TRANSPORTATION AND ASSIGNMENT PROBLEM

Transportation Problem

- The transportation problem is concerned with finding the minimum cost of transporting a single commodity from a given number of sources (e.g. factories) to a given number of destinations (e.g. warehouses).

The data of the model include

1. The level of supply at each source and the amount of demand at each destination.
2. The **unit** transportation cost of the commodity from each source to each destination.

- Since there is only one commodity, a destination can receive its demand from more than one source.
- The objective is to determine how much should be shipped from each source to each destination so as to minimize the total transportation cost.

Transportation Problem

Steps for Transportation Problem

Step 1. Determine a *starting* basic feasible solution, and go to step 2.

Step 2. Use the optimality condition of the simplex method to determine the *entering variable* from among all the nonbasic variables. If the optimality condition is satisfied, stop. Otherwise, go to step 3.

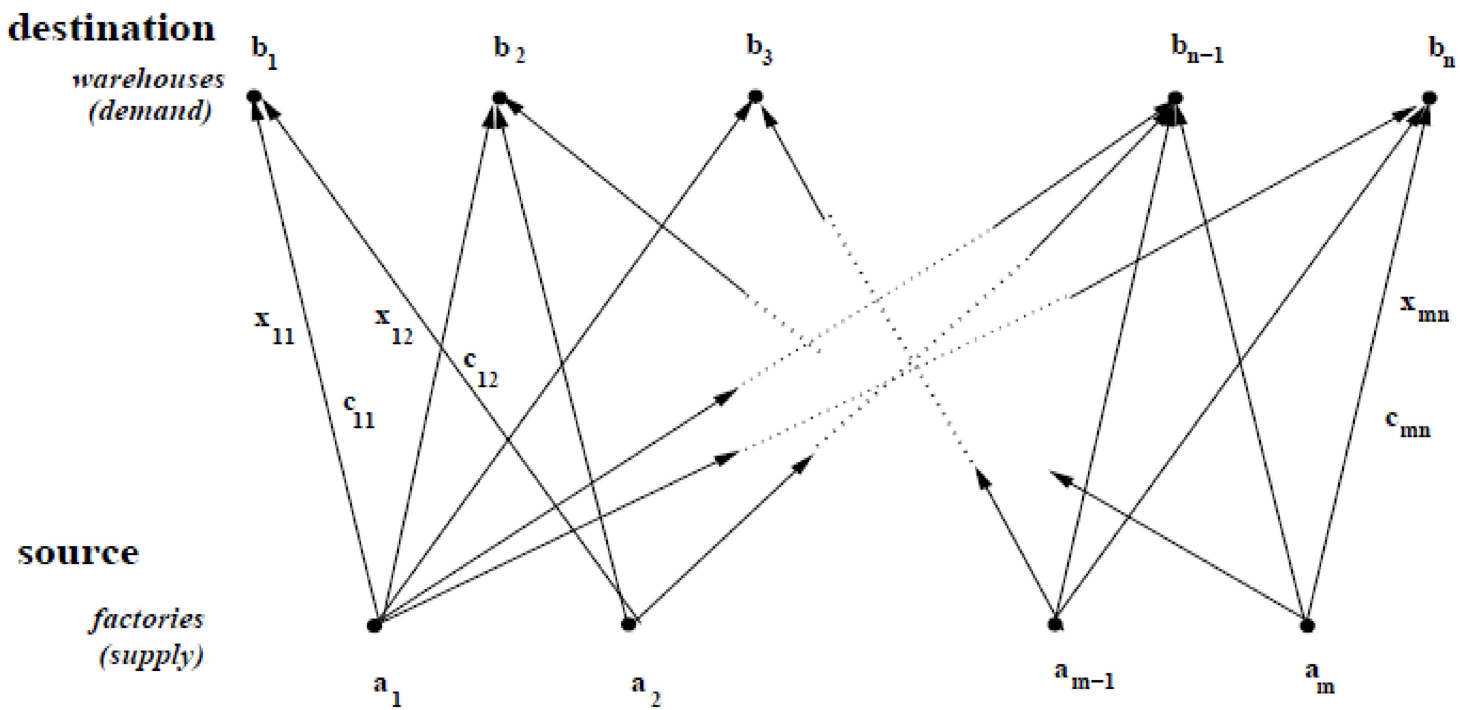
Step 3. Use the feasibility condition of the simplex method to determine the *leaving variable* from among all the current basic variables, and find the new basic solution. Return to step 2.

