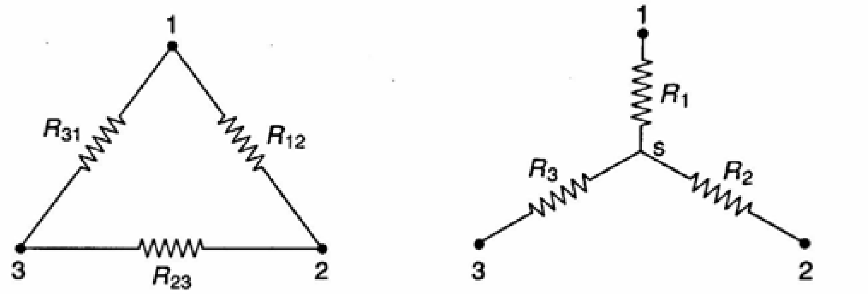
1.9 Star - Delta (Y - Δ) Transformation

We know that by using series/parallel circuit rules, we can reduce or simplify the circuit. But there are some networks in which the resistances are neither in series nor in parallel and are connected in Y- or Δ - connection. In such a situation, it is not possible to simplify the network by series/parallel circuit rules. However, converting Δ - connection into equivalent Y- connection and vice versa, a network can be simplified and application of series/parallel circuit rules is made possible. Figure (a) shows three resistances R_{12}, R_{23}, and R_{31} connected in delta, while Figure (b) shows three resistances R_1, R_2, and R_3 connected in star.



(a) Delta connection (b) Star connection

The two connections will be electrically equivalent if the resistance as measured between any pair of terminals is the same in both arrangements.

1.9.1 Delta (Δ) to Star (Y) Transformation

Refereeing to delta network in figure (a),

Resistance between terminals 1 and 2 = $R_{12} \parallel (R_{23} + R_{31}) = \dfrac{R_{12}(R_{23}+R_{31})}{R_{12}+R_{23}+R_{31}}$

Referring to star network in fig (b),

Resistance between terminals 1 and 2 = $R_1 + R_2$

$$R_1 + R_2 = \frac{R_{12}(R_{23} + R_{31})}{R_{12} + R_{23} + R_{31}} \tag{1.8}$$

Similarly, it can be shown that between terminals 2 and 3 as well as 3 and 1

$$R_2 + R_3 = \frac{R_{23}(R_{31} + R_{12})}{R_{12} + R_{23} + R_{31}} \tag{1.9}$$

and

$$R_3 + R_1 = \frac{R_{31}(R_{12} + R_{23})}{R_{12} + R_{23} + R_{31}} \tag{1.10}$$

Subtracting Eq. (1.9) from Eq. (1.8),

$$R_1 - R_3 = \frac{R_{12}R_{31} - R_{23}R_{31}}{R_{12} + R_{23} + R_{31}} \tag{1.11}$$

Adding Eqs (1.11) and (1.10),

$$R_1 = \frac{R_{12}R_{31}}{R_{12} + R_{23} + R_{31}} \tag{1.12}$$

Similarly, $$R_2 = \frac{R_{23}R_{12}}{R_{12} + R_{23} + R_{31}} \tag{1.13}$$

$$R_3 = \frac{R_{31}R_{23}}{R_{12} + R_{23} + R_{31}} \tag{1.14}$$

Thus, star resistance connected to terminal is equal to the product of the two delta resistances connected to same terminal divided by the sum of the delta resistances.

1.9.2 Star (Y) to Delta (Δ) Transformation

Multiplying Eqs (1.12) and (1.13),

$$R_1R_2 = \frac{R_{12}^2R_{23}R_{31}}{(R_{12} + R_{23} + R_{31})^2} \tag{1.15}$$

Multiplying Eqs (1.13) and (1.14),

$$R_2R_3 = \frac{R_{23}^2R_{31}R_{12}}{(R_{12} + R_{23} + R_{31})^2} \tag{1.16}$$

Multiplying Eqs (1.14) and (1.12),

$$R_3R_1 = \frac{R_{31}^2R_{12}R_{23}}{(R_{12} + R_{23} + R_{31})^2} \tag{1.17}$$

Adding Eqs (1.15), (1.16), and (1.17),

$$R_1R_2 + R_2R_3 + R_3R_1 = \frac{R_{12}R_{23}R_{31}(R_{12} + R_{23} + R_{31})}{(R_{12} + R_{23} + R_{31})^2}$$

or $$R_1R_2 + R_2R_3 + R_3R_1 = \frac{R_{12}R_{23}R_{31}}{R_{12} + R_{23} + R_{31}}$$

or $$R_1R_2 + R_2R_3 + R_3R_1 = R_{12}R_3 \quad \left[\because R_3 = \frac{R_{23}R_{31}}{R_{12} + R_{23} + R_{31}}\right]$$

