Thus the numeric function (∇a) for the given function a is

$$(\nabla a)_r = \begin{cases} 0 & , 0 \le r \le 2 \\ \frac{41}{8} & , r = 3 \\ -2^{-r} & , r \ge 4 \end{cases}$$

Example 10 : The numeric function **a** is defined as

$$a_r = \begin{cases} 1 & , 0 \le r \le 10 \\ 2 & , r \ge 11 \end{cases}$$

Determine (i) S^5a (ii) $S^{-7}a$

Solution :

(i) For the numeric function a, the numeric function S^ia is given by

$$(S^ia)_r = \begin{cases} 0 \ge 2 & , 0 \le r \le i-1 \\ a_{r-i} & , r \ge i \end{cases}$$

Hence for i = 5,

$$(S^5a)_r = \begin{cases} 0 & , 0 \le r \le 4 \\ 1 & , 5 \le r \le 15 \\ 2 & , r \ge 16 \end{cases}$$

(ii) The numeric function S^{-1} a is defined as

$$(S^{-i}a)_r = a_{r+i}, \quad r \ge 0$$

Thus for the numeric function (S^{-7} a),

$$(S^{-7}a)_r = \begin{cases} 1 & , 0 \le r \le 3 \\ 2 & , r \ge 4 \end{cases}$$

Syllabus Topic : Generating Functions

5.4 Generating Functions :

5.4.1 Introduction :

As we pointed out in Section 5.1, a numeric function can be specified by an exhaustive listing of its values. In this section, we introduce an alternative way of representing numeric functions.

Suppose we have an iron rod and wish to make a horseshoe out of it. Since hammering a cold iron rod into a horseshoe is quite tedious, we first place the iron rod in a furnace. We can then hammer the hot iron rod into a (hot) horseshoe. When we dip the hot horseshoe in a water tank to cool it, we obtain the (cold) horseshoe we want. The moral of our example is rather obvious. Our goal is to turn a (cold) iron rod into a (cold) horseshoe. However, instead of trying to achieve this goal directly, we change the cold iron rod into a hot iron rod first. Once we make a hot horse shoe out of the hot iron rod, we can cool the hot horseshoe and obtain the cold horseshoe we want. The process is shown in Fig. 5.1 (a).

Given positive number x we can compute its logarithm $\ln x$. Indeed, the logarithm of a number can be viewed as an alternative representation of the number, since from x we can compute $\ln x$, and from $\ln x$ we can compute x. Thus, when we want to compute xy, instead of multiplying the two numbers x and y directly, we can represent x as $\ln x$ and y as $\ln y$, compute $\ln x + \ln y$, which is equal to $\ln xy$, and then obtain the product xy from its alternative representation $\ln xy$. Similarly, when we want to compute x/y instead of dividing x by y directly, we can compute $\ln (x/y)$, which is equal to $\ln x - \ln y$, and then obtain x/y from its alternative representation $\ln (x/y)$. The process is depicted in Fig. 5.1 (b).

$$\begin{array}{ccc} \text{Cold iron rod} & \longrightarrow & \text{Hot iron rod} \\ & & \downarrow \\ \text{Cold horseshoe} & \longleftarrow & \text{Hot horseshoe} \end{array}$$

(a)

$$\begin{array}{ccc} 5.7 & \longrightarrow & 1.01 \ . \ 11\,1 \\ & & \downarrow \\ 12 & \longleftarrow & 1\,01 + 111 = 1100 \end{array}$$

(b)

$$\begin{array}{ccc} x, y & \longrightarrow & \ln x, \ln y \\ & & \downarrow \\ xy, \frac{x}{y} & \longleftarrow & \ln x + \ln y \\ & & \ln x \ . \ \ln y \end{array}$$

(c)

Fig. 5.1

To a computer science student, the notation of alternative representation is a most familiar one. An alternative representation for a decimal number is its corresponding binary number. Thus, instead of adding, subtracting, multiplying, and dividing decimal numbers, directly, we represent them as binary numbers use a computer to carry out all arithmetic operations on binary numbers (which a computer can do effortlessly), and then obtain the result of our computation by converting the results in binary numbers into decimal numbers. Again, the process is shown in Fig. 5.1 (c).

These examples illustrate very well the notion of alternative representation for physical as well as mathematical entities. Furthermore, we saw that a *suitably chosen* alternative representation also leads to efficiency and easiness in some operations we wish to carry out.

We introduce now an alternative way to represent numeric functions. For the numeric function

$$a = \{a_0, a_1, a_2, \ldots, a_r, \ldots\}$$

We define an infinite series

$$a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \ldots + a_r z^r + \ldots$$

which is called the **generating function** of the numeric function a. It is denoted by A(z). Therefore,

$$A(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_r z^r + \ldots = \sum_{r=0}^{\infty} (a_r z^r)$$

For example : if $a = \{4^0, 4^1, 4^2, 4^3, 4^4, \ldots, 4^r, \ldots\}$

then, the generating function of a is

$$A(z) = 4^0 + 4z + 4^2z^2 + 4^3z^3 + \ldots + 4^rz^r + \ldots$$

In $$A(z) = a_0 + a_1z + a_2z^2 + \ldots + a_rz^r + \ldots$$

The term a_0 is called the 'constant term' and the term a_rz^r is the term of degree r.

Note that A(z) generates its coefficients. If all the coefficients are zero from some point on, then A(z) is a polynomial. If $a_s = 0$ for $s \geq r + 1$, then A(z) is a polynomial of degree r.

The above infinite series can be written in the closed form as

$$A(z) = \frac{1}{1-4z}$$

Thus, we can say that using the generating functions, representation of numeric functions gives the possibility of expressing infinite series in the closed form so that the different operations can be performed conveniently on them.

Finite sequences can also be represented by generating functions.

Consider the numeric function $a_r = {}^nC_r$ for a fixed n. The generating function of a is

$$A(z) = {}^nC_0 + {}^nC_1z + {}^nC_2z^2 + \ldots {}^nC_rz^r + \ldots {}^nC_n z^n.$$

Note that $${}^nC_r = 0 \text{ for } r > n$$

$\therefore$ $$A(z) = (1+z)^n \text{ by application of binomial expansion.}$$

5.4.2 Manipulation of Generating Functions :

Now, we define some operations on generating functions.

Let $$A(z) = a_0 + a_1z + a_2z^2 + \ldots + a_rz^r + \ldots$$

and $$B(z) = b_0 + b_1z + b_2z^2 + \ldots + b_rz^r + \ldots$$ denote the generating functions.

Two generating functions are equal i.e.

$$A(z) = B(z)$$

if and only if $a_r = b_r$ for each $r \geq 0$

1. Sum of A(z) and B(z)

The sum of A(z) and B(z) is defined as

$$A(z) + B(z) = \sum_{r=0}^{\infty} (a_r + b_r)\, z^r$$

If k is any scalar then,

$$kA(z) = k(a_0 + a_1z + a_2z^2 + \ldots + a_rz^r + \ldots)$$

$$= k \sum_{r=0}^{\infty} a_rz^r$$

2. Product of A(z) and B(z) :

The product of A(z) and B(z) is given by

$$A(z)\,B(z) = a_0b_0 + (a_0b_1 + a_1b_0)\,z + (a_0b_2 + a_1b_1 + a_2b_0)\,z^2 + \ldots$$
$$+ (a_0b_r + a_1b_{r-1} + \ldots + a_r\,b_0)\,z^r + \ldots$$

In the remaining section, we will see that the relation between the numeric functions correspond to the same type of relations between their generating functions.

Let A(z), B(z), C(z) be the generating functions of the numeric functions a, b and c. Let α be any scalar

(i) If $b = \alpha a$, then $B(z) = \alpha A(z)$.

Thus for $b_r = 8 \times 9^r, \quad r \geq 0$

$= 8 \times a_r, \quad r \geq 0$

where $a_r = 9^r$

$$B(z) = 8A(z)$$

$$B(z) = 8\left(\frac{1}{1-9z}\right) = \frac{8}{1-9z}$$

It is obvious that corresponding to $c = a + b$, we have $C(z) = A(z) + B(z)$

(ii) Thus if $c_r = 3^r + 4^r, \quad r \geq 0$

then $C(z) = \dfrac{1}{(1-3z)} + \dfrac{1}{(1-4z)}$

or $C(z) = \dfrac{2-7z}{1-7z+12z^2}$

For $b_r = a^r a_r$

we have $B(z) = \alpha^0 a_0 + \alpha a_1 z + \alpha^2 a_2 z^2 + \ldots + \alpha^r a_r z^r + \ldots$

or $B(z) = a_0 + a_1\,(\alpha z) + a_2(\alpha^2 z^2) + \ldots a_r\,(\alpha^r z^r) + \ldots$

$\therefore$ $B(z) = A(\alpha z)$

(iii) For example,

if $b_r = 3^r$ then $B(z) = \dfrac{1}{1-3z}$

The generating function of $S^i a$ for any positive integer i is $z^i\,A(z)$ where A(z) is the generating function of a

(iv) Hence, if $A(z) = \dfrac{z^4}{1-2z}$

We have $a_r = \begin{cases} 0 & , 0 \leq r \leq 3 \\ 2^{r-4} & , r \geq 4 \end{cases}$

Also the generating function of $S^{-i}a$ is $z^{-i}\,[A(z) - a_0 - a_1 z - a_2 z^2 - \ldots - a_{i-1}\,z^{i-1}]$

For $b = \Delta a$, we have

$$B(z) = \frac{1}{z}\,[A(z) - a_0] - A(z)$$

and for $b = \nabla a$, we have

$$B(z) = A(z) - zA(z)$$

The generating function representation of numeric function is very useful in convolution of numeric functions.

If $c = a * b$ then $C(z) = A(z)\,B(z)$

(v) For example,

if $c = a * b$

where $a_r = 3^r, \quad r \geq 0 \qquad b_r = 5^r, \quad r \geq 0$

then $C(z) = A(z)\,B(z)$

where $A(z) = \dfrac{1}{1-3z}$

$$B(z) = \frac{1}{1-5z}$$

$$\therefore \quad C(z) = \frac{1}{(1-3z)} \times \frac{1}{(1-5z)} = \frac{1}{2}\left[\frac{5}{1-5z} - \frac{3}{1-3z}\right]$$

The closed form of the generating functions of some numeric functions are given in the table below.

Table 5.1

Numeric function	Generating function
$a_r = ka^r$	$A(z) = \dfrac{k}{1-az}$
$a_r = r$	$A(z) = \dfrac{z}{(1-z)^2}$
$a_r = b_r a^r$	$A(z) = \dfrac{abz}{(1-az)^2}$
$a_r = \dfrac{1}{r!}$	$A(z) = e^z$
$a_r = \begin{cases} {}^nC_r, & 0 \leq r \leq n \\ 0, & r > n \end{cases}$	$A(z) = (1+z)^n$

5.4.3 Types of Generating Functions :

MU - May 09

1. Ordinary Generating Function (O.G.F) :

The infinite sum

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots$$

is called the **ordinary generating function** of the sequence $\{a_n\}_{n=0}^{\infty}$

2. Exponential Generating Function (E.G.F) :

Let $\{a_n\}_{n=0}^{\infty}$ be a given sequence then $\displaystyle\sum_{n=0}^{\infty} \frac{a_n}{n!} x^n = a_0 + a_1 x + \frac{a_2}{2!} x^2 + \frac{a_3}{3!} x^3 + \ldots$

5.4.4 Exercise Set - 2 :

Example 1 : For the sequence {1, –1, 1, –1, ...} the ordinary generating function is **MU - May 04**

Solution :

$$\sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + \ldots = \frac{1}{1+x}$$

Example 2 : For the sequence $\left\{1, \frac{1}{2}, \frac{1}{2^2}, \ldots\right\}$ the O.G.F. is

Solution :

$$\sum_{n=0}^{\infty} \frac{x^n}{2^n} = 1 + \frac{x}{2} + \frac{x^2}{2^2} + \ldots$$

Example 3 : Conjecture a simple formula for a_n if the first 10 terms of the sequence $\{a_n\}$ are 1, 7, 25, 79, 214, 727, 2185, 65559, 19687, 59047. **MU - May 2000**

Solution :

The generating function is

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots\ldots$$

$$= 1 + 7x + 25 x^2 + 79 x^3 + 241 x^4 + 727 x^5 + 2185 x^6$$
$$6559 x^7 + 19687 x^8 + 59047 x^9 + \ldots\ldots\ldots$$

