For $y < 0$, it is clear that $P(Y \le y) = 0$. For $y \ge 0$, we have

$$P(Y \le y) = P(X^2 \le y) = P(-\sqrt{y} \le X \le +\sqrt{y})$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\sqrt{y}}^{+\sqrt{y}} e^{-x^2/2}\,dx = \frac{2}{\sqrt{2\pi}}\int_{0}^{+\sqrt{y}} e^{-x^2/2}\,dx$$

where the last step uses the fact that the standard normal density function is even. Making the change of variable $x = +\sqrt{t}$ in the final integral, we obtain

$$P(Y \le y) = \frac{1}{\sqrt{2\pi}}\int_0^y t^{-1/2}e^{-t/2}\,dt$$

But this is a chi-square distribution with 1 degree of freedom, as is seen by putting $v = 1$ in (39), page 115, and using the fact that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

**4.37.** Prove Theorem 4-3, page 115, for $v = 2$.

By Problem 4.36 we see that if $X_1$ and $X_2$ are normally distributed with mean 0 and variance 1, then $X_1^2$ and $X_2^2$ are chi square distributed with 1 degree of freedom each. Then, from Problem 4.35(b), we see that $Z = X_1^2 + X_2^2$ is chi square distributed with $1 + 1 = 2$ degrees of freedom if $X_1$ and $X_2$ are independent. The general result for all positive integers $v$ follows in the same manner.

**4.38.** The graph of the chi-square distribution with 5 degrees of freedom is shown in Fig. 4-18. (See the remarks on notation on page 115.) Find the values $\chi_1^2, \chi_2^2$ for which

(a) the shaded area on the right = 0.05,

(b) the total shaded area = 0.05,

(c) the shaded area on the left = 0.10,

(d) the shaded area on the right = 0.01.

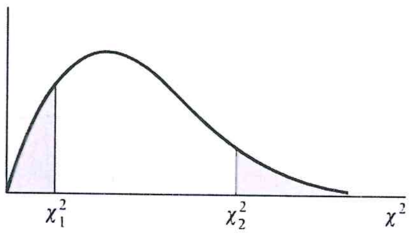



Fig. 4-18

(a) If the shaded area on the right is 0.05, then the area to the left of $\chi_2^2$ is $(1 - 0.05) = 0.95$, and $\chi_2^2$ represents the 95th percentile, $\chi_{0.95}^2$.

Referring to the table in Appendix E, proceed downward under column headed $v$ until entry 5 is reached. Then proceed right to the column headed $\chi_{0.95}^2$. The result, 11.1, is the required value of $\chi^2$.

(b) Since the distribution is not symmetric, there are many values for which the total shaded area = 0.05. For example, the right-hand shaded area could be 0.04 while the left-hand shaded area is 0.01. It is customary, however, unless otherwise specified, to choose the two areas equal. In this case, then, each area = 0.025.

If the shaded area on the right is 0.025, the area to the left of $\chi_2^2$ is $1 - 0.025 = 0.975$ and $\chi_2^2$ represents the 97.5th percentile, $\chi_{0.975}^2$, which from Appendix E is 12.8.

Similarly, if the shaded area on the left is 0.025, the area to the left of $\chi_1^2$ is 0.025 and $\chi_1^2$ represents the 2.5th percentile, $\chi_{0.025}^2$, which equals 0.831.

Therefore, the values are 0.831 and 12.8.

(c) If the shaded area on the left is 0.10, $\chi_1^2$ represents the 10th percentile, $\chi_{0.10}^2$, which equals 1.61.

(d) If the shaded area on the right is 0.01, the area to the left of $\chi_2^2$ is 0.99, and $\chi_2^2$ represents the 99th percentile, $\chi_{0.99}^2$, which equals 15.1.

**4.39.** Find the values of $\chi^2$ for which the area of the right-hand tail of the $\chi^2$ distribution is 0.05, if the number of degrees of freedom $v$ is equal to (a) 15, (b) 21, (c) 50.

Using the table in Appendix E, we find in the column headed $\chi^2_{0.95}$ the values: (a) 25.0 corresponding to $v = 15$; (b) 32.7 corresponding to $v = 21$; (c) 67.5 corresponding to $v = 50$.

**4.40.** Find the median value of $\chi^2$ corresponding to (a) 9, (b) 28, (c) 40 degrees of freedom.

Using the table in Appendix E, we find in the column headed $\chi^2_{0.50}$ (since the median is the 50th percentile) the values: (a) 8.34 corresponding to $v = 9$; (b) 27.3 corresponding to $v = 28$; (c) 39.3 corresponding to $v = 40$.

It is of interest to note that the median values are very nearly equal to the number of degrees of freedom. In fact, for $v > 10$ the median values are equal to $v - 0.7$, as can be seen from the table.

**4.41.** Find $\chi^2_{0.95}$ for (a) $v = 50$, (b) $v = 100$ degrees of freedom.

For $v$ greater than 30, we can use the fact that $(\sqrt{2\chi^2} - \sqrt{2v-1})$ is very closely normally distributed with mean zero and variance one. Then if $z_p$ is the $(100p)$th percentile of the standardized normal distribution, we can write, to a high degree of approximation,

$$\sqrt{2\chi^2_p} - \sqrt{2v-1} = z_p \qquad \text{or} \qquad \sqrt{2\chi^2_p} = z_p + \sqrt{2v-1}$$

from which

$$\chi^2_p = \tfrac{1}{2}(z_p + \sqrt{2v-1})^2$$

(a) If $v = 50$, $\chi^2_{0.95} = \frac{1}{2}(z_{0.95} + \sqrt{2(50)-1})^2 = \frac{1}{2}(1.64 + \sqrt{99})^2 = 69.2$, which agrees very well with the value 67.5 given in Appendix E.

(b) If $v = 100$, $\chi^2_{0.95} = \frac{1}{2}(z_{0.95} + \sqrt{2(100)-1})^2 = \frac{1}{2}(1.64 + \sqrt{199})^2 = 124.0$ (actual value = 124.3).

## Student's *t* distribution

**4.42.** Prove Theorem 4-6, page 116.

Since $Y$ is normally distributed with mean 0 and variance 1, its density function is

$$(1) \qquad \frac{1}{\sqrt{2\pi}}e^{-y^2/2}$$

Since $Z$ is chi-square distributed with $v$ degrees of freedom, its density function is

$$(2) \qquad \frac{1}{2^{v/2}\Gamma(v/2)}z^{(v/2)-1}e^{-z/2} \qquad z > 0$$

Because $Y$ and $Z$ are independent, their joint density function is the product of (1) and (2), i.e.,

$$\frac{1}{\sqrt{2\pi}\,2^{v/2}\Gamma(v/2)}z^{(v/2)-1}e^{-(y^2+z)/2}$$

for $-\infty < y < +\infty$, $z > 0$.

