CHAPTER 5

Module 4

Counting

Syllabus

Permutation, Combination, Elements of Probability, Discrete Probability and Conditional Probability, Generating Function and Recurrence Relations, Recursive Functions. Introduction to Functional Programming.

Syllabus Topic : Permutation, Combination

5.1 Permutations and Combinations :

5.1.1 Permutation :

If $1 \leq r \leq n$ then P(n, r) denote the number of r-permutations of a set with n elements of a particular set.

5.1.1.1 Permutation with distinct objects :

A permutation of a set of distinct objects is an ordered arrangement of these objects.

We can find P(n, r) using product rule.

Theorem 1 : If n is a positive integer and r is an integer with $1 \leq r \leq n$ then there are

$$P(n, r) = n(n-1)(n-2)\ldots(n-r+1) \text{ r - permutations of a set with n distinct elements}$$

Proof : The first elements of the permutation can be chosen in 'n' ways as there are n elements in the set. After this there are (n – 1) elements left in the set, therefore second element of the permutation can be chosen in (n – 1) ways.

Similarly, there are (n – 2) ways to choose the third element and so on until there are exactly,

$$n-(r-1) = n-r+1 \text{ ways to choose the } r^{th} \text{ element.}$$

Consequently by the product rule there are

$n(n-1)(n-2)\ldots.(n-r+1)$ r-permutations of the set

$$\therefore \; p(n-r) = n(n-1)\cdot(n-2)\ldots\ldots(n-r+1)$$

Theorem 2 : If n and r are integers with $0 \leq r \leq n$

$$\text{Then } P(n, r) = \frac{n!}{(n-r)!}$$

Proof : First of all we define factorial n as the product of first 'n' natural numbers and denoted by n!

We also define $0! = 1$

$$\therefore \; n! = n(n-1)(n-2)\times\ldots\ldots\times 3\times 2\times 1$$

$$\therefore \; (n-r)! = (n-r)(n-r-1)\times\ldots\ldots\times 3\times 2\times 1$$

We can rewrite n! in the following manner.

$$n! = n(n-1)(n-2)\ldots\ldots(n-(r-1)(n-r)(n-(r+1))\times\ldots\times 3\times 2\times 1$$

$$\therefore \; \frac{n!}{(n-r)!} = \frac{n(n-1)(n-2)\times\ldots\ldots\times(n-(r-1))(n-r)(n-(r+1))\times\ldots\times 3\times 2\times 1}{(n-r)(n-r-1)\times\ldots\ldots\times 3\times 2\times 1}$$

$$\therefore \; \frac{n!}{(n-r)!} = n(n-1)(n-2)\times\ldots\ldots\times(n-r+1)$$

$$\frac{n!}{(n-r)!} = P(n, r)$$

$$\therefore \; P(n, r) = \frac{n!}{(n-r)!}$$

We deduce following results from theorem 2

1. if $r = 0$ then $P(n, 0) = \frac{n!}{n!} = 1$
2. if $r = 1$ the $P(n, 1) = \frac{n!}{(n-1)!} = n$
3. if $r = n$ then $P(n, n) = \frac{n!}{0!} = n!$

5.1.1.2 Examples

Example 1 : Compute the number of permutations of the given set {0, 1, 2, 3, 4}.

Solution :

There are 5 elements in the set. 5 elements can be arranged in $P\{5, 5\} = 5! = 120$ ways.

Example 2 : There are six different political parties in a state. How many different orders can the names of the parties be printed on a ballot paper ?

Solution :

There are six different candidates and six different places on a ballot paper, therefore number of different ways in which the names of the candidates can be printed on a ballot paper therefore $= P(6, 6) = 6! = 720$ ways.

Example 3 : How many ways are there for eight men and five women to stand in a line so that no two women stand next to each other.

Solution :

First of all eight men can be arranged in a row in P(8, 8) ways. Five women can be arranged on 9 places in the following manner which are marked by *

* M * M * M * M * M * M * M * M *

This can be arranged in P(9, 5)

$\therefore$ By product rule, total no. of ways $= P(8, 8) \times P(9, 5) = 8! \times \frac{9!}{4!}$

$= 8! \times 9 \times 8 \times 7 \times 6 \times 5 = 609{,}638{,}400$

Example 4 : A family of 4 brothers and 3 sisters is to be seated for a photograph in one row. In how many ways they can be seated if (i) all the sisters sit together (ii) no two sisters sit together.

Solution :

(i) Taking 3 sisters as one unit, we have to arrange 4 + 1 i.e. 5 different units taking 5 at time in a row. This can be done in P(5, 5) = 5! ways.

