Contents

1 Tensors and Maxwell's Equations

1.1 Introduction

Here I'll be building up to interpreting Maxwell's equations with tensors. I'm using the following references:

- `https://www.ese.wustl.edu/~nehorai/Porat_A_Gentle_Introduction_to_Tensors_2014.pdf`
- Wikipedia

At times, I may include information not directly pertinent to the goal, but just because it may be helpful to compare definitions to other previously-studied topics.

1.2 Tensor

Definition 1 (tensor)**.** A (p, q) tensor $T \in \underbrace{V \otimes \ldots \otimes V}_{p\text{copies}} \otimes \underbrace{V^* \otimes \ldots \otimes V^*}_{q\text{copies}}$

1.3 Operations on tensors

Definition 2 (contraction)**.** Let T have type (a, b). The (i, j)th contraction of T is $T^{s_1,\ldots,s_{i-1},k,s_{i+1},\ldots,s_a}_{t_1,\ldots,t_{j-1},k,t_{j+1},\ldots,t_b}$

Definition 3 (raising and lowering)**.**

Definition 4 (Levi-Civita symbol)**.**

$$\varepsilon_{i_1,\ldots,i_n} = \begin{cases} 1 & i_1, \ldots, i_n \text{even} \\ -1 & \text{odd} \\ 0 & \text{not a permutation} \end{cases}$$

Example 1.

$$\det(A) = \varepsilon_{i_1,\ldots,i_n} A^1_{i_1} \cdots A^n_{i_n}$$

1.4 Beginning

Definition 5 (Gram Matrix)**.** Let B be a bilinear form on a vector space V. The Gram Matrix G is defined by $G_{ij} = B(v_i, v_j)$.

One property of the gram matrix is that $\{v_i\}$ are linearly independent iff $\det(G) \neq 0$. I remember learning about this matrix once in machine learning class as well in the context of kernel functions.

1.5 Equivalence relations on matrices

Definition 6. Two $n \times n$ matrices A and B are similar if there exists a P such that $B = P^{-1}AP$, ie if A and B are in the same conjugacy class of $GL(n, \mathbb{F})$.

Similar matrices are the same matrix with respect to a different basis, and this similar matrices share all non-basis-dependent properties.

Definition 7 (Matrix congruence)**.** Let A and B be $n \times n$ matrices over $\mathbb{F}$. A and B are congruent if there exists an invertible P such that $B = P^T AP$.

Matrix equivalence is defined, in general, on rectangular matrices, and is more restrictive than similarity.

1.6 Back to the program

Theorem 1 (Sylvestor)**.** *Every real symmetric matrix Gram matrix G is congruent to a diagonal matrix Λ with entries $0, \pm 1$ that is unique up to congruence.*

Proof. □

Definition 8. An $n \times n$ symmetric real matrix M is positive semidefinite if for all $x \in \mathbb{R}^n$, $x^T Mx \geq 0$. If $x^T Mx = 0 \Rightarrow x = 0$, then M is positive definite. Negative semidefinite and negative definite matrices are defined similarly.

Equivalently, a positive definite matrix has all positive eigenvalues, a positive semidefinite matrix has all nonnegative eigenvalues, and so on.

TODO proof

Definition 9 (signature)**.** Let n^+, n^-, n^0 be the number of $+1, -2, 0$, respectively, in Λ. (n^+, n^-, n^0) is called the signature of G.

When $\{V\}$ are linearly independent, then G is positive-definite, so because it is symmetric,

Theorem 2. *If G is positive, then it is positive definite.*

Proof.

$$\begin{aligned} x^T G x &= \sum_{i,j} x_i G_{ij} x_j \\ &=< \sum_i x_i v_i, \sum_j x_j v_j > \\ &= \| \sum_i v_i x_i \|^2 \\ &\geq 0 \end{aligned}$$

□

Note that B is a bilinear form, not a Hermitian inner product. It turns out that if G is positive, then the space is Euclidean.

TODO elaborate

1.7 The metric tensor

Definition 10 (metric tensor)**.** The metric tensor g_{ij} of an inner product space is a $(0,2)$ tensor with coordinates under $\{v\}$ given by the Gram matrix.

Because G is nonsingular, there is a dual metric tensor g^{ij} that satisfies $g_{ij}g^{jk} = \delta_i^k$
Why are tensors multiplies like matrices? TODO

1.8 types of potentials

Definition 11 (scalar potential)**.** A scalar potential is a vector field whose gradient $v = -\nabla\Phi$ is a vector field. Then for Φ C^1 on U, $v : U \to \mathbb{R}^n$, where $U \subset \mathbb{R}^n$ is open, is said to be conservative and curl-free as the curl vanishes everywhere.

