Ring and field
Tutorial

Ring :★ Properties to prove for a ring OR condition to be satisfied for a ring. $(A, +, \cdot)$

★ (I) $(A, +)$ is a commutative group / Abelian group :→ ie, (i) prove closure operation (ii) prove associative operation (iii) prove identity (left & right) operation (iv) Inverse operation (v) commutative operation. After proving all 5 condition we can say $(A, +)$ is a commutative / abelian group.

★ (II) $(A, \cdot)$ is a semigroup :- ie (i) $(\cdot)$ is a closed operation for multiplication (ii) $(\cdot)$ is an associavite operation for multiplication ie, $(a \times b) \times c = a \times (b \times c)$

★ (III) $(A, \cdot)$ is distributive, ie, $(\cdot)$ is distributed over $(+)$ ie $a \times (b + c) = (a \times b) + (a \times c)$

Integral Domain :- 3 necessary conditions for Ring as an integral domain.

(i) It is a commutative ring
(ii) It has a (1) multiplicative identity element
(iii) It has no zero divisors

} Defination given in notes

Field :- 3 necessary conditions for this kind of ring:-

(i) It is commutative group / Abelian group.
(ii) It has a unity (1) element (Multiplicative identity element)
(iii) Every non-zero element in the table has a multiplicative inverse

Note :- A field is an integral domain. However, not every integral domain is a field.

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Q1) $R = \{0, 2, 4, 6, 8\}$. To prove it is ring for modulo 10. Show it is integral domain? It is field? or both

Sol:- $(R, +, \cdot)$ draw the table for $+_{10}$ and $\times_{10}$

$+_{10}$	0	2	4	6	8
0	0	2	4	6	8
2	2	4	6	8	0
4	4	6	8	0	2
6	6	8	0	2	4
8	8	0	2	4	6

fig (a): Table (1)

$\times_{10}$	0	2	4	6	8
0	0	0	0	0	0
2	0	4	8	2	6
4	0	8	6	4	8
6	0	2	4	6	8
8	0	6	2	8	4

fig (b): Table (2)

(i) We have to show $(R, +_{10})$ is an abelian group:-

(a) $+_{10}$ is a closed operation. Since from fig (a) the elements of the table belong to set $\{0, 2, 4, 6, 8\}$

(b) $+_{10}$ is an associative operation, ie, $(a +_{10} b) +_{10} c = a +_{10} (b +_{10} c)$

let $a = 2$, $b = 4$, $c = 6$

$$\therefore \quad (a +_{10} b) +_{10} c = a +_{10} (b +_{10} c)$$

$$(2 +_{10} 4) +_{10} 6 = 2 +_{10} (4 +_{10} 6)$$

$$6 +_{10} 6 = 2 +_{10} 0$$

$$12 = 2$$

$$\therefore \boxed{2 = 2}$$

Ily prove for $a = 4$, $b = 6$, $c = 8$

$$(a +_{10} b) +_{10} c = a +_{10} (b +_{10} c)$$

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(c) Identity : '0' is the identity element for table 1.

So, $0+a = a = a+0$ for all $a \in Z_5$.

$0 +_5 1 = 1 = 1 +_5 0$

$0 +_5 2 = 2 = 2 +_5 0$

$0 +_5 3 = 3 = 3 +_5 0$

$0 +_5 4 = 4 = 4 +_5 0$

So '0' it is an identity is proved.

(d) Additive inverse :- $1 +_5 4 = 0$, $2 +_5 3 = 0$, $3 +_5 2 = 0$, $4 +_5 1 = 0$

So its additive inverse exists.

(e) ~~Sim~~ commutative property : $a+b = b+a$

$1+2 = 2+1$

$3 = 3$

So addition is commutative.

$\therefore$ $(Z_5, +)$ is a commutative group.

III. Distributive property :- $a \times (b+c) = (a \times b) + (a \times c)$

$1 \times (2+3) = (1 \times 2) + (1 \times 3)$

$1 \times 0 = 2 + 3 = 5$

$\boxed{0 = 0}$

So, the $(\times)$ is distributed over $(+)$.

