Basics of Linear Algebra

Vectors, Matrices, and Linear Algebra

Scalars and *Vectors*

Definition: A scalar is a number.

Examples of scalars are temperature, distance, speed, or mass – all quantities that have a magnitude but no “direction”, other than perhaps positive or negative.

Definition: A vector is *a list of numbers*.

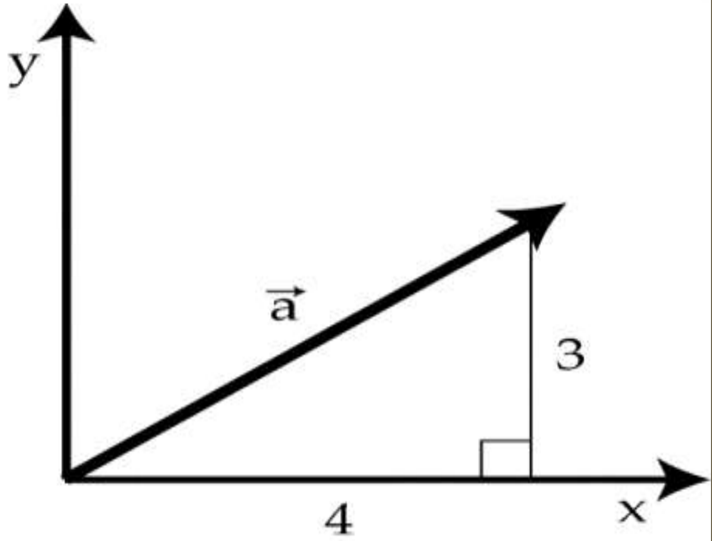
There are (at least) two ways to interpret what this list of numbers mean:

One way to think of the vector as being *a point in a space*. Then this list of numbers is a way of identifying that point in space, where each number represents the vector’s component that dimension.

Another way to think of a vector is *a magnitude and a direction*, e.g. a quantity like velocity (“the fighter jet’s velocity is 250 mph north-by-northwest”). In this way of think of it, a vector is a directed arrow pointing from the origin to the end point given by the list of numbers.

An example of a vector is $\vec{a}$ = [4,3] . Graphically, you can think of this vector as an arrow in the *x*-*y* plane, pointing from the origin to the point at *x*=3, *y*=4 (see illustration.)

The dimensionality of a vector is the length of the list.
So, our example $\vec{a}$ is 2-dimensional because it is a list of two numbers.
Not surprisingly all 2-dimentional vectors live in a plane.
A 3-dimensional vector would be a list of three numbers, and they live in a 3-D volume.
A 27-dimensional vector would be a list of twenty seven numbers, and would live in a space only Ilana's dad could visualize.



Magnitudes and direction

The "magnitude" of a vector is the distance from the endpoint of the vector to the origin – in a word, it's length.
Suppose we want to calculate the magnitude of the vector $\vec{a}$ = [4,3] .
This vector extends 4 units along the x-axis, and 3 units along the y-axis.
To calculate the magnitude $||\vec{a}||$ of the vector we can use the Pythagorean theorem ($x^2 + y^2 = z^2$). $||\vec{a}|| = \sqrt{4^2 + 3^2} = 5$.
The magnitude of a vector is a scalar value – a number representing the length of the vector independent of the direction.

Definition:

A unit vector is a vector of magnitude 1.

Unit vectors can be used to express the direction of a vector independent of its magnitude.

A unit vector is denoted by a small "carrot" or "hat" above the symbol.

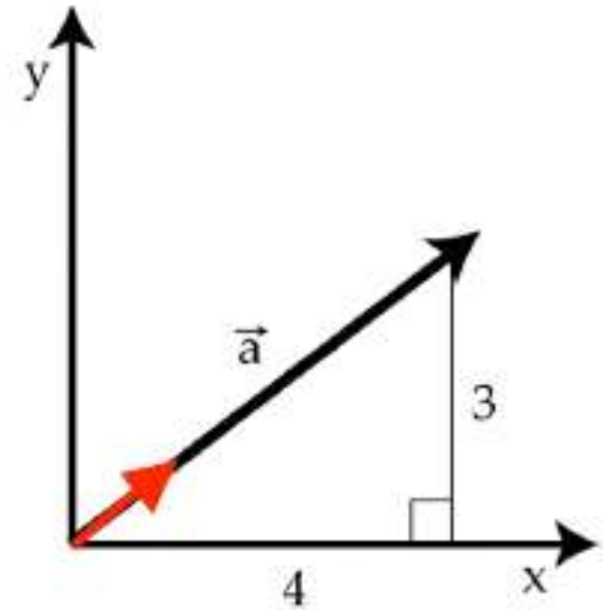
For example, $\hat{a}$ represents the unit vector associated with the vector.

To calculate the unit vector associated with a particular vector, we take the original vector and divide it by its magnitude. In mathematical terms, this process is written as:

$$\hat{a} = \frac{\vec{a}}{||\vec{a}||}\ ; \quad \text{For example, } \vec{a} = [4,3]\ and\ ||\vec{a}|| = 5.$$

When dividing a vector ($\vec{a}$) by a scalar ($||\vec{a}||$), we divide each component of the vector individually by the scalar. In the same way, when multiplying a vector by a scalar we will proceed component by component i.e.

$$\hat{a} = \frac{[4,3]}{5} = [\frac{4}{5}, \frac{3}{5}]$$



By dividing each component of the vector by the same number, we leave the direction of the vector unchanged, while we change the magnitude.
If we have done this correctly, then the magnitude of the unit vector must be equal to 1 (otherwise it would not be a unit vector). Thus,

$$||\hat{a}||^2 = \left(\frac{4}{5}\right)^2 + \left(\frac{3}{5}\right)^2 = \left(\frac{16}{25}\right) + \left(\frac{9}{25}\right) = \left(\frac{25}{25}\right) = 1$$

We can use these two components to re-create the vector $\vec{a}$ by multiplying the vector $\hat{a}$ by the scalar $||\vec{a}||$ like so:

$$\vec{a} = \hat{a} * ||\vec{a}||$$

Vector addition and subtraction

Vectors can be added and subtracted.

Graphically, we can think of adding two vectors together as placing two line segments end-to-end, maintaining distance and direction.

