Set Theory

Topics to be covered:

Definition: Sets, subsets, set operations: union, intersection, complement, difference and $\oplus$

Laws on set operations and Duality

(We will have to check the background of the students: I am assuming that the students have learnt all the set operation and done problems based on verification of laws)

Definition: Cartesian product of sets, power set.

Counting the number of elements in the Cartesian product and power set.

Definition: Partition of a set

Examples.

Problems:

1. What is the Cartesian product of $A = \{1, 2\}$ and $B = \{a, b, c\}$?
2. Suppose that $A = \{1, 2\}$, find A^2 and A^3.
3. Use a membership table to show that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
4. Prove that $(A \cap B)' = A' \cup B'$, using membership table.
5. Prove that $(A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A)$
6. Verify the theorem

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| + |A \cap B \cap C|$
where

[a] A = {a, b, c, d, e}

B = {d, e, f, g, h, i, k}
C = {a, c, d, e, k, r, s, t}

[b] A = {4, 5, 9, 15, 36, 70}

B = {5, 9, 37, 65, 81, 89}
C = {5, 6, 11, 15, 28, 65, 71, 79, 86}

7. Determine the number of positive integers n where $1 \le n \le 100$ and n is not divisible by 2, 3 or 5.
8. Suppose that 109 of the 150 computer science students at one of the Mumbai college take at least one of the following computer language:- VB, VC++ and Java. Suppose 45 study VB, 61 study V C ++, 53 study Java, 18 Study VB and VC ++, 53 study VC + + and Java, and 23 study VB and Java.

(i) How many students study all three languages?
(ii) How many students study only VC++?
(iii) How many students do not study any of the language?

9. A survey on a sample of 25 new cars being sold of a local auto dealer was conducted to see which of three popular options, air conditioning A, radio R, and power windows W, were already installed. The survey found,
   15 had air conditioning
   12 had radio
   11 had power windows
   5 had air conditioning and power window
   9 had air conditioning and radio
   4 had radio and power windows
   5 had all three options.
   Find the no. of cars having:
   1) only one of these options
   2) radio & power windows but not air conditioning
   3) none of these options.
10. If $S = \{1, 2, 3\}$, Find $P(S)$.
11. If X has 10 members, then how many proper subsets does $P(X)$ have?
12. Salad is made with combination of one or more eatables. How many different salads can be prepared from onion, tomato, carrot, cabbage and cucumber?
13. Consider the following collection of subsets of $S = \{1, 2, \ldots, 8, 9\}$.
   a) $\{\{1,3,5\}, \{2,6\}, \{4,8,9\}\}$
   b) $\{\{1,3,5\}, \{2,4,6,8\}, \{5,7,9\}\}$
   c) $\{\{1,3,5\}, \{2,4,6,8\}, \{7,9\}\}$

   Which of the above is a partition of S .
14. Find all partitions of the set $\{1, 2, 3\}$.
15. Find all partitions of the set $\{1, 2, 3, 4\}$.

Relations

Topics to be covered:
Definition and Examples (Examples are given in Schaum series book, page number 2.3)
Representation of relation using matrices and composition of relation
Types of relation and Equivalence relation
Relation between Equivalence relation and Partitions.

1. Find domain, range, matrix of relation R.

a) A= {1, 2, 3, 4}, B = {1, 4, 6, 8, 9}; a R b if and only if $b=a^2$

b) A = {1, 2, 3, 4, 8} = B; a R b if and only if a=b

c) A= {1, 2, 3, 4, 8}, B = {1, 4, 6, 9}; a R b if and only if a divides b

d) A = {1, 2, 3, 4, 6} =B; a R b if and only if a is multiple of b

e) A = {1, 2, 3, 4, 5} = B; a R b if and only if a ≤ b

f) A = {1, 3, 5, 7, 9}, B = {2, 4, 6, 8}; a R b if and only if a<b

i) A = {1, 2, 3, 4, 8} = B; a R b if and only if a + b ≤ 9

2. Let $A=Z^+$, the positive integer, and R be the relation defined by a R b if and only if there exist a k in Z^+ so that $a=b^k$ (k depends on a and b). Which of the following belongs to R?

   (a) (4, 6), (b) (1, 7), c) (8, 2), d) (3, 3), (e) (2, 8), (f) (2, 32)

3. Find the relation R defined on A if

(a) A= {1, 2, 3, 4} and $M_R = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$, (b) A= {a, b, c, d, e} and

$$M_R = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

4. Let A = {1, 2, 3, 4}, and B = {a, b, c, d}, & C = {x, y, z}.Consider the relations R from A to B and S from B to C defined by R = {(1, a), (2, d), (3, a), (3, b), (3, d)}, and

S = {(b, x), (b, z), (c, y), (d, z)}. Compute (i) find the composition relation SoR,

(ii)Find the Matrices M_R, M_S, & M_{SoR}, representing the relations R, S, SoR.

(iii) Is $M_R.M_S = M_{SoR}$?