Hence, $$R_{12} = R_1 + R_2 + \frac{R_1 R_2}{R_3} \quad (1.18)$$

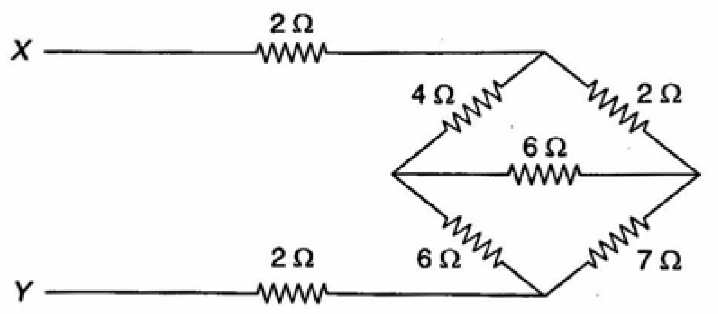
Similarly, $$R_{23} = R_2 + R_3 + \frac{R_2 R_3}{R_1} \quad (1.19)$$

and $$R_{31} = R_3 + R_1 + \frac{R_3 R_1}{R_2} \quad (1.20)$$

Thus, the delta resistance between the two terminals is the sum of the two star resistances connected to the same terminals plus the product of the two resistances divided by the remaining third star resistance.

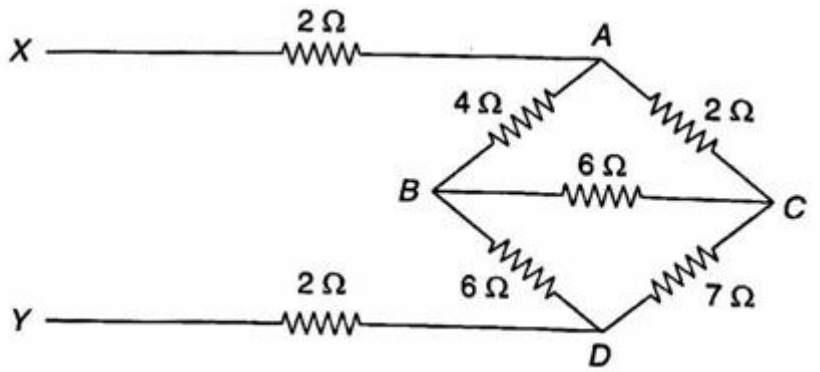
Problems

1. Find the equivalent resistance between the terminals X and Yin the network shown

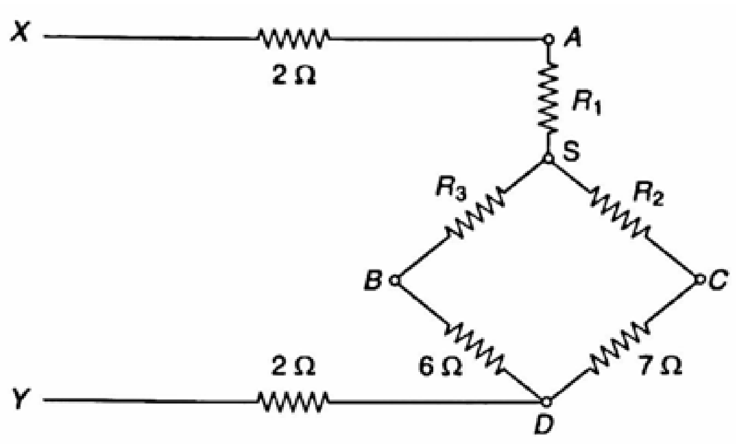


Solution

Marking different nodes we get the following circuit



Converting the delta connection formed by 4 Ω, 2 Ω, and 6 Ω resistors into equivalent star network, i.e., Δ ABC → Y ABC, we get the following circuit:

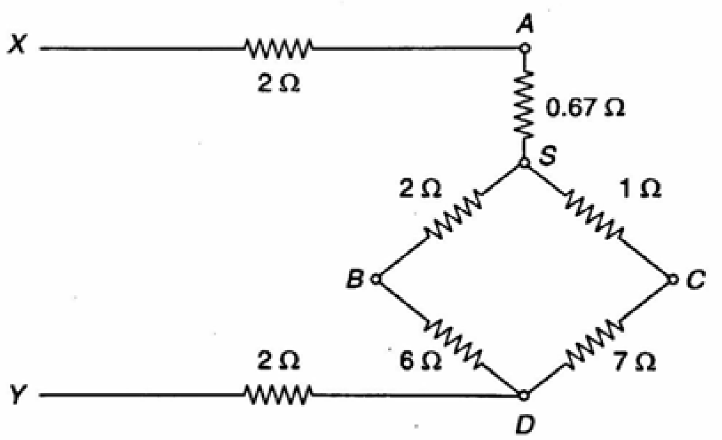


$$R_1 = \frac{4 \times 2}{4+2+6} = 0.67\ \Omega$$

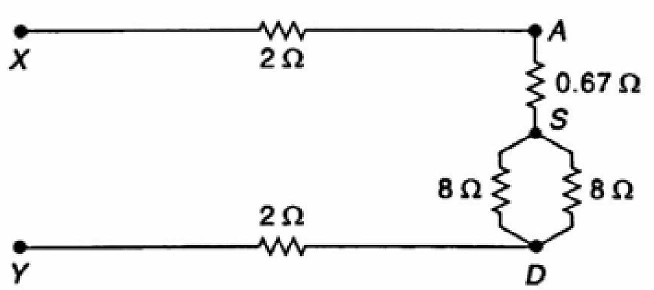
$$R_2 = \frac{6 \times 2}{4+2+6} = 1\ \Omega$$

$$R_3 = \frac{6 \times 4}{4+2+6} = 2\ \Omega$$

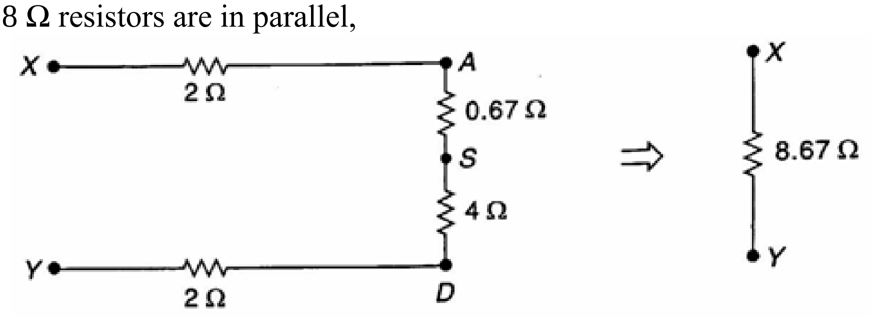
The simplified circuit will be as shown in circuit below



In Fig., resistors 6 Ω and 2 Ω are in series. Also resistors I Ω and 7 Ω are in series.

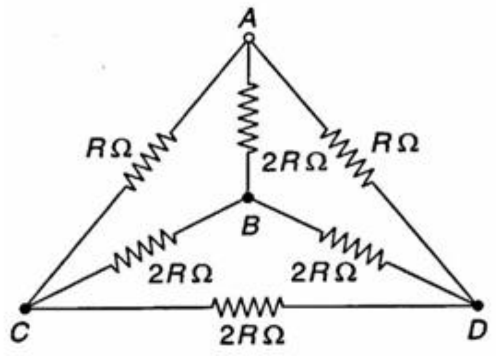


Two 8 Ω resistors are in parallel,



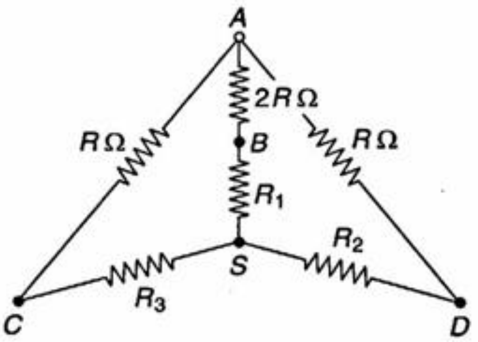
Resistance, R_{XY} = 8.67 Ω

2. Find the equivalent resistance between the terminals A and B in the network shown



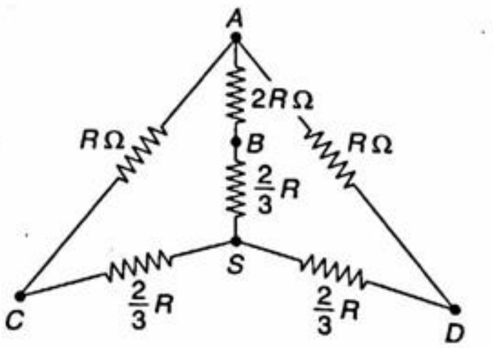
Solution

Converting delta connection formed by three '2R' Ω resistors into equivalent star network