Example 4 : Let {1, 1, 1, ...} be a given sequence. Give the exponential generating function. **MU - Dec. 02, Dec.06, May 10**

Solution :

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \ldots \text{ is E.G.F.}$$

Example 5 : Let {0, 1, 0, –1, 0, 1, 0, –1, ...} be a given sequence. Give the exponential generating function. **MU - Dec. 02, 06, 09, May 10**

Solution :

$$\sum_{n=0}^{\infty} \frac{a_n}{n!} x^n = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots \text{ is E.G.F.}$$

Example 6 : Find generating function for following numeric functions

(i) $a_r = \alpha^r$, $r \geq 0$ (ii) $(3^0, 3^1, 3^2, \ldots, 3^r, \ldots)$

(iii) $a_r = 7 \cdot 3^r$, $r \geq 0$ (iv) $a_r = 3^{r+2}$, $r \geq 0$

Solution :

(i) $a_r = \alpha^r, \quad r \geq 0$

$\therefore \quad A(z) = \dfrac{1}{1 - \alpha z}$

(ii) $(3^0, 3^1, 3^2, \ldots, 3^r, \ldots)$

$3^0 + 3z + 3^2z^2 + 3^3z^3 + \ldots + 3^r z^r + \ldots$

The infinite series can be written in closed form as,

$$= \frac{1}{1 - 3z}$$

(iii) $a_r = 7.3^r, \quad r \geq 0$

i.e. $7 \cdot 3^0 + 7 \cdot 3\,z + 7 \cdot 3^2 z^2 + \ldots 7 \cdot 3^r \cdot z^r + \ldots$

$\therefore \quad A(z) = \dfrac{7}{1 - 3z}$

(iv) $a_r = 3^{r+2}, \quad r \geq 0$

$= 3^r \cdot 3^2 = 3^2 \cdot 3^r$

$= 3^2 \cdot 3^0 + 3^2 \cdot 3^1 z^1 + 3^2 \cdot 3^2 \cdot z^2 + \ldots 3^2 \cdot 3^r z^r + \ldots$

$\therefore \quad A(z) = \dfrac{9}{1 - 3z}$

Example 7 : Find the generating function of

$$a_r = \begin{cases} 0 & , \text{ r oddd} \\ 2^{r+1} & , \text{ r even} \end{cases}$$

Solution :

According the problem, the numeric function

$$a = \{2, 0, 2^3, 0, 2^4, 0, 2^5, \ldots\}$$

This numeric function can be written as

$$a_r = 2^r + (-2)^r, \quad r \geq 0$$

Its generating function is

$$A(z) = \frac{1}{(1 - 2z)} + \frac{1}{(1 + 2z)}$$

$$A(z) = \frac{2}{(1 - 4z^2)}$$

Example 8 : Determine the generating function of the numeric function a_r where

(i) $a_r = 3^r + 4^{r+1}, \quad r \geq 0$

(ii) $a_r = 5 \quad , \quad r \geq 0$

MU - Dec. 05, Dec. 06

Solution :

(i) Let $a_r = b_r + c_r$

where $b_r = 3^r$

and $c_r = 4^{r+1}$

Let A(z), B(z) and C(z) be the generating functions of a, b and c.

For $b_r = 3^r$, the corresponding generating function is

$$B(z) = \frac{1}{1-3z}$$

For $c_r = 4^{r+1}$, $C(z) = \frac{4}{1-4z}$

$$\therefore \quad A(z) = B(z) + C(z)$$

$$A(z) = \frac{1}{1-3z} + \frac{4}{1-4z}$$

$$A(z) = \frac{(5-16z)}{(1-3z)(1-4z)}$$

(ii) Given $a_r = 5, \quad r \geq 0$

Therefore numeric function

$$a = \{5, 5, 5, \ldots\}$$

$\therefore$ It generating function

$$A(z) = 5 + 5z + 5z^2 + 5z^3 + \ldots$$

$$= 5(1 + z + z^2 + z^3 + \ldots) = 5 \times \frac{1}{(1-z)}$$

$$\therefore \quad A(z) = \frac{5}{(1-z)}$$

Example 9 : Determine the discrete numeric functions corresponding to the following generating functions

(i) $\frac{1}{(1+z)}$ (ii) $\frac{3-5z}{(1-2z-3z^2)}$

Solution :

(i) Given the generating function

$$A(z) = \frac{1}{1+z} = \frac{1}{1-(-z)}$$

It is the sum of geometric progression whose first term is 1 and common ratio is (–z)

$$A(z) = 1 - z + z^2 - z^3 + z^4 - \ldots$$

Therefore, corresponding numeric function is

$$a_r = (-1)^r$$

(ii) Given $A(z) = \frac{3-5z}{(1-2z-3z^2)}$

$$= \frac{3-5z}{(1-3z)(1+z)}$$

Here we use partial fraction method.

So let, $$\frac{3-5z}{(1-3z)(1+z)} = \frac{A}{1-3z} + \frac{B}{1+z}$$

which after simplification gives A = 1 and B = 2

Therefore, $$A(z) = \frac{3-5z}{(1-3z)(1+z)}$$
$$= \frac{1}{(1-3z)} + \frac{2}{(1+z)}$$

Corresponding to $\frac{1}{(1-3z)}$, the numeric function is 3^r and corresponding to $\frac{2}{1+z}$, the numeric function is $2(-1)^r$.

Therefore, the numeric function corresponding to the generating function A(z) is

$$a_r = 3^r + 2(-1)^r, \qquad r \geq 0$$

Example 10 : Find the numeric functions corresponding to

(i) $\frac{2+3z-6z^2}{(1-2z)}$ (ii) $\frac{z^4}{(1-2z)}$

Solution :

(i) The given generating function is

$$A(z) = \frac{2+3z-6z^2}{1-2z}$$

$$\therefore \qquad A(z) = 3z + \frac{2}{1-2z}$$

which can be written as

$$A(z) = B(z) + C(z)$$

where $$B(z) = 3z$$

and $$C(z) = \frac{2}{1-2z}$$

For B(z), the numeric function is

$$b = \{0, 3, 0, 0, 0, \ldots\}$$

and for C(z), $$c_r = 2 \times 2^r$$
$$= 2^{r+1}$$

Therefore, $$c = \{2, 2^2, 2^3, 2^4, \ldots\}$$

Hence the numeric function corresponding to A(z) is

$$a = b + c$$
$$a = \{0, 3, 0, 0, 0, \ldots\} + \{2, 2^2, 2^3, 2^4, \ldots\}$$

Thus, $a = \{2, 7, 2^3, 2^4, 2^5, 2^6, \ldots\}$

or can be written as

$$a_r = \begin{cases} 2^{r+1} & , \; r = 0 \\ 7^{r+1} & , \; r = 1 \\ 2^{r+1} & , \; r \geq 2 \end{cases}$$

(ii) $$A(z) = \frac{z^4}{1-2z}$$

Now, $$\frac{z^4}{1-2z} = -\frac{1}{2^4} - \frac{1}{2^3}z - \frac{1}{2^2}z^2 - \frac{1}{2}z^3 + \frac{1/16}{(1-2z)}$$

Therefore, $A(z) = B(z) + C(z) + D(z) + E(z) + F(z)$

Hence the numeric function corresponding to A(z) is

$$A(z) = \left\{-\frac{1}{2^4}, -\frac{1}{2^3}, -\frac{1}{2^2}, -\frac{1}{2}, 0, 0, 0, \ldots\right\}$$

$$+ \{2^{-4} \times 2^0, 2^{-4} \times 2^1, 2^{-4} \times 2^2, 2^{-4} \times 2^3, 2^{-4} \times 2^4, 2^{-4} \times 2^5, \ldots\}$$

Therefore $a = \{0, 0, 0, 0, 2^0, 2^1, 2^2, 2^3, \ldots\}$

which can be written as

$$a_r = \begin{cases} 0 \leq 3 & , \; 0 \leq r \leq 3 \\ 2^{r-4} & , \; r \geq 4 \leq 3 \end{cases}$$

Example 11 : Let $a_r = 3^r, r \geq 0$ and $b_r = 2^r, r \geq 0$

Find c_r that is $a_r * b_r$.

Solution :

$$a_r = 3^r, r \geq 0$$

$\therefore$ $$A(z) = \frac{1}{1-3z}$$

and $$b_r = 2^r, r \geq 0$$

$\therefore$ $$B(z) = \frac{1}{1-2z}$$

We have $$C(z) = A(z) \cdot B(z)$$

$$= \frac{1}{1-3z} \cdot \frac{1}{1-2z}$$

Using partial fraction

$$\frac{1}{1-3z} \cdot \frac{1}{1-2z} = \frac{A}{1-3z} + \frac{B}{1-2z}$$

$$1 = A(1-2z) + B(1-3z) \quad \ldots(i)$$

Put $z = \frac{1}{2}$ in exp (i)

$$1 = A(0) + B\left(1 - \frac{3}{2}\right)$$

$$= B\left(-\frac{1}{2}\right)$$

$\therefore$ $B = -2$

put $z = \frac{1}{3}$ in exp (i)

$\therefore$ $1 = A\left[1 - 2\left(\frac{1}{3}\right)\right] + B(0)$

$\therefore$ $A = 3$

$\therefore$ $$C(z) = \frac{3}{1-3z} - \frac{2}{1-2z}$$

$\therefore$ $c_r = (3)(3)^r - (2)(2)^r = 3^{r+1} - 2^{r+1}$

Example 12 : Find the ordinary generating functions for the given sequences :

(i) $\{0, 1, 2, 3, 4, \ldots\}$ (ii) $\{1, 2, 3, 4, \ldots\}$

(iii) $\{0, 3, 3^2, 3^3, \ldots\}$ (iv) $\{2, 2, 2, 2, \ldots\}$

(v) $\{0, 0, 0, 1, 1, 1, 1, \ldots\ldots\ldots\ldots\ldots\}$

MU - May 13, Dec. 13, May 14, May 15, May 16, Dec. 16, May 17

Solution :

(i) The generating function is

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots$$

$\therefore$ The generating function for sequence $\{0, 1, 2, 3, 4, \ldots\}$ is $0 + x + 2x^2 + 3x^3 + 4x^3 + \ldots$

(ii) The generating function is

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots$$

$\therefore$ The generating function for sequence $\{1, 2, 3, 4, \ldots\}$ is $1 + 2x + 3x^2 + 4x^3 + \ldots$

(iii) The generation function is

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots$$

$\therefore$ The generating function for sequence $\{0, 3, 3^2, 3^3, \ldots\}$ is $0 + 3x + 3^2x^2 + 3^3x^3 + \ldots$ i.e. $3x + 3^2x^2 + 3^3x^3 + \ldots$

(iv) The generating function is

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots$$

$\therefore$ The generating function for the sequence $\{2, 2, 2, 2, \ldots\}$ is $2 + 2x + 2x^2 + 2x^3 + \ldots$

(v) $$\sum_{n=3}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots.$$

$\therefore$ The generating function for sequence

$$\sum_{n=3}^{\infty} (1)^n x^n = \{0, 0, 0, 1, 1, 1, 1, \ldots\ldots\ldots\ldots\ldots\} \text{ is } 0 + 0x + x^2 + 1x^3 + 1x^4 + 1x^5 + 1x^6$$

Example 13 : Find formula for the sequence with the following first five terms

(i) 1, 3, 5, 7, 9 (ii) 1, –1, 1, –1, 1 (iii) $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$

MU - May 04

Solution :

(i) 1, 3, 5, 7, 9 This is arithmetic progression

Here $a = 1$; $d = 2$

$\therefore$ $a + (n-1)d = 1 + (n-1)2 = 2n - 1$

$\therefore$ Generating function is $\sum_{n=1}^{5} (2n-1)\, x^n$.

(In this example question is for finding formula).

i.e. $$\sum_{n=1}^{5} (2n-1)$$

(ii) 1, –1, 1, –1, 1

Here alternative terms are positive and negative

$\therefore$ $$\sum_{n=0}^{4} (-1)^n$$

(iii) $\frac{1}{1}, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16} = \frac{1}{2^0}, \frac{1}{2^1}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}$

$\therefore$ Generating function is, $\sum_{n=0}^{4} \frac{1}{2^n}$

Example 14 : What is generating functions for the following sequences ?

