Mathematical modeling

Module2

Mathematical modeling by example

Product mix

A toy company makes two types of toys: *toy soldiers* and *trains*. Each toy is produced in two stages, first it is constructed in a carpentry shop, and then it is sent to a finishing shop, where it is varnished, vaxed, and polished. To make one toy soldier costs \$10 for raw materials and \$14 for labor; it takes 1 hour in the carpentry shop, and 2 hours for finishing. To make one train costs \$9 for raw materials and \$10 for labor; it takes 1 hour in the carpentry shop, and 1 hour for finishing.

There are 80 hours available each week in the carpentry shop, and 100 hours for finishing. Each toy soldier is sold for \$27 while each train for \$21. Due to decreased demand for toy soldiers, the company plans to make and sell at most 40 toy soldiers; the number of trains is not restriced in any way.

What is the optimum (*best*) product mix (i.e., what quantities of which products to make) that *maximizes* the profit (assuming all toys produced will be sold)?

Terminology

decision variables:	$x_1, x_2, \ldots, x_i, \ldots$
variable domains: values that variables can take	$x_1, x_2 \geq 0$
goal/objective:	maximize/minimize
objective function: function to minimize/maximize	$2x_1 + 5x_2$
constraints: equations/inequalities	$3x_1 + 2x_2 \leq 10$

Example

Decision variables:

- x_1= # of toy soldiers
- x_2= # of toy trains

Objective: maximize profit

- $\$27 - \$10 - \$14 = \$3$ profit for selling one toy soldier $\Rightarrow 3x_1$ profit (in \$) for selling x_1 toy soldier
- $\$21 - \$9 - \$10 = \$2$ profit for selling one toy train $\Rightarrow 2x_2$ profit (in \$) for selling x_2 toy train

$\Rightarrow \underbrace{z = 3x_1 + 2x_2}_{\textit{objective function}}$ profit for selling x_1 toy soldiers and x_2 toy trains

Constraints:

- producing x_1 toy soldiers and x_2 toy trains requires
  (a) $1x_1 + 1x_2$ hours in the carpentry shop; there are 80 hours available
  (b) $2x_1 + 1x_2$ hours in the finishing shop; there are 100 hours available
- the number x_1 of toy soldiers produced should be at most 40

Variable domains: the numbers x_1, x_2 of toy soldiers and trains must be non-negative (sign restriction)

$$
\begin{aligned}
\text{Max } 3x_1 &+ 2x_2 \\
x_1 + x_2 &\le 80 \\
2x_1 + x_2 &\le 100 \\
x_1 &\le 40 \\
x_1, x_2 &\ge 0
\end{aligned}
$$

We call this a program. It is a linear program, because the objective is a linear function of the decision variables, and the constraints are linear inequalities (in the decision variables).

Blending

A company wants to produce a certain alloy containing 30% lead, 30% zinc, and 40% tin. This is to be done by mixing certain amounts of existing alloys that can be purchased at certain prices. The company wishes to minimize the cost. There are 9 available alloys with the following composition and prices.

Alloy	1	2	3	4	5	6	7	8	9	Blend
Lead (%)	20	50	30	30	30	60	40	10	10	30
Zinc (%)	30	40	20	40	30	30	50	30	10	30
Tin (%)	50	10	50	30	40	10	10	60	80	40
Cost ($/lb)	7.3	6.9	7.3	7.5	7.6	6.0	5.8	4.3	4.1	minimize

Designate a *decision* variables $x_1, x_2, \ldots, x_9$ where

x_i is the amount of Alloy i in a unit of blend

In particular, the decision variables must satisfy $x_1 + x_2 + \ldots + x_9 = 1$. (It is a common mistake to choose x_i the absolute amount of Alloy i in the blend. That may lead to a non-linear program.)

With that we can setup constraints and the objective function.

$$
\begin{array}{llr}
\text{Min} & 7.3x_1 + 6.9x_2 + 7.3x_3 + 7.5x_4 + 7.6x_5 + 6.0x_6 + 5.8x_7 + 4.3x_8 + 4.1x_9 = z & \text{[Cost]} \\
\text{s.t.} & x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 = 1 & \\
 & 0.2x_1 + 0.5x_2 + 0.3x_3 + 0.3x_4 + 0.3x_5 + 0.6x_6 + 0.4x_7 + 0.1x_8 + 0.1x_9 = 0.3 & \text{[Lead]} \\
 & 0.3x_1 + 0.4x_2 + 0.2x_3 + 0.4x_4 + 0.3x_5 + 0.3x_6 + 0.5x_7 + 0.3x_8 + 0.1x_9 = 0.3 & \text{[Zinc]} \\
 & 0.5x_1 + 0.1x_2 + 0.5x_3 + 0.3x_4 + 0.4x_5 + 0.1x_6 + 0.1x_7 + 0.6x_8 + 0.8x_9 = 0.4 & \text{[Tin]}
\end{array}
$$

Do we need all the four equations?

Product mix (once again)

Furniture company manufactures four models of chairs. Each chair requires certain amount of raw materials (wood/steel) to make. The company wants to decide on a production that maximizes profit (assuming all produced chair are sold). The required and available amounts of materials are as follows.

	Chair 1	Chair 2	Chair 3	Chair 4	Total available
Steel	1	1	3	9	4,4000 (lbs)
Wood	4	9	7	2	6,000 (lbs)
Profit	$12	$20	$18	$40	maximize

Decision variables:

x_i = the number of chairs of type i produced

each x_i is non-negative

Objective function:

maximize profit $z = 12x_1 + 20x_2 + 18x_3 + 40x_4$

Costraints:

at most 4,400 lbs of steel available: $x_1 + x_2 + 3x_3 + 9x_4 \leq 4,400$

at most 6,000 lbs of wood available: $4x_1 + 9x_2 + 7x_3 + 2x_4 \leq 6,000$

Resulting program:

$$
\begin{array}{llll}
\text{Max} & 12x_1 + 20x_2 + 18x_3 + 40x_4 & = z & \text{[Profit]} \\
\text{s.t.} & x_1 + x_2 + 3x_3 + 9x_4 & \leq 4,400 & \text{[Steel]} \\
& 4x_1 + 9x_2 + 7x_3 + 2x_4 & \leq 6,000 & \text{[Wood]} \\
& & x_1, x_2, x_3, x_4 \geq 0 &
\end{array}
$$

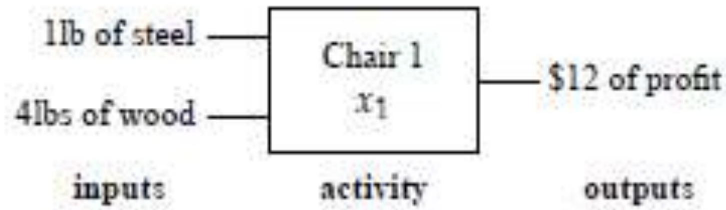
Instead of constructing the formulation as before (row-by-row), we can proceed by columns.

We can view columns of the program as **activities**. An activity has

inputs: materials consumed per unit of activity (1lb of steel and 4lbs of wood)

outputs: products produced per unit of activity ($12 of profit)

activity level: a level at which we operate the activity (indicated by a variable x_1)



Operating the activity "Chair 1" at level x_1 means that we produce x_1 chairs of type 1, each consuming 1lb of steel, 4lbs of wood, and producing $12 of profit. Activity levels are always assumed to be **non-negative**.

The materials/labor/profit consumed or produced by an activity are called **items** (correspond to rows).

The effect of an activity on items (i.e. the amounts of items that are consumed/produced by an activity) are **input-output coefficients**.

The total amount of items available/supplied/required is called the **external flow of items**.

We choose **objective** to be one of the items which we choose to maximize or minimize.

Last step is to write **material balance equations** that express the flow of items in/out of activies and with respect to the external flow.

