# Mathematical modeling

Module2

# Mathematical modeling by example

## Product mix

A toy company makes two types of toys: *toy soldiers* and *trains*. Each toy is produced in two stages, first it is constructed in a carpentry shop, and then it is sent to a finishing shop, where it is varnished, vaxed, and polished. To make one toy soldier costs \$10 for raw materials and \$14 for labor; it takes 1 hour in the carpentry shop, and 2 hours for finishing. To make one train costs \$9 for raw materials and \$10 for labor; it takes 1 hour in the carpentry shop, and 1 hour for finishing.

There are 80 hours available each week in the carpentry shop, and 100 hours for finishing. Each toy soldier is sold for \$27 while each train for \$21. Due to decreased demand for toy soldiers, the company plans to make and sell at most 40 toy soldiers; the number of trains is not restriced in any way.

What is the optimum (*best*) product mix (i.e., what quantities of which products to make) that *maximizes* the profit (assuming all toys produced will be sold)?

## Terminology

| | |
|---|---|
| **decision variables:** | $x_1, x_2, \ldots, x_i, \ldots$ |
| **variable domains:** values that variables can take | $x_1, x_2 \geq 0$ |
| **goal/objective:** | maximize/minimize |
| **objective function:** function to minimize/maximize | $2x_1 + 5x_2$ |
| **constraints:** equations/inequalities | $3x_1 + 2x_2 \leq 10$ |

## Example

Decision variables:

- $x_1$= # of toy soldiers
- $x_2$= # of toy trains

Objective: maximize profit

- $\$27 - \$10 - \$14 = \$3$ profit for selling one toy soldier $\Rightarrow 3x_1$ profit (in \$) for selling $x_1$ toy soldier
- $\$21 - \$9 - \$10 = \$2$ profit for selling one toy train $\Rightarrow 2x_2$ profit (in \$) for selling $x_2$ toy train

$\Rightarrow \underbrace{z = 3x_1 + 2x_2}_{\textit{objective function}}$ profit for selling $x_1$ toy soldiers and $x_2$ toy trains

Constraints:

- producing $x_1$ toy soldiers and $x_2$ toy trains requires
  (a) $1x_1 + 1x_2$ hours in the carpentry shop; there are 80 hours available
  (b) $2x_1 + 1x_2$ hours in the finishing shop; there are 100 hours available
- the number $x_1$ of toy soldiers produced should be at most 40

Variable domains: the numbers $x_1, x_2$ of toy soldiers and trains must be non-negative (sign restriction)

$$\begin{aligned} \text{Max } 3x_1 &+ 2x_2 \\ x_1 + x_2 &\le 80 \\ 2x_1 + x_2 &\le 100 \\ x_1 &\le 40 \\ x_1, x_2 &\ge 0 \end{aligned}$$

We call this a program. It is a linear program, because the objective is a linear function of the decision variables, and the constraints are linear inequalities (in the decision variables).

## Blending

A company wants to produce a certain alloy containing 30% lead, 30% zinc, and 40% tin. This is to be done by mixing certain amounts of existing alloys that can be purchased at certain prices. The company wishes to minimize the cost. There are 9 available alloys with the following composition and prices.

| Alloy | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Blend |
|---|---|---|---|---|---|---|---|---|---|---|
| Lead (%) | 20 | 50 | 30 | 30 | 30 | 60 | 40 | 10 | 10 | 30 |
| Zinc (%) | 30 | 40 | 20 | 40 | 30 | 30 | 50 | 30 | 10 | 30 |
| Tin (%) | 50 | 10 | 50 | 30 | 40 | 10 | 10 | 60 | 80 | 40 |
| Cost ($/lb) | 7.3 | 6.9 | 7.3 | 7.5 | 7.6 | 6.0 | 5.8 | 4.3 | 4.1 | minimize |

Designate a *decision* variables $x_1, x_2, \ldots, x_9$ where

$x_i$ **is the amount of Alloy** $i$ **in a unit of blend**

In particular, the decision variables must satisfy $x_1 + x_2 + \ldots + x_9 = 1$. (It is a common mistake to choose $x_i$ the absolute amount of Alloy $i$ in the blend. That may lead to a non-linear program.)

