
Vector
Space

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4.1 Vector Space, Linear independence of vectors, basis, dimension

VECTOR SPACE

Definition: Let V be a non-empty set of objects whose elements will be called as 'vectors' and let $\mathbb{R}$ be the set of real numbers whose elements will be called as 'scalars'. V is called as '*vector space*' over real field $\mathbb{R}$ if and only if the following conditions/axioms are satisfied

A1) Closure under vector addition :

$$\boxed{u + v \in V \quad \forall\, u, v \in V}$$

i.e. to every pair of elements u and v in V there corresponds a unique $u + v$ in V

A2) Associativity Property of vector addition:

$$\boxed{u + (v + w) = (u + v) + w \quad \forall\, u, v, w \in V}$$

A3) Existence of additive identity element w.r.t. vector addition:

There exist a vector called as 'the zero vector' denoted by '0' in V st

$$\boxed{u + 0 = u = 0 + u \quad \forall\, u \in V}$$

VECTOR SPACE

A4) Existence of inverse element w.r.t. vector addition:

Given any u in V there exist a vector denoted by '-u' in V such that

$$\boxed{u + (-u) = 0 = (-u) + u \qquad \forall\, u \in V}$$

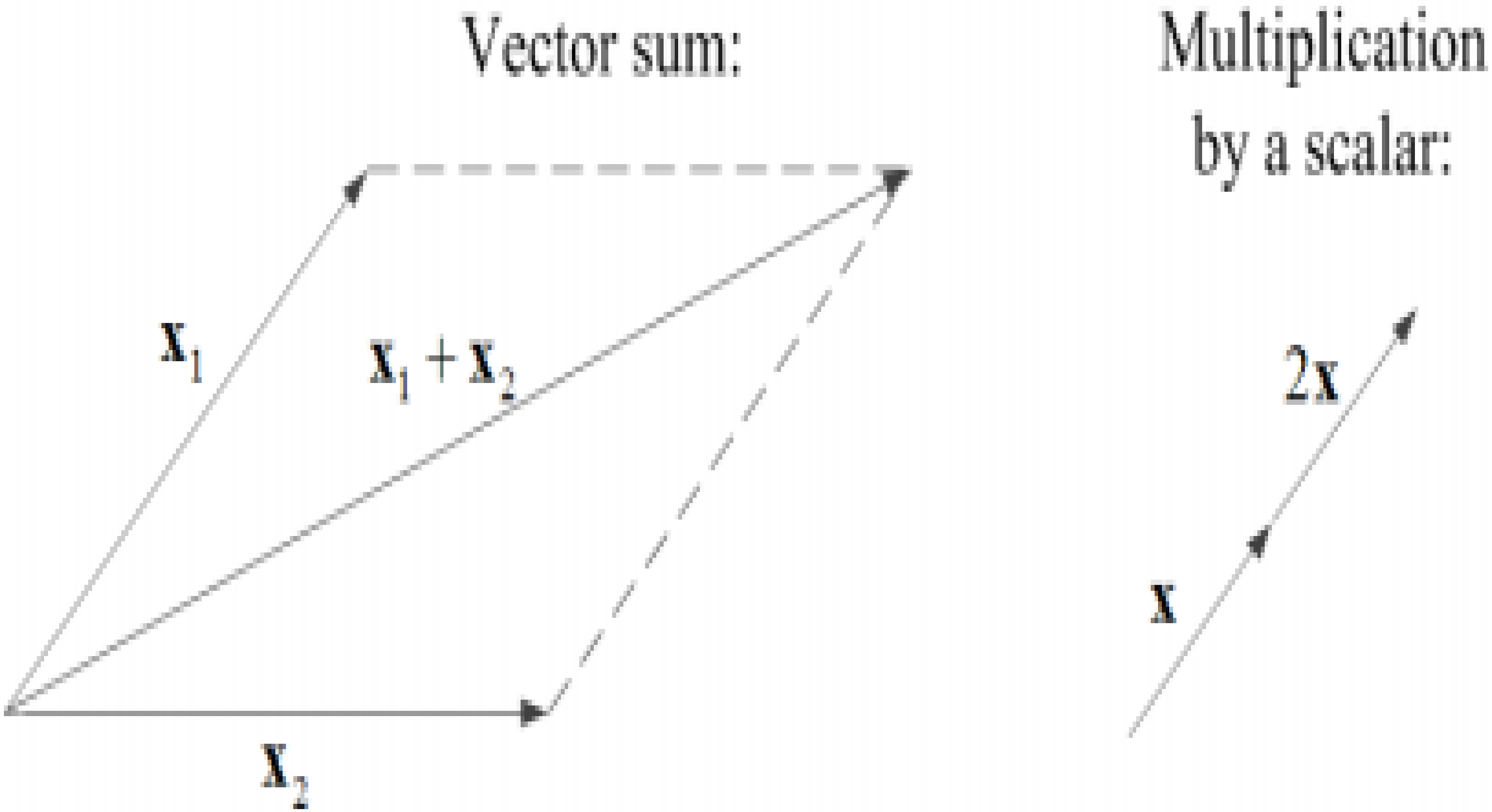
A5) Commutativity property of vector addition :