(i) 1, 1, 1, 1, 1, 1 (ii) 2, 2, 2, 2, 2

(iii) 1, 1, 1, 1 ...... **MU - Dec. 03, Dec. 05, Dec. 09, May 15, May 16**

Solution :

(i) We know $\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + \ldots$

$$= \sum_{n=0}^{\infty} (1)^n x^n$$

$\therefore$ Generating function for sequence 1, 1, 1, 1, 1, 1 is

$$\frac{1}{1-x} = \sum_{n=0}^{5} (1)^n x^n$$

(ii) Generating function for sequence 2, 2, 2, 2, 2

$$\sum_{n=0}^{5} (2)^n x^n = 2^0 x^0 + 2' x' + 2^2 x^2 + 2^3 x^3 + 2^4 x^4 + 2^5 x^5$$

(iii) Similarly generating function for sequence 1, 1, 1, 1.........

$$\sum_{n=0}^{\infty} (1)^n x^n = \frac{1}{1-x}$$

Example 15 : Let G (x) be the generating function for the sequence { a_k }. What is the generating function for the following sequences :

(i) 0, 0, 0, a_3, a_4, a_5, ... (ii) 0, 0, 0, 0, a_0, a_1, a_2, ...

Solution :

Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$

it generates sequence a_0, a_1, a_2, ...

$\therefore$ $x\,G(x) = \sum_{n=0}^{\infty} a_n x^{n+1}$

Generates a, a_0, a_1, a_2, ...

Also, $x^2 G(x) = \sum_{n=0}^{\infty} a_n x^{n+2}$

Generates 0, 0, a_0, a_1, a_2, ...

(i) $\therefore$ Generating function for 0, 0, 0, a_3, a_4, a_5, ...

$$G(x) - a_0 - a_1 x - a_2 x^2 = \sum_{n=3}^{\infty} a_n x^n$$

Generates $0, 0, 0, a_3, a_4, a_5, \ldots\ldots$

(ii) Similarly,

$$G(x) - a_1 x - a_2 x^2 - a_3 x^3 = \sum_{n=4}^{\infty} a_n x^n$$

Generates $0, 0, 0, 0, a_0, a_1, a_2, a_3, \ldots\ldots\ldots\ldots$

Example 16 : Find the generating function for the following sequence. **MU - Dec. 14**
(i) 1, 2, 3, 4, 5, 6.... (ii) 3, 3, 3, 3, 3...

Solution :

(i) 1, 2, 3, 4, 5, 6....

The generating function is $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_0 x^2 + \ldots$

The generating function for sequence $\{1, 2, 3, 4, 5, 6\}$ is $1 + 2x + 3x^2 + 3x^2 + 4x^3 + 5x^4 + 6x^5$

(ii) $\{3, 3, 3, 3, 3\}$

The generating function for the sequence $\{3, 3, 3, 3, 3\}$ is $3 + 3x + 3x^2 + 3x^3 + 3x^4 + \ldots$

Example 17 : Find the generating functions for the following sequence **MU - Dec. 15**
(i) 0, 0, 0, 1, 2, 3, 4, 5, 6, 7, ...
(ii) 6, – 6, 6, – 6, 6, – 6, 6 – ....

Solution :

Find the generating function.

The generating function is,

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots$$

(i) The generating function for sequence $\{0, 0, 0, 1, 2, 3, 4, 5, 6, 7\}$ is,

$0 + 0x + 0x^2 + 1x^3 + 2x^4 + 3x^5 + 4x^6 + 5x^7 + \ldots$

$x^3 + 2x^4 + 3x^5 + 4x^6 + 5x^7 + \ldots$

(ii) The generating function for sequence $\{6, -6, 6, -6, 6, -6, \ldots\}$

$= 6 + -6x + 6x^2 - 6x^3 + 6x^4 - 6x^5$

5.5 Combinatorial Problems :

The generating function representation of numeric functions is most useful in solving combinatorial problems. We shall present in this section some illustrative examples. Consider the numeric function a such that

$$a_r = C(n, r)$$

for a fixed n. The generating function of **a** is

$$A(z) = C(n, 0) + C(n, 1) z + C(n, z) z^2 + \ldots + C(n, r)z^r + \ldots + C(n, n) z^n$$

Let us consider a round about way to select r objects from n objects. We can first divide the n objects into two piles, one with k objects and the other with n–k objects, for some fixed k less than n. We can then select i objects from the first pile, where there are C(k, j) ways, and r – i objects from the second pile, where there are C(n – k, r – i) ways, for i = 0, 1, 2, … , r. Thus, we have the equation.

$$C(n, r) = \sum_{i=0}^{r} C(k, i)\, C(n-k, r-i)$$

It follows that we can write **a** = **d** * **e** where

$$D(z) = C(k, 0) + C(k, 1)z + C(k, 2) z^2 + \ldots + C(k, k)z^k$$

and $$E(z) = C(n-k, 0) + C(n-k, 1) z + C(n-k, 2) z^2 + \ldots + C(n-k, n-k) z^{n-k}$$

Repeating the argument a sufficient number of times, we obtain

$$A(z) = (1+z)^n \qquad \ldots(I)$$

because the generating function of the numeric function (C(1, 0), C(1, 1), 0, 0, …) is 1 + z. Let us also point out a very simple combinatorial argument that can also be used to derive expression (I). Consider the coefficient of the term z^r in the expansion of $(1 + z)^n$. In computing the product of the n 1 + z, each factor will "contribute" either a 1 or a z. In particular, to make up the term z^r, r of the factors contribute a z each and n–r of the factors contribute a 1 each. Consequently the coefficient of z^r is the number of ways of selecting r of the n 1 + z factors to make up the term z^r, which, of course, is equal to C(n, r). We show now some results that can be derived directly from expression (I).

5.5.1 Examples :

Example 1 :

Setting z to 1 in expression (I), we obtain

$$C(n, 0) + C(n, 1) + \ldots + C(n, r) + \ldots + C(n, n) = 2^n$$

That is, the number of ways to select none, one, two, … or n objects from n objects is 2^n.

Setting z to –1 in expression (I), we obtain

$$C(n, 0) - C(n, 1) + C(n, 2) + \ldots + (-1)^r C(n, r) + \ldots + (-1)^n C(n, n) = 0$$

or $$C(n, 0) + C(n, 2) + C(n, 4) + \ldots = C(n, 1) + C(n, 3) + C(n, 5) + \ldots$$

That is, the number of ways of selecting an even number of objects from n objects is equal to the number of ways of selecting an odd number of objects.

Example 2 :

The relation $$C(n, r) = C(n-1, r) + C(n-1, r-1) \qquad \ldots(II)$$

can be proved in several ways. By straight forward algebraic manipulation, we obtain

$$C(n, r) = \frac{n!}{r!\,(n-r)!} = \frac{n!}{r!(n-r)!}\left(\frac{n-r}{n} + \frac{r}{n}\right) = \frac{(n-1)!}{r!(n-r-1)!} + \frac{(n-1)!}{(r-1)!\,(n-r)!}$$

$$= C(n-1, r) + C(n-1, r-1)$$

We can also prove expression (II) using a combinatorial argument : If we are to select r objects from n objects, there are C(n – 1, r) ways to select r objects so that a particular object is always excluded, and C(n – 1, r – 1) ways so that this particular object is always included. Thus expression (II) follows. The use of generating functions provides a third proof of expression (II). Note that C(n, r) is the coefficient of z^r in $(1 + z)^n$, which can be written as $(1 + z)^{n-1} + z(1 + z)^{n-1}$. Since the coefficient of z^r in $(1 + z)^{n-1}$ is C(n – 1, r) and the coefficient of z^r in $z(1 + z)^{n-1}$ is C (n – 1, r – 1), expression (II) follows immediately.

We present now some further examples :

Example 3 :

Let $(a_0, a_1, a_2, \ldots)$ be a numeric function such that a_r is equal to the number of ways to select r objects from 10 objects among which one, which will be denoted X, can be selected at most twice, one, which will be denoted Y, can be selected at most thrice, and the others can be selected only once. We claim that the generating function of **a** is

$$A(z) = (1 + z + z^2)(1 + z + z^2 + z^3)(1 + z)^8$$

Note that the coefficient of z^r in A(z) is the number of ways to make up the term z^r from the factors $1 + z + z^2$, $1 + z + z^2 + z^3$, and the eight factors 1 + z. The contribution from the factor $1 + z + z^2$ can be 1, z or z^2, corresponding to selecting the object X zero times, once, or twice. The contribution from the factor $1 + z + z^2 + z^3$ can be 1 or z or z^2 or z^3, corresponding to selecting the object Y zero times, once, twice, or thrice. The contribution of each of the eight factors 1 + z can be 1 or z, corresponding to selecting each of the eight remaining objects zero times or once.

Example 4 :

Let a_r denote the number of ways of selecting r objects from n objects with unlimited repetitions. Since each object can be selected as many times as we wish, a_r is equal to the coefficient of z^r in $(1 + z + z^2 + \ldots)^n$

Note that the contribution from each of the factors $1 + z + z^2 + \ldots$ will correspond to the number of times one of the objects is selected. Since

$$(1 + z + z^2 + \ldots)^n = \left(\frac{1}{1-z}\right)^n = (1 - z)^{-n}$$

We obtain

$$a_r = (-1)^r \frac{(-n)(-n-1)\ldots(-n-r+1)}{r!} = \frac{(n+r-1)!}{r!\,(n-1)!}$$
$$= C(n + r - 1, r)$$

Example 5 :

Suppose we want to determine the number of ways in which 2t + 1 marbles can be distributed among three distinct boxes so that no box will contain more than t marbles. We claim that the coefficient of z^{2t+1} in

$$A(z) = (1 + z + z^2 + \ldots + z^t)^3$$

will be our answer. Note that the coefficient of z^{2t+1} in A(z) is the number of ways to make up the term z^{2t+1} from the three factors $1 + z + z^2 + \ldots + z^t$. The contribution from each factor $1 + z + z^2 + \ldots z^t$ can be 1, z, z^2, … or z^t corresponding to having none, one, two, … or, t marbles in a box. Since

$$(1 + z + z^2 + \ldots + z^t)^3 = \left(\frac{1 - z^{t+1}}{1 - z}\right)^3$$

$$= (1 - 3z^{t+1} + 3z^{2t+2} - z^{3t+3})(1 - z)^{-3} \qquad \ldots\text{(III)}$$

the coefficient of z^{2t+1} in expression (III) is the coefficient of z^{2t+1} in $(1 - z)^{-3}$ minus three times the coefficient of z^t in $(1 - z)^{-3}$. Thus, it is

$C(3 + 2t + 1 - 1, 2t + 1) - 3\,C(3 + t - 1, t)$

which is simplified as

$C(2t + 3, 2t + 1) - 3\,C(t + 2, t)$

Syllabus Topic : Recurrence Relations

5.6 Recurrence Relations :

In the previous sections, we have discussed about discrete numeric functions and generating functions. In many discrete computation problems, it is easier to obtain the numeric function in the form of a relation between its terms. The recursive formula for defining the numeric function is called a **recurrence relation.**

If $a = \{a_0, a_1, a_2, \ldots, a_r, \ldots\}$ is a numeric function, then the recurrence relation for a is an equation relating a_r, for any r, to one or more a_i s $(i < r)$. In other words, a recurrence relation on the numeric function **a** is a formula that relates all the terms of **a** to previous terms of **a**. A recurrence relation is also called **difference equation.** To define the numeric function completely using the recurrence relation, the values of the numeric function at one or more points are required to initiate the computation. These given values of the function are called initial conditions. For example, consider the recurrence relation

$$a_r = a_{r-1} + 3, \qquad r \geq 1 \text{ with } a_0 = 2$$

Here

$$a_1 = a_0 + 3 = 2 + 3 = 5$$

$$a_2 = a_1 + 3 = 5 + 3 = 8$$

$$a_3 = a_2 + 3 = 8 + 3 = 11$$

$$\vdots$$

Thus the given recurrence relation recursively defines the numeric function

$$a = \{2, 5, 8, 11, \ldots\}$$

The condition $a_0 = 2$ is the initial condition.

Another example is 'Fibonacci sequence of numbers'. It is defined by the recurrence relation

$$a_r = a_{r-2} + a_{r-1}, \qquad r \geq 2$$

with the initial conditions

$$a_0 = 1 \text{ and } a_1 = 1$$

Here

$$a_2 = a_0 + a_1 = 2$$

$$a_3 = a_1 + a_2 = 3$$

$$a_4 = a_2 + a_3 = 5$$

Thus, the Fibonacci series is given by 1, 1, 2, 3, 5, 8, 13, ...