5. Let A = {1, 2, 3, 4}, and B = {a, b, c}. Let R = {(1, a), (1, b), (2, b), (2, c), (3, b), (4, a)}, and

S = {(1, b), (2, c), (3, b), (4, b)}. Compute (a) R; (b) R ∩ S; (c)R ∪ S; and R^{-1}

6. Let A = {a, b, c, d, e} and

R = {(a, b), (a, d), (a, c), (b, c), (b, d), (b, e), (c, c), (d, c), (d, d), (e, e)}

S = {(a, a), (a, b), (b, b), (b, c), (c, b), (c, c), (c, e), (d, b), (e, d), (e, a)}

find (a) R ; (b) R^{-1}; (c) R ∩ S; (e) S ; (f) S^{-1}

7. Let A = {1, 2, 3} and let R and S be the defined on A. suppose that the matrices of R and S are

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \ M_S = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$ Find $M_{\overline{R}}, M_{\overline{S}}, M_{R^{-1}}, M_{S^{-1}}, M_{R \cap S}, M_{R \cup S}$

8. Let R and S be the given relations from A to B. Compute

R , S , R^{-1},S^{-1}, R ∩ S, and R ∪ S

(a) A=B= {1, 2, 3}, R= {(1, 1), (1, 2), (2, 3),(3, 1)}, and S= {(2, 1), (3, 1), (3, 2), (3, 3)}

(b) A= {a, b, c}; B= {1, 2, 3}, R= {(a, 1), (b, 1), (c, 2), (c, 3)}, S= {(a, 1), (a, 2), (b, 1), (b, 2)}

9. Let A= {2, 3, 6, 12} and let R and S be the following relations on A;

(i) x R y if and only if 2|(x-y); (ii) x S y if and only if 3|(x-y). Compute

R^{-1}, S^{-1}, R, S, R ∩ S, and R ∪ S

10. Let A = {a, b, c} and let R and S be relations on A whose Matrices are

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \text{ and } M_S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$ Find R o S,& S o R also find $M_{R o S}$, & $M_{S o R}$

Properties of the relation:

1) If A = {1, 2, 3, 4}. Determine whether the relation is reflexive, irreflexive, symmetric, asymmetric, anti-symmetric, or transitive.

(a) R= {(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (3, 4), (4, 3), (4, 4)}

(b) R= {(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)}

(c) R= {(1, 1), (1, 2), (1, 3), (3, 1), (3, 3), (4, 4)}

(d) R= {(1, 1), (2, 2), (3, 3)}

(e) R= {(1, 1), (1, 2), (1, 3), (1, 4), (3, 1), (3, 2), (3, 3), (3, 4), (4, 2)}

(f) R= {(1, 3), (4, 2), (2, 4), (3, 1), (2, 2)}

2) Let A= {1, 2, 3, 4, 5}. Determine whether the relation R whose length is given is reflexive, irreflexive, symmetric, asymmetric, anti-symmetric, or transitive.

(a) R= {(1, 1), (1, 2), (1, 3), (1, 5), (2, 3), (4, 2), (4, 3), (4, 4), (4, 5), (5, 3)} (figure 4.22 page 146)

(b) R= {(1, 2), (1, 3), (1, 4), (5, 2), (5, 3), (5, 4)} (figure 4.23 page 146)

3) A = {1, 2, 3, 4} Determine whether the relation R whose matrix M_R is reflexive, irreflexive, symmetric, asymmetric, anti-symmetric, or transitive.

(a) $M_R = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$, (b) $M_R = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

4) Determine whether the relation R on the set A is reflexive, irreflexive, symmetric, asymmetric, anti-symmetric, or transitive.

(a) A=Z; a R b if and only if $a \leq b + 1$

(b) $A=Z^+$; a R b if and only if $|a-b| \leq 2$

(c) $A=Z^+$; a R b if and only if $a = b^k$ for some $k \in Z^+$

(d) A=Z; a R b if and only if a + bis even

5) Define relation on set A = {a, b, c, d} which is

(i) Reflexive and symmetric but not transitive,

(ii) Reflexive and transitive but not, symmetric

(iii) Transitive symmetric, and symmetric, (iv) a Symmetric and transitive

6) Determine whether the following relation R on the set A is an equivalence relation

(a) A = {a, b, c, d}, R = {(a, a), (b, a), (b, b), (c, c), (d, d), (d, c)}.

(b) A = {1, 2, 3, 4, 5}, R= {(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (3, 1), (2, 3), (3, 3), (4, 4), (3, 2), (5, 5)}.

7) Let A be a set of non-zero integers and let R be a relation on $A \times A$ defined by

$(a, b)R(c, d)$ if $ad = bc$, Then prove that R is an equivalence relation

8) Let A be a set of non-zero integers and let R be a relation on $A \times A$ defined by

$(a, b)R(c, d)$ if $a + d = b + c$, Then prove that R is an equivalence relation

9) Let R be the relation defined on the set Z by $R = \{(a, b)/a,\ b \in Z$ & a-b divisible by 3 (or $a \equiv b \bmod 3$)$\}$. Show that the relation R is an equivalence relation on A. Determine an equivalence classes. Also find Z/R

10) If R is a relation on the set of integers such that aRb if and only if $1a + 4b$ is divisible by 5. find the equivalence classes. Find Z/R .