$$\boxed{u + v = v + u \qquad \forall\, u, v \in V}$$

A6) Closure under scalar multiplication :

$$\boxed{\alpha u \in V \quad \forall\, u \in V \text{ and } \forall \alpha \in \mathbb{R}}$$

i.e. for any scalar α in R and u in V there corresponds a unique vector 'α u' in V, this operation is known as 'scalar multiplication'


Vector sum:
Multiplication
by a scalar:
$\mathbf{x}_1$
$\mathbf{x}_1 + \mathbf{x}_2$
$\mathbf{x}_2$
$2\mathbf{x}$
$\mathbf{x}$

VECTOR SPACE

A7) Distributivity property :

$$\alpha(\mathrm{u}+\mathrm{v})=\alpha\mathrm{u}+\alpha\mathrm{v} \quad \forall\, \mathrm{u},\mathrm{v}\in \mathrm{V} \text{ and } \alpha\in \mathbb{R}$$

$$\text{and} \quad (\alpha+\beta)\mathrm{u}=\alpha\mathrm{u}+\beta\mathrm{u} \quad \forall\, \mathrm{u}\in \mathrm{V} \text{ and } \alpha,\beta\in \mathbb{R}$$

A8) $\boxed{\alpha(\beta\mathrm{u}) = (\alpha\beta)\mathrm{u} \quad \forall\, \mathrm{u}\in \mathrm{V} \text{ and } \alpha,\beta\in \mathbb{R}}$

A9) For any u in V , $\boxed{1\mathrm{u}=\mathrm{u}}$, where 1 is the unity of field $\mathbb{R}$

Let n be fixed positive integer and

$\mathbb{R}^n = \{ (x_1, x_2, \ldots, x_n) \ / \ x_i \in \mathbb{R}, 1 \le i \le n \}$ **, $\mathbb{R}^n$ is set of all n -tuples of real numbers.**

Let $u = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and

$v = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ **, define vector addition as**

$u + v = (x_1+y_1, x_2+y_2, \ldots, x_n+y_n)$ **and for any scalar**

$\alpha \in \mathbb{R}$ **define scalar multiplication as**

$\alpha u = (\alpha x_1, \alpha x_2, \ldots, \alpha x_n)$. **Prove that $\mathbb{R}^n$ is vector space over $\mathbb{R}$ under these operations.**

PROBLEMS

Let $M_{m\times n}(\mathbb{R})$ denote the set of all $m\times n$ matrices with real entries. Define for $A=(a_{ij}), B=(b_{ij})$,

$A+B=(a_{ij}+b_{ij})$ and $\alpha A=(\alpha\, a_{ij})$ where $\alpha\in\mathbb{R}$.