It is clear from above examples that according to the recurrence relation, we can carry out a step-by-step computation to determine a_r from $a_{r-1}, a_{r-2}, a_{r-3}, \ldots$ for any r using given initial conditions. Thus the numeric function which is computed using recurrence relation is known as the **solution of the recurrence relation.**

5.6.1 Linear Recurrence Relations with Constant Coefficients :

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Most of the recurrence relations we come across are linear recurrence relations. Now, we define the linear recurrence relation with constant coefficients.

Suppose r and k are non-negative integers. A recurrence relation of the form

$$C_0a_r + C_1a_{r-1} + C_2a_{r-2} + C_3a_{r-3} + \ldots + C_ka_{r-k} = f(r) \quad \text{for} \quad r \geq k \qquad \ldots(5.1)$$

where $C_0, C_1, C_2, C_3, \ldots C_k$ are constant, is called a **linear recurrence relation with constant coefficient of order** k, provided C_0 and C_k are non-zero.

The relation $a_r - 2a_{r-1} = 2r$ is a first order linear recurrence relation with constant coefficients.

Similarly, $a_r + 2a_{r-3} = r^2$ is a third order linear recurrence relation.

But the relation $a_r^2 + a_{r-1} = 5$ is not a linear recurrence relation.

To solve the k^{th} order linear recurrence relation with constant coefficients, we require k initial conditions to determine the numeric function uniquely. With fewer than k initial conditions, the numeric function computed is not unique.

Consider the second order linear relation $a_r + a_{r-1} + a_{r-2} = 4$ with only one initial condition $a_0 = 2$. The numeric functions which satisfy the given recurrence relation and initial condition are

(i) 2, 0, 2, 2, 0, 2, 2, 0, 2, 2, 0, ...

(ii) 2, 2, 0, 2, 2, 0, 2, 2, 0, 2, 2, ...

(iii) 2, 5, –3, 2, 5, –3, 2, 5, –3, ...

Thus, the numeric function described by the second order recurrence relation with only one initial condition is not unique.

5.6.2 Homogeneous Solutions :

Each linear recurrence is associated with its homogeneous equation and the solution of homogeneous equation is called **homogeneous solution** of the given recurrence relation.

Consider a k^{th} order linear recurrence relation with constant coefficients.

$$C_0a_r + C_1a_{r-1} + C_2a_{r-2} + C_3a_{r-3} + \ldots + C_ka_{r-k} = f(r)$$

Homogeneous recurrence relation of the above recurrence relation is given by

$$C_0a_r + C_1a_{r-1} + C_2a_{r-2} + C_3a_{r-3} + \ldots\ldots + C_ka_{r-k} = 0$$

This means that for any linear recurrence relation, if $f(r) = 0$ (right hand side term zero), then the given equation is homogenous recurrence relation.

For example, if the given recurrence relation is,

$$a_r - 6a_{r-1} + 11a_{r-2} + 6a_{r-3} = 2r$$

then, its homogeneous recurrence equation is

$$a_r - 6a_{r-1} + 11a_{r-2} + 6a_{r-3} = 0$$

Now we describe the method to find the solution of homogeneous recurrence relation. For this, we define the term **characteristic equation.**

The characteristic equation of the homogeneous k^{th} order linear recurrence relation.

$$C_0a_r + C_1a_{r-1} + C_2a_{r-2} + C_3a_{r-3} + \ldots + C_k\,a_{r-k} = 0 \qquad \ldots(5.2)$$

is the k^{th} degree polynomial equation

$$C_0\alpha^k + C_1\alpha^{k-1} + \ldots C_k = 0 \qquad \ldots(5.3)$$

The polynomial Equation (5.3) in α is of degree k. Therefore, it has k roots, called **characteristics roots.**

5.[illegible].1 Case of Distinct Roots :

Suppose $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_k$ are the roots of characteristic Equation (5.3). If all the roots are distinct, then the solution of the homogeneous recurrence relation (5.2) is given by

$$a_r = A_1\alpha_1^r + A_2\alpha_2^r + \ldots + A_k\alpha_k^r$$

where $A_1, A_2, \ldots A_k$ are constants which are to be determined by initial conditions.

5.[illegible].2 Case of Equal Roots :

If any characteristic root is repeated say α_1, is repeated m times, then the term $A_1\,\alpha_1^r$ is replaced by

$$(A_1r^{m-1} + A_2r^{m-2} + \ldots + A_{m-1}r + A_m)\,\alpha_1^r$$

where the constants A_i's are to be calculated using initial conditions.

5.[illegible].3 Case of Complex Roots :

Let $\alpha + i\beta$ and $\alpha - i\beta$ be the complex roots of $a_{r+2} + 2\,ca_{r+1} + da_r = 0$ (where $c^2 < d$) then the solution of the homogeneous recurrence relation is given by

$$a_r = r_1^r\,(A \cos r\theta + B \sin r\theta)$$

where $\alpha + i\beta = r_1e^{i\theta}$ and

$$\alpha - i\beta = r_1\,e^{-i\theta}$$

or $r_1 = \sqrt{\alpha^2 + \beta^2}$ and $\theta = \tan^{-1}\left(\frac{\beta}{\alpha}\right)$

5.6.3 Exercise Set - 3 :

Case of Distint Roots

Example 1 : Find the solution of $a_{r+2} - a_{r+1} - 6a_r = 0$.

Solution :

The characteristic equation is

$$\alpha^2 - \alpha - 6 = 0$$

$$\therefore \quad (\alpha - 3)(\alpha + 2) = 0$$

or $\alpha = 3, -2$ which are the characteristic roots of the equation.

Therefore, the solution of the given recurrence relation is

$$a_r = A_1 (3)^r + A_2 (-2)^r$$

Example 2 : Find the solution of $a_{r+2} + 2a_{r+1} - 3a_r = 0$ that satisfies $a_0 = 1$, $a_1 = 2$.

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Solution :

The characteristic equation is

$$\alpha^2 + 2\alpha - 3 = 0$$

$$\therefore \quad (\alpha + 3)(\alpha - 1) = 0$$

or $\alpha = -3, 1$

which are the characteristic roots of the equation.

Therefore, the solution of the given recurrence relation is

$$a_r = A_1 (-3)^r + A_2 (1)^r$$

$$= A_1 (-3)^r + A_2 \quad \ldots(5.4)$$

where A_1, A_2 are constants

To find A_1 and A_2, putting $r = 0$, we get

$$a_0 = A_1 + A_2$$

$$1 = A_1 + A_2$$

Also putting $r = 1$ in Equation (5.4), we get

$$a_1 = -3A_1 + A_2$$

$$2 = -3A_1 + A_2$$

On solving, we get

$$A_1 = -\frac{1}{4} \quad \text{and} \quad A_2 = \frac{5}{4}$$

Hence the homogeneous solution of the given recurrence relation is

$$a_r = -\frac{1}{4}(-3)^r + \frac{5}{4}$$

Example 3 : $a_r - 10\,a_{r-1} + 9\,a_{r-2} = 0$ with $a_0 = 3$ and $a_1 =$. Find homogeneous solution.

Solution :

The characteristic equation is

$$\alpha^2 - 10\alpha + 9 = 0$$

$$\therefore \quad (\alpha - 1)(\alpha - 9) = 0$$

or $$\alpha = 1, 9$$

which are the characteristic roots of the equation.

Therefore, the solution of the given recurrence relation i

$$a_r = A_1 (1)^r + A_2 (9)^r \qquad \ldots(5.5)$$

where A_1, A_2 are constants.

To find A_1 and A_2 putting $r = 0$, we get

$$a_0 = A_1 + A_2$$

$$\therefore \quad 3 = A_1 + A_2$$

Also putting $r = 1$ in Equation (5.5), we get,

$$a_1 = A_1 + 9A_2$$

$$\therefore \quad 11 = A_1 + 9A_2$$

On solving, we get

$$A_1 = 2 \quad \text{and} \quad A_2 \;\; 1$$

Hence the homogeneous solution of the given recurrenc ation is

$$a_r = 2(1)^r + (9)^r$$

Example 4 : Find the solution to the recurrence relation a_n $a_{n-1} + 11a_{n-2} - 6a_{n-3}$ with condition $a_0 = 2$, $a_1 = 5$ and $a_2 = 15$. **MU - May 03, 05, May 15**

Solution :

Given $a_n - 6a_{n-1} + 11a_{n-2} - 6a_{n-3} = 0$

$\therefore$ The characteristic equation is

$$\alpha^3 - 6\alpha^2 + 11\alpha - 6 = 0$$

$$\therefore \quad (\alpha - 1)(\alpha - 2)(\alpha - 3) = 0$$

$$\therefore \quad \alpha = 1, 2, 3$$

which are the characteristic roots of the equation.

Therefore the solution of the given recurrence relation i

$$a_n = A_1 (1)^n + A_2 (2)^n + A_3 (3)^n \qquad \ldots(5.6)$$

To find A_1, A_2 and A_3, putting $r = 0$, $r = 1$ and $r = 2$ we get

Given $a_0 = 2 \quad \therefore \quad 2 = A_1 + A_2 + A_3$

$a_1 = 5 \quad \therefore \quad 5 = A_1 + 2A_2 + 3A_3$

$a_2 = 15 \quad \therefore \quad 15 = A_1 + 4A_2 + 9A_3$

On solving we get

$A_1 = 1, A_2 = -1$ and $A_3 = 2$

Hence the homogeneous solution of the given recurrence relation is

$$a_n = 1 - 2^n + 2.3^n$$

Example 5 : Determine the sequence whose R.R. is given by $C_n = 3C_{n-1} - 2C_{n-2}$ with initial conditions $C_1 = 5, C_2 = 3$. **MU - Dec. 08**

Solution :

Let $C_n - 3\,C_{n-1} + 2\,C_{n-2} = 0$

The characteristic equation is

$$\alpha^2 - 3\alpha + 2 = 0$$

$$(\alpha - 2) \quad (\alpha - 1)$$

$\therefore$ $\alpha = 2, 1$

which are the characteristic roots of the equation. Therefore, the solution of the given recurrence relation is

$$C_r = A_1\,(2)^r + A_2\,(1)^a \qquad \text{Where } A_1, A_2 \text{ are constants} \qquad \ldots(I)$$

To find A_1 and A_2 putting $r = 1$ we get

$$C_1 = 2\,A_1 + A_2 \qquad 5 = 2\,A_1 + A_2$$

Also putting $r = 2$ in Equation (I), we get

$$C_2 = 4\,A_1 + A_2 \qquad 3 = 4\,A_1 + A_2$$

On solving we get, $A_1 = -1$ and $A_2 = 7$

Hence the homogeneous solution of the given recurrence relation is

$$C_r = -2^r + 7^r$$

Example 6 : Solve the recurrence relation

$$2a_{n+2} - 11a_{n+1} + 5a_n = 0;\ n \geq 0, a_0 = 2, a_1 = -8$$

Solution :

The characteristic equation

$$2\alpha^2 - 11\alpha + 5 = 0$$

$$\therefore 2\alpha^2 - 10\alpha - 1\alpha + 5 = 0$$

i.e. $(\alpha - 5)(2\alpha - 1)$

Thus $\alpha = 5, 1/2$

which are characteristic roots of the equation

Therefore the solution of the given recurrence relation is

$$a_n = A_1(5)^n + A_2\,(1/2)^n \qquad \ldots(5.7)$$

The constants A_1 and A_2 can be found by putting the initial values $a_0 = 2$ and $a_1 = -8$ in Equation (5.7)

For $n = 0$, $a_0 = 2$, $2 = A_1 + A_2$

For $n = 1$, $a_1 = -8$, $-8 = 5A_1 + \frac{1}{2} A_2$

On solving we get,

$$A_1 = -2 \quad \text{and} \quad A_2 = 4$$

Hence the solution of the given recurrence relation

$$a_n = (-2)(5)^n + 4\left(\frac{1}{2}\right)^n, \; n \geq 0$$

is a unique solution to

$$a_0 = 2 \quad \text{and} \quad a_1 = -8$$

Case of Equal Roots

Example 7 : Find solution of $a_{r+2} - 2\,Ca_{r+1} + C^2 a_r = 0$.

