Assignment 2
Operation Research
BTech Sem IV (20-21 Batch)

Q1. Describe graphically what the simplex method does step by step to solve the following problem. (4.1.6)

Maximize $Z = 2x_1 + 3x_2$,

subject to

$$-3x_1 + x_2 \leq 1$$
$$4x_1 + 2x_2 \leq 20$$
$$4x_1 + x_2 \leq 10$$
$$-x_1 + 2x_2 \leq 5$$

and

$$x1 \geq 0,\ x2 \geq 0.$$

Q2. Describe graphically what the simplex method does step by step to solve the following problem. (4.1.7)

Minimize $Z = 5x_1 + 7x_2$,

subject to

$$2x_1 + 3x_2, \geq 147$$
$$3x_1 + 4x_2, \geq 210$$
$$x_1 - x_2, \geq 63$$

and

$$x1 \geq 0,\ x2 \geq 0.$$

Q3. Consider the following problem. (4.2.2)

Maximize $Z = x_1 + 2x_2$,

subject to

$$x_1 + 3x_2 \leq 8$$
$$x_1 + x_2 \leq 4$$

and

$$x1 \geq 0,\ x2 \geq 0.$$

(a) Introduce slack variables in order to write the functional constraints in augmented form.
(b) For each CPF solution, identify the corresponding BF solution by calculating the values of the slack variables. For each BF solution, use the values of the variables to identify the nonbasic variables and the basic variables.
(c) For each BF solution, demonstrate (by plugging in the solution) that, after the nonbasic variables are set equal to zero, this BF solution also is the simultaneous solution of the system of equations obtained in part (*a*).
(d) Repeat part (*b*) for the corner-point infeasible solutions and the corresponding basic infeasible solutions.
(e) Repeat part (*c*) for the basic infeasible solutions.

Q4. Consider the problem in Q3. . (4.3.3)

a) Work through the simplex method (in algebraic form) step by step to solve the model
b) Verify the optimal solution you obtained by using a software package based on the simplex method.

4.3-3.

(a) maximize $Z = x_1 + 2x_2$

subject to

$$\begin{aligned} x_1 + 3x_2 + x_3 \quad\quad &= 8 \\ x_1 + x_2 \quad\quad + x_4 &= 4 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

Initialization: $x_1 = x_2 = 0 \Rightarrow x_3 = 8,\ x_4 = 4,\ z = x_1 + 2x_2 = 0$, is not optimal since the improvement rates are positive. Since it offers a rate of improvement of 2, choose to increase x_2, which becomes the entering basic variable for Iteration 1. Given $x_1 = 0$, the highest possible increase in x_2 is found by looking at:

$$x_3 = 8 - 3x_2 \geq 0 \Rightarrow x_2 \leq 8/3$$

$$x_4 = 4 - x_2 \geq 0 \Rightarrow x_2 \leq 4$$

The minimum of these two bounds is $8/3$, so x_2 can be raised to $8/3$ and $x_3 = 0$ leaves the basis. Using Gaussian elimination, we obtain:

$$\begin{aligned} Z = \quad & \tfrac{1}{3}x_1 \quad - \tfrac{2}{3}x_3 \quad + \tfrac{16}{3} \\ & \tfrac{1}{3}x_1 + x_2 + \tfrac{1}{3}x_3 \quad = \tfrac{8}{3} \\ & \tfrac{2}{3}x_1 \quad - \tfrac{1}{3}x_3 + x_4 = \tfrac{4}{3} \\ & x_1, x_2, x_3, x_4 \quad \geq 0 \end{aligned}$$

Again $(0, \frac{8}{3}, 0, \frac{4}{3})$ is not optimal since the rate of improvement for x_1 is $\frac{1}{3} > 0$ and x_1 can be increased to 2. Consequently, x_4 becomes 0. By Gaussian elimination:

$$\begin{aligned} Z = \quad & - \tfrac{1}{2}x_3 - \tfrac{1}{2}x_4 + 6 \\ & x_2 + \tfrac{1}{2}x_3 + \tfrac{1}{2}x_4 = 2 \\ & x_1 \quad - \tfrac{1}{2}x_3 + \tfrac{3}{2}x_4 = 2 \\ & x_1, x_2, x_3, x_4 \quad \geq 0 \end{aligned}$$

The current solution is optimal, since increasing x_3 or x_4 would decrease the objective value. Hence $x^* = (2, 2, 0, 0)$, $Z^* = 6$.

