7.7 GRAM–SCHMIDT ORTHOGONALIZATION PROCESS

Suppose $\{v_1, v_2, \ldots, v_n\}$ is a basis of an inner product space V. One can use this basis to construct an orthogonal basis $\{w_1, w_2, \ldots, w_n\}$ of V as follows. Set

$$w_1 = v_1$$

$$w_2 = v_2 - \frac{\langle v_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1$$

$$w_3 = v_3 - \frac{\langle v_3, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 - \frac{\langle v_3, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$w_n = v_n - \frac{\langle v_n, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 - \frac{\langle v_n, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2 - \ldots - \frac{\langle v_n, w_{n-1} \rangle}{\langle w_{n-1}, w_{n-1} \rangle} w_{n-1}$$

In other words, for $k = 2, 3, \ldots, n$, we define

$$w_k = v_k - c_{k1} w_1 - c_{k2} w_2 - \ldots - c_{k,k-1} w_{k-1}$$

where $c_{ki} = \langle v_k, w_i \rangle / \langle w_i, w_i \rangle$ is the component of v_k along w_i. By Theorem 7.8, each w_k is orthogonal to the preceeding w's. Thus $w_1, w_2, \ldots, w_n$ form an orthogonal basis for V as claimed. Normalizing each w_i will then yield an orthonormal basis for V.

The above construction is known as the *Gram–Schmidt orthogonalization process*. The following remarks are in order.

Remark 1: Each vector w_k is a linear combination of v_k and the preceding w's. Hence one can easily show, by induction, that each w_k is a linear combination of $v_1, v_2, \ldots, v_n$.

Remark 2: Since taking multiples of vectors does not affect orthgonality, it may be simpler in hand calculations to clear fractions in any new w_k, by multiplying w_k by an appropriate scalar, before obtaining the next w_{k+1}.

Remark 3: Suppose $u_1, u_2, \ldots, u_r$ are linearly independent, and so they form a basis for $U = \text{span}(u_i)$. Applying the Gram–Schmidt orthogonalization process to the u's yields an orthogonal basis for U.

The following theorem (proved in Problems 7.26 and 7.27) use the above algorithm and remarks.

V. Then one may extend S to an orthogonal basis for V, that is, one may find vectors $w_{r+1}, \ldots, w_n$ such that $\{w_1, w_2, \ldots, w_n\}$ is an orthogonal basis for V.

Example 7.10. Apply the Gram–Schmidt orthogonalization process to find an orthogonal basis and then an orthonormal basis for the subspace U of $\mathbf{R}^4$ spanned by

$$v_1 = (1, 1, 1, 1), \qquad v_2 = (1, 2, 4, 5), \qquad v_3 = (1, -3, -4, -2)$$

(1) First set $w_1 = v_1 = (1, 1, 1, 1)$.

(2) Compute

$$v_2 - \frac{\langle v_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 = v_2 - \frac{12}{4} w_1 = (-2, -1, 1, 2)$$

Set $w_2 = (-2, -1, 1, 2)$.

(3) Compute

$$v_3 - \frac{\langle v_3, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 - \frac{\langle v_3, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2 = v_3 - \frac{(-8)}{4} w_1 - \frac{(-7)}{10} w_2 = \left(\tfrac{8}{5}, -\tfrac{17}{10}, -\tfrac{13}{10}, \tfrac{7}{5}\right)$$

Clear fractions to obtain $w_3 = (-6, -17, -13, 14)$.

Thus w_1, w_2, w_3 form an orthogonal basis for U. Normalize these vectors to obtain an orthonormal basis $\{u_1, u_2, u_3\}$ of U. We have $\|w_1\|^2 = 4$, $\|w_2\|^2 = 10$, $\|w_3\|^2 = 910$, so

$$u_1 = \frac{1}{2}(1, 1, 1, 1), \qquad u_2 = \frac{1}{\sqrt{10}}(-2, -1, 1, 2), \qquad u_3 = \frac{1}{\sqrt{910}}(16, -17, -13, 14)$$

... with inner product $\langle f, g \rangle = \int_{-1}^{1} f(t)g(t)\, dt$. Apply the ...

and $\|w\|^2 = 1+4+1+4 = 10$. Then

$c = -\frac{8}{10} = -\frac{4}{5}$ and $\text{proj}(v, w) = cw = (-\frac{4}{5}, -\frac{8}{5}, -\frac{4}{5}, -\frac{8}{5})$

7.21. Consider the subspace U of $\mathbf{R}^4$ spanned by the vectors:

$$v_1 = (1, 1, 1, 1), \qquad v_2 = (1, 1, 2, 4), \qquad v_3 = (1, 2, -4, -3)$$

Find (*a*) an orthogonal basis of U; (*b*) an orthonormal basis of U.