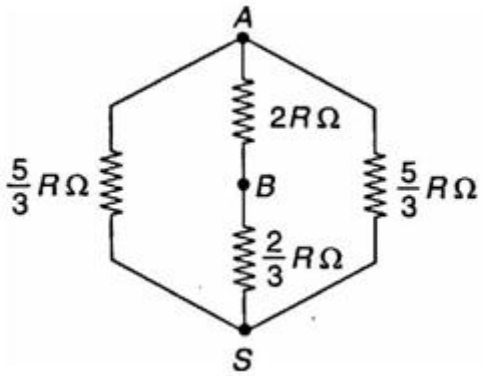


We have $R_1 = R_2 = R_3 = \dfrac{2R \times 2R}{2R + 2R + 2R} = \dfrac{2}{3}R$

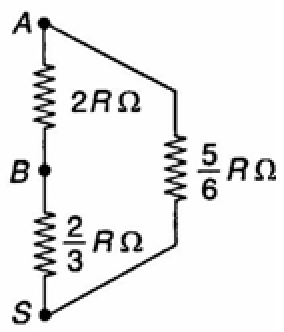
The simplified circuit is shown below



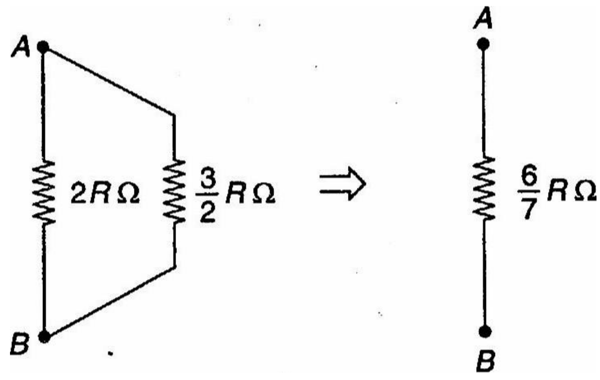
In branch ACS, R Ω and 2/3R Ω are in series. Also in branch ADS, R Ω and 2/3R Ω resistors are in series.



In Fig., two 5/3R Ω resistors are in parallel.

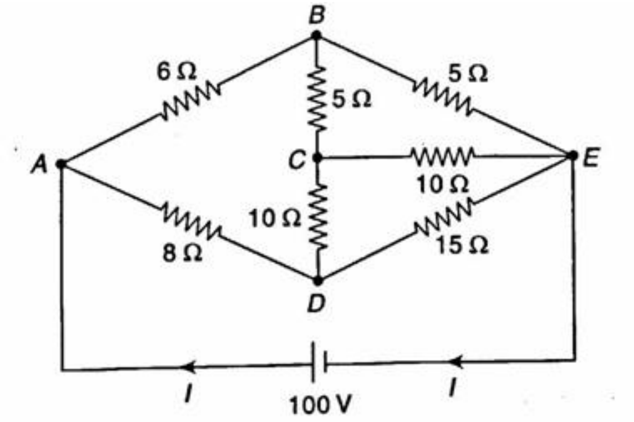


In Fig., resistors 2/3R Ω and 5/6R Ω are in series.



Resistance R_{AB} = 6/7 R Ω

3. Find the current I in the network shown



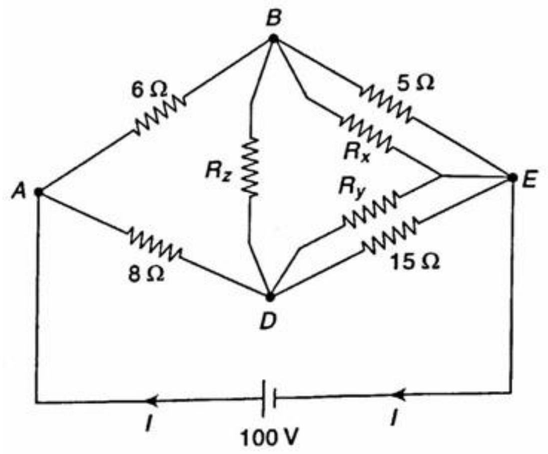
Solution

Converting connection formed by 5 Ω and two 10 Ω resistors into equivalent delta network (between nodes EBD)

