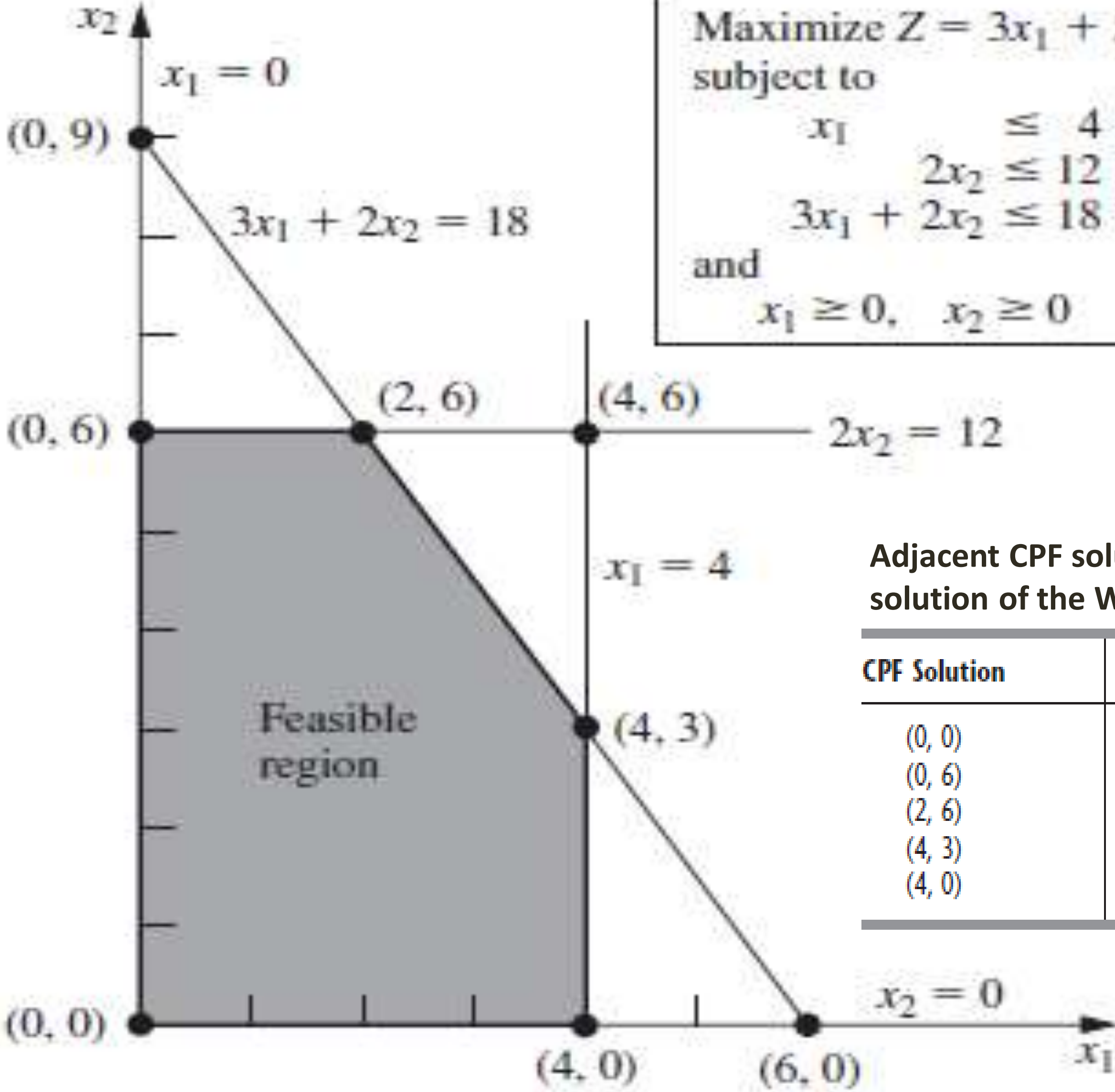
Module: THE SIMPLEX METHOD

THE ESSENCE OF THE SIMPLEX METHOD

- The simplex method is an *algebraic* procedure and the underlying concepts are *geometric*.
- To illustrate the general geometric concepts, we shall use the Wyndor Glass Co. example
- Each **constraint boundary** is a line that forms the boundary of what is permitted by the corresponding constraint
- The points of intersection are the **corner-point solutions** of the problem. The five that lie on the corners of the *feasible region*
  —(0, 0), (0, 6), (2, 6), (4, 3), and (4, 0) —are the ***corner point feasible solutions*** (**CPF solutions**).

For any linear programming problem with *n* decision variables, two CPF solutions are **adjacent** to each other if they share *n* -1 constraint boundaries. The two adjacent CPF solutions are connected by a line segment that lies on these same shared constraint boundaries. Such a line segment is referred to as an **edge** of the feasible region.



Maximize $Z = 3x_1 + 5x_2$,
subject to

$$\begin{aligned} x_1 \quad\quad\quad &\le 4 \\ 2x_2 &\le 12 \\ 3x_1 + 2x_2 &\le 18 \end{aligned}$$

and

$$x_1 \ge 0, \quad x_2 \ge 0$$

Adjacent CPF solutions for each CPF solution of the Wyndor Glass Co. problem

CPF Solution	Its Adjacent CPF Solutions
(0, 0)	(0, 6) and (4, 0)
(0, 6)	(2, 6) and (0, 0)
(2, 6)	(4, 3) and (0, 6)
(4, 3)	(4, 0) and (2, 6)
(4, 0)	(0, 0) and (4, 3)

Optimality test:

- Consider any linear programming problem that possesses at least one optimal solution.
- If a CPF solution has no *adjacent* CPF solutions that are *better* (as measured by *Z*), then it *must* be an *optimal* solution.
- Thus, for the example, (2, 6) must be optimal simply because its Z = 36 is larger than Z =30 for (0, 6) and Z 27 for (4, 3).
- Outline of what the simplex method does (from a geometric viewpoint) to solve the Wyndor Glass Co. problem.

  Initialization:
  - Choose (0, 0) as the *initial* CPF solution to examine. (This is a convenient choice because no calculations are required to identify this CPF solution.)
  - *Optimality Test:* Conclude that (0, 0) is *not* an optimal solution. (Adjacent CPF solutions are better.)

Iteration 1: Move to a better *adjacent* CPF solution, (0, 6), by performing the following three steps.

1. Considering the two edges of the feasible region that emanate from (0, 0), choose to move along the edge that leads up the x_2 axis. (With an objective function of $Z = 3x_1 + 5x_2$, moving up the x_2 axis increases Z at a faster rate than moving along the x_1 axis.)
2. Stop at the first new constraint boundary: $2x_2 = 12$.

[Moving farther in the direction selected in step 1 leaves the feasible region; e.g., moving to the second new constraint boundary hit when moving in that direction gives (0, 9), which is a corner-point *infeasible* solution.]

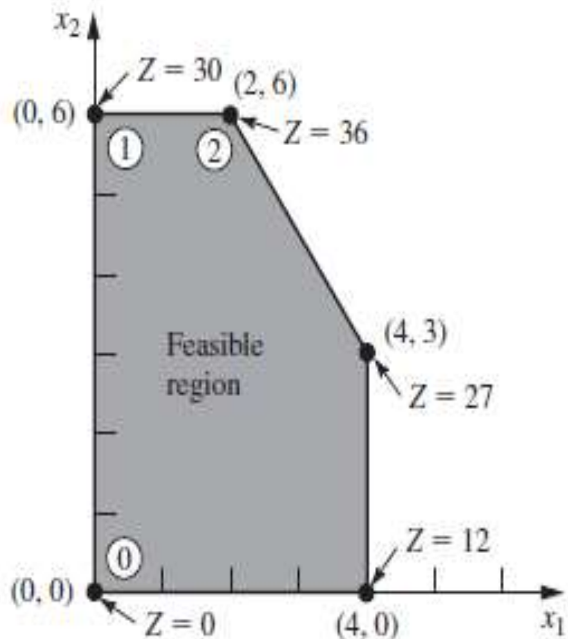
3. Solve for the intersection of the new set of constraint boundaries: (0, 6).

(The equations for these constraint boundaries, $x_1 = 0$ and $2x_2 = 12$, immediately yield this solution).

Optimality Test: Conclude that (0, 6) is *not* an optimal solution. (An adjacent CPF solution is better).

Iteration 2: Move to a better adjacent CPF solution, (2, 6), by performing the following three steps.

1. Considering the two edges of the feasible region that emanate from (0, 6), choose to move along the edge that leads to the right. (Moving along this edge increases *Z*, whereas backtracking to move back down the x_2 axis decreases *Z*.)
2. Stop at the first new constraint boundary encountered when moving in that direction: $3x_1 + 2x_2 = 12$. (Moving farther in the direction selected in step 1 leaves the feasible region.)
3. Solve for the intersection of the new set of constraint boundaries: (2, 6). (The equations for these constraint boundaries, $3x_1 + 2x_2 = 18$ and $2x_2 = 12$, immediately yield this solution.)



Optimality Test: Conclude that (2, 6) *is* an optimal solution, so stop. (None of the adjacent CPF solutions are better.)

The 6 Key Solution Concepts

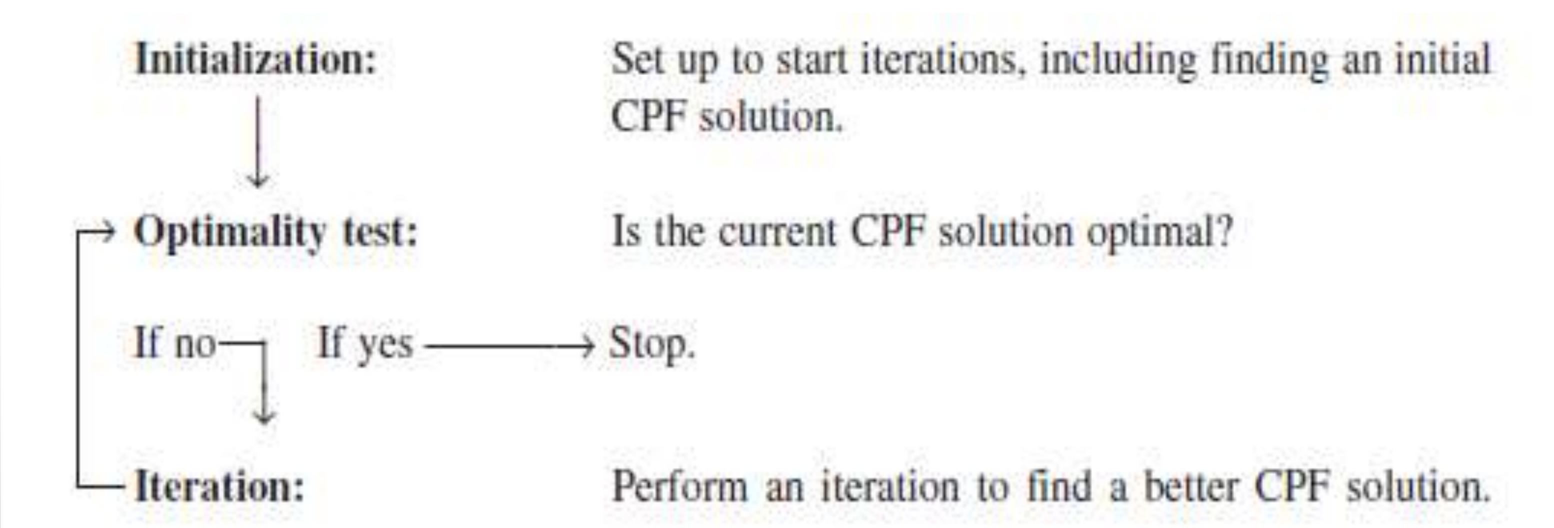
Solution concept 1:

The simplex method focuses solely on CPF solutions.

For any problem with at least one optimal solution, finding one requires only finding a best CPF solution.

Solution concept 2:

The simplex method is an *iterative algorithm* (a systematic solution procedure that keeps repeating a fixed series of steps, called an *iteration,* until a desired result has been obtained) with the following structure.



Solution concept 3:

Whenever possible, the initialization of the simplex method chooses the *origin* (all decision variables equal to zero) to be the initial CPF solution.

When there are too many decision variables to find an initial CPF solution graphically, this choice eliminates the need to use algebraic procedures to find and solve for an initial CPF solution.

Solution concept 4:

Given a CPF solution, it is much quicker computationally to gather information about its *adjacent* CPF solutions than about other CPF solutions.

Therefore, each time the simplex method performs an iteration to move from the current CPF solution to a better one, it *always* chooses a CPF solution that is *adjacent* to the current one.

No other CPF solutions are considered.

Consequently, the entire path followed to eventually reach an optimal solution is along the *edges* of the feasible region.

Solution concept 5:

After the current CPF solution is identified, the simplex method examines each of the edges of the feasible region that emanate from this CPF solution.

Each of these edges leads to an *adjacent* CPF solution at the other end, but the simplex method does not even take the time to solve for the adjacent CPF solution. Instead, it simply identifies the *rate of improvement in Z* that would be obtained by moving along the edge.

Among the edges with a *positive* rate of improvement in *Z*, it then chooses to move along the one with the *largest* rate of improvement in *Z*.

The iteration is completed by first solving for the adjacent CPF solution at the other end of this one edge and then relabeling this adjacent CPF solution as the *current* CPF solution for the optimality test and (if needed) the next iteration.

Solution concept 6:

Solution concept 5 describes how the simplex method examines each of the edges of the feasible region that emanate from the current CPF solution.

This examination of an edge leads to quickly identifying the rate of improvement in *Z* that would be obtained by moving along the edge toward the adjacent CPF solution at the other end. A *positive* rate of improvement in *Z* implies that the adjacent CPF solution is *better* than the current CPF solution, whereas a *negative* rate of improvement in *Z* implies that the adjacent CPF solution is *worse*.

Therefore, the optimality test consists simply of checking whether *any* of the edges give a *positive* rate of improvement in *Z*. If *none* do, then the current CPF solution is optimal

SETTING UP THE SIMPLEX METHOD

- First step in setting up the simplex method is to convert the functional *inequality constraints* to equivalent *equality constraints*.
- We introduce the **slack variables i. e.** the original constraint $x_1 \leq 4$ is entirely *equivalent* to the pair of constraints $x_1 + x_3 = 4$ and $x_3 \geq 0$.