Determine whether $M_{m\times n}(\mathbb{R})$ is a vector space under the above operations. Ans: yes

Let P_2 denote the set of all polynomials of degree less than or equal to 2, with real coefficients. Define Addition and Scalar multiplication in the usual way: If $p(x) = a_0 + a_1x + a_2x^2$ **and** $q(x) = b_0 + b_1x + b_2x^2$ **, then**

$$p(x) + q(x) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2$$ **and**

for any $\alpha \in \mathbb{R}$ **,** $\alpha f(x) = \alpha a_0 + \alpha a_1 x + \alpha a_2 x^2$ **.**

Show that P_2 **is a vector space over** $\mathbb{R}$ **.**

PROBLEMS

\# Let F be set of all real valued functions i.e. $F = \{ f : \mathbb{R} \to \mathbb{R} \,/\, f \text{ is a function} \}$.

Define $(f + g)(x) = f(x) + g(x)$ and $(\alpha f)(x) = \alpha f(x)$, then prove that F is a vector space over $\mathbb{R}$.

\# Prove that set of all real valued differentiable functions on (a, b) is a vector space over $\mathbb{R}$.

Vector Subspaces

Definition: Let V be a vector space over field R. A non-empty subset W of V is subspace of V iff

i) Closure under vector addition :for any $u, v \in W$, $u + v \in W$

ii) Closure under scalar multiplication : for any $a \in R$ and $u \in W$, $au \in W$

Alternatively ,

Subset W of vector space V is called as subspace iff $au + bv \in W \quad \forall\ u, v \in W\ \&\ a,b \in R$

Note that for a subset to become a subspace, it is necessary that it must contain the zero vector.

e.g. The set C of complex numbers is a vector space over R and as $R \subset C$ and R itself is vector space over R , R is a subspace of C over R

Let $W = \{ (x, y) \in \mathbb{R}^2 / ax + by = 0 \}$ **Show that W is subspace of** $\mathbb{R}^2$ **.i.e. Show that every line passing through the origin is a subspace of** $\mathbb{R}^2$**.**

Let $W = \{ (x, y, z) \in R^3 / ax + by + cz = 0 \}$ **Show that** W **is subspace of** $\mathbb{R}^3$**. i.e. Show that every plane passing through the origin is a subspace of** $\mathbb{R}^3$**.**

Let $S = \{ (x, y, z) \in \mathbb{R}^3 / ax = by = cz \}$ Show that S is subspace of $\mathbb{R}^3$.i.e. Show that every line passing through the origin is a subspace of $\mathbb{R}^3$.

Solve the system of equations: $\begin{matrix} 3x+4y+z=0 \\ x+y+z=0 \end{matrix}$ **. Let V denote the set of all solutions to the above system. Show that V is a vector subspace of $\mathbb{R}^3$.**

LINEAR SPAN OF A SET

Definition: Let $S = \{u_1, u_2, \ldots, u_n\}$ be a non-empty subset of a vector space V. The linear span of S is denoted by L(S) and it is defined as

$$\boxed{L(S) = \{a_1u_1 + a_2u_2 + \ldots + a_nu_n \,/\, a_1, a_2, \ldots, a_n \in \mathbb{R},\ u_1, u_2, \ldots, u_n \in S,\ n \in \mathbb{N}\}}$$

Definition: Let S be a non-empty subset of a vector space V. We say that S spans the vector space V iff

$$\boxed{L(S) = V}$$

i.e. if every vector in V can be expressed as linear combinations of elements of S

Linear dependence and independence of sets

Definition : Let $S = \{u_1, u_2, \ldots, u_n\}$ be a subset of vector space V. S is said to be **linearly dependent** if there exists a non-trivial (non-zero) solution to the homogeneous system

$$\boxed{a_1u_1 + a_2u_2 + \ldots + a_nu_n = 0,}$$

where $a_1, a_2, \ldots, a_n \in \mathbb{R}$

If the linear system has unique (zero) solution then the set S is known as **linearly independent** set of vectors.

PROBLEMS

Examine whether the vectors (4,9,5) (1,1,0) (1,3,2) in $\mathbb{R}^3$ are linearly dependent.

