R (b) = { x }.

R (c) = { x }.

R (d) = { x }.

R (e) = { x }.

R is a function.

Graphically we can represent this function as shown in Fig. 4.9 (c)

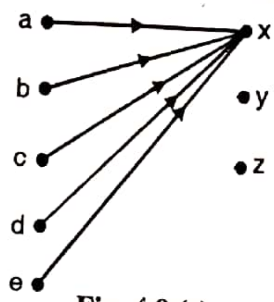

Fig. 4.9 (c)

Ran (R) = { x } i.e. Range of R is { x }.

(v) R = { (a, z), (b, y), (c, z), (c, y), (c, x), (b, x) }.

R (a) = { z }.

R (b) = { y, x }.

R (c) = { z, y, x }.

R is not a function, since R (a), R (b) and R (c) are not giving single value.

Example 5 : If f is the mod - 12 function, compute each of the following :

(i) f (1259 + 743) (ii) f (1259) + f (743)

(iii) f (2.319) (iv) 2 · f (319)

Solution :

(i) $f(1259 + 743) = f(2002)$

$= 2002 \% 12 = 10$

(ii) $f(1259) + f(743) = 1259\% \ 12 + 743\%1 = 11 + 11$

$= 22$

(iii) $f(2.319) = 2.319 \% 12$

$= 2.319$

(iv) $2 \cdot f(319) = 2 \times 319 \% 12 = 2 \times 7$

$= 14$

Syllabus Topic : Types of fucntions

4.2 Types of Functions :

4.2.1 Onto or Surjective Function :

MU - Dec. 10, Dec. 12

A function from A to B is said to be an **onto** function if every element of B is the image of one or more elements of A. **Onto function** is also called **surjective** or 'f' is **ONTO** if Ran (f) = B.

MU - May 10

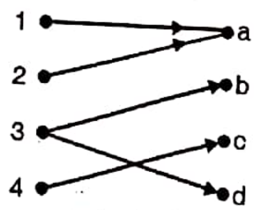

Fig. 4.10

Example : Let A = { 1, 2, 3, 4 } and B = { a, b, c, d }.

and f = { (1, a), (2, a), (3, d), (4, c), (3, b) }.

f (1) = a, f (2) = a, f (3) = d,

f (4) = c, f (3) = b.

Ran (f) = { a, b, c, d } = B.

So this function is onto or surjective function.

4.2.2 One to One or Injective Function : MU - May 10

A function from A to B is said to be a **one to one function** if no two elements of A have the same image. One to one function is also called as **injective function.**

Example : Let A = { 1, 2, 3, 4 }, and B = { a, b , c, d, e }.

and f = { (1, a), (2, e), (3, c), (4, d) }.

f (1) = a

f (2) = e

f (3) = c

f (4) = d

Given function f is one to one or injective function.

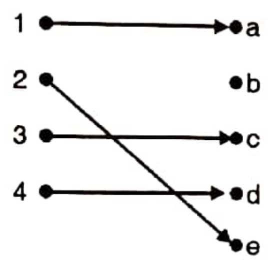

Fig. 4.11

4.2.3 One to One onto Function or Bijective Function :

A function from A to B is said to be a **one to one onto function** if it is both an onto and one to one function. One to one onto function is also called as **bijective function.**

Example : Let A = { 1, 2, 3, 4 }, B = { a, b, c, d }.

and f = { (1, b), (2, c), (3, d), (4, a) }.

f(1) = b,

f(2) = c,

f(3) = d,

f(4) = a.

Given function f is one to one onto or bijective function.

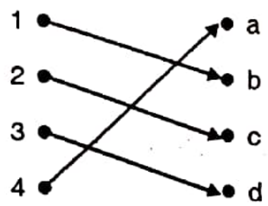

Fig. 4.12

4.2.4 Everywhere Defined Function :

A function from A to B is said to be **everywhere defined** if Dom (f) = A.

Example : Let A = {1, 2, 3 }, B = { a, b, c }.

and f = { (1, c), (2, b), (3, a) }.

f (1) = c,

f (2) = b,

f (3) = a.

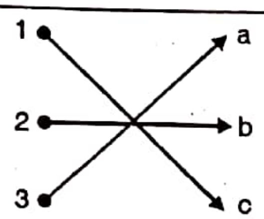

Fig. 4.13

Dom (f) = { 1, 2, 3 }. Thus given function f is everywhere defined function.

Theorem 1 : If $f : A \rightarrow B$ and $g : B \rightarrow C$ are injective functions then $A \rightarrow C$ is an injective function.

Theorem 2 : If $f : A \rightarrow B$ and $g : B \rightarrow C$ are surjective functions then $A \rightarrow C$ is a surjective function.

4.2.5 Exercise Set - 2 (Solved) :

Example 1 : Let $A = \{ a_1, a_2, a_3 \}$, $B = \{ b_1, b_2, b_3 \}$, $C = \{ c_1, c_2 \}$, $D = \{ d_1, d_2, d_3, d_4 \}$. Consider the following four functions from A to B, A to D, B to C and D to B respectively.

(a) $f_1 = \{ (a_1, b_2), (a_2, b_3), (a_3, b_1) \}$. (b) $f_2 = \{ (a_1, d_2), (a_2, d_1), (a_3, d_4) \}$.

(c) $f_3 = \{ (b_1, c_2), (b_2, c_2), (b_3, c_1) \}$. (d) $f_4 = \{ (d_1, b_1), (d_2, b_2), (d_3, b_1) \}$.

Determine whether each function is one to one, whether each function is onto and whether each function is everywhere defined.

