# Q.02 Find area common to the circles $x^2+y^2-4y=0$ and $x^2+y^2-4x-4y+4=0$

Soln:- $x^2+y^2-4y=0 \Rightarrow x^2+(y-2)^2=4$ ie circle with centre $(0,2)$ and radius 2

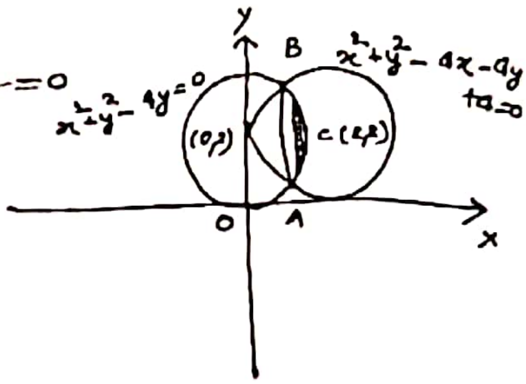
$x^2+y^2-4x-4y+4=0 \Rightarrow (x-2)^2+(y-2)^2=4$ ie circle with centre $(2,2)$ and radius 2.

By subtraction

$x^2+y^2-4y-x^2-y^2+4x+4y-4=0$

$\Rightarrow 4x-4=0$

$\Rightarrow x=1$




~~x=1~~ $\Rightarrow$ $1+y^2-4y=0$

Circle intersects where $x=1$

Consider a strip $\parallel^{al}$ to $y$-axis

$x=1$ to $x=2$

$y=\dfrac{4\pm\sqrt{16-4x^2}}{2}$ ie $y=2-\sqrt{4-x^2}$ to $y=2+\sqrt{4-x^2}$

Area = 2 x area ABC

$$= 2\int_1^2\int_{2-\sqrt{4-x^2}}^{2+\sqrt{4-x^2}} dy\,dx = 2\int_1^2 [y]_{2-\sqrt{4-x^2}}^{2+\sqrt{4-x^2}} dx$$

$$= 2\int_1^2 [2+\sqrt{4-x^2}-2+\sqrt{4-x^2}]\,dx = 2\int_1^2 2.\sqrt{4-x^2}\,dx$$

$$= 4\left[\frac{x}{2}\sqrt{4-x^2}+\frac{4}{2}\sin^{-1}\frac{x}{2}\right]_1^2 = 4\left\{2.\frac{\pi}{2}-\left(\frac{\sqrt{3}}{2}+\frac{2\pi}{6}\right)\right\}$$

$$= 4\left(\frac{2\pi}{3}-\frac{\sqrt{3}}{2}\right)$$

$\left\{ \int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\right.$



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Using double integration, find the area between parabolas $x^2 = 4ay$ and $x^2 = -4a(y-2a)$ ⑤

Sol$^n$ The two curves are parabolas. They intersect where

$$4ay = -4a(y-2a)$$

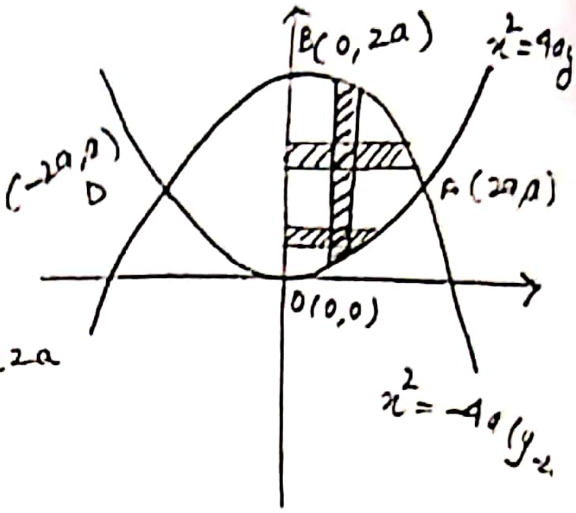
$$4ay + 4ay = 8a^2$$

$$8ay = 8a^2$$

$$y = a$$

$\therefore \quad x^2 = 4a^2 \Rightarrow x = \pm 2a$

$\therefore \quad (2a, a) \;\&\; (-2a, a)$




$\therefore$ Required area $= 2$ Area OABO

Consider a strip parallel to y-axis

$$y = \frac{x^2}{4a} \quad \text{to} \quad x^2 = -4ay + 8a^2$$

$$y = \frac{x^2}{4a} \quad \text{to} \quad y = \frac{8a^2}{4a} - \frac{x^2}{4a} = 2a - \frac{x^2}{4a}$$