With that we can setup constraints and the objective function.

$$\begin{array}{ll} \text{Min} & 7.3x_1 + 6.9x_2 + 7.3x_3 + 7.5x_4 + 7.6x_5 + 6.0x_6 + 5.8x_7 + 4.3x_8 + 4.1x_9 = z \quad \text{[Cost]} \\ \text{s.t.} & x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 = 1 \\ & 0.2x_1 + 0.5x_2 + 0.3x_3 + 0.3x_4 + 0.3x_5 + 0.6x_6 + 0.4x_7 + 0.1x_8 + 0.1x_9 = 0.3 \quad \text{[Lead]} \\ & 0.3x_1 + 0.4x_2 + 0.2x_3 + 0.4x_4 + 0.3x_5 + 0.3x_6 + 0.5x_7 + 0.3x_8 + 0.1x_9 = 0.3 \quad \text{[Zinc]} \\ & 0.5x_1 + 0.1x_2 + 0.5x_3 + 0.3x_4 + 0.4x_5 + 0.1x_6 + 0.1x_7 + 0.6x_8 + 0.8x_9 = 0.4 \quad \text{[Tin]} \end{array}$$

Do we need all the four equations?

# Product mix (once again)

Furniture company manufactures four models of chairs. Each chair requires certain amount of raw materials (wood/steel) to make. The company wants to decide on a production that maximizes profit (assuming all produced chair are sold). The required and available amounts of materials are as follows.

| | Chair 1 | Chair 2 | Chair 3 | Chair 4 | Total available |
|---|---|---|---|---|---|
| Steel | 1 | 1 | 3 | 9 | 4,4000 (lbs) |
| Wood | 4 | 9 | 7 | 2 | 6,000 (lbs) |
| Profit | \$12 | \$20 | \$18 | \$40 | maximize |

Decision variables:

$x_i$ = the number of chairs of type $i$ produced
each $x_i$ is non-negative

Objective function:

maximize profit $z = 12x_1 + 20x_2 + 18x_3 + 40x_4$

Costraints:

at most $4,400$ lbs of steel available: $x_1 + x_2 + 3x_3 + 9x_4 \leq 4,400$
at most $6,000$ lbs of wood available: $4x_1 + 9x_2 + 7x_3 + 2x_4 \leq 6,000$

Resulting program:

$$\begin{array}{llllllllll} \text{Max} & 12x_1 & + & 20x_2 & + & 18x_3 & + & 40x_4 & = z & \text{[Profit]} \\ \text{s.t.} & x_1 & + & x_2 & + & 3x_3 & + & 9x_4 & \leq 4,400 & \text{[Steel]} \\ & 4x_1 & + & 9x_2 & + & 7x_3 & + & 2x_4 & \leq 6,000 & \text{[Wood]} \\ & & & & & & & & x_1, x_2, x_3, x_4 \geq 0 & \end{array}$$

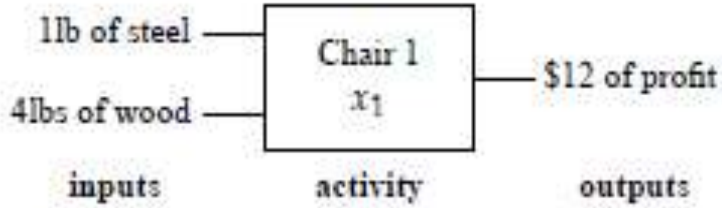
Instead of constructing the formulation as before (row-by-row), we can proceed by columns.

We can view columns of the program as **activities**. An activity has

**inputs**: materials consumed per unit of activity (1lb of steel and 4lbs of wood)

**outputs**: products produced per unit of activity ($12 of profit)

**activity level**: a level at which we operate the activity (indicated by a variable $x_1$)




Operating the activity "Chair 1" at level $x_1$ means that we produce $x_1$ chairs of type 1, each consuming 1lb of steel, 4lbs of wood, and producing $12 of profit. Activity levels are always assumed to be **non-negative**.

The materials/labor/profit consumed or produced by an activity are called **items** (correspond to rows).

The effect of an activity on items (i.e. the amounts of items that are consumed/produced by an activity) are **input-output coefficients**.

The total amount of items available/supplied/required is called the **external flow of items**.

We choose **objective** to be one of the items which we choose to maximize or minimize.

Last step is to write **material balance equations** that express the flow of items in/out of activies and with respect to the external flow.

