$\cdots + 1u = (1+1)u = 2u$

4.4. Express $v = (1, -2, 5)$ in $\mathbf{R}^3$ as a linear combination of the vectors

$$u_1 = (1, 1, 1), \qquad u_2 = (1, 2, 3), \qquad u_3 = (2, -1, 1)$$

We seek scalars x, y, z, as yet unknown, such that $v = xu_1 + yu_2 + zu_3$. Thus we require

$$\begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix} = x\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + y\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + z\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \qquad \text{or} \qquad \begin{aligned} x + y + 2z &= 1 \\ x + 2y - z &= -2 \\ x + 3y + z &= 5 \end{aligned}$$

(For notational convenience, we write the vectors in $\mathbf{R}^3$ as columns, since it is then easier to find the equivalent system of linear equations.) Reducing the system to echelon form yields the triangular system

$$x + y + 2z = 1, \qquad y - 3z = -3, \qquad 5z = 10$$

The system is consistent and has a solution. Solving by back-substitution yields the solution $x = -6$, $y = 3$, $z = 2$. Thus $v = -6u_1 + 3u_2 + 2u_3$.

Alternatively, write down the augmented matrix M of the equivalent system of linear equations, where u_1, u_2, u_3 are the first three columns of M and v is the last column, and then reduce M to echelon form:

$$M = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 2 & -1 & -2 \\ 1 & 3 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & -3 & -3 \\ 0 & 2 & -1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & -3 & -3 \\ 0 & 0 & 5 & 10 \end{bmatrix}$$

The last matrix corresponds to a triangular system, which has a solution. Solving the triangular system by back-substitution yields the solution $x = -6$, $y = 3$, $z = 2$. Thus $v = -6u_1 + 3u_2 + 2u_3$.

4.5. Express $v = (2, -5, 3)$ in $\mathbf{R}^3$ as a linear combination of the vectors

$$u_1 = (1, -3, 2), \quad u_2 = (2, -4, -1), \quad u_3 = (1, -5, 7)$$

We seek scalars x, y, z, as yet unknown, such that $v = xu_1 + yu_2 + zu_3$. Thus we require

$$\begin{bmatrix} 2 \\ -5 \\ 3 \end{bmatrix} = x\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} + y\begin{bmatrix} 2 \\ -4 \\ -1 \end{bmatrix} + z\begin{bmatrix} 1 \\ -5 \\ 7 \end{bmatrix} \qquad \text{or} \qquad \begin{aligned} x + 2y + z &= 2 \\ -3x - 4y - 5z &= -5 \\ 2x - y + 7z &= 3 \end{aligned}$$

Reducing the system to echelon form yields the system

$$x + 2y + z = 2, \qquad 2y - 2z = 1, \qquad 0 = 3$$

The system is inconsistent and so has no solution. Thus v cannot be written as a linear combination of u_1, u_2, u_3.

by (a). Finally, if $v \in W$, then $(-1)v = -v \in W$, and $v + (-v) = 0$. Thus $[A_3]$ holds.

4.9. Let $V = \mathbf{R}^3$. Show that W is not a subspace of V, where:

(a) $W = \{(a, b, c) : a \geq 0\}$, (b) $W = \{(a, b, c) : a^2 + b^2 + c^2 \leq 1\}$.

In each case, show that Theorem 4.2 does not hold.

(a) W consists of those vectors whose first entry is nonnegative. Thus $v = (1, 2, 3)$ belongs to W. Let $k = -3$. Then $kv = (-3, -6, -9)$ does not belong to W, since -3 is negative. Thus W is not a subspace of V.

(b) W consists of vectors whose length does not exceed 1. Hence $u = (1, 0, 0)$ and $v = (0, 1, 0)$ belong to W, but $u + v = (1, 1, 0)$ does not belong to W, since $1^2 + 1^2 + 0^2 = 2 > 1$. Thus W is not a subspace of V.

4.10. Let $V = \mathbf{P}(t)$, the vector space of real polynomials. Determine whether or not W is a subspace of V, where:

polynomials in W belong to W.

4.11. Let V be the vector space of functions $f : \mathbf{R} \to \mathbf{R}$. Show that W is a subspace of V, where:

(*a*) $W = \{f(x) : f(1) = 0\}$, all functions whose value at 1 is 0.