An **augmented solution** is a solution for the original variables (the *decision variables*) that has been augmented by the corresponding values of the *slack variables*. E. g. augmenting the solution (3, 2) in the example yields the augmented solution (3, 2, 1, 8, 5) because the corresponding values of the slack variables are $x_3 = 1$, $x_4 = 8$, and $x_5 = 5$.

A **basic solution** is an *augmented* corner-point solution. That is, a **basic feasible (BF) solution** is an *augmented* CPF solution.

Original Form of the Model

Maximize $Z = 3x_1 + 5x_2$,

subject to

$$x_1 \leq 4$$
$$2x_2 \leq 12$$
$$3x_1 + 2x_2 \leq 18$$

and

$$x_1 \geq 0, \quad x_2 \geq 0.$$

Augmented Form of the Model[4]

Maximize $Z = 3x_1 + 5x_2$,

subject to

$$(1) \quad x_1 + x_3 = 4$$
$$(2) \quad 2x_2 + x_4 = 12$$
$$(3) \quad 3x_1 + 2x_2 + x_5 = 18$$

and

$$x_j \geq 0, \quad \text{for } j = 1, 2, 3, 4, 5.$$

Thus the system with functional constraints has 5 variables and 3 equations, so
Number of variables number of equations   5 - 3 = 2.
This fact gives us 2 *degrees of freedom* in solving the system, since any two variables can be chosen to be set equal to any arbitrary value in order to solve the three equations in terms of the remaining three variables

A **basic solution** has the following properties:

1. Each variable is designated as either a nonbasic variable or a basic variable.
2. The *number of basic variables* equals the number of functional constraints (now equations). Therefore, the *number of nonbasic variables* equals the total number of variables *minus* the number of functional constraints. (i. e. 5-3 =2)
3. The **nonbasic variables** are set equal to zero.
4. The values of the **basic variables** are obtained as the simultaneous solution of the system of equations (functional constraints in augmented form). (The set of basic variables is often referred to as **the basis.**)
5. If the basic variables satisfy the *non-negativity constraints,* the basic solution is a **BF solution.**

E. g. choose x_1 and x_4 to be the two nonbasic variables, and so the two variables are set equal to zero. Because all three of these basic **variables are nonnegative, this *basic*** *solution* (0, 6, 4, 0, 6) is indeed a *BF solution*

$x_1 = 0$ and $x_4 = 0$ so

$$
\begin{array}{lllll}
(1) & x_1 + x_3 & = 4 & & x_3 = 4 \\
(2) & 2x_2 + x_4 & = 12 & & x_2 = 6 \\
(3) & 3x_1 + 2x_2 + x_5 & = 18 & & x_5 = 6
\end{array}
$$

- Two BF solutions are **adjacent** if *all but one* of their *nonbasic variables* are the same.

This implies that *all but one* of their *basic variables* also are the same, although perhaps with different numerical values.

- *For example, adjacent BF solutions,* consider one pair of adjacent CPF solutions, say (0, 0) and (0, 6). Their augmented solutions, (0, 0, 4, 12, 18) and (0, 6, 4, 0, 6), automatically are adjacent BF solutions.
- A signpost is that their nonbasic variables, (x_1, x_2) and (x_1, x_4), are the same with just the one exception—x_2 has been replaced by x_4.

Consequently, moving from (0, 0, 4, 12, 18) to (0, 6, 4, 0, 6) involves switching x_2 from nonbasic to basic and vice versa for x_4.

Augmented form

Consider the objective function equation at the same time as the new constraint equations.

Therefore, before we start the simplex method, the problem needs to be rewritten once again in an equivalent way:

Maximize Z,

subject to

$$
\begin{array}{llr}
(0) & Z - 3x_1 - 5x_2 & = 0 \\
(1) & x_1 + x_3 & = 4 \\
(2) & 2x_2 + x_4 & = 12 \\
(3) & 3x_1 + 2x_2 + x_5 & = 18
\end{array}
$$

and

$x_j \geq 0, \quad \text{for } j = 1, 2, \ldots, 5.$

Geometric and algebraic interpretations of how the simplex method solves the Wyndor Glass Co. problem

Method Sequence	Geometric Interpretation	Algebraic Interpretation
Initialization	Choose (0, 0) to be the initial CPF solution.	Choose x_1 and x_2 to be the nonbasic variables (= 0) for the initial BF solution: (0, 0, 4, 12, 18).
Optimality test	Not optimal, because moving along either edge from (0, 0) increases Z.	Not optimal, because increasing either nonbasic variable (x_1 or x_2) increases Z.
Iteration 1		
Step 1	Move up the edge lying on the x_2 axis.	Increase x_2 while adjusting other variable values to satisfy the system of equations.
Step 2	Stop when the first new constraint boundary ($2x_2 = 12$) is reached.	Stop when the first basic variable (x_3, x_4, or x_5) drops to zero (x_4).
Step 3	Find the intersection of the new pair of constraint boundaries: (0, 6) is the new CPF solution.	With x_2 now a basic variable and x_4 now a nonbasic variable, solve the system of equations: (0, 6, 4, 0, 6) is the new BF solution.
Optimality test	Not optimal, because moving along the edge from (0, 6) to the right increases Z.	Not optimal, because increasing one nonbasic variable (x_1) increases Z.
Iteration 2		
Step 1	Move along this edge to the right.	Increase x_1 while adjusting other variable values to satisfy the system of equations.
Step 2	Stop when the first new constraint boundary ($3x_1 + 2x_2 = 18$) is reached.	Stop when the first basic variable (x_2, x_3, or x_5) drops to zero (x_5).
Step 3	Find the intersection of the new pair of constraint boundaries: (2, 6) is the new CPF solution.	With x_1 now a basic variable and x_5 now a nonbasic variable, solve the system of equations: (2, 6, 2, 0, 0) is the new BF solution.
Optimality test	(2, 6) is optimal, because moving along either edge from (2, 6) decreases Z.	(2, 6, 2, 0, 0) is optimal, because increasing either nonbasic variable (x_4 or x_5) decreases Z.

1. Initialization

The choice of x_1 and x_2 to be the *nonbasic* variables (the variables set equal to zero) for the initial BF solution is based on solution concept 3. This choice eliminates the work required to solve for the *basic variables* (x_3, x_4, x_5) from the following system of equations:

$$
\begin{array}{llcl}
(1) & x_1 \quad\quad + x_3 & = 4 \\
(2) & \quad 2x_2 \quad\quad + x_4 & = 12 \\
(3) & 3x_1 + 2x_2 \quad\quad + x_5 & = 18
\end{array}
\qquad
\begin{array}{r}
x_1 = 0 \text{ and } x_2 = 0 \text{ so} \\
x_3 = 4 \\
x_4 = 12 \\
x_5 = 18
\end{array}
$$

Thus, the **initial BF solution** is (0, 0, 4, 12, 18).

2. Optimality Test

The objective function is $Z = 3x_1 + 5x_2$, so $Z = 0$ for the initial BF solution.

None of the basic variables (x_3, x_4, x_5) have a *nonzero* coefficient in this objective function, the coefficient of each nonbasic variable (x_1, x_2) gives the rate of improvement in Z if that variable were to be increased from zero .
These rates of improvement (3 and 5) are *positive*. Therefore, based on solution concept 6 in above, we conclude that (0, 0, 4, 12, 18) is not optimal.

3. Determining the Direction of Movement (Step 1 of an Iteration)

Based on solution concepts 4 and 5 in above, the choice of which nonbasic variable to increase is made as follows:

$$Z = 3x_1 + 5x_2$$

Increase x_1? Rate of improvement in $Z = 3$.

Increase x_2? Rate of improvement in $Z = 5$.

$5 > 3$, so choose x_2 to increase.

As indicated next, we call x_2 the *entering basic variable* for iteration 1.

4. Determining Where to Stop (Step 2 of an Iteration)

Increasing x_2 increases *Z*, so we want to go as far as possible without leaving the feasible region. That is, increasing x_2 (while keeping the nonbasic variable $x_1 = 0$) changes the values of some of the basic var

$$x_1 = 0, \quad \text{so}$$

$$\begin{array}{llr|l}
(1) & x_1 + x_3 & = 4 & x_3 = 4 \\
(2) & 2x_2 + x_4 & = 12 & x_4 = 12 - 2x_2 \\
(3) & 3x_1 + 2x_2 + x_5 & = 18 & x_5 = 18 - 2x_2.
\end{array}$$

we need to check how far x_2 can be increased without violating the nonnegativity constraints for the basic variables.

We have,

$$x_3 = 4 \geq 0 \qquad \Rightarrow \text{no upper bound on } x_2.$$

$$x_4 = 12 - 2x_2 \geq 0 \Rightarrow x_2 \leq \frac{12}{2} = 6 \quad \leftarrow \text{minimum.}$$

$$x_5 = 18 - 2x_2 \geq 0 \Rightarrow x_2 \leq \frac{18}{2} = 9.$$

Thus, x_2 can be increased just to 6, at which point $\boldsymbol{x_4}$ has dropped to 0. Increasing x_2 beyond 6 would cause $\boldsymbol{x_4}$ to become negative, which would violate feasibility. These calculations are referred to as the **minimum ratio test.** The objective of this test is to determine which basic variable drops to zero first as the entering basic variable is increased.

Thus, $\boldsymbol{x_4}$ is the leaving basic variable for iteration 1 of the example

5. Solving for the New BF Solution (Step 3 of an Iteration)

Increasing $x_2 = 0$ to $x_2 = 6$ moves us from the *initial* BF solution on the left to the *new* BF solution on the right.

	Initial BF solution	New BF solution
Nonbasic variables:	$x_1 = 0, \quad x_2 = 0$	$x_1 = 0, \quad x_4 = 0$
Basic variables:	$x_3 = 4, \quad x_4 = 12, \quad x_5 = 18$	$x_3 = ?, \quad x_2 = 6, \quad x_5 = ?$

The purpose of step 3 is to convert the system of equations to a more convenient form (proper form from Gaussian elimination) for conducting the optimality test and (if needed) the next iteration with this new BF solution.

Consider

$$\begin{array}{llllllll}
(0) & Z - 3x_1 - 5x_2 & & & & = & 0 \\
(1) & \quad x_1 & + x_3 & & & = & 4 \\
(2) & \quad\quad 2x_2 & & + x_4 & & = & 12 \\
(3) & \quad 3x_1 + 2x_2 & & & + x_5 & = & 18.
\end{array}$$

Thus, x_2 has replaced *x*4 as the basic variable in Eq. (2).
To solve this system of equations for *Z*, x_2, x_3, and x_5, we need to perform some **elementary algebraic operations** to reproduce the current pattern of coefficients of *x*4 (0, 0, 1, 0) as the new coefficients of *x*2.

We can use either of two types of elementary algebraic operations:

1. Multiply (or divide) an equation by a nonzero constant.

2. Add (or subtract) a multiple of one equation to (or from) another equation.The coefficients of x_2 in the above system of equations are -5, 0, 2, and 2, respectively.

To turn the coefficient of 2 in Eq. (2) into 1, we use the first type of elementary algebraic operation by dividing Eq. (2) by 2 to obtain

$$(2) \qquad x_2 + \frac{1}{2}x_4 = 6.$$

Add 5 times this new Eq. (2) to Eq. (0), and subtract 2 times this new Eq. (2) from Eq. (3).
The resulting complete new system of equations is:

$$
\begin{array}{llcr}
(0) & Z - 3x_1 & + \frac{5}{2}x_4 & = 30 \\
(1) & x_1 + x_3 & & = 4 \\
(2) & x_2 & + \frac{1}{2}x_4 & = 6 \\
(3) & 3x_1 & - x_4 + x_5 & = 6.
\end{array}
$$

Since $x_1 = 0$ and $x_4 = 0$, the equations in this form immediately yield the new BF solution, ($x1$, $x2$, $x3$, $x4$, $x5$) (0, 6, 4, 0, 6), which yields $Z = 30$.
This procedure for obtaining the simultaneous solution of a system of linear equations is called the *Gauss-Jordan method of elimination,* or **Gaussian elimination.**

6. Optimality Test for the New BF Solution

The current Eq. (0) gives the value of the objective function in terms of just the current nonbasic variables:

$$Z = 30 + 3x_1 - \frac{5}{2}x_4.$$

Because x_1 has a *positive* coefficient, increasing x_1 would lead to an adjacent BF solution that is better than the current BF solution, so the current solution is not optimal.

7. Iteration 2 and the Resulting Optimal Solution

Since $Z = 30 + 3x_1 - \frac{5}{2}x_4$, Z can be increased by increasing x_1, but not x_4. Therefore, step 1 chooses x_1 to be the entering basic variable.

For step 2, the current system of equations yields the following conclusions about how far x_1 can be increased (with $x_4 = 0$):

$$x_3 = 4 - x_1 \geq 0 \quad \Rightarrow x_1 \leq \frac{4}{1} = 4.$$

$$x_2 = 6 \geq 0 \qquad \Rightarrow \text{no upper bound on } x_1.$$

$$x_5 = 6 - 3x_1 \geq 0 \Rightarrow x_1 \leq \frac{6}{3} = 2 \quad \leftarrow \text{minimum.}$$

Therefore, the minimum ratio test indicates that x_5 is the leaving basic variable.

For step 3, with x_1 replacing x_5 as a basic variable, we perform elementary algebraic operations on the current system of equations to reproduce the current pattern of coefficients of x_5 (0, 0, 0, 1) as the new coefficients of x_1. This yields the following new system of equations:

$$\begin{array}{llllllll}
(0) & Z & & & + \frac{3}{2}x_4 & + x_5 & = & 36 \\
(1) & & & x_3 & + \frac{1}{3}x_4 & - \frac{1}{3}x_5 & = & 2 \\
(2) & & x_2 & & + \frac{1}{2}x_4 & & = & 6 \\
(3) & x_1 & & & - \frac{1}{3}x_4 & + \frac{1}{3}x_5 & = & 2.
\end{array}$$

Therefore, the next BF solution is $(x_1, x_2, x_3, x_4, x_5) = (2, 6, 2, 0, 0)$, yielding $Z = 36$. To apply the *optimality test* to this new BF solution, we use the current Eq. (0) to express Z in terms of just the current nonbasic variables,

$$Z = 36 - \frac{3}{2}x_4 - x_5.$$

Increasing either $x4$ or $x5$ would *decrease Z,* so neither adjacent BF solution is as good as the current one.

Therefore, based on solution concept 6 in above, the current BF solution must be optimal.

Which is similar to the original form of the problem (no slack variables), the optimal solution is $x_1 = 2$, $x_2 = 6$, which yields $Z = 3x_1 + 5x_2 = 36$.