Solution :

(a)

$$f_1 = \{ (a_1, b_2), (a_2, b_3), (a_3, b_1) \}.$$

$$f_1 (a_1) = b_2, f_1 (a_2) = b_3, f_1 (a_3) = b_1$$

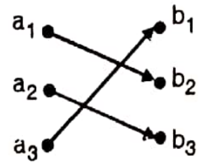

Fig. 4.14 (a)

f_1 is everywhere defined because Dom $(f_1) = A$.

f_1 is onto because Range $(f_1) = B$.

f_1 is one to one function because no two elements of set B have same image.

So f_1 is surjective, injective, bijective and everywhere defined function.

(b)

$$f_2 = \{ (a_1, d_2), (a_2, d_1), (a_3, d_4) \}.$$

$$f_2 (a_1) = d_2, f_2 (a_2) = d_1, f_2 (a_3) = d_4.$$

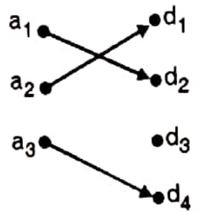

Fig. 4.14 (b)

f_2 is everywhere defined function because Dom $(f_2) = A$.

f_2 is not onto function because Ran $(f_2) \neq D$.

f_2 is one to one function or injective function because no two elements of set D have same image.

(c)

$$f_3 = \{(b_1, c_2), (b_2, c_2), (b_3, c_1) \}.$$

$$f_3 (b_1) = c_2, f_3 (b_2) = c_2, f_3 (b_3) = c_1.$$

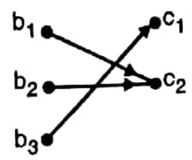

Fig. 4.14 (c)

f_3 is everywhere defined function because Dom $(f_3) = B$.

f_3 is onto surjective function because Ran $(f_3) = C$.

f_3 is not one to one injective function because two element of set B have same image.

Syllabus Topic : Identity

4.4 Identity :

Definition :

Let A be a non-empty set. Then we can always define a function
$f : A \to A$ (i. e. B = A) as $f(a) = a$ for all $a \in A$
f is called the **Identify** function on A and is denoted by I_A.

$\therefore \quad I_A = \{ (a,a) \mid a \in A \}$

Example :

Let $A = \{ 1, 2, 3 \}$ and $f : A \to A$ is identify function since

$f(1) = 1 \qquad f(2) = 2 \qquad f(3) = 3$

Syllabus Topic : Inverse

4.5 Inverse Function :

The concept of inverse of a function is a analogous to that of the converse of a relation.

4.5.1 Definition :

Let $f : A \to B$, be function, then
$f^{-1} : B \to A$ is called the **inverse** mapping of f
f^{-1} is the set defined as
$f^{-1} = \{ (b, a) \mid (a, b) \in f \}$.

Note : (i) In general the inverse f^{-1} of a function $f : A \to B$, need not be a function. It may be a relation.

Let $f : A \to A$. If there, exists a function $g : A \to A$ such that $g \circ f = f \circ g = IA$, then g is called the inverse of the function f and is denoted by f^{-1}, read as "f inverse".

Let $f : A \to A$ be such that $f(a) = b$. Then when it exists f^{-1} is a function from A to A. Such that $f^{-1}(b) = a$. Note that f^{-1} "undoes" what f does.

(ii) If $f : A \to B$ is a bijection and $f(a) = b$, then $a = f^{-1}(b)$ where $a \in A$ and $b \in B$

Example :

Let $A = \{ 1, 2, 3 \}$ and f be the function defined on A such that $f(1) = 2$, $f(2) = 3$, $f(3) = 1$. Then $f^{-1} : A \to A$ is defined by

$$f^{-1}(1) = \{ 3 \}.$$
$$f^{-1}(3) = \{ 2 \}.$$
$$f^{-1}(2) = \{ 1 \}.$$

$$f^{-1} = \{(1, 3)(3, 2), (2, 1)\}.$$

Note : A function f for which f^{-1} exists called Invertible.

General Case :

Let $f : A \rightarrow B$. If there exists a function $g : B \rightarrow A$ such that $g \circ f = I_A$ and $f \circ g = I_B$ then g is called the **inverse** of f and is denoted by f^{-1}.

Example :

Let $A = \{1, 2, 3\}$, $B = \{a, b, c\}$. Define $f : A \rightarrow B$ by $f(1) = a$, $f(2) = b$, $f(3) = c$.

Then $g : B \rightarrow A$ defined by $g(a) = 1$, $g(b) = 2$, $g(c) = 3$ is the inverse of f.

$$f = \{(1, a), (2, b), (3, c)\}.$$

$$g = \{(a, 1), (b, 2), (c, 3)\}.$$

$$\left.\begin{aligned} g \circ f(1) &= 1 \\ g \circ f(2) &= 2 \\ g \circ f(3) &= 3 \end{aligned}\right\} \quad g \circ f = I_A$$

$$\left.\begin{aligned} f \circ g(a) &= a \\ f \circ g(b) &= b \\ f \circ g(c) &= c \end{aligned}\right\} \quad f \circ g = I_B$$

$$\therefore g = f^{-1} = \{(a, 1), (b, 2), (c, 3)\}.$$

Example :

1. Let

$$A = \{a, b, c\}.$$

$$B = \{1, 2, 3\} \text{ and}$$

$$f = \{(a, 1), (b, 3), (c, 2)\}.$$

We can say that f is both one to one and onto

$$f^{-1} = \{(1, a), (2, c), (3, b)\} \text{ is a function from B to A}$$

2. Let R be a set of real numbers and $f : R \rightarrow R$ be given by

$$f(x) = x + 5 \quad \forall x \in R \text{ (i.e.)}$$

$$f = \{(x, x + 5) \mid x \in R\}.$$

then $$f^{-1} = \{(x + 5, x) \mid x \in R\} \text{ is a function from R to R.}$$

4.5.2 Theorem :

Theorem 1 : If $f : A \rightarrow B$ be both one to one and onto, then $f^{-1} : B \rightarrow A$ is both one to one and onto

MU - May 04, 05, Dec. 13, May 15

Proof :

Let $f : A \rightarrow B$ be both one to one and onto then there exists elements.