Iteration	Basic Var.		Equation	Z	x_1	x_2	x_3	x_4	RHS	Minimum Ratio Test	
1	Z		(0)	1	-1	-2	0	0	**0**		
Solution (0 0 8 4) Z = 0	x_3		(1)	0	1	**3**	1	0	**8**	**8/3**	**Min**
Not Optimal	x_4		(2)	0	1	1	0	1	**4**	**4/1**	
	Entering Variable : x_2					**Leaving Variable: x_3 (will be set to 0)**					
2			(0)	1	-1/3	0	2/3	0	**16/3**		
Solution (0 8/3 0 4/3) Z = 16/3	x_2	2*Eq (1) add to Eq. (0)	(1)	0	1/3	1	1/3	0	**8/3**	**8**	**New Equation**
Not Optimal	x_4	Eq (1) subst from Eq. 2	(2)	0	2/3	0	-1/3	1	**4/3**	**2**	**Min**
	Entering Variable : x_1					**Leaving Variable: x_4 (will be set to 0)**					
3			(0)	1	0	0	1/2	-1/2	**6**		
Solution (2 2 0 0) Z = 6	x_2		(1)	0	0	1	1/2	1/2	**2**		
Optimal	x_1	1/3*Eq (2) substract from Eq. (1), and add to Eq (0)	(2)	0	1	0	-1/2	3/2	**2**		**New Equation**

5. Work through the simplex method step by step (in tabular form) to solve the following problem. (4.4-7)

Maximize $Z = 2x_1 - x_2 + x_3$,

subject to,

$$3x_1 + x_2 + x_3 \leq 6$$
$$x_1 - x_2 + 2x_3 \leq 1$$
$$x_1 + x_2 - x_3 \leq 2$$

and

$$x_1 \geq 0,\ x_3 \geq 0\ x_3 \geq 0.$$

Iteration	**Basic Var.**		**Equation**	**Z**	x_1	x_2	x_3	x_4	x_5	x_6	**RHS**	**Minimum Ratio Test**	
1	Z		(0)	1	-2	1	-1	0	0	0	0		
Solution	x_4		(1)	0	3	1	1	1	0	0	6	2	
(0 0 0 2 1 2)	x_5		(2)	0	1	-1	2	0	1	0	1	1	Min
Z = 0	x_6		(3)	0	1	1	-1	0	0	1	2	2	
	EV	x_1								LV	x_5		
2			(0)	1	0	-1	3	0	2	0	2		
Solution	x_4		(1)	0	0	4	-5	1	-3	0	3	3/4	
(1 0 0 3 0 1)	x_1		(2)	0	1	-1	2	0	1	0	1	-1	New Equation
	x_6		(3)	0	0	2	-3	0	-1	1	1	1/2	Min
	EV	x_2								LV	x_6		
3			(0)	1	0	0	1.5	0	1.5	0.5	2.5		
Solution	x_4		(1)	0	0	0	1	1	-1	-2	1		
(1.5 0.5 0 1 0 0)	x_1		(2)	0	1	0	0.5	0	0.5	0.5	1.5		
Z = 2.5	x_2		(3)	0	0	1	-1.5	0	-0.5	0.5	0.5		New Equation

6. Consider the following problem. (4.4-6.)

$$\text{Maximize } Z = 3x_1 + 5x_2 + 6x_3,,$$

subject to

$$\begin{aligned} 2x_1 + x_2 + x_3 &\leq 4 \\ x_1 + 2x_2 + x_3 &\leq 4 \\ x_1 + x_2 + 2x_3 &\leq 4 \\ x_1 + x_2 + x_3 &\leq 3 \end{aligned}$$

and

$$x_1 \geq 0,\ x_2 \geq 0\ x_3 \geq 0$$

(a) Work through the simplex method step by step in algebraic form.

(b) Work through the simplex method in tabular form.

(c) Use a computer package based on the simplex method to solve the problem.

