

## Ring and field Tutorial.

# Ring : \* Properties to prove for a ring OR condition to be satisfied for a ring  $(A, +, \cdot)$

\* I  $(A, +)$  is a commutative group / Abelian group.  $\rightarrow$   
ie, (i) prove closure operation (ii) prove associative operation  
(iii) prove identity (left & right) operation (iv) Inverse operation  
(v) commutative operation. After proving all 5 condition  
we can say  $(A, +)$  is a commutative / abelian group

\* II  $(A, \cdot)$  is a semigroup :- ie (i)  $(\cdot)$  is a closed operation for multiplication (ii)  $(\cdot)$  is an associative operation for multiplication, ie,  $(a \times b) \times c = a \times (b \times c)$

\* III  $(A, \cdot)$  is distributive, ie,  $(\cdot)$  is distributed over  $(+)$  ie  $a \times (b + c) = (a \times b) + (a \times c)$

# Integral Domain :- 3 necessary conditions for Ring as an integral domain.

- |                                                     |                             |
|-----------------------------------------------------|-----------------------------|
| (i) It is a commutative ring                        | } Definition given in notes |
| (ii) It has a $(1)$ multiplicative identity element |                             |
| (iii) It has no zero divisors                       |                             |

# Field :- 3 necessary conditions for this kind of rings:-  
(i) It is commutative ~~ring~~ group / Abelian group.  
(ii) It has a unity  $(1)$  element (multiplicative identity element)  
(iii) Every non-zero element in the table has a multiplicative inverse

Note:- A field is an integral domain. However, not every integral domain is a field.



Q3) Show that set of  $2 \times 2$  matrices is a ring is a non-commutative ring.

Sol: The set of  $2 \times 2$  matrices:  $R = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix}; a, b, c, d \in \mathbb{Z} \right\}$

under matrix addition and multiplication is a ring with respect to additive and multiplicative ~~inverse~~ identities.

→ The ring is not commutative since in the matrix multiplication does not commute. ~~So~~ The ring of  $n \times n$  matrices with usual matrix addition and multiplication is a ring but non-commutative.

Also,  $n \times n$  matrices is a ring, but not a field. Since it does not have multiplicative inverse.

Q4)  $R = \{\mathbb{R}, +, \cdot\}$  is set of real numbers, and  $\mathbb{C} = \{\mathbb{C}, +, \cdot\}$  is a set of complex numbers is a field. <sup>is a ring</sup>

Sol: (i) Set of real numbers  $(R)$  is a ring for  $(R, +, \cdot)$ .

(a)  $R$  is closed under  $+$  and  $\times$

(b) Since  $a + (b + c) = (a + b) + c$ ; addition is associative

(c) Since  $a + 0 = 0 + a = a \in R$ , additive identity exists.

(d) Since we have every element  $a \in R$  is an inverse. So inverse property also exists

(e) Since  $a + b = b + a$ . So  $R$  is commutative.  $\therefore (R, +)$  is a commutative group.

(ii)  $(a \cdot b) \cdot c = a \cdot (b \cdot c)$  for all  $a, b, c \in R$ , Hence  $(\cdot)$  is associative.

(iii) Since  $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ . Hence  $(\cdot)$  is distributed over  $(+)$ . So set of real numbers  $(R, +, \cdot)$  is a ring.

### Integral domain:-

- (a) It is a commutative ring i.e.  $a \cdot b = b \cdot a$ .  
 $2 \times 4 = 4 \times 2$   
 $8 = 8$
- (b) with unity element:- A set of even integers with usual operation of  $+$  and  $\times$ , i.e.  $(R, +, \times)$  is a commutative ring without unity. (Can be seen in table (2)). [Given in notes on pg 16-48 ex(2)]
- (c) ~~is~~ Multiplicative inverse:- Set of even integers including zero with usual  $+$  and  $\times$  is not an integral domain because it does not have a multiplicative identity. [Given in notes on pg 16-54].

Therefore,  $(R)$  is not an integral domain. Also, the non-zero elements of table (2) are not without zero divisors.

Field:- Since the multiplicative identity does not exist, there cannot be a multiplicative inverse. Hence it is not a field.