The distribution function of $T = Y/\sqrt{Z/v}$ is

$$F(x) = P(T \le x) = P(Y \le x\sqrt{Z/v})$$

$$= \frac{1}{\sqrt{2\pi}2^{v/2}\Gamma(v/2)}\iint_{\mathcal{R}} z^{(v/2)-1}e^{-(y^2+z)/2}\,dy\,dz$$

where the integral is taken over the region $\mathcal{R}$ of the $yz$ plane for which $y \le x\sqrt{z/v}$. We first fix $z$ and integrate with respect to $y$ from $-\infty$ to $x\sqrt{z/v}$. Then we integrate with respect to $z$ from 0 to $\infty$. We therefore have

$$F(x) = \frac{1}{\sqrt{2\pi}2^{v/2}\Gamma(v/2)}\int_{z=0}^{\infty} z^{(v/2)-1}e^{-z/2}\left[\int_{y=-\infty}^{x\sqrt{z/v}} e^{-y^2/2}\,dy\right]dz$$

Letting $y = u\sqrt{z/v}$ in the bracketed integral, we find

$$F(x) = \frac{1}{\sqrt{2\pi}\,2^{v/2}\Gamma(v/2)}\int_{z=0}^{\infty}\int_{u=-\infty}^{\infty} z^{(v/2)-1}e^{-z/2}\sqrt{z/v}\,e^{-u^2z/2v}\,du\,dz$$

$$= \frac{1}{\sqrt{2\pi v}\,2^{v/2}\Gamma(v/2)}\int_{u=-\infty}^{x}\left[\int_{z=0}^{\infty} z^{(v-1)/2}\,e^{-(z/2)[1+(u^2/v)]}\,dz\right]du$$

Letting $w = \frac{z}{2}\left(1 + \frac{u^2}{v}\right)$, this can then be written

$$F(x) = \frac{1}{\sqrt{2\pi v}\,2^{v/2}\Gamma(v/2)}\cdot 2^{(v+1)/2}\int_{u=-\infty}^{x}\left[\int_{w=0}^{\infty}\frac{w^{(v-1)/2}e^{-w}}{(1 + u^2/v)^{(v+1)/2}}\,dw\right]du$$

$$= \frac{\Gamma\left(\frac{v+1}{2}\right)}{\sqrt{\pi v}\,\Gamma\left(\frac{v}{2}\right)}\int_{u=-\infty}^{x}\frac{du}{(1 + u^2/v)^{(v+1)/2}}$$

as required.

**4.43.** The graph of Student's $t$ distribution with 9 degrees of freedom is shown in Fig. 4-19. Find the value of $t_1$ for which

(a) the shaded area on the right = 0.05,

(b) the total shaded area = 0.05,

(c) the total unshaded area = 0.99,

(d) the shaded area on the left = 0.01,

(e) the area to the left of $t_1$ is 0.90.

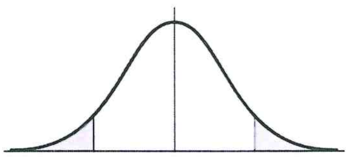



Fig. 4-19

(a) If the shaded area on the right is 0.05, then the area to the left of $t_1$ is $(1 - 0.05) = 0.95$, and $t_1$ represents the 95th percentile, $t_{0.95}$.

Referring to the table in Appendix D, proceed downward under the column headed $v$ until entry 9 is reached. Then proceed right to the column headed $t_{0.95}$. The result 1.83 is the required value of $t$.

(b) If the total shaded area is 0.05, then the shaded area on the right is 0.025 by symmetry. Therefore, the area to the left of $t_1$ is $(1 - 0.025) = 0.975$, and $t_1$ represents the 97.5th percentile, $t_{0.975}$. From Appendix D, we find 2.26 as the required value of $t$.

(c) If the total unshaded area is 0.99, then the total shaded area is $(1 - 0.99) = 0.01$, and the shaded area to the right is $0.01/2 = 0.005$. From the table we find $t_{0.995} = 3.25$.

(d) If the shaded area on the left is 0.01, then by symmetry the shaded area on the right is 0.01. From the table, $t_{0.99} = 2.82$. Therefore, the value of $t$ for which the shaded area on the left is 0.01 is $-2.82$.

(e) If the area to the left of $t_1$ is 0.90, then $t_1$ corresponds to the 90th percentile, $t_{0.90}$, which from the table equals 1.38.

**4.44.** Find the values of $t$ for which the area of the right-hand tail of the $t$ distribution is 0.05 if the number of degrees of freedom $\nu$ is equal to (a) 16, (b) 27, (c) 200.

Referring to Appendix D, we find in the column headed $t_{0.95}$ the values: (a) 1.75 corresponding to $\nu = 16$; (b) 1.70 corresponding to $\nu = 27$; (c) 1.645 corresponding to $\nu = 200$. (The latter is the value that would be obtained by using the normal curve. In Appendix D this value corresponds to the entry in the last row marked $\infty$.)

### The *F* distribution

**4.45.** Prove Theorem 4-7.

The joint density function of $V_1$ and $V_2$ is given by

$$f(v_1, v_2) = \left(\frac{1}{2^{\nu_1/2}\Gamma(\nu_1/2)} v_1^{(\nu_1/2)-1}e^{-v_1/2}\right)\left(\frac{1}{2^{\nu_2/2}\Gamma(\nu_2/2)} v_2^{(\nu_2/2)-1}e^{-v_2/2}\right)$$

$$= \frac{1}{2^{(\nu_1+\nu_2)/2}\Gamma(\nu_1/2)\Gamma(\nu_2/2)} v_1^{(\nu_1/2)-1}v_2^{(\nu_2/2)-1}e^{-(v_1+v_2)/2}$$

if $v_1 > 0$, $v_2 > 0$ and 0 otherwise. Make the transformation

$$u = \frac{v_1/\nu_1}{v_2/\nu_2} = \frac{\nu_2 v_1}{\nu_1 v_2}, \qquad w = v_2 \qquad \text{or} \qquad v_1 = \frac{\nu_1 uw}{\nu_2} \qquad v_2 = w$$

Then the Jacobian is

$$\frac{\partial(v_1, v_2)}{\partial(u, w)} = \begin{vmatrix} \partial v_1/\partial u & \partial v_1/\partial w \\ \partial v_2/\partial u & \partial v_2/\partial w \end{vmatrix} = \begin{vmatrix} \nu_1 w/\nu_2 & \nu_1 u/\nu_2 \\ 0 & 1 \end{vmatrix} = \frac{\nu_1 w}{\nu_2}$$

Denoting the density as a function of $u$ and $w$ by $g(u, w)$, we thus have

$$g(u, w) = \frac{1}{2^{(\nu_1+\nu_2)/2}\Gamma(\nu_1/2)\Gamma(\nu_2/2)}\left(\frac{\nu_1 uw}{\nu_2}\right)^{(\nu_1/2)-1} w^{(\nu_2/2)-1}e^{-[1+(\nu_1 u/\nu_2)](w/2)}\frac{\nu_1 w}{\nu_2}$$

if $u > 0$, $w > 0$ and 0 otherwise.