The tabular form of the simplex method records only the essential information, namely, (1) the coefficients of the variables, (2) the constants on the right-hand sides of the equations, and (3) the basic variable appearing in each equation.

Initial system of equations for the Wyndor Glass Co. problem

(a) Algebraic Form	(b) Tabular Form								
	Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	Right Side
(0) $Z - 3x_1 - 5x_2 = 0$	Z	(0)	1	−3	−5	0	0	0	0
(1) $x_1 + x_3 = 4$	x_3	(1)	0	1	0	1	0	0	4
(2) $2x_2 + x_4 = 12$	x_4	(2)	0	0	2	0	1	0	12
(3) $3x_1 + 2x_2 + x_5 = 18$	x_5	(3)	0	3	2	0	0	1	18

Initialization.

1. Introduce slack variables.
2. Select the *decision variables* to be the *initial nonbasic variables* (set equal to zero) and
3. the *slack variables* to be the *initial basic variables.*

This selection yields the initial simplex tableau shown in column (*b*), so the initial BF solution is (0, 0, 4, 12, 18).

Optimality Test.

The current BF solution is optimal if and only if *every* coefficient in row 0 is nonnegative (≥ 0).

If it is, stop; otherwise, go to an iteration to obtain the next BF solution, which involves changing one nonbasic variable to a basic variable (step 1) and vice versa (step 2) and then solving for the new solution (step 3).

Initial system of equations for the Wyndor Glass Co. problem

(a) Algebraic Form	(b) Tabular Form								
	Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	Right Side
(0) $Z - 3x_1 - 5x_2 = 0$	Z	(0)	1	-3	-5	0	0	0	0
(1) $x_1 + x_3 = 4$	x_3	(1)	0	1	0	1	0	0	4
(2) $2x_2 + x_4 = 12$	x_4	(2)	0	0	2	0	1	0	12
(3) $3x_1 + 2x_2 + x_5 = 18$	x_5	(3)	0	3	2	0	0	1	18

Iteration.

Step 1: Determine the *entering basic variable* by selecting the variable (automatically a nonbasic variable) with the *negative coefficient* having the largest absolute value (i.e., the "most negative" coefficient) in Eq. (0).
Put a box around the column below this coefficient, and call this the **pivot column.**

Step 2: Determine the *leaving basic variable* by applying the minimum ratio test.

Minimum Ratio Test

1. Pick out each coefficient in the pivot column that is strictly positive (>0).
2. Divide each of these coefficients into the *right side* entry for the same row.
3. Identify the row that has the *smallest* of these ratios.
4. The basic variable for that row is the leaving basic variable, so replace that variable by the entering basic variable in the basic variable column of the next simplex tableau.

Applying the minimum ratio test to determine the first leaving basic variable

Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	Right Side	Ratio
Z	(0)	1	−3	−5	0	0	0	0	
x_3	(1)	0	1	$\boxed{0}$	1	0	0	4	
x_4	(2)	0	0	$\boxed{2}$	0	1	0	12	$\to \frac{12}{2} = 6 \leftarrow$ minimum
x_5	(3)	0	3	$\boxed{2}$	0	0	1	18	$\to \frac{18}{2} = 9$

Next, put a box around this row and call it the **pivot row.**
Also call the number that is in *both* boxes the **pivot number.**

Step 3: Solve for the *new BF solution* by using **elementary row operations**

a) Multiply or divide a row by a nonzero constant;
b) add or subtract a multiple of one row to another row to construct a new simplex tableau in proper form from Gaussian elimination below the current one, and then return to the optimality test.

The specific elementary row operations that need to be performed are listed below.

1. Divide the pivot row by the pivot number. Use this *new* pivot row in steps 2 and 3.

2. For each other row (including row 0) that has a *negative* coefficient in the pivot column, *add* to this row the *product* of the absolute value of this coefficient and the new pivot row.

3. For each other row that has a *positive* coefficient in the pivot column, *subtract* from this row the *product* of this coefficient and the new pivot row.

Simplex tableaux for the Wyndor Glass Co. problem after the first pivot row is divided by the first pivot number

Iteration	Basic Variable	Eq.	Z	x_1	x_2	x_3	x_4	x_5	Right Side
			Coefficient of:						
0	Z	(0)	1	−3	−5	0	0	0	0
	x_3	(1)	0	1	0	1	0	0	4
	x_4	(2)	0	0	2	0	1	0	12
	x_5	(3)	0	3	2	0	0	1	18
1	Z	(0)	1						
	x_3	(1)	0						
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6
	x_5	(3)	0						

To start, divide the pivot row (row 2) by the pivot number (2), which gives the new row 2. (See Table previous slide)

Next, we add to row 0 the product, 5 times the new row 2.

Then we subtract from row 3 the product, 2 times the new row 2 (or equivalently, subtract from row 3 the *old* row 2).

These calculations yield the new tableau see Table below for iteration 1.

Thus, the new BF solution is (0, 6, 4, 0, 6), with $Z = 30$.

We next return to the optimality test to check if the new BF solution is optimal.

Since the new row 0 still has a negative coefficient (-3 for x_1), the solution is not optimal, and so at least one more iteration is needed.

First two simplex tableaux

Iteration	Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	Right Side
0	Z	(0)	1	−3	−5	0	0	0	0
	x_3	(1)	0	1	0	1	0	0	4
	x_4	(2)	0	0	2	0	1	0	12
	x_5	(3)	0	3	2	0	0	1	18
1	Z	(0)	1	−3	0	0	$\frac{5}{2}$	0	30
	x_3	(1)	0	1	0	1	0	0	4
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6
	x_5	(3)	0	3	0	0	−1	1	6

Following the instructions for steps 1 and 2, we find x_1 as the entering basic variable and x_5 as the leaving basic variable (See Table below)

For step 3, we start by dividing the pivot row (row 3) in Table below by the pivot number (3).
Next, we add to row 0 the product, 3 times the new row 3. Then we subtract the new row 3 from row 1.

Steps 1 and 2 of iteration 2

Iteration	Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	Right Side	Ratio
	Z	(0)	1	−3	0	0	$\frac{5}{2}$	0	30	
	x_3	(1)	0	1	0	1	0	0	4	$\frac{4}{1} = 4$
1	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6	
	x_5	(3)	0	3	0	0	−1	1	6	$\frac{6}{3} = 2 \leftarrow$ minimum

We now have the set of tableaux shown in Table below.
Therefore, the new BF solution is (2, 6, 2, 0, 0), with $Z = 36$. Going to the optimality test, we find that this solution is *optimal* because none of the coefficients in row 0 is negative, so the algorithm is finished

Complete set of simplex tableaux

Iteration	Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	Right Side
0	Z	(0)	1	−3	−5	0	0	0	0
	x_3	(1)	0	1	0	1	0	0	4
	x_4	(2)	0	0	2	0	1	0	12
	x_5	(3)	0	3	2	0	0	1	18
1	Z	(0)	1	−3	0	0	$\frac{5}{2}$	0	30
	x_3	(1)	0	1	0	1	0	0	4
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6
	x_5	(3)	0	3	0	0	−1	1	6
2	Z	(0)	1	0	0	0	$\frac{3}{2}$	1	36
	x_3	(1)	0	0	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	2
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6
	x_1	(3)	0	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	2

TIE BREAKING IN THE SIMPLEX METHOD

1. Tie for the Entering Basic Variable

- Step 1 of each iteration chooses the nonbasic variable having the *negative* coefficient with the *largest absolute value* in the current Eq. (0) as the entering basic variable.
- The answer is that the selection between these contenders may be made *arbitrarily.*
- The optimal solution will be reached eventually, regardless of the tied variable chosen, and there is no convenient method for predicting in advance which choice will lead there sooner.

2. Tie for the Leaving Basic Variable—Degeneracy

- Suppose that two or more basic variables tie for being the leaving basic variable in step 2 of an iteration. Does it matter which one is chosen?
- Theoretically it does, and in a very critical way, because of the following sequence of events that could occur.

- First, all the tied basic variables reach zero simultaneously as the entering basic variable is increased. Therefore, the one or ones *not* chosen to be the leaving basic variable also will have a value of zero in the new BF solution. (Note that basic variables with a value of *zero* are called **degenerate,** and the same term is applied to the corresponding BF solution.)

- Second, if one of these degenerate basic variables retains its value of zero until it is chosen at a subsequent iteration to be a leaving basic variable, the corresponding entering basic variable also must remain zero (since it cannot be increased without making the leaving basic variable negative), so the value of *Z* must remain unchanged.

- Third, if *Z* may remain the same rather than increase at each iteration, the simplex method may then go around in a loop, repeating the same sequence of solutions periodically rather than eventually increasing *Z* toward an optimal solution.

3. No Leaving Basic Variable—Unbounded *Z*

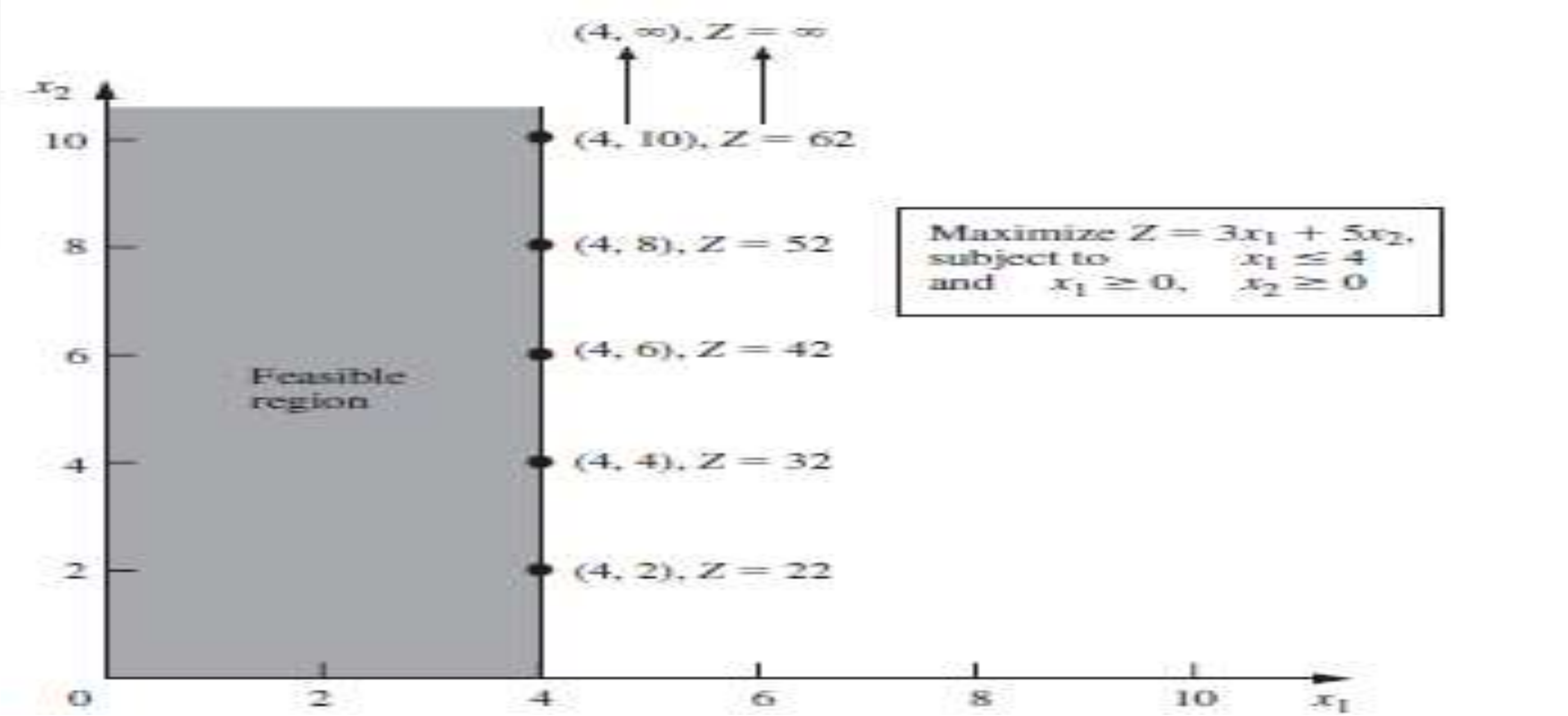
- In step 2 of an iteration, there is one other possible outcome that we have not yet discussed, namely, that *no* variable qualifies to be the leaving basic variable.
- This outcome would occur if the entering basic variable could be increased *indefinitely* without giving negative values to *any* of the current basic variables.
- In tabular form, this means that *every* coefficient in the pivot column (excluding row 0) is either negative or zero.
- Note in figure at the right how x_2 can be increased indefinitely (thereby increasing Z indefinitely) without ever leaving the feasible region.
- Then note in table below that x_2 is the entering basic variable but the only coefficient in the pivot column is zero. Because the minimum ratio test uses only coefficients that are greater than zero, there is no ratio to provide a leaving basic variable.

Interpretation of a tableau like the one shown in Table Next slide is that;

a) The constraints do not prevent the value of the objective function *Z* from increasing indefinitely, so the simplex method would stop with the message that *Z* is *unbounded.*

b) Because even linear programming has not discovered a way of making infinite profits, the real message for practical problems is that a mistake has been made!

c) The model probably has been mis-formulated, either by omitting relevant constraints or by stating them incorrectly. Alternatively, a computational mistake may have occurred.



Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	Right Side	Ratio
Z	(0)	1	−3	−5	0	0	
x_3	(1)	0	1	$\boxed{0}$	1	4	None

With $x_1 = 0$ and x_2 increasing,
$x_3 = 4 - 1x_1 - 0x_2 = 4 > 0$.