So,  $(R, +, \times)$  is a ring. But it is not an integral domain and neither a field.



(b)  $\times$  is an associative operation; i.e.,

$$(a \times_{10} b) \times_{10} c = a \times_{10} (b \times_{10} c)$$

let  $a=2, b=4, c=6$ .

$$\therefore (2 \times_{10} 4) \times_{10} 6 = 2 \times_{10} (4 \times_{10} 6)$$

$$\therefore 8 \times_{10} 6 = 2 \times_{10} 24$$

$$\therefore 48 = 48 \Rightarrow \boxed{8 = 8}$$

$\therefore (R, \times)$  is a semigroup.

III We have to show operation  $\times_{10}$  is distributive over the operation  $+_{10}$ ; i.e.,

$$a \times_{10} (b +_{10} c) = (a \times_{10} b) +_{10} (a \times_{10} c)$$

let  $a=2, b=4, c=6$ .

$$\therefore 2 \times_{10} (4 +_{10} 6) = (2 \times_{10} 4) +_{10} (2 \times_{10} 6)$$

$$\therefore 2 \times 0 = 2 +_{10} 2 \checkmark$$

$$\therefore 0 = 10$$

$$\boxed{0 = 0}$$

$\therefore \times_{10}$  is distributive over  $+_{10}$ . Thus we can say that set  $P = \{0, 2, 4, 6, 8\}$  is a ring with respect to addition & multiplication modulo 10.

(c) Identity element from fig (a) is table 1 is '0'.

$$\therefore a +_0 a = a = a +_0 0$$

Left identity

- ①  $0 +_0 0 = 0$
- ②  $0 +_0 2 = 2$
- ③  $0 +_0 4 = 4$
- ④  $0 +_0 6 = 6$
- ⑤  $0 +_0 8 = 8$

Right identity

- $0 +_0 0 = 0$
- $2 +_0 0 = 2$
- $4 +_0 0 = 4$
- $6 +_0 0 = 6$
- $8 +_0 0 = 8$

(d) Inverse :-  $0 +_0 0 = 0$ ,  $2 +_0 8 = 0$ ,  $4 +_0 6 = 0$ ,  
 $6 +_0 4 = 0$ ,  $8 +_0 2 = 0$   
 OR  $0^{-1} = 0$ ,  $2^{-1} = 8$ ,  $4^{-1} = 6$ ,  $6^{-1} = 4$ ,  $8^{-1} = 2$ .

(e) Commutative property :-  $a +_0 b +_0 c = c +_0 b +_0 a$ .  
 $a = 2, b = 4, c = 6$ .

$$\therefore 2 +_0 4 +_0 6 = 6 +_0 4 +_0 2$$

$$\therefore 12 = 12$$

$$\boxed{0 = 0}$$

$\therefore (R, +)$  is an abelian / commutative group.

II  $(R, \times)$  is a semigroup :-

(a) closure property :- from fig (b) table (2) it is clear that all the elements of the table (2) belong to the set R. So closure operation is proved.



Q10)  $R = \{0, 2, 4, 6, 8\}$ . To prove it is ring module (o) Show it is integral domain? It is field? or both

Soln:  $(R, +, \cdot)$  draw the table for  $+$  and  $\times$

$+$	0	2	4	6	8
0	0	2	4	6	8
2	2	4	6	8	0
4	4	6	8	0	2
6	6	8	0	2	4
8	8	0	2	4	6

$\times$	0	2	4	6	8
0	0	0	0	0	0
2	0	4	8	2	6
4	0	8	6	4	8
6	0	2	4	6	8
8	0	6	2	8	4

fig (a): Table (1)

fig (b): Table (2)

(i) We have to show  $(R, +)$  is an abelian group:-

(a)  $+$  is a closed operation. Since from fig (a) the elements of the table belong to set  $\{0, 2, 4, 6, 8\}$

(b)  $+$  is an associative operation, i.e.,  $(a + b) + c = a + (b + c)$

let  $a = 2, b = 4, c = 6$

$$\therefore (a + b) + c = a + (b + c)$$

$$(2 + 4) + 6 = 2 + (4 + 6)$$

$$6 + 6 = 2 + 0$$

$$12 = 2$$

$$\therefore 2 = 2$$

By prove for  $a = 4, b = 6, c = 8$

$$(a + b) + c = a + (b + c)$$

(c) Identity: '0' is the identity element for table 1.

