_n T(u_n)) \\
& + \alpha_{n+1} u_{n+1}].
\end{aligned}$$

22.17 Suppose first that T is linear. Let $u, v \in V$ and $\alpha \in \mathbb{R}$. Then $\alpha u \in V$. Since T is linear we have $T(\alpha u + v) = T(\alpha u) + T(v) = \alpha T(u) + T(v)$.
Conversely, suppose that $T(\alpha u + v) = \alpha T(u) + T(v)$ for all $u, v \in V$ and $\alpha \in \mathbb{R}$. In particular, letting $\alpha = 1$ we see that $T(u+v) = T(u) + T(v)$ for all $u, v \in V$. Now, letting $v = 0$ we see that $T(\alpha u) = \alpha T(u)$. Thus, T is linear.

22.18 Let $v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n \in V$. Then

$$\begin{aligned}
T(v) =& T(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n) \\
=& \alpha_1 T(v_1) + \alpha_2 T(v_2) + \cdots + \alpha_n T(v_n) \\
=& \alpha_1 S(v_1) + \alpha_2 S(v_2) + \cdots + \alpha_n S(v_n) \\
=& S(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n) = S(v).
\end{aligned}$$

Since this is true for any $v \in V$ then $T = S$.

22.19 (a)

$$\begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}.$$

(b) Let $p, q \in P_2$ and $\alpha \in \mathbb{R}$. Then

$$\begin{aligned}
T(\alpha p + q) =& \begin{bmatrix} (\alpha p + q)(-1) \\ (\alpha p + q)(0) \\ (\alpha p + q)(1) \end{bmatrix} \\
=& \begin{bmatrix} \alpha p(-1) + q(-1) \\ \alpha p(0) + q(0) \\ \alpha p(1) + q(1) \end{bmatrix} \\
=& \begin{bmatrix} \alpha p(-1) \\ \alpha p(0) \\ \alpha p(1) \end{bmatrix} + \begin{bmatrix} q(-1) \\ q(0) \\ q(1) \end{bmatrix} \\
=& \alpha \begin{bmatrix} p(-1) \\ p(0) \\ p(1) \end{bmatrix} + \begin{bmatrix} q(-1) \\ q(0) \\ q(1) \end{bmatrix} \\
=& \alpha T(p) + T(q).
\end{aligned}$$

22.20 (a) Let $v_1, v_2 \in \text{Ker}(T)$ and $\alpha \in \mathbb{R}$. Then $T(\alpha v_1 + v_2) = \alpha T(v_1) + T(v_2) = 0$. That is, $\alpha v_1 + v_2 \in \text{Ker}(T)$. This proves that $\text{Ker}(T)$ is a subspace of V.

(b) Let $w_1, w_2 \in \text{Im}(T)$. Then there exist $v_1, v_2 \in V$ such that $T(v_1) = w_1$ and $T(v_2) = w_2$. Let $\alpha \in \mathbb{R}$. Then $T(\alpha v_1 + v_2) = \alpha T(v_1) + T(v_2) = \alpha w_1 + w_2$. Hence, $\alpha w_1 + w_2 \in \text{Im}(T)$. This shows that $\text{Im}(T)$ is a subspace of W.

Section 23

23.1 (a) Let $A, B \in M_{nn}$ and $\alpha \in \mathbb{R}$. Then $T(\alpha A + B) = (\alpha A + B - (\alpha A + B)^T = \alpha(A - A^T) + (B - B^T) = \alpha T(A) + T(B)$. Thus, T is linear.
(b) Let $A \in \text{Ker}(T)$. Then $T(A) = \mathbf{0}$. That is $A^T = A$. This shows that A is symmetric. Conversely, if A is symmetric then $T(A) = \mathbf{0}$. It follows that $\text{Ker}(T) = \{A \in M_{nn} : A \text{ is symmetric}\}$. Now, if $B \in \text{Im}(T)$ and A is such that $T(A) = B$ then $A - A^T = B$. But then $A^T - A = B^T$. Hence, $B^T = -B$,i.e., B is skew-symmetric. Conversely, if B is skew-symmetric then $B \in \text{Im}(T)$ since $T(\frac{1}{2}B) = \frac{1}{2}(B - B^T) = B$. We conclude that $\text{Im}(T) = \{B \in M_{nn} : B \text{ is skew} - \text{symmetric}\}$.

23.2 $\text{Ker}(T) = \text{span}\left\{\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}\right\}$.

23.3 $nullity(T) = 2$ and $rank(T) = 2$.