An example of this is shown in the illustration, showing the addition of two vectors $\vec{a}$ and $\vec{b}$ to create a third vector $\vec{c}$ $\mathrm{i.e.}$

Numerically,

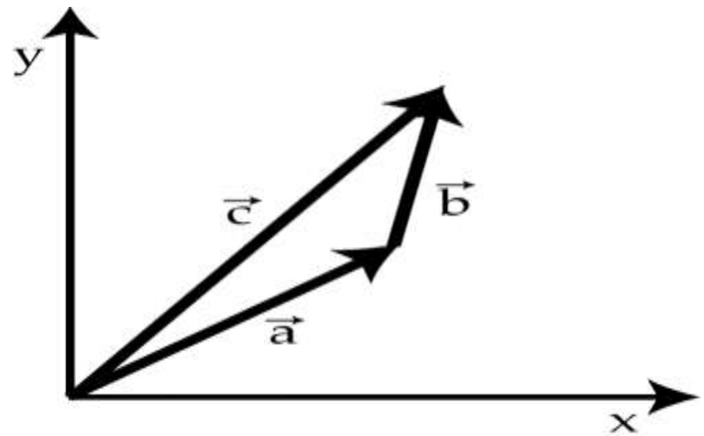
$$\vec{a} + \vec{b} = \vec{c}$$

$$\vec{c} = \vec{a} + \vec{b}$$

$$\vec{c} = [4,3] + [1,2]$$

$$\vec{c} = [4+1, 3+2]$$

$$\vec{c} = [5,5]$$



$$\vec{c} = \vec{a} - \vec{b}$$

$$\vec{c} = [4,3] - [1,2]$$

$$\vec{c} = [3,1]$$

Vector addition has a very simple interpretation in the case of things like displacement. If in the morning a ship sailed 4 miles east and 3 miles north, and then in the afternoon it sailed a further 1 mile east and 2 miles north, what was the total displacement for the whole day? 5 miles east and 5 miles north – vector addition at work.

Linear Independence

If two vectors point in different directions, even if they are not very different directions, then the two vectors are said to be *linearly independent*.

Definition: A family of vectors is linearly independent if no one of the vectors can be created by any linear combination of the other vectors in the family. For example,

$\vec{c}$ is linearly independent of $\vec{a}$ and $\vec{b}$ if and only if it is *impossible* to find scalar values of α and β such that $\vec{c} = \alpha\vec{a} + \beta\vec{b}$.

Vector multiplication: dot products

Next we move into the world of vector multiplication. There are two principal ways of multiplying vectors, called *dot products* (a.k.a. *scalar products*) and *cross products*. The dot product generates a scalar value from the product of two vectors.

$\vec{a} \cdot \vec{b} = [4,3] \cdot [1,2] = 4 * 1 + 3 * 2 = 10$. The dot product $d = \vec{a} \cdot \vec{b}$ can be expressed geometrically as: $\vec{a} \cdot \vec{b} = \|\vec{a}\|\|\vec{b}\| cos\theta$, where θ represents the angle between the two vectors.

Definition: A dot product (or scalar product) is the numerical product of the lengths of two vectors, multiplied by the cosine of the angle between them.

As the angle between the two vectors opens up to approach 90 °, the dot product of the two vectors will approach 0, regardless of the vector magnitudes $\|\vec{a}\| \ and \ \|\vec{b}\|$, the two vectors are said to be *orthogonal*.

A basis set is a linearly independent set of vectors that, when used in linear combination, can represent every vector in a given vector space.

Basis of a Vector Space

Definition: A set of vectors {*v*1,*v*2,...,*vn*} is said to be a **Basis** of the F-vector space *V* if both *V*=span(*v*1,*v*2,...,*vn*) and {*v*1,*v*2,...,*vn*} is a linearly independent set.

A set $S = \{\overrightarrow{v_1}, \overrightarrow{v_2}, \ldots \overrightarrow{v_k}\}$ of vectors in a vector space V is a **basis** for V if

(1) S spans V and

(2) S is linearly independent.

Definition: Two vectors are orthogonal to one another if the dot product of those two vectors is equal to zero.

Basis of a Vector Space

Example 1: Let U $=\{(x_1, x_2, x_3) \in \mathrm{R}^3 : x_1 = 2x_2\}$ **be a subspace of** R^3 **Find a basis of** U**.**

If $U=\{(x_1, x_2, x_3) \in \mathrm{R}^3 : x_1 = 2x_2\}$ then $U=\{(2x_2, x_2, x_3) \in \mathrm{R}^3 : x_2, x_3 \in \mathrm{R}\}$, and so one such basis of U is:

(1) $\{(2,1,0),(0,0,1)\}$

To verify this set of vectors is a basis of U we must show that $U = \mathrm{span}((2,1,0),(0,0,1))$ and that $\{(2,1,0),(0,0,1)\}$ is a linearly independent set of vectors in V.

1. Let $x=(x_1, x_2, x_3) \in U$. Then we have that:

(2) $x=(x_1, x_2, x_3) = (2x_2, x_2, x_3) = x_2\,(2,1,0) + x_3\,(0,0,1)$

So $U=\mathrm{span}((2,1,0),(0,0,1))$.

2. Now consider the following vector equation for $a_1, a_2 \in \mathrm{F}$:

(3) $a_1(2,1,0) + a_2(0,0,1) = 0$

$(2a_1, a_1, 0) + (0,0,a_2) = 0$

$(2a_1, a_1, a_2) = (0,0,0)$

The equation above implies that $2a_1 = 0$, $a_1 = 0$, and $a_2 = 0$, so $a_1 = a_2 = 0$ and $\{(2,1,0),(0,0,1)\}$ is a linearly independent set of vectors in R^3.

Thus $\mathbf{\{(2,1,0),(0,0,1)\}}$ is a basis of U.

Example 2: **Let** $U=\{(x_1, x_2, x_3, x_4, x_5) \in R^5: x_1 = 2x_2, x_3 = 4x_4, x_5 = 2x_4\}$ **be a subspace of** R5**. Find a basis for** U**.**

We can rewrite the subspace U as:

(1) $U=\{(2x_2, x_2, 4x_4, x_4, 2x_4): x_2, x_4 \in R\}$

Therefore we have that $\{(2,1,0,0,0),(0,0,4,1,2)\}$ is a basis of U. To verify this, let $x = (x_1, x_2, x_3, x_4, x_5) \in R^5$. Then we have that:

(2) $x = (x_1, x_2, x_3, x_4, x_5) = (2x_2, x_2, 4x_4, x_4, 2x_4)$

$= x_2 (2,1,0,0,0) + x_4 (0,0,4,1,2)$

So $U=\text{span}((2,1,0,0,0),(0,0,4,1,2))$.

Now consider the following vector equation for $a_1, a_2 \in R$:

(3) $a_1 (2,1,0,0,0) + a_2 (0,0,4,1,2) = 0$

$(2a_1, a_1, 0, 0, 0) + (0, 0, 4a_2, a_2, 2a_2) = 0$

$(2a_1, a_1, 4a_2, a_2, 2a_2) = (0,0,0,0,0)$

The equation above implies that:

(4) $2a_1 = 0; a_1 = 0; 4a_2 = 0; a_2 = 0; 2a_2 = 0$

Thus $a_1 = a_2 = 0$ and so $\{(2,1,0,0,0),(0,0,4,1,2)\}$ is a linearly independent set of vectors in U.

Thus $\{(2,1,0,0,0),(0,0,4,1,2)\}$ is a basis of U.

SPANNING SETS AND LINEAR INDIPENDENCE: Examples

Ex. 1

Let S = {(6, 2, 1), (−1, 3, 2)}. Determine, if S is linearly independent or dependent?

Solution: Let

$$c(6, 2, 1) + d(-1, 3, 2) = (0, 0, 0).$$

If this equation has only trivial solutions, then it is linearly independent.