The (marginal) density function of $U$ can now be found by integrating with respect to $w$ from 0 to $\infty$, i.e.,

$$h(u) = \frac{(\nu_1/\nu_2)^{\nu_1/2}u^{(\nu_1/2)-1}}{2^{(\nu_1+\nu_2)/2}\Gamma(\nu_1/2)\Gamma(\nu_2/2)}\int_0^\infty w^{[(\nu_1+\nu_2)/2]-1}e^{-[1+(\nu_1 u/\nu_2)](w/2)}\,dw$$

if $u > 0$ and 0 if $u \le 0$. But from 15, Appendix A,

$$\int_0^\infty w^{p-1}e^{-aw}\,dw = \frac{\Gamma(p)}{a^p}$$

Therefore, we have

$$h(u) = \frac{(\nu_1/\nu_2)^{\nu_1/2}u^{(\nu_1/2)-1}\Gamma\left(\dfrac{\nu_1+\nu_2}{2}\right)}{2^{(\nu_1+\nu_2)/2}\Gamma(\nu_1/2)\Gamma(\nu_2/2)\left[\dfrac{1}{2}\left(1+\dfrac{\nu_1 u}{\nu_2}\right)\right]^{(\nu_1+\nu_2)/2}}$$

$$= \frac{\Gamma\left(\dfrac{\nu_1+\nu_2}{2}\right)}{\Gamma\left(\dfrac{\nu_1}{2}\right)\Gamma\left(\dfrac{\nu_2}{2}\right)}\nu_1^{\nu_1/2}\nu_2^{\nu_2/2}u^{(\nu_1/2)-1}(\nu_2+\nu_1 u)^{-(\nu_1+\nu_2)/2}$$

if $u > 0$ and 0 if $u \le 0$, which is the required result.

**4.46.** Prove that the $F$ distribution is unimodal at the value $\left(\frac{v_1 - 2}{v_1}\right)\left(\frac{v_2}{v_2 + 2}\right)$ if $v_1 > 2$.

The mode locates the maximum value of the density function. Apart from a constant, the density function of the $F$ distribution is

$$u^{(v_1/2)-1}(v_2 + v_1 u)^{-(v_1+v_2)/2}$$

If this has a relative maximum, it will occur where the derivative is zero, i.e.,

$$\left(\frac{v_1}{2} - 1\right)u^{(v_1/2)-2}(v_2 + v_1 u)^{-(v_1+v_2)/2} - u^{(v_1/2)-1}v_1\left(\frac{v_1 + v_2}{2}\right)(v_2 + v_1 u)^{-[(v_1+v_2)/2]-1} = 0$$

Dividing by $u^{(v_1/2)-2}(v_2 + v_1 u)^{-[(v_1+v_2)/2]-1}$, $u \neq 0$, we find

$$\left(\frac{v_1}{2} - 1\right)(v_2 + v_1 u) - uv_1\left(\frac{v_1 + v_2}{2}\right) = 0 \quad \text{or} \quad u = \left(\frac{v_1 - 2}{v_1}\right)\left(\frac{v_2}{v_2 + 2}\right)$$

Using the second-derivative test, we can show that this actually gives the maximum.

**4.47.** Using the table for the $F$ distribution in Appendix F, find (a) $F_{0.95,10,15}$, (b) $F_{0.99,15,9}$, (c) $F_{0.05,8,30}$, (d) $F_{0.01,15,9}$.

(a) From Appendix F, where $v_1 = 10$, $v_2 = 15$, we find $F_{0.95,10,15} = 2.54$.

(b) From Appendix F, where $v_1 = 15$, $v_2 = 9$, we find $F_{0.99,15,9} = 4.96$.

(c) By Theorem 4-8, page 117, $F_{0.05,8,30} = \dfrac{1}{F_{0.95,30,8}} = \dfrac{1}{3.08} = 0.325$.

(d) By Theorem 4-8, page 117, $F_{0.01,15,9} = \dfrac{1}{F_{0.99,9,15}} = \dfrac{1}{3.89} = 0.257$.

## Relationships among $F$, $\chi^2$, and $t$ distributions

**4.48.** Verify that (a) $F_{0.95} = t^2_{0.975}$, (b) $F_{0.99} = t^2_{0.995}$.

(a) Compare the entries in the first column of the $F_{0.95}$ table in Appendix F with those in the $t$ distribution under $t_{0.975}$. We see that

$$161 = (12.71)^2, \qquad 18.5 = (4.30)^2, \qquad 10.1 = (3.18)^2, \qquad 7.71 = (2.78)^2, \qquad \text{etc.}$$

(b) Compare the entries in the first column of the $F_{0.99}$ table in Appendix F with those in the $t$ distribution under $t_{0.995}$. We see that

$$4050 = (63.66)^2, \qquad 98.5 = (9.92)^2, \qquad 34.1 = (5.84)^2, \qquad 21.2 = (4.60)^2, \qquad \text{etc.}$$

**4.49.** Prove Theorem 4-9, page 117, which can be briefly stated as

$$F_{1-p} = t^2_{1-(p/2)}$$

and therefore generalize the results of Problem 4.48.

Let $v_1 = 1$, $v_2 = v$ in the density function for the $F$ distribution [(45), page 116]. Then

$$f(u) = \frac{\Gamma\left(\frac{v+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{v}{2}\right)} v^{v/2}u^{-1/2}(v + u)^{-(v+1)/2}$$

$$= \frac{\Gamma\left(\frac{v+1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{v}{2}\right)} v^{v/2}u^{-1/2}v^{-(v+1)/2}\left(1 + \frac{u}{v}\right)^{-(v+1)/2}$$

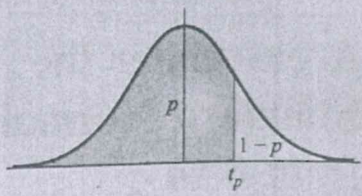
$$= \frac{\Gamma\left(\frac{v+1}{2}\right)}{\sqrt{v\pi}\,\Gamma\left(\frac{v}{2}\right)} u^{-1/2}\left(1 + \frac{u}{v}\right)^{-(v+1)/2}$$

EWG – 01 Lab.
EWS – 01 Lab.

EVS 01 Th.
DLD 01 Th.
M–III 01 Th.
OOPS
Dis. 06 – Th + Tut
D.S.

