Orthogonality (unit 3)

Defn: Let V be an inner product space. The vectors $u, v \in V$ are said to be orthogonal and u is said to be orthogonal to v if

$$\langle u, v \rangle = 0.$$

Ex (1) Consider the vectors $u = (1,1,1)$, $v = (1,2,-3)$, $w = (1,-4,3)$ in R^3;

Then
$$\langle u, v \rangle = 1 + 2 - 3 = 0$$
$$\langle u, w \rangle = 1 - 4 + 3 = 0$$
$$\langle v, w \rangle = 1 - 8 - 9 = -16.$$

Thus u is orthogonal to v & w.
but v & w are not orthogonal.

Ex (2) Consider the functions $\sin t$ & $\cos t$ in the vector space $C[-\pi, \pi]$ of continuous functions on the closed interval $[-\pi, \pi]$. Then

$$\langle \sin t, \cos t \rangle = \int_{-\pi}^{\pi} \sin t \cos t \, dt = \frac{1}{2} \sin^2 t \Big)_{-\pi}^{\pi} = 0.$$

Thus $\sin t$ & $\cos t$ are orthogonal functions in the vector space $C[-\pi, \pi]$.

Ex In. R^3, having Euclidean inner product. For which values of k, u & v are orthogonal?

(i) $u = (1, 4, 2)$, $v = (3, -2, k)$.

(ii) $u = (k, -2, 4)$, $v = (k, k, -2)$

Sol^n (i) If u & v are orthogonal

$\langle u, v \rangle = 0$

$\Rightarrow \langle u, v \rangle = 3 - 8 + 2k = 0.$

$-5 + 2k = 0$

$k = 5/2$.

(ii) $\langle u, v \rangle = 0$

$\Rightarrow k^2 - 2k - 8 = 0.$

$k = -2, 4.$

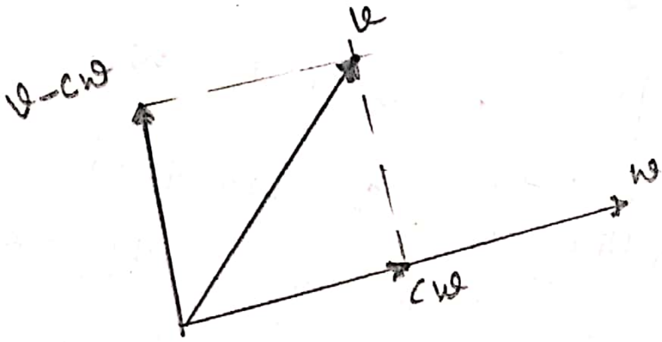
Projections.

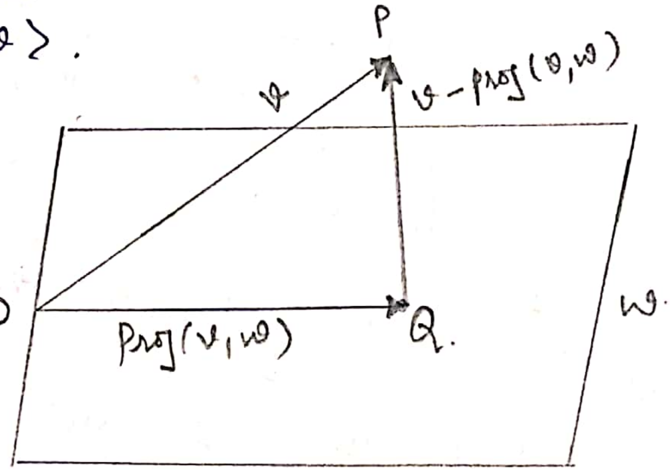
Let V be an inner product space. Suppose w is a given non zero vector in V, and suppose v is another vector. The 'projection of v along w' will be the multiple cw of w such that $v' = v - cw$ is orthogonal to w. This means

$$\langle v - cw, w \rangle = 0.$$

or $\langle v, w \rangle - c \langle w, w \rangle = 0.$

or $c = \dfrac{\langle v, w \rangle}{\langle w, w \rangle}.$





The projection of v along w is denoted & defined by –

$$proj(v, w) = cw = \frac{\langle v, w \rangle}{\langle w, w \rangle} w.$$

Ex Let w be the plane in $\mathbb{R}^3$, with equation $x-y+2z=0$, and let $v=(3,-1,2)$. Find the orthogonal projection of v onto w & let the component of v orthogonal to w.