Solution :

The characteristic equation is

$$\alpha^2 - 2C\alpha + C^2 = 0$$

$$\therefore \quad (\alpha - C)^2 = 0$$

$$\therefore \quad \alpha = C \text{ is a repeated root}$$

Therefore the solution is

$$a_r = (A_1 + A_2 r)\, C^r$$

Example 8 : Find solution of $a_r + 6a_{r-1} + 12a_{r-2} + 8a_{r-3} = 0$

Solution :

The characteristic equation is

$$\alpha^3 + 6\alpha^2 + 12\alpha + 8 = 0$$

$$\therefore \quad (\alpha + 2)^3 = 0$$

$$\therefore \quad \alpha = -2, -2, -2$$

which is repeated 3 times

Therefore the solution is

$$a_r = (A_1 r^2 + A_2 r + A_3)(-2)^r$$

Example 9 : Consider $a_r - 8a_{r-1} + 16a_{r-2} = 0$ where $a_2 = 16$ and $a_3 = 80$. Find solution.

Solution :

The characteristic equation is

$$\alpha^2 - 8\alpha + 16 = 0$$

$$\therefore \quad (\alpha - 4)^2 = 0$$

$$\therefore \quad \alpha = 4, 4$$

which is repeated twice.

Therefore the solution is

$$a_r = (A_1 r + A_2)(4)^r \quad \ldots(5.8)$$

Now, given $a_2 = 16$ and $a_3 = 80$. So by putting $r = 2$ and $r = 3$ in Equation (5.8) we get,

$$(2A_1 + A_2)\, 4^2 = 16$$

and $$(3A_1 + A_2)\, 4^3 = 80$$

On solving we get

$$A_1 = \frac{1}{4}, \qquad A_2 = \frac{1}{2}$$

Hence $$a_r = \left(\frac{1}{4}r + \frac{1}{2}\right)(4)^r$$

or $$a_r = (r+2)(4)^{r-1}$$

Example 10 : What is the solution of the recurrence relation $a_n = 6a_{n-1} - 9a_{n-2}$ with initial conditions $a_0 = 1$ and $a_1 = 6$? **MU - May 04.**

Solution :

Given $$a_n - 6a_{n-1} + 9a_{n-2} = 0$$

The characteristic equation is

$$\alpha^2 - 6\alpha + 9 = 0$$

$$\therefore \quad (\alpha - 3)^2 = 0$$

$$\therefore \quad \alpha = 3, 3$$ which is repeated twice.

Therefore the solution of the given recurrence relation is

$$a_n = (A_1 n + A_2)(3)^n \quad \ldots(5.9)$$

To find A_1, A_2, put $n = 0$ and 1 in Equation (5.9), we get

For $n = 0$, $a_0 = 1 \quad \therefore \quad 1 = A_2$

For $n = 1$, $a_1 = 6 \quad \therefore \quad 6 = (A_1 + A_2)(3)$

$$\therefore \quad 3A_1 + 3A_2 = 6$$

On solving we get $A_1 = 1$, $A_2 = 1$

Hence the homogeneous solution of the given recurrence relation

$$a_n = (1n + 1)(3)^n.$$

Example 11 : Solve the recurrence relation :

$d_n = 4(d_{n-1} - d_{n-2})$

Subject to Initial conditions $d_0 = 1 = d_1$ **MU - Dec. 05, May 06, May 07, Dec. 07**

Solution :

$$d_n = 4d_{n-1} - 4d_{n-2}$$

$$d_n - 4d_{n-1} + 4d_{n-2} = 0$$

The characteristic equation is,

$$\alpha^2 - 4\alpha + 4 = 0$$

$$\therefore \quad (\alpha - 2)(\alpha - 2) = 0$$

$$\therefore \quad \alpha = 2, 2$$

Which are the characteristic roots of the equation. Therefore solution of the given recurrence relation is

$$d_n = (A_1 n + A_2)(2)^n \ldots\ldots$$

Now, given $d_0 = 1$ and $d_1 = 1$. So by putting $n = 0$ and 1 in above equation we get,

$$1 = (A_1(0) + A_2(2))^0$$

$$1 = A_2$$

$$1 = (A_1(1) + A_2)(2)^1$$

$$= (A_1 + A_2)(2)$$

$$1 = 2A_1 + 2A_2$$

$$\text{put } A_2 = 1$$

$$1 = 2A_1 + 2$$

$$-1 = 2A_1$$

$$-\frac{1}{2} = A_1$$

$$A_2 = 1 \text{ and } A_1 = -\frac{1}{2}$$

Hence the homogeneous solution of the given recurrence relation is,

$$d_n = \left[\left(-\frac{1}{2}\right)n + (1)\right](2)^n$$

Example 12 : Find the solution to the recurrence relation $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$ with initial conditions $a_0 = 1$, $a_1 = -2$ and $a_2 = -1$. **MU - Dec. 03**

Solution :

Given $\quad a_n + 3a_{n-1} + 3a_{n-2} + a_{n-3} = 0$

$\therefore$ The characteristic equation is

$$\alpha^3 + 3\alpha^2 + 3\alpha + 1 = 0$$

$$\therefore \quad (\alpha + 1)^3 = 0$$

$$\therefore \quad \alpha = -1, -1, -1$$

Thus, the roots are $-1, -1, -1$ (tripple or 3–fold roots)

Therefore the solution of the given recurrence relation is

$$a_n = (A_1 n^2 + A_2 n + A_3)(-1)^n \quad \text{...(5.10)}$$

To find A_1, A_2 and A_3 putting $n = 0, 1, 2$ in Equation (5.10) we get

For $n = 0$, $a_0 = 1$ $\quad\therefore\quad 1 = A_3$

For $n = 1$, $a_1 = -2$ $\quad\therefore$

$$\begin{aligned} -2 &= (A_1 + A_2 + A_3)(-1)^1 \\ &= -A_1 - A_2 - A_3 \\ &= -A_1 - A_2 - 1 \\ -2 + 1 &= -A_1 - A_2 \\ -1 &= -A_1 - A_2 \end{aligned}$$

For $n = 2$, $a_2 = -1$ $\quad\therefore$

$$\begin{aligned} -1 &= (4A_1 + 2A_2 + A_3)(-1)^2 \\ &= 4A_1 + 2A_2 + 1 \\ -2 &= 4A_1 + 2A_2 \end{aligned}$$

Solving we get,

$$A_1 = -2, \; A_2 = 3, \; A_3 = 1$$

Hence the homogeneous solution of the given recurrence relation is

$$a_n = (-2n^2 - 3n + 1)(-1)^n$$

Example 13 : Determine whether the sequence $\{a_n\}$ is solution of recurrence relation

$a_n = 2a_{n-1} - a_{n-2}$ for $n = 2, 3, 4, \ldots$

where $a_n = 3n$ for every non–negative integers n. Answer the same question for $a_n = 5$.

MU - May 03

Solution :

Given $\quad a_n - 2a_{n-1} + a_{n-2} = 0$

$\therefore$ The characteristic equation is

$$\alpha^2 - 2\alpha + 1 = 0$$

$$\therefore \quad (\alpha - 1)^2 = 0$$

$$\therefore \quad \alpha = 1, 1$$ which is repeated twice.

Therefore the solution is,

$$a_n = (A_1 n + A_2)(1)^n \qquad \ldots(5.11)$$

Now given $a_n = 3n$ for $n = 2, 3, 4, \ldots$, so by putting $n = 2$, and $n = 3$ in Equation (5.11), we get

For $n = 2, \quad 6 = 2A_1 + A_2$

$n = 3, \quad 9 = 3A_1 + A_2$

On solving, we get

$$A_1 = 3 \quad \text{and} \quad A_2 = 0$$

Hence the Homogeneous solution of the given recurrence relation is,

$$a_n = 3n$$

Now for $a_n = 5$ we get

For $n = 2, \quad 5 = 2A_1 + A_2$

$n = 3, \quad 5 = 3A_1 + A_2$

On solving we get

$$A_1 = 0 \quad \text{and} \quad A_2 = 5$$

$\therefore \quad a_n = 5$ is solution if $A_1 = 0$

Example 14 : Solve the recurrence relation –

$d_n = 2\,d_n - 1 - d_{n-2}$ with initial conditions.

$d_1 = 1.5$ and $d_2 = 3$. **MU - May 08. May 09**

Solution :

Solve as per above Example 13.

Example 15 : Find the solution of

$$4a_r - 20a_{r-1} + 17a_{r-2} - 4a_{r-3} = 0$$

Solution :

The characteristic equation is

$$4\alpha^3 - 20\alpha^2 + 17\alpha - 4 = 0$$

Observe that $\alpha = 4$ satisfies the equation, hence by synthetic division

4	4	–20	17	–4
		16	–16	4
	4	–4	1	0

$$\therefore \quad (\alpha - 4)(4\alpha^2 - 4\alpha + 1) = 0$$

$$\therefore \quad (\alpha - 4)(2\alpha - 1)^2 = 0$$

$$\therefore \quad \alpha = 4, \frac{1}{2}, \frac{1}{2}$$

Hence the homogeneous solution of the given recurrence relation is

$$a_r = (A_1 r + A_2)\left(\frac{1}{2}\right)^r + A_3 (4)^r$$

Example 16 : Solve the recurrence relation

$a_n = -3\,a_{n-1}\,a_{n-2} - a_{n-3}$ with $a_0 = 5$, $a_1 = -9$, $a_2 = 15$. **MU - Dec. 05, May 14**

Solution :

Given : $a_n + 3a_{n-1} + 3\,a_{n-2} + a_{n-3} = 0$

The characteristic equation is

$$\alpha^3 + 3\alpha^2 + 3\alpha + 1 = 0$$

$$(\alpha + 1)_3 = 0$$

$$\therefore \quad \alpha = -1, -1, -1$$

Thus, the roots are –1, –1, –1 (triple or 3-fold roots). Therefore the solution of the given recurrence relation is,

$$a_n = (A_1 n^2 + A_2 n + A_3)\,(-1)^n$$

To find A_1, A_2 and A_3 putting $n = 0, 1, 2$ in above equation we get,

For $n = 0$, $a_0 = 5$ $\quad\therefore\quad 5 = A_3$

For $n = 1$, $a_1 = -9$ $\quad\therefore\quad -9 = (A_1 + A_2 + A_3)(-1)^1$

$$= -A_1 - A_2 - A_3$$

$$= -A_1 - A_2 - 5$$

$$-9 + 5 = -A_1 - A_2$$

$$-4 = -A_1 - A_2$$

For $n = 2$, $a_2 = 15$ $\quad\therefore\quad 15 = (4A_1 + 2A_2 + A_3)\,(-1)^2$

$$= 4A_1 + 2A_2 + 1$$

$$14 = 4A_1 + 2A_2$$

Soling we get, $A_1 = 3$, $A_2 = 1$, $A_3 = 5$.