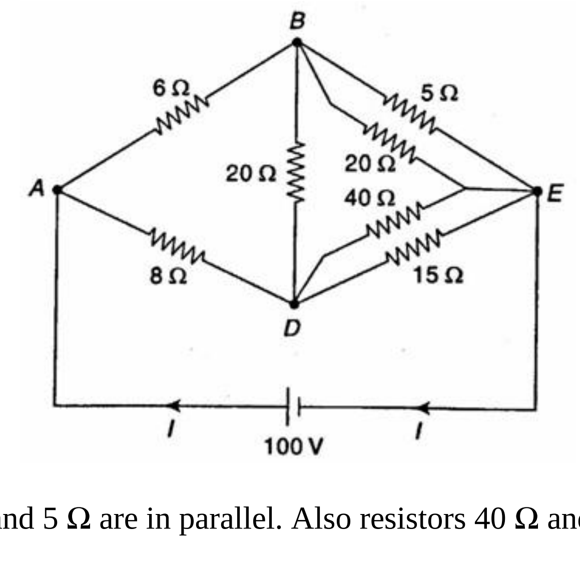
$$R_x = 5 + 10 + \frac{5\times10}{10} = 20\ \Omega,$$

$$R_y = 10 + 10 + \frac{10\times10}{5} = 40\ \Omega,$$

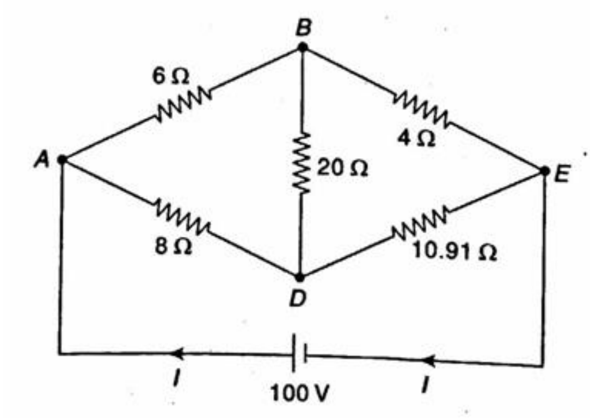
$$R_z = 5 + 10 + \frac{5\times10}{10} = 20\ \Omega$$



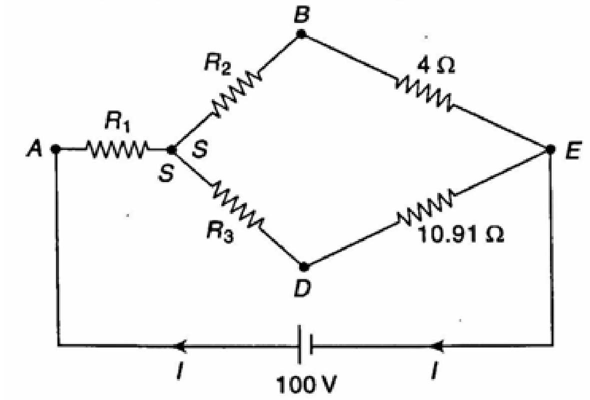
The simplified network is shown below



Resistors 20 Ω and 5 Ω are in parallel. Also resistors 40 Ω and 15 Ω are in parallel.



Converting the delta connection formed by 6 Ω, 20 Ω, and 8 Ω resistors into equivalent star network, i.e., Δ ABD → Y ABD, we get the following network:

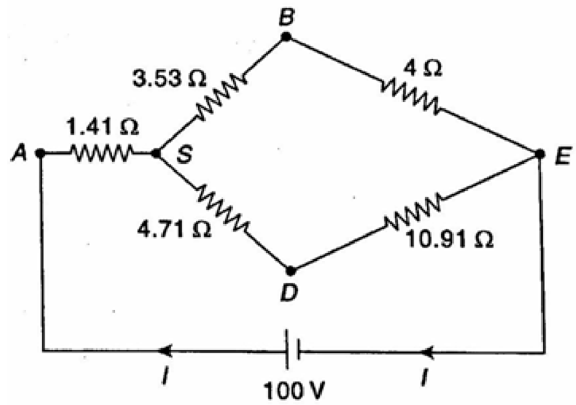


$$R_1 = \frac{6 \times 8}{6+20+8} = 1.41\ \Omega,$$

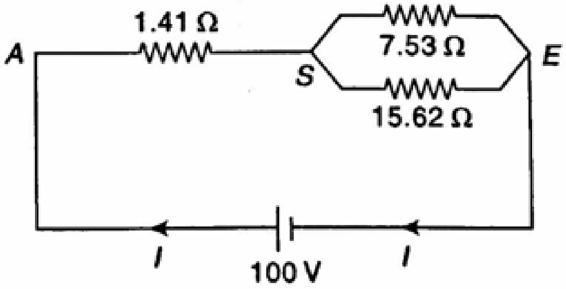
$$R_2 = \frac{6 \times 20}{6+20+8} = 3.53\ \Omega,$$