In the vector space P_2 , determine whether the vectors $3 - 2x + 4x^2$, $4 - x + 6x^2$ and $7 - 8x + 8x^2$ are linearly dependent. Ans: L.D.

Basis and dimension of vector space

Let V be a vector space. A finite subset S of V is called a **basis** of V iff

i) S is linearly independent set

and ii) S spans V i.e. $L(S) = V$

The number of elements in basis set (i.e. cardinality of basis set) of vector space V is known as **dimension** of vector space V.

PROBLEMS

Show that the set $S = \{ (1, 0), (0, 1) \}$ is a basis of vector space $\mathbb{R}^2$ [Infact it is the standard/Euclidean basis of $\mathbb{R}^2$]

Show that the set $S = \{ (1, 0, 0), (0, 1, 0), (0, 0, 1) \}$ is a basis of vector space $\mathbb{R}^3$ [Infact it is the standard/Euclidean basis of $\mathbb{R}^3$]

Show that the set $S = \{1, x, x^2\}$ **is basis of vector space** P_2 **[Infact it is the standard basis of** P_2**]**

Show that $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ **is basis of** $M_{2\times 2}(\mathbb{R})$

PROBLEMS

Show that the set $S = \left\{-4 + x + 3x^2, 6 + 5x + 2x^2, 8 + 4x + x^2\right\}$ is a basis of P_2.

Determine a basis and dimension of the solution space of following homogeneous systems

$$\begin{aligned}
x_1 + 3x_2 + x_3 + x_4 &= 0\\
2x_1 - 2x_2 + x_3 + 2x_4 &= 0\\
x_1 + 11x_2 + 2x_3 + x_4 &= 0
\end{aligned}$$

4.2 Linear transformations (maps)

LINEAR TRANSFORMATION

A Linear Transformation from a vector space V to vector space W is a mapping $T: V \to W$ such that, for all v_1 and v_2 in V and for all scalars c $(c \in \mathbb{R})$,

$$\boxed{T(v_1 + v_2) = T(v_1) + T(v_2)}$$

&

$$\boxed{T(cv_1) = cT(v_1)}$$

PROBLEMS

\# **Show that the following mappings are linear transformations:**

1. $T:\mathbb{R}\to\mathbb{R}, T(x)=4x$.

2. $T:\mathbb{R}^2\to\mathbb{R}, T(x,y)=2x+3y$.

3. $T:\mathbb{R}^3\to\mathbb{R}^2, T(x,y,z)=(2x+z, x-3y)$.

4. $T:\mathbb{R}^3\to\mathbb{R}^3, T(x,y,z)=(x+2y-3z, 2x-y+z, -x+2y+z)$.

5. **Let A be a 3×3 matrix. Let $T:\mathbb{R}^3\to\mathbb{R}^3, T(X)=AX$ where $X=(x,y,z)$.**

6. **Let F denote the set of all differentiable real valued functions defined on the real line $\mathbb{R}$. Let $T:F\to F$, $T(f)=\dfrac{df}{dx}$.**

7. **Let C denote the set of all continuous real valued functions defined on the real line. Let $T:C\to C, T(f)=\int_0^1 f(x)\,dx$.**

PROBLEMS

#

. Let $T:\mathbb{R}^2 \to \mathbb{R}^2$ be a linear map such that $T(4,1)=(1,1)$, $T(1,1)=(3,-2)$. Compute $T(1,0)$. Find $T(x,y)$, where $(x,y)\in\mathbb{R}^2$.

4.3 Matrix associated with a linear map.

Matrix associated with a linear map.

Let V and W be vector spaces of dimension n and m respectively

Let $\{v_1, v_2, \ldots\ldots, v_n\}$ be a basis of V and $\{w_1, w_2, \ldots. w_m\}$ be a basis of W

Let $T : V \to W$ be a linear map. As $T(v_1), T(v_2), \ldots. T(v_n)$ are elements of W, we have

$$\begin{aligned}
T(v_1) &= a_{11}w_1 + a_{21}w_2 + \ldots. + a_{m1}w_m \\
T(v_2) &= a_{12}w_1 + a_{22}w_2 + \ldots. + a_{m2}w_m \\
&\vdots \qquad\quad \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \\
T(v_n) &= a_{1n}w_1 + a_{2n}w_2 + \ldots. + a_{mn}w_m
\end{aligned}$$

where a_{ij} are scalars.