(*b*) $W = \{f(x) : f(3) = f(1)\}$, all functions assigning the same value to 3 and 1.

(*c*) $W = \{f(t) : f(-x) = -f(x)\}$, all *odd functions*.

Let $\hat{0}$ denote the zero polynomial, so $\hat{0}(x) = 0$ for every value of x.

(*a*) $\hat{0} \in W$, since $\hat{0}(1) = 0$. Suppose $f, g \in W$. Then $f(1) = 0$ and $g(1) = 0$. Also, for scalars a and b, we have

$$(af + bg)(1) = af(1) + bg(1) = a0 + b0 = 0$$

Thus $af + bg \in W$, and hence W is a subspace.

(*b*) $\hat{0} \in W$, since $\hat{0}(3) = 0 = \hat{0}(1)$. Suppose $f, g \in W$. Then $f(3) = f(1)$ and $g(3) = g(1)$. Thus, for any scalars a and b, we have

$$(af + bg)(3) = af(3) + bg(3) = af(1) + bg(1) = (af + bg)(1)$$

Thus $af + bg \in W$, and hence W is a subspace.

(*c*) $0 \in W$, since $\hat{0}(-x) = 0 = -0 = -\hat{0}(x)$. Suppose $f, g \in W$. Then $f(-x) = -f(x)$ and $g(-x) = -g(x)$. Also, for scalars a and b,

$$(af + bg)(-x) = af(-x) + bg(-x) = -af(x) - bg(x) = -(af + bg)(x)$$

Thus $ab + gf \in W$, and hence W is a subspace of V.

each W_i is a subspace, $au + bv \in W_i$ for every ...

LINEAR SPANS

4.13. Show that the vectors $u_1 = (1, 1, 1)$, $u_2 = (1, 2, 3)$, $u_3 = (1, 5, 8)$ span $\mathbf{R}^3$.

We need to show that an arbitrary vector $v = (a, b, c)$ in $\mathbf{R}^3$ is a linear combination of u_1, u_2, u_3. Set $v = xu_1 + yu_2 + zu_3$, that is, set

$$(a, b, c) = x(1, 1, 1) + y(1, 2, 3) + z(1, 5, 8) = (x + y + z, \quad x + 2y + 5z, \quad x + 3y + 8z)$$

Form the equivalent system and reduce it to echelon form:

$$\begin{aligned} x + y + z &= a \\ x + 2y + 5z &= b \\ x + 3y + 8z &= c \end{aligned} \qquad \text{or} \qquad \begin{aligned} x + y + z &= a \\ y + 4z &= b - a \\ 2y + 7c &= c - a \end{aligned} \qquad \text{or} \qquad \begin{aligned} x + y + z &= a \\ y + 4z &= b - a \\ -z &= c - 2b + a \end{aligned}$$

The above system is in echelon form and is consistent; in fact,

$$x = -a + 5b - 3c, \quad y = 3a - 7b + 4c, \quad z = a + 2b - c$$

is a solution. Thus u_1, u_2, u_3 span $\mathbf{R}^3$.

4.14. Find conditions on a, b, c so that ...

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Example 5.4

(*a*) Let $F : \mathbf{R}^3 \to \mathbf{R}^3$ be the "projection" mapping into the xy-plane, that is, F is the mapping defined by $F(x, y, z) = (x, y, 0)$. We show that F is linear. Let $v = (a, b, c)$ and $w = (a', b', c')$. Then

$$F(v + w) = F(a + a', \; b + b', \; c + c') = (a + a', \; b + b', \; 0)$$
$$= (a, b, 0) + (a', b', 0) = F(v) + F(w)$$

and, for any scalar k,

$$F(kv) = F(ka, kb, kc) = (ka, kb, 0) = k(a, b, 0) = kF(v)$$

Thus F is linear.

... 2) [That is, G adds the vector

which maps each vector $v \in V$ into its coordinate vector $[v]_S$, is an isomorphism between V and K^n.

5.4 KERNEL AND IMAGE OF A LINEAR MAPPING

We begin by defining two concepts.

Definition: Let $F: V \to U$ be a linear mapping. The *kernel* of F, written Ker F, is the set of elements in V that map into the zero vector 0 in U; that is,

$$\text{Ker } F = \{v \in V : F(v) = 0\}$$

The *image* (or *range*) of F, written Im F, is the set of image points in U; that is,

$$\text{Im } F = \{u \in U : \text{there exists } v \in V \text{ for which } F(v) = u\}$$

The following theorem is easily proved (Problem 5.22).

