Sorting

SFWRENG 2CO3: Data Structures and Algorithms

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McMaster University

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Why sorting

Most computational problems involve *data processing*.

Processing data is typically much simpler if that data is *sorted*.

Example

Finding values: BinarySearch versus LinearSearch.

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Finding values: BinarySearch versus LinearSearch.

The analysis of *sorting* will require universal tools and techniques.
Sort algorithms utilize common design strategies for algorithms.

The power of sorting: The two-sum problem

Problem

Given a list $L[0 \ldots N)$ of distinct weights and a target weight w, find all distinct values $v_1, v_2 \in L$ with $w = v_1 + v_2$.

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L:	1	3	7	9	8	4	10	5

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Target weight: $w = 11$.

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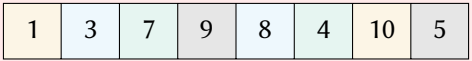

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Algorithm SimpleTwoSum(L, w):

Input: List $L[0 \ldots N)$ of N distinct weights, target weight w.

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1: result := empty bag.
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Complexity of SimpleTwoSum

For a rough estimate, we can count the number of times Line 4 is executed.

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For a rough estimate, we can count the number of times Line 4 is executed.

$$\sum_{i=0}^{N-2} (N - (i+1)) = \sum_{i=0}^{N-2} (N-1) - \sum_{i=0}^{N-2} i$$

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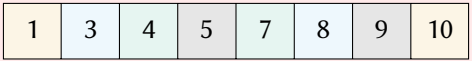

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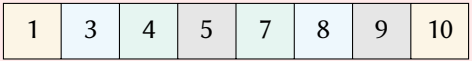

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For a rough estimate, we can count the cost of each BinarySearch call.

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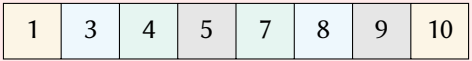

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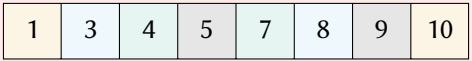

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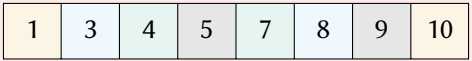

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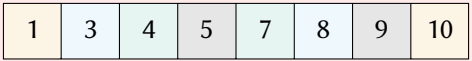

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- $L[i] < L[k]$ implies $L[j] > L[m]$.

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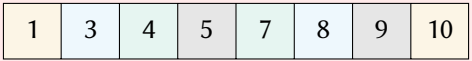

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We can search from both ends in L: position i as a *lower bound* and j as an *upper bound*.

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Algorithm BestTwoSum(L, w)**:**

Input: *Ordered* list $L[0 \ldots N)$ of N distinct weights, target weight w.

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1:  result := empty bag.
2:  i, j := 0, N − 1.
3:  while i < j do
4:      if L[i] + L[j] = w then
5:          add (L[i], L[j]) to result.
6:          i, j := i + 1, j − 1.
7:      else if L[i] + L[j] < w then
8:          i := i + 1.
9:      else
10:         j := j − 1.
11: return result.
```

The power of sorting: The two-sum problem

Can we do better than $\Theta(N \log_2(N))$ if L is ordered?

We can search from both ends in L: position i as a *lower bound* and j as an *upper bound*.

Algorithm BestTwoSum(L, w):

Input: *Ordered* list $L[0 \ldots N)$ of N distinct weights, target weight w.

```
1:  result := empty bag.
2:  i, j := 0, N − 1.
3:  while i < j do
4:      if L[i] + L[j] = w then
5:          add (L[i], L[j]) to result.
6:          i, j := i + 1, j − 1.
7:      else if L[i] + L[j] < w then
8:          i := i + 1.
9:      else
10:         j := j − 1.
11: return result.
```

$\Theta(N)$

Intermezzo: Correctness of BestTwoSum

Warning

Proving the correctness of BestTwoSum in all details is *tricky*!

Intermezzo: Correctness of BestTwoSum

High-level proof steps

1: *result* := empty bag.
2: $i, j := 0, N - 1$.
3: **while** $i < j$ **do**
4: **if** $L[i] + L[j] = w$ **then**
5: add $(L[i], L[j])$ to *result*.
6: $i, j := i + 1, j - 1$.
7: **else if** $L[i] + L[j] < w$ **then**
8: $i := i + 1$.
9: **else**
10: $j := j - 1$.
11: **return** *result*.

Intermezzo: Correctness of BestTwoSum

High-level proof steps

1. Specify what the *result* should be.

