NORM of a VECTOR

- **Length: The length of a vector** $\mathbf{v}=(v_1,v_2,\cdots,v_n)$ **in R^n is given by**

$$\|\mathbf{v}\|=\sqrt{v_1^{\,2}+v_2^{\,2}+\cdots+v_n^{\,2}} \quad (\|\mathbf{v}\| \text{ is a real number})$$

- **Notes: The length of a vector is also called its norm**

- **Properties of length (or norm)**

(1) $\|\mathbf{v}\|\geq 0$

(2) $\|\mathbf{v}\|=1 \Rightarrow$ $\mathbf{v}$ **is called a unit vector**

(3) $\|\mathbf{v}\|=0$ if and only if $\mathbf{v}=\mathbf{0}$

(4) $\|c\mathbf{v}\|=|c|\,\|\mathbf{v}\|$

5.

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- **Ex:**

(a) In R^5, the length of $\mathbf{v} = (0\,,-2\,,1\,,4\,,-2)$ **is given by**

$$\|\mathbf{v}\| = \sqrt{0^2 + (-2)^2 + 1^2 + 4^2 + (-2)^2} = \sqrt{25} = 5$$

(b) In R^3, the length of $\mathbf{v} = (\frac{2}{\sqrt{17}}\,, \frac{-2}{\sqrt{17}}\,, \frac{3}{\sqrt{17}})$ **is given by**

$$\|\mathbf{v}\| = \sqrt{\left(\frac{2}{\sqrt{17}}\right)^2 + \left(\frac{-2}{\sqrt{17}}\right)^2 + \left(\frac{3}{\sqrt{17}}\right)^2} = \sqrt{\frac{17}{17}} = 1$$

(If the length of v is 1, then v is a unit vector)

- **A standard unit vector in R^n: only one component of the vector is 1 and the others are 0 (thus the length of this vector must be 1)**

$$R^2 : \{\mathbf{e}_1, \mathbf{e}_2\} = \{(1,0),(0,1)\}$$

$$R^3 : \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} = \{(1,0,0),(0,1,0),(0,0,1)\}$$

$$R^n : \{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\} = \{(1,0,\cdots,0),(0,1,\cdots,0),\cdots,(0,0,\cdots,1)\}$$

Dot product in Rn

- The dot product of $\mathbf{u} = (u_1, u_2, \cdots, u_n)$ and $\mathbf{v} = (v_1, v_2, \cdots, v_n)$ returns a scalar quantity

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

- Ex: Finding the dot product of two vectors

  The dot product of **u** = (1, 2, 0, –3) and **v** = (3, –2, 4, 2) is

$$\mathbf{u} \cdot \mathbf{v} = (1)(3) + (2)(-2) + (0)(4) + (-3)(2) = -7$$

Distance between two vectors

$$\boxed{d(u,v) = \| u - v \|}$$

Angle between two vectors

$$\boxed{\operatorname{Cos}\theta = \frac{< u,v >}{\| u \| . \| v \|}}$$

ORTHOGONALITY

- **Orthogonal vectors:**

  Two vectors u and v in R^n are orthogonal (perpendicular) if

$$\mathbf{u} \cdot \mathbf{v} = 0$$

- **Note:**

  The vector 0 is said to be orthogonal to every vector

INNER PRODUCT SPACE

- **Inner product: represented by angle brackets** $\langle \mathbf{u}, \mathbf{v} \rangle$

Let u, v, and w be vectors in a vector space *V*, and let *c* be any scalar. An inner product on *V* is a function that associates a real number with each pair of vectors u and v and satisfies the following axioms

(1) $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$ **(commutative property)**

(2) $\langle \mathbf{u}, \mathbf{v}+\mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$ **(distributive property)**

(3) $c\langle \mathbf{u}, \mathbf{v} \rangle = \langle c\mathbf{u}, \mathbf{v} \rangle$ **(associative property of the scalar multiplication)**

(4) $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$

(5) $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ **if and only if** $\mathbf{v} = \mathbf{0}$

- **Note:**

$$\mathbf{u}\cdot\mathbf{v} = \text{dot product (Euclidean inner product for } R^n)$$

$$<\mathbf{u}\,,\,\mathbf{v}> = \text{general inner product for a vector space } V$$

- **Note:**

A vector space *V* with an inner product is called an inner product space

Vector space: $(V,\ +,\ \cdot)$

Inner product space: $(V,\ +,\ \cdot\ ,\ <,\ >)$

Let   be the Euclidean inner product on R^2 .

Let $u=(1,1), v=(3,2), w=(-1,0)$ **and k=5.**

Compute the following:

$\langle v,w\rangle$

$\langle ku,v\rangle$

$\langle u+v,w\rangle$

$\|u\|$

$d(u,v)$

$\|u-kv\|$

Angle between v and w

Normalizing vectors

(1) If $\|\mathbf{v}\| = 1$, then v is called a unit vector

(Note that $\|\mathbf{v}\|$ is defined as $\sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$)

(if v is not a zero vector)

(2) $\mathbf{v} \neq \mathbf{0} \xrightarrow{\text{Normalizing}} \dfrac{\mathbf{v}}{\|\mathbf{v}\|}$

(the unit vector in the direction of v)

Orthonormal Bases

- Orthogonal set

A set S of vectors in an inner product space V is called an orthogonal set if every pair of vectors in the set is orthogonal

$$S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\} \subseteq V$$

$$\langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0, \text{ for } i \neq j$$