This equation gives the following system of linear equations:

$$6c - d = 0$$
$$2c + 3d = 0$$
$$c + 2d = 0$$

The augmented matrix for this system is

$$\begin{bmatrix} 6 & -1 & 0 \\ 2 & 3 & 0 \\ 1 & 2 & 0 \end{bmatrix}. \quad its\ gauss - Jordan\ form: \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So, c = 0, d = 0. The system has only trivial (i.e. zero) solution. We conclude that S is linearly independent.

Gaussian Elimination

A method of solving a linear system of equations. This is done by transforming the system's augmented matrix into row-echelon form by means of row operations. Then the system is solved by back-substitution.

Example: The system of equations $\begin{cases} x + y + z = 3 \\ 2x + 3y + 7z = 0 \\ x + 3y - 2z = 17 \end{cases}$ has augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 2 & 3 & 7 & 0 \\ 1 & 3 & -2 & 17 \end{array}\right].$$

Row operations can be used to express the matrix in row-echelon form.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 2 & 3 & 7 & 0 \\ 1 & 3 & -2 & 17 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 5 & -6 \\ 0 & 2 & -3 & 14 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 5 & -6 \\ 0 & 0 & -13 & 26 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 5 & -6 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

The system has become $\begin{cases} x + y + z = 3 \\ y + 5z = -6 \\ z = -2 \end{cases}$. By back-substitution we find that $x = 1$, $y = 4$, and $z = -2$.

Gauss-Jordan Elimination

A method of solving a linear system of equations.

This is done by transforming the system's augmented matrix into reduced row-echelon form by means of row operations.

Example: The system of equations $\begin{cases} x + y + z = 3 \\ 2x + 3y + 7z = 0 \\ x + 3y - 2z = 17 \end{cases}$ has augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 2 & 3 & 7 & 0 \\ 1 & 3 & -2 & 17 \end{array}\right].$$

Row operations can be used to express the matrix in reduced row-echelon form.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 2 & 3 & 7 & 0 \\ 1 & 3 & -2 & 17 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 5 & -6 \\ 0 & 2 & -3 & 14 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & -4 & 9 \\ 0 & 1 & 5 & -6 \\ 0 & 0 & -13 & 26 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & -4 & 9 \\ 0 & 1 & 5 & -6 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

The augmented matrix now says that $x = 1$, $y = 4$, and $z = -2$.

Ex. 2

Let $$S=\left\{\left(\frac{3}{4},\frac{5}{2},\frac{3}{2}\right),\left(3,4,\frac{7}{2}\right),\left(-\frac{3}{2},6,2\right)\right\}.$$

Determine, if S is linearly independent or dependent?

Solution: Let $$a\left(\frac{3}{4},\frac{5}{2},\frac{3}{2}\right)+b\left(3,4,\frac{7}{2}\right)+c\left(-\frac{3}{2},6,2\right)=(0,0,0).$$

If this equation has only trivial solutions, then it is linearly independent.
This equation gives the following system of linear equations:

$$\begin{array}{llll} \frac{3}{4}a & +3b & -\frac{3}{2}c & =0 \\ \frac{5}{2}a & +4b & +6c & =0 \\ \frac{3}{2}a & +\frac{7}{2}b & +2c & =0 \end{array}$$

The augmented matrix for this system is

$$\begin{bmatrix} \frac{3}{4} & 3 & -\frac{3}{2} & 0 \\ \frac{5}{2} & 4 & 6 & 0 \\ \frac{3}{2} & \frac{7}{2} & 2 & 0 \end{bmatrix}. \quad its\ Gaus-Jordan\ form \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

So, a = 0, b = 0, c = 0. The system has only trivial (i.e. zero) solution.
We conclude that S is linearly independent

Ex.3 Let

$$S = \{(1,0,0), (0,4,0), (0,0,-6), (1,5,-3)\}.$$

Solution: Let

$$c_1(1,0,0) + c_2(0,4,0) + c_3(0,0,-6) + c_4(1,5,-3) = (0,0,0).$$

If this equation has only trivial solutions, then it is linearly independent.
This equation gives the following system of linear equations:

$$\begin{aligned} c_1 \qquad\qquad\qquad\qquad +c_4 &= 0 \\ 4c_2 \qquad\qquad\quad 5c_4 &= 0 \\ -6c_3 \quad -3c_4 &= 0 \end{aligned}$$

The augmented matrix for this system is

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 4 & 0 & 5 & 0 \\ 0 & 0 & -6 & -3 & 0 \end{bmatrix}. \quad its\ Gaus\text{-}Jordan\ form \quad \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1.25 & 0 \\ 0 & 0 & 1 & .5 & 0 \end{bmatrix}$$

Correspondingly:

$$c_1 + c_4 = 0, \quad c_2 + 1.25c_4 = 0, \quad c_3 + .5c_4 = 0.$$

With $c_4 = t$ as parameter, we have

$$c_1 = -t, \quad c_2 = -1.25t, \quad c_3 = .5t, \quad c_4 = t.$$

The equation above has nontrivial (i.e. nonzero) solutions. So, S is linearly dependent.

Basis and Dimension

Consider $$\mathbf{v_1} = (1,1,1), \mathbf{v_2} = (1,-1,1), \mathbf{v_3} = (1,1,-1) \quad in \quad \mathbb{R}^3.$$

Then v1, v2, v3 form a basis for R^3.

1. First, we prove that v1, v2, v3 are linearly independent.

Let $$c_1\mathbf{v_1}+c_2\mathbf{v_2}+c_3\mathbf{v_3} = 0. \quad OR \quad c_1(1,1,1)+c_2(1,-1,1)+c_3(1,1,-1) = (0,0,0).$$

We have to prove c1 = c2 = c3 = 0. The equations give the following system of linear equations:

$$\begin{array}{llll} c_1 & +c_2 & +c_3 & = 0 \\ c_1 & -c_2 & +c_3 & = 0 \\ c_1 & +c_2 & -c_3 & = 0 \end{array}$$

The augmented matrix is

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 1 & 0 \\ 1 & 1 & -1 & 0 \end{bmatrix} \quad its\ Gauss-Jordan\ form \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

So, c1 = c2 = c3 = 0 and this establishes that v1, v2, v3 are linearly independent.

Now to show that v1, v2, v3 spans R3, let v = (x1, x2, x3) be a vector in R^3. We have to show that, we can find c1, c2, c3 such that

$$(x_1, x_2, x_3) = c_1\mathbf{v_1} + c_2\mathbf{v_2} + c_3\mathbf{v_3}$$

OR

$$(x_1, x_2, x_3) = c_1(1,1,1) + c_2(1,-1,1) + c_3(1,1,-1).$$

This gives the system of linear equations:

$$\begin{bmatrix} c_1 & +c_2 & +c_3 \\ c_1 & -c_2 & +c_3 \\ c_1 & +c_2 & -c_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad OR \quad \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

The coefficient matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \quad has\ inverse \quad A^{-1} = \begin{bmatrix} 0 & .5 & .5 \\ .5 & -.5 & 0 \\ .5 & 0 & -.5 \end{bmatrix}.$$

So, the above system has tha solution:

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = A^{-1} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & .5 & .5 \\ .5 & -.5 & 0 \\ .5 & 0 & -.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

So, each vector (x_1, x_2, x_3) is in the span of $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$. So, they form a basis of $\mathbb{R}^3$. The proof is complete. ■

Ex. 1

Explain, why the set $S = \{(-1, 2), (1, -2), (2, 4)\}$ is not a basis of R^2?