and $x = 0$ to $x = 2a$

$$\therefore \text{Area} = 2\int_0^{2a}\int_{x^2/4a}^{2a - x^2/4a} dy\,dx$$

$$= 2\int_0^{2a} [y]_{x^2/4a}^{2a - x^2/4a}$$



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$$= 2\int_0^{2a}\left(2a - \frac{x^2}{4a} - \frac{x^2}{4a}\right)dx$$

$$= 2\int_0^{2a}\left(2a - \frac{2x^2}{4a}\right)dx$$

$$= 4\left[\cancel{\frac{a^2}{2}} - \frac{x^3}{12a}\right]_0^{2a} = 2\left[2ax - \frac{2x^3}{4 \cdot 3a}\right]_0^{2a}$$

$$= 4\left[\cancel{2a}\right.$$

$$= 4\left[ax - \frac{x^3}{12a}\right]_0^{2a}$$

$$= 4\left[2a^2 - \frac{8a^3}{12a}\right]$$

$$= 4\left[2a^2 - \frac{2a^2}{3}\right]$$

$$= 4\left[\frac{4a^2}{3}\right]$$

$$= \frac{16a^2}{3}$$



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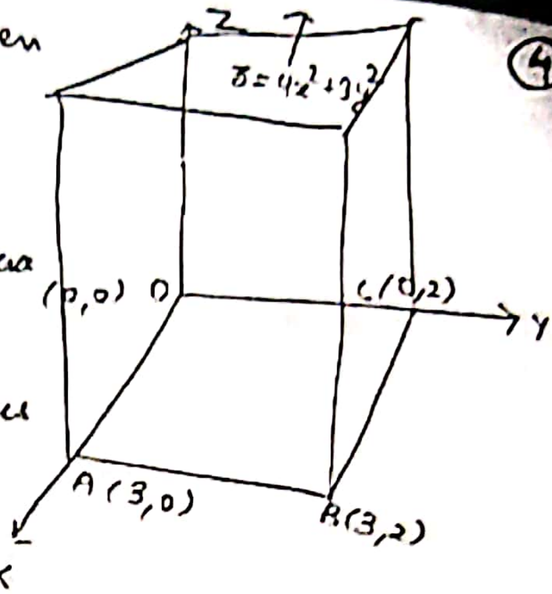
Q.3 Find by double integral, the volume of the region benath $z = 4x^2 + 9y^2$ and above the rectangle (0, 0) (3, 0) (3, 2) (0, 2) in the X – Y plane.

∴ surface of the given ... is

$z = 4x^2 + 9y^2$

The projection R of the surface on the XY-plane into the square itself with coordinates

O (0,0) A (3,0) B (3,2) C (0,2)




∴ $V = \iint_R z \, dx dy$

$$\Rightarrow V = \int_0^3 \int_0^2 (4x^2 + 9y^2) \, dx \, dy$$

$$= \int_0^3 \left[ 4x^2 y + 3y^3 \right]_0^2 dx$$

$$= \int_0^3 \left[ 8x^2 + 24 \right] dx$$

$$= \left[ \frac{8x^3}{3} + 24x \right]_0^3$$

$$= \frac{8(3)^3}{3} + 24(3)$$

$$= 72 + 72$$

$= 144$ cubic units.