Hence the homogeneous solution of the given recurrence relation is,

$$a_n = (3n^2 + n + 5)\,(-1)^n$$

Case of Complex Roots

Example 17 : Find the solution of $a_{r+2} + a_r = 0$

Solution :

The characteristic equation is

$$\alpha^2 + 1 = 0$$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$a = 1, b = 0, c = 1$

$$\therefore \frac{-0 \pm \sqrt{(0)^2 - 4(1)(1)}}{2(1)} = \frac{0 \pm \sqrt{-4}}{2} = \frac{\sqrt{-1}\sqrt{4}}{2} = \pm\frac{2i}{2}$$

Roots are $\pm i$

$\alpha = 0, \beta = 1$

$$\therefore \quad r_1 = \sqrt{\alpha^2 + \beta^2} = \sqrt{0+1} = 1$$

$$\therefore \quad \theta = \tan^{-1}\left(\frac{\beta}{\alpha}\right) = \frac{\pi}{2}$$

$$\therefore \quad a_r = (1)^r\left[A_1 \cos\frac{r\pi}{2} + A_2 \sin\frac{r\pi}{2}\right]$$

$$= A_1 \cos\left(\frac{r\pi}{2}\right) + A_2 \sin\left(\frac{r\pi}{2}\right)$$

Example 18 : Find the solution of Fibonacci relation $a_r = a_{r-1} + a_{r-2}$ with the initial conditions $a_0 = 0$, $a_1 = 1$. **MU - Dec. 02**

Solution :

The characteristic equation is

$$\alpha^2 - \alpha - 1 = 0$$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$a = 1, b = -1, c = -1$

$$\therefore \frac{-(-1) \pm \sqrt{(-1)^2 + 4(1)(-1)}}{2(1)} = \frac{1 \pm \sqrt{1+4}}{2}$$

$$= \frac{1 \pm \sqrt{5}}{2}$$

$\therefore$ Roots are $\frac{1+\sqrt{5}}{2}$ and $\frac{1-\sqrt{5}}{2}$

Therefore, the solution of the given recurrence relation is

$$a_r = A_1\left(\frac{1+\sqrt{5}}{2}\right)^r + A_2\left(\frac{1-\sqrt{5}}{2}\right)^r \qquad \text{...(5.12)}$$

To find A_1 and A_2, putting $r = 0$ in Equation (5.12), we get

$$a_0 = A_1\left(\frac{1+\sqrt{5}}{2}\right)^0 + A_2\left(\frac{1-\sqrt{5}}{2}\right)^0$$

$$\therefore \quad 0 = A_1 + A_2$$

Also putting r = 1 in Equation (5.12), we get

$$a_1 = A_1\left(\frac{1+\sqrt{5}}{2}\right)^1 + A_2\left(\frac{1-\sqrt{5}}{2}\right)^1$$

$$\therefore \quad 1 = A_1\left(\frac{1+\sqrt{5}}{2}\right)^1 + A_2\left(\frac{1-\sqrt{5}}{2}\right)^1$$

$$\therefore \quad 2 = A_1(1+\sqrt{5})^1 + A_2(1-\sqrt{5})^1$$

On solving we get,

$$A_1 = \frac{1}{\sqrt{5}}, \qquad A_2 = -\frac{1}{\sqrt{5}}$$

Hence the homogeneous solution of the given recurrence relation is

$$a_r = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^r - \left(\frac{1-\sqrt{5}}{2}\right)^r\right]$$

Example 19 : Solve the recurrence relation

$$a_n - 2a_{n-1} + 2a_{n-2} - a_{n-3} = 0$$

Solution :

The characterstic equation is

$$\alpha^3 - 2\alpha^2 + 2\alpha - 1 = 0$$

i.e. $$(\alpha - 1)(\alpha^2 - \alpha + 1) = 0$$

Thus, the roots are

$$\alpha = 1, \frac{1+3i}{2}, \frac{1-3i}{2}$$

Hence the Homogeneous solution of the given recurrence relaiton

$$a_n = A_1(1)^n + A_2\left(\frac{1+3i}{2}\right)^n + A_3\left(\frac{1-3i}{2}\right)^n$$

5.6.4 Inhomogeneous Equations :

Consider ak^{th} order linear recurrence relation with constant coefficients.

$$C_0a_r + C_1a_{r-1} + C_2a_{r-2} + \ldots + C_ka_{r-k} = f(r)$$

For any linear recurrence relation, if $f(r) \neq 0$, then the equation is said to be **inhomogeneous.**

For example

$$a_r - a_{r-1} - 6a_{r-2} = -30.$$

$$a_r + 5a_{r-1} + 6a_{r-2} = 3r^2.$$

5.7 Total Solutions :

In the previous section, we have seen the method to find the homogeneous solution of the given homogeneous recurrence relation. A homogeneous relation is obtained by putting $f(r) = 0$ (Right hand side zero) in the given k^{th} order linear recurrence relation. The solution which satisfies the linear recurrence relation with f (r) on right hand side is called the **particular solution.** There is no general procedure for determining the particular solution. It depends on the nature of f (r). The homogeneous solution of the linear recurrence relation is denoted by $a_r^{(h)}$ and the particular solution is denoted by $a_r^{(p)}$. The addition of these two solutions is called a **total solution** of a given linear recurrence relation with the constant coefficients.

Thus, the total solution a_r is given by

$$a_r = a_r^{(h)} + a_r^{(p)}$$

The homogeneous solution $a_r^{(h)}$ is found out by finding characteristic roots of the characteristic equation of corresponding homogeneous recurrence relation. However, there is no general procedure for determining the particular solution of the given recurrence relation. In some cases, the particular solution can be obtained by the method of inspection of f (r).

For difference functions f (r) (right hand side of the recurrence relation), there are different forms of particular solutions. These forms are given in the following Table 5.2.

Table 5.2 : Particular solutions for given Right hand sides

Sr. No.	f (r) (Right hand side)	Form of particular solution
1.	A constant, d	A constant, P
2.	A linear function, $d_0 + d_1 r$	A linear function $P_0 + P_1 r$
3.	d^n	Pd^n or $P_n d^n$ if Pd^n fails
4.	An n^{th} degree polynomial $d_0 + d_1 r + d_2 r^2 + \ldots\ldots d_n r^n$	An n^{th} degree polynomial $P_0 + P_1 r + P_2 r^2 + \ldots + P_n r^n$
5.	An exponential function db^r, provided b is not the characteristic root	An exponential function Pb^r
6.	An exponential function db^r where b is the characteristic root of the equation with multiplicity (m – 1)	An exponential function $Pr^{m-1} b^r$
7.	Sin dn or cos dn	$P_1 \cos dn + P_2 \sin dn$

Imposing initial conditions, we can find the values of constants $P_0, P_1, \ldots$ etc.

5.7.1 Algorithm for Inhomogeneous Linear Recurrence Relation :

Now, we give algorithm for solving inhomogeneous (Refer section 5.4.4) linear recurrence relation,

$$C_0 a_r + C_1 a_{r-1} + C_2 a_{r-2} + \ldots + C_k a_{r-k} = f(r)$$

Algorithm :

Step I : Write the associated homogeneous relation by putting f (r) = 0 and find the homogeneous solution $a_r^{(h)}$.

Step II : According to the function f(r) (right hand side) given in the recurrence relation, choose the form of a particular solution with unknown coefficients which is most suited to the given recurrence relation.

Step III : Substitute the form of a particular solution selected in the step II into the given recurrence relation. From the substitution, determine the unknown coefficients involved in the selected form of particular solution.

If all the known coefficients are not evaluated then go back to the step II for choosing the form of particular solution, otherwise go to next step IV.

Step IV : Write the particular solution $a_r^{(p)}$ with the values evaluated in the step III.

Step V : Write the general solution of the given recurrence relation as the sum of homogeneous and particular solutions.

$$a_r = a_r^{(h)} + a_r^{(p)}$$

Step VI : If no initial conditions are given, then stop the procedure. Otherwise, for given k initial conditions, obtain k linear equations in k unknowns and solve the system to get a complete solution.

5.7.2 Exercise Set - 4 :

Example 1 : Solve $a_{r+2} - a_{r+1} - 6a_r = 4$ **MU - Dec. 14**

Solution :

The corresponding homogeneous recurrence relation of the given recurrence relation is given by

$$a_{r+2} - a_{r+1} - 6\,a_r = 0$$

The characteristic equation is

$$\alpha^2 - \alpha - 6 = 0$$

$$\therefore \quad \alpha = 3, -2$$

Therefore, the homogeneous solution of the given recurrence relation is

$$a_r^{(h)} = A_1\,(3)^r + A_2\,(-2)^r$$

To find the particular solution, we consider the term f(r) (right hand side of the given equation). Since f (r) is a constant, the particular solution will also be a constant P.

i.e. $\quad a_r = P, \quad$ for all r

$\Rightarrow \quad a_{r+1} = a_{r+2} = P$ (constant)

Substituting the values of a_r, a_{r+1}, and a_{r+2} in the given recurrence relation, we get,

$$P - P - 6\,P = 4$$

$$\therefore \quad P = -\frac{2}{3}$$

Hence, the particular solution is

$$a_r^{(p)} = -\frac{2}{3}$$

Thus the total solution or the general solution of the given recurrence relation is

$$a_r = a_r^{(h)} + a_r^{(p)}$$

$$\Rightarrow \quad a_r = A_1 (3)^r + A_2 (-2)^r - \frac{2}{3}$$

Example 2 : Find total solution of $a_{r+2} + 2a_{r+1} - 3a_r = 4$

Solution :

The corresponding homogeneous recurrence relation of the given recurrence relation is given by,

$$a_{r+2} + 2a_{r+1} - 3a_r = 0$$

The characteristic equation is

$$\alpha^2 + 2\alpha - 3 = 0$$

$$\therefore \quad \alpha = -3, 1$$

Therefore, the homogeneous solution of the given recurrence relation is

$$a_r^{(h)} = A_1 (-3)^r + A_2 (1)^r$$

To find the particular solution, we consider the term f (r) (right hand side of the given equation). Since f (r) is a constant, the particular solution will also be a constant P.

i.e. $\quad a_r = P$, for all r

$$\Rightarrow \quad a_{r+1} = a_{r+2} = P \text{ (constant)}$$

Substituting the values of a_r, a_{r+1} and a_{r+2} in the given recurrence relation, we get,

$$P + 2P - 3P = 4$$

$$\therefore \quad 0 = 4 \text{ is absurd}$$

Let $\quad a_r = P \cdot r$

Substituting in the given equation,

$$P(r+2) + 2P(r+1) - 3Pr = 4$$

$$\therefore \quad 4P = 4$$

$$\therefore \quad P = 1$$

Hence the particular solution is,

$$a_r^{(p)} = 1$$

Thus the total solution or the general solution of the given recurrence relation is,

$$a_r = a_r^{(h)} + a_r^{(p)}$$

$$\Rightarrow \quad a_r = A_1 (-3)^r + A_2(1)^r + r$$

Example 3 : Consider the difference equation

$$a_r - 5a_{r-1} + 6a_{r-2} = 1$$

Find total solution.

Solution :

The corresponding homogeneous recurrence relation of the given recurrence relation is given by

$$a_r - 5a_{r-1} + 6a_{r-2} = 0$$

The characteristic equation is

$$\alpha^2 - 5\alpha + 6 = 0$$

$$\therefore \quad \alpha = 3, 2$$

Therefore, the homogeneous solution of the given recurrence relation is

$$a_r^{(h)} = A_1 (3)^r + A_2 (2)^r$$

To find the particular solution, we consider the term f (r). Since f (r) is a constant, the particular solution will also be a constant P.

i.e. $\quad a_r = P$, for all r.

$$\Rightarrow \quad a_{r-1} = a_{r-2} = P \text{ (constant)}$$

Substituting the values of a_r, a_{r-1} and a_{r-2} in the given recurrence relation, we get

$$P - 5P + 6P = 1$$

$$\therefore \quad 2P = 1$$

$$\therefore \quad P = \frac{1}{2}$$

Hence, the particular solution is

$$a_r^{(p)} = \frac{1}{2}$$

Thus the total solution of the given recurrence relation is,

$$a_r = a_r^{(h)} + a_r^{(p)}$$

$$\therefore \quad a_r = A_1 (3)^r + A_2 (2)^r + \frac{1}{2}$$

Example 4 : Solve the recurrence relation $a_{n+2} - 5a_{n+1} + 6a_n = 2$ with initial conditions $a_0 = 1$, $a_1 = -1$.

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Solution :

The corresponding homogeneous recurrence relation of the given recurrence is given by

$$a_{n+2} - 5a_{n+1} + 6a_n = 0$$

The characteristics equation is

$$\alpha^2 - 5\alpha + 6 = 0$$

$$\alpha = 3, 2$$

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Therefore, the homogeneous solution of the given recurrence relation, we get

$$a_n^{(h)} = A_1(3)^n + A_2(2)^n$$

To find particular solution, we consider the term f(n). Since f(n) is a constant, the particular solution will also be a constant P.

i.e. $a_n = P$, for all n

$\Rightarrow$ $a_{n+1} = a_{n+2} = P$ (constant)

Substituting the values of a_n, a_{n+1}, a_{n+2} in the given recurrence relation, we get

$$P - 5P + 6P = 2$$

$$2P = 2$$

$$P = 1$$

Hence, the particular solution is,

$$a_n^{(p)} = 1$$

Thus the total solution of the given recurrence relation is,

$$a_n = a_n^{(h)} + a_n^{(p)}$$

$$a_n = A1(3)^n + A2(2)^n + 1$$

To determine A1 and A2, use given initial conditions, i.e. $a_0 = 1$, $a_1 = -1$.