Solution: Note $(-1, 2) + (1, -2) + 0(2, 4) = (0, 0)$.

So, these three vectors are not linearly independent. So, S is not a basis of R^2.

We have dim(R2) = 2 and S has 3 elements. So, (by theorem) S cannot be a basis.

Ex 2.

Explain, why the set $S = \{(2, 1, -2), (-2, -1, 2), (4, 2, -4)\}$ is not a basis of R^3?

Solution: Note $(4, 2, -4) = (2, 1, -2) - (-2, -1, 2)$

OR $(2, 1, -2) - (-2, -1, 2) - (4, 2, -4) = (0, 0, 0)$.

So, these three vectors are linearly dependent. So, S is not a basis of R^3.

Ex 3. Explain, why the set $S = \{6x - 3, 3x^2, 1 - 2x - x^2\}$ is not a basis of P_2?

Note

$$1 - 2x - x^2 = -\frac{1}{3}(6x - 3) - \frac{1}{3}(3x^2)$$

OR

$$(1 - 2x - x^2) + \frac{1}{3}(6x - 3) + \frac{1}{3}(3x^2) = 0.$$

So, these three vectors are linearly dependent. So, S is not a basis of P_2.

Ex 4

Determine, whether $S = \{(1, 2), (1,-1)\}$ is a basis of R^2 or not?

Solution: We will show that S is linearly independent.

Let $a(1, 2) + b(1,-1) = (0, 0)$.

Then $a + b = 0$, and $2a - b = 0$.

Solving, we get $a = 0$, $b = 0$.

So, these two vectors are linearly independent.

We have dim(R2) = 2. Therefore, by theorem 4.5.8, S is a basis of R^2.

Ex 5.

Determine, whether $S = \{(0, 0, 0), (1, 5, 6), (6, 2, 1)\}$ is a basis of R^3 or not?

Solution:

We have $1.(0, 0, 0) + 0.(1, 5, 6) + 0.(6, 2, 1) = (0, 0, 0)$.

So, S is linearly dependent and hence is not a basis of R3.

Ex.1 Is $S = \{(1, 2,3), (0, 1,2), (-1, 0,1)\}$ a basis for R^3?

Ex.2 In R^3 the vector (1, 2, 3) is not a linear combination of the vectors (1, 1, 0) and (1, −1, 0).

Ex. 3 In R^2 the vector (8, 2) is a linear combination of the vectors (1, 1) and (1, −1) because (8, 2) = 5(1, 1) + 3(1, −1).

Ex.4) Show that in the space R^3 the vectors
$x = (1, 1, 0)$, $y = (0, 1, 2)$, and $z = (3, 1, −4)$ are linearly dependent by finding scalars α and β such that $\alpha x + \beta y + z = 0$.
Answer: α = ___ , β = ____ .

Basis

Definition. Let V be a vector space. A linearly independent spanning set for V is called a **basis**.

Equivalently, a subset $S \subset V$ is a basis for V if any vector $\mathbf{v} \in V$ is *uniquely represented* as a linear combination

$$\mathbf{v} = r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k,$$

where $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are distinct vectors from S and $r_1, \ldots, r_k \in \mathbb{R}$.

Examples. • Standard basis for $\mathbb{R}^n$:
$\mathbf{e}_1 = (1,0,0,\ldots,0,0)$, $\mathbf{e}_2 = (0,1,0,\ldots,0,0),\ldots,$
$\mathbf{e}_n = (0,0,0,\ldots,0,1)$.

• Matrices $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$
form a basis for $\mathcal{M}_{2,2}(\mathbb{R})$.

• Polynomials $1, x, x^2, \ldots, x^{n-1}$ form a basis for
$\mathcal{P}_n = \{a_0 + a_1 x + \cdots + a_{n-1}x^{n-1} : a_i \in \mathbb{R}\}$.

• The infinite set $\{1, x, x^2, \ldots, x^n, \ldots\}$ is a basis for $\mathcal{P}$, the space of all polynomials.

Bases for $\mathbb{R}^n$

Theorem Every basis for the vector space $\mathbb{R}^n$ consists of n vectors.

Theorem For any vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in \mathbb{R}^n$ the following conditions are equivalent:

(i) $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for $\mathbb{R}^n$;

(ii) $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a spanning set for $\mathbb{R}^n$;

(iii) $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a linearly independent set.

Dimension

Theorem Any vector space V has a basis. All bases for V are of the same cardinality.

Definition. The **dimension** of a vector space V, denoted $\dim V$, is the cardinality of its bases.

Remark. By definition, two sets are of the same cardinality if there exists a one-to-one correspondence between their elements.

For a finite set, the cardinality is the number of its elements.

For an infinite set, the cardinality is a more sophisticated notion. For example, $\mathbb{Z}$ and $\mathbb{R}$ are infinite sets of different cardinalities while $\mathbb{Z}$ and $\mathbb{Q}$ are infinite sets of the same cardinality.

Examples. • $\dim \mathbb{R}^n = n$

• $\mathcal{M}_{2,2}(\mathbb{R})$: the space of 2×2 matrices
$\dim \mathcal{M}_{2,2}(\mathbb{R}) = 4$

• $\mathcal{M}_{m,n}(\mathbb{R})$: the space of $m\times n$ matrices
$\dim \mathcal{M}_{m,n}(\mathbb{R}) = mn$

• $\mathcal{P}_n$: polynomials of degree less than n
$\dim \mathcal{P}_n = n$

• $\mathcal{P}$: the space of all polynomials
$\dim \mathcal{P} = \infty$

• $\{\mathbf{0}\}$: the trivial vector space
$\dim \{\mathbf{0}\} = 0$

Problem. Find the dimension of the plane $x + 2z = 0$ in $\mathbb{R}^3$.

The general solution of the equation $x + 2z = 0$ is

$$\begin{cases} x = -2s \\ y = t \qquad\qquad (t, s \in \mathbb{R}) \\ z = s \end{cases}$$

That is, $(x, y, z) = (-2s, t, s) = t(0, 1, 0) + s(-2, 0, 1)$.

Hence the plane is the span of vectors $\mathbf{v}_1 = (0, 1, 0)$ and $\mathbf{v}_2 = (-2, 0, 1)$. These vectors are linearly independent as they are not parallel.

Thus $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis so that the dimension of the plane is 2.

How to find a basis?

Theorem Let S be a subset of a vector space V. Then the following conditions are equivalent:

(i) S is a linearly independent spanning set for V, i.e., a basis;

(ii) S is a minimal spanning set for V;

(iii) S is a maximal linearly independent subset of V.