4. Multiple Optimal Solutions

- We can see in figure below, (under the definition of **optimal solution**) that a problem can have more than one optimal solution. This fact was illustrated by changing the objective function in the Wyndor Glass Co. problem to $Z = 3x_1 + 2x_2$, so that every point on the line segment between (2, 6) and (4, 3) is optimal.
- Thus, all optimal solutions are a *weighted average* of these two optimal CPF solutions

$$(x_1, x_2) = w_1(2, 6) + w_2(4, 3),$$

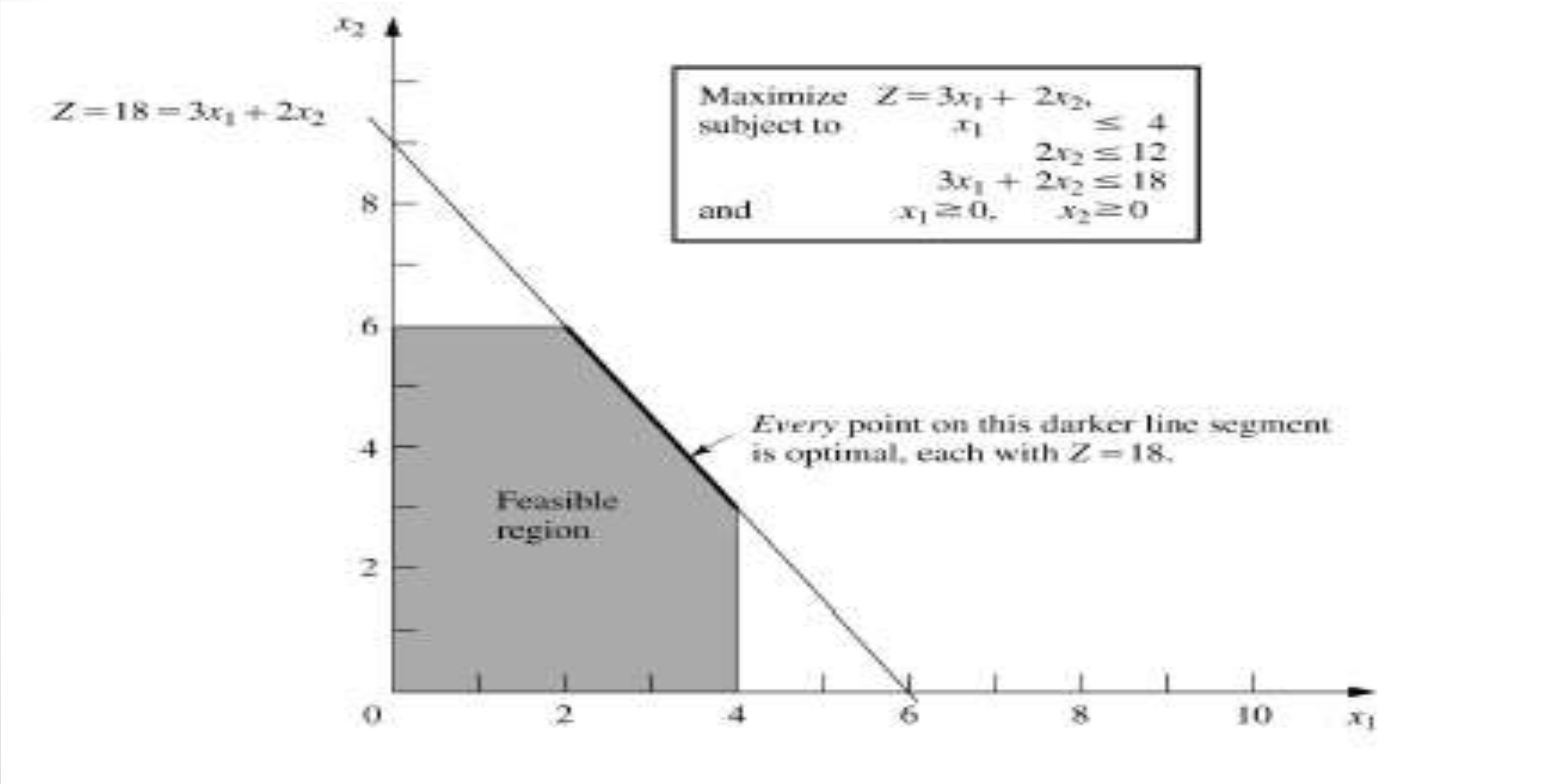
where the weights w_1 and w_2 are numbers that satisfy the relationships

$$w_1 + w_2 = 1 \qquad \text{and} \qquad w_1 \geq 0, \qquad w_2 \geq 0.$$

For example, $w_1 = \frac{1}{3}$ and $w_2 = \frac{2}{3}$ give

$$(x_1, x_2) = \frac{1}{3}(2, 6) + \frac{2}{3}(4, 3) = \left(\frac{2}{3} + \frac{8}{3}, \ \frac{6}{3} + \frac{6}{3}\right) = \left(\frac{10}{3}, \ 4\right)$$

- However, because a nonbasic variable (x_3) then has a zero coefficient in row 0, we perform one more iteration in table (see next slide) to identify the other optimal BF solution.
- Thus, the two optimal BF solutions are (4, 3, 0, 6, 0) and (2, 6, 2, 0, 0), each yielding $Z = 18$.
- Notice that the last tableau also has a *nonbasic* variable (x_4) with a zero coefficient in row 0. This situation is inevitable because the extra iteration does not change row 0, so this leaving basic variable necessarily retains its zero coefficient.
- Making x_4 an entering basic variable now would only lead back to the third tableau



Therefore, these two are the only BF solutions that are optimal, and all *other* optimal solutions are a convex combination of these two.

$$(x_1, x_2, x_3, x_4, x_5) = w_1(2, 6, 2, 0, 0) + w_2(4, 3, 0, 6, 0),$$
$$w_1 + w_2 = 1, \qquad w_1 \geq 0, \qquad w_2 \geq 0.$$

Iteration	Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	Right Side	Solution Optimal?
0	Z	(0)	1	−3	−2	0	0	0	0	No
	x_3	(1)	0	1	0	1	0	0	4	
	x_4	(2)	0	0	2	0	1	0	12	
	x_5	(3)	0	3	2	0	0	1	18	
1	Z	(0)	1	0	−2	3	0	0	12	No
	x_1	(1)	0	1	0	1	0	0	4	
	x_4	(2)	0	0	2	0	1	0	12	
	x_5	(3)	0	0	2	−3	0	1	6	
2	Z	(0)	1	0	0	0	0	1	18	Yes
	x_1	(1)	0	1	0	1	0	0	4	
	x_4	(2)	0	0	0	3	1	−1	6	
	x_2	(3)	0	0	1	$-\frac{3}{2}$	0	$\frac{1}{2}$	3	
Extra	Z	(0)	1	0	0	0	0	1	18	Yes
	x_1	(1)	0	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	2	
	x_3	(2)	0	0	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	2	
	x_2	(3)	0	0	1	0	$\frac{1}{2}$	0	6	

Ex 1: Consider the following problem.

$$\text{Maximize } Z = x_1 + 2x_2,$$

subject to

$$x_1 \le 5$$

$$x_2 \le 6$$

$$x_1 + x_2 \le 8$$

and

$$x_1 \ge 0, x_2 \ge 0.$$

(a) Plot the feasible region and circle all the CPF solutions.

(b) For each CPF solution, identify the pair of constraint boundary equations that it satisfies.

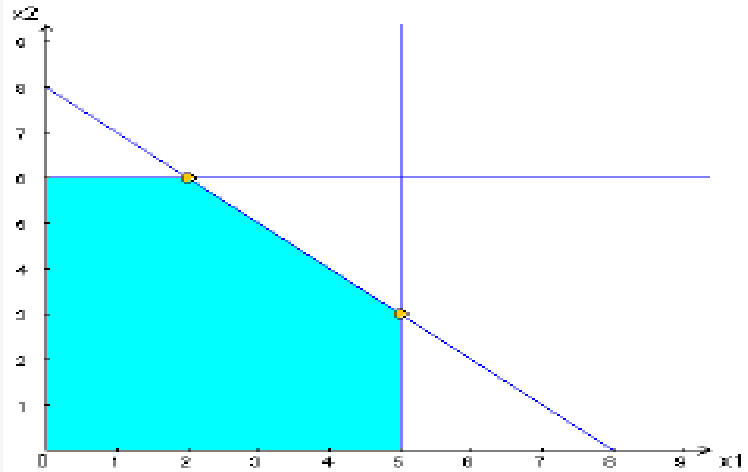
(c) For each CPF solution, use this pair of constraint boundary equations to solve algebraically for the values of $x1$ and $x2$ at the corner point.

(d) For each CPF solution, identify its adjacent CPF solutions.

(e) For each pair of adjacent CPF solutions, identify the constraint boundary they share by giving its equation

4.1-1.

(a) Label the corner points as A, B, C, D, and E in the clockwise direction starting from $(0, 6)$.



(b)

A:	$x_1 = 0$ and $x_2 = 6$
B:	$x_2 = 6$ and $x_1 + x_2 = 8$
C:	$x_1 + x_2 = 8$ and $x_1 = 5$
D:	$x_1 = 5$ and $x_2 = 0$
E:	$x_2 = 0$ and $x_1 = 0$

(c)

A:	$(x_1, x_2) = (0, 6)$
B:	$(x_1, x_2) = (6, 2)$
C:	$(x_1, x_2) = (5, 3)$
D:	$(x_1, x_2) = (5, 0)$
E:	$(x_1, x_2) = (0, 0)$

(d)

Corner Point	Adjacent Points
A	E, B
B	A, C
C	B, D
D	C, E
E	D, A

(e)

A and B:	$x_2 = 6$
B and C:	$x_1 + x_2 = 8$
C and D:	$x_1 = 5$
D and E:	$x_2 = 0$
E and A:	$x_1 = 0$

Ex. 2 Consider the following problem.

$$\text{Maximize } Z = 5x_1 + 9x_2 + 7x_3,$$

subject to

$$x_1 + 3x_2 + 2x_3 \leq 10$$
$$3x_1 + 4x_2 + 2x_3 \leq 12$$
$$2x_1 + x_2 + 2x_3 \leq 8$$

and

$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0.$$

(a) Work through the simplex method step by step in algebraic form.

(b) Work through the simplex method step by step in tabular form.

(c) Use a software package based on the simplex method to solve the problem.

4.4-5.

(a) Set $x_1 = x_2 = x_3 = 0$.

(0) $\quad Z - 5x_1 - 9x_2 - 7x_3 = 0$

(1) $\quad x_1 + 3x_2 + 2x_3 + x_4 = 10 \Rightarrow x_4 = 10$

(2) $\quad 3x_1 + 4x_2 + 2x_3 + x_5 = 12 \Rightarrow x_5 = 12$

(3) $\quad 2x_1 + x_2 + 2x_3 + x_6 = 8 \Rightarrow x_6 = 8$

Optimality Test: The coefficients of all nonbasic variables are positive, so the solution $(0, 0, 0, 10, 12, 8)$ is not optimal.

Choose x_2 as the entering basic variable, since it has the largest coefficient.

(1) $\quad x_1 + 3x_2 + 2x_3 + x_4 = 10 \Rightarrow x_4 = 10 - 3x_2 \Rightarrow x_2 \leq 10/3$

(2) $\quad 3x_1 + 4x_2 + 2x_3 + x_5 = 12 \Rightarrow x_5 = 12 - 4x_2 \Rightarrow x_2 \leq 3 \leftarrow$ minimum

(3) $\quad 2x_1 + x_2 + 2x_3 + x_6 = 8 \Rightarrow x_6 = 8 - x_2 \Rightarrow x_2 \leq 8$

We choose x_5 as the leaving basic variable. Set $x_1 = x_5 = x_3 = 0$.

(0) $\quad Z + 1.75x_1 - 2.5x_3 + 2.25x_5 = 27$

(1) $\quad -1.25x_1 + 0.5x_3 + x_4 - 0.75x_5 = 1 \Rightarrow x_4 = 1$

(2) $\quad 0.75x_1 + x_2 + 0.5x_3 + 0.25x_5 = 3 \Rightarrow x_2 = 3$

(3) $\quad 1.25x_1 + 1.5x_3 - 0.25x_5 + x_6 = 5 \Rightarrow x_6 = 5$

Optimality Test: The coefficient of x_3 is positive, so the solution $(0, 3, 0, 1, 0, 5)$ is not optimal.

Let x_3 be the entering basic variable.

(1)
$-1.25x_1 + 0.5x_3 + x_4 - 0.75x_5 = 1 \Rightarrow x_4 = 1 - 0.5x_3 \Rightarrow x_3 \le 2 \leftarrow$ minimum

(2) $\quad 0.75x_1 + x_2 + 0.5x_3 + 0.25x_5 = 3 \Rightarrow x_2 = 3 - 0.5x_3 \Rightarrow x_3 \le 6$

(3) $\quad 1.25x_1 + 1.5x_3 - 0.25x_5 + x_6 = 5 \Rightarrow x_6 = 5 - 1.5x_3 \Rightarrow x_3 \le 10/3$

We choose x_4 as the leaving basic variable. Set $x_1 = x_5 = x_4 = 0$.

(0) $\quad Z - 4.5x_1 + 5x_4 - 1.5x_5 = 32$

(1) $\quad -2.5x_1 + x_3 + 2x_4 - 1.5x_5 = 2 \Rightarrow x_3 = 2$

(2) $\quad 2x_1 + x_2 - x_4 + x_5 = 2 \Rightarrow x_2 = 2$

(3) $\quad 5x_1 - 3x_4 + 2x_5 + x_6 = 2 \Rightarrow x_6 = 2$

Optimality Test: The coefficient of x_1 is positive, so the solution $(0, 2, 2, 0, 0, 2)$ is not optimal.

Let x_1 be the entering basic variable.

(1) $\quad -2.5x_1 + x_3 + 2x_4 - 1.5x_5 = 2 \Rightarrow x_3 = 2 + 2.5x_1$

(2) $\quad 2x_1 + x_2 - x_4 + x_5 = 2 \Rightarrow x_2 = 2 - 2x_1 \Rightarrow x_1 \leq 1$

(3) $\quad 5x_1 - 3x_4 + 2x_5 + x_6 = 2 \Rightarrow x_6 = 2 - 5x_1 \Rightarrow x_1 \leq 0.4 \leftarrow$ minimum

We choose x_6 as the leaving basic variable. Set $x_6 = x_5 = x_4 = 0$.

(0) $\quad Z + 2.3x_4 + 0.3x_5 + 0.9x_6 = 33.8$

(1) $\quad x_3 + 0.5x_4 - 0.5x_5 + 0.5x_6 = 3 \Rightarrow x_3 = 3$

(2) $\quad x_2 + 0.2x_4 + 0.2x_5 - 0.4x_5 = 1.2 \Rightarrow x_2 = 1.2$

(3) $\quad x_1 - 0.6x_4 + 0.4x_5 + 0.2x_6 = 0.4 \Rightarrow x_1 = 0.4$

Optimality Test: The coefficients of all nonbasic variables are nonpositive, so the solution $(0.4, 1.2, 3, 0, 0, 0)$ is optimal.