```
1:  result := empty bag.
2:  i, j := 0, N − 1.
3:  while i < j do
4:      if L[i] + L[j] = w then
5:          add (L[i], L[j]) to result.
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7:      else if L[i] + L[j] < w then
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Intermezzo: Correctness of BestTwoSum

High-level proof steps

1. Specify what the *result* should be.

Let $\mathsf{TS}(start, end) = \{(L[i], L[j]) \mid (L[i] + L[j] = w) \wedge (start \leq i < j \leq end)\}$.

1: *result* := empty bag.
2: $i, j := 0, N - 1$.
3: **while** $i < j$ **do**
4: **if** $L[i] + L[j] = w$ **then**
5: add $(L[i], L[j])$ to *result*.
6: $i, j := i + 1, j - 1$.
7: **else if** $L[i] + L[j] < w$ **then**
8: $i := i + 1$.
9: **else**
10: $j := j - 1$.
11: **return** *result*. /* *result* = $\mathsf{TS}(L, 0, N - 1)$. */

Intermezzo: Correctness of BestTwoSum

High-level proof steps

1. Specify what the *result* should be. The *invariant* must establish this result!

Let $\mathsf{TS}(start, end) = \{(L[i], L[j]) \mid (L[i] + L[j] = w) \wedge (start \leq i < j \leq end)\}$.

1: $result$:= empty bag.
2: $i, j := 0, N - 1$.
3: **while** $i < j$ **do**
4: **if** $L[i] + L[j] = w$ **then**
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7: **else if** $L[i] + L[j] < w$ **then**
8: $i := i + 1$.
9: **else**
10: $j := j - 1$.
11: **return** $result$. /* $result = \mathsf{TS}(L, 0, N - 1)$. */

Intermezzo: Correctness of BestTwoSum

High-level proof steps

2. Specify the *invariant*.

Let $\mathsf{TS}(start, end) = \{(L[i], L[j]) \mid (L[i] + L[j] = w) \wedge (start \leq i < j \leq end)\}$.

1: $result$:= empty bag.
2: $i, j := 0, N - 1$.
3: **while** $i < j$ **do**
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11: **return** $result$. /* $result = \mathsf{TS}(L, 0, N - 1)$. */

Intermezzo: Correctness of BestTwoSum

High-level proof steps

2. Specify the *invariant*. Look at what you need *after* the loop!

Let $\mathsf{TS}(start, end) = \{(L[i], L[j]) \mid (L[i] + L[j] = w) \wedge (start \leq i < j \leq end)\}$.

1: $result$:= empty bag.
2: $i, j := 0, N - 1$.
3: **while** $i < j$ **do** /* inv: $result = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i, j)$ */
4: **if** $L[i] + L[j] = w$ **then**
5: add $(L[i], L[j])$ to $result$.
6: $i, j := i + 1, j - 1$.
7: **else if** $L[i] + L[j] < w$ **then**
8: $i := i + 1$.
9: **else**
10: $j := j - 1$.
11: **return** $result$. /* $result = \mathsf{TS}(L, 0, N-1)$. */

Intermezzo: Correctness of BestTwoSum

High-level proof steps

3. Prove the *invariant* right *before the loop*.

Let $\mathsf{TS}(start, end) = \{(L[i], L[j]) \mid (L[i] + L[j] = w) \wedge (start \leq i < j \leq end)\}$.

1: $result$:= empty bag.
2: $i, j := 0, N - 1$.

Base case: prove that the invariant holds before the loop.

3: **while** $i < j$ **do** /* inv: $result = \mathsf{TS}(L, 0, N - 1) \setminus \mathsf{TS}(L, i, j)$ */

Intermezzo: Correctness of BestTwoSum

High-level proof steps

3. Prove the *invariant* right *before the loop*. Use facts established *before* the loop.

Let $\mathsf{TS}(start, end) = \{(L[i], L[j]) \mid (L[i] + L[j] = w) \wedge (start \leq i < j \leq end)\}$.

1: *result* := empty bag.
2: $i, j := 0, N - 1$.
Known: we have $i = 0$, $j = N - 1$, and $result = \emptyset$ (due to assignments).