$a_1, a_2 \in A$ and elements

$b_1, b_2 \in B$

Such that $f(a_1) = b_1$ and $f(a_2) = b_2$

$$a_1 = f^{-1}(b_1) \text{ and } \quad a_2 = f^{-1}(b_2)$$

Now let $f^{-1}(b_1) = f^{-1}(b_2)$

$\Rightarrow \quad a_1 = a_2$

$\Rightarrow \quad f(a_1) = f(a_2)$

$b_1 = b_2$

$\therefore$ f^{-1} is one to one, again since f is onto, for $b \in B$, there is some element $a \in A$, such that

$f(a) = b$

$\Rightarrow \quad a = f^{-1}(b)$

$\Rightarrow \quad f^{-1}$ is onto

Hence f^{-1} is both one-one and onto

Theorem 2 : The inverse of an invertible mapping is unique.

Proof :

Let $f : A \rightarrow B$

By any invertible mapping. If possible let

$g : B \rightarrow A$ and

$h : B \rightarrow A$

be two different inverse mappings of f.

Let $b \in B$ and

$g(b) = a_1, \quad a_1 \in A$

$h(b) = a_2, \quad a_2 \in A$

Now $g(b) = a_1$

$\Rightarrow \quad b = f(a_1)$

and $h(b) = a_2$

$\Rightarrow \quad b = f(a_2)$

Further more $b = f(a_1)$ and $b = f(a_2)$

$\Rightarrow \quad f(a_1) = f(a_2)$

$\Rightarrow \quad a_1 = a_2 \quad (\because f \text{ is one to one})$

This proves that $g(b) = h(b)\ \forall\ b \in B$ thus, the inverse of f is unique. This completes the proof of the theorem.

Theorem 3 : If $f : A \rightarrow B$ is an invertible mapping, then

$f \circ f^{-1} = I_B$ and

$f^{-1} \circ f = I_A$

Proof :

f is invertible, then f^{-1} is defined by

$f(a) = b \quad \Leftrightarrow \quad f^{-1}(b) = a$

where $a \in A$ and $b \in B$

To prove that $f \circ f^{-1} = I_B$

Let $b \in B$ and $f^{-1}(b) = a, a \in A$

then $f \circ f^{-1}(b) = f[f^{-1}(b)] = f(a) = b$

$\therefore$ $f \circ f^{-1}(b) = b \quad \forall\, b \in B$

$\Rightarrow$ $f \circ f^{-1} = I_B$

Now $f^{-1} f(a) = f^{-1}[f(a)] = f^{-1}(b) = a$

$\therefore$ $f^{-1} \circ f(a) = a \quad \forall\, a \in A$

$\Rightarrow$ $f^{-1} \circ f = I_A$

Theorem 4 : If $f : A \to B$ and $g : B \to C$ are both one-one and onto, then $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$

MU - Dec. 08, Dec. 15

Proof :

$f : A \to B$ is one to one and onto

$g : B \to C$ is one to one and onto

$\therefore$ $g \circ f : A \to C$ is one to one and onto

$\Rightarrow$ $(g \circ f)^{-1} : C \to A$ is one to one and onto.

Let $a \in A$, then there exists an element $b \in B$ such that $f(a) = b$

$\Rightarrow$ $a = f^{-1}(b)$

Now $b \in B \Rightarrow$ there exists an element $c \in C$ such that $g(b) = C$

$\Rightarrow$ $b = g^{-1}(c)$

Then $(g \circ f)(a) = g[f(a)] = g[b] = c$

$\Rightarrow$ $a = (g \circ f)^{-1}(c)$ ...(i)

$(f^{-1} \circ g^{-1})(c) = f^{-1}[g^{-1}(c)]$

$= f^{-1}(b) = a$

$\Rightarrow$ $a = (f^{-1} \circ g^{-1})(c)$ ...(ii)

Combining (i) and (ii) we have,

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}$$

4.5.3 Exercise Set - 4 (Solved) :

Example 1 : Let f be a function, from A = { 1, 2, 3, 4 } to B = { a, b, c, d }. Determine whether f^{-1} is a function.

(i) f = { (1, a), (2, a), (3, c), (4, d) }. (ii) f = { (1, a), (2, c), (3, b), (4, d) }.

Solution :

$f(1) = \{a\}$ $\quad$ $f(2) = \{a\}$.

$f(3) = \{c\}$ $\quad$ $f(4) = \{d\}$.

$f^{-1}(a) = \{1, 2\}$

$$f^{-1}(c) = \{3\}.$$
$$f^{-1}(d) = \{4\}.$$

f^{-1} is not a function, since $f^{-1}(a) = \{1, 2\}$. Hence f is not invertable.

(ii) $f(1) = \{a\}$ $f(2) = \{c\}$.

$f(3) = \{b\}$ $f(4) = \{d\}$.

$f^{-1}(a) = \{1\}$ $f^{-1}(c) = \{2\}$.

$f^{-1}(b) = \{3\}$ $f^{-1}(d) = \{4\}$.

f^{-1} is a function. Hence f is invertible.

Example 2 : Let $f : Z \rightarrow Z$ where $f(x) = x^2 - 1$. Is f Invertible?

Solution :

We have $f(x) = x^2 - 1$;

For $x = 1$ and -1

$f(1) = 0$ and $f(-1) = 0$

$\therefore \quad f^{-1}(0) = \{1, -1\}$.