Example 5 : Solve $a_r - a_{r-1} - 6a_{r-2} = -30$ given $a_0 = 20$, $a_1 = -5$

Solution :

The corresponding homogeneous recurrence relation of the given recurrence relation is given by,

$$a_r - a_{r-1} - 6a_{r-2} = 0$$

The characteristic equation is

$$\alpha^2 - \alpha - 6 = 0$$

$$\alpha = -2, 3$$

Therefore, the homogeneous solution of the given recurrence relation is

$$a_r^{(h)} = A_1(-2)^r + A_2(3)^r$$

To find the particular solution, we consider the term f (r) (right hand side of the given equation). Since f (r) is a constant, the particular solution will also be a constant P.

i.e. $a_r = P$, for all r

$\Rightarrow$ $a_{r-1} = a_{r-2} = P$ (constant)

Substituting the values of a_r, a_{r-1} and a_{r-2} in the given recurrence relation,

We get,

$$P - P - 6P = -30$$

$\therefore$ $P = 5$

Hence, the particular solution is

$$a_r^{(p)} = 5$$

Thus the total solution or the general solution of the given recurrence relation is

$$a_r = a_r^{(h)} + a_r^{(p)}$$

$$\Rightarrow \quad a_r = A_1(-2)^r + A_2(3)^r + 5$$

To determine A_1, and A_2, we use the initial conditions. Putting $r = 0$ in the above equation

We get, $\quad a_0 = A_1 + A_2 + 5$

Given $\quad a_0 = 20$

$$\Rightarrow \quad A_1 + A_2 = 15$$

Also using $\quad a_1 = -5$, we get,

$$-2A_1 + 3A_2 = -10$$

which on solution gives $A_1 = 11$, $A_2 = 4$

Hence the complete solution is

$$a_r = 11(-2)^r + 4(3)^r + 5$$

Example 6 : Find the General solution of

$$a_{r+2} - 4a_r = r$$

Solution :

The characteristic equation is

$$\alpha^2 - 4 = 0$$

$\therefore \quad \alpha = \pm 2$ are roots

Thus homogeneous solution is,

$$a_r^{(h)} = A_1(2)^r + A_2(-2)^r$$

The General form of particular solution is

$$P_0 r + P_1$$

Substitute $a_r = P_0 r + P_1$ in the given recurrence relation, we get

$$[P_0(r+2) + P_1] - 4[P_0 r + P_1] = r$$

On solving, we get

$$P_0 = -\frac{1}{3},\ P_1 = -\frac{2}{9}$$

Therefore particular solution is

$$a_r^{(p)} = -\frac{1}{3}r - \frac{2}{9}$$

and the general solution is

$$a_r = a_r^{(h)} + a_r^{(p)}$$

$$= A_1(2)^r + A_2(-2)^r - \frac{1}{3}r - \frac{2}{9}$$

Example 7 : Find the general solution of $a_r + 5a_{r-1} + 6a_{r-2} = 3r^2$

Solution :

The characteristic equation of the corresponding equation is

$$\alpha^2 + 5\alpha + 6 = 0$$

$$\therefore \quad \alpha = -2, -3$$

Thus, the homogeneous solution is

$$a_r^{(h)} = A_1(-2)^r + A_2(-3)^r$$

For a particular solution, since f(r) (right hand side) of the given recurrence relation is a quadratic polynomial, the form of the particular solution will be $P_0 + P_1 r + P_2 r^2$

i.e.

$$a_r = P_0 + P_1 r + P_2 r^2$$

$$a_{r-1} = P_0 + P_1(r-1) + P_2(r-1)^2$$

$$a_{r-2} = P_0 + P_1(r-2) + P_2(r-2)^2$$

Substituting in the given recurrence relation, we get,

$$(P_0 + P_1 r + P_2 r^2) + 5[P_0 + P_1(r-1) + P_2(r-1)^2] + 6[P_0 + P_1(r-2) + P_2(r-2)^2] = 3r^2$$

Equating the coefficients of powers of r, we get,

$$12 P_2 = 3$$

$$34 P_2 - 12 P_1 = 0$$

$$12 P_0 - 17 P_1 + 29 P_2 = 0$$

Solving, we get,

$$P_0 = \frac{115}{288}, \; P_1 = \frac{17}{24}, \; P_2 = \frac{1}{4}$$

The particular solution is

$$a_r^{(p)} = \frac{115}{288} + \frac{17}{24} r + \frac{1}{4} r^2$$

Thus the solution is

$$a_r = A_1(-2)^r + A_2(-3)^r + \frac{115}{288} + \frac{17}{24} r + \frac{1}{4} r^2$$

Example 8 : Find total solution of equation

$$a_r + 5a_{r-1} + 6a_{r-2} = 3r^2 - 2r + 1$$

Solution :

The characteristic equation is

$$\alpha^2 + 5\alpha + 6 = 0$$

$$\therefore \quad \alpha = -2, -3$$

Thus, the homogeneous solution is

$$a_r^{(h)} = A_1(-2)^r + A_2(-3)^r$$

For a particular solution, since f (r) (right hand side) of the given recurrence relation is quadratic polynomial the form of the particular solution will be $P_0 + P_1 r + P_2 r^2$

i.e.

$$a_r = P_0 + P_1 r + P_2 r^2$$

$$a_{r-1} = P_0 + P_1 (r-1) + P_2 (r-1)^2$$

$$a_{r-2} = P_0 + P_1 (r-2) + P_2 (r-2)^2$$

Substituting in the given recurrence relation, we get

$$(P_0 + P_1 r + P_2 r^2) + 5 [P_0 + P_1 (r-1) + P_2 (r-1)^2] + 6 [P_0 + P_1 (r-2) + P_2 (r-2)^2] = 3 r^2 - 2 r + 1$$

Equating the coefficients of powers of r, we get,

$$12 P_2 = 3$$

$$34 P_2 - 12 P_1 = 2$$

$$12 P_0 - 17 P_1 + 29 P_2 = 1$$

Which yield $P_0 = \frac{71}{288}$, $P_1 = \frac{13}{24}$, $P_2 = \frac{1}{4}$

Therefore, the particular solution is

$$a_r^{(p)} = \frac{71}{288} + \frac{13}{24} r + \frac{1}{4} r^2$$

Thus the solution is

$$a_r = A_1 (-2)^r + A_2 (-3)^r + \frac{71}{288} + \frac{13}{24} r + \frac{1}{4} r^2$$

Example 9 : Solve $a_r - 7a_{r-1} + 10a_{r-2} = 6 + 8r$ with $a_0 = 1$, $a_1 = 2$

Solution :

The corresponding homogeneous equation is

$$a_r - 7a_{r-1} + 10a_{r-2} = 0$$

Thus the characteristic equation is

$$\alpha^2 - 7\alpha + 10 = 0$$

$$\therefore \quad \alpha = 2, 5$$

The homogeneous solution is

$$a_r^{(h)} = A_1 (2)^r + A_2 (5)^r$$

For particular solution, since f (r) (right hand side) is a linear polynomial, therefore, the particular solution will be of the form $(P_0 + P_1 r)$

i.e.

$$a_r = P_0 + P_1 r$$

$$a_{r-1} = P_0 + P_1 (r-1)$$

$$a_{r-2} = P_0 + P_1 (r-2)$$

Substituting these values in the given recurrence relation, we get,

$$(P_0 + P_1 r) - 7 [P_0 + P_1 (r-1)] + 10 [P_0 + P_1 (r-2)] = 6 + 8r$$

$$\Rightarrow \quad (4 P_0 - 13 P_1) + (4P_1) r = 6 + 8r$$

On comparing the coefficients of polynomials, we get,

$$4 P_0 - 13P_1 = 6$$

and $$4P_1 = 8$$

Which on solving give, $P_0 = 8$ and $P_1 = 2$

Hence, the particular solution is

$$a_r^{(p)} = 8 + 2r$$

Thus the general solution is

$$a_r = a_r^{(h)} + a_r^{(p)}$$

$$\therefore \quad a_r = A_1 (2)^r + A_2 (5)^r + 8 + 2r$$

Using the initial conditions $a_0 = 1$ and $a_1 = 2$,

We get, $A_1 = -9$ and $A_2 = 2$

Therefore, $a_r = -9 (2)^r + 2(5)^r + 8 + 2r$

Example 10 : Find the complete solution of the recurrence relation

$a_n + 2 a_{n-1} = n + 3$ for $n \geq 1$ and with $a_0 = 3$ **MU - Dec. 07**

Solution :

The characteristic equation is

$$\alpha + 2 = 0$$

$$\therefore \quad \alpha = -2$$

Hence, Homogeneous solution is

$$a_n^{(h)} = A_1 (-2)^n$$

For particular solution, since f (r) (Right hand side) is a linear polynomial, therefore, the particular solution will be of the form $(P_o + P_{1n})$.

i.e. $$a_n = P_0 + P_{1n}$$

$$a_{n-1} = P_0 + P_1 (n-1)$$

Substituting these values in the given recurrence relation, we get,

$$(P_0 + P_{1n}) + 2 [P_0 + P_1 (n-1)] = n + 3$$

$$P_0 + P_1 n + 2P_0 + 2P_1 n - 2P_1 = n + 3$$

$$(P_0 + 2P_0 - 2P_1) + n (P_1 + 2P_1) = n + 3$$

$$(3 P_0 - 2P_1) + n (3P_1) = n + 3$$

On comparing the coefficients of polynomials, we get

and $$3P_0 - 2P_1 = 3$$

$$3P_1 = 1$$

Which on solving give $P_1 = 1/3$, $P_0 = 11/9$

Hence, the particular solution is

$$a_n^{(p)} = \frac{11}{9} + \frac{1}{3} n$$

Thus the general solution is

$$a_n = a_n^{(h)} + a_n^{(p)}$$

$$\therefore \quad a_n = A_1 (-2)^n + \frac{11}{9} + \frac{1}{3}$$

Using the initial conditions $a_0 = 3$ we get

$$a_0 = A_1 + \frac{11}{9}$$

Given $$a_0 = 3$$

$$3 = A_1 + \frac{11}{9}$$

$$3 - \frac{11}{9} = A_1$$

$$A_1 = 1.78$$

Therefore

$$a_n = 1.78 (-2)^n + \frac{11}{9} + \frac{1}{3} n$$

Example 11 : Find the general solution of

$$a_r + 5a_{r-1} + 6a_{r-2} = 42 \cdot 4^r$$

MU - May 06

Solution :

The characteristic equation is

$$\alpha^2 + 5\alpha + 6 = 0$$

$$\therefore \quad \alpha = -2, -3$$

Thus, the homogeneous solution is

$$a_r^{(h)} = A_1 (-2)^r + A_2 (-3)^r$$

For a particular solution, since f (r) (right hand side) of the given recurrence relation is a $42 \cdot 4^r$, the form of the particular solution will be, $P4^r$

Substituting in the given recurrence relation, we get,

$$P4^r + 5P4^{r-1} + 6P4^{r-2} = 42 \cdot 4^r$$

Which simplifies to

$$\frac{21}{8} P4^r = 42 \cdot 4^r$$

$$\therefore \quad P = 16$$

Therefore, the particular solution is

$$a_r^{(p)} = 16 \cdot 4^r$$

Thus the general solution is

$$a_r = a_r^{(h)} + a_r^{(p)}$$

$$\therefore \quad a_r = A_1 (-2)^r + A_2 (-3)^r + 16 \cdot 4^r$$

Example 12 : Find all solutions of the recurrence relation $a_n = 5a_{n-1} - 6a_{n-2} + 7^n$

MU - May 06, Dec. 06, Dec. 10, May 11, May 12, Dec. 13, Dec. 15

Solution :

This is nonhomogeneous recurrence relation. Solution of its associated homogeneous recurrence relation.

Given : $a_n = 5a_{n-1} - 6a_{n-2} + 7^n$

$$\therefore \quad a_n - 5a_{n-1} + 6a_{n-2} = 7^n$$

The characteristic equation is

$$\alpha^2 - 5\alpha + 6 = 0$$

$$\therefore \quad \alpha = 3, 2$$

The homogeneous solution is

$$a_n^{(h)} = A_1 (3)^n + A_2 (2)^n$$

Now, consider the form of particular solution.

Here f (n) is 7^n, the form of particular solution is, $P7^n$.