Base case: prove that the invariant holds before the loop.
3: **while** $i < j$ **do** /* inv: $result = \mathsf{TS}(L, 0, N - 1) \setminus \mathsf{TS}(L, i, j)$ */

Intermezzo: Correctness of BestTwoSum

High-level proof steps

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Let $\mathsf{TS}(start, end) = \{(L[i], L[j]) \mid (L[i] + L[j] = w) \wedge (start \leq i < j \leq end)\}$.

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2: $i, j := 0, N - 1$.
   Known: we have $i = 0$, $j = N - 1$, and $result = \emptyset$ (due to assignments).
   Hence, $\mathsf{TS}(L, 0, N - 1) \setminus \mathsf{TS}(L, i, j) = \emptyset = result$.
   Base case: prove that the invariant holds before the loop.
3: **while** $i < j$ **do** /* inv: $result = \mathsf{TS}(L, 0, N - 1) \setminus \mathsf{TS}(L, i, j)$ */

Intermezzo: Correctness of BestTwoSum

High-level proof steps

3. Prove the *invariant* right *before the loop*. Use facts established *before* the loop.

Let $\mathsf{TS}(start, end) = \{(L[i], L[j]) \mid (L[i] + L[j] = w) \wedge (start \leq i < j \leq end)\}$.

1: $result$:= empty bag.
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   Known: we have $i = 0$, $j = N - 1$, and $result = \emptyset$ (due to assignments).
   Hence, $\mathsf{TS}(L, 0, N - 1) \setminus \mathsf{TS}(L, i, j) = \emptyset = result$.
   Base case: the invariant holds before the loop.
3: **while** $i < j$ **do** /* inv: $result = \mathsf{TS}(L, 0, N - 1) \setminus \mathsf{TS}(L, i, j)$ */

Intermezzo: Correctness of BestTwoSum

High-level proof steps

4. Prove that the *invariant* is maintaned by the loop.

Let $\mathsf{TS}(start, end) = \{(L[i], L[j]) \mid (L[i] + L[j] = w) \wedge (start \leq i < j \leq end)\}$.

Given: invariant and $i < j \rightarrow result = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i, j)$ and $i < j$.

4: **if** $L[i] + L[j] = w$ **then**
5: add $(L[i], L[j])$ to *result*.
6: $i, j := i + 1, j - 1$.
7: **else if** $L[i] + L[j] < w$ **then**
8: $i := i + 1$.
9: **else**
10: $j := j - 1$.

Induction step: prove that the invariant holds after each step of the loop.

Intermezzo: Correctness of BestTwoSum

High-level proof steps

5. An if-statement introduces a case distinction: prove each branch separately.

Let $\mathsf{TS}(start, end) = \{(L[i], L[j]) \mid (L[i] + L[j] = w) \wedge (start \leq i < j \leq end)\}$.

Given: invariant and $i < j \rightarrow result = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i, j)$ and $i < j$.

4: **if** $L[i] + L[j] = w$ **then**

Given: $result = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i, j)$, $i < j$, and $L[i] + L[j] = w$.

5: add $(L[i], L[j])$ to $result$.

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Induction step: prove that the invariant holds after each step of the loop.

Intermezzo: Correctness of BestTwoSum

High-level proof steps

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Let $\mathsf{TS}(start, end) = \{(L[i], L[j]) \mid (L[i] + L[j] = w) \wedge (start \leq i < j \leq end)\}$.

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4: **if** $L[i] + L[j] = w$ **then**

Given: $result = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i, j)$, $i < j$, and $L[i] + L[j] = w$.

By $L[i] + L[j] = w$ and the Definition of TS, we have: $(L[i], L[j]) \in \mathsf{TS}(L, i, j)$.

5: add $(L[i], L[j])$ to $result$.

6: $i, j := i + 1, j - 1$.

Induction step: prove that the invariant holds after each step of the loop.

Intermezzo: Correctness of BestTwoSum

High-level proof steps

6. Carry over all facts obtained via the assignments.

Let $\mathrm{TS}(start, end) = \{(L[i], L[j]) \mid (L[i] + L[j] = w) \wedge (start \leq i < j \leq end)\}$.

Given: invariant and $i < j \rightarrow result = \mathrm{TS}(L, 0, N-1) \setminus \mathrm{TS}(L, i, j)$ and $i < j$.