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Sol

**Q.1** Change to polar coordinates and evaluate -

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} (x^2+y^2)\,dx\,dy.$$

Sol$^n$ Limits are

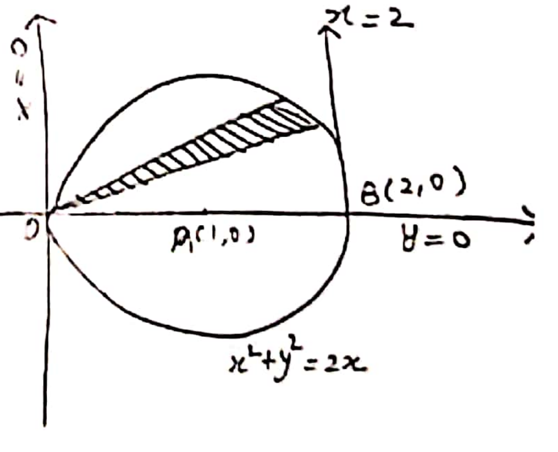
$y=0$ to $y=\sqrt{2x-x^2}$ and $x=0$ to $x=2$

$$x^2+y^2=2x$$

$$\Rightarrow (x-1)^2+y^2=1$$

ie circle with center $(1,0)$ and radius 1.

Region of integration is OABO




If we put

$x = r\cos\theta$, $y = r\sin\theta$

Then $y=\sqrt{2x-x^2}$ becomes.

$$x^2+y^2=2x$$

$$= 2\left[\frac{3}{2}+5\sin^{-1}\frac{3}{\sqrt{10}}\right]$$



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③

$r^2 = 2r\cos\theta$

$\Rightarrow \boxed{r = 2\cos\theta}$

$\therefore$ $r = 0$ to $r = 2\cos\theta$.

& $\theta = 0$ to $\theta = \frac{\pi}{2}$.

$$\therefore I = \int_0^{\pi/2} \int_0^{2\cos\theta} r^2 \cdot r \, dr \, d\theta$$

$$= \int_0^{\pi/2} \left[\frac{r^4}{4}\right]_0^{2\cos\theta} d\theta$$

$$= \frac{1}{4} \int_0^{\pi/2} 2^4 \cdot \cos^4\theta \, d\theta$$

$$= \frac{16}{4} \int_0^{\pi/2} \cos^4\theta \, d\theta.$$

$m = 0 \quad n = 4$

$$= \cancel{\frac{16}{4}}^{2} \times \frac{1}{\cancel{2}} \frac{\Gamma(\frac{1}{2})\Gamma(\frac{5}{2})}{\Gamma(3)}$$

$$= \frac{2 \cdot \sqrt{\pi} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi}}{2!}$$

$$= \frac{3\pi}{4}.$$

$$\left[ \int_0^{\pi/2} \sin^m\theta \cos^n\theta \, d\theta = \frac{1}{2} \beta\left(\frac{m+1}{2}, \frac{n+1}{2}\right) = \frac{1}{2} \frac{\Gamma(\frac{m+1}{2})\Gamma(\frac{n+1}{2})}{\Gamma(\frac{m+n+2}{2})} \right.$$



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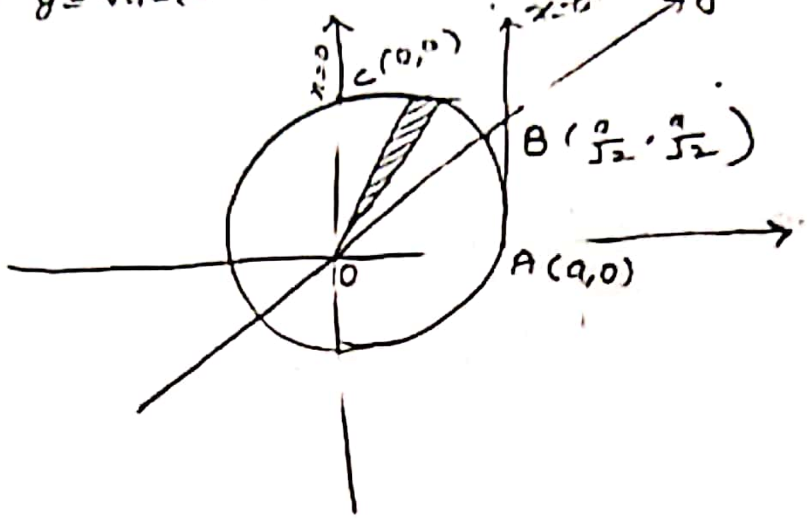
(4)

Change to polar coordinates and evaluate
$$\int_0^a \int_x^{\sqrt{a^2-x^2}} y\sqrt{x^2+y^2}\,dy\,dx$$