# APPENDIX D

## Percentile Values $t_p$ for Student's $t$ Distribution with $\nu$ Degrees of Freedom




| $\nu$ | $t_{.55}$ | $t_{.60}$ | $t_{.70}$ | $t_{.75}$ | $t_{.80}$ | $t_{.90}$ | $t_{.95}$ | $t_{.975}$ | $t_{.99}$ | $t_{.995}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | .158 | .325 | .727 | 1.000 | 1.376 | 3.08 | 6.31 | 12.71 | 31.82 | 63.66 |
| 2 | .142 | .289 | .617 | .816 | 1.061 | 1.89 | 2.92 | 4.30 | 6.96 | 9.92 |
| 3 | .137 | .277 | .584 | .765 | .978 | 1.64 | 2.35 | 3.18 | 4.54 | 5.84 |
| 4 | .134 | .271 | .569 | .741 | .941 | 1.53 | 2.13 | 2.78 | 3.75 | 4.60 |
| 5 | .132 | .267 | .559 | .727 | .920 | 1.48 | 2.02 | 2.57 | 3.36 | 4.03 |
| 6 | .131 | .265 | .553 | .718 | .906 | 1.44 | 1.94 | 2.45 | 3.14 | 3.71 |
| 7 | .130 | .263 | .549 | .711 | .896 | 1.42 | 1.90 | 2.36 | 3.00 | 3.50 |
| 8 | .130 | .262 | .546 | .706 | .889 | 1.40 | 1.86 | 2.31 | 2.90 | 3.36 |
| 9 | .129 | .261 | .543 | .703 | .883 | 1.38 | 1.83 | 2.26 | 2.82 | 3.25 |
| 10 | .129 | .260 | .542 | .700 | .879 | 1.37 | 1.81 | 2.23 | 2.76 | 3.17 |
| 11 | .129 | .260 | .540 | .697 | .876 | 1.36 | 1.80 | 2.20 | 2.72 | 3.11 |
| 12 | .128 | .259 | .539 | .695 | .873 | 1.36 | 1.78 | 2.18 | 2.68 | 3.06 |
| 13 | .128 | .259 | .538 | .694 | .870 | 1.35 | 1.77 | 2.16 | 2.65 | 3.01 |
| 14 | .128 | .258 | .537 | .692 | .868 | 1.34 | 1.76 | 2.14 | 2.62 | 2.98 |
| 15 | .128 | .258 | .536 | .691 | .866 | 1.34 | 1.75 | 2.13 | 2.60 | 2.95 |
| 16 | .128 | .258 | .535 | .690 | .865 | 1.34 | 1.75 | 2.12 | 2.58 | 2.92 |
| 17 | .128 | .257 | .534 | .689 | .863 | 1.33 | 1.74 | 2.11 | 2.57 | 2.90 |
| 18 | .127 | .257 | .534 | .688 | .862 | 1.33 | 1.73 | 2.10 | 2.55 | 2.88 |
| 19 | .127 | .257 | .533 | .688 | .861 | 1.33 | 1.73 | 2.09 | 2.54 | 2.86 |
| 20 | .127 | .257 | .533 | .687 | .860 | 1.32 | 1.72 | 2.09 | 2.53 | 2.84 |
| 21 | .127 | .257 | .532 | .686 | .859 | 1.32 | 1.72 | 2.08 | 2.52 | 2.83 |
| 22 | .127 | .256 | .532 | .686 | .858 | 1.32 | 1.72 | 2.07 | 2.51 | 2.82 |
| 23 | .127 | .256 | .532 | .685 | .858 | 1.32 | 1.71 | 2.07 | 2.50 | 2.81 |
| 24 | .127 | .256 | .531 | .685 | .857 | 1.32 | 1.71 | 2.06 | 2.49 | 2.80 |
| 25 | .127 | .256 | .531 | .684 | .856 | 1.32 | 1.71 | 2.06 | 2.48 | 2.79 |
| 26 | .127 | .256 | .531 | .684 | .856 | 1.32 | 1.71 | 2.06 | 2.48 | 2.78 |
| 27 | .127 | .256 | .531 | .684 | .855 | 1.31 | 1.70 | 2.05 | 2.47 | 2.77 |
| 28 | .127 | .256 | .530 | .683 | .855 | 1.31 | 1.70 | 2.05 | 2.47 | 2.76 |
| 29 | .127 | .256 | .530 | .683 | .854 | 1.31 | 1.70 | 2.04 | 2.46 | 2.76 |
| 30 | .127 | .256 | .530 | .683 | .854 | 1.31 | 1.70 | 2.04 | 2.46 | 2.75 |
| 40 | .126 | .255 | .529 | .681 | .851 | 1.30 | 1.68 | 2.02 | 2.42 | 2.70 |
| 60 | .126 | .254 | .527 | .679 | .848 | 1.30 | 1.67 | 2.00 | 2.39 | 2.66 |
| 120 | .126 | .254 | .526 | .677 | .845 | 1.29 | 1.66 | 1.98 | 2.36 | 2.62 |
| $\infty$ | .126 | .253 | .524 | .674 | .842 | 1.28 | 1.645 | 1.96 | 2.33 | 2.58 |

*Source:* R. A. Fisher and F. Yates, *Statistical Tables for Biological, Agricultural and Medical Research*, published by Longman Group Ltd., London (previously pulished by Oliver and Boyd, Edinburgh), and by permission of the authors and publishers.

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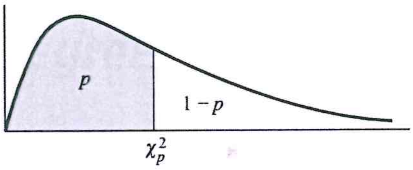
# APPENDIX E