4: **if** $L[i] + L[j] = w$ **then**

Given: $result = \mathrm{TS}(L, 0, N-1) \setminus \mathrm{TS}(L, i, j)$, $i < j$, and $L[i] + L[j] = w$.

By $L[i] + L[j] = w$ and the Definition of TS, we have: $(L[i], L[j]) \in \mathrm{TS}(L, i, j)$.

5: add $(L[i], L[j])$ to $result$.

6: $i, j := i + 1, j - 1$.

Known: $result_{\text{new}} = result_{\text{old}} \cup \{(L[i_{\text{old}}], L[j_{\text{old}}])\}$, $i_{\text{new}} = i_{\text{old}} + 1$, $j_{\text{new}} = j_{\text{old}} - 1$, $result_{\text{old}} = \mathrm{TS}(L, 0, N-1) \setminus \mathrm{TS}(L, i_{\text{old}}, j_{\text{old}})$, and $(L[i_{\text{old}}], L[j_{\text{old}}]) \in \mathrm{TS}(L, i_{\text{old}}, j_{\text{old}})$.

Induction step: prove that the invariant holds after each step of the loop.

Intermezzo: Correctness of BestTwoSum

High-level proof steps

7. Complete the proof for this case using all provided facts.

Let $\mathsf{TS}(start, end) = \{(L[i], L[j]) \mid (L[i] + L[j] = w) \wedge (start \leq i < j \leq end)\}$.

Given: invariant and $i < j \rightarrow result = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i, j)$ and $i < j$.

4: **if** $L[i] + L[j] = w$ **then**

Given: $result = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i, j)$, $i < j$, and $L[i] + L[j] = w$.

By $L[i] + L[j] = w$ and the Definition of TS, we have: $(L[i], L[j]) \in \mathsf{TS}(L, i, j)$.

5: add $(L[i], L[j])$ to $result$.

6: $i, j := i + 1, j - 1$.

Known: $result_{\text{new}} = result_{\text{old}} \cup \{(L[i_{\text{old}}], L[j_{\text{old}}])\}$, $i_{\text{new}} = i_{\text{old}} + 1$, $j_{\text{new}} = j_{\text{old}} - 1$, $result_{\text{old}} = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i_{\text{old}}, j_{\text{old}})$, and $(L[i_{\text{old}}], L[j_{\text{old}}]) \in \mathsf{TS}(L, i_{\text{old}}, j_{\text{old}})$.

Need to prove: $result_{\text{new}} = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i_{\text{new}}, j_{\text{new}})$.

Induction step: prove that the invariant holds after each step of the loop.

Intermezzo: Correctness of BestTwoSum

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4: **if** $L[i] + L[j] = w$ **then**

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$result_{\text{new}} = (\mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i_{\text{old}}, j_{\text{old}})) \cup \{(L[i_{\text{old}}], L[j_{\text{old}}])\}$.

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Intermezzo: Correctness of BestTwoSum

High-level proof steps

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$result_{\text{new}} = \mathsf{TS}(L, 0, N-1) \setminus (\mathsf{TS}(L, i_{\text{old}}, j_{\text{old}}) \setminus \{(L[i_{\text{old}}], L[j_{\text{old}}])\})$.

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Intermezzo: Correctness of BestTwoSum

High-level proof steps

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Let $\mathsf{TS}(start, end) = \{(L[i], L[j]) \mid (L[i] + L[j] = w) \wedge (start \leq i < j \leq end)\}$.

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Intermezzo: Correctness of BestTwoSum

High-level proof steps

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Induction step: the invariant holds after each step of the loop.

Intermezzo: Correctness of BestTwoSum

High-level proof steps

8. Next, the *else if* case of the case distinction.

Let $\mathsf{TS}(start, end) = \{(L[i], L[j]) \mid (L[i] + L[j] = w) \wedge (start \leq i < j \leq end)\}$.

Given: invariant and $i < j \rightarrow result = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i, j)$ and $i < j$.

Given: $result = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i, j)$, $i < j$, and $L[i] + L[j] = w$.

7: **else if** $L[i] + L[j] < w$ **then**

Given: $result = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i, j)$, $i < j$, and $L[i] + L[j] < w$.

8: $i := i + 1$.

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Intermezzo: Correctness of BestTwoSum

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Given: $result = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i, j)$, $i < j$, and $L[i] + L[j] < w$.