Sol^n

$W: x-y+2z=0$.

The subspace w is a plane through the origin in $\mathbb{R}^3$. From the equation of the plane, we have $x=y-2z$, so W consists of vectors of the form

$$\begin{bmatrix} y-2z \\ y \\ z \end{bmatrix} = y\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}.$$

$\Rightarrow u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ & $v = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$ are basis of W.

but they are not orthogonal. So we find another non-zero vector in W, which is orthogonal to either one of these.

Let $w = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is a vector is W, which is orthogonal to u_1. Then $x-y+2z=0$,

Since w is in the plane W.

Since $u_1 \cdot w = 0$, we have $x+y=0$.

Solving

$$x-y+2z=0$$
$$x+y=0.$$

$\Rightarrow x=-z, \quad y=z. \quad \Rightarrow w = \begin{bmatrix} -z \\ z \\ z \end{bmatrix}$

$\Rightarrow w = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$.

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So $w_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ is an orthogonal sol^n in W

& hence an orthogonal basis.

So, the orthogonal basis for W are -

$u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ & $u_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$.

we have. $u_1 \cdot v = 2$, $u_2 \cdot v = -2$

$u_1 \cdot u_1 = 2$, $u_2 \cdot u_2 = 3$

$$\text{Proj}_W(v) = \left(\frac{u_1 \cdot v}{u_1 \cdot u_1}\right) u_1 + \left(\frac{u_2 \cdot v}{u_2 \cdot u_2}\right) u_2.$$

$$= \frac{2}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{2}{3}\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5/3 \\ 1/3 \\ -2/3 \end{bmatrix}$$

The component of v orthogonal to W is the vector.

$$\text{Perp}_W(v) = v - \text{proj}_W(v)$$

$$= \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} - \begin{bmatrix} 5/3 \\ 1/3 \\ -2/3 \end{bmatrix} = \begin{bmatrix} 4/3 \\ -4/3 \\ 8/3 \end{bmatrix}$$

It is easy to see that $\text{Proj}_W(v)$ is in W, since it satisfies the equation of the plane. It is equally easy to see that $\text{Perp}_W(v)$ is orthogonal to W, since it is a scalar multiple of the normal vector $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ to W.

The Gram Schmidth Process

Ex Find an orthogonal basis for the subspace W of $\mathbb{R}^3$ given by –

$$W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x - y + 2z = 0 \right\}.$$

Sol^n we have $x = y - 2z$, so W consists of vectors of the form

$$\begin{bmatrix} y-2z \\ y \\ z \end{bmatrix} = y \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}.$$

Let W · span (x_1, x_2) where

$$x_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \; \& \; x_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}.$$

* by Gram Schmidth process: To construct an orthogonal basis for W.

Starting with x_1, we get a second vector that is orthogonal to it by taking the component of x_2 orthogonal to x_1

we set $v_1 = x_1$, so.

$$V_2 = \text{perp}_{x_1}(x_2) = x_2 - \text{proj}_{x_1}(x_2)$$

$$= x_2 - \left(\frac{x_1 \cdot x_2}{x_1 \cdot x_1}\right) x_1$$

$$v_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} - \frac{(-2)}{(2)} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Thus $\{v_1, v_2\}$ is an orthogonal set of vectors in W. They are L.I. so form a **basis**.

Ex Using Gram Schmidt process. find an orthogonal basis for $\mathbb{R}^3$ that contains the vector

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

Sol^n → we first find any basis for $\mathbb{R}^3$ containing v_1, if we take

$$x_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \& \quad x_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

then $\{v_1, x_2, x_3\}$ is clearly a basis for $\mathbb{R}^3$. Since using Gram Schmidth process.

$$v_2 = x_2 - \left(\frac{v_1 \cdot x_2}{v_1 \cdot v_1}\right) v_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \left(\frac{2}{14}\right)\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1/7 \\ 5/7 \\ -3/7 \end{bmatrix}$$

$$v_2' = \begin{bmatrix} -1 \\ 5 \\ -3 \end{bmatrix}$$

and finally, $v_3 = x_3 - \left(\frac{v_1 \cdot x_3}{v_1 \cdot v_1}\right) v_1 - \left(\frac{v_2' \cdot x_3}{v_2' \cdot v_2'}\right) v_2'$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \left(\frac{3}{14}\right)\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \left(\frac{-3}{35}\right)\begin{bmatrix} -1 \\ 5 \\ -3 \end{bmatrix} = \begin{bmatrix} -3/10 \\ 0 \\ 1/10 \end{bmatrix}$$

$$v_3' = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

Then $\{v_1, v_2', v_3'\}$ is an orthogonal basis for $\mathbb{R}^3$, that contain v_1.