$$R_3 = \frac{20 \times 8}{6+20+8} = 4.71\ \Omega$$

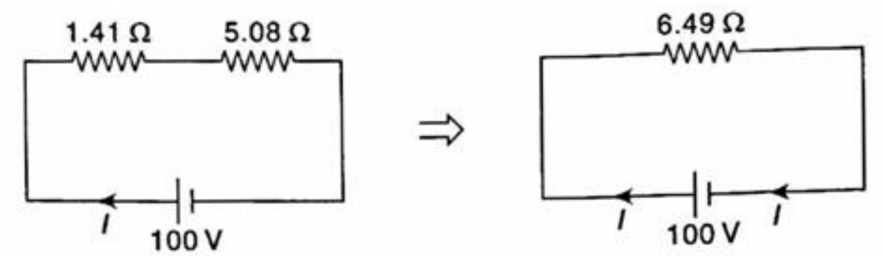
The simplified network is shown



Resistors 3.53 Ω and 4 Ω are in series. Also resistors 4.71 Ω and 10.91 Ω are in series.



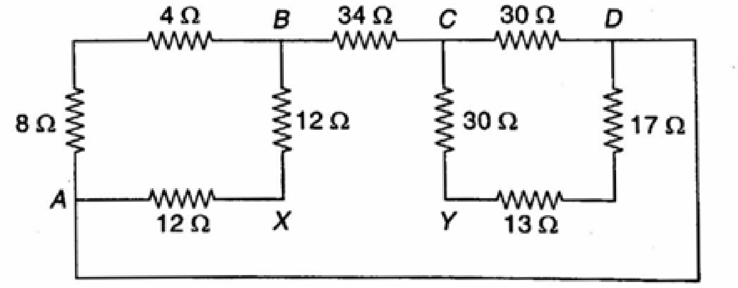
Resistors 7.53 Ω and 15.62 Ω are in parallel, both in series with 1.41 Ω.



By Ohm's Law

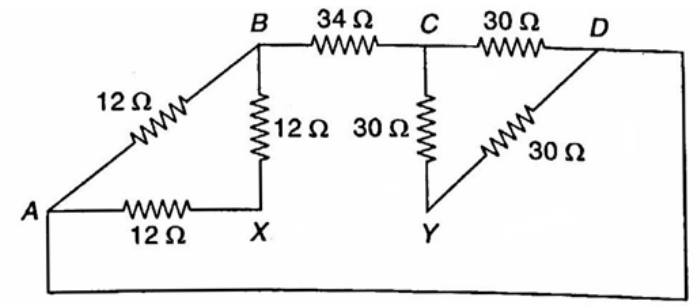
$$I = \frac{100}{6.49} = 15.41\ A$$

4. Find the equivalent resistance between the terminals X and Y in the network shown

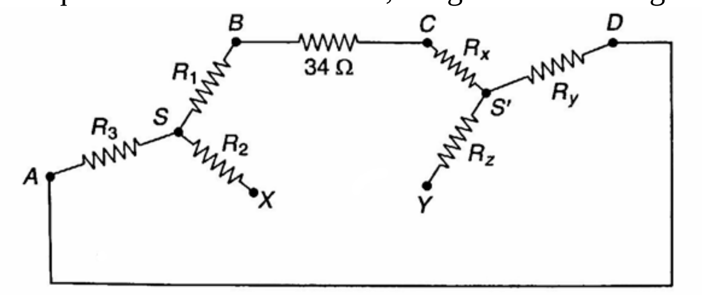


Solution

Resistors 8 Ω and 4 Ω are in series. Also resistors 17 Ω and 13 Ω are in series.



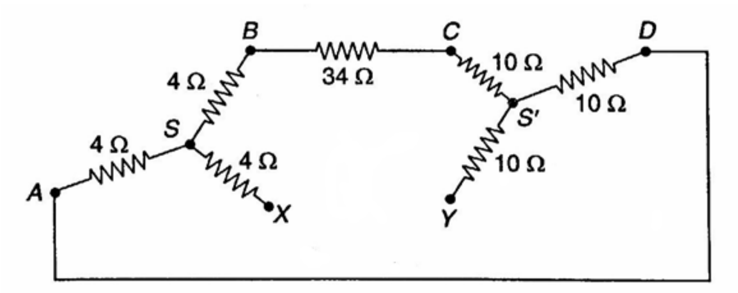
Converting the delta connections formed by three 12 Ω resistors (ΔABX) and three 30 Ω resistors (ΔCDY) into equivalent star connections, we get the following network:



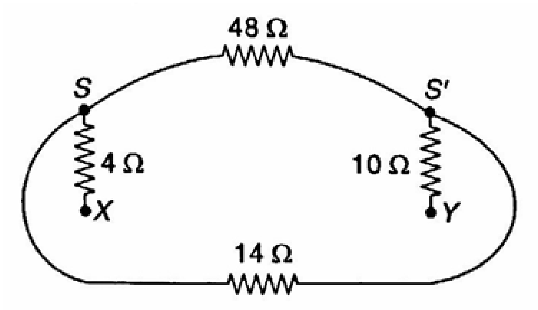
$$R_1 = R_2 = R_3 = \frac{12 \times 12}{12 + 12 + 12} = 4\ \Omega$$

$$R_x = R_y = R_z = \frac{30 \times 30}{30 + 30 + 30} = 10\ \Omega$$

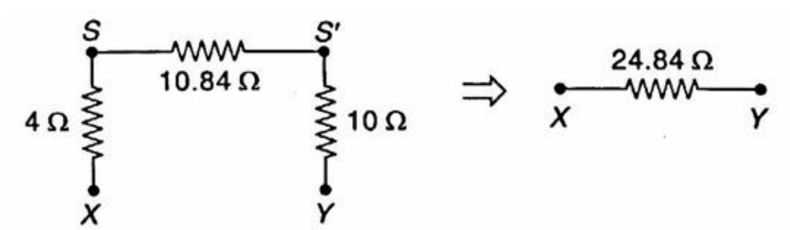
The simplified network is shown below



In branch SBCS', 4 Ω, 34 Ω, and 10 Ω 48 Ω are in series. Also in branch SADS', 4 Ω and 10 Ω are in series.



In Fig, 48 Ω and 14 Ω are in parallel.



Resistance R_{XY} = 24.84 Ω

Maximum Power Transfer Theorem

Although applicable to all branches of electrical engineering, this theorem is particularly useful for analysing communication networks. The overall efficiency of a network supplying maximum power to any branch is 50 per cent. For this reason, the application of this theorem to power transmission and distribution networks is limited because, in their case, the goal is high efficiency and not maximum power transfer.

However, in the case of electronic and communication networks, very often, the goal is either to receive or transmit maximum power (through at reduced efficiency) specially when power involved is only a few milliwatts or microwatts. Frequently, the problem of maximum power transfer is of crucial significance in the operation of transmission lines and antennas.

As applied to d.c. networks, this theorem may be stated as follows :

A resistive load will abstract maximum power from a network when the load resistance is equal to the resistance of the network as viewed from the output terminals, with all energy sources removed leaving behind their internal resistances.

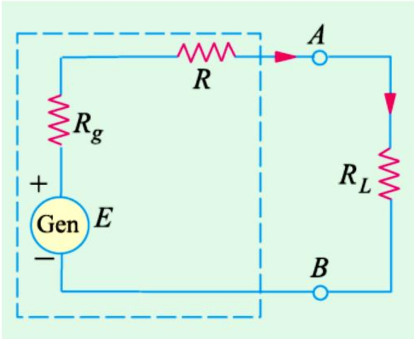


Fig. 2.230

In Fig. 2.230 (*a*), a load resistance of R_L is connected across the terminals *A* and *B* of a network which consists of a generator of e.m.f. *E* and internal resistance R_g and a series resistance *R* which, in fact, represents the lumped resistance of the connecting wires. Let $R_i = R_g + R$ = internal resistance of the network as viewed from *A* and *B*.

According to this theorem, R_L will abstract maximum power from the network when $R_L = R_i$.

Proof. Circuit current $\quad I = \dfrac{E}{R_L + R_i}$

Power consumed by the load is

$$P_L = I^2 R_L = \frac{E^2 R_L}{(R_L + R_i)^2} \qquad \text{...}(i)$$

For P_L to be maximum, $\dfrac{dP_L}{dR_L} = 0.$

Differentiating Eq. (*i*) above, we have

$$\frac{dP_L}{dR_L} = E^2\left[\frac{1}{(R_L+R_i)^2} + R_L\left(\frac{-2}{(R_L+R_i)^3}\right)\right] = E^2\left[\frac{1}{(R_L+R_i)^2} - \frac{2R_L}{(R_L+R_i)^3}\right]$$

$$\therefore \quad 0 = E^2\left[\frac{1}{(R_L+R_i)^2} - \frac{2R_L}{(R_L+R_i)^3}\right] \quad \text{or} \quad 2R_L = R_L + R_i \quad \text{or} \quad R_L = R_i$$

It is worth noting that under these conditions, the voltage across the load is hold the open-circuit voltage at the terminals *A* and *B*.

