Tutorial Unit - III

1) a) $\{(2,1,0), (3,1,-1), (0,-1,1)\}$

$2k_1 + 3k_2 = 0$
$k_1 + k_2 - k_3 = 0$
$-k_2 + k_3 = 0$

$$\begin{bmatrix} 2 & 3 & 0 \\ 1 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = 0$$

$R_2 \to R_1 - 2R_2$

$$\begin{bmatrix} 2 & 3 & 0 \\ 0 & 1 & 2 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = 0$$

$R_3 \to R_2 + R_3$

$$\begin{bmatrix} 2 & 3 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = 0$$

$3k_3 = 0 \Rightarrow \boxed{k_3 = 0}$

$k_2 + 2k_3 = 0$
$\Rightarrow \boxed{k_2 = 0}$

$2k_1 + 3k_2 = 0$
$\boxed{k_1 = 0}$

$\because k_1 = k_2 = k_3 = 0$

$\therefore$ linearly independent

b) $\{(1,1,-1,1), (1,-1,2,-1), (3,1,0,1)\}$

$k_1 + k_2 + 3k_3 = 0$
$k_1 - k_2 + k_3 = 0$
$-k_1 + 2k_2 = 0$
$k_1 - k_2 + k_3 = 0$

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & -1 & 1 \\ -1 & 2 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = 0$$

$R_2 \to R_1 - R_2$
$R_3 \to R_1 + R_3$
$R_4 \to R_1 - R_4$

$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & 2 & 2 \\ 0 & 3 & 3 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = 0$$

$R_3 \to 3R_2 - 2R_3$
$R_4 \to R_2 - R_4$

$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = 0$$

$\boxed{k_2 = -t}$

$k_1 + k_2 + 3k_3 = 0$
$2k_2 + 2k_3 = 0$

$\boxed{k_3 = t}$

$\boxed{k_1 = -3t + t = -2t}$

$$-2tv_1 - tv_2 + tv_3 = 0$$

$$2v_1 + v_2 - v_3 = 0$$

$\therefore$ linearly dependent.

2) $\{(1, 2, -1), (1, -1, 2), (2, -1, 1)\}$

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & -1 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = 0$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_1 + R_2$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & -3 & -5 \\ 0 & 3 & 3 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = 0$$

$R_3 \to R_2 + R_3$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & -3 & -5 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = 0$$

$$\boxed{k_3 = 0}$$

$$-3k_2 - 5k_3 = 0$$

$$\boxed{k_2 = 0}$$

$$k_1 + k_2 + 2k_3 = 0$$

$$\boxed{k_1 = 0}$$

$\therefore$ linearly independent.

~~Basis~~ $\therefore$ the set S is a basis of vector space $\mathbb{R}^3$.

3) $S = \{-4 + x + 3x^2, 6 + 5x + 2x^2, 8 + 4x + x^2\}$

$-4k_1 + 6k_2 + 8k_3 = 0$
$k_1 + 5k_2 + 4k_3 = 0$
$3k_1 + 2k_2 + k_3 = 0$

$$\begin{bmatrix} -4 & 6 & 8 \\ 1 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} = 0$$

$R_2 \to R_1 + 4R_2$
$R_3 \to 3R_1 + 4R_3$

$$\begin{bmatrix} -4 & 6 & 8 \\ 0 & 26 & 24 \\ 0 & 26 & 28 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = 0$$

$R_3 \to R_2 - R_3$

$$\begin{bmatrix} -4 & 6 & 8 \\ 0 & 26 & 24 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = 0$$

$k_3 = 0$
$k_2 = 0$
$k_1 = 0$

$\therefore$ linearly independent $\Rightarrow$ S must be basis of P_2

4) $W = \{(x, y) \in R^2 \mid ax + by = 0\}$

Let $(0,0) \in W$

$a \cdot 0 + b \cdot 0 = 0$

$0 \in W$

Let $u = \{(x_1, y_1,) \mid ax_1 + by_1 = 0 \in W\}$

$V = \{(x_2, y_2) \mid ax_2 + by_2 = 0 \in W\}$.

$\alpha u + \beta V = \alpha(x_1, y_1,) + \beta(x_2, y_2)$

$= (\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2)$

$a(\alpha x_1 + \beta x_2) + b(\alpha y_1 + \beta y_2) = 0$

$\alpha(ax_1 + by_1) + \beta(ax_2 + by_2) = 0$

$\alpha \cdot 0 + \beta \cdot 0 = 0$

$\alpha u + \beta V \in W$.

$\therefore$ W is a subspace of R^2.