"Minimal spanning set" means "remove any element from this set, and it is no longer a spanning set".

"Maximal linearly independent subset" means "add any element of V to this set, and it will become linearly dependent".

Theorem Let V be a vector space. Then

(i) any spanning set for V can be reduced to a minimal spanning set;

(ii) any linearly independent subset of V can be extended to a maximal linearly independent set.

Equivalently, any spanning set contains a basis, while any linearly independent set is contained in a basis.

Corollary A vector space is finite-dimensional if and only if it is spanned by a finite set.

How to find a basis?

Approach 1. Get a spanning set for the vector space, then reduce this set to a basis.

Proposition Let $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_k$ be a spanning set for a vector space V. If $\mathbf{v}_0$ is a linear combination of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ then $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is also a spanning set for V.

Indeed, if $\mathbf{v}_0 = r_1\mathbf{v}_1 + \cdots + r_k\mathbf{v}_k$, then

$$t_0\mathbf{v}_0 + t_1\mathbf{v}_1 + \cdots + t_k\mathbf{v}_k =$$
$$= (t_0r_1 + t_1)\mathbf{v}_1 + \cdots + (t_0r_k + t_k)\mathbf{v}_k.$$

How to find a basis?

Approach 2. Build a maximal linearly independent set adding one vector at a time.

If the vector space V is trivial, it has the empty basis.

If $V \neq \{\mathbf{0}\}$, pick any vector $\mathbf{v}_1 \neq \mathbf{0}$.
If $\mathbf{v}_1$ spans V, it is a basis. Otherwise pick any vector $\mathbf{v}_2 \in V$ that is not in the span of $\mathbf{v}_1$.

If $\mathbf{v}_1$ and $\mathbf{v}_2$ span V, they constitute a basis.
Otherwise pick any vector $\mathbf{v}_3 \in V$ that is not in the span of $\mathbf{v}_1$ and $\mathbf{v}_2$.

And so on...

Problem. Find a basis for the vector space V spanned by vectors $\mathbf{w}_1 = (1, 1, 0)$, $\mathbf{w}_2 = (0, 1, 1)$, $\mathbf{w}_3 = (2, 3, 1)$, and $\mathbf{w}_4 = (1, 1, 1)$.

To pare this spanning set, we need to find a relation of the form $r_1\mathbf{w}_1 + r_2\mathbf{w}_2 + r_3\mathbf{w}_3 + r_4\mathbf{w}_4 = \mathbf{0}$, where $r_i \in \mathbb{R}$ are not all equal to zero. Equivalently,

$$\begin{pmatrix} 1 & 0 & 2 & 1 \\ 1 & 1 & 3 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

To solve this system of linear equations for r_1, r_2, r_3, r_4, we apply row reduction.

$$\begin{pmatrix} 1 & 0 & 2 & 1 \\ 1 & 1 & 3 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{(reduced row echelon form)}$$

$$\begin{cases} r_1 + 2r_3 = 0 \\ r_2 + r_3 = 0 \\ r_4 = 0 \end{cases} \iff \begin{cases} r_1 = -2r_3 \\ r_2 = -r_3 \\ r_4 = 0 \end{cases}$$

General solution: $(r_1, r_2, r_3, r_4) = (-2t, -t, t, 0)$, $t \in \mathbb{R}$.

Particular solution: $(r_1, r_2, r_3, r_4) = (2, 1, -1, 0)$.

Problem. Find a basis for the vector space V spanned by vectors $\mathbf{w}_1 = (1, 1, 0)$, $\mathbf{w}_2 = (0, 1, 1)$, $\mathbf{w}_3 = (2, 3, 1)$, and $\mathbf{w}_4 = (1, 1, 1)$.

We have obtained that $2\mathbf{w}_1 + \mathbf{w}_2 - \mathbf{w}_3 = \mathbf{0}$. Hence any of vectors $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ can be dropped. For instance, $V = \mathrm{Span}(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_4)$.

Let us check whether vectors $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_4$ are linearly independent:

$$\begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1 \neq 0.$$

They are!!! It follows that $V = \mathbb{R}^3$ and $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_4\}$ is a basis for V.

Vectors $\mathbf{v}_1 = (0, 1, 0)$ and $\mathbf{v}_2 = (-2, 0, 1)$ are linearly independent.

Problem. Extend the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ to a basis for $\mathbb{R}^3$.

Our task is to find a vector $\mathbf{v}_3$ that is not a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

Then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ will be a basis for $\mathbb{R}^3$.

Hint 1. $\mathbf{v}_1$ and $\mathbf{v}_2$ span the plane $x + 2z = 0$.

The vector $\mathbf{v}_3 = (1, 1, 1)$ does not lie in the plane $x + 2z = 0$, hence it is not a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. Thus $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for $\mathbb{R}^3$.

Vectors $\mathbf{v}_1 = (0, 1, 0)$ and $\mathbf{v}_2 = (-2, 0, 1)$ are linearly independent.

Problem. Extend the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ to a basis for $\mathbb{R}^3$.

Our task is to find a vector $\mathbf{v}_3$ that is not a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

Hint 2. At least one of vectors $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, and $\mathbf{e}_3 = (0, 0, 1)$ is a desired one.

Let us check that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{e}_1\}$ and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{e}_3\}$ are two bases for $\mathbb{R}^3$:

$$\begin{vmatrix} 0 & -2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 1 \neq 0, \qquad \begin{vmatrix} 0 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 2 \neq 0.$$

Line Segments

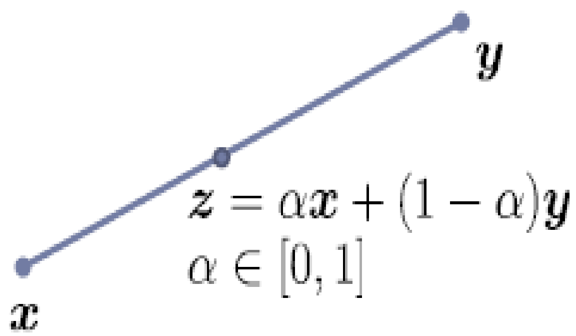
- The line segment between two points $\boldsymbol{x}$ and $\boldsymbol{y}$ in R^n is the set of points on the straight line joining points $\boldsymbol{x}$ and $\boldsymbol{y}$. If $\boldsymbol{z}$ lies on the line segment, then

$$\boldsymbol{z} - \boldsymbol{y} = \alpha(\boldsymbol{x} - \boldsymbol{y})$$

$$\boldsymbol{z} = \alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y}$$

- Hence, the line segment between $\boldsymbol{x}$ and $\boldsymbol{y}$ can be represented as

$$\{\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y} : \alpha \in [0,1]\}$$



