Task 1

My first step is to translate a real-world problem into a mathematical model. I saw this with the Blue Ridge Hot Tubs example. This involves:

- Defining the decision variables (e.g., X1 for Aqua-Spas, X2 for Hydro-Luxes).
- Creating an objective function that I want to maximize or minimize
- Listing all the constraints as mathematical inequalities or equalities (e.g., limits on pumps, labor hours, and tubing). I realize that correctly formulating the problem is the most critical and challenging part of the process.

Solving the Problem: Two Approaches

Once I have the LP formulation, I have two ways to solve it: a visual method for simple cases and an algorithmic one for more complex problems.

The Graphical Method (for 2 Variables)

If my problem only has two variables, I can solve it visually:

1. I plot each constraint on a graph to define the boundaries.
2. The area that satisfies all constraints at once is the Feasible Region. My optimal solution must be somewhere in this region.
3. I then plot the objective function as a line (a "level curve"). By pushing this line as far as possible while still touching the feasible region, I can find the optimal point.

I also learned to watch out for special conditions:

- Infeasibility: The constraints contradict each other, so there's no feasible region at all.
- Unbounded Solution: The objective function can be increased infinitely, which usually means I've missed a constraint.
- Alternate Optimal Solutions: More than one solution gives the same best outcome.

The Algorithmic Method (for Many Variables)

For problems with more than two variables, I need a systematic algorithm. This requires preparing the LP problem in a specific format first.

Step A: Convert to a Standardized Form

Algorithms need the problem to be structured consistently. I learned about two key forms:

- Standard Form: This requires the problem to be a maximization, with all primary constraints being 'less than or equal to' (≤) inequalities and all variables being non-negative.
- Slack Form: This is what the Simplex algorithm uses. I convert every inequality constraint into an equality by adding a unique "slack variable." This turns the problem into a system of linear equations.

Step B: Solve the System of Equations

The Slack Form is a system of linear equations, which can be written as $Ax=b$. I learned that simply calculating the inverse of the matrix ($x=A^{-1}b$) is a bad idea because it's slow and numerically unstable.

A much better approach is LU Decomposition. This breaks the main matrix A into two simpler triangular matrices: a Lower-triangular (L) and an Upper-triangular (U). I can then solve the system efficiently in two steps:

1. Solve $Ly=b$ using forward substitution.
2. Solve $Ux=y$ using backward substitution to get my final answer, x.

Step C: Apply the Simplex Algorithm

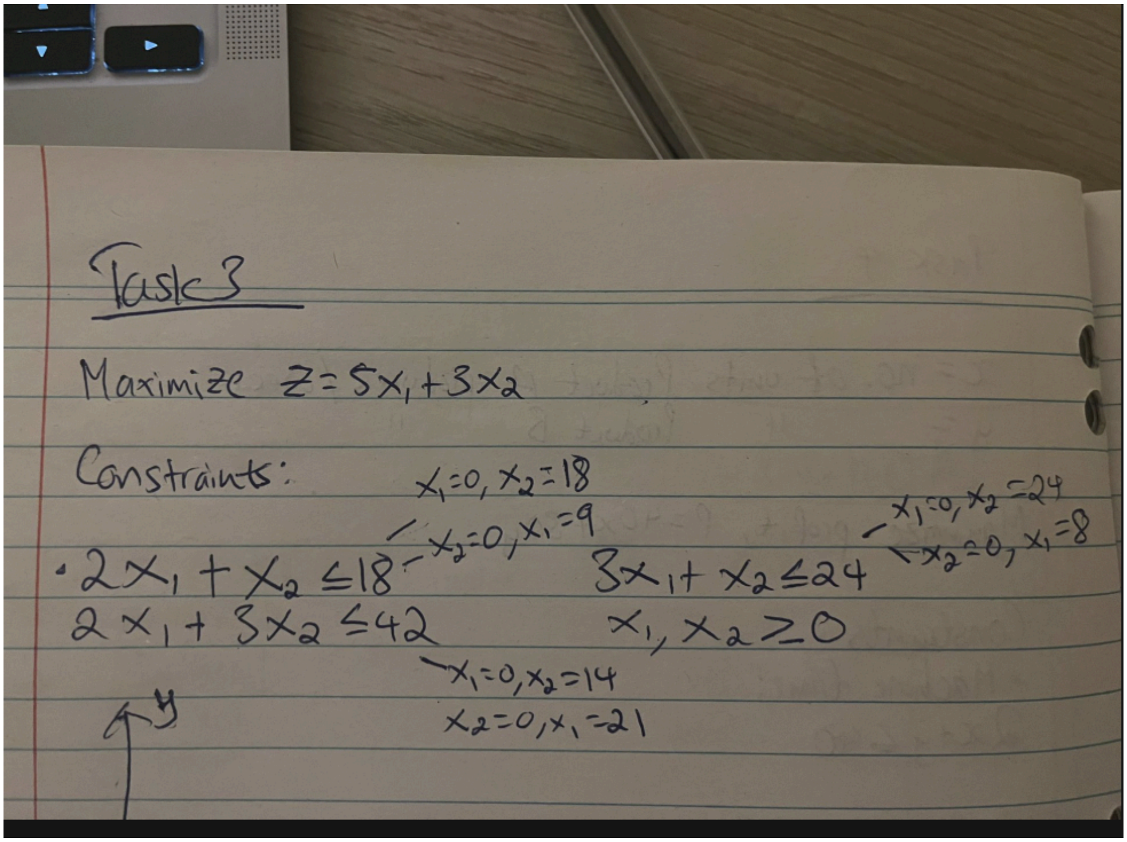
Finally, I learned about the Simplex Algorithm itself, which is the engine that solves the LP problem. It operates on the Slack Form. My understanding is that it's an iterative process:

1. It starts with an initial feasible solution (usually the origin).
2. It then repeatedly performs a pivot operation. This involves intelligently choosing a variable to enter the solution and one to leave, which mathematically corresponds to moving from one vertex of the feasible region to an adjacent, better one.
3. The algorithm stops when it finds a vertex where no adjacent vertex has a better objective function value, meaning it has found the optimal solution.

Task 2:

Provided in code

Task 3:



Task 3

Maximize $Z = 5x_1 + 3x_2$

Constraints:

$x_1 = 0, x_2 = 18$

$x_2 = 0, x_1 = 9$

$2x_1 + x_2 \le 18$

$x_1 = 0, x_2 = 24$

$x_2 = 0, x_1 = 8$

$3x_1 + x_2 \le 24$

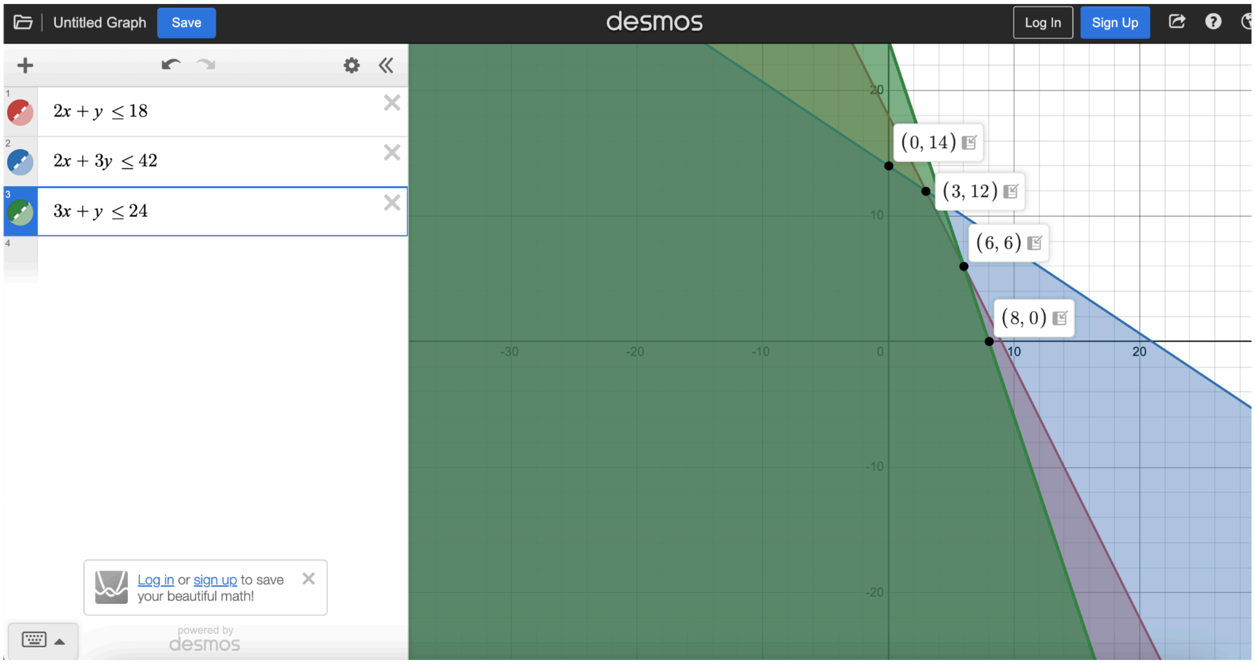
$2x_1 + 3x_2 \le 42$

$x_1, x_2 \ge 0$

$x_1 = 0, x_2 = 14$

$x_2 = 0, x_1 = 21$

y



$(0,0)$, $z=0$
$(8,0)$, $z=40$
$(6,6)$, $z=48$
$(3,12)$, $z=51$
$(0,14)$, $z=42$

Max value of $z=51$, $x_1=3$, $x_2=12$

Task 4:

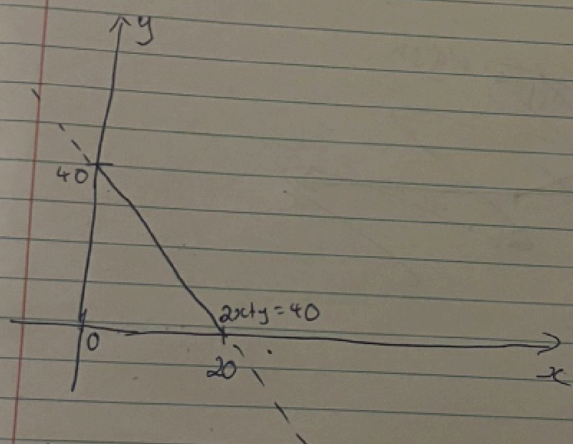
Task 4

x = no. of units Product A produces / week
y = " Product B "

Maximize profit, $P = 40x + 30y$

Constraints.
- Machine time:
$2x + y \leq 40$

- $x \geq 0$, $y \geq 0$



$2x + y = 40$
$x = 0$, $y = 40$
$y = 0$, $x = 20$

Evaluating corner points,

$(0, 40)$, $P = 1200$
$(0, 0)$, $P = 0$
$(20, 0)$, $P = 800$

Max Profit occurs at vertex $(0, 40)$
The factory should produce 0 units of Product A and 40 units of Product B, generating a maximum profit of $1200.

Task 5:

$Z - 18X1 - 12.5X2 = 0$

$X1 + X2 + s1 = 20$

$X1 + s2 = 12$

$X2 + s3 = 16$

We start with the initial tableau and perform pivot operations until no negative indicators remain in the objective function row (the Z-row)

Basic	Z	X1	X2	s1	s2	s3	RHS
Z	1	-18	-12.5	0	0	0	0
s1	0	1	1	1	0	0	20
s2	0	1	0	0	1	0	12
s3	0	0	1	0	0	1	16

The most negative value in the Z-row is -18. The X1 column is the pivot column (X1 enters the basis).

We perform the minimum ratio test (RHS / Pivot Column value):

- Row s1: 20/1=20
- Row s2: 12/1=12 (Minimum)

The s2 row is the pivot row (s2 leaves the basis). The pivot element is 1.

Pivot Operations We use row operations to make all other elements in the pivot column zero.

1. New Z Row = (Old Z Row) + 18 * (Row s2)
2. New s1 Row = (Old s1 Row) - (Row s2)

Basic	Z	X1	X2	s1	s2	s3	RHS
Z	1	0	-12.5	0	18	0	216
s1	0	0	1	1	-1	0	8
X1	0	1	0	0	1	0	12
s3	0	0	1	0	0	1	16

The only negative indicator in the Z-row is -12.5. The X2 column is the pivot column (X2 enters).

Minimum ratio test:

- Row s1: 8/1=8 (Minimum)
- Row s3: 16/1=16

The s1 row is the pivot row (s1 leaves). The pivot element is 1.

Pivot Operations

1. New Z Row = (Old Z Row) + 12.5 * (Row s1)
2. New s3 Row = (Old s3 Row) - (Row s1)

After the second iteration, there are no negative indicators in the Z-row, so we have reached the optimal solution.

Basic	Z	X1	X2	s1	s2	s3	RHS
Z	1	0	0	12.5	5.5	0	316
X2	0	0	1	1	-1	0	8
X1	0	1	0	0	1	0	12
s3	0	0	0	-1	1	1	8

X1=12, X2=8, Maximum Z=316