Regional Planning

The SOUTHERN CONFEDERATION OF KIBBUTZIM is a group of three kibbutzim (communal farming communities) in Israel. Overall planning for this group is done in its Coordinating Technical Office. This office currently is planning agricultural production for the coming year.

The agricultural output of each kibbutz is limited by both the amount of available irrigable land and the quantity of water allocated for irrigation by the Water Commissioner (a national government official). These data are given in Table 1. The crops suited for this region include sugar beets, cotton, and sorghum, and these are the three being considered for the upcoming season. These crops differ primarily in their expected net return per acre and their consumption of water. In addition, the Ministry of Agriculture has set a maximum quota for the total acreage that can be devoted to each of these crops by the Southern Confederation of Kibbutzim, as shown in Table 2.

Table 1: **Resource data for the Southern Confederation of Kibbutzim**

Kibbutz	Usable Land (Acres)	Water Allocation (Acre Feet)
1	400	600
2	600	800
3	300	375

Table 2: **Crop data for the Southern Confederation of Kibbutzim**

Crop	Maximum Quota (Acres)	Water Consumption (Acre Feet/Acre)	Net Return ($/Acre)
Sugar beets	600	3	1,000
Cotton	500	2	750
Sorghum	325	1	250

Because of the limited water available for irrigation, the Southern Confederation of Kibbutzim will not be able to use all its irrigable land for planting crops in the upcoming season.

To ensure equity between the three kibbutzim, it has been agreed that every kibbutz will plant the same proportion of its available irrigable land.

For example,

a. If kibbutz 1 plants 200 of its available 400 acres,

b. Then kibbutz 2 must plant 300 of its 600 acres,

c. While kibbutz 3 plants 150 acres of its 300 acres. However, any combination of the crops may be grown at any of the kibbutzim.

The job facing the Coordinating Technical Office is to plan how many acres to devote to each crop at the respective kibbutzim while satisfying the given restrictions.

The objective is to maximize the total net return to the Southern Confederation of Kibbutzim as a whole.

Formulation as a Linear Programming Problem

The quantities to be decided upon are the number of acres to devote to each of the three crops at each of the three kibbutzim.
The decision variables x_j (j _ 1, 2, . . . ,9) represent these nine quantities, as shown in Table 3.
Since the measure of effectiveness Z is the total net return, the resulting linear programming model for this problem is:

	Allocation (Acres)		
	Kibbutz		
Crop	**1**	**2**	**3**
Sugar beets	x_1	x_2	x_3
Cotton	x_4	x_5	x_6
Sorghum	x_7	x_8	x_9

Maximize $Z = 1{,}000(x_1 + x_2 + x_3) + 750(x_4 + x_5 + x_6) + 250(x_7 + x_8 + x_9)$,

subject to the following constraints:

1. Usable land for each kibbutz:

$$x_1 + x_4 + x_7 \leq 400$$
$$x_2 + x_5 + x_8 \leq 600$$
$$x_3 + x_6 + x_9 \leq 300$$

2. Water allocation for each kibbutz:

$$3x_1 + 2x_4 + x_7 \leq 600$$
$$3x_2 + 2x_5 + x_8 \leq 800$$
$$3x_3 + 2x_6 + x_9 \leq 375$$

3. Total acreage for each crop:

$$x_1 + x_2 + x_3 \leq 600$$
$$x_4 + x_5 + x_6 \leq 500$$
$$x_7 + x_8 + x_9 \leq 325$$

4. Equal proportion of land planted:

$$\frac{x_1 + x_4 + x_7}{400} = \frac{x_2 + x_5 + x_8}{600}$$
$$\frac{x_2 + x_5 + x_8}{600} = \frac{x_3 + x_6 + x_9}{300}$$
$$\frac{x_3 + x_6 + x_9}{300} = \frac{x_1 + x_4 + x_7}{400}$$