Example

Items: Steel
Wood
Profit

External flow of items:
Steel: 4,400lbs of available (flowing in)
Wood: 6,000lbs of available (flowing in)

Objective:

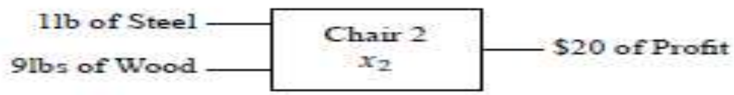
Profit: maximize (flowing out)

Activities:

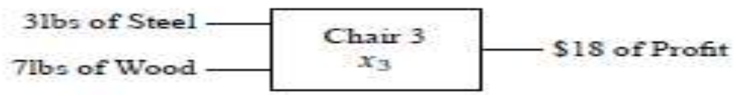
producing a chair of type i where $i = 1, 2, 3, 4$, each is assigned an activity level x_i

Chair 1: Producing 1 chair of type 1
consumes 1 lb of Steel
4 lbs of Wood
produces $12 of Profit

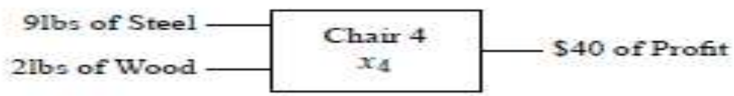
Chair 2: Producing 1 chair of type 2
consumes 1 lb of Steel
9 lbs of Wood
produces $20 of Profit



Chair 3: Producing 1 chair of type 3
consumes 3 lbs of Steel
7 lbs of Wood
produces $18 of Profit

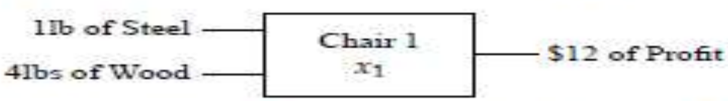


Chair 4: Producing 1 chair of type 4
consumes 9 lbs of Steel
2 lbs of Wood
produces $40 of Profit



The material balance equations:

To see how to do this, consider activity Chair 1: consumes 1lb of Steel, 4lbs of Wood, and produces $12 of Profit. Thus at level x_1, we consume $1x_1$ lbs of Steel, $4x_1$ lbs of Wood, and produce $12x_1$ dollars of Profit.



$$\begin{array}{lcl} \ldots + \; 12x_1 \; + \ldots & & \text{[Profit]} \\ \ldots + \; 1x_1 \; + \ldots & & \text{[Steel]} \\ \ldots + \; 4x_1 \; + \ldots & & \text{[Wood]} \end{array}$$

On the right, you see the effect of operating the activity at level x_1. (Note in general we will adopt a different sign convention; we shall discuss is in a later example.)

Thus considering all activities we obtain:

$$\begin{array}{rcrcrcrl} 12x_1 & + & 20x_2 & + & 18x_3 & + & 40x_4 & \quad \text{[Profit]} \\ x_1 & + & x_2 & + & 3x_3 & + & 9x_4 & \quad \text{[Steel]} \\ 4x_1 & + & 9x_2 & + & 7x_3 & + & 2x_4 & \quad \text{[Wood]} \end{array}$$

Finally, we incorporate the external flow and objective: 4,400lbs of Steel available, 6,000lbs of Wood available, maximize profit:

$$\begin{array}{llcrcrcrcll} \text{Max} & 12x_1 & + & 20x_2 & + & 18x_3 & + & 40x_4 & = & z & \quad \text{[Profit]} \\ \text{s.t.} & x_1 & + & x_2 & + & 3x_3 & + & 9x_4 & \leq & 4,400 & \quad \text{[Steel]} \\ & 4x_1 & + & 9x_2 & + & 7x_3 & + & 2x_4 & \leq & 6,000 & \quad \text{[Wood]} \end{array}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Linear Programming

Linear program (LP) in a standard form (maximization)

$$
\begin{array}{llllllllll}
\max & c_1x_1 & + & c_2x_2 & + \dots + & c_nx_n & & & \text{Objective function} \\
\text{subject to} & a_{11}x_1 & + & a_{12}x_2 & + \dots + & a_{1n}x_n & \le & b_1 & \\
& a_{21}x_1 & + & a_{22}x_2 & + \dots + & a_{2n}x_n & \le & b_2 & \\
& \vdots & + & \vdots & & \vdots & & \vdots & \text{Constraints} \\
& a_{m1}x_1 & + & a_{m2}x_2 & + \dots + & a_{mn}x_n & \le & b_m & \\
& & & & & x_1, x_2, \dots, x_n & \ge & 0 & \text{Sign restrictions}
\end{array}
$$

Feasible solution (point) $P = (p_1, p_2, \dots, p_n)$ is an assignment of values to the $p_1, \dots, p_n$ to variables $x_1, \dots, x_n$ that satisfies all constraints and all sign restrictions.

Feasible region $\equiv$ the set of all feasible points.

Optimal solution $\equiv$ a feasible solution with maximum value of the objective function.

Formulating a linear program

1. Choose decision variables
2. Choose an objective and an objective function – linear function in variables
3. Choose constraints – linear inequalities
4. Choose sign restrictions

Example

You have \$100. You can make the following three types of investments:

Investment A. Every dollar invested now yields \$0.10 a year from now, and \$1.30 three years from now.

Investment B. Every dollar invested now yields \$0.20 a year from now and \$1.10 two years from now.

Investment C. Every dollar invested a year from now yields \$1.50 three years from now.

During each year leftover cash can be placed into money markets which yield 6% a year. The most that can be invested a single investment (A, B, or C) is \$50.

Formulate an LP to maximize the available cash three years from now.

Decision variables: x_A, x_B, x_C, amounts invested into Investments A, B, C, respectively
y_0, y_1, y_2, y_3 cash available/invested into money markets now, and in 1,2,3 years.

$$
\begin{array}{llllllllllll}
\text{Max} & y_3 & & & & & & & & \\
\text{s.t.} & x_A & + & x_B & & & + & y_0 & = & 100 \\
& 0.1x_A & + & 0.2x_B & - & x_C & + & 1.06y_0 & = & y_1 \\
& & & 1.1x_B & & & + & 1.06y_1 & = & y_2 \\
& 1.3x_A & & & + & 1.5x_C & + & 1.06y_2 & = & y_3 \\
& x_A & & & & & & & \leq & 50 \\
& & & x_B & & & & & \leq & 50 \\
& & & & & x_C & & & \leq & 50 \\
& & & & & x_A, x_B, x_C, y_0, y_1, y_2, y_3 & & & \geq & 0
\end{array}
$$

Post office problem

Post office requires different numbers of full-time employees on different days. Each full time employee works 5 consecutive days (e.g. an employee may work from Monday to Friday or, say from Wednesday to Sunday). Post office wants to hire minimum number of employees that meet its daily requirements, which are as follows.

Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Sunday
17	13	15	19	14	16	11

Let x_i denote the number of employees that start working in day i where $i = 1, \ldots, 7$ and work for 5 consecutive days from that day. How many workers work on Monday? Those that start on Monday, or Thursday, Friday, Saturday, or Sunday. Thus $x_1 + x_4 + x_5 + x_6 + x_7$ should be at least 17.

Then the formulation is thus as follows:

$$
\begin{array}{lcccccccccccccccc}
\min & x_1 & + & x_2 & + & x_3 & + & x_4 & + & x_5 & + & x_6 & + & x_7 & & \\
\text{s.t.} & x_1 & & & & & + & x_4 & + & x_5 & + & x_6 & + & x_7 & \geq & 17 \\
& x_1 & + & x_2 & & & & & + & x_5 & + & x_6 & + & x_7 & \geq & 13 \\
& x_1 & + & x_2 & + & x_3 & & & & & + & x_6 & + & x_7 & \geq & 15 \\
& x_1 & + & x_2 & + & x_3 & + & x_4 & & & & & + & x_7 & \geq & 19 \\
& x_1 & + & x_2 & + & x_3 & + & x_4 & + & x_5 & & & & & \geq & 14 \\
& & & x_2 & + & x_3 & + & x_4 & + & x_5 & + & x_6 & & & \geq & 16 \\
& & & & & x_3 & + & x_4 & + & x_5 & + & x_6 & + & x_7 & \geq & 11 \\
& & & & & & & & & & & \multicolumn{3}{r}{x_1, x_2, \ldots, x_7} & \geq & 0
\end{array}
$$

	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Sunday		
Total	1	1	1	1	1	1	1		minimize
Monday	1			1	1	1	1	≥	17
Tuesday	1	1			1	1	1	≥	13
Wednesday	1	1	1			1	1	≥	15
Thursday	1	1	1	1			1	≥	19
Friday	1	1	1	1	1			≥	14
Saturday		1	1	1	1	1		≥	16
Sunday			1	1	1	1	1	≥	11

Example 2.1-1 (The Reddy Mikks Company)

Reddy Mikks produces both interior and exterior paints from two raw materials, $M1$ and $M2$. The following table provides the basic data of the problem:

	Tons of raw material per ton of		Maximum daily availability (tons)
	Exterior paint	*Interior paint*	
Raw material, $M1$	6	4	24
Raw material, $M2$	1	2	6
Profit per ton ($1000)	5	4	

A market survey indicates that the daily demand for interior paint cannot exceed that for exterior paint by more than 1 ton. Also, the maximum daily demand for interior paint is 2 tons.