## Example

Items: Steel
Wood
Profit

External flow of items:
Steel: 4,400lbs of available (flowing in)
Wood: 6,000lbs of available (flowing in)

Objective:

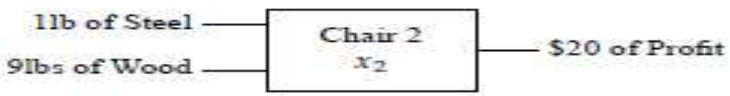
Profit: maximize (flowing out)

Activities:

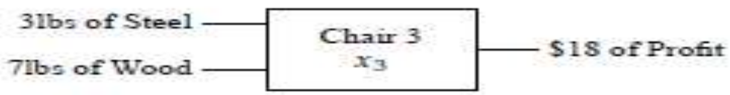
producing a chair of type $i$ where $i = 1, 2, 3, 4$, each is assigned an activity level $x_i$

Chair 1: Producing 1 chair of type 1
consumes 1 lb of Steel
4 lbs of Wood
produces \$12 of Profit

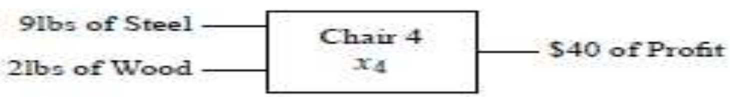

Chair 2: Producing 1 chair of type 2
consumes 1 lb of Steel
9 lbs of Wood
produces \$20 of Profit




Chair 3: Producing 1 chair of type 3
consumes 3 lbs of Steel
7 lbs of Wood
produces \$18 of Profit

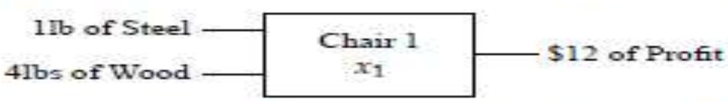



Chair 4: Producing 1 chair of type 4
consumes 9 lbs of Steel
2 lbs of Wood
produces \$40 of Profit




The material balance equations:

To see how to do this, consider activity Chair 1: consumes 1lb of Steel, 4lbs of Wood, and produces \$12 of Profit. Thus at level $x_1$, we consume $1x_1$ lbs of Steel, $4x_1$ lbs of Wood, and produce $12x_1$ dollars of Profit.




$$\begin{array}{lcl} \ldots + & 12x_1 & + \ldots \quad \text{[Profit]} \\ \ldots + & 1x_1 & + \ldots \quad \text{[Steel]} \\ \ldots + & 4x_1 & + \ldots \quad \text{[Wood]} \end{array}$$

On the right, you see the effect of operating the activity at level $x_1$. (Note in general we will adopt a different sign convention; we shall discuss is in a later example.)

Thus considering all activities we obtain:

$$\begin{array}{rcrcrcrl} 12x_1 & + & 20x_2 & + & 18x_3 & + & 40x_4 & \quad \text{[Profit]} \\ x_1 & + & x_2 & + & 3x_3 & + & 9x_4 & \quad \text{[Steel]} \\ 4x_1 & + & 9x_2 & + & 7x_3 & + & 2x_4 & \quad \text{[Wood]} \end{array}$$

Finally, we incorporate the external flow and objective: 4,400lbs of Steel available, 6,000lbs of Wood available, maximize profit:

$$\begin{array}{lrcrcrcrcll} \text{Max} & 12x_1 & + & 20x_2 & + & 18x_3 & + & 40x_4 & = & z & \quad \text{[Profit]} \\ \text{s.t.} & x_1 & + & x_2 & + & 3x_3 & + & 9x_4 & \leq & 4,400 & \quad \text{[Steel]} \\ & 4x_1 & + & 9x_2 & + & 7x_3 & + & 2x_4 & \leq & 6,000 & \quad \text{[Wood]} \end{array}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

# Linear Programming

Linear program (LP) in a standard form (maximization)

$$
\begin{array}{llcllcllcllcl}
\max & c_1x_1 & + & c_2x_2 & + & \dots & + & c_nx_n & & & \text{Objective function} \\
\text{subject to} & a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & \leq & b_1 & \\
 & a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & \leq & b_2 & \text{Constraints} \\
 & \vdots & + & \vdots & & & & \vdots & & \vdots & \\
 & a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & + & a_{mn}x_n & \leq & b_m & \\
 & & & & & & & x_1, x_2, \dots, x_n & \geq & 0 & \text{Sign restrictions}
\end{array}
$$

**Feasible solution (point)** $P = (p_1, p_2, \dots, p_n)$ is an assignment of values to the $p_1, \dots, p_n$ to variables $x_1, \dots, x_n$ that satisfies all constraints and all sign restrictions.