Determination of the Starting Solution

- The transportation model with m sources and n destinations has m + n equations, one for each source and destination.

Table 1: Parameter table for the transportation problem

		Cost per Unit Distributed				
		Destination				
		1	2	…	n	Supply
Source	1	c_{11}	c_{12}	…	c_{1n}	s_1
	2	c_{21}	c_{22}	…	c_{2n}	s_2
	⋮	⋮	⋮		⋮	⋮
	m	c_{m1}	c_{m2}	…	c_{mn}	s_m
Demand		d_1	d_2	…	d_n	


destination
warehouses
(demand)
source
factories
(supply)
b_1
b_2
b_3
b_{n-1}
b_n
x_{11}
c_{11}
x_{12}
c_{12}
x_{mn}
c_{mn}
a_1
a_2
a_{m-1}
a_m

- Let x_{ij} denote the quantity transported from source i to destination j.
- The cost associated with this movement is cost X quantity = $c_{ij}\, x_{ij}$.
- The cost of transporting the commodity from source i to all destinations is given by

$$\sum_{j=1}^{n} c_{ij}x_{ij} = c_{i1}x_{i1} + c_{i2}x_{i2} + \cdots + c_{in}x_{in}.$$

- Thus, the total cost of transporting the commodity from all the sources to all the destinations is:

$$
\begin{aligned}
\text{Total Cost} &= \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} \\
&= c_{11}x_{11} + c_{12}x_{12} + \cdots + c_{1n}x_{1n} + \\
&\quad c_{21}x_{21} + c_{22}x_{22} + \cdots + c_{2n}x_{2n} + \\
&\qquad\qquad \vdots \\
&\quad c_{m1}x_{m1} + c_{m2}x_{m2} + \cdots + c_{mn}x_{mn}
\end{aligned}
$$

In order to minimize the transportation costs, the following problem must be solved:

$$\text{Minimise} \quad z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij},$$

$$\text{subject to} \quad \sum_{j=1}^{n} x_{ij} \leq a_i \text{ for } i = 1, \ldots, m$$

$$\text{and} \quad \sum_{i=1}^{m} x_{ij} \geq b_j \text{ for } j = 1, \ldots, n$$

$$\text{where} \quad x_{ij} \geq 0 \text{ for all } i \text{ and } j.$$

First constraint says; that the sum of all shipments from a source cannot exceed the available supply.

Second constraint specifies; that the sum of all shipments to a destination must be at least as large as the demand.

Finding an initial basic feasible solution

- We consider two ways of constructing initial basic feasible solutions for a transportation problem, i.e. allocations with $n + m - 1$ basic variables which satisfy all the constraint equations.
- **Example 1** Balanced transportation model.

Consider the following problem with 2 factories and 3 warehouses:

	Warehouse 1	Warehouse 2	Warehouse 3	Supply
Factory 1	c_{11}	c_{12}	c_{13}	20
Factory 2	c_{21}	c_{22}	c_{23}	10
Demand	7	10	13	

$$\begin{aligned} \text{Total supply} &= 20 + 10 = 30 \\ \text{Total demand} &= 7 + 10 + 13 = 30 \\ &= \text{Total supply} \end{aligned}$$

Since Total supply = Total demand, the problem is balanced.

Example 2: Unbalanced transportation model.

There are two cases to consider, namely excess demand and excess supply.

Case 1:

- Suppose the demand at warehouse 1 above is 9 units. Then the total supply and total demand are unequal, and the problem is unbalanced.
- In this case it is not possible to satisfy all the demand at each destination simultaneously.
- We reformulate the model as follows: since demand exceeds supply by 2 units, we introduce a **dummy source** (i.e. a fictitious factory) which has a capacity of 2.
- The amount shipped from this dummy source to a destination represents the shortage quantity at that destination.