Matrix associated with a linear map.

The matrix

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

is called the matrix associated with T with respect to the bases $\{v_1, v_2, \ldots\ldots, v_n\}$ and $\{w_1, w_2, \ldots. w_m\}$.

PROBLEMS

**Let**

$$T:R^2 \to R^2, T(x,y) = (2x-3y, -x+y)$$

Find the matrix associated with T wrt standard basis.

**Let**

$$T:R^3 \to R^2, T(x,y,z) = (2x-z, y+2z)$$

Find the matrix associated with T wrt standard basis.

PROBLEMS

Let $T:\mathbb{R}^3 \to \mathbb{R}^2, T(x,y,z)=\left(x-2y, x+y-3z\right)$. **Let** $B=\{e_1,e_2,e_3\}$ **and** $C=\{e_2,e_1\}$ **be bases for** $\mathbb{R}^3$ **and** $\mathbb{R}^2$ **respectively. Find the matrix** M **with respect to** B **and** C.

Verify that $M\begin{pmatrix}1\\3\\-2\end{pmatrix}=T(1,3,-2)$.

$$Ans:\begin{pmatrix}1 & 1 & -3\\1 & -2 & 0\end{pmatrix}, T(1,3,-2)=(-5,10)$$

PROBLEMS

#

Let $D: P_3 \to P_2$ be the differential operator $D(P(x)) = p'(x)$. Let $B = \{1, x, x^2, x^3\}$, $C = \{1, x, x^2\}$ be bases for P_3 and P_2 respectively.

a) Find the matrix M of D with respect to B and C.

b) Compute $D(2x^3 - x + 5)$ using part a).

Let $T:P_3 \to P_5$ be the linear transformation given by $T\left(p(x)\right)=\left(1+2x-x^2\right)p(x)$. Find the matrix of T relative to the standard bases of P_3 and P_5 .

$$Ans:\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 2 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & -1 \end{pmatrix}$$

4.4 Range and kernel of a linear map, Rank-nullity theorem

Range of a Linear Transformation

The range of T, denoted as range(T), is the set of all vectors in W that are images of vectors in V under T. That is

$$range(T) = \{T(v) : v \in V\}$$

$$= \{w \in W : w = T(v) \text{ for some } v$$

Kernal of a Linear Transformation

Let V and W be vector spaces. Let $T : V \rightarrow W$ be any linear transformation. The *Kernel* of T, denoted as $\ker(T)$, is the set of all vectors in V, that are mapped by T to 0 in W. That is

$$\boxed{ker(T) = \{v \in V : T(v) = 0\}}$$

Null Space and Column Space

Let A be a $m \times n$ matrix.

- The **null space** of A is the subspace of $\mathbb{R}^n$, consisting of solutions of the homogeneous linear system $AX = 0$. It is denoted by $null(A)$.

- The **column space** of A is the subspace $col(A)$ of $\mathbb{R}^m$ spanned by the columns of A.

Note: The kernel of a linear transformation T, ($\ker(T)$) is a subspace of V; and the range of T (range(T)) is a subspace of W.

RANK AND NULLITY

- Let $T: V \to W$ be any linear transformation. The **rank** of T is the dimension of the range of T, and is denoted by rank(T).
- The nullity of T is the dimension of the **kernel** of T, and is denoted by nullity(T).

RANK NULLITY THEOREM

Let T : V→W be any linear transformation , then:

$$\boxed{Dim V = Rank(T) + Nullity(T)}$$

PROBLEMS

\# Let

$$T : R^2 \to R^3, T(x, y) = (x - y, x + y, y)$$

Find the kernal and range. Also verify Rank nullity theorem..

\#Let

$$T : R^3 \to R^3,$$

$$T(x, y, z) = (x + z, x + y + 2z, 2x + y + 3z)$$

Find the kernal and range. Also verify Rank nullity theorem..