**Feasible region** $\equiv$ the set of all feasible points.

**Optimal solution** $\equiv$ a feasible solution with maximum value of the objective function.

**Formulating a linear program**

1. Choose decision variables
2. Choose an objective and an objective function – linear function in variables
3. Choose constraints – linear inequalities
4. Choose sign restrictions

# Example

You have \$100. You can make the following three types of investments:

**Investment A.** Every dollar invested now yields \$0.10 a year from now, and \$1.30 three years from now.

**Investment B.** Every dollar invested now yields \$0.20 a year from now and \$1.10 two years from now.

**Investment C.** Every dollar invested a year from now yields \$1.50 three years from now.

During each year leftover cash can be placed into money markets which yield 6% a year. The most that can be invested a single investment (A, B, or C) is \$50.

Formulate an LP to maximize the available cash three years from now.

**Decision variables:** $x_A, x_B, x_C$, amounts invested into Investments A, B, C, respectively
$y_0, y_1, y_2, y_3$ cash available/invested into money markets now, and in 1,2,3 years.

$$
\begin{array}{llcccccccc}
\text{Max} & y_3 \\
\text{s.t.} & x_A & + & x_B & & & + & y_0 & = & 100 \\
& 0.1x_A & + & 0.2x_B & - & x_C & + & 1.06y_0 & = & y_1 \\
& & & 1.1x_B & & & + & 1.06y_1 & = & y_2 \\
& 1.3x_A & & & + & 1.5x_C & + & 1.06y_2 & = & y_3 \\
& x_A & & & & & & & \le & 50 \\
& & & x_B & & & & & \le & 50 \\
& & & & & x_C & & & \le & 50 \\
& & & & & & & x_A, x_B, x_C, y_0, y_1, y_2, y_3 & \ge & 0
\end{array}
$$

## Post office problem

Post office requires different numbers of full-time employees on different days. Each full time employee works 5 consecutive days (e.g. an employee may work from Monday to Friday or, say from Wednesday to Sunday). Post office wants to hire minimum number of employees that meet its daily requirements, which are as follows.

| Monday | Tuesday | Wednesday | Thursday | Friday | Saturday | Sunday |
|---|---|---|---|---|---|---|
| 17 | 13 | 15 | 19 | 14 | 16 | 11 |

Let $x_i$ denote the number of employees that start working in day $i$ where $i = 1, \ldots, 7$ and work for 5 consecutive days from that day. How many workers work on Monday? Those that start on Monday, or Thursday, Friday, Saturday, or Sunday. Thus $x_1 + x_4 + x_5 + x_6 + x_7$ should be at least 17.

Then the formulation is thus as follows:

$$
\begin{array}{llllllllllllllll}
\min & x_1 & + & x_2 & + & x_3 & + & x_4 & + & x_5 & + & x_6 & + & x_7 & & \\
\text{s.t.} & x_1 & & & & & + & x_4 & + & x_5 & + & x_6 & + & x_7 & \geq & 17 \\
& x_1 & + & x_2 & & & & & + & x_5 & + & x_6 & + & x_7 & \geq & 13 \\
& x_1 & + & x_2 & + & x_3 & & & & & + & x_6 & + & x_7 & \geq & 15 \\
& x_1 & + & x_2 & + & x_3 & + & x_4 & & & & & + & x_7 & \geq & 19 \\
& x_1 & + & x_2 & + & x_3 & + & x_4 & + & x_5 & & & & & \geq & 14 \\
& & & x_2 & + & x_3 & + & x_4 & + & x_5 & + & x_6 & & & \geq & 16 \\
& & & & & x_3 & + & x_4 & + & x_5 & + & x_6 & + & x_7 & \geq & 11 \\
& & & & & & & & & & & \multicolumn{3}{r}{x_1, x_2, \ldots, x_7} & \geq & 0
\end{array}
$$