$f^{-1}(n)$ is not a single value function. Hence f is not invertible.

Example 3 : Show that function $f(x) = ax + b$ from R to R is inversible. Where a and b are constant with $a \neq 0$. Find the inverse of f. **MU - May 02**

Solution :

For if $x, x_2 \in R$. $f(x_1) = f(x_2) \Rightarrow ax_1 + b = ax_2 + b$

$\Rightarrow \quad ax_1 = ax_2$

$\Rightarrow \quad x_1 = x_2$

$\therefore$ f is one to one

Also if $y \in R$ Then

$$y = f(x) = ax + b$$

$$\Rightarrow \quad x = \frac{y-b}{a}$$

$\therefore$ for $x \in R$ there exists $\left(\frac{y-b}{a}\right) \in R$

Such that $f\left(\frac{y-b}{a}\right) = a\left(\frac{y-b}{a}\right) + b = y$

$\therefore$ f is onto

$\therefore f^{-1}$ exists and is defined by

$$f^{-1}(y) = \frac{1}{a}(y-b)$$

Example 4 : Let $f : N \rightarrow N$ where $f(x) = x + 3$. Is f invertible?

Solution :

An inverse function would have to take any $x \in N$ back to something. So in particular, it would have to take say 1 back to something. But $f(x) \neq 1$ for any $x \in N$. Hence f is not invertible.

Example 5 : Function $f(x) = (4x + 3) / (5x - 2)$. Find f^{-1} **MU - Dec. 99**

Solution :

To find f^{-1}

Set $y = f(x)$ and then interchange x and y as follow.

$$y = (4x + 3) / (5x - 2)$$

One interchanging x and y.

$$x = (4y + 3) / (5y - 2)$$

$$5xy - 2x = 4y + 3$$

$$5xy - 4y = 2x + 3$$

$$y(5x - 4) = (2x + 3)$$

$$y = \frac{(2x+3)}{(5x-4)}$$

$$f^{-1}(x) = \frac{(2x+3)}{(5x-4)}$$

Example 6 : Let $f : A \to B$ be one to one and onto then prove that

$f^{-1} \circ f = I_A$

$f \circ f^{-1} = I_B$

where I_A and I_B are identical mapping an set A and set B. **MU - Dec. 02, 03, 06**

Solution :

Let $f : A \to B$ be defined by

$$f(a) = b.$$

Then as f is one to one and onto. Therefore f^{-1}, the inverse function exists and is defined as $f^{-1}(b) = a$.

Since f is onto

$\therefore \quad (f^{-1} \circ f)(a) = f^{-1}[f(a)] = f^{-1}(b) = a.$

i.e. $\quad f^{-1} \circ f(a) = a.$

$\Rightarrow \quad f^{-1} \circ f =$ identify function $= I_A$ say.

Also $\quad (f \circ f^{-1})(b) = f[f^{-1}(b)] = f(a) = b.$

$\therefore \quad f \circ f^{-1}(b) = b.$

$\therefore \quad f \circ f^{-1} =$ Identity function $= I_B.$

$\therefore \quad f^{-1} \circ f = I_A.$ and $f \circ f^{-1} = I_B.$

Example 7 : Show that the function $f : R - \{2\} \to R - \{0\}$ where R is set of real numbers defined by $f(x) = \frac{1}{x-2}$ is a bijection. Find its inverse.

MU - Dec. 09, May 11

Solution :

To find f^{-1},

Set $y = f(x)$ and then interchange x and y as follows.

$$y = \frac{1}{x-2}$$

One interchanging x and y

$$x = \frac{1}{y-2}$$

$$x(y-2) = 1$$

$$xy - 2x = 1$$

$$xy = 1 + 2x$$

$$y = \frac{1+2x}{x}$$

$$f^{-1}(x) = \frac{1+2x}{x}$$

Example 8 : Function $f(x) = \frac{(4x+3)}{(5x-2)}$. Find f^{-1} **MU - May 14**

Solution :

$$f(x) = \frac{(4x+3)}{(5x-2)}$$

interchange x and y to get :

$$x = \frac{(4y+3)}{(5y-2)}$$

solve for "y"

$$5xy - 2x = 4y + 3$$

$$5xy - 4y = 2x + 3$$

$$(5x-4)y = 2x + 3$$

$$y = \frac{(2x+3)}{(5x-4)}$$

$$f^{-1} = \frac{(2x+3)}{(5x-4)}$$

Syllabus Topic : The Pigeonhole Principle

4.6 The Pigeonhole Principle :

In this section we discuss the pigeonhole principle. It is also known as dirihlet drawer principle or the shoe box principle and it can be stated as follows.

4.6.1 Theorem of Pigeonhole Principle : MU - Dec. 14, May 15, May 16

If n pigenons are assigned to m pigeonholes, and $m < n$, then at least one pigeonhole contains two or more pigeons.

Proof : Consider labeling the m plgeonholes with the numbers 1 through m and the n pigeons with the numbers 1 through n. Now, beginning with pigeon 1, assign each pigeon in order to the pigeonhole with the same number. This assigns as many pigeons as possible to individual pigeon holes, but because $m < n$, there are $n - m$ pigeons that have not yet been assigned to a pigeonhole. At least one pigeonhole will be assigned a second pigeon.

4.6.2 The Extended Pigeonhole Principle : MU - Dec. 14, May 15, May 16

If there are m pigeonholes and more than 2m pigeons, then three or more pigeons will have to be assigned to at least. One of the pigeonholes. In general if the number at pigeons is much larger than the number at pigeonholes, previous theorm can be restated to give a stronger conclusion.