5. Nonnegativity:

$$x_j \geq 0, \qquad \text{for } j = 1, 2, \ldots, 9.$$

This completes the model, except that the equality constraints are not yet in an appropriate form for a linear programming model because some of the variables are on the right-hand side. Hence, their final form is

$$3(x_1 + x_4 + x_7) - 2(x_2 + x_5 + x_8) = 0$$
$$(x_2 + x_5 + x_8) - 2(x_3 + x_6 + x_9) = 0$$
$$4(x_3 + x_6 + x_9) - 3(x_1 + x_4 + x_7) = 0$$

The Coordinating Technical Office formulated this model and then applied the simplex to find an optimal solution. The resulting optimal value of the objective function is Z=633, 3331/3that is, a total net return of $633,333.33.

$$(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9) = \left(133\frac{1}{3}, 100, 25, 100, 250, 150, 0, 0, 0\right),$$

Personnel Scheduling

UNION AIRWAYS is adding more flights to and from its hub airport, and so it needs to hire additional customer service agents. However, it is not clear just how many more should be hired. Management recognizes the need for cost control while also consistently providing a satisfactory level of service to customers. Therefore, an OR team is studying how to schedule the agents to provide satisfactory service with the smallest personnel cost. Based on the new schedule of flights, an analysis has been made of the *minimum* number of customer service agents that need to be on duty at different times of the day to provide a satisfactory level of service.

Table: **Data for the Union Airways personnel scheduling problem**

	Time Periods Covered					
	Shift					
Time Period	1	2	3	4	5	Minimum Number of Agents Needed
6:00 A.M. to 8:00 A.M.	✔					48
8:00 A.M. to 10:00 A.M.	✔	✔				79
10:00 A.M. to noon	✔	✔				65
Noon to 2:00 P.M.	✔	✔	✔			87
2:00 P.M. to 4:00 P.M.		✔	✔			64
4:00 P.M. to 6:00 P.M.			✔	✔		73
6:00 P.M. to 8:00 P.M.			✔	✔		82
8:00 P.M. to 10:00 P.M.				✔		43
10:00 P.M. to midnight				✔	✔	52
Midnight to 6:00 A.M.					✔	15
Daily cost per agent	$170	$160	$175	$180	$195	

The provision is that each agent work an 8-hour shift 5 days per week, and the authorized shifts are and the rightmost column of Table1 shows the number of agents needed for the time periods given in the first column.
Shift 1: 6:00 A.M. to 2:00 P.M.
Shift 2: 8:00 A.M. to 4:00 P.M.
Shift 3: Noon to 8:00 P.M.
Shift 4: 4:00 P.M. to midnight
Shift 5: 10:00 P.M. to 6:00 A.M.

Checkmarks in the main body of Table 1 show the hours covered by the respective shifts. Because some shifts are less desirable than others, the wages specified in the contract differ by shift. For each shift, the daily compensation (including benefits) for each agent is shown in the bottom row.

Objective Function:
The problem is to determine how many agents should be assigned to the respective shifts each day to minimize the *total* personnel cost for agents, based on this bottom row, while meeting (or surpassing) the service requirements given in the rightmost column.

Formulation as a Linear Programming Problem

Linear programming problems always involve finding the best *mix of activity levels.* The key to formulating this particular problem is to recognize the nature of the activities.
Activities correspond to shifts, where the *level* of each activity is the number of agents assigned to that shift.

Thus, this problem involves finding the *best mix of shift sizes.*
Since the decision variables always are the levels of the activities, the five decision variables here are

$$x_j = \text{number of agents assigned to shift } j, \quad \text{for } j = 1, 2, 3, 4, 5.$$