Reddy Mikks wants to determine the optimum (best) product mix of interior and exterior paints that maximizes the total daily profit.

The LP model, as in any OR model, has three basic components.

1. Decision **variables** that we seek to determine.
2. **Objective** (goal) that we need to optimize (maximize or minimize).
3. **Constraints** that the solution must satisfy.

For the Reddy Mikks problem, we need to determine the daily amounts to be produced of exterior and interior paints. Thus the variables of the model are defined as

$$x_1 = \text{Tons produced daily of exterior paint}$$

$$x_2 = \text{Tons produced daily of interior paint}$$

To construct the objective function, note that the company wants to *maximize* (i.e., increase as much as possible) the total daily profit of both paints. Given that the profits per ton of exterior and interior paints are 5 and 4 (thousand) dollars, respectively, it follows that

$$\text{Total profit from exterior paint} = 5x_1 \text{ (thousand) dollars}$$

$$\text{Total profit from interior paint} = 4x_2 \text{ (thousand) dollars}$$

Letting z represent the total daily profit (in thousands of dollars), the objective of the company is

$$\text{Maximize } z = 5x_1 + 4x_2$$

Next, we construct the constraints that restrict raw material usage and product demand. The raw material restrictions are expressed verbally as

$$\begin{pmatrix}\text{Usage of a raw material}\\ \text{by both paints}\end{pmatrix} \leq \begin{pmatrix}\text{Maximum raw material}\\ \text{availability}\end{pmatrix}$$

The daily usage of raw material $M1$ is 6 tons per ton of exterior paint and 4 tons per ton of interior paint. Thus

$$\text{Usage of raw material } M1 \text{ by exterior paint} = 6x_1 \text{ tons/day}$$

$$\text{Usage of raw material } M1 \text{ by interior paint} = 4x_2 \text{ tons/day}$$

Hence

$$\text{Usage of raw material } M1 \text{ by both paints} = 6x_1 + 4x_2 \text{ tons/day}$$

In a similar manner,

$$\text{Usage of raw material } M2 \text{ by both paints} = 1x_1 + 2x_2 \text{ tons/day}$$

Because the daily availabilities of raw materials $M1$ and $M2$ are limited to 24 and 6 tons, respectively, the associated restrictions are given as

$$\begin{aligned} 6x_1 + 4x_2 &\leq 24 \quad \text{(Raw material } M1\text{)} \\ x_1 + 2x_2 &\leq 6 \quad \text{(Raw material } M2\text{)} \end{aligned}$$

The first demand restriction stipulates that the excess of the daily production of interior over exterior paint, $x_2 - x_1$, should not exceed 1 ton, which translates to

$$x_2 - x_1 \leq 1 \quad \text{(Market limit)}$$

The second demand restriction stipulates that the maximum daily demand of interior paint is limited to 2 tons, which translates to

$$x_2 \leq 2 \text{ (Demand limit)}$$

An implicit (or "understood-to-be") restriction is that variables x_1 and x_2 cannot assume negative values. The **nonnegativity restrictions,** $x_1 \geq 0$, $x_2 \geq 0$, account for this requirement.

The complete Reddy Mikks model is

$$\text{Maximize } z = 5x_1 + 4x_2$$

subject to

$$6x_1 + 4x_2 \leq 24 \quad (1)$$
$$x_1 + 2x_2 \leq 6 \quad (2)$$
$$-x_1 + x_2 \leq 1 \quad (3)$$
$$x_2 \leq 2 \quad (4)$$
$$x_1, x_2 \geq 0 \quad (5)$$

Any values of x_1 and x_2 that satisfy *all* five constraints constitute a **feasible solution**. Otherwise, the solution is **infeasible**. For example, the solution, $x_1 = 3$ tons per day and $x_2 = 1$ ton per day, is feasible because it does not violate *any* of the constraints, including the nonnegativity restrictions. To verify this result, substitute $(x_1 = 3, x_2 = 1)$ in the left-hand side of each constraint. In constraint (1) we have $6x_1 + 4x_2 = 6 \times 3 + 4 \times 1 = 22$, which is less than the right-hand side of the constraint (= 24). Constraints 2 through 5 will yield similar conclusions (verify!). On the other hand, the solution $x_1 = 4$ and $x_2 = 1$ is infeasible because it does not satisfy constraint (1)—namely, $6 \times 4 + 4 \times 1 = 28$, which is larger than the right-hand side (= 24).

1. Proportionality: This property requires the contribution of each decision variable in both the objective function and the constraints to be *directly proportional* to the value of the variable. For example, in the Reddy Mikks model, the quantities $5x_1$ and $4x_2$ give the profits for producing x_1 and x_2 tons of exterior and interior paint, respectively, with the unit profits per ton, 5 and 4, providing the constants of proportionality. If, on the other hand, Reddy Mikks grants some sort of quantity discounts when sales exceed certain amounts, then the profit will no longer be proportional to the production amounts, x_1 and x_2, and the profit function becomes nonlinear.

2. Additivity: This property requires the total contribution of all the variables in the objective function and in the constraints to be the direct sum of the individual contributions of each variable. In the Reddy Mikks model, the total profit equals the sum of the two individual profit components. If, however, the two products *compete* for market share in such a way that an increase in sales of one adversely affects the other, then the additivity property is not satisfied and the model is no longer linear.

3. Certainty: All the objective and constraint coefficients of the LP model are deterministic. This means that they are known constants—a rare occurrence in real life, where data are more likely to be represented by probabilistic distributions. In essence, LP coefficients are average-value approximations of the probabilistic distributions. If the standard deviations of these distributions are sufficiently small, then the approximation is acceptable. Large standard deviations can be accounted for directly by using stochastic LP algorithms , or indirectly by applying sensitivity analysis to the optimum solution (Section 3.6).

Example 1: Reddy Mikks Model

Determine the best *feasible* solution among the following (feasible and infeasible) solutions of the Reddy Mikks model:

(a) $x_1 = 1, x_2 = 4.$

(b) $x_1 = 2, x_2 = 2.$

(c) $x_1 = 3, x_2 = 1.5.$

(d) $x_1 = 2, x_2 = 1.$

(e) $x_1 = 2, x_2 = -1.$

For the Reddy Mikks model, construct each of the following constraints and express it with a linear left-hand side and a constant right-hand side:

(a) The daily demand for interior paint exceeds that of exterior paint by *at least* 1 ton.

(b) The daily usage of raw material *M2* in tons is *at most* 6 and *at least* 3.

(c) The demand for interior paint cannot be less than the demand for exterior paint.

(d) The minimum quantity that should be produced of both the interior and the exterior paint is 3 tons.

(e) The proportion of interior paint to the total production of both interior and exterior paints must not exceed .5.

Ex.1

(a) $x_2 - x_1 \geq 1$ or $-x_1 + x_2 \geq 1$

(b) $x_1 + 2x_2 \geq 3$ and $x_1 + 2x_2 \leq 6$

(c) $x_2 \geq x_1$ or $x_1 - x_2 \leq 0$

(d) $x_1 + x_2 \geq 3$

(e) $\dfrac{x_2}{x_1 + x_2} \leq .5$ or $.5x_1 - .5x_2 \geq 0$

Ex. 1

(a) $(x_1, x_2) = (1, 4)$

$(x_1, x_2) \geq 0$

$6 \times 1 + 4 \times 4 = 22 < 24$

$1 \times 1 + 2 \times 4 = 9 \not\leq 6$ infeasible

(b) $(x_1, x_2) = (2, 2)$

$(x_1, x_2) \geq 0$

$$\left.\begin{array}{lll} 6 \times 2 + 4 \times 2 = 20 & < 24 \\ 1 \times 2 + 2 \times 2 = 6 & = 6 \\ -1 \times 2 + 1 \times 2 = 0 & < 1 \\ 1 \times 2 = 2 & = 2 \end{array}\right\} \text{feasible}$$

$Z = 5 \times 2 + 4 \times 2 = \$18$

(c) $(x_1, x_2) = (3, 1.5)$

$x_1, x_2 \geq 0$

$$\left.\begin{array}{lll} 6 \times 3 + 4 \times 1.5 = 24 & = 24 \\ 1 \times 3 + 2 \times 1.5 = 6 & = 6 \\ -1 \times 3 + 1 \times 1.5 = -1.5 & < 1 \\ 1 \times 1.5 = 1.5 & < 2 \end{array}\right\} \text{feasible}$$

$Z = 5 \times 3 + 4 \times 1.5 = \$21$

(d) $(x_1, x_2) = (2, 1)$

$$\left.\begin{array}{lll} x_1, x_2 \geq 0 \\ 6 \times 2 + 4 \times 1 = 16 & < 24 \\ 1 \times 2 + 2 \times 1 = 4 & < 6 \\ -1 \times 2 + 1 \times 1 = -1 & < 1 \\ 1 \times 1 = 1 & < 2 \end{array}\right\} \text{feasible}$$

$Z = 5 \times 2 + 4 \times 1 = \$14$

(e) $(x_1, x_2) = (2, -1)$

$x_1 \geq 0$, $x_2 < 0$, infeasible

Conclusion: (c) gives the best feasible solution

Suppose that Reddy Mikks sells its exterior paint to a single wholesaler at a quantity discount. The profit per ton is \$5000 if the contractor buys no more than 2 tons daily and \$4500 otherwise. Express the objective function mathematically. Is the resulting function linear?