By $L[i] + L[j] < w$ and the Definition of TS, we have: $(L[i], v) \notin \mathsf{TS}(L, i, j), \forall v$.

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Intermezzo: Correctness of BestTwoSum

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8: $i := i + 1$.

Known: $i_{\text{new}} = i_{\text{old}} + 1$,
$result = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i_{\text{old}}, j)$, and $(L[i_{\text{old}}], v) \notin \mathsf{TS}(L, i_{\text{old}}, j)$.

Induction step: prove that the invariant holds after each step of the loop.

Intermezzo: Correctness of BestTwoSum

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Known: $i_{\text{new}} = i_{\text{old}} + 1$,
$result = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i_{\text{old}}, j)$, and $(L[i_{\text{old}}], v) \notin \mathsf{TS}(L, i_{\text{old}}, j)$.

Need to prove: $result = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i_{\text{new}}, j)$.

Induction step: prove that the invariant holds after each step of the loop.

Intermezzo: Correctness of BestTwoSum

High-level proof steps

8. Next, the *else if* case of the case distinction.

Let $\mathrm{TS}(start, end) = \{(L[i], L[j]) \mid (L[i] + L[j] = w) \wedge (start \leq i < j \leq end)\}$.

Given: invariant and $i < j \rightarrow result = \mathrm{TS}(L, 0, N-1) \setminus \mathrm{TS}(L, i, j)$ and $i < j$.

Given: $result = \mathrm{TS}(L, 0, N-1) \setminus \mathrm{TS}(L, i, j)$, $i < j$, and $L[i] + L[j] = w$.

7: **else if** $L[i] + L[j] < w$ **then**

Given: $result = \mathrm{TS}(L, 0, N-1) \setminus \mathrm{TS}(L, i, j)$, $i < j$, and $L[i] + L[j] < w$.

By $L[i] + L[j] < w$ and the Definition of TS, we have: $(L[i], v) \notin \mathrm{TS}(L, i, j), \forall v$.

8: $i := i + 1$.

Known: $i_{\text{new}} = i_{\text{old}} + 1$,
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Intermezzo: Correctness of BestTwoSum

High-level proof steps

8. Next, the *else if* case of the case distinction.

Let $\mathsf{TS}(start, end) = \{(L[i], L[j]) \mid (L[i] + L[j] = w) \wedge (start \leq i < j \leq end)\}$.

Given: invariant and $i < j \rightarrow result = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i, j)$ and $i < j$.

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Given: $result = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i, j)$, $i < j$, and $L[i] + L[j] < w$.

By $L[i] + L[j] < w$ and the Definition of TS, we have: $(L[i], v) \notin \mathsf{TS}(L, i, j), \forall v$.

8: $i := i + 1$.

Known: $i_{\text{new}} = i_{\text{old}} + 1$,
$result = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i_{\text{old}}, j)$, and $(L[i_{\text{old}}], v) \notin \mathsf{TS}(L, i_{\text{old}}, j)$.

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Induction step: prove that the invariant holds after each step of the loop.

Intermezzo: Correctness of BestTwoSum

High-level proof steps

8. Next, the *else if* case of the case distinction.

Let $\mathsf{TS}(start, end) = \{(L[i], L[j]) \mid (L[i] + L[j] = w) \wedge (start \leq i < j \leq end)\}$.

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7: **else if** $L[i] + L[j] < w$ **then**

Given: $result = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i, j)$, $i < j$, and $L[i] + L[j] < w$.

By $L[i] + L[j] < w$ and the Definition of TS, we have: $(L[i], v) \notin \mathsf{TS}(L, i, j), \forall v$.

8: $i := i + 1$.

Known: $i_{\text{new}} = i_{\text{old}} + 1$,
$result = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i_{\text{old}}, j)$, and $(L[i_{\text{old}}], v) \notin \mathsf{TS}(L, i_{\text{old}}, j)$.

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$result = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i_{\text{new}}, j)$.

Induction step: the invariant holds after each step of the loop.

Intermezzo: Correctness of BestTwoSum

High-level proof steps

9. Finally, the *else* case of the case distinction (analogous).

Let $\mathsf{TS}(start, end) = \{(L[i], L[j]) \mid (L[i] + L[j] = w) \wedge (start \leq i < j \leq end)\}$.

Given: invariant and $i < j \rightarrow result = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i, j)$ and $i < j$.