First , a word about notation. If n and m are positive integers, then $\lfloor n / m \rfloor$ stands for the largest integer less than or equal to the rational number n/m. Thus $\lfloor 3/2 \rfloor$ is 1, $\lfloor 9/4 \rfloor$ is 2 and $\lfloor 6/3 \rfloor$ is 2.

Theorem : (The extended pigeonhole principle)

If n pigeons are assigned to m, pigeonholes, then one of the pigeonholes must obtain at least $\lfloor (n-1)/m \rfloor + 1$ pigeons.

Proof : (by contradiction) :

If each pigeonhole contains no more than $\lfloor (n-1)/m \rfloor$ pigeons, then there are at most m. $\lfloor (n-1)/m \rfloor \leq$ m. $(n-1)/m = n-1$ pigeons in all. This contradicts our assumption, so one of the pigeonholes must contain at least $\lfloor (n-1)/m \rfloor + 1$ pigeons.

4.6.3 Examples :

Example 1 : If eight people are chosen in anyway from some group, at least two of them will have been born on the same day of the week. $n = 8$, $m = 7$

Solution :

Here each person (pigeon) is assigned to the day of the week (pigeonhole) on which he or she was born. Since there are eight people and only seven days of the week, the pigeon hole principle tells us least two people must be assigned to the same day of the week.

Example 2 : Show that if any five numbers from 1 to 8 are chosen, then two of them will add upto 9.

MU - Dec. 96, Dec. 09, Dec. 12

Solution :

Construct four different sets, each containing two numbers that add up to 9 as follows :

$$A_1 = \{1, 8\}, A_2 = \{2, 7\}, A_3 = \{3, 6\}, A_4 = \{4, 5\}$$

Each of the five numbers chosen must belong to one of these sets. Since there are only four sets, the pigeonhole principle tells us that two of the chosen numbers belong to the same set. These numbers add upto 9.

Example 3 : Show that if any 11 numbers are chosen from the set {1, 2, ...20 }, then one of them will be a multiple of another. **MU - May 05, Dec. 08, Dec. 15**

Solution :

Every positive integer n can be written as $n = 2^k$, where m is odd and $k \geq 0$. This can be seen by simply factoring all powers of 2 (if any) out of n. In this case let us call m the odd part of n. If 11 numbers are chosen from the set { 1, 2, ...20 }, then two of them must have the same odd parts. This follows from the pigeonhole principle since there are 11 numbers (pigeons), but only 10 odd numbers between 1 and 20 (pigeonholes) that can be odd parts of these numbers.

Let n_1 and n_2 be two chosen numbers with the same odd part we must have $n_1 = 2^{k1}$ m and $n_2 = 2^{k2}$ m, for some k_1 and k_2. If $k_1 \geq k_2$, then n_1 is a multiple at n_2; otherwise n_2 is a multiple of n_1.

Example 4 : Consider the region shown in Fig. 4.22 (a)

It is bounded by a regular hexagon whose sides are of length 1 unit. Show that if any seven points are chosen in his region, then two of them must be no farther apart than 1 unit. **MU - May 13, Dec. 15**

Fig. 4.22 (a)

Solution :

Divide the region into six equilateral triangles as shown in Fig. 4.22(b).

If seven points are chosen in the region, we can assign each of them to a triangle that contains it. If the point belongs to several triangles arbitrarily assign to one at them. Then the seven points are assigned to six triangular regions. So by the pigeonhole principle at least two points must belong to the same region. These two cannot be more than 1 unit apart.

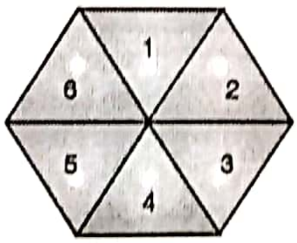

Fig. 4.22 (b)

Example 5 : Shirts numbered consecutively from 1 to 20 are worn by the 20 members of a bow leaque. When any 3 of these members are chosen to be a team, the sum of their s numbers is used as a code number for the team show that if any 8 of the 20 are selected, then from these 8 we may form at least two different teams having the same code number. **MU - May 03**

Solution :

From the 8 selected bowlers, we can form a total of ${}^8C_3$ or 56 different teams. These will play the role at pigeons. The largest possible team code number is 18 + 19 + 20 or 57, and the smallest possible is 1 + 2 + 3 or 6. Thus only the 52 code numbers (pigeonholes) between 6 and 57 inclusive are available for the 56 possible teams. By the pigeonhole principle at least two teams will have the same code number.

Example 6 : Show that it any 30 people are selected, then we may choose a subset at 5 so that all 5 were born on the same day of the week.

Solution :

Assign each person to the day of the week on which she or he was born. Then 30 pigeons are being assigned to 7 pigeonholes. By the extended pigeonhole principle with n = 30 and m = 7, at least $\lfloor (30-1)/7 \rfloor + 1$ or 5 of the people must have been born on the same day of the week.

Example 7 : Show that if 30 dictionaries in a library contain a total of 61,327 pages, then one of the dictionaries must have at least 2045 pages. **MU - Dec. 11**

Solution :

Let the pages be the pigeons and the dictionaries the pigeonholes. Assign each page to the dictionary in which it appears. Then by the extended pigeonhole principle, one dictionary must contain at least. $\lfloor 61,326/30 \rfloor + 1$ or 2045 pages.

Example 8 : Six friends discover that they have total of 2161 Rs. with them on a trip to the movies. Show that one or more of them must have at least 361 Rs.

Solution :

Let the rupees be the pigeons and the number of friends is number of pigeonholes. Then by the extended pigeonhole principle one friend must have at least. $\lfloor 2160/6 \rfloor + 1$ or 361 rupees.