IV. Since (1) is the multiplicative identity element as seen from table (2), so the multiplicative inverse also exist, ie, $1 \times 1 = 1$, $2 \times 3 = 1$, $3 \times 2 = 1$, $4 \times 4 = 1$,

ie, $1^{-1} = 1$, $2^{-1} = 3$, $3^{-1} = 2$, $4^{-1} = 4$.

$\therefore$ $(Z_5, +, \times)$ is a field.

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(b) $(\times)$ is an associative operation; ie,

$$(a \times_{10} b) \times_{10} c = a \times_{10} (b \times_{10} c)$$

let $a = 2$, $b = 4$, $c = 6$

$$\therefore (2 \times_{10} 4) \times_{10} 6 = 2 \times_{10} (4 \times_{10} 6)$$

$$\therefore 8 \times_{10} 6 = 2 \times_{10} 24$$

$$\therefore 48 = 48 \Rightarrow i.e. \boxed{8 = 8}$$

$\therefore (R, \times)$ is a semigroup.

III We have to show operation $\times_{10}$ is distributive over the operation $+_{10}$; ie,

$$a \times_{10} (b +_{10} c) = (a \times_{10} b) +_{10} (a \times_{10} c)$$

let $a = 2$, $b = 4$, $c = 6$

$$\therefore 2 \times_{10} (4 +_{10} 6) = (2 \times_{10} 4) +_{10} \underbrace{(2 \times_{10} 6)}_{12/10}$$

$$\therefore 2 \times 0 = 8 +_{10} 2$$

$$\therefore 0 = 10$$

$$\boxed{0 = 0}$$

So $\times_{10}$ is distributive over $+_{10}$. Thus we can say that set $R = \{0, 2, 4, 6, 8\}$ is a ring with respect to addition & multiplication modulo 10.

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Integral domain:-

(a) It is a commutative ring ie, $a \cdot b = b \cdot a$
$2 \times 4 = 4 \times 2$
$\boxed{8 = 8}$

(b) With unity element:- A set of even integers with usual operation of + and x, ie, (R, +, x) is a commutative ring without unity. Can be seen in table(2). [~~Notes~~ Given in notes ex(2) on pg 16-48]

(c) Multiplicative inverse:- Set of even integers including zero with usual + and x is not an integral domain because it does not have a multiplicative identity. [Given in notes on pg 16-54).

Therefore, (R) is not an integral domain. Also, the non-zero elements of table (2) are not without zero divisors

Field:- Since the multiplicative identity does not exist, there cannot be a multiplicative inverse. Hence it is not a field.

So, (R, +, x) is a ring. But it is not an integral domain and neither a field.

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classmate

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Page

(3) Show that set of 2x2 matrices is a ring is a non-commutative Ring.

Sol: The set of 2x2 matrices: $R = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} ; a, b, c, d \in Z \right\}$

under matrix addition and multiplication is a ring with respect to additive and multiplicative ~~inverse~~ identities.

⟹ The ring is not commutative since in the matrix multiplication does not commute. So, The ring of $n \times n$ matrices with usual matrix addition and multiplication is a ring but non-commutative.

Also, $n \times n$ matrices is a ring, but not a field. Since it does not have multiplicative inverse.

(4) $R = \{R, +, \cdot\}$ is set of real numbers is a ring, and $C = \{C, +, \cdot\}$ is a set of complex numbers is a field.

Sol: (I) Set of real numbers (R) is a ring for $(R, +, \cdot)$.

(a) R is closed under $+$ and $\times$

(b) Since $a + (b + c) = (a + b) + c$, addition is associative

(c) Since $a + 0 = 0 + a = a \in R$, additive identity exists.

(d) Since we have every element $a \in R$ is an inverse. So inverse property also exists

(e) Since $a + b = b + a$. So R is commutative. $\therefore$ $(R, +)$ is a commutative group.