PROBLEMS

Let

$$D : P_3 \rightarrow P_2, D(p(x) = p'(x)$$

Find the kernal and range. Also verify Rank nullity theorem..

#Let

$$T : M_{22} \rightarrow M_{22}, T\begin{bmatrix} a, b \\ c, d \end{bmatrix} = \begin{bmatrix} a-b, & 0 \\ 0, & c-d \end{bmatrix}$$

Find the kernal and range. Also verify Rank nullity theorem..

4.5 Composition of linear maps, Inverse of a linear transformation

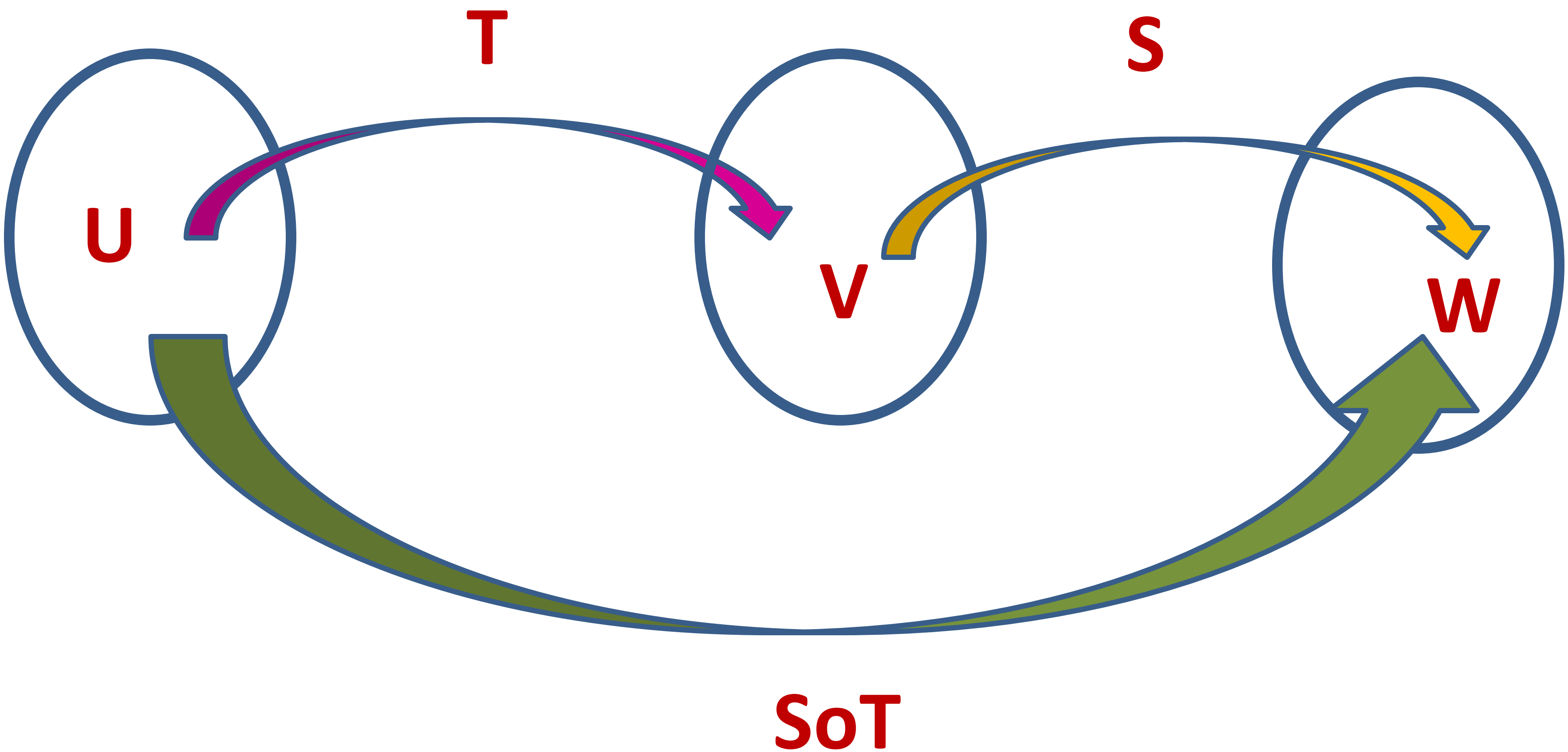
Composition of Linear Transformation

Let U, V and W be vector spaces of dimension n, m and r respectively.