## Percentile Values $\chi_p^2$ for the Chi-Square Distribution with $\nu$ Degrees of Freedom

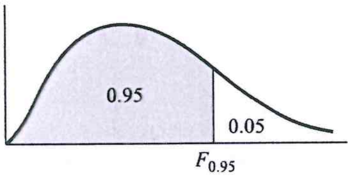



| $\nu$ | $\chi^2_{.005}$ | $\chi^2_{.01}$ | $\chi^2_{.025}$ | $\chi^2_{.05}$ | $\chi^2_{.10}$ | $\chi^2_{.25}$ | $\chi^2_{.50}$ | $\chi^2_{.75}$ | $\chi^2_{.90}$ | $\chi^2_{.95}$ | $\chi^2_{.975}$ | $\chi^2_{.99}$ | $\chi^2_{.995}$ | $\chi^2_{.999}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | .0000 | .0002 | .0010 | .0039 | .0158 | .102 | .455 | 1.32 | 2.71 | 3.84 | 5.02 | 6.63 | 7.88 | 10.8 |
| 2 | .0100 | .0201 | .0506 | .103 | .211 | .575 | 1.39 | 2.77 | 4.61 | 5.99 | 7.38 | 9.21 | 10.6 | 13.8 |
| 3 | .0717 | .115 | .216 | .352 | .584 | 1.21 | 2.37 | 4.11 | 6.25 | 7.81 | 9.35 | 11.3 | 12.8 | 16.3 |
| 4 | .207 | .297 | .484 | .711 | 1.06 | 1.92 | 3.36 | 5.39 | 7.78 | 9.49 | 11.1 | 13.3 | 14.9 | 18.5 |
| 5 | .412 | .554 | .831 | 1.15 | 1.61 | 2.67 | 4.35 | 6.63 | 9.24 | 11.1 | 12.8 | 15.1 | 16.7 | 20.5 |
| 6 | .676 | .872 | 1.24 | 1.64 | 2.20 | 3.45 | 5.35 | 7.84 | 10.6 | 12.6 | 14.4 | 16.8 | 18.5 | 22.5 |
| 7 | .989 | 1.24 | 1.69 | 2.17 | 2.83 | 4.25 | 6.35 | 9.04 | 12.0 | 14.1 | 16.0 | 18.5 | 20.3 | 24.3 |
| 8 | 1.34 | 1.65 | 2.18 | 2.73 | 3.49 | 5.07 | 7.34 | 10.2 | 13.4 | 15.5 | 17.5 | 20.1 | 22.0 | 26.1 |
| 9 | 1.73 | 2.09 | 2.70 | 3.33 | 4.17 | 5.90 | 8.34 | 11.4 | 14.7 | 16.9 | 19.0 | 21.7 | 23.6 | 27.9 |
| 10 | 2.16 | 2.56 | 3.25 | 3.94 | 4.87 | 6.74 | 9.34 | 12.5 | 16.0 | 18.3 | 20.5 | 23.2 | 25.2 | 29.6 |
| 11 | 2.60 | 3.05 | 3.82 | 4.57 | 5.58 | 7.58 | 10.3 | 13.7 | 17.3 | 19.7 | 21.9 | 24.7 | 26.8 | 31.3 |
| 12 | 3.07 | 3.57 | 4.40 | 5.23 | 6.30 | 8.44 | 11.3 | 14.8 | 18.5 | 21.0 | 23.3 | 26.2 | 28.3 | 32.9 |
| 13 | 3.57 | 4.11 | 5.01 | 5.89 | 7.04 | 9.30 | 12.3 | 16.0 | 19.8 | 22.4 | 24.7 | 27.7 | 29.8 | 34.5 |
| 14 | 4.07 | 4.66 | 5.63 | 6.57 | 7.79 | 10.2 | 13.3 | 17.1 | 21.1 | 23.7 | 26.1 | 29.1 | 31.3 | 36.1 |
| 15 | 4.60 | 5.23 | 6.26 | 7.26 | 8.55 | 11.0 | 14.3 | 18.2 | 22.3 | 25.0 | 27.5 | 30.6 | 32.8 | 37.7 |
| 16 | 5.14 | 5.81 | 6.91 | 7.96 | 9.31 | 11.9 | 15.3 | 19.4 | 23.5 | 26.3 | 28.8 | 32.0 | 34.3 | 39.3 |
| 17 | 5.70 | 6.41 | 7.56 | 8.67 | 10.1 | 12.8 | 16.3 | 20.5 | 24.8 | 27.6 | 30.2 | 33.4 | 35.7 | 40.8 |
| 18 | 6.26 | 7.01 | 8.23 | 9.39 | 10.9 | 13.7 | 17.3 | 21.6 | 26.0 | 28.9 | 31.5 | 34.8 | 37.2 | 42.3 |
| 19 | 6.84 | 7.63 | 8.91 | 10.1 | 11.7 | 14.6 | 18.3 | 22.7 | 27.2 | 30.1 | 32.9 | 36.2 | 38.6 | 43.8 |
| 20 | 7.43 | 8.26 | 9.59 | 10.9 | 12.4 | 15.5 | 19.3 | 23.8 | 28.4 | 31.4 | 34.2 | 37.6 | 40.0 | 45.3 |
| 21 | 8.03 | 8.90 | 10.3 | 11.6 | 13.2 | 16.3 | 20.3 | 24.9 | 29.6 | 32.7 | 35.5 | 38.9 | 41.4 | 46.8 |
| 22 | 8.64 | 9.54 | 11.0 | 12.3 | 14.0 | 17.2 | 21.3 | 26.0 | 30.8 | 33.9 | 36.8 | 40.3 | 42.8 | 48.3 |
| 23 | 9.26 | 10.2 | 11.7 | 13.1 | 14.8 | 18.1 | 22.3 | 27.1 | 32.0 | 35.2 | 38.1 | 41.6 | 44.2 | 49.7 |
| 24 | 9.89 | 10.9 | 12.4 | 13.8 | 15.7 | 19.0 | 23.3 | 28.2 | 33.2 | 36.4 | 39.4 | 43.0 | 45.6 | 51.2 |
| 25 | 10.5 | 11.5 | 13.1 | 14.6 | 16.5 | 19.9 | 24.3 | 29.3 | 34.4 | 37.7 | 40.6 | 44.3 | 46.9 | 52.6 |
| 26 | 11.2 | 12.2 | 13.8 | 15.4 | 17.3 | 20.8 | 25.3 | 30.4 | 35.6 | 38.9 | 41.9 | 45.6 | 48.3 | 54.1 |
| 27 | 11.8 | 12.9 | 14.6 | 16.2 | 18.1 | 21.7 | 26.3 | 31.5 | 36.7 | 40.1 | 43.2 | 47.0 | 49.6 | 55.5 |
| 28 | 12.5 | 13.6 | 15.3 | 16.9 | 18.9 | 22.7 | 27.3 | 32.6 | 37.9 | 41.3 | 44.5 | 48.3 | 51.0 | 56.9 |
| 29 | 13.1 | 14.3 | 16.0 | 17.7 | 19.8 | 23.6 | 28.3 | 33.7 | 39.1 | 42.6 | 45.7 | 49.6 | 52.3 | 58.3 |
| 30 | 13.8 | 15.0 | 16.8 | 18.5 | 20.6 | 24.5 | 29.3 | 34.8 | 40.3 | 43.8 | 47.0 | 50.9 | 53.7 | 59.7 |
| 40 | 20.7 | 22.2 | 24.4 | 26.5 | 29.1 | 33.7 | 39.3 | 45.6 | 51.8 | 55.8 | 59.3 | 63.7 | 66.8 | 73.4 |
| 50 | 28.0 | 29.7 | 32.4 | 34.8 | 37.7 | 42.9 | 49.3 | 56.3 | 63.2 | 67.5 | 71.4 | 76.2 | 79.5 | 86.7 |
| 60 | 35.5 | 37.5 | 40.5 | 43.2 | 46.5 | 52.3 | 59.3 | 67.0 | 74.4 | 79.1 | 83.3 | 88.4 | 92.0 | 99.6 |
| 70 | 43.3 | 45.4 | 48.8 | 51.7 | 55.3 | 61.7 | 69.3 | 77.6 | 85.5 | 90.5 | 95.0 | 100 | 104 | 112 |
| 80 | 51.2 | 53.5 | 57.2 | 60.4 | 64.3 | 71.1 | 79.3 | 88.1 | 96.6 | 102 | 107 | 112 | 116 | 125 |
| 90 | 59.2 | 61.8 | 65.6 | 69.1 | 73.3 | 80.6 | 89.3 | 98.6 | 108 | 113 | 118 | 124 | 128 | 137 |
| 100 | 67.3 | 70.1 | 74.2 | 77.9 | 82.4 | 90.1 | 99.3 | 109 | 118 | 124 | 130 | 136 | 140 | 149 |

*Source*: E. S. Pearson and H. O. Hartley, *Biometrika Tables for Statisticians*, Vol. 1 (1966), Table 8, pages 137 and 138, by permission.