23.4 $\text{Ker}(T) = \text{Im}(T) = \left\{\begin{bmatrix} 0 \\ a \end{bmatrix} : a \in \mathbb{R}\right\}$.

23.5 By Theorem 23.2, the rank of T is 1 so that the range of T is generated by a single vector.

23.6 $a \neq 2$.

23.7 $\text{Ker}(T) = \text{span}\left\{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$ and $\text{Im}(T) = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$.

23.8 $\text{Ker}(T) = \left\{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right\}$ and $\text{Im}(T) = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}\right\}$.

23.9 $\text{Ker}(T) = \text{span}\{-t^2 + t\}$ and $\text{Im}(T) = \text{span}\{t, t^2\}$.

23.10 $nullity(T) = 9,\ \text{rank}(S) = 6$.

23.11 $\text{Ker}(T) = \text{span}\{t^3 - t, t^2 - t\}$ and $\text{Im}(T) = \text{span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$.

23.12 $nullity(T) = 1$ and $\text{rank}(T) = n$.

23.13 $\mathrm{Ker}(T) = \mathrm{span}\{\cos t, \sin t\}$.

23.14 If S is a basis of $\mathrm{Ker}(T)$ then S is linearly independent. By Problem 18.18, S can be extended to a basis of V.

23.15 Let $w \in \mathrm{Im}(T)$. Then there is $v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k + \alpha_{k+1} v_{k+1} + \cdots + \alpha_n v_n \in V$ such that $T(v) = w$. Hence,

$$w = T(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k + \alpha_{k+1} v_{k+1} + \cdots + \alpha_n v_n) = \alpha_{k+1} T(v_{k+1}) + \cdots + \alpha_n T(v_n)$$

since $T(v_1) = \cdots = T(v_k) = 0$. This shows that $\{T(v_{k+1}, \cdots, T(v_n)\}$ is a span of $\mathrm{Im}(T)$.

23.16 Suppose not. Then one of the vectors of $\{T(v_{k+1}), \cdots, T(v_n)\}$ is a linear combination of the remaining vectors. Without loss of generality, suppose that $T(v_{k+1}) = \alpha_{k+2} T(v_{k+2}) + \cdots + \alpha_n T(v_n)$. Then $T(\alpha_{k+2} v_{k+2} + \cdots \alpha_n v_n - v_{k+1}) = 0$. Hence, $\alpha_{k+2} v_{k+2} + \cdots \alpha_n v_n - v_{k+1} \in \mathrm{Ker}(T) = \mathrm{span}\{v_1, v_2, \cdots, v_k\}$. It follows that $\alpha_{k+2} v_{k+2} + \cdots \alpha_n v_n - v_{k+1} = \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_k v_k$ which can be written as

$$\beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_k v_k + v_{k+1} - \alpha_{k+2} v_{k+2} - \cdots \alpha_n v_n = 0.$$

Since the coefficient of v_{k+1} is 1, the previous equality says that $\{v_1, v_2, \cdots, v_k, v_{k+1}, \cdots, v_n\}$ is linearly independent and this contradicts the fact that $\{v_1, v_2, \cdots, v_k, v_{k+1}, \cdots, v_n\}$ is a basis of V. Hence, $\{T(v_{k+1}), \cdots, T(v_n)\}$ is linearly independent.

23.17 According to Problems 23.15 - 23.16, the basis $\{v_1, \cdots, v_k\}$ of $\mathrm{Ker}(T)$ can be extended to a basis $\{v_1, \cdots, v_k, v_{k+1}, \cdots, v_n\}$ of V with $\{T(v_{k+1}), \cdots, T(v_n)\}$ being a basis of $\mathrm{Im}(T)$. Hence, $n = k + (n-k) = \dim(\mathrm{Ker}(T)) + \dim(\mathrm{Im}(T))$.

23.18 (a) Let $v \in \mathrm{Ker}(T)$. Then $T(v) = 0$ and $S(T(v)) = S(0) = 0$. Hence, $v \in \mathrm{Ket}(S \circ T)$.
(b) Let $x \in \mathrm{Im}(S \circ T)$. Then there is a $v \in V$ such that $(S \circ T)(v) = x$. Hence, $S(T(v)) = x$. Let $w = T(v) \in W$. Then $S(w) = x$. Thus, $x \in \mathrm{Im}(S)$.

23.19 $\mathrm{Ker}(T) = \mathrm{span}\{1\}$ and $\mathrm{Im}(T) = \mathrm{span}\{x, x^2\}$.