Let $T : U \to V$ and $S : V \to W$ be linear maps.

Then, $S \circ T : U \to W$ is also a linear map.

Composition of Linear Transformation



Matrix associated with the composite linear transformation

Let U, V and W be a vector spaces with bases $B_1 = \{u_1, u_2, \ldots\ldots, u_n\}$, $B_2 = \{v_1, v_2, \ldots\ldots, v_m\}$ and $B_3 = \{w_1, w_2, \ldots. w_r\}$ respectively. Let $T : U \to V$ and $S : V \to W$ be linear maps. Let us denote the matrix associated with the linear map, $T : U \to V$ with respect to the bases B_1 and B_2 as $[A]_{B_2 \leftarrow B_1}$; and the matrix associate with $S : V \to W$ with respect to the bases B_2 and B_3 as $[B]_{B_3 \leftarrow B_2}$ respectively. Then, the matrix associated with the linear map $S \circ T : U \to W$, denoted by $[C]_{B_3 \leftarrow B_1}$ satisfies

$$[C]_{B_3 \leftarrow B_1} = [B]_{B_3 \leftarrow B_2} [A]_{B_2 \leftarrow B_1} .$$

(Right hand side of the above equation is the product of two matrices.)

PROBLEMS

#

Let

$$T : R^2 \to R^2, T(x, y) = (x + y, x - y)$$

$$S : R^2 \to R, S(a, b) = a + b$$

Find SoT(x,y)

PROBLEMS

#

Let

$$T: R^2 \to P_1, T(y,z) = y + (y+z)x$$

$$S: P_1 \to P_2, S(p(x)) = xp(x)$$

Find SoT(x,y)

PROBLEMS

#

Let

$$T : R^2 \to R^2, T(x, y) = (2x + y, -y)$$

$$S : R^2 \to M_{22}, S(a,b) = \begin{bmatrix} a+b, & b \\ 0, & a-b \end{bmatrix}$$

Find SoT(x,y)

PROBLEMS

#

Let

$$T : R^3 \to R^2, T(x, y, z) = (x + 2z, 3x - z)$$

$$S : R^2 \to R^3, S(x, y) = (x, x + y, y)$$

Find SoT(x,y) and ToS(x,y).

Write matrices of T, S, SoT and ToS wrt standard basis.

Inverse of a linear transformation

- Let U and V be vector spaces and $T:U\to V$ be a linear map. A linear map $T^{-1}:V\to U$ is the inverse of T if

  - $T\circ T^{-1}=T^{-1}\circ T=I$.

- $T^{-1}:V\to U$ is also linear.

- A linear map $T:U\to V$ which has inverse is called **invertible or nonsingular transformation or an isomorphism.**

- A linear transformation is said to be invertible if the map $T:U\to V$ is one- one and onto.

Inverse of a linear transformation

T is one to one if Nullity (T)=0

and

T is onto if Rank(T)=Dim U

PROBLEMS

#

Let $T : R^3 \to R^3$,

$$T(x, y, z) = (2x - y + z, x + y, y + 3x + z)$$

Show that T is invertible.

PROBLEMS

#

Let

$$T : R^3 \to R^3,$$

$$T(x, y, z) = (x + y + z, x + 2y - z, 5y + 3x - z)$$

Is T is invertible?

Find u if T is nonsingular. If not find u such that Tu=0

PROBLEMS

**Find a linear map**

$$T : R^3 \to R^3,$$

For which

T(1,2)=(2,3)

And

T(0,1)=(1,4).

Is T invertible?.

If so find inverse of T.


time for questions