So,  $0 + a = a = a + 0$  for all  $a \in \mathbb{Z}_5$ .

$$0 +_5 1 = 1 = 1 +_5 0$$

$$0 +_5 2 = 2 = 2 +_5 0$$

$$0 +_5 3 = 3 = 3 +_5 0$$

$$0 +_5 4 = 4 = 4 +_5 0$$

So '0' is an identity is proved.

(d) Additive inverse:  $1 +_5 4 = 0$ ,  $2 +_5 3 = 0$ ,  $3 +_5 2 = 0$ ,  $4 +_5 1 = 0$   
So it's additive inverse exists.

(e) ~~Since~~ commutative property:  $a + b = b + a$

$$1 + 2 = 2 + 1$$

$$3 = 3$$

so addition is commutative.

$\therefore (\mathbb{Z}_5, +)$  is a commutative group.

III. Distributive property:  $a \times (b + c) = (a \times b) + (a \times c)$

$$1 \times (2 + 3) = (1 \times 2) + (1 \times 3)$$

$$1 \times 0 = 2 + 3 = 5$$

$$0 = 0$$

So, the ~~(x)~~ is distributed over (+).

IV. Since (1) is the multiplicative identity element as seen from table (2), so the multiplicative inverse also exist, i.e.,  $1 \times 1 = 1$ ,  $2 \times 3 = 1$ ,  $3 \times 2 = 1$ ,  $4 \times 4 = 1$ ,  
i.e.,  $1^{-1} = 1$ ,  $2^{-1} = 3$ ,  $3^{-1} = 2$ ,  $4^{-1} = 4$ .

$\therefore (\mathbb{Z}_5, +, \times)$  is a field.



Q3) Is  $(\mathbb{Z}_5, +, \times)$  a field? Modulo 5.

Soln:

$+$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

fig (a): Table 1.

$\times$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

fig (b): Table 2

(I) This is a commutative ring with unity. As (1) is the identity element given in table (2) for multiplication.

It is a commutative ring, i.e.,  $a \cdot b = b \cdot a$ ,  $a = 1, b = 2$   
 $1 \times 2 = 2 \times 1$   
 $2 = 2$

(II) Also, To prove it is a commutative group:-

(a) Closure property: It is clear from table that  $\mathbb{Z}_5$  is closed under addition and multiplication mod 5.

(b) Since:  $a + (b + c) = (a + b) + c$ , for  $a = 1, b = 2, c = 3$   
 $1 + (2 + 3) = (1 + 2) + 3$   
 $1 + 0 = 3 + 3 = 6$   
 $1 = 1$

$\therefore$  Addition is also associative.



\* Same sol<sup>n</sup>  
for Q2-

Now, for  $C = \{a+bi; a, b \in \mathbb{Z}\}$  where  $i = \sqrt{-1}$   
This is a field and extension of  $\mathbb{R}$  and is called  
complex numbers denoted by  $C = \mathbb{R}[i]$  and  
has binary operations.

$$(a+bi) + (c+di) = (a+c) + (b+d)i$$

$$(a+bi)(c+di) = (ac-bd) + (ad+bc)i$$

proves the addition and multiplication associative  
property. So it is a commutative group. It has  
a identity element and  $a+bi \in C$  has a  
multiplicative inverse, i.e.,

$$(a+bi) \frac{1}{(a+bi)} = 1.$$

$$\text{i.e., } \frac{1}{a+bi} = \left( \frac{1}{a+bi} \right) \left( \frac{a-bi}{a-bi} \right)$$

$$= \frac{a-bi}{a^2+b^2}$$

$$= \frac{a}{a^2+b^2} + i \left( \frac{-b}{a^2+b^2} \right) \in C$$

Therefore it satisfies all the properties of field.



classmate

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Q6) Is  $(\mathbb{Z}_6, +, \times)$  a field?

Ans:- This question is same as previous problem.  
But it is not a field because the multiplicative  
inverse does not exist even though there exist  
an identity element in multiplication table (2)

$+$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

$\times$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1