Basics of Linear Algebra

Vectors, Matrices, and Linear Algebra

Scalars and *Vectors*

Definition: A scalar is a number.

Examples of scalars are temperature, distance, speed, or mass – all quantities that have a magnitude but no "direction", other than perhaps positive or negative.

Definition: A vector is *a list of numbers*.

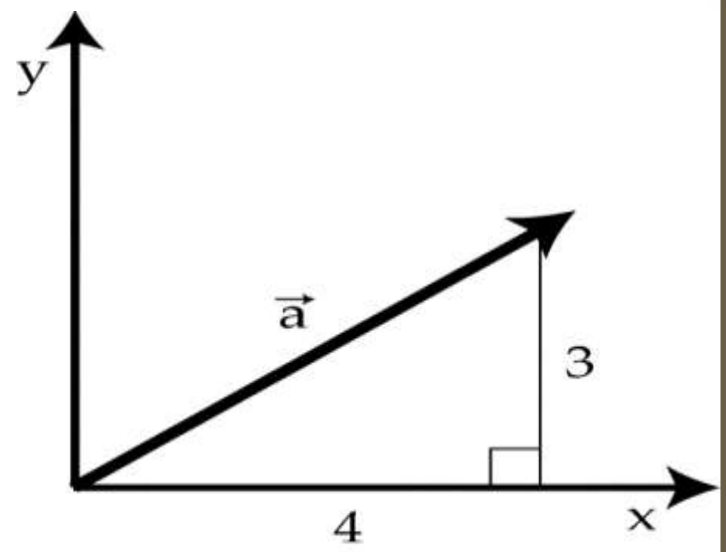
There are (at least) two ways to interpret what this list of numbers mean:

One way to think of the vector as being *a point in a space*. Then this list of numbers is a way of identifying that point in space, where each number represents the vector's component that dimension.

Another way to think of a vector is *a magnitude and a direction*, e.g. a quantity like velocity ("the fighter jet's velocity is 250 mph north-by-northwest"). In this way of think of it, a vector is a directed arrow pointing from the origin to the end point given by the list of numbers.

An example of a vector is $\vec{a}$ = [4,3] . Graphically, you can think of this vector as an arrow in the *x-y* plane, pointing from the origin to the point at *x*=3, *y*=4 (see illustration.)

The dimensionality of a vector is the length of the list.
So, our example $\vec{a}$ is 2-dimensional because it is a list of two numbers.
Not surprisingly all 2-dimentional vectors live in a plane.
A 3-dimensional vector would be a list of three numbers, and they live in a 3-D volume.
A 27-dimensional vector would be a list of twenty seven numbers, and would live in a space only Ilana's dad could visualize.



Magnitudes and direction

The "magnitude" of a vector is the distance from the endpoint of the vector to the origin – in a word, it's length.
Suppose we want to calculate the magnitude of the vector $\vec{a}$ = [4,3] .
This vector extends 4 units along the x-axis, and 3 units along the y-axis.
To calculate the magnitude $||\vec{a}||$ of the vector we can use the Pythagorean theorem ($x^2 + y^2 = z^2$). $||\vec{a}|| = \sqrt{4^2 + 3^2} = 5$.
The magnitude of a vector is a scalar value – a number representing the length of the vector independent of the direction.

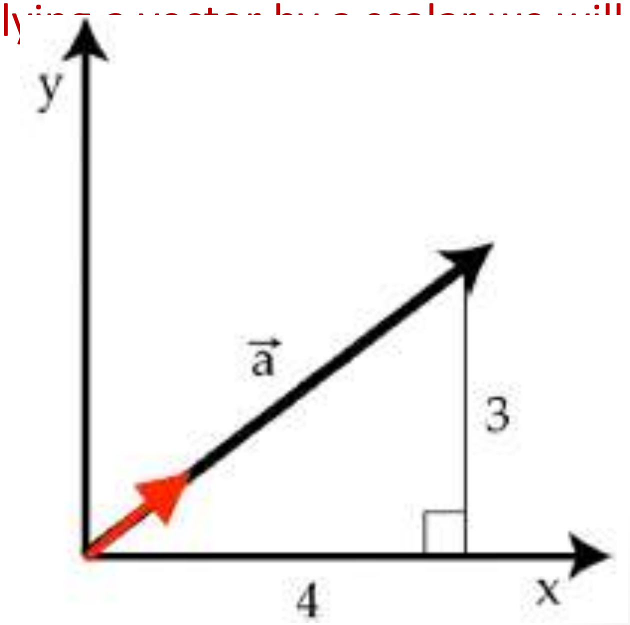
Definition:

A unit vector is a vector of magnitude 1.
Unit vectors can be used to express the direction of a vector independent of its magnitude.
A unit vector is denoted by a small "carrot" or "hat" above the symbol.
For example, $\hat{a}$ represents the unit vector associated with the vector.
To calculate the unit vector associated with a particular vector, we take the original vector and divide it by its magnitude. In mathematical terms, this process is written as:

$$\hat{a} = \frac{\vec{a}}{\|\vec{a}\|} \text{ ; For example, } \vec{a} = [,4,3] \; and \; \|\vec{a}\| = 5.$$

When dividing a vector ($\vec{a}$) by a scalar ($\|\vec{a}\|$), we divide each component of the vector individually by the scalar. In the same way, when multiplying a vector by a scalar we will proceed component by component i.e.

$$\hat{a} = \frac{[4,3]}{5} = [\frac{4}{5}, \frac{3}{5}]$$



By dividing each component of the vector by the same number, we leave the direction of the vector unchanged, while we change the magnitude. If we have done this correctly, then the magnitude of the unit vector must be equal to 1 (otherwise it would not be a unit vector). Thus,

$$\|\hat{a}\|^2 = \left(\frac{4}{5}\right)^2 + \left(\frac{3}{5}\right)^2 = \left(\frac{16}{25}\right) + \left(\frac{9}{25}\right) = \left(\frac{25}{25}\right) = 1$$

We can use these two components to re-create the vector $\vec{a}$ by multiplying the vector $\hat{a}$ by the scalar $||\vec{a}||$ like so:

$$\vec{a} = \hat{a} * ||\vec{a}||$$

Vector addition and subtraction

Vectors can be added and subtracted.

Graphically, we can think of adding two vectors together as placing two line segments end-to-end, maintaining distance and direction.

An example of this is shown in the illustration, showing the addition of two vectors $\vec{a}$ and $\vec{b}$ to create a third vector $\vec{c}$ i.e.

Numerically,

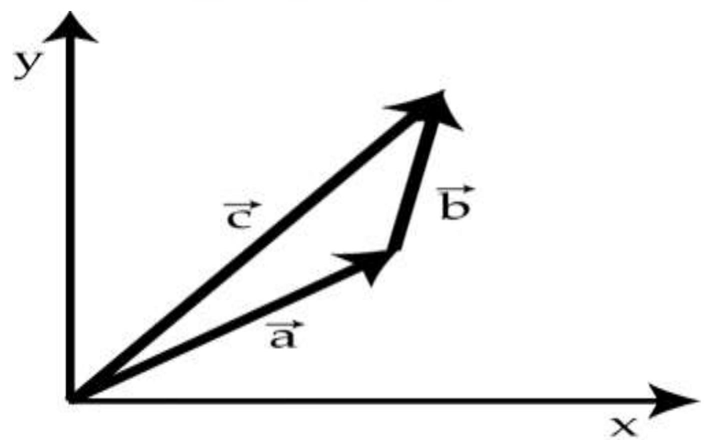
$$\vec{a} + \vec{b} = \vec{c}$$

$$\vec{c} = \vec{a} + \vec{b}$$

$$\vec{c} = [4,3] + [1,2]$$

$$\vec{c} = [4+1, 3+2]$$

$$\vec{c} = [5,5]$$



$$\vec{c} = \vec{a} - \vec{b}$$

$$\vec{c} = [4,3] - [1,2]$$

$$\vec{c} = [3,1]$$

Vector addition has a very simple interpretation in the case of things like displacement. If in the morning a ship sailed 4 miles east and 3 miles north, and then in the afternoon it sailed a further 1 mile east and 2 miles north, what was the total displacement for the whole day? 5 miles east and 5 miles north – vector addition at work.