Hyperplanes and Linear Varieties

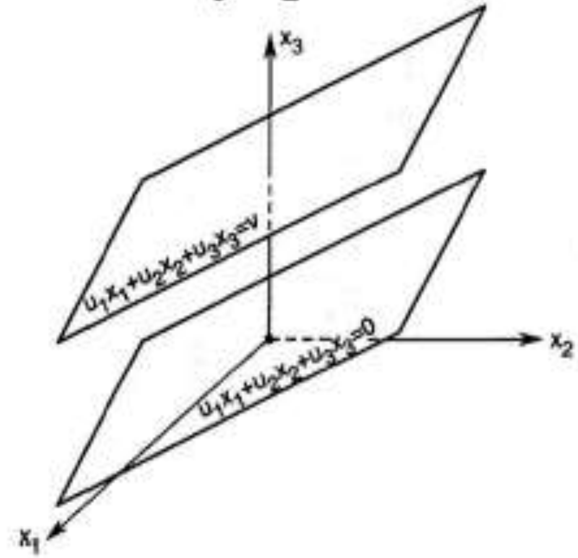
- Let $u_1, u_2, ..., u_n, v \in R$ where at least one of the u_i is nonzero. The set of all points $\boldsymbol{x} = [x_1, x_2, ..., x_n]^T$ that satisfy the linear equation $u_1x_1 + u_2x_2 + \cdots + u_nx_n = v$ is called a ***hyperplane*** of the space R^n.
- We may describe the hyperplane by

$$\{\boldsymbol{x} \in R^n : \boldsymbol{u}^T\boldsymbol{x} = v\} \qquad \boldsymbol{u} = [u_1, u_2, ..., u_n]^T$$

- A hyperplane is not necessarily a subspace of R^n since, in general, it does not contain the origin.
- For $n = 2$, the hyperplane has the form $u_1x_1 + u_2x_2 = v$, which is a straight line. In R^3, hyperplanes are ordinary planes.

Hyperplanes and Linear Varieties

- By translating a hyperplane so that it contains the origin of R^n, it becomes a subspace of R^n. Because the dimension of this subspace is n-1, we say that the hyperplane has dimension n-1.
- The hyperplane $H = \{\boldsymbol{x} : u_1x_1 + \cdots + u_nx_n = v\}$ divides R^n into two ***half-spaces***. One satisfies the inequality $u_1x_1 + \cdots + u_nx_n \geq v$ denoted by $H_+ = \{\boldsymbol{x} \in R^n : \boldsymbol{u}^T\boldsymbol{x} \geq v\}$, and the another one satisfies $u_1x_1 + \cdots + u_nx_n \leq v$, denoted by $H_- = \{\boldsymbol{x} \in R^n : \boldsymbol{u}^T\boldsymbol{x} \leq v\}$
- The half-spaces H_+ and H_- are called ***positive half-space*** and ***negative half-space***, respectively.

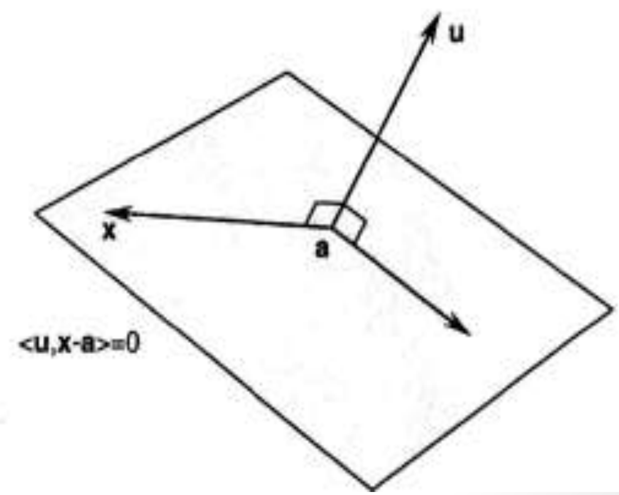


Hyperplanes and Linear Varieties

- Let $\boldsymbol{a} = [a_1, a_2, \ldots, a_n]^T$ be an arbitrary point of the hyperplane H. Thus, $\boldsymbol{u}^T\boldsymbol{a} - v = 0$. We can write

$$\begin{aligned}\boldsymbol{u}^T\boldsymbol{x} - v &= \boldsymbol{u}^T\boldsymbol{x} - v - (\boldsymbol{u}^T\boldsymbol{a} - v)\\ &= \boldsymbol{u}^T(\boldsymbol{x} - \boldsymbol{a})\\ &= u_1(x_1 - a_1) + u_2(x_2 - a_2) + \cdots + u_n(x_n - a_n) = 0\end{aligned}$$

- The hyperplane H consists of the points $\boldsymbol{x}$ for which $\langle \boldsymbol{u}, \boldsymbol{x} - \boldsymbol{a}\rangle = 0$ In other words, the hyperplane H consists of the points $\boldsymbol{x}$ for which the vectors $\boldsymbol{u}$ and $\boldsymbol{x} - \boldsymbol{a}$ are orthogonal. The vector $\boldsymbol{u}$ is the ***normal*** to the hyperplane H.
- The set H_+ consists of those points $\boldsymbol{x}$ for which $\langle \boldsymbol{u}, \boldsymbol{x} - \boldsymbol{a}\rangle \geq 0$ and H_- consists of those points $\boldsymbol{x}$ for which $\langle \boldsymbol{u}, \boldsymbol{x} - \boldsymbol{a}\rangle \leq 0$



Hyperplanes and Linear Varieties

- A ***linear variety*** is a set of the form

$$\{\boldsymbol{x} \in R^n : \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}\}$$

  for some matrix $\boldsymbol{A} \in R^{m \times n}$ and vector $\boldsymbol{b} \in R^m$

- If $\dim \mathcal{N}(\boldsymbol{A}) = r$, we say that the linear variety has dimension r. A linear variety is a subspace if and only if $\boldsymbol{b} = \boldsymbol{0}$. If $\boldsymbol{A} = \boldsymbol{O}$, the linear variety is R^n.

- If the dimension of the linear variety is less than n, then it is the intersection of a finite number of hyperplanes.

Linear Varieties

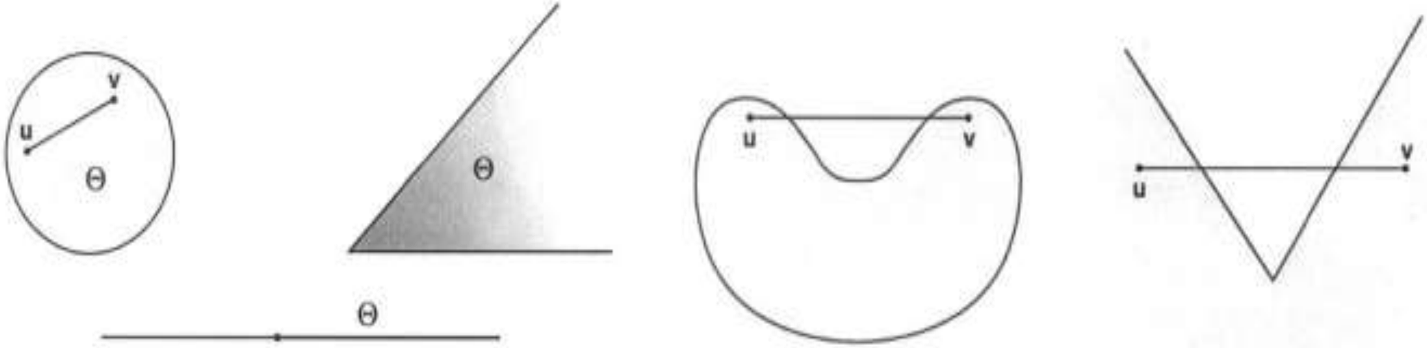
- A linear variety is also called a *linear manifold* or *flat*.
- A linear variety can be described by a system of linear equations. For example, a line in two-dimensional space