(II) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in R$, Hence $(\cdot)$ is associative.

(III) Since $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$. Hence $(\cdot)$ is distributed over $(+)$. So set of real numbers $(R, +, \cdot)$ is a ring.

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*Same sol^n for Q2 -

Now, for $C = \{a+bi;\ a,b \in \mathbb{R}\}$ where $i = \sqrt{-1}$
This is a ~~field and~~ extension of R and is called complex numbers denoted by $C = R[i]$ and has binary operations.

$$(a+bi)+(c+di) = (a+c)+(b+d)i$$
$$(a+bi)(c+di) = (ac-bd)+(ad+bc)i$$

proves the addition and multiplication associative property. So it is a commutative group. It has a identity element and $a+bi \in C$ has a multiplicative inverse, ie,

$$(a+bi)\frac{1}{(a+bi)} = 1.$$

ie, $$\frac{1}{a+bi} = \left(\frac{1}{a+bi}\right)\left(\frac{a-bi}{a-bi}\right)$$

$$= \frac{a-bi}{a^2+b^2}$$

$$= \frac{a}{a^2+b^2} + i\left(\frac{-b}{a^2+b^2}\right) \in C$$

Therefore it satisfies all the properties of field.

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(5) Is $(Z_5, +, \times)$ a field? Modulo 5.

Soln:

$+_5$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

fig (a): Table 1.

$\times_5$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

fig (b): table 2

(I) This is a commutative ring with unity. As (1) is the identity element given in table(2) for multiplication.

It is a commutative ring, i.e, $a \cdot b = b \cdot a$, $a = 1, b = 2$

$1 \times 2 = 2 \times 1$

$\boxed{2 = 2}$

(II) Also, To prove it is a commutative group:-

(a) Closure property: It is clear from table that Z_5 is closed under addition and multiplication mod 5.

(b) Since: $a +_5 (b +_5 c) = (a +_5 b) + c$. for $a = 1, b = 2, c = 3$

$1 +_5 (2 +_5 3) = (1 +_5 2) +_5 3$

$1 + 0 = 3 +_5 3 = 6$

$\boxed{1 = 1}$

$\therefore$ Addition is also associative.

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(c) Identity element from fig (a) i.e table 1 is '0'.

$\therefore a +_{10} a = a = a +_{10} a$

left identity | Right identity

① $0 +_{10} 0 = 0$ | $0 +_{10} 0 = 0$

② $0 +_{10} 2 = 2$ | $2 +_{10} 0 = 2$

③ $0 +_{10} 4 = 4$ | $4 +_{10} 0 = 4$

④ $0 +_{10} 6 = 6$ | $6 +_{10} 0 = 6$

⑤ $0 +_{10} 8 = 8$ | $8 +_{10} 0 = 8$

(d) Inverse :- $0 +_{10} 0 = 0$, $2 +_{10} 8 = 0$, $4 +_{10} 6 = 0$, $6 +_{10} 4 = 0$, $8 +_{10} 2 = 0$

OR $0^{-1} = 0$, $2^{-1} = 8$, $4^{-1} = 6$, $6^{-1} = 4$, $8^{-1} = 2$.

(e) Commutative property :- $a +_{10} b +_{10} c = c +_{10} b +_{10} a$.

$a = 2, b = 4, c = 6$.

$\therefore 2 +_{10} 4 +_{10} 6 = 6 +_{10} 4 +_{10} 2$

$\therefore 12 = 12$

$\boxed{2 = 2}$

$\therefore (R, +)$ is an abelian / commutative group.

II $(R, \times)$ is a semigroup :-

(a) closure property :- from fig (b) table (2) it is clear that all the elements of the table (2) belong to the set R. So closure operation is proved.

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Q6) Is $(Z_6, +, \times)$ a field?

Sol:- This question is same as previous problem. But it is not a field because the multiplicative inverse does not exist even though there exist an identity element in multiplication table(2)

$+_6$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

$\times_6$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

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