Sol$^n$ limits are
$y=x$ to $y=\sqrt{a^2-x^2}$ and $x=0$ to $x=a$




Point of intersection
$$x^2+y^2=a^2$$
$$2x^2=a^2$$
$$x^2=a^2/2$$
$$x=\pm\sqrt{\tfrac{a}{}}\ \pm\frac{a}{\sqrt{2}}$$

$B(\frac{a}{\sqrt{2}}, \frac{a}{\sqrt{2}})$

region of integration is ~~OABC~~ O. OBCO

$x = r\cos\theta \quad y = r\sin\theta$

$x^2+y^2=a^2$
$r^2=a^2$
$r=a$

$r=0$ to $r=a$

$\theta =$ ~~$\pi/4$~~ $0$ to $\theta=\pi/2$, $dy\,dx = r\,dr\,d\theta$

$$\therefore I = \int_{\pi/4}^{\pi/2}\int_0^a r\sin\theta \cdot r \cdot r\,dr\,d\theta .$$

$$I = \int_{0}^{\pi/2} \sin\theta \cdot \left[ \frac{r^4}{4} \right]_0^a d\theta$$

$$= \frac{1}{4} a^4 \int_{0}^{\pi/2} \sin\theta \, d\theta$$

$$= \frac{1}{4} a^4 \left[ -\cos\theta \right]_0^{\pi/2}$$

$$= -\frac{1}{4} a^4 \left[ \cos\frac{\pi}{2} - \cos 0 \right]$$

$$= -\frac{1}{4} a^4 \left[ 0 - 1 \right]$$

$$= \frac{a^4}{4}$$

...ordinates an



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## Triple Integration

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$$\int_{x=a}^{b}\int_{y=\phi_1(x)}^{y=\phi_2(x)}\int_{z=f_1(x,y)}^{f_2(x,y)} f(x,y,z)\,dx\,dy\,dz$$

Q.01. Evaluate $\int_0^2\int_1^2\int_0^{yz} xyz\,dx\,dy\,dz$ May 2016 MU:06

Soln:-

$$I = \int_0^2\int_1^2 zy\left[\frac{x^2}{2}\right]_0^{yz} dy\,dz = \frac{1}{2}\int_0^2\int_1^2 zy\,(y^2z^2-0)\,dy\,dz$$

$$= \frac{1}{2}\int_0^2\int_1^2 y^3z^3\,dy\,dz = \frac{1}{2}\int_0^2 z^3\left[\frac{y^4}{4}\right]_1^2 dz = \frac{1}{8}\int_0^2 15z^3\,dz$$

$$= \frac{15}{8}\left[\frac{z^4}{4}\right]_0^2 = \frac{15}{8}\left(\frac{16}{4}-0\right) = \frac{15}{2}.$$



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6. Evaluate $\int_1^e \int_1^{\log y} \int_1^{e^x} \log z \, dz\, dx\, dy$

$I = \int_1^e \int_1^{\log y} [z \log z - z]_1^{e^x} dx\, dy$

$\int_1^e \int_1^{\log y} (xe^x - e^x + 1)\, dx\, dy$

$\int_1^e \int_1^{\log y} [(x-1)e^x - e^x + x]_1^{\log y} dy$

$\int_1^e [(\log y - 1)y - y + \log y - 0 + e - 1]\, dy$

$\int_1^e [\log y\,(y+1) - 2y + e - 1]\, dy$

$= \int_1^e \left[ \left(\frac{y^2}{2} + y\right)\log y - \int \frac{1}{y}\left(\frac{y^2}{2}+y\right)dy - y^2 + (e-1)y \right]_1^e$

$= \left[ \left(\frac{y^2}{2} + y\right)\log y - \frac{y^2}{4} - y - y^2 + (e-1)y \right]_1^e$

$= \left(\frac{e^2}{2} + e\right)\log e - \frac{e^2}{4} - e - e^2 + (e-1)e - 0 + \frac{1}{4} + 1 + 1 -$