$$3x + 5y = 8$$

- In three-dimensional space, a single linear equation involving x, y, and z defines a plane, while a pair of linear equations can be used to describe a line.
- In general, a linear equation in n variables describes a hyperplane, and a system of linear equations describes the intersection of those hyperplanes.
- Assuming the equations are consistent and linearly independent, a system of k equations describes a flat of dimension n-k.

Convex Sets

- Recall that the line segment between two points $\boldsymbol{u}, \boldsymbol{v} \in R^n$ is the set $\{\boldsymbol{w} \in R^n : \boldsymbol{w} = \alpha\boldsymbol{u} + (1-\alpha)\boldsymbol{v}, \alpha \in [0,1]\}$. A point $\boldsymbol{w} = \alpha\boldsymbol{u} + (1-\alpha)\boldsymbol{v}$ (where $\alpha \in [0,1]$) is called a ***convex combination*** of the points $\boldsymbol{u}$ and $\boldsymbol{v}$
- A set $\Theta \subset R^n$ is ***convex*** if for all $\boldsymbol{u}, \boldsymbol{v} \in \Theta$, the line segment between $\boldsymbol{u}$ and $\boldsymbol{v}$ is in Θ

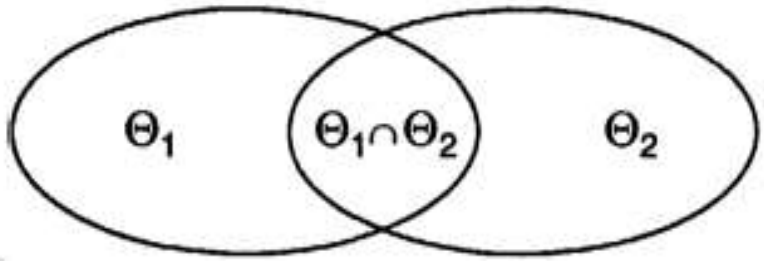


Convex sets

Sets that are not convex

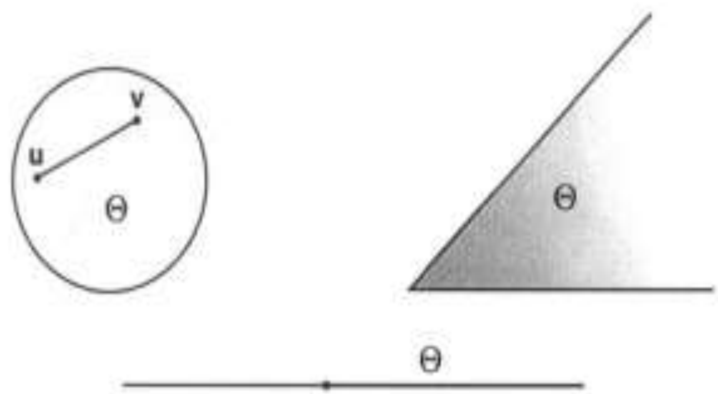
Convex Sets

- Examples of convex sets
  - The empty set; a set consisting of a single point; a line or a line segment; a subspace; a hyperplane; a linear variety; a half-space; R^n
- Theorem 4.1: Convex subsets of R^n have the following properties:
  - If Θ is a convex set and β is a real number, then the set $\beta\Theta = \{\boldsymbol{x} : \boldsymbol{x} = \beta\boldsymbol{v}, \boldsymbol{v} \in \Theta\}$ is also convex.
  - If Θ_1 and Θ_2 are convex sets, then the set $\Theta_1 + \Theta_2 = \{\boldsymbol{x} : \boldsymbol{x} = \boldsymbol{v}_1 + \boldsymbol{v}_2, \boldsymbol{v}_1 \in \Theta_1, \boldsymbol{v}_2 \in \Theta_2\}$ is also convex.
  - The intersection of any collection of convex sets is convex.



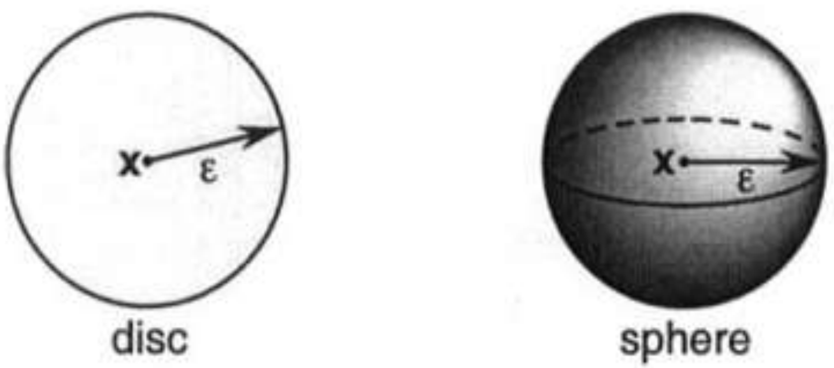
Convex Sets

- A point $\boldsymbol{x}$ in a convex set Θ is said to be an ***extreme point*** of Θ if there are no two distinct points $\boldsymbol{u}$ and $\boldsymbol{v}$ in Θ such that $\boldsymbol{x} = \alpha\boldsymbol{u} + (1-\alpha)\boldsymbol{v}, \alpha \in [0, 1]$
- For example, any point on the boundary of the disk is an extreme point, the vertex (corner) of the set on the right is an extreme point, and the endpoint of the half-line is also an extreme point.



Neighborhoods

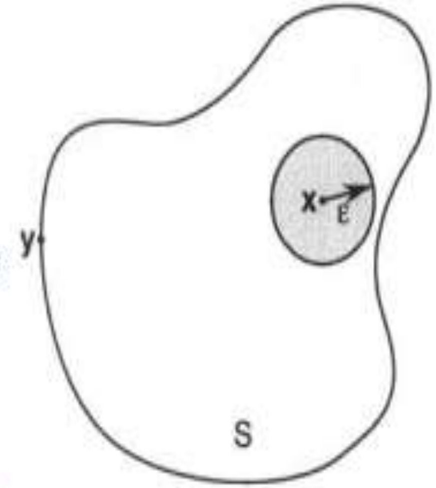
- A ***neighborhood*** of a point $\boldsymbol{x} \in R^n$ is the set $\{\boldsymbol{y} \in R^n : \|\boldsymbol{y} - \boldsymbol{x}\| < \epsilon\}$ where ϵ is some positive number. The neighborhood is also called a ***ball*** with radius ϵ and center $\boldsymbol{x}$.
- In the plane R^2, a neighborhood of $\boldsymbol{x} = [x_1, x_2]^T$ consists of all the points inside a disk centered at $\boldsymbol{x}$. In R^3, a neighborhood of $\boldsymbol{x} = [x_1, x_2, x_3]^T$ consists of all the points inside a sphere centered at $\boldsymbol{x}$.