In each of these ways, 3 sisters can be arranged among themselves in P(3, 3) = 3! ways. Hence the required number of ways = $5! \times 3! = 120 \times 6 = 720$

(ii) 4 brothers can be arranged in P(4, 4) = 4! = 24 ways

In between and at the extremities of 4 brothers there are 5 places in which 3 sisters can be arranged in $P(5, 3) = \frac{5!}{3!} = \frac{5 \times 4 \times 3 \times \cancel{2!}}{\cancel{2!}} = 60$ ways

Hence the required no. of ways $= 24 \times 60 = 1440$

Example 5 : How many different arrangements can be made by using all the letters of the word COMPUTER ? How many of these begin with VOWEL and end with a consonant ?

Solution :

There are 8 letters in the word COMPUTER, without any restrictions 8 letters can be arranged in P(8, 8) = 8! = 40320 ways. In the word COMPUTER, there are three vowels namely O, U, E and remaining 5 consonants.

The arrangement can be shown as under

V							C

First place can be filled in by a vowel in P(3, 1) = 3 ways. Last place can be filled in by a consonant in P(5, 1) = 5 ways and the remaining 6 places can be filled in by remaining 6 letters in P(6, 6) = 6! ways

$\therefore$ Total no. of words which can be formed

$= 3 \times 5 \times 6! = 15 \times 720 = 10800$ ways

Example 6 : A bookshelf is used to display 4 books of computer science, 3 books of discrete mathematics and 2 books of electronics. Find the number of ways in which these books can be arranged so that 2 books on electronics are together.

Solution :

Here we consider 2 books on electronics as one unit, we have to arrange 7 + 1 i.e. 8 different units taking 8 at a time. This can be done in P(8, 8) = 8! Ways. In each of these ways, 2 books on electronics can be arranged among themselves in P(2, 2) = 2! Ways. Hence the required no. of ways = $8! \times 2! = 40320 \times 2 = 80640$ ways.

5.1.1.3 Permutation of Indistinct Objects :

We have studied permutation of objects which are all different. Now suppose we have to find permutations of objects which are all not different then we use following formula in general.

If 'n' objects are to be arranged where the first object appears k_1 times the second object k_2 times and so on then the number of distinct permutations that can be formed is $\frac{n!}{k_1!\, k_2! \ldots k_t!}$

Example 1 : Find the number of distinct permutations of the letter in the word "DISCRETE".

Solution :

The word discrete has 8 letters of which E is repeated 2 times and the rest are different.

Hence the number of distinct arrangements are

$$\frac{8!}{2!} = 20160$$

Example 2 : In how many arrangements of the word 'letter' contains the two T's together ?

Solution :

If two T's must be together then they can be considered as one unit. Hence we have 4 + 1 i.e. 5 different units of which E is repeated twice

$$\therefore \text{ Number of distinct arrangements } = \frac{5!}{2!} = \frac{5 \times 4 \times 3 \times \cancel{2!}}{\cancel{2!}} = 60$$

Example 3 : A library has 5 copies of one book, 4 copies of each of two books, 6 copies of each of three books and single copy of each of 8 books. In how many ways can all the books be arranged ?

Solution :

In library there are

5	copies of one book	$\therefore 5 \times 1 = 5$
4	copies of two books	$\therefore 4 \times 2 = 8$
6	copies of three books	$\therefore 6 \times 3 = 18$
1	copy of 8 books	$\therefore 1 \times 8 = 8$

$\therefore$ Total number of books in library = 39

In this case we have permutations of indistinct objects.

$\therefore$ Total number of ways in which all the books can be arranged

$$= \frac{39!}{5! \times (4!)^2 \times (6!)^3} \text{ Ways}$$

5.1.2 Combinations :

Another important counting technique which is based on selection of objects without taking into account the order of objects is called combination.

Let A be any set with n elements denoted by

$|A| = n$ and $1 \leq r \leq n$. Then the no. of combinations of the elements of A, taken r at a time is the number of r element subsets of A denoted by C (n, r) or $\binom{n}{r}$, ... $\binom{n}{r}$ is also called binomial coefficient.

5.1.2.1 Combinations with Distinct Objects :

Theorem 3 : The number of r combinations of set with n elements, where n is a non negative integer and r is an integer with $0 \leq r \leq n$.

$$C(n, r) = \frac{n!}{r!\,(n-r)!}$$

Proof : The r permutations of the set can be obtained by first selecting r objects from n objects in C(n, r) ways i.e. r combinations of the set. These r combinations of the set are then ordered by taking the elements in each r combination.