Iteration	**Basic Var.**		**Equation**	**Z**	x_1	x_2	x_3	x_4	x_5	x_6	**RHS**	**Minimum Ratio Test**	
1	Z		(0)										
			(1)										
			(2)										
			(3)										

7. Consider the following statements about linear programming and the simplex method. Label each statement as true or false, and then justify your answer. (4.5-1)

(a) In a particular iteration of the simplex method, if there is a tie for which variable should be the leaving basic variable, then the next BF solution must have at least one basic variable equal to zero.

(b) If there is no leaving basic variable at some iteration, then the problem has no feasible solutions.

(c) If at least one of the basic variables has a coefficient of zero in row 0 of the final tableau, then the problem has multiple optimal solutions.

(d) If the problem has multiple optimal solutions, then the problem must have a bounded feasible region.

8. Consider the following problem. 4.6-17

$$\text{Maximize } Z = 4x_1 + 5x_2 + 3x_3,$$

subject to

$$x_1 + x_2 + 2x_3 \geq 20$$
$$15x_1 + 6x_2 - 5x_3 \leq 50$$
$$x_1 + 3x_2 + 5x_3 \leq 30$$

and

$$x_1 \geq 0, x_2 \geq 0\ x_3 \geq 0$$

Work through the simplex method step by step to demonstrate that this problem does not possess any feasible solutions.

9. Label each of the following statements as true or false, and then justify your answer. 4.6-11.

(a) When a linear programming model has an equality constraint, an artificial variable is introduced into this constraint in order to start the simplex method with an obvious initial basic solution that is feasible for the original model.

(b) When an artificial problem is created by introducing artificial variables and using the Big *M* method, if all artificial variables in an optimal solution for the artificial problem are equal to zero, then the real problem has no feasible solutions.

(c) The two-phase method is commonly used in practice because it usually requires fewer iterations to reach an optimal solution than the Big *M* method does.

10. Consider the following problem. 4.6-5.

Maximize $Z = 5x_1 + 4x_2$,

subject to

$$3x_1 + 2x_2 \leq 6$$
$$2x_1 - x_2 \geq 6$$

and

$$x_1 \geq 0,\ x_2 \geq 0.$$

(a) Demonstrate graphically that this problem has no feasible solutions.
(b) Use a computer package based on the simplex method to determine that the problem has no feasible solutions.
(c) Using the Big *M* method, work through the simplex method step by step to demonstrate that the problem has no feasible solutions.

11. Consider the following problem. 4.6-9.

Minimize $Z = 3x_1 + 2x_2 + 4x_3$

subject to

$$2x_1 + x_2 + 3x_3 = 60$$
$$3x_1 + 3x_2 + 5x_3 \geq 120$$

and

$$x_1 \geq 0,\ x_2 \geq 0\ x_3 \geq 0$$

(a) Using the Big *M* method, work through the simplex method step by step to solve the problem.
(b) Use a software package based on the simplex method to solve the problem.

12. Consider the following problem 4.6-10.

Minimize $Z = 3x_1 + 2x_2 + 7x_3$,

subject to

$$-x_1 + x2 = 10$$
$$2x_1 + x_2 + x3 \geq 10$$

and

$$x_1 \geq 0,\ x_2 \geq 0\ x_3 \geq 0$$

(a)Using the Big *M* method, work through the simplex method step by step to solve the problem
(b) Use a software package based on the simplex method to solve the problem.

13. Consider the following problem. 4.7-4.

$$\text{Maximize } Z = x_1 - 7x_2 + 3x_3,$$

subject to

$$2x_1 + x_2 - x_3 \leq 4 \text{ (resource 1)}$$
$$4x_1 - 3x_2 \leq 2 \text{ (resource 2)}$$
$$-3x_1 + 2x_2 + x_3 \leq 3 \text{ (resource 3)}$$

and

$$x_1 \geq 0, x_2 \geq 0\ x_3 \geq 0.$$

(a) Work through the simplex method step by step to solve the problem.
(b) Identify the shadow prices for the three resources and describe their significance.
(c) Use a software package based on the simplex method to solve the problem and then to generate sensitivity information.

Use this information to identify the shadow price for each resource, the allowable range for each objective function coefficient, and the allowable range for each right hand side.