23.20 Since $T(0) = 0$, $0 \in \mathrm{Ker}(T)$. If $v \in \mathrm{Ker}(T)$ and $v \neq 0$ then $T(v) = 0 =$

$T(0)$. But this contradicts the definition of T. Hence, 0 is the only vector in $\mathrm{Ker}(T)$.

Section 24

24.1 We first show that T is linear. Indeed, let $X, Y \in M_{mn}$ and $\alpha \in \mathbb{R}$. Then $T(\alpha X + Y) = A(\alpha X + Y) = \alpha AX + AY = \alpha T(X) + T(Y)$. Thus, T is linear. Next, we show that T is one-one. Let $X \in \text{Ker}(T)$. Then $AX = \mathbf{0}$. Since A is invertible, $X = \mathbf{0}$. This shows that $\text{Ker}(T) = \{\mathbf{0}\}$ and thus T is one-one. Finally, we show that T is onto. Indeed, if $B \in \text{Im}(T)$ then $T(A^{-1}B) = B$. This shows that T is onto.

24.2 Since $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \in \text{Ker}(T)$, by Theorem 24.1, T is not one-to-one.

24.3 Suppose that T is one-one. Then $\text{Ker}(T) = \{0\}$ and therefore $dim(\text{Ker}(T)) = 0$. By Theorem 24.2, $dim(\text{Im}(T)) = dimV$. The converse is similar.

24.4 If $A \in \text{Ker}(T)$ then $T(A) = \mathbf{0} = A^T$. This implies that $A = \mathbf{0}$ and consequently $\text{Ker}(T) = \{\mathbf{0}\}$. So T is one-one. Now suppose that $A \in M_{mn}$. Then $T(A^T) = A$ and $A^T \in M_{nn}$. This shows that T is onto. It follows that T is an isomorphism.

24.5 Let $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \text{Ker}(T)$. Then

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ x_1 + 2x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This implies that $x_1 = x_2 = 0$ so that T is one-to-one.

24.6 Let $S = \{v_1, v_2, \cdots, v_n\}$ consists of linearly independent vectors. Then $T(S) = \{T(v_1), T(v_2), \cdots, T(v_n)\}$. Suppose that $\alpha_1 T(v_1) + \alpha_2 T(v_2) + \cdots + \alpha_n T(v_n) = 0$. Then we have

$$\begin{aligned} T(0) = 0 =& \alpha_1 T(v_1) + \alpha_2 T(v_2) + \cdots + \alpha_n T(v_n) \\ =& T(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n) \end{aligned}$$

Since T is one-to-one, we must have $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$. Since the vectors $v_1, v_2, \cdots, v_n$ are linearly independent, we have $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$. This shows that $T(S)$ consists of linearly independent vectors.

24.7 Suppose that $T(S)$ is linearly independent for any linearly independent set S. Let v be a nonzero vector of V. Since $\{v\}$ is linearly independent, $\{T(v)\}$ is linearly independent. That is, $T(v) \neq 0$. Hence, $\text{Ker}(T) = \{0\}$ and by Theorem 24.1, T is one-to-one.

24.8 (a) Let $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ be two vectors in $\mathbb{R}^2$. Then for any $\alpha \in \mathbb{R}$ we have

$$\begin{aligned}
T\left(\alpha \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right) =& T\left(\begin{bmatrix} \alpha x_1 + y_1 \\ \alpha x_2 + y_2 \end{bmatrix}\right) \\
=& \begin{bmatrix} \alpha x_1 + y_1 + \alpha x_2 + y_2 \\ \alpha x_1 + y_1 - \alpha x_2 - y_2 \\ \alpha x_1 + y_1 \end{bmatrix} \\
=& \begin{bmatrix} \alpha(x_1 + x_2) \\ \alpha(x_1 - x_2) \\ \alpha x_1 \end{bmatrix} + \begin{bmatrix} y_1 + y_2 \\ y_1 - y_2 \\ y_1 \end{bmatrix} \\
=& \alpha T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) + T\left(\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right).
\end{aligned}$$

Hence, T is linear.

(b) If $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \text{Ker}(T)$ then $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ x_1 - x_2 \\ x_1 \end{bmatrix}$ and this leads to $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Hence, $\text{Ker}(T) = \left\{\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\}$ so that T is one-to-one.

To show that T is not onto, take the vector $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^3$. Suppose that $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$ is such that $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. This leads to $x_1 = 1$ and $x_1 = 0$ which is impossible. Thus, T is not onto.