Which can be done in P(r, r) ways.

$$\therefore\ P(n, r) = C(n, r)\, P(r, r)$$

$$\therefore\ C(n, r) = \frac{P(n, r)}{P(r, r)} = \frac{\frac{n!}{(n-r)!}}{\frac{r!}{\theta!}} = \frac{n!}{r!\,(n-r)!} = \frac{n!}{r!(n-r)!}$$

Examples 1 : Prove that :

(i) $C(n, 0) = 1$ (ii) $C(n, 1) = n$,

(iii) $C(n, r) - C(n, n-r)$

Solution :

(i) $\text{LHS} = C(n, 0) = \frac{n!}{0!(n-0)!} = \frac{n!}{1 \times n!} = 1 = \text{RHS}$

(ii) $\text{LHS} = C(n, 1) = \frac{n!}{1!(n-1)!} = \frac{n(n-1)!}{(n-1)!} = n = \text{RHS}$

(iii) $\text{LHS} = C(n, r) = \frac{n!}{r!(n-r)!}$ ...(1)

$$\text{RHS} = C(n, n-r) = \frac{n!}{(n-r)!\,(n-(n-r)!)}$$

$$= \frac{n!}{(n-r)!\,r!} \qquad \ldots(2)$$

$\therefore$ From (1) and (2) LHS = RHS

Example 2 : Out of 4 officers and 10 clerks in an office, a committee consisting of 2 officers and 3 clerks is to be formed. In how many ways can this be done if any officer and any clerk can be include ?

Solution :

2 officers can be selected from 4 officers in C(4, 2) ways and 3 clerks can be selected from 10 clerks in C(10, 3) ways

$$\therefore \text{ Total number of ways} = C(4, 2) \times C(10, 3) = \frac{4 \times 3}{2 \times 1} \times \frac{10 \times 9 \times 8}{3 \times 2 \times 1}$$

$$= 6 \times 120 = 720 \text{ ways}$$

Example 3 : Find the number of subsets of the set {1, 2, 3 . . . . 11} having 4 elements.

Solution :

We have to find the number of ways of choosing 4 elements of this set which has 11 elements. This can be done in

$$^{11}C_4 = \frac{11 \times 10 \times 9 \times 8}{4 \times 2 \times 3 \times 4} = 330 \text{ ways}$$

Example 4 : 12 points lie on a circle. How many cyclic quadrilaterals can be drawn by using these points ?

Solution :

For any set of 4 points we get a cyclic quadrilateral. Number of ways of choosing 4 points out of 12 points is $^{12}C_4 = 495$

Therefore we can draw 495 quadrilaterals.

Example 5 : A question paper consists of 10 questions divided into two parts A and B. Each part contains 5 questions. A candidate is required to attempt 6 questions in all of which at least 2 should be from part A and at least 2 from part B.

In how many ways can the candidate select the questions if he can answer all questions equally well ?

Solution :

The candidate has to select 6 questions in of which at least two should be from part A and two should be from part B. he can select questions in any of the following way :

	Part A	Part B
(i)	2	4
(ii)	3	3
(iii)	4	2

If the candidate follows choice (i) the number of ways in which he can do so is

$$^{5}C_2 \times {}^{5}C_4 = 10 \times 5 = 50$$

If the candidate follows choice (ii) the number of ways in which he can do so is

$$^{5}C_3 \times {}^{5}C_3 = 10 \times 10 = 100$$

If the candidate follows choice (iii) then the number of ways in which he can do so is

$$^{5}C_4 \times {}^{5}C_2 = 50$$

Therefore the candidate can select the question in 50 + 100 + 50 = 200 ways

5.1.2.2 Combination with indistinct objects :

Suppose K selections are to be made from 'n' objects in any order and that the repetitions are allowed assuming at least k copies of each of the n objects. The number of ways these selection can be made is $C(n + k - 1, k)$.

Example 1 : A gift voucher at a book store allows the recipients to select 5 books from the combined lots of 10 copies of books on software. In how many ways can the selection of 5 books be made.

Solution :

Here total number of books is

$$n = 10 + 10 = 20$$

Number of books to be selected is $k = 5$

Here repetition of books is allowed

$$\therefore \text{ Number of selections } = C(n + k - 1, k)$$

$$= C(20 + 5 - 1, 5) = C(24, 5)$$

$$= \frac{24!}{5!\ 19!} = \frac{24 \times 23 \times 22 \times 21 \times 20}{5 \times 4 \times 3 \times 2 \times 1} = 42504$$

Example 2 : Mr. X invites a party of 20 guests to a dinner and places 12 of them at a round table and the other guests at another round table. Find the total number of ways in which he can arrange all the guests.

Solution :

Here Mr. X would like to select 12 guests out of 20 in $^{20}C_{12}$ ways. Then these selected 12 persons will be asked to occupy their chairs at a round table in (12 – 1)! i.e. 11! Ways and remaining 8 guests another round table in (8 – 1)! i.e. 7! ways.

$$\therefore \text{ Total number of ways } = {}^{20}C_{12} \times 11! \times 7!$$

Q 1d) Addⁿ

Q2b)

Q3a) using small x and itallics.

3c) Volume (same question) using {}

Q4a)

Q4b) set theory add membershiptable.

Q4c)