14. Consider the following problem. 4.7-6.

$$\text{Maximize } Z = 5x_1 + 4x_2 - x_3 + 3x_4,$$

subject to

$$3x_1 + 2x_2 - 3x_3 + x_4 \leq 24 \text{ (resource 1)}$$
$$3x_1 + 3x_2 + x_3 + 3x_4 \leq 36 \text{ (resource 2)}$$

and

$$x_1 \geq 0, x_2 \geq 0\ x_3 \geq 0, x_4 \geq 0$$

(a) Work through the simplex method step by step to solve the problem.
(b) Identify the shadow prices for the two resources and describe their significance.
(c) Use a software package based on the simplex method to solve the problem and then to generate sensitivity information.

Use this information to identify the shadow price for each resource, the allowable range for each objective function coefficient, and the allowable range for each right-hand side.

(a) Optimal Solution: $(x_1^*, x_2^*, x_3^*, x_4^*) = (11, 0, 3, 0)$ and $Z^* = 52$

Bas Var	Eq No	Z	X_1	X_2	X_3	X_4	X_5	X_6	Right Side
Z	0	1	-5	-2	1	-3	0	0	0
X_5	1	0	3	2	-3	1	1	0	24
X_6	2	0	3	3	1	3	0	1	36

Bas Var	Eq No	Z	X_1	X_2	X_3	X_4	X_5	X_6	Right Side
Z	0	1	0	1.3333	-4	-1.333	1.6667	0	40
X_1	1	0	1	0.6667	-1	0.3333	0.3333	0	8
X_6	2	0	0	1	4	2	-1	1	12

Bas Var	Eq No	Z	X_1	X_2	X_3	X_4	X_5	X_6	Right Side
Z	0	1	0	2.3333	0	0.6667	0.6667	1	52
X_1	1	0	1	0.9167	0	0.8333	0.0833	0.25	11
X_3	2	0	0	0.25	1	0.5	-0.25	0.25	3

(b) The shadow prices are $y_1^* = 0.6667$ and $y_2^* = 1$. They are the marginal values of resources 1 and 2 respectively.

(c)

	5	4	-1	3			
Variables	11	0	3	0		52	Optimal Value
Constraints						RHS	
	3	2	-3	1	24 <=	24	
	3	3	1	3	36 <=	36	

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
C3	Variables	11	0	5	1E+30	0.363636
D3	Variables	0	-0.33333333	4	0.33333333	1E+30
E3	Variables	3	0	-1	2.66666667	1.333333
F3	Variables	0	-0.66666667	3	0.66666667	1E+30

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
H5	Constraints	24	0.66666667	24	12	132
H6		36	1	36	1E+30	12

Matrix Method LP

1. Consider the following problem.

$$\text{Maximize } Z = 8x_1 + 4x_2 + 6x_3 + 3x_4 + 9x,$$

subject to

$$x_1 + 2x_2 + 3x_3 + 3x_4 \leq 180 \text{ (resource 1)}$$
$$4x_1 + 3x_2 + 2x_3 + x_4 + x_5 \leq 270 \text{ (resource 2)}$$
$$x_1 + 3x_2 + x_4 + 3x_5 \leq 180 \text{ (resource 3)}$$

and

$$x_j \geq 0, \; j = 1, 2, 3, 4, 5.$$

You are given the facts that the basic variables in the optimal solution are x_3, x_1, and x_5 and that

$$\begin{bmatrix} 3 & 1 & 0 \\ 2 & 4 & 1 \\ 0 & 1 & 3 \end{bmatrix}^{-1} = \frac{1}{27}\begin{bmatrix} 11 & -3 & 1 \\ -6 & 9 & -3 \\ 2 & -3 & 10 \end{bmatrix}.$$

(a) Use the given information to identify the optimal solution.
(b) Use the given information to identify the shadow prices for the three resources.