Example 9 : If 13 people are assembled in a room, show that of least 2 of them must have their birthday in the same month.

Solution :

Let the birth months play the role of the pigeons and the calendar months, the pigeonholes. Then there are 13 pigeons and 12 pigeonholes. By the pigeonhole principle, at least 2 people were born in the same month.

Example 10 : Show that if seven numbers from 1 to 12 are chosen, then 2 of them will add upto 13.

Solution :

Construct 6 different sets each containing two numbers that add upto 13 as follows.

$$A_1 = \{1, 12\}, A_2 = \{8, 5\}, A_3 \{7, 6\},$$
$$A_4 = \{11, 2\}, A_5 = \{9, 4\}, A_6 = \{10, 3\}.$$

Each of the seven numbers chosen must belong to one of the sets. Since there are only 6 sets, the pigeonhole principle tells us that two of the chosen numbers belong to the same set. These numbers add upto 13.

Example 11 : How many numbers must be selected from the set {1, 2, 3, 4, ,5 ,6} to guarantee that at least one pair of these numbers add up to 7 ? **MU - Dec. 06, May 10, Dec. 14**

Solution :

Solve by extended Pigeonhole principle 4 numbers must be selected from the set $\{1, 2, 3, 4, 5, 6\}$ to guarantee that at least one pair of these numbers add upto 7.

Example 12 : How many people do you need in a school to gurantee that there are two people who have the same initial. (first and last names only).

Solution :

There are 26 letters for each of the first and the last names number at possible sets of initials are $26^2 = 676$. So by pigeonhole principle, provided there are more than 676 people there must be atleast two people with the same initials.

Example 13 : Show that in any room of people who have been doing some handsaking there will always be atleast two people who have shaken hands the same number of times.

Solution :

Suppose there are n people, then since people shake hands only once, the labels on the pigeonhole will go from 0 to (n – 1). That is we have n people and n holes. But it is not possible say for 0^{th} and $(n-1)^{th}$ holes both to be occupied. Thus we have at most (n – 1) holes occupied at any one time. Hence, by the principle at least one of the holes has two occupants, which shows that there are atleast two people who have shaken hands the same number of times.

Example 14 : Among six people there are either three who all know each other or three who are complete strangers.

Solution :

Consider one of the people, say X. of the other five people we can put them in either of two pigeonhole depednding on whether they know. X or not. If we put n = 2 and k = 2 so that there must be

$k + 1 = 3$ people who do not know X. Suppose there are three people A, B, C who all know X. Then if any two of A, B and C know each other, say B and C, then X, B, C all know each other.

On the other hand, if no two A, B and C know each other then A, B and C form a set of three complete strangers.

Example 15 : Show that among $n + 1$ arbitrarily chosen positive integers, there are two whose difference is divisible by n.

Solution :

We use Eculid's division algorithm. Given positive integers a and b, we can divide a by b and get a quotient q and remainder r, i.e.

$$a = bq + r$$

Let $S = \{ a_1, a_2, \ldots a_{n+1} \}$ be the set of $n + 1$ arbitrarily chosen positive integers.

Define, $f : S \rightarrow \{ 0, 1, 2, \ldots n - 1 \}$.

by $f(ai) = ri$, the remainder left after dividing by n.

Here $|s| = n + 1$ and cordinality of the co-domain is n. Hence by pigeonhole principle, $f(a_i) = f(a_1)$ for $i \neq j$. This means that $r_i = r_j$. Hence $a_i - a_j = n(q_i - q_j)$. Thus means that there are two integers a_i and a_j in s whose difference is divisible by n.

Example 16 : A sports tournament consisting of 45 events is spread over 30 days. There is atleast one event per day. Prove that no matter how the events are arranged there will be a period of consecutive days during which exactly 14 events will take place.

Solution :

Let a_i denote the total number of events that takes place upto and including the i^{th} day. Hence $a_1 \geq 1$ and $a_{30} = 45$, and we have a sequence $a_1, a_2, \ldots a_{30} = 45$, which is strictly increasing since there is atleast one event per day. Adding 14 to each term in the sequence, we obtain

$$a_1 + 14, a_2 + 14, \ldots a_{30} + 14 = 59$$

Now consider the sequence $a_1, a_2, \ldots a_{30}, a_1 + 14, a_2 + 14, \ldots a_{30} + 14$, which consists of 60 numbers ranging from 1 to 59. Hence by pigeonhole principle, two of these numbers must be the same. Since $a_i \neq a_j$ for $i \neq j$, it follows that $a_j = a_i + 14$ for some $j > i$. Hence $a_j - a_i = 14$, which means that there is a peirod of consecutive days from the i^{th} day during which exactly 14 games take place.

Example 17 : If a set of 16 numbers is selected from $\{ 2, \ldots, 50 \}$, atleast two numbers will be in the set with a common divisor greater than 1.

Solution :

In the set $\{ 2, \ldots 50 \}$, there are 15 prime numbers viz $\{ 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47 \}$.

Suppose in any set of 16 numbers from 2 to 50, no two have a common divisor greator than one. Consider the prime factors of these numbers . By our assumption no two numbers will have common prime factor. This would mean that there should be atleast 16 different prime numbers. This contradicts the fact that there are only 15 prime numbers. Hence our assumption is wrong Therefore, in any set of 16 numbers from 0 to 50, 2 numbers will have a common divisor greater than 1.

Example 18 : Show that among any n + 1 positive integers not exceeding 2n, there must be an integer that divides one of the other integers.