Definition 12 (Vector potential)**.** A vector potential is a vector field A whose curl $\nabla \times A$ is a vecctor field.

Theorem 3. *Let $v : \mathbb{R}^3 \to \mathbb{R}^3$ be a twice continuously differentiable solenoidal vector field, and that $v(x)$ decreases sufficiently fast as $||x|| \to \infty$. Then $A(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{\nabla_y \times v(y)}{\|x-y\|} d^3y$ is a vector potential for v.*

This contrustuction can be generalized for an arbitrary Helmholtz decomposition. This is not unique, as for any scalar function $m \in C^1$, $v = A + \nabla m$.

Question 1.1. Do there exist scalar potentials that are not of the above form, $A + \nabla m$?

This also relates to "gauge fixing".

Theorem 4 (Helmholtz decomposition)**.** *Let F be a vector field on a bounded $V \subseteq \mathbb{R}^3$ that is twice continuosly differentiable. Then there exists a scalar potential Φ and vector potential A such that $F = -\nabla\Phi + \nabla \times A$.*

Proof. TODO

- what is differentiability over a vector field
- `https://en.wikipedia.org/wiki/Helmholtz_decomposition`

□

$-\nabla\Phi$ is also called the longitudinal partof F, and $\nabla \times A$ is also called the transverge part of F.

1.9 electromagnetic potentials

Definition 13. The magnetic vector potential A is a vector potential of the magnetic field B

Definition 14 (electric potential)**.** Let $F = E + \frac{\partial A}{\partial t}$. By Faraday's law, F is conservative. TODO why is this? `https://en.wikipedia.org/wiki/Electric_potential` Therefore, $E = -\nabla V - \frac{\partial A}{\partial t}$, where V is the scalar potential defined by F.

Note that in the electrostatic case, we have that V is a scalar potential of E. Also, we will be using both V and φ to refer to the electric potential unless these symbols are otherwise defined.

Definition 15 (electromagnetic four-potential)**.** $A^\alpha = (\varphi/c, A)$.

Definition 16 (four-gradient)**.** $\partial_\mu = (\frac{1}{c}\frac{\partial}{\partial t}, \nabla) = (\frac{\partial_t}{c}, \nabla)$

Definition 17 (category of manifolds)**.** Man^p is the category whose objects are manifolds of smoothness class C^p and whose morphisms are p-times differentiable maps. Similarly, the category of smooth manifolds is Man^∞ and the category of analytic manifolds is Man^ω.

TODO what does it mean looking at manifolds modeled on a fixed caegory A?

1.10 category theory

Here I'll try to build up to categorically describing a differential form.

`https://ncatlab.org/nlab/show/differentiable+manifold` `https://ncatlab.org/nlab/show/tangent+bundle`

Definition 18 (bundle)**.** A bundle over an object B in a category C is an object E of C together with a morphism $p : E \to B$.

Definition 19 (section)**.** `https://ncatlab.org/nlab/show/section`

Definition 20 (tangency relation)**.**

Definition 21 (tangent vector)**.** A tangent vector on X at x is an equivalence class of the tangency equivalence relation, and the set of all tangent vectors at $x \in X$ is denoted T_xX.

Definition 22. A coordinate chart is a map $\varphi : \mathbb{R}^n \xrightarrow[\cong]{X} |_{\mathrm{im}\,\varphi} \hookrightarrow X$

Theorem 5. *T_xX is a real vector space.*

Proof. □

Definition 23 (exterior algebra)**.** `https://ncatlab.org/nlab/show/exterior+algebra`

Definition 24 (differential form)**.** `https://ncatlab.org/nlab/show/differential+form`

This content is necessary for seeing pullbacks generally, but might not be immediately necessary for our goal.

Definition 25 (fiber)**.** The fiber of a morphism of bundle $f : E \to B$ over a point x of B is the collection of generalized elements of E that are mapped by f to x.

Definition 26 (pullback)**.**

TODO

- `https://ncatlab.org/nlab/show/pullback`
- `https://ncatlab.org/nlab/show/fiber`
- `https://ncatlab.org/nlab/show/bundle`
- `https://ncatlab.org/nlab/show/differential+form`

1.11 TODO

- `https://en.wikipedia.org/wiki/Electromagnetic_tensor`
- `https://en.wikipedia.org/wiki/Electromagnetic_four-potential`
- `https://en.wikipedia.org/wiki/Exterior_derivative#Exterior_derivative_of_a_k-form`