24.9 Suppose first that T is one-to-one. Let $v \in \text{Ker}(T)$. Then $T(v) = 0 = T(0)$. Since T is one-to-one, $v = 0$. Hence, $\text{Ker}(T) = \{0\}$.
Conversely, suppose that $\text{Ker}(T) = \{0\}$. Let $u, v \in V$ be such that $T(u) =$

$T(v)$, i.e., $T(u-v)=0$. This says that $u-v \in \text{Ker}(T)$, which implies that $u-v=0$ or $u=v$. Thus, T is one-to-one.

24.10 Suppose first that T is invertible. Then there is a unique function $T^{-1}: W \to V$ such that $T^{-1} \circ T = \text{id}_v$. So if $T(v_1) = T(v_2)$ then $T^{-1}(T(v_1)) = T^{-1}(T(v_2))$ and this implies $v_1 = v_2$. Hence, T is one-to-one. Next, we show that T is onto. Let $w \in W$. Then $w = \text{id}_W = T(T^{-1}(w))$ and $T^{-1}(w) \in V$. Hence, $\text{Im}(T) = W$.
Conversely, suppose that T is a linear transformation that is both one-to-one and onto. Then for each $v \in V$, there is a unique $w \in W$ such that $T(v) = w$. Define the function $S : W \to V$ by $S(w) = v$. Clearly, $S \circ T = \text{id}_V$ and $T \circ S = \text{id}_W$. Hence, T is invertible with inverse S.

24.11 (a) Since $T(e_1) = \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix}$ and $T(e_2) = \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix}$, the standard matrix of T is

$$\begin{bmatrix} 5 & -3 \\ -7 & 8 \\ 2 & 0 \end{bmatrix}.$$

(b) Let $x = [x_1, x_2]^T \in \text{Ker}(T)$. Then $T(x) = 0$ and this leads to the syste

$$\begin{cases} 5x_1 - 3x_2 = 0 \\ -7x_1 + 8x_2 = 0 \\ 2x_1 \qquad\quad = 0. \end{cases}$$

Solving this system, we find $x_1 = x_2 = 0$. Hence, $\text{Ker}(T) = \{0\}$ and T is one-to-one.

24.12 Let $q(t) = a + bt \in P_1$. Then $p(t) = c + at + \frac{b}{2}t^2 \in P_2$ and $T(p) = q$. Hence, T is onto. Since $p(t) = a \in P_2$ where $a \neq 0$ and $T(p) = 0$, $\text{Ker}(T) \neq \{0\}$ so that T is not one-to-one.

24.13 Since A is singular, the homogeneous system $Ax = 0$ has non-trivial solutions so that $\text{Ker}(T) \neq \{0\}$. Hence, T is not one-to-one. Now by the dimension theorem, $\dim(\text{Im}(T)) < n$ since $\dim(\text{Ker}(T)) > 0$. This implies that $\text{Im}(T) \neq \mathbb{R}^n$ and therefore T is not onto.

24.14 Neither one-to-one nor onto.

24.15

$$T\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right)=\begin{bmatrix}4x_1-3x_2\\-x_1+x_2\end{bmatrix}.$$

24.16 Suppose that $(S\circ T)(v_1)=(S\circ T)(v_2)$. Then $S(T(v_1))=S(T(v_2))$. Since S is one-to-one, $T(v_1)=T(v_2)$. Since T is one-to-one, $v_1=v_2$. Hence, $S\circ T$ is one-to-one.

24.17 $T^{-1}\left(\begin{bmatrix}x_1 & x_2\\x_3 & x_4\end{bmatrix}\right)=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}.$

24.18 Let $x\in X$. Since S is onto, there is a $w\in W$ such that $S(w)=x$. Since T is onto, there is a $v\in V$ such that $T(v)=w$. Hence, $S(T(v))=S(w)=x$. This shows that $S\circ T$ is onto.

24.19 Since $\dim(M_{22})=4$, it suffices to show that T is one-to-one. So let $A=[a_{ij}]\in M_{22}$ be such that $T(A)=0$. This implies that

$$\begin{bmatrix}2&0\\0&3\end{bmatrix}\begin{bmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{bmatrix}-\begin{bmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{bmatrix}\begin{bmatrix}4&0\\0&5\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}.$$

Simplifying, one finds

$$\begin{bmatrix}-2a_{11}&-3a_{12}\\-a_{21}&-2a_{22}\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}.$$

Hence, $a_{11}=a_{12}=a_{21}=a_{22}=0$ and T is one-to-one. We conclude that T is an isomorphism.

24.20 This follows from Example 24.2.

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