Solution :

Let us denote the n +1 positive integers as $a_1, a_2, \ldots a_{n+1}$. Then we can write each integer a_i as $a_i = 2^{kj}\, b_j$ for $j = 1, 2 \ldots n + 1$, where k_j is a non negative integer and b_j odd positive integer. For example, $1 = 2^0 \cdot 1$, $2 = 2^1 \cdot 1$, $4 = 2^2 \cdot 1$, $6 = 2^1 \cdot 3$ and so on. Now $b_1, b_2 \ldots b_{n+1}$ are all odd positive integers less than 2n. Since there are only n odd positive integers which are less than 2n, it follows from pigeonhole principle that $b_i = b_j$ for some i and j. Then $a_i = 2^{kj}$ q and $a_j = 2^{kj}$ q, where $a = a_i = a_j$. If $k_i < k_j$, a_i divides a_j, otherwise a_j divides a_i. Hence the result.

Example 19 : Show that 7 colors are used to paint 50 bicycles, at least 8 bicycles will be of same color.

MU - May 01, May 07, May 10, May 15, May 17

Solution :

If n pigeons are assigned to m pigeonholes, and m < n, then at least one pigeonhole contains two or more pigeons.

By the extended pigeonhole principle at least. $\lfloor (50 - 1) / 7 \rfloor + 1 = 8$ will be of the same color.

Example 20 : What is the minimum number of students required in a discrete structures class to be sure that at least six will receive the same grade, if there are five possible grades A, B, C, D, E

MU - Dec. 03, Dec. 10

Solution :

By extended pigeon hole principle

$$\left\lfloor \frac{(n-1)}{5} \right\rfloor + 1 = 6$$

$$\therefore \frac{n-1}{5} = 6 - 1$$

$$\therefore \frac{n-1}{5} = 5$$

$$\therefore n - 1 = 25$$

$$\therefore n = 26$$

26 Students are required in a discrete structures class.

Example 21 : How many friends must you have to gurantee that at least five of them will have birthday in the same month. **MU - May 04, May 10**

Solution :

Let n be the no. of friends

If number of months are to be pigeonhole then number of friends n will be pigeon

$\therefore$ By extended pigeonhole principle

$$\left\lfloor \frac{(n-1)}{12} \right\rfloor + 1 = 5$$

$$\therefore \quad \frac{n-1}{12} = 4$$

$$\therefore \quad n - 1 = 48$$

$$\therefore \quad n = 49$$

Thus among 49 friends, at least five of them will have birthdays in the same month.

Example 22 : Prove that among 1,00,000 people, there are two who are born at exactly the same time (hour, minute, second)

Solution :

Let A be the set of people. (pigeons) and B, the set of seconds (pigeonholes) of one day.

$$\therefore \quad |A| = 100,000 = n$$

$$|B| = 24 \times 3600 = 86400 = m$$

Then

$$k = \lfloor (n-1)/m \rfloor + 1$$

$$k = \lfloor (100000-1)/86400 \rfloor + 1$$

$$k = 1 + 1$$

$$k = 2.$$

Hence there are atleast two who are born on the same day.

Example 23 : Show that there must be atleat 90 ways to choose six numbers from 1 to 15. So that all the choices have the same sum

Solution :

$$\Rightarrow \quad n = {}^{15}C_6 = 5005$$

The lowest sum of 6 numbers chosen from 1 to 15

$$= 1 + 2 + 3 + 4 + 5 + 6$$

$$= 21$$

$$\text{Highest sum} = 10 + 11 + 12 + 13 + 14 + 15 = 75$$

Hence

$$m = 75 - 21 + 1 = 55$$

Hence by the pigeonhole principle

$$k = \left\lfloor \frac{(n-1)}{m} \right\rfloor + 1$$

$$= \left\lfloor \frac{5004}{55} \right\rfloor + 1$$

$$= 91$$

Hence in atleast 90 ways, we can choose six numbers from 1 to 15 that all the choices have the same sum.

Example 24 : Prove that if any 14 integers from 1 to *25* are chosen, then one of them is a multiple of another. **Dec. 05, May 08, May 09, May 12**

Solution :

Every positive integer n can be written as $n = 2^k$, where n is odd and $k \geq 0$. This can be seen by simply factoring all powers of 2 (if any) out of n. In this caselet us all m the odd part of n. If 14 numbers are chosen from the set {1, 2, ……, 25} then 2 of them must have the same odd parts. This follows from the pigeonhole principle since there are 14 numbers (pigeons), but only B odd numbers between 1 and 25 (pigeonholes) that can be odd parts of these numbers.

Let n_1 and n_2 be 2 chosen numbers with the same odd part we must have $n_1 = 2^{k1}$ m and $n_2 = 2^{k2}$ m. for some k and k_2. If $k_2 \geq k_2$ then n_1 is a multiple at n_2 ; otherwise n_2 is a multiple of n_1.

Example 25 : There are 3000 students in a college which offers 7 disfinct courses of 4 years duration. A student who has taken a course in Discrete Mathematics learns that the largest classroom can hold only 100 students. She at once realizes there is a problem. What is the problem ?

Solution :

Since, there are 7 distinct classes of 4 years duration, we have $7 \times 4 = 28$ different classes. Hence, by extended pigeonhole principle, each classroom must hold atleast $\lfloor (3000 - 1) / 28 \rfloor + 1 = 107 + 1 = 108$ students. But since the capacity of the largest classroom is only 100, this is obviously a problem.

Example 26 : Let T be an equilateral triangle whose sides are of length 1 unit. Show that if any five points are chosen lying on or inside the triangle, then two of them must be no more than 1/2 unit apart. **MU - May 00, Dec. 07, May 11, Dec. 13**

Solution :

m_1 , m_2 , m_3 are midpoints of sides AC, AB and BC, respectively. Let the four small triangles created be the pigeonholes. For any five points in or on triangle ABC, at least two must be in or on the small triangle and thus are no more than 1/2 unit apart.