Quantity discount results in the following nonlinear objective function:

$$Z = \begin{cases} 5x_1 + 4x_2, & 0 \le x_1 \le 2 \\ 4.5x_1 + 4x_2, & x_1 > 2 \end{cases}$$

The situation cannot be treated as a linear program. Nonlinearity can be accounted for in this case using mixed integer programming (chapter 9).

GRAPHICAL LP SOLUTION

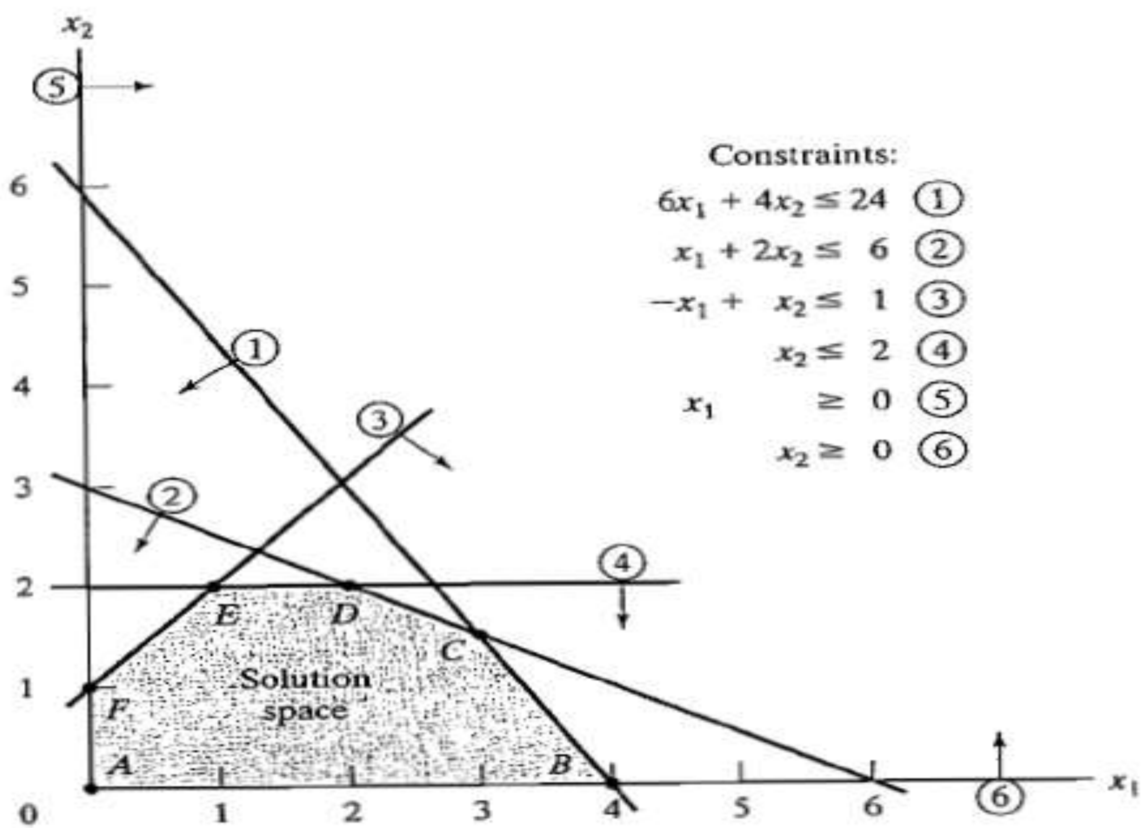
The graphical procedure includes two steps:

1. Determination of the feasible solution space.
2. Determination of the optimum solution from among all the feasible points in the solution space.

The procedure uses two examples to show how maximization and minimization objective functions are handled.

FIGURE 2.1

Feasible space of the Reddy Mikks model



Determination of the Feasible Solution Space:

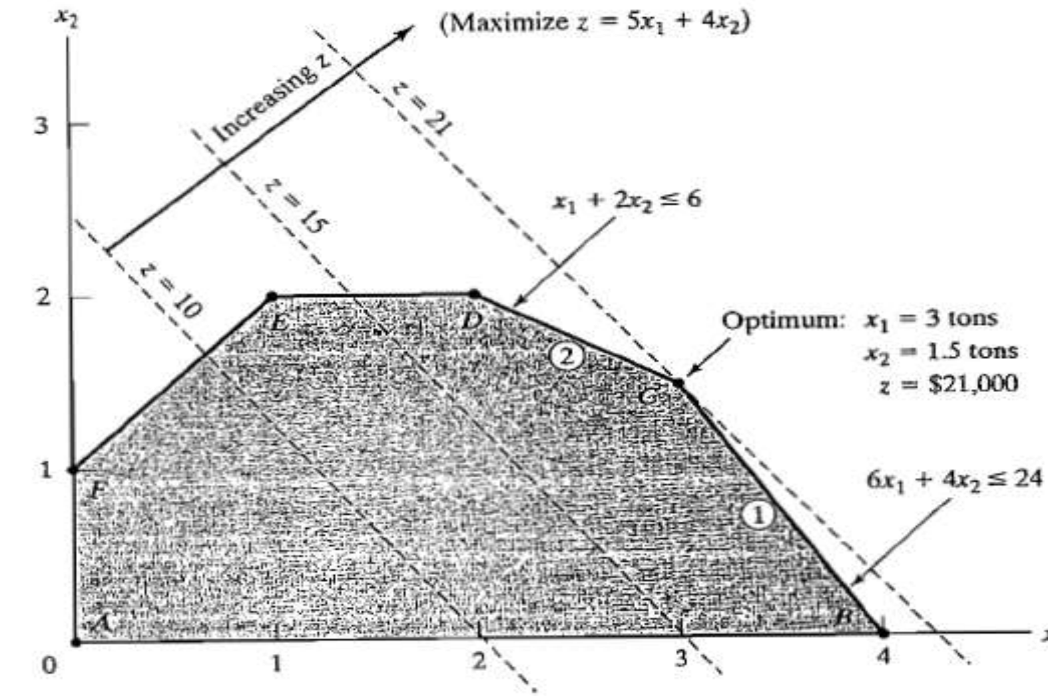
First, we account for the nonnegativity constraints $x_1 \geq 0$ and $x_2 \geq 0$. In Figure 2.1, the horizontal axis x_1 and the vertical axis x_2 represent the exterior- and interior-pain variables, respectively. Thus, the nonnegativity of the variables restricts the solution-space area to the first quadrant that lies above the x_1-axis and to the right of the x_2-axis.

To account for the remaining four constraints, first replace each inequality with an equation and then graph the resulting straight line by locating two distinc points on it. For example, after replacing $6x_1 + 4x_2 \leq 24$ with the straight line $6x_1 + 4x_2 = 24$, we can determine two distinct points by first setting $x_1 = 0$ to obtain $x_2 = \frac{24}{4} = 6$ and then setting $x_2 = 0$ to obtain $x_1 = \frac{24}{6} = 4$. Thus, the line passes through the two points $(0, 6)$ and $(4, 0)$, as shown by line (1) in Figure 2.1.

Next, consider the effect of the inequality. All it does is divide the (x_1, x_2)-plane into two half-spaces, one on each side of the graphed line. Only one of these two halves satisfies the inequality. To determine the correct side, choose $(0, 0)$ as a *reference point*. If it satisfies the inequality, then the side in which it lies is the feasible half-space, otherwise the other side is. The use of the reference point $(0,0)$ is illustrated with the constraint $6x_1 + 4x_2 \leq 24$. Because $6 \times 0 + 4 \times 0 = 0$ is less than 24, the half-space representing the inequality includes the origin (as shown by the arrow in Figure 2.1).