Subsituting in the given recurrence relation, we get,

$$P7^n - 5P7^{n-1} + 6P7^{n-2} = 7^n$$

which simplifies to

$$P = 2.44$$

Therefore, the particular solution is,

$$a_n^{(P)} = 2.447^n$$

Thus the general solution is

$$a_n = A_1 (3)^n + A_2 (2)^n + 2.447^n$$

Example 13 : Solve $a_r + a_{r-1} = 3r\,2^r$

Solution :

The characteristic equation is

$$\alpha + 1 = 0$$

$$\therefore \quad \alpha = -1$$

So, homogeneous solution is

$$a_r^{(h)} = A_1 (-1)^r$$

Here, the form of the particular solution is $(P_0 + P_1 r)\,2^r$

Substituting in the given recurrence relation,

We get,

$(P_0 + P_1 r)\,2^r + [P_0 + P_1 (r-1)]\,2^{r-1} = 3r\,2^r$

which simplifies to

$$\frac{3}{2} P_1 r\,2^r + \left(-\frac{1}{2}P_1 + \frac{3}{2}P_0\right) 2^r = 3r\,2^r$$

Comparing the two sides, we obtain the equations

$$\frac{3}{2} P_1 = 3$$

and $$-\frac{1}{2} P_1 + \frac{3}{2} P_0 = 0$$

Thus $$P_0 = \frac{2}{3},\ P_1 = 2$$

The particular solution is

$$a_r^{(p)} = \left(\frac{2}{3} + 2r\right) 2^r$$

Hence the general solution is

$$a_r = A_1 (-1)^r + \left(\frac{2}{3} + 2r\right) 2^r$$

Example 14 : Find the general solution of

$$a_r - 3a_{r-1} - 4a_{r-2} = 4^r$$

Solution :

The characteristic equation is

$$\alpha^2 - 3\alpha - 4 = 0$$

$$\therefore \quad \alpha = -1, 4$$

The homogeneous solution is

$$a_r^{(h)} = A_1 (-1)^r + A_2 (4)^r$$

Now, consider the form of particular solution. Here f (r) is 4^r, but the form of particular solution is not $P4^r$ because 4 is the characteristic root also. Since the characteristic root 4 is repeated only once, the form of the particular solution is $P\, r\, 4^r$

Substitute $a_r = P\, r\, 4^r$ in the given recurrence relation, we get,

$$P\, r\, 4^r - 3P\,(r-1)\, 4^{r-1} - 4P\,(r-2)\, 4^{r-2} = 4^r$$

$$\therefore \quad P\, r\, 4^r - \frac{3}{4}\, P\,(r-1)\, 4^r - \frac{4}{4^2}\, P\,(r-2)\, 4^r = 4^r$$

Compare the coefficients of 4^r,

$$\frac{3P}{4} + \frac{2P}{4} = 1$$

$$\therefore \quad P = \frac{4}{5} = 0.8$$

Therefore particular solution is

$$a_r^{(p)} = 0.8\, r\, 4^r$$

and the general solution is

$$a_r = A_1\,(-1)^r + A_2\, 4^r + 0.8\, r\, 4^r$$

Example 15 : Solve $a_r - 2\, a_{r-1} = 3 \cdot 2^r$

Solution :

The characteristic equation is

$$\alpha - 2 = 0$$

$$\therefore \quad \alpha = 2$$

Hence, Homogeneous solution is

$$a_r^{(h)} = A_1\,(2)^r$$

Because 2 is a characteristic root (of multiplicity 1), the general form of the particular solution is $P\, r\, 2^r$

Substitute $a_r = P\, r\, 2^r$ in the given recurrence relation, we get

$$P\, r\, 2^r - 2P\,(r-1)\, 2^{r-1} = 3 \cdot 2^r$$

that is, $$P\, 2^r = 3 \cdot 2^r$$

$$\therefore \quad P = 3$$

Thus, the particular solution is

$$a_r^{(p)} = 3r\, 2^r$$

and the total solution is

$$a_r = A_1\,(2)^r + 3r2^r$$

Example 16 : Solve : $a_r - 4a_{r-1} + 4a_{r-2} = (r+1)\,2^r$

Solution :

The characteristic equation is

$$\alpha^2 - 4\alpha + 4 = 0$$

$$\therefore \quad (\alpha - 2)^2 = 2$$

$$\therefore \quad \alpha = 2, 2$$

So the homogeneous solution is

$$a_r^{(h)} = (A_1 r + A_2)\,2^r$$

For particular solution, f(r) is of the form $(r + 1)\,2^r$ and 2 is the characteristic root with multiplicity 2, therefore, the particular solution will be of the form

$$(P_0 + P_1 r)\,r^2\,2^r,$$

Substitute $a_r = (P_0 + P_1 r)\,r^2\,2^r$ into the given recurrence relation we get,

$$(P_0 + P_1 r)\,r^2\,2^r - 4\,[\,\{P_0 + P_1 (r-1)\}\,(r-1)^2\,2^{r-1}\,] + 4\,[\,\{P_0 + P_1 (r-2)\}\,(r-2)^2\,2^{r-2}\,]$$

$$= (r+1)\,2^r$$

On simplification, we get,

$$6P_1\,r\,2^r = r\,2^r$$

and $$(-6P_1 + 2P_0)\,2^r = 2^r$$

Which gives $$P_1 = \frac{1}{6}, \quad P_0 = 1$$

Hence, the particular solution is

$$a_r^{(p)} = \left(1 + \frac{1}{6}r\right) r^2\,2^r$$

Thus the complete solution is

$$a_r = (A_1 + A_2 r)\,2^r + \left(1 + \frac{1}{6}r\right) r^2\,2^r$$

Example 17 : Solve the following recurrence relation :

$a_n - 5a_{n-1} + 6a_{n-2} = 2^n$ with initial conditions $a_0 = -1$ and $a_1 = 1$ **MU - Dec. 15**

Solution :

This is nonhomogeneous recurrence relation. Solution of its associated homogeneous recurrence relation.

Given : $$a_n = 5a_{n-1} - 6a_{n-2} + 7^n$$

$$\therefore \quad a_n - 5a_{n-1} + 6a_{n-2} = 7^n$$

The characteristic equation is

$$\alpha^2 - 5\alpha + 6 = 0$$

$\therefore \quad \alpha = 3, 2$

The homogeneous solution is

$$a_n^{(h)} = A_1 (3)^n + A_2 (2)^n$$

Now, consider the form of particular solution. Here f (n) is 7^n, the form of particular solution is, $P7^n$.

Substituting in the given recurrence relation, we get,

$$P7^n - 5P7^{n-1} + 6P7^{n-2} = 7^n$$

which simplifies to

$$P = 2.44$$

Therefore, the particular solution is,

$$a_n^{(P)} = 2.447^n$$

Thus the general solution is

$$a_n = A_1 (3)^n + A_2 (2)^n + 2.447^n$$

Example 18 : Solve the following recurrence relation: $a_n - 7a_{n-1} + 10 a_{n-2} = 0$ with initial condition $a_0 = 1$, $a_1 = 6$ **MU - May 16, 8 Marks**

Solution :

Given : $a_n - 7a_{n-1} + 10a_{n-2} = 0$

Initial condition : $a_0 = 1, \quad a_1 = 6$

The characteristic equation is,

$$\alpha^2 - 7\alpha + 10 = 0$$

$$\therefore \quad (\alpha - 5)(\alpha - 2) = 0$$

Or $\quad \therefore \quad \alpha = 5, 2$

Which are the characteristic roots of the equation.

$\therefore$ The solution of the given recurrence relation is,

$$a_n = A_1 (5)^n + A_2 (2)^n$$

Where, A_1, A_2 are constants.

To find A_1 and A_2,

Putting n = 0, we get,

$$a_0 = A_1 + A_2$$

$$1 = A_1 + A_2 \quad \text{...(1)}$$

Putting n = 1, we get,

$$a_1 = A_1 5^1 + A_2 2^1$$

$$6 = 5A_1 + 2A_2 \quad \text{...(2)}$$

Solving two equations, we get,

$$A_1 = \frac{4}{3} \qquad A_2 = \frac{-1}{3}$$

∴ Solution of given recurrence relation is,

$$a_n = \frac{4}{3}(5)^n - \frac{1}{3}(2)^n$$

5.8 Method of Generating Functions :

5.8.1 Solution of Recurrence Relation by the Method of Generating Functions :

Recurrence relations can also be solved by using the generating functions. For this, we directly determine the generating function of the numeric function from the given recurrence relation. Once the generating function is determined, an expression for the value of the numeric function can easily be obtained.

Closed form expression for some generating functions and their numeric functions (sequences) are given in the following Table 5.3.

Table 5.3 : Numeric functions for the closed form expression of generating functions

Generating Functions	Numeric Functions
$A(z) = \frac{1}{1-az}$	a^r
$A(z) = \frac{1}{(1-z)^2}$	$(r+1)$
$A(z) = \frac{1}{(1-az)^2}$	$(r+1)\,a^r$
$A(z) = \frac{z}{(1-z)^2}$	r
$A(z) = \frac{az}{(1-az)^2}$	ra^r
$A(z) = e^z$	$\frac{1}{n!}$
$A(z) = (1+z)^n$	$\begin{cases} {}^nC_r, & 0 \le r \le n \\ 0, & r > n \end{cases}$

Syllabus Topic : Recursive Function

5.8.2 Recursive Function :

Recursive function "builds" on itself. A recursive definition has two parts :

Definition of the smallest argument (usually f(0) or f(1)

Definition of f(n), given f(n – 1), f(n – 2), etc.

Example : Example of a recursively defined function

(1)
$$f(0) = 5$$
$$f(n) = f(n-1) + 2$$

we can calculate the values of this function

$$f(0) = 5$$
$$f(1) = f(0) + 2 = 5 + 2 = 7$$
$$f(2) = f(1) + 2 = 7 + 2 = 9$$
$$f(3) = f(2) + 2 = 9 + 2 = 11$$

....

This recursively defined function is equivalent to the explicitly defined function $f(n) = 2n + 5$. However, the recursive function is defined only for non-negative integers.

(2)
$$f(0) = 0$$
$$f(n) = f(n-1) + 2n + 1$$

The values of this function are :

$$f(0) = 0$$
$$f(1) = f(0) + (2)(1) - 1 = 0 + 2 - 1 = 1$$
$$f(2) = f(1) + (2)(2) - 1 = 1 + 4 - 1 = 4$$
$$f(3) = f(1) + (2)(3) - 1 = 4 + 6 - 1 = 9$$
$$f(4) = f(3) + (2)(4) - 1 = 9 + 8 - 1 = 16$$

....

(3)
$$f(0) = 1$$
$$f(n) = n \cdot f(n-1)$$

The values of this function are :

$$f(0) = 1$$
$$f(1) = 1\ f\ f(0) = 2\ f\ 1 = 1$$
$$f(2) = 2\ ff(1) = 2\ f\ 1 = 2$$
$$f(3) = 3\ f\ f(2) = 3\ f\ 2 = 6$$
$$f(4) = 4\ f\ f(3) = 4\ f\ 6 = 24$$
$$f(5) = 5\ f\ f(4) = 5\ f\ 24 = 120$$

This is the recursive definition of the factorial function, $F(n) = n!$

To define a function on the set of non negative integers

(1) Specify the value of the function at 0

(2) Give a rule for finding the function's value at $n + 1$ in terms of the function's value at integers $i \leq n$.

Example :

factorial function definition

$0! = 1$

$n! = n\,(n-1)\,!$

5.8.3 Exercise Set - 5 :

Example 1 : Solve $a_r - 3a_{r-1} = 2$, $r \geq 1$ with $a_0 = 1$ using the generating functions.

MU - Dec. 02, Dec. 13

Solution : Given recurrence relation is

$$a_r - 3a_{r-1} = 2$$

Multiplying both sides by z^r, we obtain

$$a_r z^r - 3a_{r-1} z^r = 2z^r$$

Since $r \geq 1$, summing for all r, we get,

$$\sum_{r=1}^{\infty} a_r z^r - 3\sum_{r=1}^{\infty} a_{r-1} z^r = 2\sum_{r=1}^{\infty} z^r$$

Consider the first term,

$$\sum_{r=1}^{\infty} a_r z^r = a_1 z + a_2 z^2 + a_3 z^3 + \ldots$$

Since the generating function

$$A(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \ldots$$

$$\Rightarrow \quad \sum_{r=1}^{\infty} a_r z^r = A(z) - a_0$$

For the second term,

$$\sum_{r=1}^{\infty} a_{r-1} z^r = z\sum_{r=1}^{\infty} a_{r-1} z^{r-1}$$

$$= z\,A(z)$$

Also the third term gives

$$\sum_{r=1}^{\infty} z^r = \frac{z}{(1-z)}$$

Hence we obtain

$$[A(z) - a_0] - 3z\,A(z) = \frac{2z}{1-z}$$

or $$(1-3z)\,A(z) = \frac{2z}{1-z} + a_0$$

But, $a_0 = 1$