4: **if** $L[i] + L[j] = w$ **then**
5: add $(L[i], L[j])$ to *result*.
6: $i, j := i + 1, j - 1$.
7: **else if** $L[i] + L[j] < w$ **then**
8: $i := i + 1$.
9: **else**
10: $j := j - 1$.

Induction step: prove that the invariant holds after each step of the loop.

Intermezzo: Correctness of BestTwoSum

High-level proof steps

10. The invariant holds!

Let $\mathsf{TS}(start, end) = \{(L[i], L[j]) \mid (L[i] + L[j] = w) \wedge (start \leq i < j \leq end)\}$.

1: $result$:= empty bag.
2: $i, j := 0, N - 1$.
3: **while** $i < j$ **do** /* inv: $result = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i, j)$ */
4: **if** $L[i] + L[j] = w$ **then**
5: add $(L[i], L[j])$ to $result$.
6: $i, j := i + 1, j - 1$.
7: **else if** $L[i] + L[j] < w$ **then**
8: $i := i + 1$.
9: **else**
10: $j := j - 1$.

Known: invariant and $\neg(i < j) \rightarrow result = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i, j)$ and $i \geq j$.

11: **return** $result$. /* $result = \mathsf{TS}(L, 0, N-1)$. */

Intermezzo: Correctness of BestTwoSum

High-level proof steps

10. The invariant holds! Do not forget termination of the while-loop.

Let $\mathsf{TS}(start, end) = \{(L[i], L[j]) \mid (L[i] + L[j] = w) \wedge (start \leq i < j \leq end)\}$.

1: $result$:= empty bag.
2: $i, j := 0, N - 1$.
3: **while** $i < j$ **do** /* inv: $result = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i, j)$; bf: $j - i$ */
4: **if** $L[i] + L[j] = w$ **then**
5: add $(L[i], L[j])$ to $result$.
6: $i, j := i + 1, j - 1$.
7: **else if** $L[i] + L[j] < w$ **then**
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Known: invariant and $\neg(i < j) \rightarrow result = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i, j)$ and $i \geq j$.

11: **return** $result$. /* $result = \mathsf{TS}(L, 0, N-1)$. */

Intermezzo: Correctness of BestTwoSum

High-level proof steps

11. Prove the post-condition.

Let $\mathsf{TS}(start, end) = \{(L[i], L[j]) \mid (L[i] + L[j] = w) \wedge (start \leq i < j \leq end)\}$.

Known: invariant and $\neg(i < j) \rightarrow result = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i, j)$ and $i \geq j$.

11: **return** *result*. /* $result = \mathsf{TS}(L, 0, N-1)$. */

Intermezzo: Correctness of BestTwoSum

High-level proof steps

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Known: invariant and $\neg(i < j) \rightarrow result = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i, j)$ and $i \geq j$.

By $i \geq j$ and the Definition of TS, we have $\mathsf{TS}(L, i, j) = \emptyset$.

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Intermezzo: Correctness of BestTwoSum

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By $i \geq j$ and the Definition of TS, we have $\mathsf{TS}(L, i, j) = \emptyset$.

Hence, $result = \mathsf{TS}(L, 0, N-1) \setminus \mathsf{TS}(L, i, j) = \mathsf{TS}(L, 0, N-1) \setminus \emptyset = \mathsf{TS}(L, 0, N-1)$.

11: **return** *result*. /* $result = \mathsf{TS}(L, 0, N-1)$. */

Intermezzo: Correctness of BestTwoSum

Warning

You cannot learn correctness proofs from slides: practice on simple algorithms yourself!

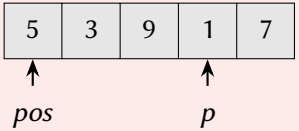
A first sort algorithm: SelectionSort

Algorithm SelectionSort(L):

Input: List $L[0 \ldots N)$ of N values.

1: **for** $pos := 0$ **to** $N - 2$ **do**
2: Find the position p of the *minimum value* in $L[pos \ldots N)$.
3: Exchange $L[pos]$ and $L[p]$.