(b) Optimal solution: $(x_1^*, x_2^*, x_3^*) = (0.4, 1.2, 3)$ and $Z^* = 33.8$

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
Z	0	1	-5	-9	-7	0	0	0	0
X4	1	0	1	3	2	1	0	0	10
X5	2	0	3	4*	2	0	1	0	12
X6	3	0	2	1	2	0	0	1	8

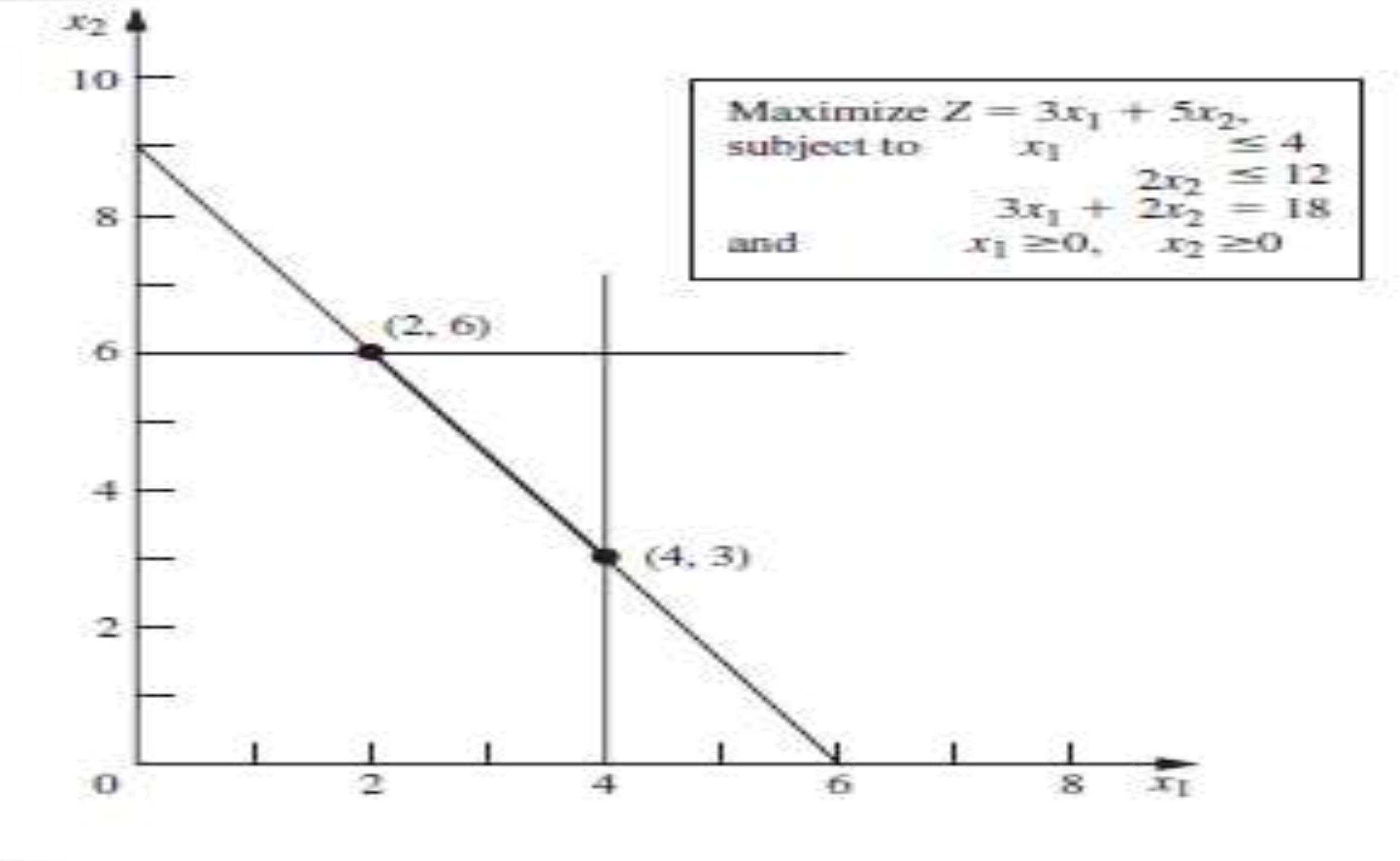
Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
Z	0	1	1.75	0	-2.5	0	2.25	0	27
X4	1	0	-1.25	0	0.5*	1	-0.75	0	1
X2	2	0	0.75	1	0.5	0	0.25	0	3
X6	3	0	1.25	0	1.5	0	-0.25	1	5

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
Z	0	1	-4.5	0	0	5	-1.5	0	32
X3	1	0	-2.5	0	1	2	-1.5	0	2
X2	2	0	2	1	0	-1	1	0	2
X6	3	0	5*	0	0	-3	2	1	2

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
Z	0	1	0	0	0	2.3	0.3	0.9	33.8
X3	1	0	0	0	1	0.5	-0.5	0.5	3
X2	2	0	0	1	0	0.2	0.2	-0.4	1.2
X1	3	0	1	0	0	-0.6	0.4	0.2	0.4

	Coefficient of					
	X1	X2	X3	Total		
Constraint 1	1	3	2	10	≤	10
Constraint 2	3	4	2	12	≤	12
Constraint 3	2	1	2	8	≤	8
Objective	5	9	7	33.8		
Solution	0.4	1.2	3			

Artificial Variable technique



$$
\begin{array}{lrcl}
(0) & Z - 3x_1 - 5x_2 & & = 0 \\
(1) & x_1 \quad\quad + x_3 & & = 4 \\
(2) & 2x_2 \quad\quad + x_4 & & = 12 \\
(3) & 3x_1 + 2x_2 & & = 18.
\end{array}
$$

Obtaining an Initial BF Solution. The procedure is to construct an **artificial problem** that has the same optimal solution as the real problem by making two modifications of the real problem

1. Apply the **artificial-variable technique** by introducing a *nonnegative* **artificial variable** (call it *x*5) into Eq. (3), just as if it were a slack variable

$$(3) \qquad 3x_1 + 2x_2 + \overline{x_5} = 18.$$

2. Assign an *overwhelming penalty* to having $\overline{x_5} > 0$ by changing the objective function

$$Z = 3x_1 + 5x_2 \quad \text{to}$$

$$Z = 3x_1 + 5x_2 - M\overline{x_5}$$

where *M* symbolically represents a *huge* positive number. (This method of forcing $\overline{x_5}$ to be $\overline{x_5} = 0$ in the optimal solution is called the **Big *M* method.**)

Now find the optimal solution for the real problem by applying the simplex method to the artificial problem, starting with the following initial BF solution:

Initial BF Solution Nonbasic variables: $x_1 = 0$, $x_2 = 0$

Basic variables: $x_3 = 4$, $x_4 = 12$, $\overline{x_5} = 18$.

Because *x*5 plays the role of the slack variable for the third constraint in the artificial problem, this constraint is equivalent to $3x_1 + 2x_2 \leq 18$

The Real Problem

Maximize $Z = 3x_1 + 5x_2$,

subject to

$$x_1 \le 4$$

$$2x_2 \le 12$$

$$3x_1 + 2x_2 = 18$$

and

$$x_1 \ge 0, \quad x_2 \ge 0.$$

The Artificial Problem

Define $\bar{x}_5 = 18 - 3x_1 - 2x_2$.

Maximize $Z = 3x_1 + 5x_2 - M\bar{x}_5$,

subject to

$$x_1 \le 4$$

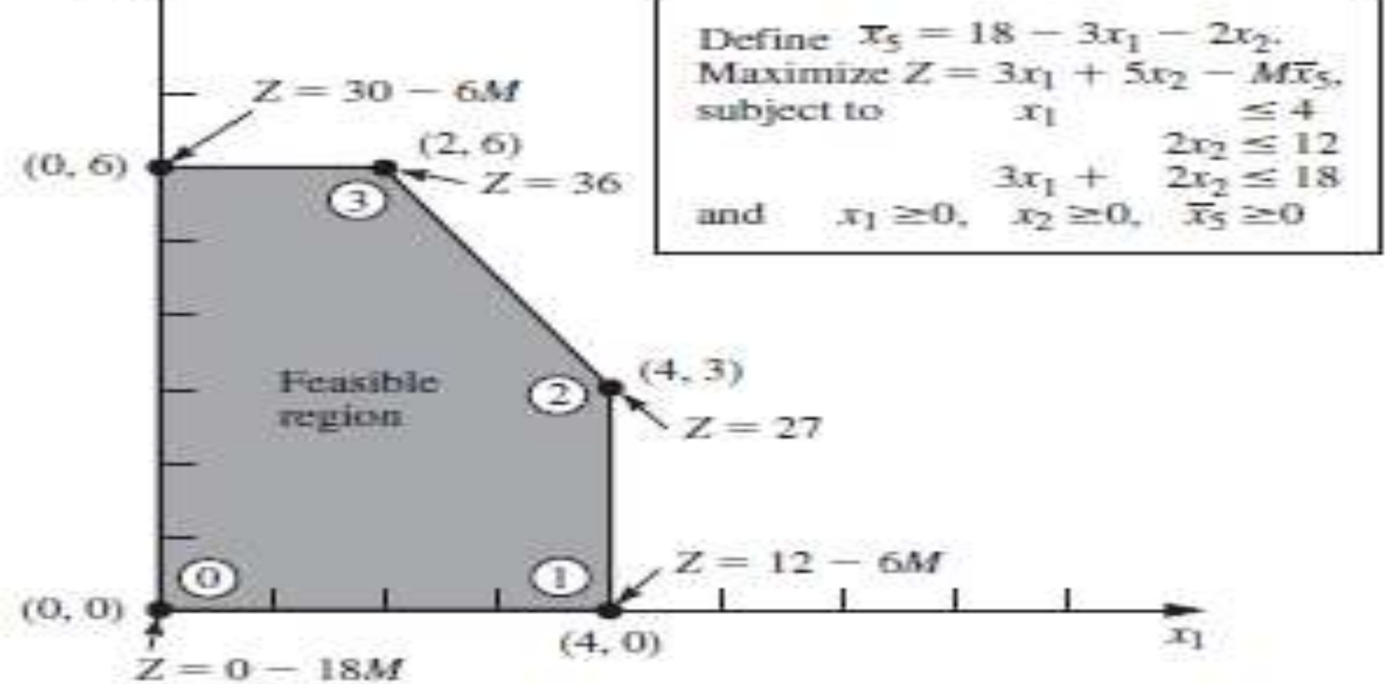
$$2x_2 \le 12$$

$$3x_1 + 2x_2 \le 18$$

$$(\text{so} \quad 3x_1 + 2x_2 + \bar{x}_5 = 18)$$

and

$$x_1 \ge 0, \quad x_2 \ge 0, \quad \bar{x}_5 \ge 0.$$



Converting Equation (0) to Proper Form. The system of equations after the artificial problem is augmented is:

$$\begin{array}{lllllll}
(0) & Z - 3x_1 - 5x_2 & & & + M\overline{x}_5 & = & 0 \\
(1) & x_1 & + \boldsymbol{x_3} & & & = & 4 \\
(2) & 2x_2 & & + \boldsymbol{x_4} & & = & 12 \\
(3) & 3x_1 + 2x_2 & & & + \boldsymbol{\overline{x}_5} & = & 18
\end{array}$$

where the initial basic variables (x_3, x_4, $\overline{x_5}$) are shown in bold type.

To algebraically eliminate $\overline{x_5}$ from Eq. (0), we need to subtract from Eq. (0) the product, *M* times Eq. (3).

$$
\begin{array}{ll}
 & Z - 3x_1 - 5x_2 + M\bar{x}_5 = 0 \\
 & -M(3x_1 + 2x_2 + \bar{x}_5 = 18) \\
\hline
\text{New (0)} & Z - (3M + 3)x_1 - (2M + 5)x_2 = -18M.
\end{array}
$$

Application of the Simplex Method. This new Eq. (0) gives *Z* in terms of *just* the nonbasic variables (x_1, x_2)

$$Z = -18M + (3M + 3)x_1 + (2M + 5)x_2.$$

Since $3M + 3 > 2M + 5$ (remember that *M* represents a huge number), increasing x_1 increases *Z* at a faster rate than increasing x_2 does, so x_1 is chosen as the entering basic variable.
This leads to the move from (0, 0) to (4, 0) at iteration 1, shown in figure above , thereby increasing *Z* by $4(3M + 3)$.

Complete set of simplex tableaux for the problem

Iteration	Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	Right Side
0	Z	(0)	1	$-3M-3$	$-2M-5$	0	0	0	$-18M$
	x_3	(1)	0	1	0	1	0	0	4
	x_4	(2)	0	0	2	0	1	0	12
	$\bar{x}_5$	(3)	0	3	2	0	0	1	18
1	Z	(0)	1	0	$-2M-5$	$3M+3$	0	0	$-6M+12$
	x_1	(1)	0	1	0	1	0	0	4
	x_4	(2)	0	0	2	0	1	0	12
	$\bar{x}_5$	(3)	0	0	2	-3	0	1	6
2	Z	(0)	1	0	0	$-\frac{9}{2}$	0	$M+\frac{5}{2}$	27
	x_1	(1)	0	1	0	1	0	0	4
	x_4	(2)	0	0	0	3	1	-1	6
	x_2	(3)	0	0	1	$-\frac{3}{2}$	0	$\frac{1}{2}$	3
3	Z	(0)	1	0	0	0	$\frac{3}{2}$	$M+1$	36
	x_1	(1)	0	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	2
	x_3	(2)	0	0	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	2
	x_2	(3)	0	0	1	0	$\frac{1}{2}$	0	6

Negative Right-Hand Sides

- If the right-hand side is negative, multiply through both sides by -1 first.