Linear Independence

If two vectors point in different directions, even if they are not very different directions, then the two vectors are said to be *linearly independent*.

Definition: A family of vectors is linearly independent if no one of the vectors can be created by any linear combination of the other vectors in the family. For example,

$\vec{c}$ is linearly independent of $\vec{a}$ and $\vec{b}$ if and only if it is *impossible* to find scalar values of α and β such that $\vec{c} = \alpha\vec{a} + \beta\vec{b}$.

Vector multiplication: dot products

Next we move into the world of vector multiplication. There are two principal ways of multiplying vectors, called *dot products* (a.k.a. *scalar products*) and *cross products*. The dot product generates a scalar value from the product of two vectors.

$\vec{a} \cdot \vec{b} = [4,3] \cdot [1,2] = 4 * 1 + 3 * 2 = 10$. The dot product can be expr

$d = \vec{a} \cdot \vec{b}$

geometrically as: $\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| cos\theta$, where θ represents the angle bet

the two vectors.

Definition: A dot product (or scalar product) is the numerical product of the lengths of two vectors, multiplied by the cosine of the angle between them.

As the angle between the two vectors opens up to approach $90°$, the dot product of the two vectors will approach 0, regardless of the vector magnitudes $\|\vec{a}\|$ and $\|\vec{b}\|$, the two vectors are said to be *orthogonal*.

A basis set is a linearly independent set of vectors that, when used in linear combination, can represent every vector in a given vector space.

Basis of a Vector Space

Definition: A set of vectors {*v*1,*v*2,...,*vn*} is said to be a **Basis** of the F-vector space *V* if both *V*=span(*v*1,*v*2,...,*vn*) and {*v*1,*v*2,...,*vn*} is a linearly independent set.

A set $S = \{\overrightarrow{v_1}, \overrightarrow{v_2}, \ldots \overrightarrow{v_k}\}$ of vectors in a vector space V is a **basis** for V if

(1) S spans V and

(2) S is linearly independent.

Definition: Two vectors are orthogonal to one another if the dot product of those two vectors is equal to zero.

Basis of a Vector Space

Example 1: Let U $=\{(x_1, x_2, x_3) \in \mathrm{R}^3 : x_1 = 2x_2\}$ **be a subspace of** R^3 **Find a basis of** U**.**

If $U=\{(x_1, x_2, x_3) \in \mathrm{R}^3: x_1 = 2x_2\}$ then $U=\{(2x_2, x_2, x_3) \in \mathrm{R}^3: x_2, x_3 \in \mathrm{R}\}$, and so one such basis of U is:

$$\textbf{(1)}\ \{(2,1,0),(0,0,1)\}$$

To verify this set of vectors is a basis of U we must show that $U = \operatorname{span}((2,1,0),(0,0,1))$ and that $\{(2,1,0),(0,0,1)\}$ is a linearly independent set of vectors in V.

1. Let $x=(x_1, x_2, x_3) \in U$. Then we have that:

$$\textbf{(2)}\ x=(x_1, x_2, x_3) = (2x_2, x_2, x_3) = x_2\,(2,1,0) + x_3\,(0,0,1)$$

So $U=\operatorname{span}((2,1,0),(0,0,1))$.

2. Now consider the following vector equation for $a_1, a_2 \in \mathrm{F}$:

$$\textbf{(3)}\ a_1(2,1,0) + a_2(0,0,1) = 0$$

$$(2a_1, a_1, 0) + (0,0,a_2) = 0$$

$$(2a_1, a_1, a_2) = (0,0,0)$$

The equation above implies that $2a_1 = 0$, $a_1 = 0$, and $a_2 = 0$, so $a_1 = a_2 = 0$ and $\{(2,1,0),(0,0,1)\}$ is a linearly independent set of vectors in R^3.