It is necessary to specify the costs associated with the dummy source.
There are two situations to consider.

(a) Since the source does not exist, no shipping from the source will occur, so the unit transportation costs can be set to zero.
(b) It is necessary to specify the costs associated with the dummy source. There are two situations to consider.

In effect we are allocating the shortage to different destinations.

	Warehouse 1	Warehouse 2	Warehouse 3	Supply
Factory 1	c_{11}	c_{12}	c_{13}	20
Factory 2	c_{21}	c_{22}	c_{23}	10
dummy	P	P	P	2
Demand	7	10	13	

Case 2:

- If supply exceeds demand then a dummy destination is added which absorbs the surplus units.
- Any units shipped from a source to a dummy destination represent a surplus at that source.
- Again, there are two cases to consider for how the unit transportation costs should be determined.

(a) Since no shipping takes place, the unit transportation costs can be set to zero.

(b) If there is a **cost for storing** , S, the surplus production then the unit transportation costs should be set equal to the unit storage costs.

	Warehouse 1	Warehouse 2	Warehouse 3	dummy	Supply
Factory 1	c_{11}	c_{12}	c_{13}	S	20
Factory 2	c_{21}	c_{22}	c_{23}	S	10
Demand	7	10	13	4	

Here we are allocating the excess supply to the different destinations.

Starting the algorithm: finding an initial basic feasible solution

- We consider three ways of constructing initial basic feasible solutions for a transportation problem, i.e. allocations with $n + m - 1$ basic variables which satisfy all the constraint equations.

Method 1: The North-West Corner Method

Consider the problem represented by the following **transportation tableau**.

The number in the bottom right of cell $(i; j)$ is c_{ij} , the cost of transporting 1 unit from source *I* to destination *j*.

Values of x_{ij} , the quantity actually transported from source *i* to destination *j*, will be entered in the top left of each cell.

For example there are 3 factories and 4 warehouses and so m = 3, n = 4.

	W_1	W_2	W_3	W_4	Supply
F_1	10	0	20	11	20
F_2	12	7	9	20	25
F_3	0	14	16	18	15
Demand	10	15	15	20	

The **north-west corner method** generates an initial allocation according to the following procedure:

1. Allocate the maximum amount allowable by the supply and demand constraints to the variable $x11$ (i.e. the cell in the top left corner of the transportation tableau).
2. If a column (or row) is satisfied, cross it out. The remaining decision variables in that column (or row) are non-basic and are set equal to zero. If a row and column are satisfied simultaneously, cross only one out (it does not matter which).
3. Adjust supply and demand for the non-crossed out rows and columns.
4. Allocate the maximum feasible amount to the first available non-crossed out element in the next column (or row).
5. When exactly one row or column is left, all the remaining variables are basic and are assigned the only feasible allocation.

North-West Corner method

m + n – 1 = 4 + 3 - 1 = 6; 10x10 + 10x0 + 7x5 + 15x 9 + 5x20 + 15x 18 = 640

	W1	W2	W3	W4	S
F1	10 10x	10 0x	20x	11x	20/10/0
F2	12x	5 7x	15 9x	5 20x	25/20/5
F3	0x	14x	16x	15 18x	15
D	10/0	15/5/0	15/0	20/15	60

For the above example:

x_{10} = 10. Cross out column 1. The amount left in row 1 is 10.

x_{12} = 10. Cross out row 1. 5 units are left in column 2.

x_{22} = 5. Cross out column 2. 20 units are left in row 2.

x_{23} = 15. Cross out column 3. 5 units are left in row 2.

Only column 4 is now left and so both the remaining variables x_{24} and x_{34} will be basic.

The only feasible allocation of the 5 units in row 2 and the 15 units in row 3 is to allocate *x*24 = 5 and *x*34 = 15, which also ensures that the demand in column 4 is satisfied.

