STA511 Homework #4

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1. (a) The log likelihood function of the poisson pdf, $f(x) = \frac{\lambda^x e^{-\lambda}}{x!}$ was derived to be $log(\lambda) = \sum X_i log(\lambda) \cdot -n\lambda$. The log likelihood function of the given observations are plotted in Figure 1.
   The code can be seen below:

```
counts <- c(rep(0,7840), rep(1,1327), rep(2,239), rep(3,42), rep(4,14), rep(5,4), rep(6,4), rep(7,1))

n <- length(counts)

neglikefun <- function(lambda){
        n*lambda-sum(counts)*(log(lambda))
}

lambda <- seq(0.00001, 1, length=1000)

library(ggplot2)
qplot(lambda,
      sapply(lambda, function(lambda){-1*neglikefun(lambda)}),
      geom = 'line',
      main = paste('Log Likelihood as a Function of', sep=" ", expression(lambda)),
      xlab = expression(lambda),
      ylab = 'Log Likelihood')
```

   (b) The MLE of λ was computed as 0.2151832. The R code that was used to compute this answer can be seen below:

```
nlminb(1,function(lambda){neglikefun(lambda)})
$par
[1] 0.2151832
```

   (c) The probability that a randomly selected policy has 2 claims (g(λ)) = Pr($Y_i = 2$)) was estimated to be 0.1866955. This was computed by simply plugging in λ_{MLE} into the Poisson pdf. The R code that was used to compute this answer can be seen below:

```
> ((0.2151832^2)*(exp(-0.2151832)))/factorial(2)
[1] 0.01866955
```

2. (a) θ_{MLE} was computed by taking the log likelihood of the normal distribution pdf, $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$. The calculations for finding θ_{MLE} can be seen below:

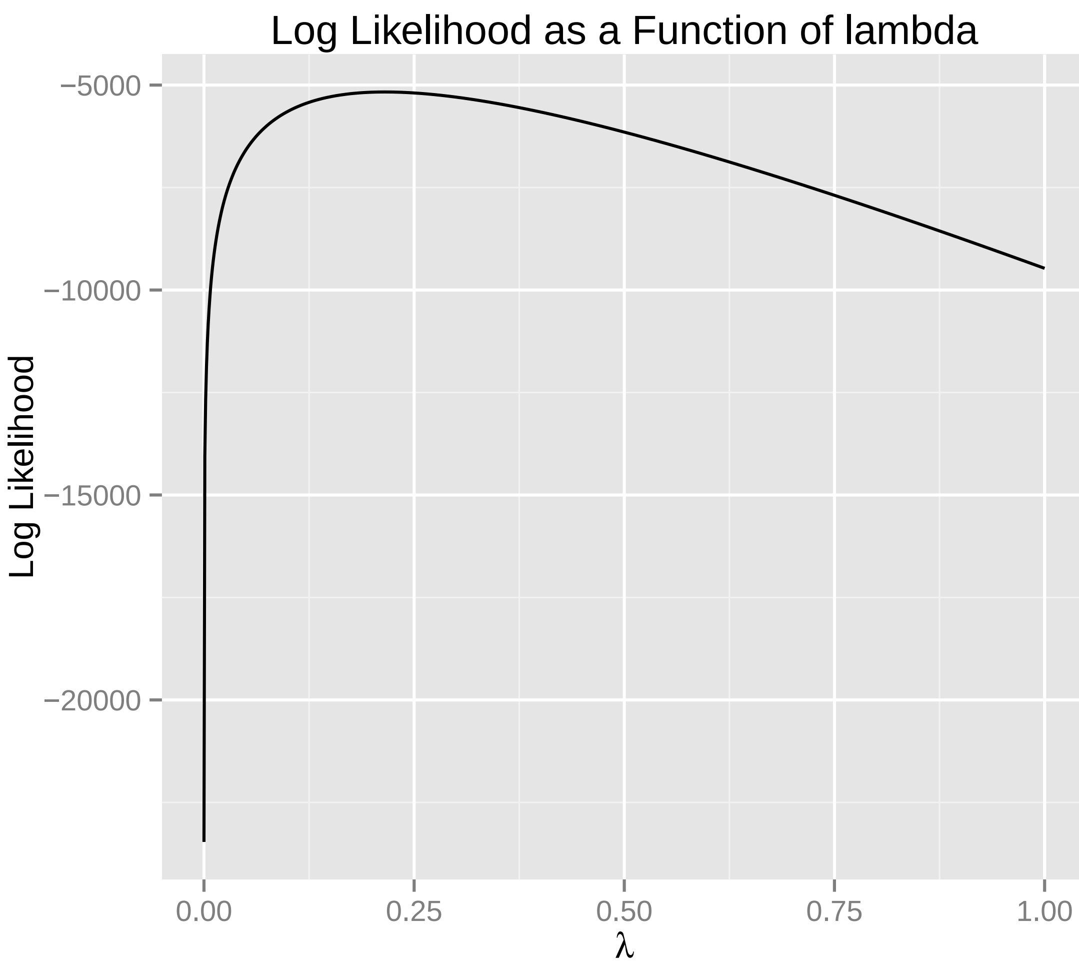
$$L(\theta) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-(x-\theta)^2/2}$$