The main restrictions on the values of these decision variables are that the number of agents working during each time period must satisfy the minimum requirement given in the rightmost column of Table 1.
For example, for 2:00 P.M. to 4:00 P.M., the total number of agents assigned to the shifts that cover this time period (shifts 2 and 3) must be at least 64, so is the functional constraint for this time period.
Because the objective is to minimize the total cost of the agents assigned to the five shifts, the coefficients in the objective function are given by the last row of Table 1.Therefore, the complete linear programming model is

$$\text{Minimize} \quad Z = 170x_1 + 160x_2 + 175x_3 + 180x_4 + 195x_5,$$

subject to

$$\begin{array}{lll}
x_1 & \geq 48 & \text{(6–8 A.M.)} \\
x_1 + x_2 & \geq 79 & \text{(8–10 A.M.)} \\
x_1 + x_2 & \geq 65 & \text{(10 A.M. to noon)} \\
x_1 + x_2 + x_3 & \geq 87 & \text{(Noon–2 P.M.)} \\
x_2 + x_3 & \geq 64 & \text{(2–4 P.M.)} \\
x_3 + x_4 & \geq 73 & \text{(4–6 P.M.)} \\
x_3 + x_4 & \geq 82 & \text{(6–8 P.M.)} \\
x_4 & \geq 43 & \text{(8–10 P.M.)} \\
x_4 + x_5 & \geq 52 & \text{(10 P.M.–midnight)} \\
x_5 & \geq 15 & \text{(Midnight–6 A.M.)}
\end{array}$$

and

$$x_j \geq 0, \quad \text{for } j = 1, 2, 3, 4, 5.$$

The optimal solution for this model is (x1, x2, x3, x4, x5) = (48, 31, 39, 43, 15). This yields Z = 30,610, that is, a total daily personnel cost of $30,610.

Note:

1. This problem is an example where the divisibility assumption of linear programming actually is not satisfied. The number of agents assigned to each shift needs to be an integer. Strictly speaking, the model should have an additional constraint for each decision variable specifying that the variable must have an integer value. Adding these constraints would convert the linear programming model to an integer programming model
2. Without these constraints, the optimal solution given above turned out to have integer values anyway, so no harm was done by not including the constraints. (The form of the functional constraints made this outcome a likely one.) If some of the variables had turned out to be noninteger, the easiest approach would have been to *round up* to integer values.

ASSUMPTIONS OF LINEAR PROGRAMMING

Proportionality

Proportionality is an assumption about both the objective function and the functional constraints, as summarized below.

Proportionality assumption: The contribution of each activity to the *value of the objective function Z* is *proportional* to the *level of the activity* x_j, as represented by the $c_j\, x_j$ term in the objective function.

Similarly, the contribution of each activity to the *left-hand side of each functional constraint* is *proportional* to the *level of the activity* x_j, as represented by the $a_{ij}\, x_j$ term in the constraint.

Consequently, this assumption rules out any exponent other than 1 for any variable in any term of any function (whether the objective function or the function on the left-hand side of a functional constraint) in a linear programming model.

Table1: **Data for the Wyndor Glass Co. problem**

Plant	Production Time per Batch, Hours — Product 1	Product 2	Production Time Available per Week, Hours
1	1	0	4
2	0	2	12
3	3	2	18
Profit per batch	$3,000	$5,000	

$$\text{Maximize} \quad Z = 3x_1 + 5x_2,$$

subject to the restrictions

$$\begin{aligned} x_1 \qquad &\le 4 \\ 2x_2 &\le 12 \\ 3x_1 + 2x_2 &\le 18 \end{aligned}$$

and

$$x_1 \ge 0, \quad x_2 \ge 0.$$

To illustrate this assumption, consider the first term ($3x_1$) in the objective function ($Z = 3x_1 + 5x_2$) for the Wyndor Glass Co. problem. (Table 1)

This term represents the profit generated per week (in thousands of dollars) by producing product 1 at the rate of x_1 batches per week.

The *proportionality satisfied* column of Table 2 shows the case that was assumed, namely, that this profit is indeed proportional to x_1 so that $3x_1$ is the appropriate term for the objective function. By contrast, the next three columns show different hypothetical cases where the proportionality assumption would be violated.

Table.2 **Examples of satisfying or violating proportionality**

	Profit from Product 1 ($000 per Week)			
		Proportionality Violated		
x_1	Proportionality Satisfied	Case 1	Case 2	Case 3
0	0	0	0	0
1	3	2	3	3
2	6	5	7	5
3	9	8	12	6
4	12	11	18	6

Case 1 See column in Table 2.