# APPENDIX F

## 95th Percentile Values (0.05 Levels), $F_{0.95}$, for the F Distribution

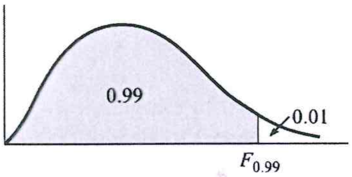



$\nu_1$ degrees of freedom in numerator
$\nu_2$ degrees of freedom in denominator

| $\nu_2$ \ $\nu_1$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 12 | 15 | 20 | 24 | 30 | 40 | 60 | 120 | ∞ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 161 | 200 | 216 | 225 | 230 | 234 | 237 | 239 | 241 | 242 | 244 | 246 | 248 | 249 | 250 | 251 | 252 | 253 | 254 |
| 2 | 18.5 | 19.0 | 19.2 | 19.2 | 19.3 | 19.3 | 19.4 | 19.4 | 19.4 | 19.4 | 19.4 | 19.4 | 19.4 | 19.5 | 19.5 | 19.5 | 19.5 | 19.5 | 19.5 |
| 3 | 10.1 | 9.55 | 9.28 | 9.12 | 9.01 | 8.94 | 8.89 | 8.85 | 8.81 | 8.79 | 8.74 | 8.70 | 8.66 | 8.64 | 8.62 | 8.59 | 8.57 | 8.55 | 8.53 |
| 4 | 7.71 | 6.94 | 6.59 | 6.39 | 6.26 | 6.16 | 6.09 | 6.04 | 6.00 | 5.96 | 5.91 | 5.86 | 5.80 | 5.77 | 5.75 | 5.72 | 5.69 | 5.66 | 5.63 |
| 5 | 6.61 | 5.79 | 5.41 | 5.19 | 5.05 | 4.95 | 4.88 | 4.82 | 4.77 | 4.74 | 4.68 | 4.62 | 4.56 | 4.53 | 4.50 | 4.46 | 4.43 | 4.40 | 4.37 |
| 6 | 5.99 | 5.14 | 4.76 | 4.53 | 4.39 | 4.28 | 4.21 | 4.15 | 4.10 | 4.06 | 4.00 | 3.94 | 3.87 | 3.84 | 3.81 | 3.77 | 3.74 | 3.70 | 3.67 |
| 7 | 5.59 | 4.74 | 4.35 | 4.12 | 3.97 | 3.87 | 3.79 | 3.73 | 3.68 | 3.64 | 3.57 | 3.51 | 3.44 | 3.41 | 3.38 | 3.34 | 3.30 | 3.27 | 3.23 |
| 8 | 5.32 | 4.46 | 4.07 | 3.84 | 3.69 | 3.58 | 3.50 | 3.44 | 3.39 | 3.35 | 3.28 | 3.22 | 3.15 | 3.12 | 3.08 | 3.04 | 3.01 | 2.97 | 2.93 |
| 9 | 5.12 | 4.26 | 3.86 | 3.63 | 3.48 | 3.37 | 3.29 | 3.23 | 3.18 | 3.14 | 3.07 | 3.01 | 2.94 | 2.90 | 2.86 | 2.83 | 2.79 | 2.75 | 2.71 |
| 10 | 4.96 | 4.10 | 3.71 | 3.48 | 3.33 | 3.22 | 3.14 | 3.07 | 3.02 | 2.98 | 2.91 | 2.85 | 2.77 | 2.74 | 2.70 | 2.66 | 2.62 | 2.58 | 2.54 |
| 11 | 4.84 | 3.98 | 3.59 | 3.36 | 3.20 | 3.09 | 3.01 | 2.95 | 2.90 | 2.85 | 2.79 | 2.72 | 2.65 | 2.61 | 2.57 | 2.53 | 2.49 | 2.45 | 2.40 |
| 12 | 4.75 | 3.89 | 3.49 | 3.26 | 3.11 | 3.00 | 2.91 | 2.85 | 2.80 | 2.75 | 2.69 | 2.62 | 2.54 | 2.51 | 2.47 | 2.43 | 2.38 | 2.34 | 2.30 |
| 13 | 4.67 | 3.81 | 3.41 | 3.18 | 3.03 | 2.92 | 2.83 | 2.77 | 2.71 | 2.67 | 2.60 | 2.53 | 2.46 | 2.42 | 2.38 | 2.34 | 2.30 | 2.25 | 2.21 |
| 14 | 4.60 | 3.74 | 3.34 | 3.11 | 2.96 | 2.85 | 2.76 | 2.70 | 2.65 | 2.60 | 2.53 | 2.46 | 2.39 | 2.35 | 2.31 | 2.27 | 2.22 | 2.18 | 2.13 |
| 15 | 4.54 | 3.68 | 3.29 | 3.06 | 2.90 | 2.79 | 2.71 | 2.64 | 2.59 | 2.54 | 2.48 | 2.40 | 2.33 | 2.29 | 2.25 | 2.20 | 2.16 | 2.11 | 2.07 |
| 16 | 4.49 | 3.63 | 3.24 | 3.01 | 2.85 | 2.74 | 2.66 | 2.59 | 2.54 | 2.49 | 2.42 | 2.35 | 2.28 | 2.24 | 2.19 | 2.15 | 2.11 | 2.06 | 2.01 |
| 17 | 4.45 | 3.59 | 3.20 | 2.96 | 2.81 | 2.70 | 2.61 | 2.55 | 2.49 | 2.45 | 2.38 | 2.31 | 2.23 | 2.19 | 2.15 | 2.10 | 2.06 | 2.01 | 1.96 |
| 18 | 4.41 | 3.55 | 3.16 | 2.93 | 2.77 | 2.66 | 2.58 | 2.51 | 2.46 | 2.41 | 2.34 | 2.27 | 2.19 | 2.15 | 2.11 | 2.06 | 2.02 | 1.97 | 1.92 |
| 19 | 4.38 | 3.52 | 3.13 | 2.90 | 2.74 | 2.63 | 2.54 | 2.48 | 2.42 | 2.38 | 2.31 | 2.23 | 2.16 | 2.11 | 2.07 | 2.03 | 1.98 | 1.93 | 1.88 |
| 20 | 4.35 | 3.49 | 3.10 | 2.87 | 2.71 | 2.60 | 2.51 | 2.45 | 2.39 | 2.35 | 2.28 | 2.20 | 2.12 | 2.08 | 2.04 | 1.99 | 1.95 | 1.90 | 1.84 |
| 21 | 4.32 | 3.47 | 3.07 | 2.84 | 2.68 | 2.57 | 2.49 | 2.42 | 2.37 | 2.32 | 2.25 | 2.18 | 2.10 | 2.05 | 2.01 | 1.96 | 1.92 | 1.87 | 1.81 |
| 22 | 4.30 | 3.44 | 3.05 | 2.82 | 2.66 | 2.55 | 2.46 | 2.40 | 2.34 | 2.30 | 2.23 | 2.15 | 2.07 | 2.03 | 1.98 | 1.94 | 1.89 | 1.84 | 1.78 |
| 23 | 4.28 | 3.42 | 3.03 | 2.80 | 2.64 | 2.53 | 2.44 | 2.37 | 2.32 | 2.27 | 2.20 | 2.13 | 2.05 | 2.01 | 1.96 | 1.91 | 1.86 | 1.81 | 1.76 |
| 24 | 4.26 | 3.40 | 3.01 | 2.78 | 2.62 | 2.51 | 2.42 | 2.36 | 2.30 | 2.25 | 2.18 | 2.11 | 2.03 | 1.98 | 1.94 | 1.89 | 1.84 | 1.79 | 1.73 |
| 25 | 4.24 | 3.39 | 2.99 | 2.76 | 2.60 | 2.49 | 2.40 | 2.34 | 2.28 | 2.24 | 2.16 | 2.09 | 2.01 | 1.96 | 1.92 | 1.87 | 1.82 | 1.77 | 1.71 |
| 26 | 4.23 | 3.37 | 2.98 | 2.74 | 2.59 | 2.47 | 2.39 | 2.32 | 2.27 | 2.22 | 2.15 | 2.07 | 1.99 | 1.95 | 1.90 | 1.85 | 1.80 | 1.75 | 1.69 |
| 27 | 4.21 | 3.35 | 2.96 | 2.73 | 2.57 | 2.46 | 2.37 | 2.31 | 2.25 | 2.20 | 2.13 | 2.06 | 1.97 | 1.93 | 1.88 | 1.84 | 1.79 | 1.73 | 1.67 |
| 28 | 4.20 | 3.34 | 2.95 | 2.71 | 2.56 | 2.45 | 2.36 | 2.29 | 2.24 | 2.19 | 2.12 | 2.04 | 1.96 | 1.91 | 1.87 | 1.82 | 1.77 | 1.71 | 1.65 |
| 29 | 4.18 | 3.33 | 2.93 | 2.70 | 2.55 | 2.43 | 2.35 | 2.28 | 2.22 | 2.18 | 2.10 | 2.03 | 1.94 | 1.90 | 1.85 | 1.81 | 1.75 | 1.70 | 1.64 |
| 30 | 4.17 | 3.32 | 2.92 | 2.69 | 2.53 | 2.42 | 2.33 | 2.27 | 2.21 | 2.16 | 2.09 | 2.01 | 1.93 | 1.89 | 1.84 | 1.79 | 1.74 | 1.68 | 1.62 |
| 40 | 4.08 | 3.23 | 2.84 | 2.61 | 2.45 | 2.34 | 2.25 | 2.18 | 2.12 | 2.08 | 2.00 | 1.92 | 1.84 | 1.79 | 1.74 | 1.69 | 1.64 | 1.58 | 1.51 |
| 60 | 4.00 | 3.15 | 2.76 | 2.53 | 2.37 | 2.25 | 2.17 | 2.10 | 2.04 | 1.99 | 1.92 | 1.84 | 1.75 | 1.70 | 1.65 | 1.59 | 1.53 | 1.47 | 1.39 |
| 120 | 3.92 | 3.07 | 2.68 | 2.45 | 2.29 | 2.18 | 2.09 | 2.02 | 1.96 | 1.91 | 1.83 | 1.75 | 1.66 | 1.61 | 1.55 | 1.50 | 1.43 | 1.35 | 1.25 |
| ∞ | 3.84 | 3.00 | 2.60 | 2.37 | 2.21 | 2.10 | 2.01 | 1.94 | 1.88 | 1.83 | 1.75 | 1.67 | 1.57 | 1.52 | 1.46 | 1.39 | 1.32 | 1.22 | 1.00 |