| | Monday | Tuesday | Wednesday | Thursday | Friday | Saturday | Sunday | | |
|---|---|---|---|---|---|---|---|---|---|
| Total | 1 | 1 | 1 | 1 | 1 | 1 | 1 | | minimize |
| Monday | 1 | | | 1 | 1 | 1 | 1 | $\geq$ | 17 |
| Tuesday | 1 | 1 | | | 1 | 1 | 1 | $\geq$ | 13 |
| Wednesday | 1 | 1 | 1 | | | 1 | 1 | $\geq$ | 15 |
| Thursday | 1 | 1 | 1 | 1 | | | 1 | $\geq$ | 19 |
| Friday | 1 | 1 | 1 | 1 | 1 | | | $\geq$ | 14 |
| Saturday | | 1 | 1 | 1 | 1 | 1 | | $\geq$ | 16 |
| Sunday | | | 1 | 1 | 1 | 1 | 1 | $\geq$ | 11 |

## Example 2.1-1 (The Reddy Mikks Company)

Reddy Mikks produces both interior and exterior paints from two raw materials, $M1$ and $M2$. The following table provides the basic data of the problem:

| | Tons of raw material per ton of | | Maximum daily availability (tons) |
|---|---|---|---|
| | *Exterior paint* | *Interior paint* | |
| Raw material, $M1$ | 6 | 4 | 24 |
| Raw material, $M2$ | 1 | 2 | 6 |
| Profit per ton ($1000) | 5 | 4 | |

A market survey indicates that the daily demand for interior paint cannot exceed that for exterior paint by more than 1 ton. Also, the maximum daily demand for interior paint is 2 tons.

Reddy Mikks wants to determine the optimum (best) product mix of interior and exterior paints that maximizes the total daily profit.

The LP model, as in any OR model, has three basic components.

1. Decision **variables** that we seek to determine.
2. **Objective** (goal) that we need to optimize (maximize or minimize).
3. **Constraints** that the solution must satisfy.

For the Reddy Mikks problem, we need to determine the daily amounts to be produced of exterior and interior paints. Thus the variables of the model are defined as

$$x_1 = \text{Tons produced daily of exterior paint}$$

$$x_2 = \text{Tons produced daily of interior paint}$$

To construct the objective function, note that the company wants to *maximize* (i.e., increase as much as possible) the total daily profit of both paints. Given that the profits per ton of exterior and interior paints are 5 and 4 (thousand) dollars, respectively, it follows that

$$\text{Total profit from exterior paint} = 5x_1 \text{ (thousand) dollars}$$

$$\text{Total profit from interior paint} = 4x_2 \text{ (thousand) dollars}$$

Letting $z$ represent the total daily profit (in thousands of dollars), the objective of the company is

$$\text{Maximize } z = 5x_1 + 4x_2$$

Next, we construct the constraints that restrict raw material usage and product demand. The raw material restrictions are expressed verbally as

$$\begin{pmatrix}\text{Usage of a raw material}\\ \text{by both paints}\end{pmatrix} \leq \begin{pmatrix}\text{Maximum raw material}\\ \text{availability}\end{pmatrix}$$

The daily usage of raw material $M1$ is 6 tons per ton of exterior paint and 4 tons per ton of interior paint. Thus

$$\text{Usage of raw material } M1 \text{ by exterior paint} = 6x_1 \text{ tons/day}$$

$$\text{Usage of raw material } M1 \text{ by interior paint} = 4x_2 \text{ tons/day}$$

Hence

$$\text{Usage of raw material } M1 \text{ by both paints} = 6x_1 + 4x_2 \text{ tons/day}$$

In a similar manner,

$$\text{Usage of raw material } M2 \text{ by both paints} = 1x_1 + 2x_2 \text{ tons/day}$$

Because the daily availabilities of raw materials $M1$ and $M2$ are limited to 24 and 6 tons, respectively, the associated restrictions are given as

$$\begin{aligned} 6x_1 + 4x_2 &\leq 24 \quad \text{(Raw material } M1\text{)} \\ x_1 + 2x_2 &\leq 6 \quad \text{(Raw material } M2\text{)} \end{aligned}$$

The first demand restriction stipulates that the excess of the daily production of interior over exterior paint, $x_2 - x_1$, should not exceed 1 ton, which translates to

$$x_2 - x_1 \leq 1 \quad \text{(Market limit)}$$

The second demand restriction stipulates that the maximum daily demand of interior paint is limited to 2 tons, which translates to

$$x_2 \leq 2 \quad \text{(Demand limit)}$$

An implicit (or "understood-to-be") restriction is that variables $x_1$ and $x_2$ cannot assume negative values. The **nonnegativity restrictions,** $x_1 \geq 0$, $x_2 \geq 0$, account for this requirement.