This case would arise if there were *start-up costs* associated with initiating the production of product 1.

For example, there might be costs involved with setting up the production facilities. There might also be costs associated with arranging the distribution of the new product. Because these are one-time costs, they would need to be amortized on a per-week basis to be commensurable with Z (profit in thousands of dollars per week). Suppose that this amortization were done and that the total start-up cost amounted to reducing Z by 1, but that the profit without considering the start-up cost would be $3x1$.

This would mean that the contribution from product 1 to Z should be $3x_1 - 1$ for $x1 > 0$, whereas the contribution would be $3x_1 = 0$ when $x_1 = 0$ (no start-up cost). This profit function, 3 which is given by the solid curve in Fig. 1, certainly is *not* proportional to x_1.

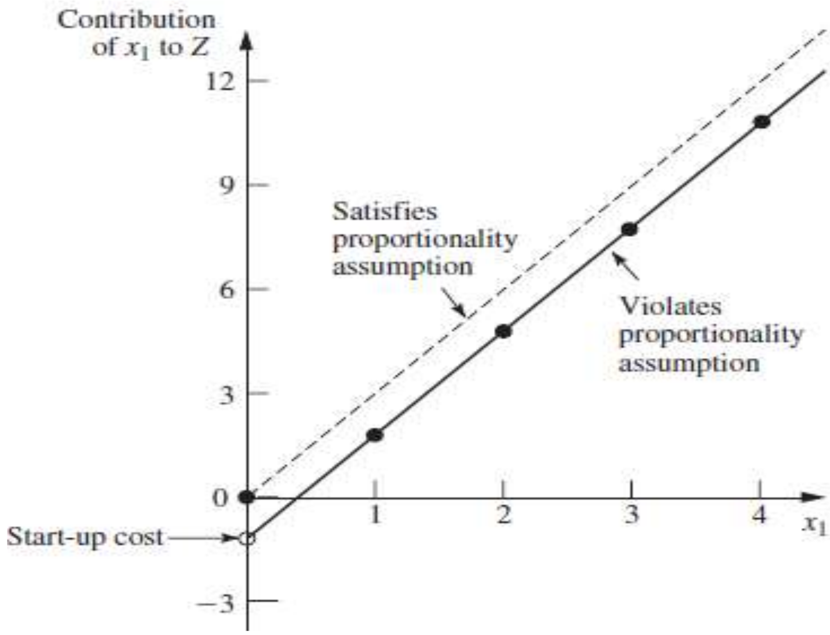


Fig. 1: for Case 1

Case 2 in Table 2 is quite similar to Case 1.
However, Case 2 actually arises in a very different way. There no longer is a start-up cost, and the profit from the first unit of product 1 per week is indeed 3, as originally assumed.
However, there now is an *increasing marginal return;* i.e., the *slope* of the *profit function* for product 1 (see the solid curve in Fig. 2) keeps increasing as x_1 is increased. This violation of proportionality might occur because of economies of scale that can sometimes be achieved at higher levels of production, e.g., through the use of more efficient high-volume machinery, longer production runs, quantity discounts for large purchases of raw materials, and the learning-curve effect whereby workers become more efficient as they gain experience with a particular mode of production. As the incremental cost goes down, the incremental profit will go up (assuming constant marginal revenue).