$$\ell(\theta) = -\frac{n}{2} log(2\pi) - \frac{\sum_{i=1}^{n} (X_i - \theta)^2}{2}$$

$$\ell'(\theta) = \sum_{i=1}^{n} X_i - n\theta = 0$$

$$\hat{\theta}_{MLE} = \frac{\sum_{i=1}^{n} X_i}{n} = \bar{X}$$

Figure 1: Question 1



(b) The question prompt said $\Phi = Pr(Y_i = 1)$ – and according the given piecewise function of Y_i, $Y_i = 1$ is true only when $X_i > 0$. Computing the cdf of a normal distribution is not trivial using integration. As such, using the rules of probability we can represent the normal distribution function as a cdf. $\Phi_{MLE} = g(\hat{\theta}_{MLE})$ where $g(\theta) = 1 - F(0)$. Where F(x) is the cdf of a standard normal distribution N(0,1).

$$P(X \leq 0) = P\left(\frac{X-\theta}{1} \leq \frac{0-\theta}{1}\right)$$

And because $F(x) = g(\frac{x-\mu}{\sigma}) = g(Z)$ when $X \sim N(0,1)$

$$= P(Z \leq -\theta)$$

Therefore,

$$\Phi_{MLE} = 1 - F(-\hat{\theta})$$

(c) We were asked to compute the asymptotic standard error for θ and for Φ.
In order to compute $se(\theta)$, the theorem that of asymptotic normality was utilized, that states that the standard error is the square root of 1 / Fisher Information ($se = \frac{1}{\sqrt{I_n(\theta)}}$).

$$\hat{se}(\hat{\theta}_{MLE}) = \sqrt{\frac{1}{I_n(\hat{\theta}_{MLE})}}$$

$$I_n(\theta) = -nE\left[\frac{\delta^2}{\delta\theta^2}logf(x|\theta)\right] = -nE\left[\frac{\delta^2}{\delta\theta^2} - \frac{1}{2}log(2\pi) - \frac{(x-\theta)^2}{2}\right] = -nE\left[\frac{\delta}{\delta(\theta)}(x-\theta)\right] = -nE\big[-1\big] = n$$

$$\hat{se}(\hat{\theta}_{MLE}) = \sqrt{\frac{1}{n}}$$

We can use the Delta Method to compute $\hat{se}(\hat{\Phi}_{MLE})$.

$$\hat{se}(\hat{\Phi}_{MLE}) = \big|g'(\hat{\theta}_{MLE})\big| \cdot \hat{se}(\hat{\theta}_{MLE})$$

$$g'(\theta) = -\frac{1}{1}\frac{1}{\sqrt{2\pi}}e^{-(\theta-\mu)^2/2\cdot 1}$$

Standard $X_i \sim N(0,1)$ was assumed for $g'(\theta)$, so 0 were plugged in for μ and 1 was plugged in for σ^2.

$$= \left|\frac{1}{\sqrt{2\pi}}e^{-(\theta)^2/2}\right|$$

And finally,

$$\hat{se}(\hat{\theta}_{MLE}) = \left|\frac{1}{\sqrt{2\pi}}e^{-\bar{X}^2/2}\right| \cdot \sqrt{\frac{1}{n}}$$

3. (a) Since the likelihood is the joint distribution of the data which is equivalent to the product of marginal distributions, we can multiply the marginal distributions:

$$L(\theta) = \prod_{i=1}^{n}\frac{1}{\theta}e^{-X_i/\theta} \cdot \prod_{i=1}^{m}e^{-5Y_i/\theta} \cdot (1 - e^{5/\theta})^{1-Y_i}$$

$$L(\theta) = \frac{1}{\theta^n}e^{-\sum\limits_{i=1}^{n}X_i/\theta} \cdot e^{\sum\limits_{i=1}^{m}-5Y_i/\theta} \cdot \left(1 - e^{-5/\theta}\right)^{\sum\limits_{i=1}^{m}(1-Y_i)}$$

(b) To MLE for θ was calculated to be 5.971734. The MLE was computed by taking the logarithm of the likelihood function as follows:
From part 3(a):

$$l(\theta) = -nlog(\theta) - \sum_{i=1}^{n}\frac{X_i}{\theta} + \sum_{i=1}^{m}\frac{5Y_i}{\theta} + \sum_{i=1}^{m}(1-Y_i)log(1 - e^{-5/\theta})$$

θ_{MLE} was then calculated by taking this log likelihood function and using the optimize function in R. The code can be seen below:

```
> f.obs <- c(2.8, 5.6, 24.7, 6.5, 1.6, 10.6, 1.0, 7.8, 7.2, 13.9)
> n <- length(f.obs)
> g.obs <- c(0,0,0,1,1,1,0,0,1,0,0,0,0,0,0)
> m <- length(g.obs)
>
> likefun <- function(theta){
+         -n*log(theta)-sum(f.obs)/theta-5*sum(g.obs)/theta+(m-sum(g.obs))*log(1-exp(-5/theta))
+ }
>
> optimize(likefun, c(10,0), maximum=TRUE)
$maximum
[1] 5.971734

$objective
[1] -41.13977
```