The complete Reddy Mikks model is

$$\text{Maximize } z = 5x_1 + 4x_2$$

subject to

$$\begin{aligned} 6x_1 + 4x_2 &\leq 24 && (1) \\ x_1 + 2x_2 &\leq 6 && (2) \\ -x_1 + x_2 &\leq 1 && (3) \\ x_2 &\leq 2 && (4) \\ x_1, x_2 &\geq 0 && (5) \end{aligned}$$

Any values of $x_1$ and $x_2$ that satisfy *all* five constraints constitute a **feasible solution**. Otherwise, the solution is **infeasible**. For example, the solution, $x_1 = 3$ tons per day and $x_2 = 1$ ton per day, is feasible because it does not violate *any* of the constraints, including the nonnegativity restrictions. To verify this result, substitute $(x_1 = 3, x_2 = 1)$ in the left-hand side of each constraint. In constraint (1) we have $6x_1 + 4x_2 = 6 \times 3 + 4 \times 1 = 22$, which is less than the right-hand side of the constraint (= 24). Constraints 2 through 5 will yield similar conclusions (verify!). On the other hand, the solution $x_1 = 4$ and $x_2 = 1$ is infeasible because it does not satisfy constraint (1)—namely, $6 \times 4 + 4 \times 1 = 28$, which is larger than the right-hand side (= 24).

**1. Proportionality:** This property requires the contribution of each decision variable in both the objective function and the constraints to be *directly proportional* to the value of the variable. For example, in the Reddy Mikks model, the quantities $5x_1$ and $4x_2$ give the profits for producing $x_1$ and $x_2$ tons of exterior and interior paint, respectively, with the unit profits per ton, 5 and 4, providing the constants of proportionality. If, on the other hand, Reddy Mikks grants some sort of quantity discounts when sales exceed certain amounts, then the profit will no longer be proportional to the production amounts, $x_1$ and $x_2$, and the profit function becomes nonlinear.

**2. Additivity:** This property requires the total contribution of all the variables in the objective function and in the constraints to be the direct sum of the individual contributions of each variable. In the Reddy Mikks model, the total profit equals the sum of the two individual profit components. If, however, the two products *compete* for market share in such a way that an increase in sales of one adversely affects the other, then the additivity property is not satisfied and the model is no longer linear.

**3. Certainty:** All the objective and constraint coefficients of the LP model are deterministic. This means that they are known constants—a rare occurrence in real life, where data are more likely to be represented by probabilistic distributions. In essence, LP coefficients are average-value approximations of the probabilistic distributions. If the standard deviations of these distributions are sufficiently small, then the approximation is acceptable. Large standard deviations can be accounted for directly by using stochastic LP algorithms , or indirectly by applying sensitivity analysis to the optimum solution (Section 3.6).

# Example 1: Reddy Mikks Model

Determine the best *feasible* solution among the following (feasible and infeasible) solutions of the Reddy Mikks model:

**(a)** $x_1 = 1, x_2 = 4.$

**(b)** $x_1 = 2, x_2 = 2.$

**(c)** $x_1 = 3, x_2 = 1.5.$

**(d)** $x_1 = 2, x_2 = 1.$

**(e)** $x_1 = 2, x_2 = -1.$

For the Reddy Mikks model, construct each of the following constraints and express it with a linear left-hand side and a constant right-hand side:

**(a)** The daily demand for interior paint exceeds that of exterior paint by *at least* 1 ton.

**(b)** The daily usage of raw material *M2* in tons is *at most* 6 and *at least* 3.

**(c)** The demand for interior paint cannot be less than the demand for exterior paint.

**(d)** The minimum quantity that should be produced of both the interior and the exterior paint is 3 tons.

**(e)** The proportion of interior paint to the total production of both interior and exterior paints must not exceed .5.