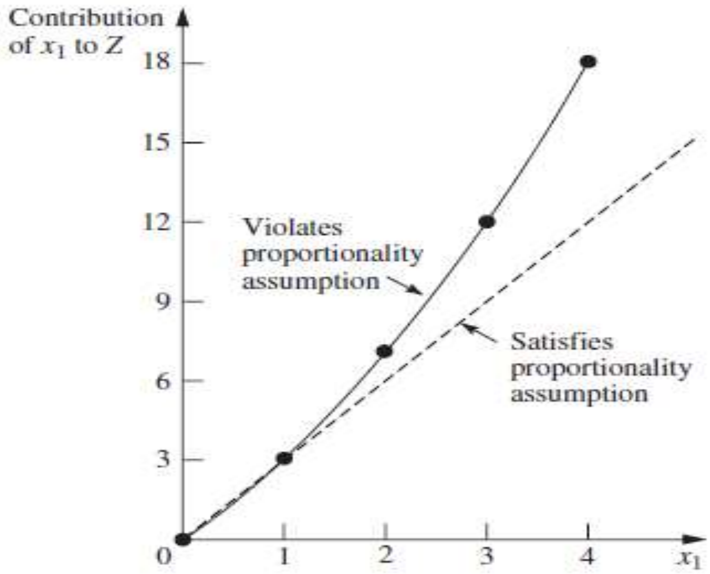


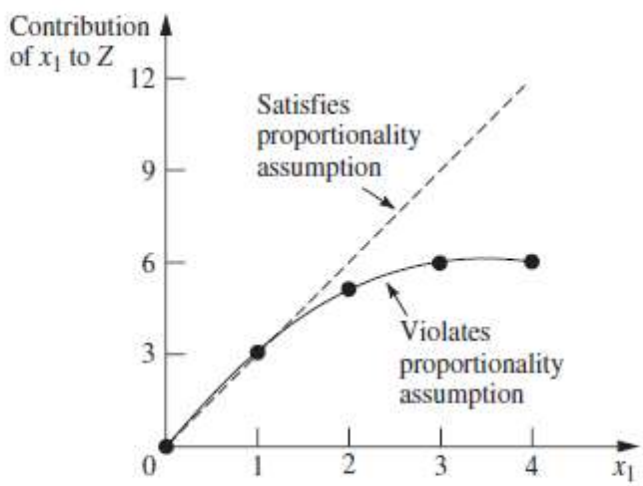
Fig 2: for Case 2

Referring again to Table 2, the reverse of Case 2 is *Case 3,* where there is a *decreasing marginal return.*

In this case, the *slope* of the *profit function* for product 1 (given by the solid curve in Fig. 3) keeps decreasing as $x1$ is increased.

This violation of proportionality might occur because the *marketing costs* need to go up more than proportionally to attain increases in the level of sales.

For example, it might be possible to sell product 1 at the rate of 1 per week (x_1 - 1) with no advertising, whereas attaining sales to sustain a production rate of $x_1 = 2$ might require a moderate amount of advertising, $x_1 = 3$ might necessitate an extensive advertising campaign, and $x_1 = 4$ might require also lowering the price.



Note:

1. All three cases are hypothetical examples of ways in which the proportionality assumption could be violated. What is the actual situation? The actual profit from producing product 1 (or any other product) is derived from the sales revenue minus various direct and indirect costs. Inevitably, some of these cost components are not strictly proportional to the production rate, perhaps for one of the reasons illustrated above.
2. In reality is whether, after all the components of profit have been accumulated, proportionality is a reasonable approximation for practical modeling purposes. For the Wyndor Glass Co. problem, the OR team checked both the objective function and the functional constraints. The conclusion was that proportionality could indeed be assumed without serious distortion.
3. For other problems, what happens when the proportionality assumption does not hold even as a reasonable approximation? In most cases, this means you must use *non-linear programming* instead.

Additivity assumption:

Every function in a linear programming model (whether the objective function or the function on the left-hand side of a functional constraint) is the *sum* of the *individual contributions* of the respective activities.

Table 2 shows some possible cases for the objective function for the Wyndor Glass Co. problem. In each case, the *individual contributions* from the products are just as assumed, namely, $3x_1$ for product 1 and $5x_2$ for product 2.
The difference lies in the last row, which gives the *function value* for Z when the two products are produced jointly.
The *additivity satisfied* column shows the case where this *function value* is obtained simply by adding the first two rows (3 + 5 = 8), so that $Z = 3x_1 + 5x_2$ as previously assumed.
By contrast, the next two columns show hypothetical cases where the additivity assumption would be violated (but not the proportionality assumption).