A neighbourhood of a point is a set containing the point where you can move that point some amount without leaving the set.

Neighborhoods

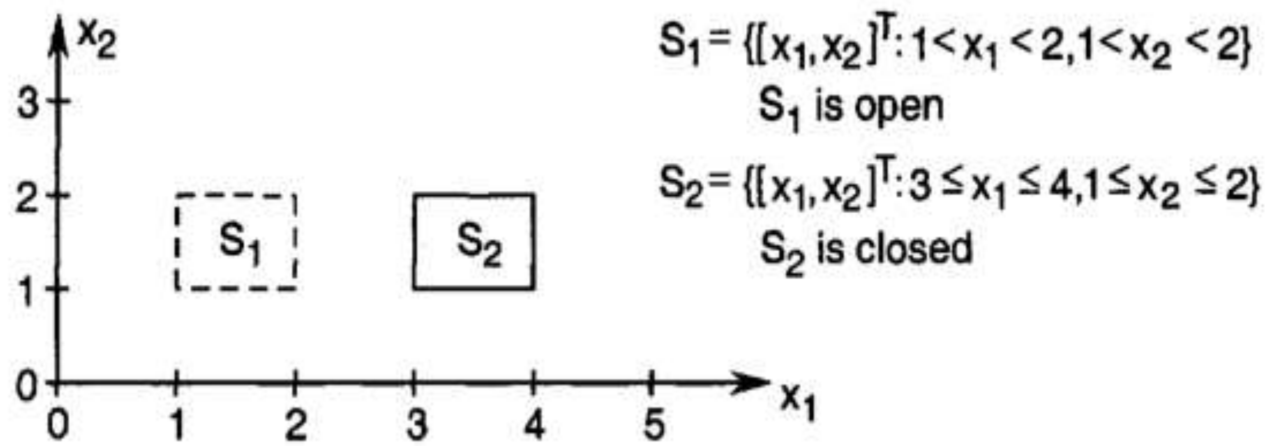
- A point $x \in S$ is said to be an ***interior point*** of the set S if the set S contains some neighborhood of x; that is, if all points within some neighborhood of x are also in S.
  The set of all the interior points of S is called the ***interior*** of S.



- A point x is said to be a ***boundary point*** of the set S if every neighborhood of x contains a point in S and a point not in S. The set of all boundary points of S is said the ***boundary*** of S.

Neighborhoods

- A set S is said to be ***open*** if it contains a neighborhood of each of its points; that is, if each of its points is an interior point, or equivalently, if S contains no boundary points.
- A set S is said to be ***closed*** if it contains its boundary.



Neighborhoods

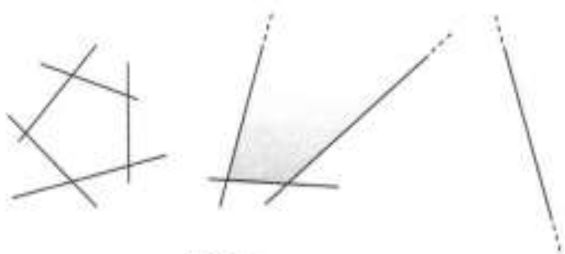
- A set that is contained in a ball of finite radius is said to be ***bounded***. A set is ***compact*** if it is both closed and bounded. Compact sets are important in optimization problems.
- **Theorem 4.2: Theorem of Weierstrass**: Let $f : \Omega \to R$ be a continuous function, where $\Omega \subset R^n$ is a compact set. Then, there exists a point $\boldsymbol{x}_0 \in \Omega$ such that $f(\boldsymbol{x}_0) \leq f(\boldsymbol{x})$ for all $\boldsymbol{x} \in \Omega$. In other words, f achieves its minimum on Ω.

Polytopes and Polyhedra

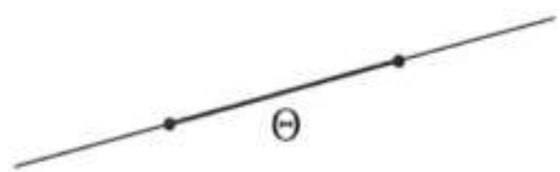
- Let Θ be a convex set, and suppose that $\boldsymbol{y}$ is a boundary point of Θ A hyperplane passing through $\boldsymbol{y}$ is called a ***hyperplane of support*** (or ***supporting hyperplane***) of the set Θ if the entire set Θ lies completely in one of the two half-spaces into which this hyperplane divides the space R^n.
- Recall that the intersection of any number of convex sets is convex. Because every half-space H_+ or H_- is convex in R^n, the intersection of any number of half-spaces is a convex set.

Polytopes and Polyhedra

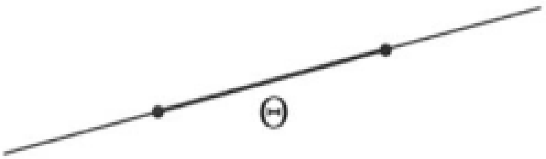
- A set that can be expressed as the intersection of a finite number of half-spaces is called a ***convex polytope*** (凸多胞形).
- A nonempty bounded polytope is called a ***polyhedron*** (多面體).
- For every convex polyhedron $\Theta \subset R^n$, there exists a nonnegative integer $k \leq n$ such that Θ is contained in a linear variety of dimension k, but is not entirely contained in any $(k-1)$-dimensional linear variety of R^n.



Polytopes



One-dimensional polyhedron



One-dimensional polyhedron

Polytopes and Polyhedra

- There exists only one k-dimensional linear variety containing Θ, called the ***carrier*** of the polyhedron Θ, and k is called the ***dimension*** of Θ.
- For example, a zero-dimensional polyhedron is a point of R^n, and its carrier is itself. A one-dimensional polyhedron is a segment, and its carrier is the straight line on which it lies.
- The boundary of any k-dimensional polyhedron, $k > 0$, consists of a finite number of $(k-1)$-dimensional polyhedra. For example, the boundary of a one-dimensional polyhedron consists of two points that are the endpoints of the segment.

Polytopes and Polyhedra

- The $(k-1)$-dimensional polyhedra forming the boundary of a k-dimensional polyhedron are called the ***faces*** of the polyhedron. Each of these faces has, in turn, $(k-2)$-dimensional faces.
- We also consider each of these $(k-2)$-dimensional faces to be faces of the original k-dimensional polyhedron. Thus, every k-dimensional polyhedron has faces of dimensions $k-1, k-2, \dots, 1, 0$
- A zero-dimensional face of a polyhedron is called a ***vertex***, and a one-dimensional face is called an ***edge***.