This provides the initial basic feasible solution $x_{11} = 10$, $x_{12} = 10$, $x_{22} = 5$, $x_{23} = 15$, $x_{24} = 5$, $x_{34} = 15$.

The remaining variables are non-basic and therefore equal to zero.

The solution has $m + n - 1 = 6$ basic variables as required.

- The values of the basic variables *xij* are entered in the top left of each cell.
- There should always be $m + n - 1$ of these; in certain (degenerate) cases some of them may be zero.

They must always add up to the total supply and demand in each row and column.

Method 2: The Least-Cost Method

This method usually provides a better initial basic feasible solution than the North-West Corner method since it takes into account the cost variables in the problem.

1. Assign as much as possible to the cell with the smallest unit cost in the entire tableau. If there is a tie then choose arbitrarily.
2. Cross out the row or column which has satisfied supply or demand. If a row and column are both satisfied then cross out only one of them.
3. Adjust the supply and demand for those rows and columns which are not crossed out.
4. When exactly one row or column is left, all the remaining variables are basic and are assigned the only feasible allocation.

	W_1	W_2	W_3	W_4	Supply
F_1	10	0	20	11	20
F_2	12	7	9	20	25
F_3	0	14	16	18	15
Demand	10	15	15	20	

The Least-Cost Method

m + n -1 = 6; 10x0 +15x0 + 15x9 + 5x11+ 10x20 +5x18 = 480

	W1	W2	W3	W4	S
F1	10x	15 0	20	5 11	20/5
F2	12x	7x	15 9x	10 20	25/10
F3	10 0x	14x	16x	5 18	15/5
D	10/0	15/0	15/0	20	60

For the above example:
Cells (1; 2) and (3; 1) both have zero cost so we arbitrarily choose the first and assign x_{12} = 15. Cross out column 2. The amount left in row 1 is 5.
x_{31} = 10. Cross out column 1. The amount left in row 3 is 5.
x_{23} = 15. Cross out column 3. The amount left in row 2 is 10.

Only column 4 is now left and so all the variables in this column will be basic.

The only feasible allocation is x_{14} = 5, x_{24} = 10 and x_{34} = 5.

This provides the initial basic feasible solution x_{12} = 15, x_{31} = 10, x_{23} = 15, x_{14} = 5, x_{24} = 10, x_{34} = 5.
All the other variables are non-basic and are therefore equal to zero.
Again, we have 6 basic variables as required.

Vogel Approximation Method (VAM). VAM is an improved version of the least-cost method that generally, but not always, produces better starting solutions.

Step 1. For each row (column), determine a penalty measure by subtracting the *smallest* unit cost element in the row (column) from the *next smallest* unit cost element in the same row (column).

Step 2. Identify the row or column with the largest penalty. Break ties arbitrarily. Allocate as much as possible to the variable with the least unit cost in the selected row or column. Adjust the supply and demand, and cross out the satisfied row *or* column. If a row and a column are satisfied simultaneously, only one of the two is crossed out, and the remaining row (column) is assigned zero supply (demand).

Step 3. **(a)** If exactly one row or column with zero supply or demand remains uncrossed out, stop.

(b) If one row (column) with *positive* supply (demand) remains uncrossed out, determine the basic variables in the row (column) by the least-cost method. Stop.

(c) If all the uncrossed out rows and columns have (remaining) zero supply and demand, determine the *zero* basic variables by the least-cost method. Stop.

(d) Otherwise, go to step 1.

Note:

If both a row and a column are satisfied simultaneously, *only one is crossed out*, the same as in the northwest-corner method. Next, look for the uncrossed-out cell with the smallest unit cost and repeat the process until exactly one row or column is left uncrossed out.

	W1	W2	W3	W4	S	Row Diff
F1	10	15 0	20	5 11	20/5/0	10/11/9
F2	12	7	15 9	10 20	25/15/0	2/2/11
F3	10 0	14	16	5 18	15/5/0	14/2/2
D	10/0	15/0	15/0	20/0	60	Max
Col. Diff	10/-/-	7/7/-	7/7/7	7/7/7		14R 11R 11R

15x0 + 5x11+ 15x9 + 15x20 + 10x0 + 5x18 = 580

Checking for optimality

Suppose that the cost c_{ij} of transporting 1 unit from source *i* to destination *j* is made up of a **dispatch cos** λ_i and a **reception cost** μ_j so that

$$\lambda_i + \mu_j = c_{ij}$$

whenever x_{ij} is a basic variable.