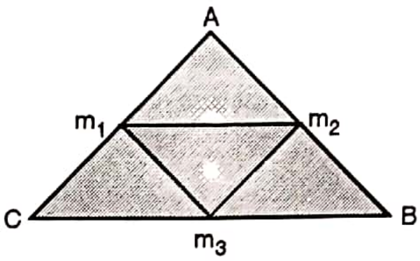

Fig. 4.23

Example 27 : Let S be a square whose sides have length 2 units. Show that for any five points on or inside S, there must be two points whose distance apart is almost $\sqrt{2}$ units.

Solution :

Divide the square into 4 equal squares, as shown in Fig. 4.24. If five points are choosen in the square, we can assign each of them to square that contains it. If a point belongs to more than one square, we assign it to one of them arbitrarily. Then the five points are assigned to 4 square regions, so by the pigeonhole principle at least two points must belong to the same region. These two cannot be more than $\sqrt{2}$ units aparts, as the side of each square being 1 units, the length at the diagonal is $\sqrt{2}$ units, which is the maximum distance, that the two points can be apart.

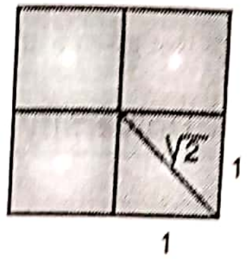

Fig. 4.24

Example 28 : In a group of six people at a party, each pair of individuals consists of two mutual acquaintances or two strangers. Show that there are either three mutual acquaintances or three mutual strangers in the group.

Solution :

Let A be one of six people. Divide the remaining five into two sets. One consisting only of acquaintances of A the other only of strangers to A. By extended Pigeonhole principle, cardinality of one of the sets must be atleast $\lfloor 5/2 \rfloor + 1 = 3$. Hence it follows that in the group there are either 3 or more Who are acquaintances of A, or there are 3 or more who are strangers to A. Let us assume the former i.e. say B, C, D, are acquaintances of A. Since any pair of individuals are either acquaintances or strangers, if say B, C are acquaintances, then together, with A, A, B, C, form a group of 3 mutual acquaintances on the other hand if B, C, D are mulual strangers they form a group of three mutual strangers.

If we assume the latter, when B, C, D are strangers to A, the proof follows in similar manner.

Example 29 : A man hiked for 10 hours and covered a total distance of 45 miles. It is known that he hiked 6 miles in the first hour and only 3 miles in the last hour. Show that he must have hiked atleast 9 miles within a certain period of two consecutive hours.

Solution :

Let a_i, $1 \leq i \leq 10$, denote the number of miles hiked by the man during the i^{th} hour. Then $a_1 = 6$, $a_{10} = 3$. Hence $a_2 + a_3 + \ldots + a_9 = 45 - (6 + 3) = 36$ miles. Consider the set

$$A = \{ (a_2, a_3), (a_4, a_5), (a_6, a_7), (a_8, a_9) \}.$$

at pairs of consecutive hours

Apply pigeonhole principle to the sum

$$(a_2 + a_3) + (a_4 + a_5) + (a_6 + a_7) + (a_8 + a_9) = 36$$

Since sum of 4 nunbers is 36, value of one number should be atleast 9.

Hence the man must have hiked atleast 9 miles within a certain period of two consecutive hours.

Example 30 : Show that among any group of five (not necessarily consecutive) integers, there are two with the same remainder when divided by 4. **MU - May 06**

Solution :

Since there are 4 possible remainder when an integer is divided by 4. The pigeon hole principle implies that given 5 integers at least two have the same remainder.

4.7 University Questions and Answers :

Dec. 2008

Q. 1 If 11 numbers are chosen from a set = {1, 2,.......20}. Prove that one of them is multiple of other. **(Section 4.6.3, Example 3)** **(6 Marks)**

Q. 2 If $f : A \rightarrow B$ and $g : B \rightarrow C$ are both one-one and onto, the,

$(g \circ f)^{-1} = f^{-1} \circ g^{-1}$ **(Section 4.5.2 Theorem 4)** **(6 Marks)**

May 2009

Q. 3 Prove that if any 14 integers from 1 to 25 are chosen, then one of them is a multiple of another. **(Section 4.6.3 Example 24)** **(4 Marks)**

Q. 4 Explain Primitive Recursive Function. Every primitive recursive function is a total function, Justify. **(Section 4.1.7)** **(4 Marks)**

Q. 5 Let $A = B = Z$ and let $f : A \rightarrow B$ be defined by $f(a) = a + 1$, for $a \in A$. Which of the special properties does f possess? **(Section 4.2.5, Example 8)** **(4 Marks)**

Dec. 2009

Q. 6 If $A = B = C = R$ where R is set of real number and $f : A \rightarrow B$, $g : B \rightarrow C$ are functions defined by $f(x) = x + 1$, $g(x) = x^2 + 2$, then find $(g \circ f)(x)$ and $(f \circ g)(2)$.

(Section 4.3.2, Example 7) **(5 Marks)**

Q. 7 Show that if any five integers from 1 to 8 are selected, then the sum of at least two of them will be 9. **(Section 4.6.3, Example 2)** **(4 Marks)**

Q. 8 Show that the function $f : R - \{2\} \rightarrow R - \{0\}$ where R is set of real numbers defined by $f(x) = \frac{1}{x-2}$ is a bijection. Find its inverse. **(Section 4.2.5, Example 7)** **(8 Marks)**

May 2010

Q. 9 Define a pigeonhole principle. Show that if seven colours are used to paint 50 bicycles, at least 8 bicycles will be of same colour.

(Sections 4.6.1 and 4.6.3, Example 19) **(4 Marks)**