$$\therefore \quad \text{Max. power is } P_{L\,\text{max.}} = \frac{E^2 R_L}{4R_L^2} = \frac{E^2}{4R_L} = \frac{E^2}{4R_i}$$

Let us consider an a.c. source of internal impedance $(R_1 + j\,X_1)$ supplying power to a load impedance $(R_L + jX_L)$. It can be proved that maximum power transfer will take place when the modules of the load impedance is equal to the modulus of the source impedance *i.e.* $|Z_L| = |Z_1|$

Where there is a completely free choice about the load, the maximum power transfer is obtained when load impedance is the complex conjugate of the source impedance. For example, if source impedance is $(R_1 + jX_1)$, then maximum transfer power occurs, when load impedance is $(R_1 - jX_1)$. It can be shown that under this condition, the load power is $= E^2/4R_1$.

Power Transfer Efficiency

If P_L is the power supplied to the load and P_T is the total power supplied by the voltage source, then power transfer efficiency is given by $\eta = P_L/P_T$.

Now, the generator or voltage source E supplies power to both the load resistance R_L and to the internal resistance $R_i = (R_g + R)$.

$$P_T = P_L + P_i \quad \text{or} \quad E \times I = I^2 R_L + I^2 R_i$$

$$\therefore \qquad \eta = \frac{P_L}{P_T} = \frac{I^2 R_L}{I^2 R_L + I^2 R_i} = \frac{R_L}{R_L + R_i} = \frac{1}{1 + (R_i / R_L)}$$

The variation of η with R_L is shown in Fig. 2.235 (*a*). The maximum value of η is unity when $R_L = \infty$ and has a value of 0.5 when $R_L = R_i$. It means that under maximum power transfer conditions, the power transfer efficiency is only 50%. As mentioned above, maximum power transfer condition is important in communication applications but in most power systems applications, a 50% efficiency is undesirable because of the wasted energy. Often, a compromise has to be made between the load power and the power transfer efficiency. For example, if we make $R_L = 2\,R_i$, then

$$P_L = 0.222\ E^2/R_i \quad \text{and} \quad \eta = 0.667.$$

It is seen that the load power is only 11% less than its maximum possible value, whereas the power transfer efficiency has improved from 0.5 to 0.667 *i.e.* by 33%.

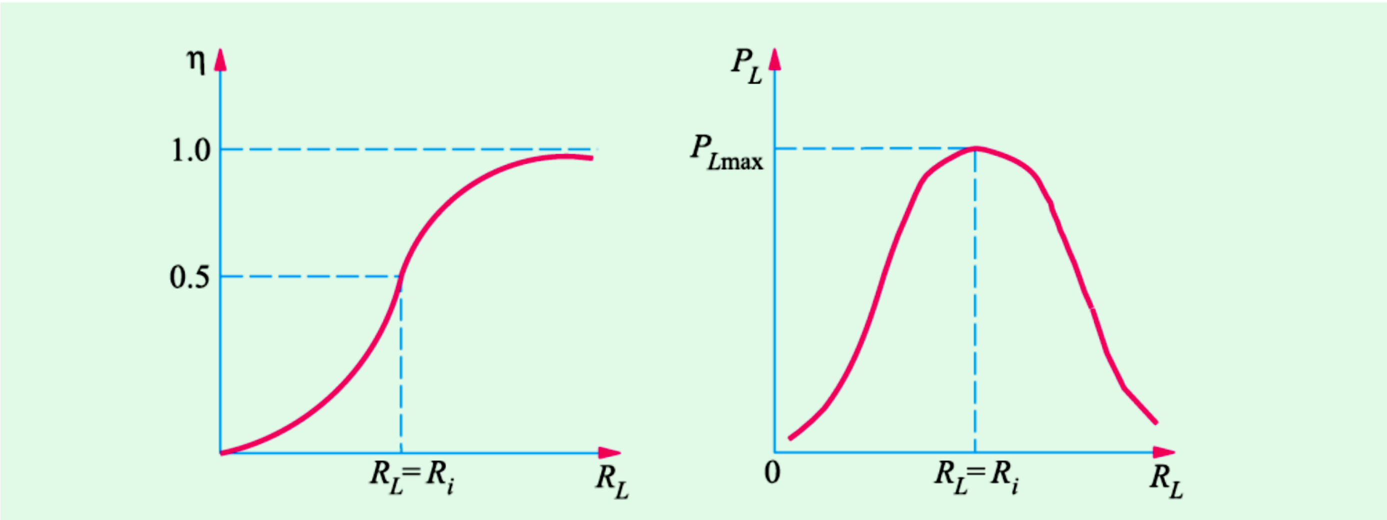


Fig. 2.235