(*a*) Use the Gram–Schmidt algorithm. Begin by setting $w_1 = u = (1, 1, 1, 1)$. Next find

$$v_2 - \frac{\langle v_2, w_1\rangle}{\langle w_1, w_1\rangle} w_1 = (1, 1, 2, 4) - \frac{8}{4}(1, 1, 1, 1) = (-1, -1, 0, 2)$$

Set $w_2 = (-1, -1, 0, 2)$. Then find

$$v_3 - \frac{\langle v_3, w_1\rangle}{\langle w_1, w_1\rangle} w_1 - \frac{\langle v_3, w_2\rangle}{\langle w_2, w_2\rangle} w_2 = (1, 2, -4, -3) - \frac{(-4)}{4}(1, 1, 1, 1) - \frac{(-9)}{6}(-1, -1, 0, 2)$$
$$= (\tfrac{1}{2}, \tfrac{3}{2}, -3, 1)$$

Clear fractions to obtain $w_3 = (1, 3, -6, 2)$. Then w_1, w_2, w_3 form an orthogonal basis of U.

(*b*) Normalize the orthogonal basis consisting of w_1, w_2, w_3. Since $\|w_1\|^2 = 4$, $\|w_2\|^2 = 6$, and $\|w_3\|^2 = 50$, the following vectors form an orthonormal basis of U:

$$u_1 = \frac{1}{2}(1, 1, 1, 1), \qquad u_2 = \frac{1}{\sqrt{6}}(-1, -1, 0, 2), \qquad u_3 = \frac{1}{5\sqrt{2}}(1, 3, -6, 2)$$

7.22. Consider the vector space $\mathbf{P}(t)$ with inner product $\langle f, g\rangle = \int_0^1 f(t)g(t)\,dt$. Apply the Gram–Schmidt algorithm to the set $\{1, t, t^2\}$ to obtain an orthogonal set $\{f_0, f_1, f_2\}$ with integer coefficients

7.23. Suppose $v = (1, 3, 5, 7)$. Find the projection of v onto W or, in other words, find $w \in W$ that minimizes $\|v - w\|$, where W is the subspance of $\mathbf{R}^4$ spanned by:

(a) $u_1 = (1, 1, 1, 1)$ and $u_2 = (1, -3, 4, -2)$,
(b) $v_1 = (1, 1, 1, 1)$ and $v_2 = (1, 2, 3, 2)$

(a) Since u_1 and u_2 are orthogonal, we need only compute the Fourier coefficients:

$$c_1 = \frac{\langle v, u_1 \rangle}{\langle u_1, u_1 \rangle} = \frac{1+3+5+7}{1+1+1+1} = \frac{16}{4} = 4$$

$$c_2 = \frac{\langle v, u_2 \rangle}{\langle u_2, u_2 \rangle} = \frac{1-9+20-14}{1+9+16+4} = \frac{-2}{30} = -\frac{1}{15}$$

Then $w = \text{proj}(v, W) = c_1 u_1 + c_2 u_2 = 4(1, 1, 1, 1) - \frac{1}{15}(1, -3, 4, -2) = (\frac{59}{15}, \frac{63}{5}, \frac{56}{15}, \frac{62}{15})$.

(b) Since v_1 and v_2 are not orthogonal, first apply the Gram–Schmidt algorithm to find an orthogonal basis for W. Set $w_1 = v_1 = (1, 1, 1, 1)$. Then find

$$v_2 - \frac{\langle v_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 = (1, 2, 3, 2) - \frac{8}{4}(1, 1, 1, 1) = (-1, 0, 1, 0)$$

Set $w_2 = (-1, 0, 1, 0)$. Now compute

$$c_1 = \frac{\langle v, w_1 \rangle}{\langle w_1, w_1 \rangle} = \frac{1+3+5+7}{1+1+1+1} = \frac{16}{4} = 4$$

$$c_2 = \frac{\langle v, w_2 \rangle}{\langle w_2, w_2 \rangle} = \frac{-1+0+5+0}{1+0+1+0} = \frac{-6}{2} = -3$$

Then $w = \text{proj}(v, W) = c_1 w_1 + c_2 w_2 = 4(1, 1, 1, 1) - 3(-1, 0, 1, 0) = (7, 4, 1, 4)$.

7.24. Suppose w_1 and w_2 are nonzero orthogonal vectors. Let v be any vector in V. Find ... so that