Ex Apply Gram-Schmidth process to construct an orthonormal basis for the subspace $W = \text{span}(x_1, x_2, x_2)$ of $\mathbb{R}^4$, where

$$x_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad x_3 = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 2 \end{bmatrix}$$

Soln First we note that $\{x_1, x_2, x_3\}$ is a L.I set, so it forms a basis for W.

we set $v_1 = x_1$.

To compute the component of x_2 orthogonal to $W_1 = \text{span}(v_1)$

$$v_2 = \text{perp}_{W_1}(x_2) = x_2 - \left(\frac{v_1 \cdot x_2}{v_1 \cdot v_1}\right) v_1$$

$$= \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \left(\frac{2}{4}\right) \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 3/2 \\ 1/2 \\ 1/2 \end{bmatrix}$$

(rescaling) $\leftarrow v_2' = 2v_2 = \begin{bmatrix} 3 \\ 3 \\ 1 \\ 1 \end{bmatrix}$.

To find component of x_3 orthogonal to $W_2 = \text{span}(x_1, x_2)$

$W_2 = \text{span}(x_1, x_2) = \text{span}(v_1, v_2) = \text{span}(v_1, v_2')$

using the orthogonal basis $\{v_1, v_2'\}$

$$v_3 = \text{perp}_{W_2}(x_3) = x_3 - \left(\frac{v_1 \cdot x_3}{v_1 \cdot v_1}\right) v_1 - \left(\frac{v_2' \cdot x_3}{v_2' \cdot v_2'}\right) v_2'$$

$$= \begin{bmatrix} 2 \\ 2 \\ 1 \\ 2 \end{bmatrix} - \left(\frac{1}{4}\right) \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} - \left(\frac{15}{20}\right) \begin{bmatrix} 3 \\ 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 0 \\ 1/2 \\ 1 \end{bmatrix}$$

(rescaling) $\leftarrow v_3' = 2v_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 2 \end{bmatrix}$

$\{v_1, v_2', v_3'\}$ forms an orthogonal basis for W.

To obtain an orthonormal basis, we normalize each vector –

$$q_1 = \frac{v_1}{\|v_1\|} = \frac{1}{2}\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix} = \begin{bmatrix}1/2\\-1/2\\-1/2\\1/2\end{bmatrix}$$

$$q_2 = \frac{v_2'}{\|v_2\|} = \frac{1}{2\sqrt{5}}\begin{bmatrix}3\\3\\1\\1\end{bmatrix} = \begin{bmatrix}3/2\sqrt{5}\\3/2\sqrt{5}\\1/2\sqrt{5}\\1/2\sqrt{5}\end{bmatrix} = \begin{bmatrix}3\sqrt{5}/10\\3\sqrt{5}/10\\\sqrt{5}/10\\\sqrt{5}/10\end{bmatrix}$$

$$q_3 = \frac{v_3'}{\|v_3'\|} = \left(\frac{1}{\sqrt{6}}\right)\begin{bmatrix}-1\\0\\1\\2\end{bmatrix} = \begin{bmatrix}-1/\sqrt{6}\\0\\1/\sqrt{6}\\2/\sqrt{6}\end{bmatrix} = \begin{bmatrix}-\sqrt{6}/6\\0\\\sqrt{6}/6\\\sqrt{6}/3\end{bmatrix}.$$

$\{q_1, q_2, q_3\}$ is an orthonormal basis for W.

QR Factorization.