Remarks

The total number of *i* and *j* variables is *n*+*m*. However, there are only *n*+*m* -1 basic variables. Thus, we are free to choose *one* of the λ_i's or μ_j's rbitrarily.

It is usual to set $\lambda_1 = 0$.

These "costs" can take negative values if required.

Thus the procedure is as follows:

1. Assign values of λ_i and μ_j to the columns.
2. Enter the values $s_{ij} = c_{ij} - \lambda_i - \mu_j$ in every cell.
3. If all the s_{ij}'s are non-negative, we have an optimal solution.

	10	0	2	13
0	10 10	10 0	18 20	−2 11
7	−5 12	5 7	15 9	5 20
5	−15 0	9 14	9 16	15 18

Adding the *sij* variables to each cell, we find three negative values and so the solution is not optimal.

Iterating the algorithm

- If the current solution is not optimal, we need a method for moving to a better basic feasible solution.
- Thus changing only one variable in the basis so again we must identify an entering variable and a departing variable in the basis.

Determining the entering variable

- If the current solution is not optimal, choose the cell with the **most negative** value of *sij* as the entering variable, as the cost will be reduced most by using this route.

For our example, the most negative value is s_{31} and so the entering variable is x_{31}.

Determining the leaving variable

- We construct a **closed loop** that starts and ends at the entering variable and comprises successive horizontal and vertical segments whose end points must be basic variables (except those associated with the entering variable). It does not matter whether the loop is clockwise or anticlockwise.

Starting Tableau

$$\lambda_i + \mu_j = C_{ij}$$

10 10	10 0	20	11	λ_1 0
12	5 7	15 9	5 20	λ_2 7
0	14	16	15 18	λ_3 5
μ_1 10	μ_2 0	μ_3 2	μ_4 13	60

10 10 -	10 0 +	20	11	λ_1 0
12	5 - 7	15 9	5 + 20	λ_2 7
0 +	14	16	15 18 -	λ_3 5
μ_1 10	μ_2 0	μ_3 2	μ_4 13	60

$$S_{ij} = C_{ij} - \lambda_i - \mu_j$$

$S_{13} = C_{13} - \lambda_1 - \mu_3 = 20 - 0 - 2 = 18$

$S_{14} = C_{14} - \lambda_1 - \mu_4 = 11 - 0 - 13 = -2$

$S_{21} = C_{21} - \lambda_2 - \mu_1 = 12 - 7 - 10 = -5$

$S_{31} = C_{31} - \lambda_3 - \mu_1 = 0 - 5 - 10 = -15$

$S_{32} = C_{32} - \lambda_3 - \mu_2 = 14 - 5 - 0 = 9$

$S_{33} = C_{33} - \lambda_3 - \mu_3 = 16 - 5 - 2 = 9$

Second Tableau

$$\lambda_i + \mu_j = C_{ij}$$

5 10	15 0	20	11	λ_1 0
12	0 7	15 9	10 20	λ_2 -8
5 0	14	16	10 18	λ_3 10
μ_1 10	μ_2 0	μ_3 17	μ_4 28	60