Thus **{(2,1,0),(0,0,1)}** is a basis of U.

Example 2: **Let** $U=\{(x_1, x_2, x_3, x_4, x_5) \in \mathrm{R}^5: x_1 = 2x_2, x_3 = 4x_4, x_5 = 2x_4\}$ **be a subspace of** R5**. Find a basis for** U**.**

We can rewrite the subspace U as:

(1) $U=\{(2x_2, x_2, 4x_4, x_4, 2x_4): x_2, x_4 \in \mathrm{R}\}$

Therefore we have that {(2,1,0,0,0),(0,0,4,1,2)} is a basis of U. To verify this, let $x = (x_1, x_2, x_3, x_4, x_5) \in \mathrm{R}^5$. Then we have that:

(2) $x = (x_1, x_2, x_3, x_4, x_5) = (2x_2, x_2, 4x_4, x_4, 2x_4)$

$= x_2\,(2,1,0,0,0) + x_4\,(0,0,4,1,2)$

So U=span((2,1,0,0,0),(0,0,4,1,2)).

Now consider the following vector equation for $a_1, a_2 \in \mathrm{R}$:

(3) $a_1\,(2,1,0,0,0) + a_2\,(0,0,4,1,2) = 0$

$(2a_1, a_1, 0, 0, 0) + (0, 0, 4a_2, a_2, 2a_2) = 0$

$(2a_1, a_1, 4a_2, a_2, 2a_2) = (0,0,0,0,0)$

The equation above implies that:

(4) $2a_1 = 0;\ a_1 = 0;\ 4a_2 = 0;\ a_2 = 0;\ 2a_2 = 0$

Thus $a_1 = a_2 = 0$ and so {(2,1,0,0,0),(0,0,4,1,2)} is a linearly independent set of vectors in U.

Thus {(2,1,0,0,0),(0,0,4,1,2)} is a basis of U.

Ex.1 Is $S = \{(1, 2,3), (0, 1,2), (-1, 0,1)\}$ a basis for R^3?

Ex.2 In R^3 the vector (1, 2, 3) is not a linear combination of the vectors (1, 1, 0) and (1, −1, 0).

Ex. 3 In R^2 the vector (8, 2) is a linear combination of the vectors (1, 1) and (1, −1) because (8, 2) = 5(1, 1) + 3(1, −1).

Ex.4) Show that in the space R^3 the vectors

$x = (1, 1, 0)$, $y = (0, 1, 2)$, and $z = (3, 1, -4)$ are linearly dependent by finding scalars α and β such that $\alpha x + \beta y + z = 0$.

Answer: α = ___ , β = ____ .

Basis

Definition. Let V be a vector space. A linearly independent spanning set for V is called a **basis**.

Equivalently, a subset $S \subset V$ is a basis for V if any vector $\mathbf{v} \in V$ is *uniquely represented* as a linear combination

$$\mathbf{v} = r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k,$$

where $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are distinct vectors from S and $r_1, \ldots, r_k \in \mathbb{R}$.

Examples. • Standard basis for $\mathbb{R}^n$:
$\mathbf{e}_1 = (1, 0, 0, \ldots, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0, \ldots, 0, 0), \ldots,$
$\mathbf{e}_n = (0, 0, 0, \ldots, 0, 1)$.

• Matrices $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$
form a basis for $\mathcal{M}_{2,2}(\mathbb{R})$.

• Polynomials $1, x, x^2, \ldots, x^{n-1}$ form a basis for
$\mathcal{P}_n = \{a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} : a_i \in \mathbb{R}\}$.

• The infinite set $\{1, x, x^2, \ldots, x^n, \ldots\}$ is a basis for $\mathcal{P}$, the space of all polynomials.

Bases for $\mathbb{R}^n$

Theorem Every basis for the vector space $\mathbb{R}^n$ consists of n vectors.