- For any inequality constraint with a negative right-hand side. Multiplying through both sides of an inequality by -1 also reverses the direction of the inequality

i.e. $x_1 - x_2 \leq -1$ (that is, $x_1 \leq x_2 - 1$) gives the equivalent constraint . $-x_1 + x_2 \geq -1$ (that is, . $x_2 - 1 \geq x_1$) but now the right-hand side is positive

Functional Constraints in ≥ Form

Minimize $\quad Z = 0.4x_1 + 0.5x_2,$

subject to

$$0.3x_1 + 0.1x_2 \leq 2.7$$

$$0.5x_1 + 0.5x_2 = 6$$

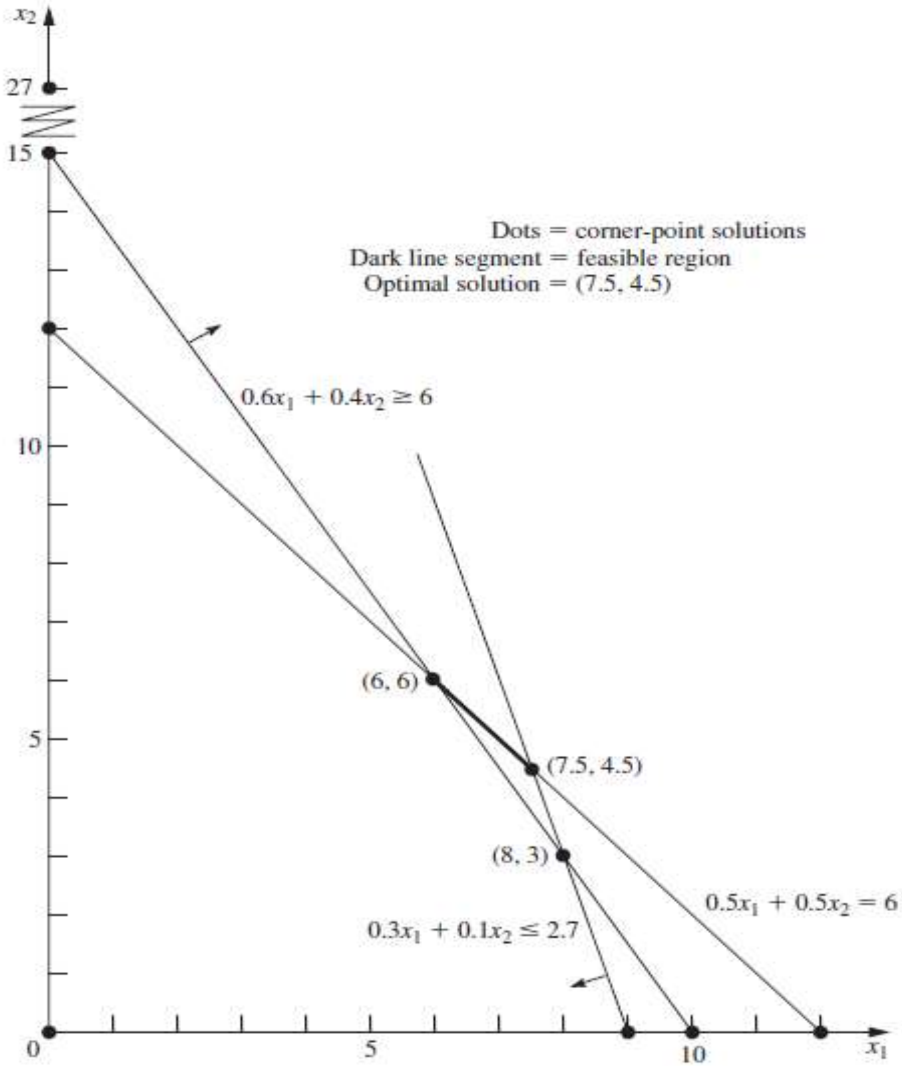
$$\boxed{0.6x_1 + 0.4x_2 \geq 6}$$

and

$$x_1 \geq 0, \quad x_2 \geq 0.$$

$$\begin{aligned}
& 0.6x_1 + 0.4x_2 && \geq 6 && \\
\rightarrow\quad & 0.6x_1 + 0.4x_2 - x_5 && = 6 && (x_5 \geq 0) \\
\rightarrow\quad & 0.6x_1 + 0.4x_2 - x_5 + \bar{x}_6 && = 6 && (x_5 \geq 0, \bar{x}_6 \geq 0).
\end{aligned}$$

Here x_5 is called a **surplus variable** because it subtracts the surplus of the left-hand side over the right-hand side to convert the inequality constraint to an equivalent equality constraint. Once this conversion is accomplished, the artificial variable is introduced just as for any equality constraint.



Minimize $\quad Z = 0.4x_1 + 0.5x_2,$

subject to

$$0.3x_1 + 0.1x_2 \leq 2.7$$

$$0.5x_1 + 0.5x_2 = 6$$

$$\boxed{0.6x_1 + 0.4x_2 \geq 6}$$

and

$$x_1 \geq 0, \quad x_2 \geq 0.$$

Note that the coefficients of the artificial variables in the objective function are *M,* instead of -- *M,* because we now are minimizing *Z*.

Minimize $\quad Z = 0.4x_1 + 0.5x_2 + M\bar{x}_4 + M\bar{x}_6,$

subject to

$$\begin{aligned} 0.3x_1 + 0.1x_2 + x_3 \qquad\qquad\qquad &= 2.7 \\ 0.5x_1 + 0.5x_2 \qquad + \bar{x}_4 \qquad\quad &= 6 \\ 0.6x_1 + 0.4x_2 \qquad\qquad - x_5 + \bar{x}_6 &= 6 \end{aligned}$$

and $\quad x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0, \quad \bar{x}_4 \geq 0, \quad x_5 \geq 0, \quad \bar{x}_6 \geq 0.$

Minimization Problem

- Simple way of converting any minimization problem to an equivalent maximization problem:

$$\textit{Minimizing} \qquad Z = \sum_{j=1}^{n} c_j x_j$$

is equivalent to

$$\textit{maximizing} \qquad -Z = \sum_{j=1}^{n} (-c_j) x_j;$$

- The two formulations are equivalent because the smaller *Z* is, the larger *Z* is, so the solution that gives the *smallest* value of *Z* in the entire feasible region must also give the *largest* value of *Z* in this region.
- By using the maximization form just obtained, the entire system of equations is now

$$\begin{aligned}
&(0) \quad -Z + 0.4x_1 + 0.5x_2 \qquad + M\bar{x}_4 \qquad\qquad + M\bar{x}_6 = 0\\
&(1) \qquad\quad 0.3x_1 + 0.1x_2 + x_3 \qquad\qquad\qquad\qquad = 2.7\\
&(2) \qquad\quad 0.5x_1 + 0.5x_2 \qquad + \bar{x}_4 \qquad\qquad\qquad = 6\\
&(3) \qquad\quad 0.6x_1 + 0.4x_2 \qquad\qquad\qquad - x_5 + \bar{x}_6 = 6.
\end{aligned}$$

The basic variables (x_3, $\bar{x}_4$, $\bar{x}_6$) for the initial BF solution (for this artificial problem) are shown in bold type.

Row 0:

$$
\begin{array}{rlllllll}
 & [0.4, & 0.5, & 0, & M, & 0, & M, & 0] \\
 & -M[0.5, & 0.5, & 0, & 1, & 0, & 0, & 6] \\
 & -M[0.6, & 0.4, & 0, & 0, & -1, & 1, & 6] \\
\hline
\text{New row } 0 = & [-1.1M + 0.4, & -0.9M + 0.5, & 0, & 0, & M, & 0, & -12M]
\end{array}
$$

The Big *M* method for the radiation therapy example

Iteration	Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	$\bar{x}_4$	x_5	$\bar{x}_6$	Right Side
0	Z	(0)	-1	$-1.1M+0.4$	$-0.9M+0.5$	0	0	M	0	$-12M$
	x_3	(1)	0	0.3	0.1	1	0	0	0	2.7
	$\bar{x}_4$	(2)	0	0.5	0.5	0	1	0	0	6
	$\bar{x}_6$	(3)	0	0.6	0.4	0	0	-1	1	6
1	Z	(0)	-1	0	$-\frac{16}{30}M+\frac{11}{30}$	$\frac{11}{3}M-\frac{4}{3}$	0	M	0	$-2.1M-3.6$
	x_1	(1)	0	1	$\frac{1}{3}$	$\frac{10}{3}$	0	0	0	9
	$\bar{x}_4$	(2)	0	0	$\frac{1}{3}$	$-\frac{5}{3}$	1	0	0	1.5
	$\bar{x}_6$	(3)	0	0	0.2	-2	0	-1	1	0.6
2	Z	(0)	-1	0	0	$-\frac{5}{3}M+\frac{7}{3}$	0	$-\frac{5}{3}M+\frac{11}{6}$	$\frac{8}{3}M-\frac{11}{6}$	$-0.5M-4.7$
	x_1	(1)	0	1	0	$\frac{20}{3}$	0	$\frac{5}{3}$	$-\frac{5}{3}$	8
	$\bar{x}_4$	(2)	0	0	0	$\frac{5}{3}$	1	$\frac{5}{3}$	$-\frac{5}{3}$	0.5
	x_2	(3)	0	0	1	-10	0	-5	5	3
3	Z	(0)	-1	0	0	0.5	$M-1.1$	0	M	-5.25
	x_1	(1)	0	1	0	5	-1	0	0	7.5
	x_5	(2)	0	0	0	1	0.6	1	-1	0.3
	x_2	(3)	0	0	1	-5	3	0	0	4.5

Example 1: Consider the following problem.

$$\text{Maximize } Z = 2x_1 + 3x_2,$$

subject to

$$x_1 + 2x_2 \le 4$$
$$x_1 + x_2 = 3$$

and

$$x_1 \ge 0, x_2 \ge 0.$$

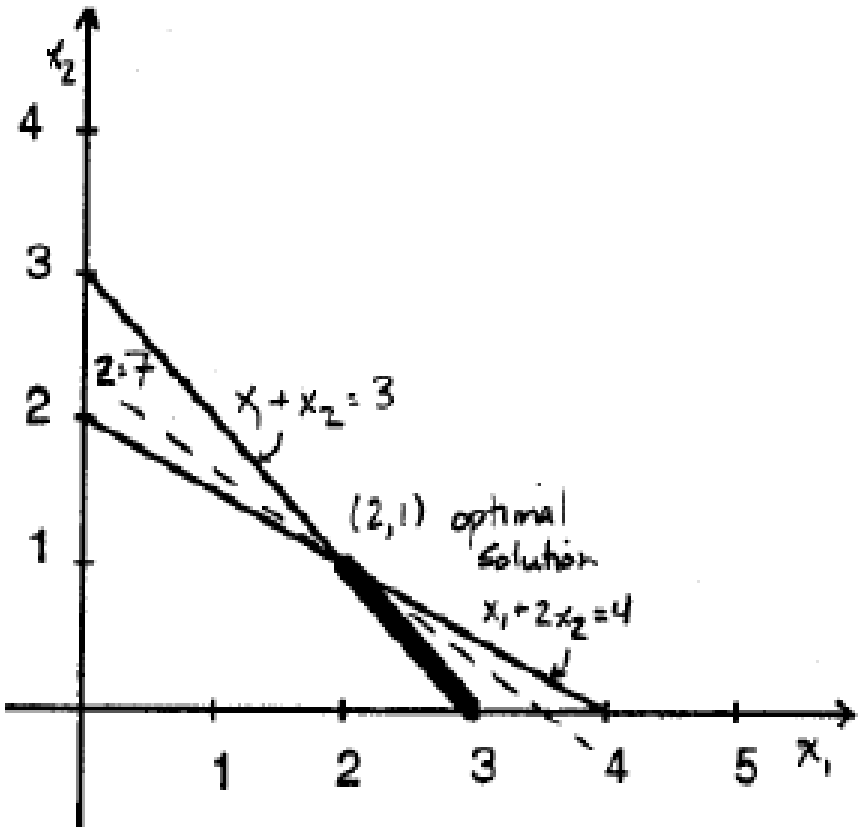
(a) Solve this problem graphically.

(b) Using the Big *M* method, construct the complete first simplex tableau for the simplex method and identify the corresponding initial (artificial) BF solution.

Also identify the initial entering basic variable and the leaving basic variable.

(c) Continue from part (*b*) to work through the simplex method step by step to solve the problem.

(a) Optimal Solution: $(x_1^*, x_2^*) = (2, 1)$ and $Z^* = 7$



Augmented Problem

$$Z \quad - \; 2x_1 - \; 3x_2 \; + M\overline{x_5} \quad = 0$$
$$x_1 + \; 2x_2 \; + \quad x_3 \; = \; 4$$
$$x_1 + \; x_2 \; + \overline{x_5} \; = \; 3$$

$$Z \quad - \; 2x_1 - \; 3x_2 + M\overline{x_5} = 0$$

Add $\quad -M(x_1 + x_2 + \overline{x_5} \; = 3)$

$$Z \quad - (M+2)x_1 - (M+3)x_2 \; + 0 \; = -3M$$

(b) Initial artificial BF solution: $(0, 0, 4, 3)$

Bas Var	Eq No	Coefficient of					Right Side
		Z	X_1	X_2	X_3	$\bar{X}_4$	
			-1M	-1M			
Z	0	1	-2	-3	0	0	-3M
X_3	1	0	1	2	1	0	4
$\bar{X}_4$	2	0	1	1	0	1	3

(c) Optimal Solution: $(x_1^*, x_2^*) = (2, 1)$ and $Z^* = 7$

Bas Var	Eq No	Coefficient of					Right Side
		Z	X_1	X_2	X_3	$\bar{X}_4$	
			-0.5M		0.5M		-1M
Z	0	1	-0.5	0	+1.5	0	+6
X_2	1	0	0.5	1	0.5	0	2
$\bar{X}_4$	2	0	0.5	0	-0.5	1	1

Bas Var	Eq No	Coefficient of					Right Side
		Z	X_1	X_2	X_3	$\bar{X}_4$	
						1M	
Z	0	1	0	0	1	+1	7
X_2	1	0	0	1	1	-1	1
X_1	2	0	1	0	-1	2	2

Example 2: Consider the following problem (4.6.3).

$$\text{Minimize } Z = 2x_1 + 3x_2 + x_3,$$

subject to

$$x_1 + 4x_2 + 2x_3 \geq 8$$
$$3x_1 + 2x_2 + 2x_3 \geq 6$$

and

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0.$$

(a) Reformulate this problem to fit our standard form for a linear programming model.

(b) Using the Big *M* method, work through the simplex method step by step to solve the problem.

(c) Using the two-phase method, work through the simplex method step by step to solve the problem.

(d) Compare the sequence of BF solutions obtained in parts (*b*) and (*c*). Which of these solutions are feasible only for the artificial problem obtained by introducing artificial variables and which are actually feasible for the real problem?

(e) Use a software package based on the simplex method to solve the problem.