Second Tableau

$$\lambda_i + \mu_j = C_{ij}$$

5 10 -	15 0	20	11 +	λ_1 0
12	0 7	15 9	10 20	λ_2 -8
5 0 +	14	16	10 18 --	λ_3 10
μ_1 10	μ_2 0	μ_3 17	μ_4 28	60

$$S_{ij} = C_{ij} - \lambda_i - \mu_j$$

$$S_{13} = C_{13} - \lambda_1 - \mu_3 = 20 - 0 - 17 = 3$$

$$S_{14} = C_{14} - \lambda_1 - \mu_4 = 11 - 0 - 28 = -17$$

$$S_{21} = C_{21} - \lambda_2 - \mu_1 = 12 - (-8) - 10 = 10$$

$$S_{22} = C_{22} - \lambda_2 - \mu_2 = 7 - (-8) - 0 = 15$$

$$S_{32} = C_{32} - \lambda_3 - \mu_2 = 14 - (-10) - 0 = 24$$

$$S_{33} = C_{33} - \lambda_3 - \mu_3 = 16 - (-10) - 17 = 9$$

Third Tableau

$$\lambda_i + \mu_j = C_{ij}$$

0 10	15 0	20	5 11	λ_1 0
12	7	5 9	10 20	λ_2 9
10 0	14	16	5 18	λ_3 7
μ_1 -7	μ_2 0	μ_3 0	μ_4 11	60

Third Tableau

$$\lambda_i + \mu_j = C_{ij}$$

0 10	15 0 -	20	5 11 +	λ_1 0
12	7 +	5 9	10 20 -	λ_2 9
10 0	14	16	5 18	λ_3 7
μ_1 -7	μ_2 0	μ_3 0	μ_4 11	60

$$S_{ij} = C_{ij} - \lambda_i - \mu_j$$

$$S_{11} = C_{11} - \lambda_1 - \mu_1 = 10 - (-7) - 0 = 17$$

$$S_{13} = C_{13} - \lambda_1 - \mu_3 = 20 - 0 - 0 = 20$$

$$S_{21} = C_{21} - \lambda_2 - \mu_1 = 12 - 9 - (-7) = 10$$

$$S_{22} = C_{22} - \lambda_2 - \mu_2 = 7 - 9 - 0 = -2$$

$$S_{32} = C_{32} - \lambda_3 - \mu_2 = 14 - 7 - 0 = 7$$

$$S_{33} = C_{33} - \lambda_3 - \mu_3 = 16 - -7 - 0 = 9$$

Fourth Tableau

$$\lambda_i + \mu_j = C_{ij}$$

10	5 0	20	15 11	λ_1 0
12	10 7	5 9	0 20	λ_2 7
10 0	14	16	5 18	λ_3 7
μ_1 -7	μ_2 0	μ_3 2	μ_4 11	60

Fourth Tableau

$$\lambda_i + \mu_j = C_{ij}$$

10	5 0	20	15 11	λ_1 0
12	10 7	5 9	20	λ_2 7
10 0	14	16	5 18	λ_3 7
μ_1 -7	μ_2 0	μ_3 2	μ_4 11	60

$$S_{ij} = C_{ij} - \lambda_i - \mu_j$$

$$S_{11} = C_{11} - \lambda_1 - \mu_1 = 10 - (-7) - 0 = 17$$

$$S_{13} = C_{13} - \lambda_1 - \mu_3 = 20 - 0 - 2 = 20$$

$$S_{21} = C_{21} - \lambda_2 - \mu_1 = 12 - (-7) - 7 = 12$$

$$S_{24} = C_{24} - \lambda_2 - \mu_4 = 20 - 7 - 11 = 2$$

$$S_{32} = C_{32} - \lambda_3 - \mu_2 = 14 - 7 - 0 = 7$$

$$S_{33} = C_{33} - \lambda_3 - \mu_3 = 16 - -7 - 2 = 7$$

All Positive

Optimal Solution Reached = 260

Pblm No. 1: Compare the solution for the following problems obtained by

a) Northwest corner,
b) b) Least cost and
c) c) Vogel's method:

*(a)

0	2	1	6
2	1	5	7
2	4	3	7
5	5	10	

(b)

1	2	6	7
0	4	2	12
3	1	5	11
10	10	10	

(c)

5	1	8	12
2	4	0	14
3	6	7	4
9	10	11	

Degeneracy- Balanced: Least cost Method

	A	B	C	D	E	Supply
1	10	20 2	15 3	15	9	35/15/0
2	5	10	15	10 2	30 4	40/30/0
3	20 15	5	14	7	15	20/0
4	20	15	25 13	25	5 8	30/25/0
Demand	20/0	20/0	40/25/0	10/0	35/5/0	125

m+n-1= 4+5-1=8 but allocated cells are 7. Resolve Degeneracy

Solution = 890

1. Convert the unallocated cells to allocated cells

	A	B	C	D	E
1	10	20 2	15 3	15	9
2	5	10	15	10 2	30 4
3	20 15	ϵ 5	14	7	15
4	20	15	25 13	25	5 8

Find a minimum cell and form a no-closed loop from the cell.