It is convenient computationally to select $(0,0)$ as the reference point, unless the line happens to pass through the origin, in which case any other point can be used. For example, if we use the reference point $(6,0)$, the left-hand side of the first constraint is $6 \times 6 + 4 \times 0 = 36$, which is larger than its right-hand side $(= 24)$, which means that the side in which $(6,0)$ lies is not feasible for the inequality $6x_1 + 4x_2 \leq 24$. The conclusion is consistent with the one based on the reference point $(0,0)$.

Application of the reference-point procedure to all the constraints of the model produces the constraints shown in Figure 2.1 (verify!). The **feasible solution space** of the problem represents the area in the first quadrant in which all the constraints are satisfied simultaneously. In Figure 2.1, any point in or on the boundary of the area *ABCDEF* is part of the feasible solution space. All points outside this area are infeasible.



- LP solution is always associated with the corner points of the solution space.
- This is true even if the constraint function is parallel to the objective function (we will see this observation in further example).
- You may test the optimum is associated with the corner point, with following objective functions:

  (a) $z = 5x_1 + x_2$

  (b) $z = 5x_1 + 4x_2$

  (c) $z = x_1 + 3x_2$

  (d) $z = -x_1 + 2x_2$

  (e) $z = -2x_1 + x_2$

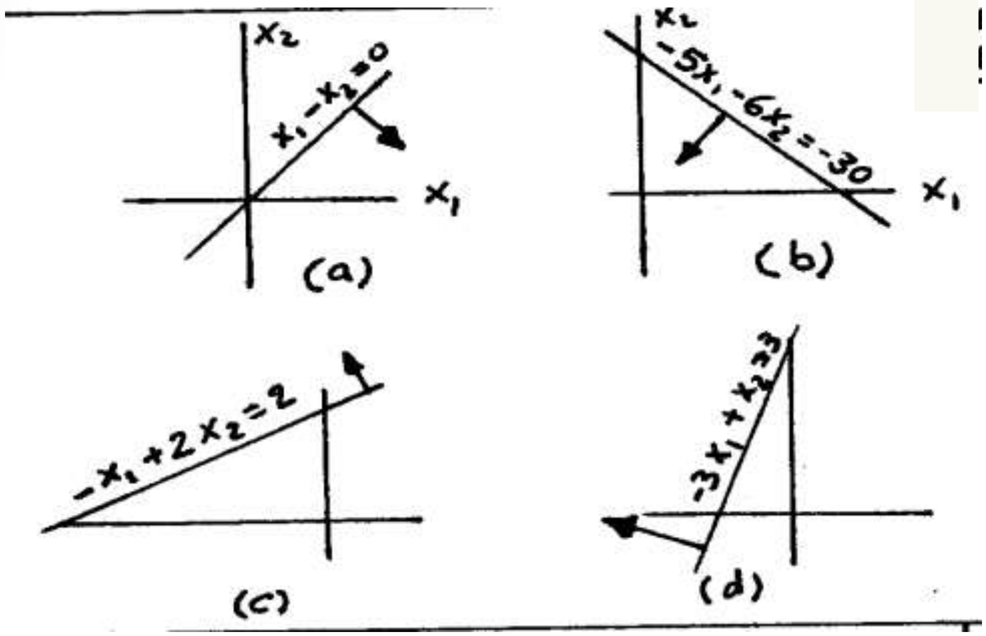
  (f) $z = -x_1 - x_2$

- Feasible solution shown by the line segments joining at points ABCDEF.
- The shaded area consists of infinite feasible point. What is the method to find the optimal solution
- Optimal solution is determined by identifying the increasing direction of the variable z (Profit). The optimal solution occurs at point C,
- Beyond this line the value of z, will lie outside the solution space.
- The values of x_1 *and* x_2 will decided by the lines 1 and 2.

$$6x_1 + 4x_2 = 24$$

$$x_1 + 2x_2 = 6$$

The solution is $x_1 = 3 \text{ tons and } x_2 = 1.5 \text{ tons and } z = \$21{,}000.$



Identify the increasing direction in z in the following cases:

$a)\ Maximize\ z = x_1 - x_2$

b) $Maximize\ z = -5x_1 - 6x_2$

c) $Maximize\ z = -x_1 + 2x_2$

d) $Maximize\ z = -3x_1 + x_2$

Practice Examples

1. Determine the solution space and the optimum solution of the Reddy Mikks model for each of the following independent changes:
   (a) The maximum daily demand for exterior paint is at most 2.5 tons.
   (b) The daily demand for interior paint is at least 2 tons.
   (c) The daily demand for interior paint is exactly 1 ton higher than that for exterior paint.
   (d) The daily availability of raw material *M*1 is at least 24 tons.
   (e) The daily availability of raw material *M*1 is at least 24 tons, and the daily demand for interior paint exceeds that for exterior paint by at least 1 ton.

Answers

a) $x_1 \leq 2.5$ Optimum value is (2.5, 1.75) and z = \$19.50

b) $x_2 \geq 2$ Optimum value is (2, 2) and z = \$18

c) $-x_1 + x_2 = 1$ Optimum value is (1,2) and z = \$13

d) $6x_1 + 4x_2 \geq 24$ Optimum value is (6,0) and z = \$30

2. Alumco manufactures aluminum sheets and bars. The maximum production capacity is estimated at either 800 sheets or 600 bars per day. The maximum daily demand is 550 sheets and 580 bars. The profit per ton is $40 per sheet and $35 per bar. Determine the daily production mix.

x_1 = No. of sheets per day

x_2 = No. of sheets per day

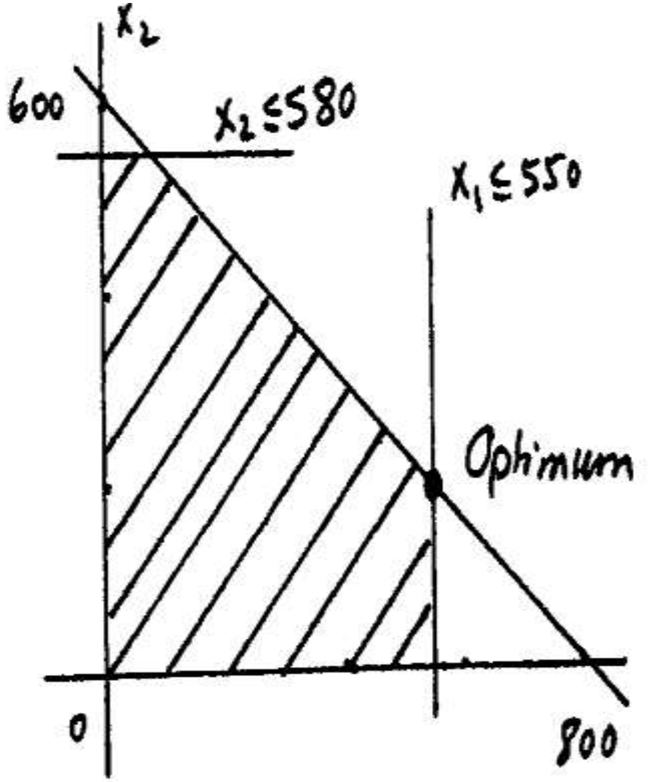
Maximize $z = 40x_1 + 35x_2$

s.t.

$$\frac{x_1}{800} + \frac{x_2}{600} \leq 1$$

$$0 \leq x_1 \leq 550$$

$$0 \leq x_2 \leq 580$$



Optimal Solution

$x_1 = 550$ sheets

$x_2 = 187.13$ bars

z= $ 28549.40

3. The company operates ten hours a day manufactures two products on three sequential process. The following table summarizes the data of the problem:

	Minutes per unit			
Product	*Process 1*	*Process 2*	*Process 3*	Unit profit
1	10	6	8	\$2
2	5	20	10	\$3

Determine the optimal mix of the two product.