Theorem For any vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in \mathbb{R}^n$ the following conditions are equivalent:

(i) $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for $\mathbb{R}^n$;
(ii) $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a spanning set for $\mathbb{R}^n$;
(iii) $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a linearly independent set.

Dimension

Theorem Any vector space V has a basis. All bases for V are of the same cardinality.

Definition. The **dimension** of a vector space V, denoted $\dim V$, is the cardinality of its bases.

Remark. By definition, two sets are of the same cardinality if there exists a one-to-one correspondence between their elements.

For a finite set, the cardinality is the number of its elements.

For an infinite set, the cardinality is a more sophisticated notion. For example, $\mathbb{Z}$ and $\mathbb{R}$ are infinite sets of different cardinalities while $\mathbb{Z}$ and $\mathbb{Q}$ are infinite sets of the same cardinality.

Examples. • $\dim \mathbb{R}^n = n$

• $\mathcal{M}_{2,2}(\mathbb{R})$: the space of 2×2 matrices
$\dim \mathcal{M}_{2,2}(\mathbb{R}) = 4$

• $\mathcal{M}_{m,n}(\mathbb{R})$: the space of $m\times n$ matrices
$\dim \mathcal{M}_{m,n}(\mathbb{R}) = mn$

• $\mathcal{P}_n$: polynomials of degree less than n
$\dim \mathcal{P}_n = n$

• $\mathcal{P}$: the space of all polynomials
$\dim \mathcal{P} = \infty$

• $\{\mathbf{0}\}$: the trivial vector space
$\dim \{\mathbf{0}\} = 0$

Problem. Find the dimension of the plane $x + 2z = 0$ in $\mathbb{R}^3$.

The general solution of the equation $x + 2z = 0$ is

$$\begin{cases} x = -2s \\ y = t \\ z = s \end{cases} \qquad (t, s \in \mathbb{R})$$

That is, $(x, y, z) = (-2s, t, s) = t(0, 1, 0) + s(-2, 0, 1)$.

Hence the plane is the span of vectors $\mathbf{v}_1 = (0, 1, 0)$ and $\mathbf{v}_2 = (-2, 0, 1)$. These vectors are linearly independent as they are not parallel.

Thus $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis so that the dimension of the plane is 2.

How to find a basis?

Theorem Let S be a subset of a vector space V. Then the following conditions are equivalent:

(i) S is a linearly independent spanning set for V, i.e., a basis;

(ii) S is a minimal spanning set for V;

(iii) S is a maximal linearly independent subset of V.

"Minimal spanning set" means "remove any element from this set, and it is no longer a spanning set".

"Maximal linearly independent subset" means "add any element of V to this set, and it will become linearly dependent".

Theorem Let V be a vector space. Then

(i) any spanning set for V can be reduced to a minimal spanning set;

(ii) any linearly independent subset of V can be extended to a maximal linearly independent set.

Equivalently, any spanning set contains a basis, while any linearly independent set is contained in a basis.

Corollary A vector space is finite-dimensional if and only if it is spanned by a finite set.

How to find a basis?

Approach 1. Get a spanning set for the vector space, then reduce this set to a basis.

Proposition Let $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_k$ be a spanning set for a vector space V. If $\mathbf{v}_0$ is a linear combination of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ then $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is also a spanning set for V.

Indeed, if $\mathbf{v}_0 = r_1\mathbf{v}_1 + \cdots + r_k\mathbf{v}_k$, then

$$t_0\mathbf{v}_0 + t_1\mathbf{v}_1 + \cdots + t_k\mathbf{v}_k =$$
$$= (t_0r_1 + t_1)\mathbf{v}_1 + \cdots + (t_0r_k + t_k)\mathbf{v}_k.$$

How to find a basis?

Approach 2. Build a maximal linearly independent set adding one vector at a time.

If the vector space V is trivial, it has the empty basis.

If $V \neq \{\mathbf{0}\}$, pick any vector $\mathbf{v}_1 \neq \mathbf{0}$.
If $\mathbf{v}_1$ spans V, it is a basis. Otherwise pick any vector $\mathbf{v}_2 \in V$ that is not in the span of $\mathbf{v}_1$.