- Orthonormal set:

An orthogonal set in which each vector is a unit vector is called orthonormal set

$$S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\} \subseteq V$$

$$\begin{cases} \text{For } i = j, \ \langle \mathbf{v}_i, \mathbf{v}_j \rangle = \langle \mathbf{v}_i, \mathbf{v}_i \rangle = \|\mathbf{v}_i\|^2 = 1 \\ \text{For } i \neq j, \ \langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0 \end{cases}$$

- **Ex: A nonstandard orthonormal basis for R^3**

  Show that the following set is an orthonormal basis

$$S = \left\{ \underset{\mathbf{v}_1}{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)}, \quad \underset{\mathbf{v}_2}{\left(-\frac{\sqrt{2}}{6}, \frac{\sqrt{2}}{6}, \frac{2\sqrt{2}}{3}\right)}, \quad \underset{\mathbf{v}_3}{\left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right)} \right\}$$

Sol: First, show that the three vectors are mutually orthogonal

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = -\tfrac{1}{6} + \tfrac{1}{6} + 0 = 0$$

$$\mathbf{v}_1 \cdot \mathbf{v}_3 = \frac{2}{3\sqrt{2}} - \frac{2}{3\sqrt{2}} + 0 = 0$$

$$\mathbf{v}_2 \cdot \mathbf{v}_3 = -\frac{\sqrt{2}}{9} - \frac{\sqrt{2}}{9} + \frac{2\sqrt{2}}{9} = 0$$

Second, show that each vector is of length 1

$$\|\mathbf{v}_1\| = \sqrt{\mathbf{v}_1 \cdot \mathbf{v}_1} = \sqrt{\tfrac{1}{2} + \tfrac{1}{2} + 0} = 1$$

$$\|\mathbf{v}_2\| = \sqrt{\mathbf{v}_2 \cdot \mathbf{v}_2} = \sqrt{\tfrac{2}{36} + \tfrac{2}{36} + \tfrac{8}{9}} = 1$$

$$\|\mathbf{v}_3\| = \sqrt{\mathbf{v}_3 \cdot \mathbf{v}_3} = \sqrt{\tfrac{4}{9} + \tfrac{4}{9} + \tfrac{1}{9}} = 1$$

Thus S is an orthonormal set

Because these three vectors are linearly independent (you can check by solving $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = 0$) in R^3 (of dimension 3), by Theorem (given a vector space with dimension n, then n linearly independent vectors can form a basis for this vector space), these three linearly independent vectors form a basis for R^3.

$\Rightarrow$ S is a (nonstandard) orthonormal basis for R^3

Let R^3 have the Euclidean inner product. For which values of k, u and v are orthogonal?

$$u = (1, 4, 2), v = (3, -2, k)$$

$$u = (k, -2, 4), v = (k, k, -2)$$

ORTHOGONAL PROJECTION

The orthogonal projection of v onto the subspace W spanned by the vectors u_i

$$\operatorname{Pr}oj_W v = \sum \left(\frac{u_i . v}{u_i . u_i} \right) u_i$$

Let W be the plane in R^3 , with equation $x-y+2z=0$, and let $v=(3,-1,2)$. Find the orthogonal projection of v onto W and the component of v orthogonal to W

Find the orthogonal projection of v onto the subspace W spanned by the vectors u_i.

$$v = (7, -4), u_1 = (1, 1)$$

$$v = (3, 1, -2), u_1 = (1, 1, 1), u_2 = (1, -1, 0)$$

$$v = (1, 2, 3), u_1 = (2, -2, 1), u_2 = (-1, 1, 4)$$

GRAM-SCHMIDT PROCESS

Given the basis: {v1,v2,v3}

To find the orthogonal basis {u1,u2,u3}

Step I:

$$\boxed{u_1 = v_1}$$

Step II:

$$\boxed{u_2 = v_2 - \frac{< v_{2,} u_1 >}{\| u_1 \|^2} . u_1}$$

Step III

$$\boxed{u_3 = v_3 - \frac{< v_{3,} u_1 >}{\| u_1 \|^2} . u_1 - \frac{< v_{3,} u_2 >}{\| u_2 \|^2} . u_2}$$

To find the orthonormal basis {q1,q2,q3}

$$\boxed{q_1 = \frac{u_1}{\| u_1 \|}, q_2 = \frac{u_2}{\| u_2 \|}, q_3 = \frac{u_3}{\| u_3 \|}}$$

Let R^3 have the Euclidean inner product. Use Gram Schmidt process to transform $\{u_1,u_2,u_3\}$ into an orthonormal basis.

- $\{(1,1,1), (1,0,-1), (2,1,-1)\}$

- $\{(0,1,0),(-7,4,2), (-3,0,-1)\}$

QR Decomposition

- QR decomposition is the matrix version of the Gram-Schmidt orthonormalization process.

- QR decomposition is widely used in many fields as data processing, image processing, communication systems, multiple input multiple output (MIMO), radar systems, linear algebra and so on. ...

  - ❑ A=QR
  - ❑ Q=[q1,q2,q3]
  - ❑ $R=Q^T.A$

- where Q is an orthogonal matrix and R is an upper triangular matrix. So-called QR-decompositions are useful for solving linear systems, eigenvalue problems and least squares approximations.

Find QR decomposition of the matrix,

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \qquad \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 1 & 2 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 1 \end{pmatrix}$$