5) $W = \{(x, y, z) \in R^3 / ax + by + cz = 0\}$

Let $(0,0,0) \in W$

$a \cdot 0 + b \cdot 0 + c \cdot 0 = 0$

$0 \in W$

Let $u = \{(x_1, y_1, z_1) / ax_1 + by_1 + cz_1 = 0 \in W\}$

$V = \{(x_2, y_2, z_2) / ax_2 + by_2 + cz_2 = 0 \in W\}$

$\alpha u + \beta V = \alpha(x_1, y_1, z_1) + \beta(x_2, y_2, z_2)$

$= (\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2, \alpha z_1 + \beta z_2)$

$a(\alpha x_1 + \beta x_2) + b(\alpha y_1 + \beta y_2) + c(\alpha z_1 + \beta z_2)$

$\alpha(ax_1 + by_1 + cz_1) + \beta(ax_2 + by_2 + cz_2) = 0$

$\alpha \cdot 0 + \beta \cdot 0$

$\alpha u + \beta V \in W$

$\therefore$ W is a subspace of R^3.

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7) $S = \{(x, y, z) \in R^3 \mid ax = by = cz\}$.

Let $\bar{0} = (0,0,0) \in W$

$a \cdot 0 = b \cdot 0 = c \cdot 0 = 0$

$0 \in W$

Let $u = \{x_1, y_1, z_1 \mid ax_1 = by_1 = cz_1\}$ — ①

$v = \{x_2, y_2, z_2 \mid ax_2 = by_2 = cz_2\}$ — ②

$\alpha u + \beta v = \alpha(x_1, y_1, z_1) + \beta(x_2, y_2, z_2)$

$= (\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2, \alpha z_1 + \beta z_2)$

$a(\alpha x_1 + \beta x_2) = b(\alpha y_1 + \beta y_2) = c(\alpha z_1 + \beta z_2)$

$\alpha(ax_1) + \beta(ax_2) = \alpha(by_1) + \beta(by_2) = \alpha(cz_1) + \beta(cz_2)$

This holds true by ① & ②

$\alpha u + \beta v \in W$

$\therefore$ ~~W~~ S is subspace of R

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10) $T: R^3 \to R^3$

$T(x, y, z) = (x - y + z, y - z, 2x - 5y + 5z)$

To find kernel

$x - y + z = 0$

$y - z = 0$

$2x - 5y + 5z = 0$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 2 & -5 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$R_3 \to 2R_1 - R_3$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 3 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$R_3 \to 3R_2 - R_3$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

rank = 2

For nullity

$x - y + z = 0$

$y - z = 0 \Rightarrow y = z$

$\Rightarrow x - z + z = 0 \Rightarrow x = 0$

$\ker = \{0, y, y / y \in R\}$

nullity = 1.

Range $= \{(r, s, 2r - 3s) / r, s \in R\}$

$\left(\begin{matrix} r = x - y + z \\ s = y - z \end{matrix} \right)$

13) $T: R^3 \to R^3$

$T(x, y) = (x, x+y, y)$

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$R_2 \to R_2 - R_1$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

$R_3 \to R_2 - R_3$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

rank $= 2$

To find kernel

$x = 0$

$x + y = 0$

$y = 0$

$\ker = \{0, 0\}$

nullity $= 0$

Range $= \{(r, s, s-r) / r, s \in R\}$

12) $T: R^3 \to R^3$

$T(x, y, z) = (x+z, x+y+2z, 2x+y+3z)$

To find kernel

$x + z = 0$

$x + y + 2z = 0$

$2x + y + 3z = 0$

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix}$$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - 2R_1$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$R_3 \to R_2 - R_3$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

rank $= 2$

For nullity

$x + z = 0 \Rightarrow x = -z$
$y + z = 0 \Rightarrow y = -z$

$x = y = -z$

$\ker = \{ -z, -z, z \mid z \in R \}$

nullity $= 1$

Range $= \{ r, s, r+s \mid r, s \in R \}$

$r = x + z, \; s = x + y + 2z$

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16) $T: R^2 \to R^2$

$T(x, y) = (2x+y,\ 3x-5y)$

$$T = \begin{bmatrix} 2 & 1 \\ 3 & -5 \end{bmatrix}$$

$|T| = -10 - 3 = -13 \neq 0$

T is non-singular

$T(x, y) = (2x+y,\ 3x-5y)$

$(x, y) = T^{-1}(2x+y,\ 3x-5y)$

$(x, y) = T^{-1}(u, v)$

$2x + y = u \qquad 3x - 5y = v$

$y = u - 2x \qquad 3x - 5(u - 2x) = v.$

$3x - 5u + 10x = v$

$13x - 5u = v$

$x = \dfrac{5u+v}{13}$

$y = u - 2\left(\dfrac{5u+v}{13}\right) = \dfrac{13u - 10u - 2v}{13}$

$= \dfrac{3u - 2v}{13}$

$T^{-1}(u, v) = \left(\dfrac{5u+v}{13},\ \dfrac{3u-2v}{13}\right)$