x_1= daily unit of product 1

x_2= daily unit of product 2

Maximize z = $2x_1 + 3x_2$

s.t. $10x_1 + 5x_2 \leq 600$

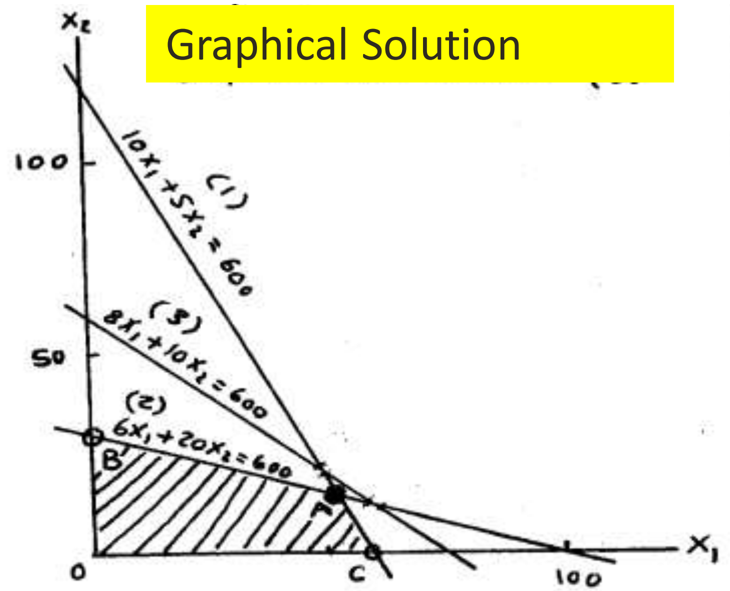
$6x_1 + 20x_2 \leq 600$

$8x_1 + 10x_2 \leq 600$

Optimum Value occurs at corner point

$x_1 = 52.94$, $x_2 = 14.12$ *and* $z = 148.24$

Graphical Solution



4. A company produces two products A and B. The sales volume of A is at least 80% of the total sales of both A and B. However the company cannot sell more than 100 units of A per day. Both products use one raw material , of which the maximum daily availability is 240 lbs. The usage rate of the raw materials are 2 lb per unit of A and 4 lb per unit of B. The profit for A and B are \$20 and \$50, respectively. Determine the optimal product mix for the company.

Answer

$x_1 =$ number of units of A

$x_2 =$ number of units of B

Maximize $z = 20x_1 + 50x_2$

$\dfrac{x_1}{x_1 + x_2} \geq 0.8 \; or \; -0.2x_1 + 0.8x_2 \leq 0$

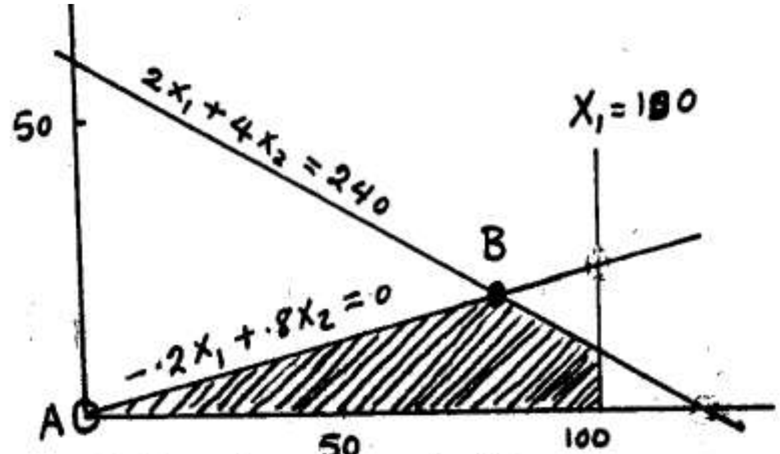
$x_1 \leq 100$

$2x_1 + 4x_2 \leq 240$

$x_1, x_2 \geq 0$

Optimum Value is

$x_1 = 80$ and $x_2 = 20 \; and \; z = 2600$



Optimal occurs at B:

$x_1 = 80$ units

$x_2 = 20$ units

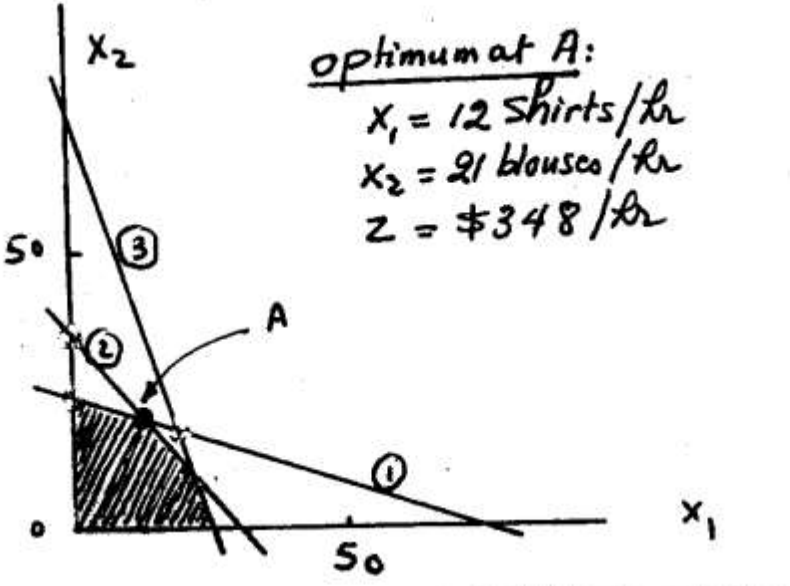
$z = \$2,600$

5.
The Burroughs Garment Company manufactures men's shirts and women's blouses for Walmark Discount Stores. Walmark will accept all the production supplied by Burroughs. The production process includes cutting, sewing, and packaging. Burroughs employs 25 workers in the cutting department, 35 in the sewing department, and 5 in the packaging department. The factory works one 8-hour shift, 5 days a week. The following table gives the time requirements and profits per unit for the two garments:

	Minutes per unit			
Garment	*Cutting*	*Sewing*	*Packaging*	Unit profit ($)
Shirts	20	70	12	8
Blouses	60	60	4	12

Determine the optimal weekly production schedule for Burroughs.

Answer
x_1 = No. of shirts per hour
x_2 = No. of blouses per hour
Max. $z = 8x_1 + 12x_2$
s.t. $20x_1 + 60x_2 \leq 25 \times 60 = 1500$
$70x_1 + 60x_2 \leq 35 \times 60 = 2100$
$12x_1 + 4x_2 \leq 5 \times 60 = 300$
$x_1, x_2 \geq 0$
Optimum value
$x_1 = 12$ *shirts per hour*
$x_2 = 21$ *blouses per hour and* $z = \$348$



Ex. 6. A furniture company manufactures desks and chairs. The sawing department cuts the lumber for both products, which is then sent to separate assembly departments. Assembly items are sent for finishing to the painting department. The daily capacity of the sawing department is 200 chairs and 80 desks. The chair assembly department can produce 120 chairs daily and the desk assembly department 60 desks. The paint department has daily has capacity of either 150 chairs or 110 desks. Given that the profit per chair id $50 and that of desk is $100, determine optimal product mix for the company.

Answer

$x_1 = No.of\ desks\ per\ day$

$x_2 = No.of\ chairs\ per\ day$

Max. $z = 50x_1 + 100x_2$

$\frac{x_1}{200} + \frac{x_2}{80} \leq 1$

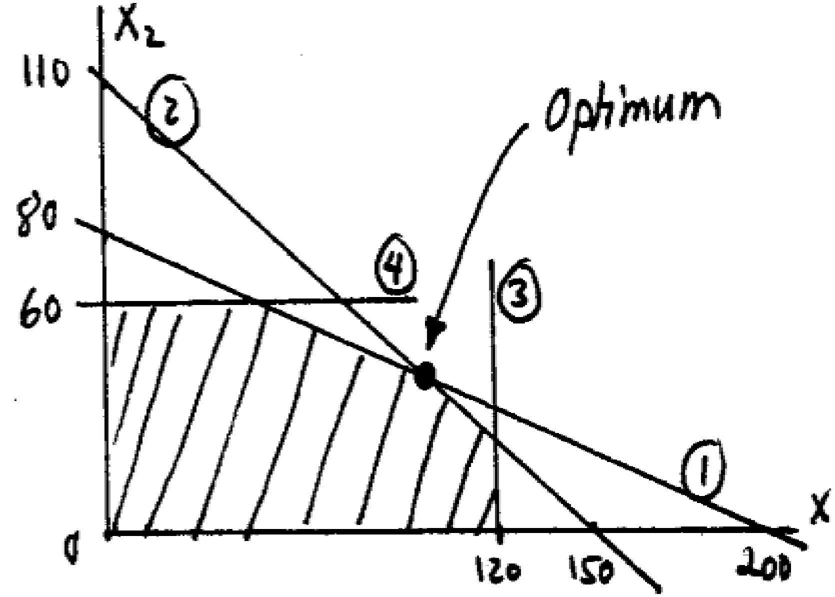
$\frac{x_1}{150} + \frac{x_2}{110} \leq 1$

$x_1 \leq 120, x_2 \leq 60$

Optimum solution

$x_1 = 90\ desks\ and\ x_2 = 44\ chairs$

$z = \$\ 8900$



D 3.1-3. Consider the following objective function for a linear programming model:

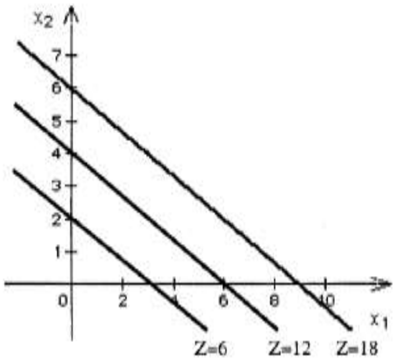
Maximize $Z = 2x_1 + 3x_2$

(a) Draw a graph that shows the corresponding objective function lines for $Z = 6$, $Z = 12$, and $Z = 18$.