(a) maximize $\quad -Z = -2x_1 - 3x_2 - x_3$

subject to

$$
\begin{aligned}
-x_1 - 4x_2 - 2x_3 &\le -8 \\
-3x_1 - 2x_2 &\le -6 \\
x_1, x_2, x_3 &\ge 0
\end{aligned}
$$

(b) Optimal Solution: $(x_1^*, x_2^*, x_3^*) = (0.8, 1.8, 0)$ and $Z^* = 7$

Bas Var	Eq No	Z	Coefficient of X_1	X_2	X_3	X_4	X_5	$\bar{X}_6$	$\bar{X}_7$	Right Side
Z	0	1	-4M +2	-6M +3	-2M +1	1M	1M	0	0	-14M
$\bar{X}_6$	1	0	1	4	2	-1	0	1	0	8
$\bar{X}_7$	2	0	3	2	0	0	-1	0	1	6

Bas Var	Eq No	Z	Coefficient of X_1	X_2	X_3	X_4	X_5	$\bar{X}_6$	$\bar{X}_7$	Right Side
Z	0	1	-2.5M +1.25	0	1M -0.5	-0.5M +0.75	1M	1.5M -0.75	0	-2M -6
X_2	1	0	0.25	1	0.5	-0.25	0	0.25	0	2
$\bar{X}_7$	2	0	2.5	0	-1	0.5	-1	-0.5	1	2

Bas Var	Eq No	Z	Coefficient of X_1	X_2	X_3	X_4	X_5	$\bar{X}_6$	$\bar{X}_7$	Right Side
Z	0	1	0	0	0	0.5	0.5	1M -0.5	1M -0.5	-7
X_2	1	0	0	1	0.6	-0.3	0.1	0.3	-0.1	1.8
X_1	2	0	1	0	-0.4	0.2	-0.4	-0.2	0.4	0.8

Pivoting x_3 for x_2 gives an alternate optimal BF solution, $(2, 0, 3)$.

Variables with a Bound on the Negative Values Allowed

Suppose we have $x_j \geq L_j$, where L_j is a negative constant. This constraint can be converted to a nonnegativity constraint by making the change of variables $x_j' = x_j - L_j$, so $x_j' \geq 0$.

$$\boxed{\begin{array}{l} Z = 3x_1 + 5x_2 \\ x_1 \leq 4 \\ 2x_2 \leq 12 \\ 3x_1 + 2x_2 \leq 18 \\ x_1 \geq -10, \quad x_2 \geq 0 \end{array}} \rightarrow \boxed{\begin{array}{l} Z = 3(x_1' - 10) + 5x_2 \\ x_1' - 10 \leq 4 \\ 2x_2 \leq 12 \\ 3(x_1' - 10) + 2x_2 \leq 18 \\ x_1' - 10 \geq -10, \quad x_2 \geq 0 \end{array}} \rightarrow \boxed{\begin{array}{l} Z = -30 + 3x_1' + 5x_2 \\ x_1' \leq 14 \\ 2x_2 \leq 12 \\ 3x_1' + 2x_2 \leq 48 \\ x_1' \geq 0, \quad x_2 \geq 0 \end{array}}$$

Variables with No Bound on the Negative Values Allowed

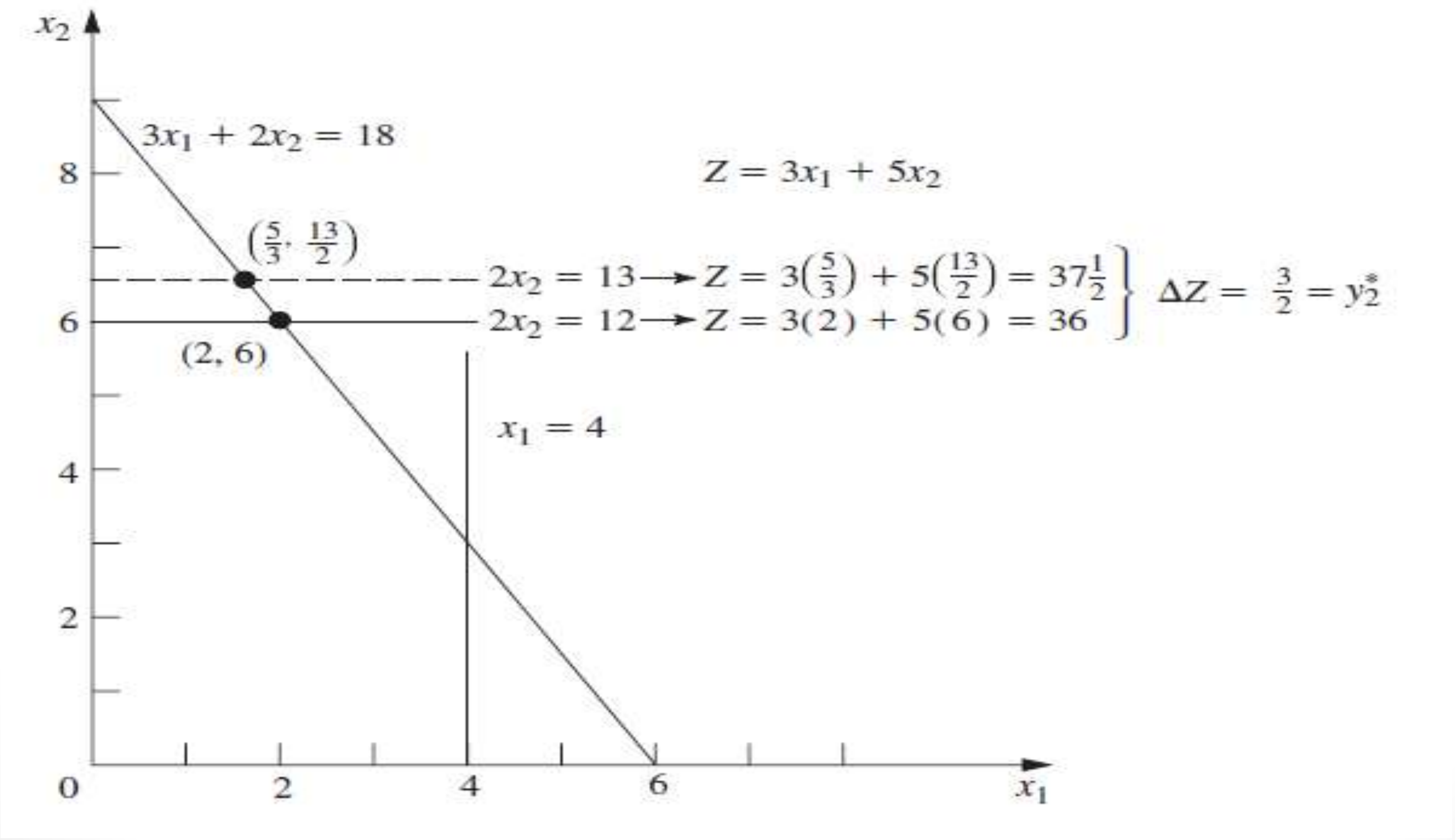
Suppose the decision variable is not bounded then we have

$$x_1 = x_1^+ - x_1^-, \quad \text{where } x_1^+ \geq 0, x_1^- \geq 0,$$

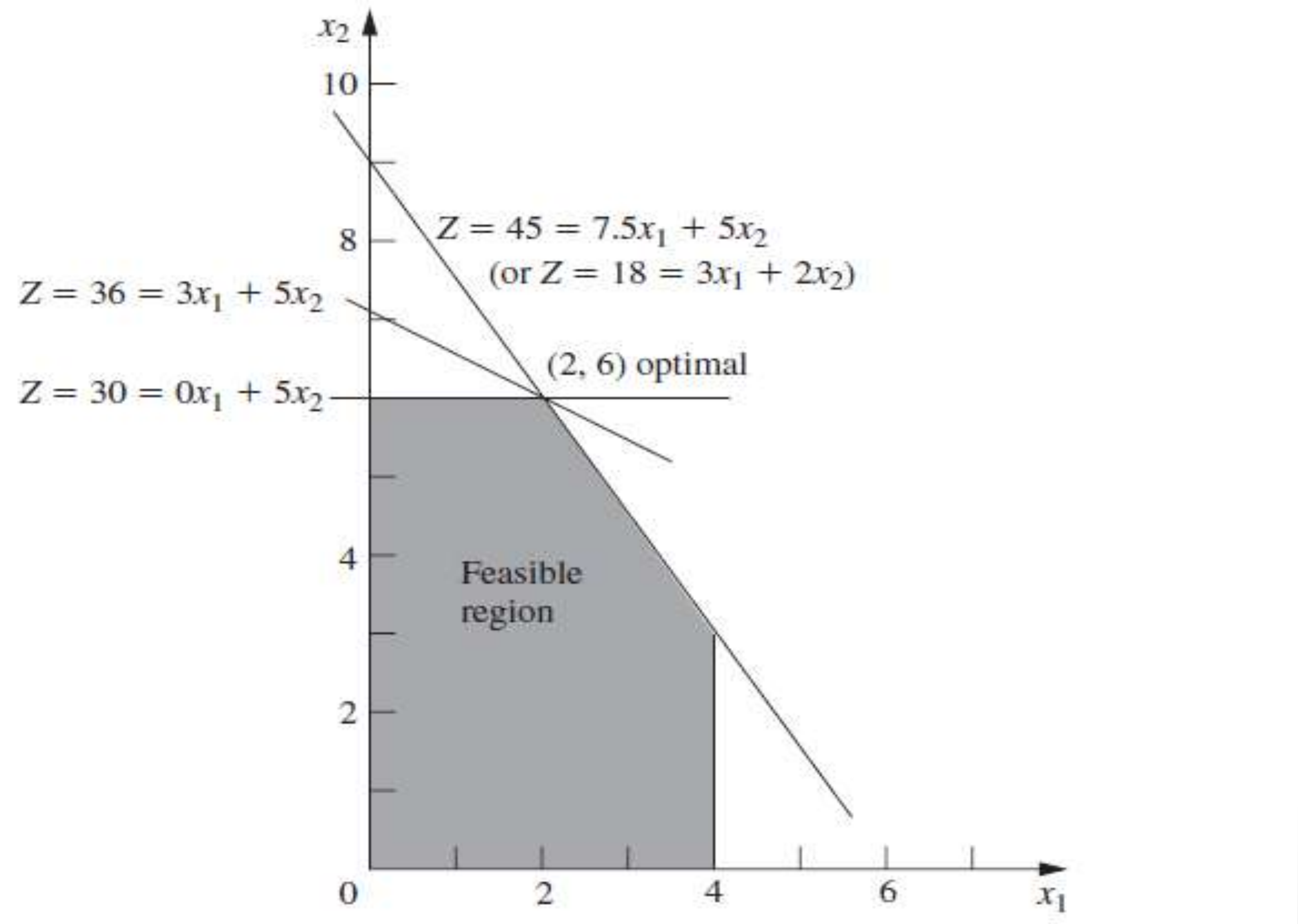
$$\boxed{\begin{array}{ll} \text{Maximize} & Z = 3x_1 + 5x_2, \\ \text{subject to} & x_1 \leq 4 \\ & 2x_2 \leq 12 \\ & 3x_1 + 2x_2 \leq 18 \\ & x_2 \geq 0 \text{ (only)} \end{array}} \rightarrow \boxed{\begin{array}{ll} \text{Maximize} & Z = 3x_1^+ - 3x_1^- + 5x_2, \\ \text{subject to} & x_1^+ - x_1^- \leq 4 \\ & 2x_2 \leq 12 \\ & 3x_1^+ - 3x_1^- + 2x_2 \leq 18 \\ & x_1^+ \geq 0, \quad x_1^- \geq 0, \quad x_2 \geq 0 \end{array}}$$

Shadow Prices

The **shadow price** for resource *i* (denoted by y_i *) measures the *marginal value* of this resource, i.e., the rate at which *Z* could be increased by (slightly) increasing the amount of this resource (b_i) being made available. The simplex method identifies this shadow price by y_i * coefficient of the *i*th slack variable in row 0 of the final simplex tableau



Sensitivity Analysis



Pblm 1: Consider the following problem.(4.6.12)

Maximize $Z = 3x_1 + 7x_2 + 5x_3$,

subject to

$$3x_1 + x_2 + 2x_3 \le 9$$
$$2x_1 + x_2 + 3x_3 \le 12$$

and

$$x_2 \ge 0,\ x_3 \ge 0.$$

(no nonnegativity constraint for $x1$).

(a) Reformulate this problem so all variables have nonnegativity constraints.
(b) Work through the simplex method step by step to solve the problem.
(c) Use a software package based on the simplex method to solve the problem.

(a) Substitute $x_1 = x_1^+ - x_1^-$, where both x_1^+ and x_1^- are nonnegative.

$$\text{maximize } Z = 3x_1^+ - 3x_1^- + 7x_2 + 5x_3$$

$$\begin{array}{lrl} \text{subject to} & 3x_1^+ - 3x_1^- + x_2 + 2x_3 & \le 9 \\ & -2x_1^+ + 2x_1^- + x_2 + 3x_3 & \le 12 \\ & x_1^+, x_1^-, x_2, x_3 & \ge 0 \end{array}$$

(b) Optimal Solution: $(x_1^*, x_2^*, x_3^*) = (-0.6, 10.8, 0)$ and $Z^* = 73.8$

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
Z	0	1	-3	3	-7	-5	0	0	0
X5	1	0	3	-3	1*	2	1	0	9
X6	2	0	-2	2	1	3	0	1	12

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
Z	0	1	18	-18	0	9	7	0	63
X3	1	0	3	-3	1	2	1	0	9
X6	2	0	-5	5*	0	1	-1	1	3

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
Z	0	1	0	0	0	12.6	3.4	3.6	73.8
X3	1	0	0	0	1	2.6	0.4	0.6	10.8
X2	2	0	-1	1	0	0.2	-0.2	0.2	0.6

Note that x_1^+, x_1^-, x_2, and x_3 are renamed as X_1, X_2, X_3 and X_4 respectively.

(c)

	Coefficient of						
	X1	X2	X3	X4	Total		
Constraint 1	3	-3	1	2	9	≤	9
Constraint 2	-2	2	1	3	12	≤	12
Objective	3	-3	7	5	73.8		
Solution	0	0.6	10.8	0			

Pblm 2: Consider the following problem.(4.7.4)

$$\text{Maximize } Z = x_1 - 7x_2 + 3x_3,$$

subject to

$$2x_1 + x_2 + x_3 \le 4 \text{ (resource 1)}$$
$$4x_1 + 3x_2 + x_3 \le 2 \text{ (resource 2)}$$
$$3x_1 + 2x_2 + x_3 \le 3 \text{ (resource 3)}$$

and

$$x_1 \ge 0, x_2 \ge 0, x_3 \ge 0.$$

a) Work through the simplex method step by step to solve the problem.