Referring to the *Case 1* column of Table 3 this case corresponds to an objective function of $Z = 3x_1 + 5x_2 + x_1 x_2$, so that $Z = 3 + 5 + 1 = 9$ for $(x_1, x_2) = (1, 1)$, thereby violating the additivity assumption that $Z = 3 + 5$. (The proportionality assumption still is satisfied since after the value of one variable is fixed, the increment in Z from the other variable is proportional to the value of that variable.)

This case would arise if the two products were *complementary* in some way that *increases* profit.

For example, suppose that a major advertising campaign would be required to market either new product produced by it but that the same single campaign can effectively promote both products if the decision is made to produce both. Because a major cost is saved for the second product, their joint profit is somewhat more than the *sum* of their individual profits when each is produced by itself.

Table 3: **Examples of satisfying or violating additivity for the objective function**

	Value of Z		
		Additivity Violated	
(x_1, x_2)	Additivity Satisfied	Case 1	Case 2
(1, 0)	3	3	3
(0, 1)	5	5	5
(1, 1)	8	9	7

Corresponding objective function, $Z = 3x_1 + 5x_2 - x_1 x_2$, so that $Z = 3 + 5 - 1 = 7$ for $(x_1, x_2) = (1, 1)$.

As the reverse of the first case, Case 2 would arise if the two products were *competitive* in some way that *decreased* their joint profit.

For example, suppose that both products need to use the same machinery and equipment. If either product were produced by itself, this machinery and equipment would be dedicated to this one use. However, producing both products would require switching the production processes back and forth, with substantial time and cost involved in temporarily shutting down the production of one product and setting up for the other. Because of this major extra cost, their joint profit is somewhat less than the *sum* of their individual profits when each is produced by itself.

Table 4: **Examples of satisfying or violating additivity for a functional constraint**

	Amount of Resource Used		
		Additivity Violated	
(x_1, x_2)	Additivity Satisfied	Case 3	Case 4
(2, 0)	6	6	6
(0, 3)	6	6	6
(2, 3)	12	15	10.8

The same kinds of interaction between activities can affect the additivity of the constraint functions. For example, consider the third functional constraint of the Wyndor Glass Co. problem: $3x_1 + 2x_2 = 18$. (This is the only constraint involving both products.)
This constraint concerns the production capacity of Plant 3, where 18 hours of production time per week is available for the two new products, and the function on the left-hand side ($3x_1 + 2x_2$) represents the number of hours of production time per week that would be used by these products.
The *additivity satisfied* column of Table 4 shows this case as is, whereas the next two columns display cases where the function has an extra cross-product term that violates additivity.
For all three columns, the *individual contributions* from the products toward using the capacity of Plant 3 are just as assumed previously, namely, $3x_1$ for product 1 and $2x_2$ for product 2, or $3(2) = 6$ for $x_1 = 2$ and $2(3) = 6$ for $x_2 = 3$.
As was true for Table 3, the difference lies in the last row, which now gives the *total function value* for production time used when the two products are produced jointly.

For Case 3 (see Table 4), the production time used by the two products is given by the function $3x_1 + 2x_2 - 0.5x_1 x_2$, so the *total function value* is 6 + 6 + 3 = 15 when $(x_1, x_2) = (2, 3)$, which violates the additivity assumption that the value is just 6 + 6 = 12.
This case can arise in exactly the same way as described for Case 2 in Table 3; namely, extra time is wasted switching the production processes back and forth between the two products. The extra cross-product term ($0.5x_1x_2$) would give the production time wasted in this way.

For Case 4 in Table 4, the function for production time used is $3x_1 + 2x_2 - 0.1x_1^2 x_2$, so the *function value* for $(x_1, x_2) = (2, 3)$ is 6 + 6 - 1.2 = 10.8. This case could arise in the following way.

As in Case 3, suppose that the two products require the same type of machinery and equipment. But suppose now that the time required to switch from one product to the other would be relatively small. Because each product goes through a sequence of production operations, individual production facilities normally dedicated to that product would incur occasional idle periods. During these otherwise idle periods, these facilities can be used by the other product. Consequently, the total production time used (including idle periods) when the two products are produced jointly would be less than the *sum* of the production times used by the individual products when each is produced by itself.