(b) Find the slope-intercept form of the equation for each of these three objective function lines. Compare the slope for these three lines. Also compare the intercept with the x_2 axis.

3.1-3.

(a)



(b)

	Slope-Intercept Form	Slope	Intercept
$Z = 6$	$x_2 = -\frac{2}{3}x_1 + 2$	$-\frac{2}{3}$	2
$Z = 12$	$x_2 = -\frac{2}{3}x_1 + 4$	$-\frac{2}{3}$	4
$Z = 18$	$x_2 = -\frac{2}{3}x_1 + 6$	$-\frac{2}{3}$	6

3.2-1. The following table summarizes the key facts about two products, A and B, and the resources, Q, R, and S, required to produce them.

Resource	Resource Usage per Unit Produced		Amount of Resource Available
	Product A	Product B	
Q	2	1	2
R	1	2	2
S	3	3	4
Profit per unit	3	2	

All the assumptions of linear programming hold.

(a) Formulate a linear programming model for this problem.

D,I (b) Solve this model graphically.

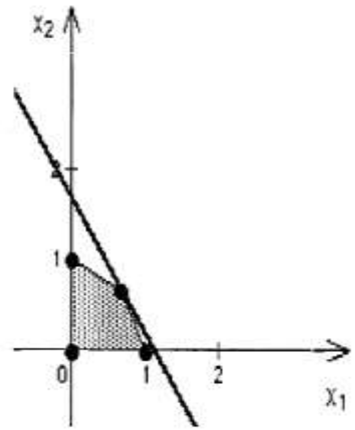
(c) Verify the exact value of your optimal solution from part (*b*) by solving algebraically for the simultaneous solution of the relevant two equations.

3.2-1.

(a) m aximize $P = 3A + 2B$

$$\begin{aligned} \text{subject to} \quad 2A + \ \ B &\leq 2 \\ A + 2B &\leq 2 \\ 3A + 3B &\leq 4 \\ A, B &\geq 0 \end{aligned}$$

(b) Optimal Solution: $(A, B) = (x_1^*, x_2^*) = (2/3, 2/3)$ and $P^* = 3.33$



(c) We have to solve $2A + B = 2$ and $A + 2B = 2$. By subtracting the second equation from the first one, we obtain $A - B = 0$, so $A = B$. Plugging this in the first equation, we get $2 = 2A + B = 3A$, hence $A = B = 2/3$.

D,I **3.1-6.** Use the graphical method to solve the problem:

Maximize $\quad Z = 10x_1 + 20x_2,$

subject to

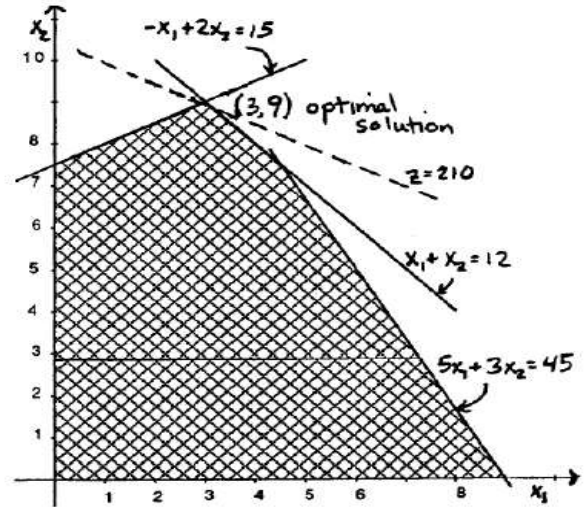
$$\begin{aligned} -x_1 + 2x_2 &\le 15 \\ x_1 + x_2 &\le 12 \\ 5x_1 + 3x_2 &\le 45 \end{aligned}$$

and

$$x_1 \ge 0, \; x_2 \ge 0.$$

3.1-6.

Optimal Solution: $(x_1^*, x_2^*) = (3, 9)$ and $Z^* = 210$



D,I **3.4-5.** Use the graphical method to solve this problem:

Minimize $\quad Z = 15x_1 + 20x_2,$

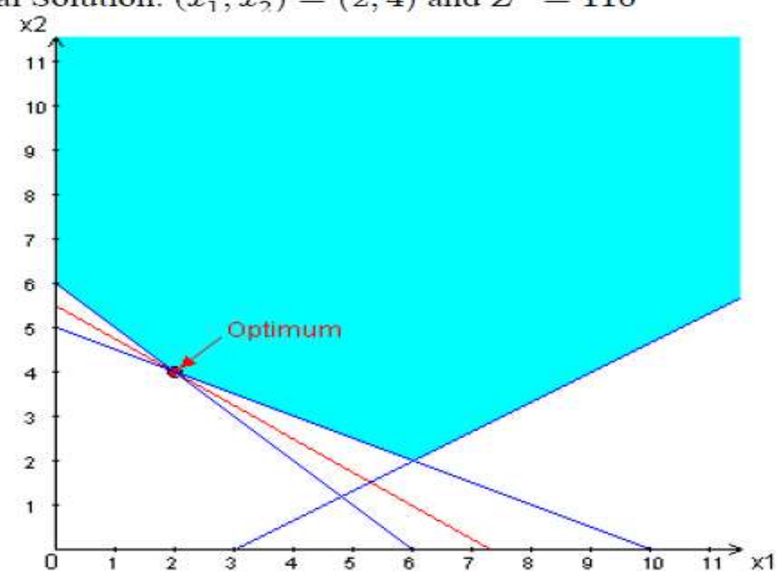
subject to

$$\begin{aligned} x_1 + 2x_2 &\ge 10 \\ 2x_1 - 3x_2 &\le 6 \\ x_1 + x_2 &\ge 6 \end{aligned}$$

and

$$x_1 \ge 0, \quad x_2 \ge 0.$$

Optimal Solution: $(x_1^*, x_2^*) = (2, 4)$ and $Z^* = 110$



3.1-9. The Primo Insurance Company is introducing two new product lines: special risk insurance and mortgages. The expected profit is \$5 per unit on special risk insurance and \$2 per unit on mortgages.

Management wishes to establish sales quotas for the new product lines to maximize total expected profit. The work requirements are as follows:

	Work-Hours per Unit		
Department	**Special Risk**	**Mortgage**	**Work-Hours Available**
Underwriting	3	2	2400
Administration	0	1	800
Claims	2	0	1200

(a) Formulate a linear programming model for this problem.

D,1 (b) Use the graphical method to solve this model.

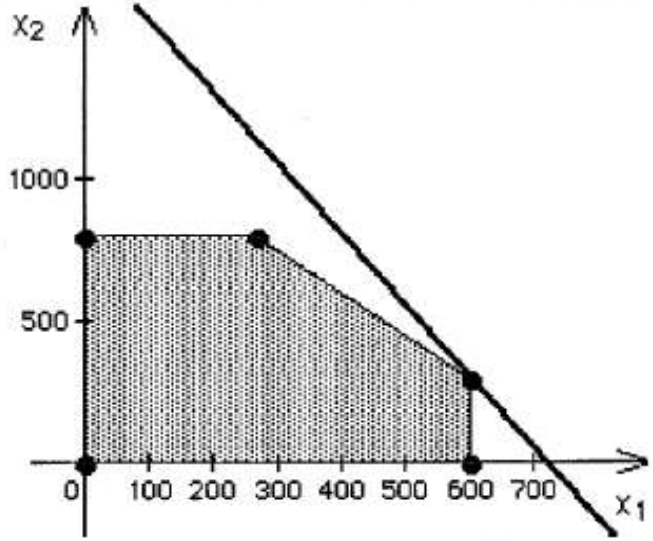
(c) Verify the exact value of your optimal solution from part (*b*) by solving algebraically for the simultaneous solution of the relevant two equations.

3.1-9.