$= \frac{e^2}{4} - 2e + \frac{13}{4}$. //

# Q.06. Evaluate $\int_0^{\log 2}\int_0^{x}\int_0^{x+y} e^{x+y+z}\, dz\,dy\,dx$ A

Sol$^n$:-

$$I = \int_0^{\log 2}\int_0^{x} e^{x+y}\left[e^{z}\right]_0^{x+y} dx\,dy$$

$$= \int_0^{\log 2}\int_0^{x} e^{x+y}\left(e^{x+y}-1\right) dx\,dy = \int_0^{\log 2}\int_0^{x}\left[e^{2(x+y)} - e^{x+y}\right] dy\,dx$$

$$= \int_0^{\log 2}\left[e^{2x}\cdot\frac{e^{2y}}{2} - e^{x}e^{y}\right]_0^{x} dx = \int_0^{\log 2}\left[\frac{e^{4x}}{2} - e^{2x} - \frac{e^{2x}}{2} + e^{x}\right] dx$$

$$= \left[\frac{e^{4x}}{8} - \frac{e^{2x}}{2} - \frac{e^{2x}}{4} + e^{x}\right]_0^{\log 2} = \left[\frac{16}{8} - \frac{4}{2} - \frac{4}{4} + 2\right] - \left[\frac{1}{8} - \frac{1}{2} - \frac{1}{4} + 1\right] = \frac{5}{8}$$

# Q.07. $\int_{-1}^{1}\int_0^{z}\int_{x-z}^{x+z} (x+y+z)\, dz\,dx\,dy$ A

May 2014, MH:0

Sol$^n$: —

$$I = \int_{-1}^{1}\int_0^{z}\left[(x+z)\cdot y + \frac{y^2}{2}\right]_{x-z}^{x+z} dx\,dz$$

$$= \int_{-1}^{1}\int_0^{z}\left[(x+z)^2 + \frac{(x+z)^2}{2} - (x+z)(x-z) - \frac{(x-z)^2}{2}\right] dx\,dz$$

$$\left[\frac{3}{2}(x+z)^2 - x^2 + z^2 - (x-z)^2/2\right]$$

$$= \int_{-1}^{1}\left[\frac{3}{2}\frac{(x+z)^3}{3} - \frac{x^3}{3} + z^2 x - \frac{(x-z)^3}{6}\right]_0^{z} dz$$

$$= \int_{-1}^{1}\left(4z^3 - \frac{z^3}{3} + z^3 - \frac{z^3}{2} - \frac{z^3}{6}\right) dz = \int_{-1}^{1} 4z^3\, dz = \left[z^4\right]_{-1}^{1} = 0$$



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Q.17 Evaluate $\int_0^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_0^y z^2 \, dx\,dy\,dz$

Soln

$$I = \int_0^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \left[\frac{z^3}{3}\right]_0^y dx\,dy$$

$$= \frac{1}{3}\int_0^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} y^3 \, dx\,dy$$

$$= \frac{1}{3}\int_0^a \left[\frac{y^4}{4}\right]_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dx$$



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# Q.08. $\int_0^{\pi/2} \int_0^{a\sin\theta} \int_0^{(a^2-r^2)/a} r \, d\theta \, dr \, dz$

June 2014, 44 0
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Sol$^n$

$$I = \int_0^{\pi/2} \int_0^{a\sin\theta} [rz]_0^{a^2-r^2/a} \, d\theta \, dr$$

$$= \int_0^{\pi/2} \int_0^{a\sin\theta} (a^2 - r^2)\frac{r}{a} \, d\theta \, dr = \frac{1}{a}\int_0^{\pi/2} \left[\frac{a^2 r^2}{2} - \frac{r^4}{4}\right]_0^{a\sin\theta} d\theta$$

$$= \int_0^{\pi/2} \left(\frac{a^3}{2}\sin^2\theta - \frac{a^3}{4}\sin^4\theta\right) d\theta = \frac{a^3}{2}\cdot\frac{1}{2}\cdot\frac{\pi}{2} - \frac{a^3}{4}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{5a}{6.}$$



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Find the volume of the tetrahedron bounded by $x=0$, $y=0$, $z=0$ and $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ (5)