Divisibility assumption:

Decision variables in a linear programming model are allowed to have *any* values, including *noninteger* values, that satisfy the functional and nonnegativity constraints. Thus, these variables are *not* restricted to just integer values. Since each decision variable represents the level of some activity, it is being assumed that the activities can be run at *fractional levels.*

For the Wyndor Glass Co. problem, the decision variables represent production rates (the number of batches of a product produced per week). Since these production rates can have *any* fractional values within the feasible region, the divisibility assumption does hold.

Certainty assumption:

The value assigned to each parameter of a linear programming model is assumed to be a *known constant.*

In real applications, the certainty assumption is seldom satisfied precisely.

Linear programming models usually are formulated to select some future course of action. Therefore, the parameter values used would be based on a prediction of future conditions, which inevitably introduces some degree of uncertainty.

Problem 1

Consider a problem with two decision variables, x_1 and x_2, which represent the levels of activities 1 and 2, respectively.

For each variable, the permissible values are 0, 1, and 2, where the feasible combinations of these values for the two variables are determined from a variety of constraints.

The objective is to maximize a certain measure of performance denoted by Z. The values of Z for the possibly feasible values of (x_1, x_2) are estimated to be those given in the following table:

	x_2		
x_1	0	1	2
0	0	4	8
1	3	8	13
2	6	12	18

Based on this information, indicate whether this problem completely satisfies each of the four assumptions of linear programming. Justify your answers.

Answer

Proportionality: If either variable is fixed, the objective value grows proportionally to the increase in the other variable, so proportionality is reasonable.

Additivity: It is not a reasonable assumption, since the activities interact with each other.
For example, the objective value at [1, 1] is not equal to the sum of the objective values at [0, 1] and [1, 0].

Divisibility: It is not justified, since activity levels are not allowed to be fractional.

Certainty: It is reasonable, since the data provided is accurate.

Problem 2

The Primo Insurance Company is introducing two new product lines: special risk insurance and mortgages. The expected profit is \$5 per unit on special risk insurance and \$2 per unit on mortgages. Management wishes to establish sales quotas for the new product lines to maximize total expected profit. The work requirements are as follows:

	Work-Hours per Unit		
Department	**Special Risk**	**Mortgage**	**Work-Hours Available**
Underwriting	3	2	2400
Administration	0	1	800
Claims	2	0	1200

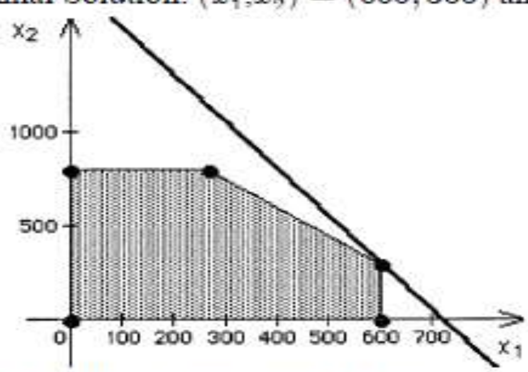
(a) Formulate a linear programming model for this problem.
(b) Use the graphical method to solve this model.
(c) Verify the exact value of your optimal solution from part (*b*) by solving algebraically for the simultaneous solution of the relevant two equations.

Answer:

(a) Let x_1 be the number of units on special risk insurance and x_2 be the number of units on mortgages.

$$\begin{array}{lrl} \text{maximize} & z = 5x_1 + 2x_2 & \\ \text{subject to} & 3x_1 + 2x_2 & \leq 2400 \\ & x_2 & \leq 800 \\ & 2x_1 & \leq 1200 \\ & x_1 \geq 0,\ x_2 \geq 0 & \end{array}$$

(b) Optimal Solution: $(x_1^*, x_2^*) = (600, 300)$ and $Z^* = 3600$



(c) The relevant two equations are $3x_1 + 2x_2 = 2400$ and $2x_1 = 1200$, so $x_1 = 600$ and $x_2 = \frac{1}{2}(2400 - 3x_1) = 300$, $z = 5x_1 + 2x_2 = 3600$