(a) Optimal Solution: $\begin{pmatrix} x_3 \\ x_1 \\ x_5 \end{pmatrix} = B^{-1}b = \frac{1}{27}\begin{pmatrix} 11 & -3 & 1 \\ -6 & 9 & -3 \\ 2 & -3 & 10 \end{pmatrix}\begin{pmatrix} 180 \\ 270 \\ 180 \end{pmatrix} = \begin{pmatrix} 50 \\ 30 \\ 50 \end{pmatrix}$

$$Z = cx = \begin{pmatrix} 8 & 4 & 6 & 3 & 9 \end{pmatrix}\begin{pmatrix} 30 \\ 0 \\ 50 \\ 0 \\ 50 \end{pmatrix} = 990$$

(b) Shadow prices: $c_B B^{-1} = \frac{1}{27}\begin{pmatrix} 6 & 8 & 9 \end{pmatrix}\begin{pmatrix} 11 & -3 & 1 \\ -6 & 9 & -3 \\ 2 & -3 & 10 \end{pmatrix} = \begin{pmatrix} 1.33 \\ 1 \\ 2.67 \end{pmatrix}$

2. Work through the matrix form of the simplex method step by step to solve the following problem.

$$\text{Maximize } Z = 5x_1 + 8x_2 + 7x_3 + 4x_4 + 6x_5,$$

subject to

$$2x_1 + 3x_2 + 3x_3 + 2x_4 + 2x_5 \le 20$$
$$3x_1 + 5x_2 + 4x_3 + 2x_4 + 4x_5 \le 30$$

and

$$x_j \ge 0, j = 1, 2, 3, 4, 5.$$

$$c = \begin{pmatrix} 5 & 8 & 7 & 4 & 6 & 0 & 0 \end{pmatrix}, A = \begin{pmatrix} 2 & 3 & 3 & 2 & 2 & 1 & 0 \\ 3 & 5 & 4 & 2 & 4 & 0 & 1 \end{pmatrix}, b = \begin{pmatrix} 20 \\ 30 \end{pmatrix}$$

Iteration 0: $B = B^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, x_B = \begin{pmatrix} x_6 \\ x_7 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 20 \\ 30 \end{pmatrix} = \begin{pmatrix} 20 \\ 30 \end{pmatrix}$

$c_B = \begin{pmatrix} 0 & 0 \end{pmatrix}, -c = \begin{pmatrix} -5 & -8 & -7 & -4 & -6 & 0 & 0 \end{pmatrix}$, so x_2 enters.

Revised x_2 coefficients: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$, so x_7 leaves.

Iteration 1: $B_{\text{new}}^{-1} = \begin{pmatrix} 1 & 3 \\ 0 & 5 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -3/5 \\ 0 & 1/5 \end{pmatrix}$,

$$x_B = \begin{pmatrix} x_6 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & -3/5 \\ 0 & 1/5 \end{pmatrix} \begin{pmatrix} 20 \\ 30 \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \end{pmatrix}, c_B = \begin{pmatrix} 0 & 8 \end{pmatrix}$$

Revised row 0: $\begin{pmatrix} 0 & 8/5 \end{pmatrix} \begin{pmatrix} 2 & 3 & 3 & 2 & 2 & 1 & 0 \\ 3 & 5 & 4 & 2 & 4 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 5 & 8 & 7 & 4 & 6 & 0 & 0 \end{pmatrix}$

$= \begin{pmatrix} -1/5 & 0 & -3/5 & -4/5 & -2/5 & 0 & 8/5 \end{pmatrix}$, so x_4 enters.

Revised x_4 coefficients: $\begin{pmatrix} 1 & -3/5 \\ 0 & 1/5 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 4/5 \\ 2/5 \end{pmatrix}$, so x_6 leaves.

Iteration 2: $B_{\text{new}}^{-1} = \begin{pmatrix} 2 & 3 \\ 2 & 5 \end{pmatrix}^{-1} = \begin{pmatrix} 5/4 & -3/4 \\ -1/2 & 1/2 \end{pmatrix}$

$$x_B = \begin{pmatrix} x_4 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5/4 & -3/4 \\ -1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 20 \\ 30 \end{pmatrix} = \begin{pmatrix} 5/2 \\ 5 \end{pmatrix}, c_B = \begin{pmatrix} 4 & 8 \end{pmatrix}$$

Revised row 0: $\begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 & 3 & 2 & 2 & 1 & 0 \\ 3 & 5 & 4 & 2 & 4 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 5 & 8 & 7 & 4 & 6 & 0 & 0 \end{pmatrix}$

$= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$, so the current solution is optimal.

Optimal Solution: $(x_1^*, x_2^*, x_3^*, x_4^*, x_5^*) = (0, 5, 0, 5/2, 0)$ and $Z^* = 50$