(b) Identify the shadow prices for the three resources and describe their significance.

(c) Use a software package based on the simplex method to solve the problem and then to generate sensitivity information.

Use this information to identify the shadow price for each resource, the allowable range for each objective function coefficient, and the allowable range for each right hand side.

(a) Optimal Solution: $(x_1^*, x_2^*, x_3^*) = (0.5, 0, 4.5)$ and $Z^* = 14$

Bas Var	Eq No	Z	Coefficient of X_1	X_2	X_3	X_4	X_5	X_6	Right Side
Z	0	1	-1	7	-3	0	0	0	0
X_4	1	0	2	1	-1	1	0	0	4
X_5	2	0	4	-3	0	0	1	0	2
X_6	3	0	-3	2	1	0	0	1	3

Bas Var	Eq No	Z	X_1	X_2	X_3	X_4	X_5	X_6	Right Side
Z	0	1	-10	13	0	0	0	3	9
X_4	1	0	-1	3	0	1	0	1	7
X_5	2	0	4	-3	0	0	1	0	2
X_3	3	0	-3	2	1	0	0	1	3

Bas Var	Eq No	Z	X_1	X_2	X_3	X_4	X_5	X_6	Right Side
Z	0	1	0	5.5	0	0	2.5	3	14
X_4	1	0	0	2.25	0	1	0.25	1	7.5
X_1	2	0	1	-0.75	0	0	0.25	0	0.5
X_3	3	0	0	-0.25	1	0	0.75	1	4.5

(b) The shadow prices for the three resources are given by the reduced costs (in the objective function) for the corresponding slack variables. These values are circled in the table above. The shadow prices for resources 1, 2 and 3 are 0, 2.5 and 3 respectively.

They represent the rate at which the objective function value Z increases as the corresponding resource is increased.

For instance, increasing resource 3 by one unit increases Z by 3, provided that no other constraints cause any trouble.

THE SIMPLEX METHOD IN MATRIX FORM

Using matrices, our standard form for the general linear programming model

$$\boxed{\begin{array}{lll} \text{Maximize} & Z = \mathbf{cx}, & \\ \text{subject to} & & \\ \mathbf{Ax} \leq \mathbf{b} & \text{and} & \mathbf{x} \geq \mathbf{0}, \end{array}}$$

where $\mathbf{c}$ is the row vector

$$\mathbf{c} = [c_1, c_2, \ldots, c_n],$$

$\mathbf{x}$, $\mathbf{b}$, and $\mathbf{0}$ are the column vectors such that

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, \qquad \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

and $\mathbf{A}$ is the matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

To obtain the *augmented form* of the problem, introduce the column vector of slack variables

$$x_s = \begin{bmatrix} x_{n+1} \\ x_{n+2} \\ \vdots \\ x_{n+m} \end{bmatrix}$$

so that the constraints become

$$[\mathbf{A}, \mathbf{I}] \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_s \end{bmatrix} = \mathbf{b} \quad \text{and} \quad \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_s \end{bmatrix} \geq \mathbf{0},$$

where **I** is the $m \quad m$ identity matrix, and the null vector **0** now has $n \quad m$ elements.

Solving for a Basic Feasible Solution

Given the variables, the resulting basic solution is the solution of the m equations

$$[\mathbf{A}, \mathbf{I}] \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_s \end{bmatrix} = \mathbf{b},$$

in which the n *nonbasic variables* from the $n + m$ elements of are set equal to zero.

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{x}_s \end{bmatrix}$$

Eliminating these n variables by equating them to zero leaves a set of m equations in m unknowns (the *basic variables*). This set of equations can be denoted by

$$\mathbf{B}\mathbf{x}_B = \mathbf{b},$$

where the vector of basic variables

$$\mathbf{x}_B = \begin{bmatrix} x_{B1} \\ x_{B2} \\ \vdots \\ x_{Bm} \end{bmatrix}$$

is obtained by eliminating the nonbasic variables from

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{x}_s \end{bmatrix},$$

and the basis matrix

$$\mathbf{B} = \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1m} \\ B_{21} & B_{22} & \cdots & B_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ B_{m1} & B_{m2} & \cdots & B_{mm} \end{bmatrix}$$

is obtained by eliminating the columns corresponding to coefficients of nonbasic variables from [**A, I**].
(In addition, the elements of $\mathbf{x}_B$ and, therefore, the columns of **B** may be placed in a different order when the simplex method is executed.)

The simplex method introduces only basic variables such that **B** is *nonsingular,* so that $\mathbf{B}^{-1}$ always will exist. Therefore, to solve $\mathbf{B}\mathbf{x}_B = \mathbf{b}$, both sides are premultiplied by $\mathbf{B}^{-1}$:

$$\mathbf{B}^{-1}\ \mathbf{B}\ \mathbf{x}_B = \mathbf{B}^{-1}\ \mathbf{b}.$$

Since $\mathbf{B}^{-1}\ \mathbf{B} = \mathbf{I}$, the desired solution for the basic variables is

$$\boxed{\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}.}$$

Let $\mathbf{c}_B$ be the vector whose elements are the objective function coefficients (including zeros for slack variables) for the corresponding elements of $\mathbf{x}_B$. The value of the objective function for this basic solution is then

$$\boxed{Z = \mathbf{c}_B\mathbf{x}_B = \mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}.}$$

$$
\begin{aligned}
&\text{Maximize} \quad Z = 3x_1 + 5x_2, \\
&\text{subject to} \\
&\qquad x_1 \qquad\quad\; \leq 4 \\
&\qquad\qquad\quad 2x_2 \leq 12 \\
&\qquad 3x_1 + 2x_2 \leq 18 \\
&\text{and} \quad x_1 \geq 0, \quad x_2 \geq 0.
\end{aligned}
$$

Maximize	$Z = 3x_1 + 5x_2,$	
subject to		
	$x_1 \leq 4$	
	$2x_2 \leq 12$	
	$3x_1 + 2x_2 \leq 18$	
and	$x_1 \geq 0,$	$x_2 \geq 0.$

$$\mathbf{c} = [3, 5], \quad [\mathbf{A}, \mathbf{I}] = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 3 & 2 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mathbf{x}_s = \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix}.$$

Referring to Table 4.8, we see that the sequence of BF solutions obtained by the simplex method is the following:

Iteration 0

$$\mathbf{x}_B = \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{B}^{-1}, \quad \text{so} \quad \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix} = \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix},$$

$$\mathbf{c}_B = [0, 0, 0], \quad \text{so} \quad Z = [0, 0, 0] \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix} = 0.$$

Iteration 1

$$\mathbf{x}_B = \begin{bmatrix} x_3 \\ x_2 \\ x_5 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 2 & 1 \end{bmatrix}, \quad \mathbf{B}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix},$$

so

$$\begin{bmatrix} x_3 \\ x_2 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 6 \end{bmatrix},$$

$$\mathbf{c}_B = [0, 5, 0], \quad \text{so} \quad Z = [0, 5, 0] \begin{bmatrix} 4 \\ 6 \\ 6 \end{bmatrix} = 30.$$

Iteration 2

$$\mathbf{x}_B = \begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 2 & 3 \end{bmatrix}, \qquad \mathbf{B}^{-1} = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix},$$

so

$$\begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix},$$

$$\mathbf{c}_B = [0, 5, 3], \qquad \text{so} \qquad Z = [0, 5, 3] \begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix} = 36.$$

Matrix Form of the Current Set of Equations

For the *original* set of equations, the matrix form is

$$\boxed{\begin{bmatrix} 1 & -\mathbf{c} & \mathbf{0} \\ \mathbf{0} & \mathbf{A} & \mathbf{I} \end{bmatrix} \begin{bmatrix} Z \\ \mathbf{x} \\ \mathbf{x}_s \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{b} \end{bmatrix}.}$$

$$\begin{bmatrix} Z \\ \mathbf{x}_B \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{c}_B\mathbf{B}^{-1} \\ \mathbf{0} & \mathbf{B}^{-1} \end{bmatrix} \begin{bmatrix} 0 \\ \mathbf{b} \end{bmatrix} = \begin{bmatrix} \mathbf{c}_B\mathbf{B}^{-1}\mathbf{b} \\ \mathbf{B}^{-1}\mathbf{b} \end{bmatrix}.$$

$$\begin{bmatrix} 1 & \mathbf{c}_B\mathbf{B}^{-1} \\ \mathbf{0} & \mathbf{B}^{-1} \end{bmatrix} \begin{bmatrix} 1 & -\mathbf{c} & \mathbf{0} \\ \mathbf{0} & \mathbf{A} & \mathbf{I} \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{c}_B\mathbf{B}^{-1}\mathbf{A} - \mathbf{c} & \mathbf{c}_B\mathbf{B}^{-1} \\ \mathbf{0} & \mathbf{B}^{-1}\mathbf{A} & \mathbf{B}^{-1} \end{bmatrix},$$

Initial and later simplex tableaux in matrix form

Iteration	Basic Variable	Eq.	Coefficient of: Z	Coefficient of: Original Variables	Coefficient of: Slack Variables	Right Side
0	Z $\mathbf{x}_B$	(0) (1, 2, . . . , m)	1 0	$-\mathbf{c}$ $\mathbf{A}$	$\mathbf{0}$ $\mathbf{I}$	0 $\mathbf{b}$
Any	Z $\mathbf{x}_B$	(0) (1, 2, . . . , m)	1 0	$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{A}-\mathbf{c}$ $\mathbf{B}^{-1}\mathbf{A}$	$\mathbf{c}_B\mathbf{B}^{-1}$ $\mathbf{B}^{-1}$	$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}$ $\mathbf{B}^{-1}\mathbf{b}$

$$\begin{bmatrix} 1 & \mathbf{c}_B\mathbf{B}^{-1}\mathbf{A}-\mathbf{c} & \mathbf{c}_B\mathbf{B}^{-1} \\ 0 & \mathbf{B}^{-1}\mathbf{A} & \mathbf{B}^{-1} \end{bmatrix} \begin{bmatrix} Z \\ \mathbf{x} \\ \mathbf{x}_s \end{bmatrix} = \begin{bmatrix} \mathbf{c}_B\mathbf{B}^{-1}\mathbf{b} \\ \mathbf{B}^{-1}\mathbf{b} \end{bmatrix}.$$

$$\mathbf{B}^{-1}\mathbf{A} = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$\mathbf{c}_B\mathbf{B}^{-1} = [0, 5, 3] \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix} = [0, \tfrac{3}{2}, 1],$$

$$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{A} - \mathbf{c} = [0, 5, 3] \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} - [3, 5] = [0, 0].$$

Also, by using the values of $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}$ and $Z = \mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}$ calculated at the end of the preceding subsection, these results give the following set of equations:

$$\left[\begin{array}{c|cc|ccc} 1 & 0 & 0 & 0 & \frac{3}{2} & 1 \\ \hline 0 & 0 & 0 & 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{3} & \frac{1}{3} \end{array}\right] \begin{bmatrix} Z \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 36 \\ 2 \\ 6 \\ 2 \end{bmatrix},$$

Consider the following problem. 4.7-6.

Maximize $Z = 5x1 + 4x2 - x3 + 3x4$,

subject to

$3x1 + 2x2 - 3x3 + x4 \leq 24$ (resource 1)

$3x1 + 3x2 + x3 + 3x4 \leq 36$ (resource 2)

and

$x1 \geq 0, x2 \geq 0\ x3 \geq 0, x4 \geq 0$

Solution:

$$c = \begin{pmatrix} 5 & 4 & -1 & 3 & 0 & 0 \end{pmatrix}, A = \begin{pmatrix} 3 & 2 & -3 & 1 & 1 & 0 \\ 3 & 3 & 1 & 3 & 0 & 1 \end{pmatrix}, b = \begin{pmatrix} 24 \\ 36 \end{pmatrix}$$

$$\text{Iteration 0: } B = B^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, x_B = \begin{pmatrix} x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 24 \\ 36 \end{pmatrix} = \begin{pmatrix} 24 \\ 36 \end{pmatrix}$$

$c_B = \begin{pmatrix} 0 & 0 \end{pmatrix}$, Row 0: $\begin{pmatrix} -5 & -4 & 1 & -3 & 0 & 0 \end{pmatrix}$, so x_1 enters the basis.

Revised x_1 coefficients: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$, so x_5 leaves the basis.

Iteration 1: $B_{\text{new}}^{-1} = \begin{pmatrix} 3 & 0 \\ 3 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1/3 & 0 \\ -1 & 1 \end{pmatrix}$

$$x_B = \begin{pmatrix} x_1 \\ x_6 \end{pmatrix} = \begin{pmatrix} 1/3 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 24 \\ 36 \end{pmatrix} = \begin{pmatrix} 8 \\ 12 \end{pmatrix}, c_B = (5 \quad 0)$$

Revised row 0: $(5/3 \quad 0) \begin{pmatrix} 3 & 2 & -3 & 1 & 1 & 0 \\ 3 & 3 & 1 & 3 & 0 & 1 \end{pmatrix} - (5 \quad 4 \quad -1 \quad 3 \quad 0 \quad 0)$

$= (0 \quad -2/3 \quad -4 \quad -4/3 \quad 5/3 \quad 0)$, so x_3 enters the basis.

Revised x_3 coefficients: $\begin{pmatrix} 1/3 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} -3 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$, so x_6 leaves.

Iteration 2: $B_{\text{new}}^{-1} = \begin{pmatrix} 3 & -3 \\ 3 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1/12 & 1/4 \\ -1/4 & 1/4 \end{pmatrix}$

$$x_B = \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1/12 & 1/4 \\ -1/4 & 1/4 \end{pmatrix} \begin{pmatrix} 24 \\ 36 \end{pmatrix} = \begin{pmatrix} 11 \\ 9 \end{pmatrix}, c_B = (5 \quad -1)$$

Revised row 0: $(2/3 \quad 1) \begin{pmatrix} 3 & 2 & -3 & 1 & 1 & 0 \\ 3 & 3 & 1 & 3 & 0 & 1 \end{pmatrix} - (5 \quad 4 \quad -1 \quad 3 \quad 0 \quad 0)$

$= (0 \quad 1/3 \quad 0 \quad 2/3 \quad 2/3 \quad 1)$, so current solution is optimal.

Optimal Solution: $(x_1^*, x_2^*, x_3^*, x_4^*) = (11, 0, 3, 0)$ and $Z^* = 52$