If cell satisfy the condition then select the cell and allocate a value

$\epsilon = 0$

Next calculate values for row and column using $\boldsymbol{\lambda_i + \mu_j = C_{ij}}$

$$\lambda_i + \mu_j = C_{ij}$$

	A	B	C	D	E	
1	10	20 2	15 3	15	9	$\lambda_1=0$
2	5	10	15	10 2	30 4	$\lambda_2=6$
3	20 15	ϵ 5	14	7	15	$\lambda_3=3$
4	20	15	25 13	25	5 8	$\lambda_4=10$
	$\mu_1=12$	$\mu_2=2$	$\mu_3=3$	$\mu_4=-4$	$\mu_5=-2$	

$$S_{ij} = C_{ij} - \lambda_i - \mu_j$$

	A	B	C	D	E	
1	10	20 2 -	15 3 +	15	9	λ_1=0
2	5 +	10	15	10 2	30 4 -	λ_2=6
3	20 15 -	ϵ 5 +	14	7	15	λ_3=3
4	20	15	25 13 -	25	5 8 +	λ_4=10
	μ_1= 12	μ_2= 2	μ_3 = 3	μ_4= - 4	μ_5= -2	

Min negative value = 20

$S_{11} = 10 - 0 - 12 = -2$

$S_{14} = 15 - 0 - (-4) = 19$

$S_{15} = 9 - 0 - (-2) = 11$

$\boldsymbol{S_{21}} = 5 - 6 - 12 = -13$

$S_{22} = 10 - 6 - 2 = 2$

$S_{23} = 15 - 6 - 3 = 6$

$S_{33} = 14 - 3 - 3 = 8$

$S_{34} = 7 - 3 - (-4) = 8$

$S_{35} = 15 - 3 - (-2) = 14$

$S_{41} = 20 - 10 - 12 = -2$

$S_{42} = 15 - 10 - 2 = 3$

$S_{44} = 25 - 10 - (-2) = 17$

$$\boldsymbol{S_{ij} = C_{ij} - \lambda_i - \mu_j}$$

$$\boldsymbol{\lambda_i + \mu_j = C_{ij}}$$

	A	B	C	D	E		
1	10	ϵ 2	35 3	15	9	35+ ϵ	$\lambda_1=0$
2	20 5	10	15	10 2	10 4	40	$\lambda_2= 6$
3	15	20+ϵ 5	14	7	15	20+ ϵ	$\lambda_3= 3$
4	20	15	5 13	25	25 8	30	$\lambda_4=$ 10
	20	20+2 ϵ	40	10	35		
	$\mu_1=$ -1	$\mu_2=$ 2	$\mu_3=$ 3	$\mu_4=$ -4	$\mu_5=$ -2		

m + n -1 = 7 allocate ϵ in cell (1, 2)

$S_{11} = 10 - 0 - (-1) = 11$

$S_{14} = 15 - 0 - (-4) = 19$

$S_{15} = 9 - 0 - (-2) = 11$

$S_{22} = 10 - 6 - 2 = 2$

$S_{23} = 15 - 6 - 3 = 6$

$S_{31} = 15 - 3 - (-1) = 13$

$S_{33} = 14 - 3 - 3 = 8$

$S_{34} = 7 - 3 - (-4) = 8$

$S_{35} = 15 - 3 - (-2) = 14$

$S_{41} = 20 - 10 - (-1) = 11$

$S_{42} = 15 - 10 - 2 = 3$

$S_{44} = 25 - 10 - (-4) = 19$

All positive - Optimal Solution reached = 630

Thank You