Theorem 5.3: Let $F: V \to U$ be a linear mapping. Then the kernel of F is a subspace of V and the

Thus F is linear.

5.11. Show that the following mappings are not linear:

(*a*) $F: \mathbf{R}^2 \to \mathbf{R}^2$ defined by $F(x, y) = (xy, x)$

(*b*) $F: \mathbf{R}^2 \to \mathbf{R}^3$ defined by $F(x, y) = (x+3, \ 2y, \ x+y)$

(*c*) $F: \mathbf{R}^3 \to \mathbf{R}^2$ defined by $F(x, y, z) = (|x|, \ y+z)$

(*a*) Let $v = (1, 2)$ and $w = (3, 4)$; then $v + w = (4, 6)$. Also,

$$F(v) = (1(2), 1) = (2, 1) \quad \text{and} \quad F(w) = (3(4), 3) = (12, 3)$$

Hence

$$F(v+w) = (4(6), 4) = (24, 6) \neq F(v) + F(w)$$

(*b*) Since $F(0, 0) = (3, 0, 0) \neq (0, 0, 0)$, F cannot be linear.

(*c*) Let $v = (1, 2, 3)$ and $k = -3$. Then $kv = (-3, -6, -9)$. We have

$$F(v) = (1, 5) \quad \text{and} \quad kF(v) = -3(1, 5) = (-3, -15).$$

Thus

$$F(kv) = F(-3, -6, -9) = (3, -15) \neq kF(v)$$

Accordingly, F is not linear.

5.12. Let V be the vector space of n-square real matrices. Let M be an arbitrary but fixed matrix in V.

KERNEL AND IMAGE OF LINEAR MAPPINGS

5.16. Let $F: \mathbf{R}^4 \to \mathbf{R}^3$ be the linear mapping defined by

$$F(x, y, z, t) = (x - y + z + t, \quad x + 2z - t, \quad x + y + 3z - 3t)$$

Find a basis and the dimension of: (*a*) the image of F, (*b*) the kernel of F.

(*a*) Find the images of the usual basis of $\mathbf{R}^4$:

$$F(1, 0, 0, 0) = (1, 1, 1), \qquad F(0, 0, 1, 0) = (1, 2, 3)$$
$$F(0, 1, 0, 0) = (-1, 0, 1), \qquad F(0, 0, 0, 1) = (1, -1, -3)$$

By Proposition 5.4, the image vectors span Im F. Hence form the matrix whose rows are these image vectors, and row reduce to echelon form:

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 2 & 3 \\ 1 & -1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & -2 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus $(1, 1, 1)$ and $(0, 1, 2)$ form a basis for Im F; hence $\dim(\text{Im } F) = 2$.

(*b*) Set $F(v) = 0$, where $v = (x, y, z, t)$; that is, set

$$F(x, y, z, t) = (x - y + z + t, \; x + 2z - t, \; x + y + 3z - 3t) = (0, 0, 0)$$

Set corresponding entries equal to each other to form the following homogeneous system whose solution space is Ker F:

$$\begin{aligned} x - y + z + t &= 0 \\ x \quad\; + 2z - t &= 0 \\ x + y + 3z - 3t &= 0 \end{aligned} \quad \text{or} \quad \begin{aligned} x - y + z + t &= 0 \\ y + z - 2t &= 0 \\ 2y + 2z - 4t &= 0 \end{aligned} \quad \text{or} \quad \begin{aligned} x - y + z + t &= 0 \\ y + z - 2t &= 0 \end{aligned}$$

The free variables are z and t. Hence $\dim(\text{Ker } F) = 2$.

(i) Set $z = -1$, $t = 0$ to obtain the solution $(2, 1, -1, 0)$.

(ii) Set $z = 0$, $t = 1$ to obtain the solution $(1, 2, 0, 1)$.

Thus $(2, 1, -1, 0)$ and $(1, 2, 0, 1)$ form a basis of Ker F.
[As expected, $\dim(\text{Im } F) + \dim(\text{Ker } F) = 2 + 2 = 4 = \dim \mathbf{R}^4$, the domain of F.]