Sol$^n$ Put $x = au$ $\quad y = bv$ $\quad z = cz$

$dx = a\,du$ $\quad dy = b\,dv$ $\quad dz = c\,dz$

$$V = \iiint dx\,dy\,dz$$

$$= \iiint abc\,du\,dv\,dw$$

$u = 0$ to $u = 1$

$v = 0$ to $v = 1-u$

$w = 0$ to $w = 1-u-v$

$$\therefore V = \int_0^1 \int_0^{1-u} \int_0^{1-u-v} abc\,du\,dv\,dw$$

$$= abc \int_0^1 \int_0^{1-u} [1-u-v]\,du\,dv$$

$$= abc \int_0^1 \left[v - uv - \frac{v^2}{2}\right]_0^{1-u} du$$

$$= abc \int_0^1 \left[1-u-u(1-u) - \frac{(1-u)^2}{2}\right] du$$

$$= abc \int_0^1 \left[\frac{u^2}{2} - u + \frac{1}{2}\right] du$$

$$= abc \left[\frac{u^3}{6} - \frac{u^2}{2} + \frac{1}{2}u\right]_0^1$$

$$= abc\left[\frac{1}{6} - \frac{1}{2} + \frac{1}{2}\right]$$

$$= \frac{abc}{6}$$



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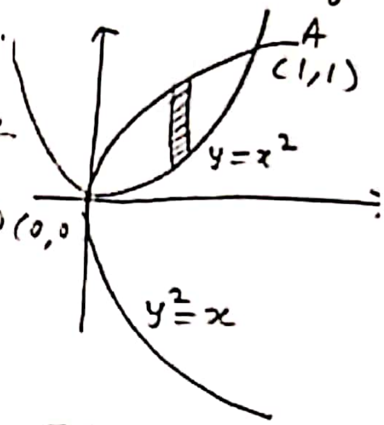
Q.1 Find the volume bounded by $y^2 = x$, $x^2 = y$ and the planes $z = 0$ and $x + y + z = 1$

Sol$^n$ The solid is bounded by parabolas $y^2 = x$ & $x^2 = y$ in the $xy$-plane which is its base and by the plane $x + y + z = 1$ at the top.




$$V = \iint_R z\,dx\,dy = \iint_R (1 - x - y)\,dx\,dy$$

Pt of intersection of parabola are $O(0,0)$ and $A(1,1)$.

$$\therefore V = \int_0^1 \int_{x^2}^{\sqrt{x}} (1 - x - y)\,dx\,dy.$$

On strip
$y = x^2$ to $y = \sqrt{x}$
$x = 0$ to $x = 1$

$$\Rightarrow V = \int_0^1 \left[ y - xy - \frac{y^2}{2} \right]_{x^2}^{\sqrt{x}} dx$$

$$\Rightarrow V = \int_0^1 \left[ \sqrt{x} - x^{3/2} - \frac{x}{2} - x^2 + x^3 + \frac{x^4}{2} \right] dx$$

$$\Rightarrow V = \left[ \frac{2x^{3/2}}{3} - \frac{2x^{5/2}}{2} - \frac{x^2}{4} + \frac{x^4}{4} + \frac{x^5}{10} \right]_0^1 = \frac{1}{30}$$



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Ex Find the volume of a sphere of radius a.

Sol$^n$ Sphere is $x^2+y^2+z^2=a^2$

$$V=\iiint dx\,dy\,dz.$$

Spherical coordinates

$$x=r\cos\phi\sin\theta$$

$$y=r\sin\phi\sin\theta$$

$$z=r\cos\theta$$

$$dx\,dy\,dz=r^2\sin\theta\,dr\,d\theta\,d\phi$$

In I octant $\quad r=0$ to $a$

$\theta=0$ to $\pi/2$

$\phi=0$ to $\pi/2$

$$\therefore\; V=8\int_0^a\int_0^{\pi/2}\int_0^{\pi/2} r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$=8\int_0^a r^2\,dr\cdot\int_0^{\pi/2}\sin\theta\,d\theta\int_0^{\pi/2}d\phi$$

$$=8\left[\frac{r^3}{3}\right]_0^a\cdot\left[-\cos\theta\right]_0^{\pi/2}\cdot\left[\phi\right]_0^{\pi/2}$$

$$=8\cdot\frac{a^3}{3}\cdot 1\cdot\frac{\pi}{2}$$

$$=\frac{4\pi a^3}{3}$$



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