If $\mathbf{v}_1$ and $\mathbf{v}_2$ span V, they constitute a basis.
Otherwise pick any vector $\mathbf{v}_3 \in V$ that is not in the span of $\mathbf{v}_1$ and $\mathbf{v}_2$.

And so on...

Problem. Find a basis for the vector space V spanned by vectors $\mathbf{w}_1 = (1,1,0)$, $\mathbf{w}_2 = (0,1,1)$, $\mathbf{w}_3 = (2,3,1)$, and $\mathbf{w}_4 = (1,1,1)$.

To pare this spanning set, we need to find a relation of the form $r_1\mathbf{w}_1+r_2\mathbf{w}_2+r_3\mathbf{w}_3+r_4\mathbf{w}_4 = \mathbf{0}$, where $r_i \in \mathbb{R}$ are not all equal to zero. Equivalently,

$$\begin{pmatrix} 1 & 0 & 2 & 1 \\ 1 & 1 & 3 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

To solve this system of linear equations for r_1, r_2, r_3, r_4, we apply row reduction.

$$\begin{pmatrix} 1 & 0 & 2 & 1 \\ 1 & 1 & 3 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{(reduced row echelon form)}$$

$$\begin{cases} r_1 + 2r_3 = 0 \\ r_2 + r_3 = 0 \\ r_4 = 0 \end{cases} \iff \begin{cases} r_1 = -2r_3 \\ r_2 = -r_3 \\ r_4 = 0 \end{cases}$$

General solution: $(r_1, r_2, r_3, r_4)=(-2t, -t, t, 0)$, $t \in \mathbb{R}$.

Particular solution: $(r_1, r_2, r_3, r_4) = (2, 1, -1, 0)$.

Problem. Find a basis for the vector space V spanned by vectors $\mathbf{w}_1 = (1, 1, 0)$, $\mathbf{w}_2 = (0, 1, 1)$, $\mathbf{w}_3 = (2, 3, 1)$, and $\mathbf{w}_4 = (1, 1, 1)$.

We have obtained that $2\mathbf{w}_1 + \mathbf{w}_2 - \mathbf{w}_3 = \mathbf{0}$. Hence any of vectors $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ can be dropped. For instance, $V = \mathrm{Span}(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_4)$.

Let us check whether vectors $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_4$ are linearly independent:

$$\begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1 \neq 0.$$

They are!!! It follows that $V = \mathbb{R}^3$ and $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_4\}$ is a basis for V.

Vectors $\mathbf{v}_1 = (0, 1, 0)$ and $\mathbf{v}_2 = (-2, 0, 1)$ are linearly independent.

Problem. Extend the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ to a basis for $\mathbb{R}^3$.

Our task is to find a vector $\mathbf{v}_3$ that is not a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

Then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ will be a basis for $\mathbb{R}^3$.

Hint 1. $\mathbf{v}_1$ and $\mathbf{v}_2$ span the plane $x + 2z = 0$.

The vector $\mathbf{v}_3 = (1, 1, 1)$ does not lie in the plane $x + 2z = 0$, hence it is not a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. Thus $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for $\mathbb{R}^3$.

Vectors $\mathbf{v}_1 = (0,1,0)$ and $\mathbf{v}_2 = (-2,0,1)$ are linearly independent.

Problem. Extend the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ to a basis for $\mathbb{R}^3$.

Our task is to find a vector $\mathbf{v}_3$ that is not a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

Hint 2. At least one of vectors $\mathbf{e}_1 = (1,0,0)$, $\mathbf{e}_2 = (0,1,0)$, and $\mathbf{e}_3 = (0,0,1)$ is a desired one.

Let us check that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{e}_1\}$ and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{e}_3\}$ are two bases for $\mathbb{R}^3$:

$$\begin{vmatrix} 0 & -2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 1 \neq 0, \qquad \begin{vmatrix} 0 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 2 \neq 0.$$