Ex.1

(a) $x_2 - x_1 \geq 1$ or $-x_1 + x_2 \geq 1$

(b) $x_1 + 2x_2 \geq 3$ and $x_1 + 2x_2 \leq 6$

(c) $x_2 \geq x_1$ or $x_1 - x_2 \leq 0$

(d) $x_1 + x_2 \geq 3$

(e) $\dfrac{x_2}{x_1 + x_2} \leq .5$ or $.5x_1 - .5x_2 \geq 0$

Ex. 1

(a) $(x_1, x_2) = (1, 4)$

$(x_1, x_2) \geq 0$

$6 \times 1 + 4 \times 4 = 22 < 24$

$1 \times 1 + 2 \times 4 = 9 \nleq 6$ infeasible

(b) $(x_1, x_2) = (2, 2)$

$(x_1, x_2) \geq 0$

$$\left.\begin{array}{lll} 6 \times 2 + 4 \times 2 = 20 & < 24 \\ 1 \times 2 + 2 \times 2 = 6 & = 6 \\ -1 \times 2 + 1 \times 2 = 0 & < 1 \\ 1 \times 2 = 2 & = 2 \end{array}\right\} \text{feasible}$$

$Z = 5 \times 2 + 4 \times 2 = \$18$

(c) $(x_1, x_2) = (3, 1.5)$

$x_1, x_2 \geq 0$

$$\left.\begin{array}{lll} 6 \times 3 + 4 \times 1.5 = 24 & = 24 \\ 1 \times 3 + 2 \times 1.5 = 6 & = 6 \\ -1 \times 3 + 1 \times 1.5 = -1.5 & < 1 \\ 1 \times 1.5 = 1.5 & < 2 \end{array}\right\} \text{feasible}$$

$Z = 5 \times 3 + 4 \times 1.5 = \$21$

(d) $(x_1, x_2) = (2, 1)$

$x_1, x_2 \geq 0$

$$\left.\begin{array}{lll} 6 \times 2 + 4 \times 1 = 16 & < 24 \\ 1 \times 2 + 2 \times 1 = 4 & < 6 \\ -1 \times 2 + 1 \times 1 = -1 & < 1 \\ 1 \times 1 = 1 & < 2 \end{array}\right\} \text{feasible}$$

$Z = 5 \times 2 + 4 \times 1 = \$14$

(e) $(x_1, x_2) = (2, -1)$

$x_1 \geq 0$, $x_2 < 0$, infeasible

Conclusion: (c) gives the best feasible solution

Suppose that Reddy Mikks sells its exterior paint to a single wholesaler at a quantity discount. The profit per ton is \$5000 if the contractor buys no more than 2 tons daily and \$4500 otherwise. Express the objective function mathematically. Is the resulting function linear?

Quantity discount results in the following nonlinear objective function:

$$Z = \begin{cases} 5x_1 + 4x_2, & 0 \le x_1 \le 2 \\ 4.5x_1 + 4x_2, & x_1 > 2 \end{cases}$$

The situation cannot be treated as a linear program. Nonlinearity can be accounted for in this case using mixed integer programming (chapter 9).

## GRAPHICAL LP SOLUTION

The graphical procedure includes two steps:

1. Determination of the feasible solution space.
2. Determination of the optimum solution from among all the feasible points in the solution space.

The procedure uses two examples to show how maximization and minimization objective functions are handled.

**Step 1.** *Determination of the Feasible Solution Space:*
First, we account for the nonnegativity constraints $x_1 \geq 0$ and $x_2 \geq 0$. In Figure 2.1, the horizontal axis $x_1$ and the vertical axis $x_2$ represent the exterior- and interior-paint variables, respectively. Thus, the nonnegativity of the variables restricts the solution-space area to the first quadrant that lies above the $x_1$-axis and to the right of the $x_2$-axis.

To account for the remaining four constraints, first replace each inequality with an equation and then graph the resulting straight line by locating two distinct points on it. For example, after replacing $6x_1 + 4x_2 \leq 24$ with the straight line $6x_1 + 4x_2 = 24$, we can determine two distinct points by first setting $x_1 = 0$ to obtain $x_2 = \frac{24}{4} = 6$ and then setting $x_2 = 0$ to obtain $x_1 = \frac{24}{6} = 4$. Thus, the line passes through the two points (0, 6) and (4, 0), as shown by line (1) in Figure 2.1.

Next, consider the effect of the inequality. All it does is divide the $(x_1, x_2)$-plane into two half-spaces, one on each side of the graphed line. Only one of these two halves satisfies the inequality. To determine the correct side, choose (0, 0) as a *reference point*. If it satisfies the inequality, then the side in which it lies is the

FIGURE 2.1

Feasible space of the Reddy Mikks model