(a) Let x_1 be the number of units on special risk insurance and x_2 be the number of units on mortgages.

$$\begin{array}{ll} \text{maximize} & z = 5x_1 + 2x_2 \\ \text{subject to} & 3x_1 + 2x_2 \leq 2400 \\ & x_2 \leq 800 \\ & 2x_1 \leq 1200 \\ & x_1 \geq 0,\ x_2 \geq 0 \end{array}$$

(b) Optimal Solution: $(x_1^*, x_2^*) = (600, 300)$ and $Z^* = 3600$



(c) The relevant two equations are $3x_1 + 2x_2 = 2400$ and $2x_1 = 1200$, so $x_1 = 600$ and $x_2 = \frac{1}{2}(2400 - 3x_1) = 300$, $z = 5x_1 + 2x_2 = 3600$

3.1-10. Weenies and Buns is a food processing plant which manufactures hot dogs and hot dog buns. They grind their own flour for the hot dog buns at a maximum rate of 200 pounds per week. Each hot dog bun requires 0.1 pound of flour. They currently have a contract with Pigland, Inc., which specifies that a delivery of 800 pounds of pork product is delivered every Monday. Each hot dog requires $\frac{1}{4}$ pound of pork product. All the other ingredients in the hot dogs and hot dog buns are in plentiful supply. Finally, the labor force at Weenies and Buns consists of 5 employees working full time (40 hours per week each). Each hot dog requires 3 minutes of labor, and each hot dog bun requires 2 minutes of labor. Each hot dog yields a profit of \$0.80, and each bun yields a profit of \$0.30.

Weenies and Buns would like to know how many hot dogs and how many hot dog buns they should produce each week so as to achieve the highest possible profit.

(a) Formulate a linear programming model for this problem.

D,I (b) Use the graphical method to solve this model.

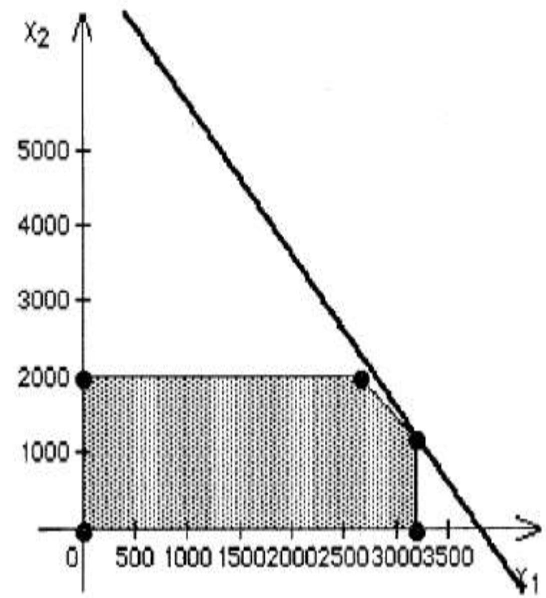
3.1-10.

(a) maximize $\quad P = 0.8H + 0.3B$

subject to

$$0.1B \leq 200$$
$$0.25H \leq 800$$
$$3H + 2B \leq 12{,}000$$
$$H, B \geq 0$$

(b) Optimal Solution: $(x_1^*, x_2^*) = (3200, 1200)$ and $P^* = 2920$



D,I **3.2-4.** Use the graphical method to find all optimal solutions for the following model:

Maximize $Z = 500x_1 + 300x_2$,

subject to

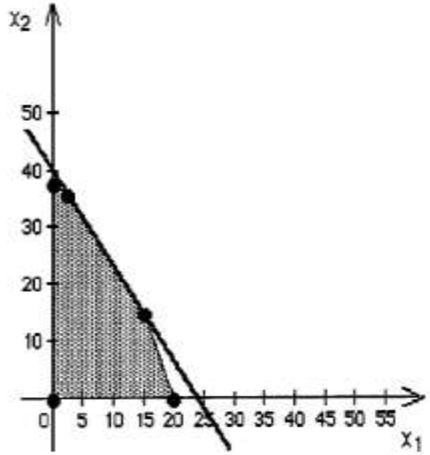
$$15x_1 + 5x_2 \le 300$$
$$10x_1 + 6x_2 \le 240$$
$$8x_1 + 12x_2 \le 450$$

and

$$x_1 \ge 0, \quad x_2 \ge 0.$$

3.2-4.

Optimal Solutions: $(x_1^*, x_2^*) = (15, 15)$, $(2.5, 35.833)$ and all points lying on the line connecting these two points, $Z^* = 12,000$



D **3.2-5.** Use the graphical method to demonstrate that the following model has no feasible solutions.

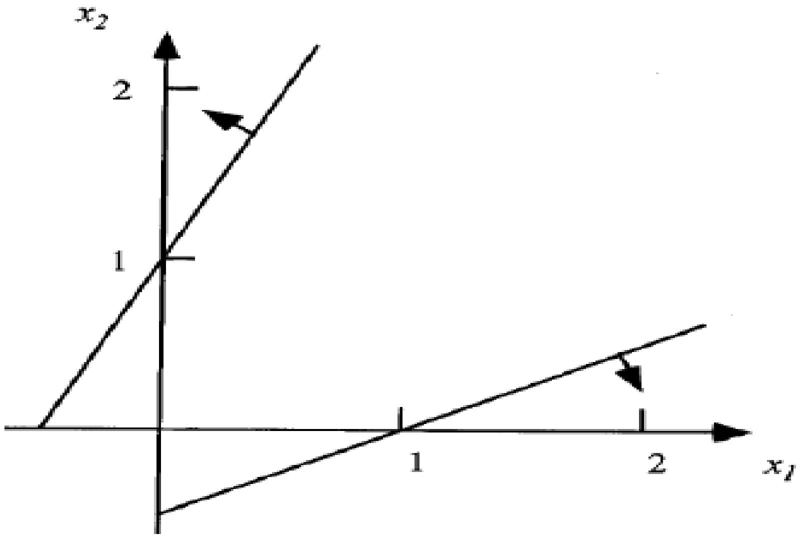
Maximize $Z = 5x_1 + 7x_2$,

subject to

$$2x_1 - x_2 \le -1$$
$$-x_1 + 2x_2 \le -1$$

and

$$x_1 \ge 0, \quad x_2 \ge 0.$$



D **3.4-7.** Consider the following problem, where the value of c_1 has not yet been ascertained.

Maximize $Z = c_1 x_1 + 2x_2$,

subject to

$$4x_1 + x_2 \le 12$$
$$x_1 - x_2 \ge 2$$

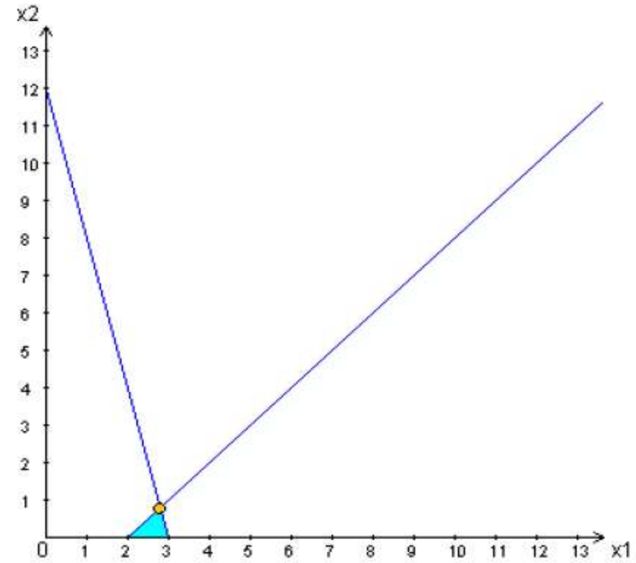
and

$$x_1 \ge 0, \quad x_2 \ge 0.$$

Use graphical analysis to determine the optimal solution(s) for (x_1, x_2) for the various possible values of c_1.

3.4-7.

The feasible region can be represented as follows:



Given $c_2 = 2 > 0$, various cases that may arise are summarized in the following table:

c_1	slope $= -\frac{c_1}{c_2}$	optimal solution (x_1^*, x_2^*)
$c_1 < -2$	$1 < -\frac{c_1}{c_2}$	$(2, 0)$
$c_1 = -2$	$-\frac{c_1}{c_2} = 1$	$(2, 0)$, $\left(\frac{14}{5}, \frac{4}{5}\right)$ and all points on the line connecting these two
$-2 < c_1 < 24$	$-12 < -\frac{c_1}{c_2} < 1$	$\left(\frac{14}{5}, \frac{4}{5}\right)$
$c_1 = 24$	$-\frac{c_1}{c_2} = -12$	$\left(\frac{14}{5}, \frac{4}{5}\right)$, $(3, 0)$ and all points on the line connecting these two
$24 < c_1$	$-\frac{c_1}{c_2} < -12$	$(3, 0)$