*Source*: E. S. Pearson and H. O. Hartley, *Biometrika Tables for Statisticians*, Vol. 2 (1972), Table 5, page 178, by permission.

# 99th Percentile Values (0.01 Levels), $F_{0.99}$, for the F Distribution




$\nu_1$ degrees of freedom in numerator
$\nu_2$ degrees of freedom in denominator

| $\nu_2$ \ $\nu_1$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 12 | 15 | 20 | 24 | 30 | 40 | 60 | 120 | ∞ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 4052 | 5000 | 5403 | 5625 | 5764 | 5859 | 5928 | 5981 | 6023 | 6056 | 6106 | 6157 | 6209 | 6235 | 6261 | 6287 | 6313 | 6339 | 6366 |
| 2 | 98.5 | 99.0 | 99.2 | 99.2 | 99.3 | 99.3 | 99.4 | 99.4 | 99.4 | 99.4 | 99.4 | 99.4 | 99.4 | 99.5 | 99.5 | 99.5 | 99.5 | 99.5 | 99.5 |
| 3 | 34.1 | 30.8 | 29.5 | 28.7 | 28.2 | 27.9 | 27.7 | 27.5 | 27.3 | 27.2 | 27.1 | 26.9 | 26.7 | 26.6 | 26.5 | 26.4 | 26.3 | 26.2 | 26.1 |
| 4 | 21.2 | 18.0 | 16.7 | 16.0 | 15.5 | 15.2 | 15.0 | 14.8 | 14.7 | 14.5 | 14.4 | 14.2 | 14.0 | 13.9 | 13.8 | 13.7 | 13.7 | 13.6 | 13.5 |
| 5 | 16.3 | 13.3 | 12.1 | 11.4 | 11.0 | 10.7 | 10.5 | 10.3 | 10.2 | 10.1 | 9.89 | 9.72 | 9.55 | 9.47 | 9.38 | 9.29 | 9.20 | 9.11 | 9.02 |
| 6 | 13.7 | 10.9 | 9.78 | 9.15 | 8.75 | 8.47 | 8.26 | 8.10 | 7.98 | 7.87 | 7.72 | 7.56 | 7.40 | 7.31 | 7.23 | 7.14 | 7.06 | 6.97 | 6.88 |
| 7 | 12.2 | 9.55 | 8.45 | 7.85 | 7.46 | 7.19 | 6.99 | 6.84 | 6.72 | 6.62 | 6.47 | 6.31 | 6.16 | 6.07 | 5.99 | 5.91 | 5.82 | 5.74 | 5.65 |
| 8 | 11.3 | 8.65 | 7.59 | 7.01 | 6.63 | 6.37 | 6.18 | 6.03 | 5.91 | 5.81 | 5.67 | 5.52 | 5.36 | 5.28 | 5.20 | 5.12 | 5.03 | 4.95 | 4.86 |
| 9 | 10.6 | 8.02 | 6.99 | 6.42 | 6.06 | 5.80 | 5.61 | 5.47 | 5.35 | 5.26 | 5.11 | 4.96 | 4.81 | 4.73 | 4.65 | 4.57 | 4.48 | 4.40 | 4.31 |
| 10 | 10.0 | 7.56 | 6.55 | 5.99 | 5.64 | 5.39 | 5.20 | 5.06 | 4.94 | 4.85 | 4.71 | 4.56 | 4.41 | 4.33 | 4.25 | 4.17 | 4.08 | 4.00 | 3.91 |
| 11 | 9.65 | 7.21 | 6.22 | 5.67 | 5.32 | 5.07 | 4.89 | 4.74 | 4.63 | 4.54 | 4.40 | 4.25 | 4.10 | 4.02 | 3.94 | 3.86 | 3.78 | 3.69 | 3.60 |
| 12 | 9.33 | 6.93 | 5.95 | 5.41 | 5.06 | 4.82 | 4.64 | 4.50 | 4.39 | 4.30 | 4.16 | 4.01 | 3.86 | 3.78 | 3.70 | 3.62 | 3.54 | 3.45 | 3.36 |
| 13 | 9.07 | 6.70 | 5.74 | 5.21 | 4.86 | 4.62 | 4.44 | 4.30 | 4.19 | 4.10 | 3.96 | 3.82 | 3.66 | 3.59 | 3.51 | 3.43 | 3.34 | 3.25 | 3.17 |
| 14 | 8.86 | 6.51 | 5.56 | 5.04 | 4.70 | 4.46 | 4.28 | 4.14 | 4.03 | 3.94 | 3.80 | 3.66 | 3.51 | 3.43 | 3.35 | 3.27 | 3.18 | 3.09 | 3.00 |
| 15 | 8.68 | 6.36 | 5.42 | 4.89 | 4.56 | 4.32 | 4.14 | 4.00 | 3.89 | 3.80 | 3.67 | 3.52 | 3.37 | 3.29 | 3.21 | 3.13 | 3.05 | 2.96 | 2.87 |
| 16 | 8.53 | 6.23 | 5.29 | 4.77 | 4.44 | 4.20 | 4.03 | 3.89 | 3.78 | 3.69 | 3.55 | 3.41 | 3.26 | 3.18 | 3.10 | 3.02 | 2.93 | 2.84 | 2.75 |