Ex Find a QR factorization of

$$A = \begin{bmatrix}1 & 2 & 2\\-1 & 1 & 2\\-1 & 0 & 1\\1 & 1 & 2\end{bmatrix}$$

Sol^n The orthonormal basis for Col(A) produced by the Gram-Schmidt process was - (refer previous example)

$$q_1 = \begin{bmatrix}1/2\\-1/2\\-1/2\\1/2\end{bmatrix},\quad q_2 = \begin{bmatrix}3\sqrt{5}/10\\3\sqrt{5}/10\\\sqrt{5}/10\\\sqrt{5}/10\end{bmatrix},\quad q_3 = \begin{bmatrix}-\sqrt{6}/6\\0\\\sqrt{6}/6\\\sqrt{6}/3\end{bmatrix}$$

$$Q = [q_1, q_2, q_3] = \begin{bmatrix}1/2 & 3\sqrt{5}/10 & -\sqrt{6}/6\\-1/2 & 3\sqrt{5}/10 & 0\\-1/2 & \sqrt{5}/10 & \sqrt{6}/6\\1/2 & \sqrt{5}/10 & \sqrt{6}/3\end{bmatrix}$$

Since $A = QR$ for some upper triangular matrix R. To find R, we use the fact that Q has orthonormal columns & hence $Q^TQ = I$

Therefore, $Q^TA = Q^TQR = IR = R$.

$$R = Q^TA = \begin{bmatrix} 1/2 & -1/2 & -1/2 & 1/2 \\ 3\sqrt{5}/10 & 3\sqrt{5}/10 & \sqrt{5}/10 & \sqrt{5}/10 \\ -\sqrt{6}/6 & 0 & \sqrt{6}/6 & \sqrt{6}/3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ -1 & 1 & 2 \\ -1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 1/2 \\ 0 & \sqrt{5} & 3\sqrt{5}/2 \\ 0 & 0 & \sqrt{6}/2 \end{bmatrix}$$

Singular Value Decomposition (SVD)

Def^n Singular Values: If A is an $m \times n$ matrix, the singular values of A are the square roots of the eigen values of A^TA and are denoted by $\sigma_1, \sigma_2, \ldots \sigma_n$. It is conventional to arrange the singular values, so that $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n$.

Ex Find the singular values of

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Sol^n The matrix.

$$A^T A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

has eigen values

$$(2-\lambda)(2-\lambda) - 1 = 0$$

$$\lambda = 3, 1.$$

$$\lambda_1 = 3, \quad \lambda_2 = 1.$$

Singular values are $\sigma_1 = \sqrt{\lambda_1} = \sqrt{3}$ and

$\sigma_2 = \sqrt{\lambda_2} = 1.$

Def^n: SVD (Singular Value Decomposition)

Let A be an $m \times n$ matrix with singular values $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r > 0$ and $\sigma_{r+1} = \sigma_{r+2} = \dots \sigma_n = 0$

Then there exist an $m \times m$ orthogonal matrix U, an $n \times n$ orthogonal matrix V, and an $m \times n$ matrix Σ of the form,

$$\Sigma = \begin{array}{c} \begin{array}{cc} r & (n-r) \end{array} \\ \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} \end{array} \begin{array}{l} r \\ m-r. \end{array}, \quad \text{where } D = \begin{bmatrix} \sigma_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ \cdot & \cdots & \sigma_r \end{bmatrix}$$

Such that $A = U \Sigma V^T.$, is called a Singular Value Decomposition (SVD) of A

[The columns of U are called left singular vectors of A,
Columns of V are called right singular vectors of A,

The matrices U & V are not uniquely determined by A, but Σ must contain the singular values of A.

Ex Find a Singular Value decomposition of -

(a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Sol^n We compute

$$A^T A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Eigen values are $\lambda_1 = 2$, $\lambda_2 = 1$, $\lambda_3 = 0$ &

corresponding eigen vectors $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

These vector are orthogonal, so we normalize them to obtain

$$v_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$ since $\|v_1\| = \sqrt{1+1+0} = \sqrt{2}$ (Dividing each element by $\|v_1\|$.)

$$v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

Singular Values of A are $\sigma_1 = \sqrt{2}$, $\sigma_2 = \sqrt{1} = 1$, $\sigma_3 = 0$, Thus,

$$V = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix} \quad \& \quad \Sigma = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

To find U, we compute $\longrightarrow$

$$u_1 = \frac{Av_1}{\sigma_1} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$u_2 = \frac{Av_2}{\sigma_2} = \frac{1}{1}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$\therefore \quad U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. These vectors u_1 & u_2 form an orthonormal basis (std basis) for $\mathbb{R}^2$.

$\therefore$ SVD $\longrightarrow$ $A = U \Sigma V^T$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix}$$

(Which can be easily checked, Note that V had to be transposed. Also note that the singular value σ_3 does not appear in Σ.)

Ex (2) Find SVD for $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$.

[Try yourself]

Ans: $A = U \Sigma V^T$.

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}\begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$