| 17 | 8.40 | 6.11 | 5.19 | 4.67 | 4.34 | 4.10 | 3.93 | 3.79 | 3.68 | 3.59 | 3.46 | 3.31 | 3.16 | 3.08 | 3.00 | 2.92 | 2.83 | 2.75 | 2.65 |
| 18 | 8.29 | 6.01 | 5.09 | 4.58 | 4.25 | 4.01 | 3.84 | 3.71 | 3.60 | 3.51 | 3.37 | 3.23 | 3.08 | 3.00 | 2.92 | 2.84 | 2.75 | 2.66 | 2.57 |
| 19 | 8.18 | 5.93 | 5.01 | 4.50 | 4.17 | 3.94 | 3.77 | 3.63 | 3.52 | 3.43 | 3.30 | 3.15 | 3.00 | 2.92 | 2.84 | 2.76 | 2.67 | 2.58 | 2.49 |
| 20 | 8.10 | 5.85 | 4.94 | 4.43 | 4.10 | 3.87 | 3.70 | 3.56 | 3.46 | 3.37 | 3.23 | 3.09 | 2.94 | 2.86 | 2.78 | 2.69 | 2.61 | 2.52 | 2.42 |
| 21 | 8.02 | 5.78 | 4.87 | 4.37 | 4.04 | 3.81 | 3.64 | 3.51 | 3.40 | 3.31 | 3.17 | 3.03 | 2.88 | 2.80 | 2.72 | 2.64 | 2.55 | 2.46 | 2.36 |
| 22 | 7.95 | 5.72 | 4.82 | 4.31 | 3.99 | 3.76 | 3.59 | 3.45 | 3.35 | 3.26 | 3.12 | 2.98 | 2.83 | 2.75 | 2.67 | 2.58 | 2.50 | 2.40 | 2.31 |
| 23 | 7.88 | 5.66 | 4.76 | 4.26 | 3.94 | 3.71 | 3.54 | 3.41 | 3.30 | 3.21 | 3.07 | 2.93 | 2.78 | 2.70 | 2.62 | 2.54 | 2.45 | 2.35 | 2.26 |
| 24 | 7.82 | 5.61 | 4.72 | 4.22 | 3.90 | 3.67 | 3.50 | 3.36 | 3.26 | 3.17 | 3.03 | 2.89 | 2.74 | 2.66 | 2.58 | 2.49 | 2.40 | 2.31 | 2.21 |
| 25 | 7.77 | 5.57 | 4.68 | 4.18 | 3.86 | 3.63 | 3.46 | 3.32 | 3.22 | 3.13 | 2.99 | 2.85 | 2.70 | 2.62 | 2.54 | 2.45 | 2.36 | 2.27 | 2.17 |
| 26 | 7.72 | 5.53 | 4.64 | 4.14 | 3.82 | 3.59 | 3.42 | 3.29 | 3.18 | 3.09 | 2.96 | 2.82 | 2.66 | 2.58 | 2.50 | 2.42 | 2.33 | 2.23 | 2.13 |
| 27 | 7.68 | 5.49 | 4.60 | 4.11 | 3.78 | 3.56 | 3.39 | 3.26 | 3.15 | 3.06 | 2.93 | 2.78 | 2.63 | 2.55 | 2.47 | 2.38 | 2.29 | 2.20 | 2.10 |
| 28 | 7.64 | 5.45 | 4.57 | 4.07 | 3.75 | 3.53 | 3.36 | 3.23 | 3.12 | 3.03 | 2.90 | 2.75 | 2.60 | 2.52 | 2.44 | 2.35 | 2.26 | 2.17 | 2.06 |
| 29 | 7.60 | 5.42 | 4.54 | 4.04 | 3.73 | 3.50 | 3.33 | 3.20 | 3.09 | 3.00 | 2.87 | 2.73 | 2.57 | 2.49 | 2.41 | 2.33 | 2.23 | 2.14 | 2.03 |
| 30 | 7.56 | 5.39 | 4.51 | 4.02 | 3.70 | 3.47 | 3.30 | 3.17 | 3.07 | 2.98 | 2.84 | 2.70 | 2.55 | 2.47 | 2.39 | 2.30 | 2.21 | 2.11 | 2.01 |
| 40 | 7.31 | 5.18 | 4.31 | 3.83 | 3.51 | 3.29 | 3.12 | 2.99 | 2.89 | 2.80 | 2.66 | 2.52 | 2.37 | 2.29 | 2.20 | 2.11 | 2.02 | 1.92 | 1.80 |
| 60 | 7.08 | 4.98 | 4.13 | 3.65 | 3.34 | 3.12 | 2.95 | 2.82 | 2.72 | 2.63 | 2.50 | 2.35 | 2.20 | 2.12 | 2.03 | 1.94 | 1.84 | 1.73 | 1.60 |
| 120 | 6.85 | 4.79 | 3.95 | 3.48 | 3.17 | 2.96 | 2.79 | 2.66 | 2.56 | 2.47 | 2.34 | 2.19 | 2.03 | 1.95 | 1.86 | 1.76 | 1.66 | 1.53 | 1.38 |
| ∞ | 6.63 | 4.61 | 3.78 | 3.32 | 3.02 | 2.80 | 2.64 | 2.51 | 2.41 | 2.32 | 2.18 | 2.04 | 1.88 | 1.79 | 1.70 | 1.59 | 1.47 | 1.32 | 1.00 |

*Source*: E. S. Pearson and H. O. Hartley, *Biometrika Tables for Statisticians*, Vol. 2 (1972), Table 5, page 180, by permission.