A first sort algorithm: SelectionSort



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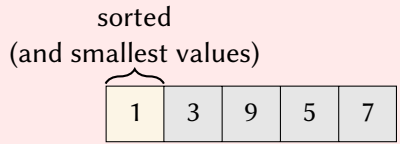

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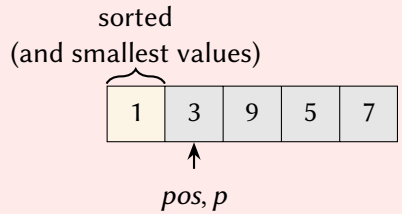

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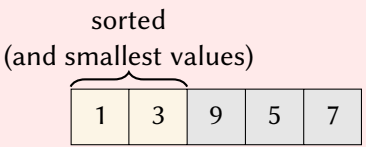

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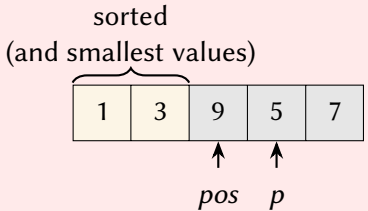

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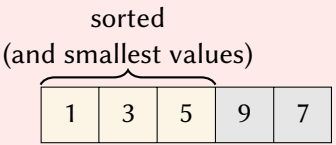

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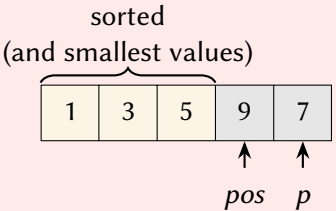

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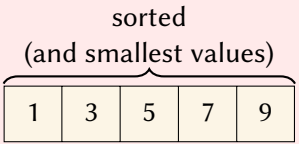

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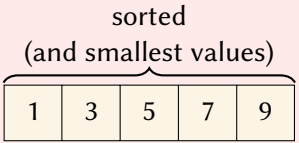
Runtime complexity of SelectionSort

A good estimate: number of comparisons and changes to list values.

A first sort algorithm: SelectionSort



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Input: List $L[0 \ldots N)$ of N values.

1: **for** $pos := 0$ **to** $N - 2$ **do**
   Find the position p of the *minimum value* in $L[pos \ldots N)$.
2: $p := pos$.
3: **for** $i := pos + 1$ **to** $N - 1$ **do**
4: **if** $L[i] < L[p]$ **then**
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6: Exchange $L[pos]$ and $L[p]$.

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3:     for i := pos + 1 to N − 1 do
4:         if L[i] < L[p] then
5:             p := i.
6:     Exchange L[pos] and L[p].
```

Comparisons: $N - 1 - pos$.
Changes: 0.

Runtime complexity of SelectionSort

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Input: List $L[0 \ldots N)$ of N values.

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1: for pos := 0 to N − 2 do
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4:         if L[i] < L[p] then
5:             p := i.
6:     Exchange L[pos] and L[p].
```

Comparisons: $\sum_{pos=0}^{N-2} (N - 1 - pos)$.

Changes: $2(N - 1)$.

Runtime complexity of SelectionSort

A good estimate: number of comparisons and changes to list values.

A first sort algorithm: SelectionSort

Algorithm SelectionSort(L):
Input: List $L[0 \ldots N)$ of N values.

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Runtime complexity of SelectionSort

A good estimate: number of comparisons and changes to list values.

$$\text{Comparisons: } \sum_{pos=0}^{N-2} (N - 1 - pos) = \sum_{j=1}^{N-1} j$$

A first sort algorithm: SelectionSort

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Changes: $2(N - 1)$.

Runtime complexity of SelectionSort

A good estimate: number of comparisons and changes to list values.

$$\text{Comparisons: } \sum_{pos=0}^{N-2} (N - 1 - pos) = \sum_{j=1}^{N-1} j = \frac{N(N-1)}{2}$$

A first sort algorithm: SelectionSort

Algorithm SelectionSort(L):
Input: List $L[0 \ldots N)$ of N values.

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1: for pos := 0 to N − 2 do
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Comparisons: $\sum_{pos=0}^{N-2} (N - 1 - pos) = \Theta(N^2)$.

Changes: $2(N - 1) = \Theta(N)$.

Runtime complexity of SelectionSort

A good estimate: number of comparisons and changes to list values.

$$\text{Comparisons: } \sum_{pos=0}^{N-2} (N - 1 - pos) = \sum_{j=1}^{N-1} j = \frac{N(N-1)}{2} = \Theta(N^2).$$

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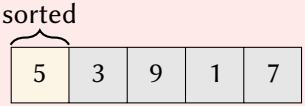
Correctness of SelectionSort: Some tips

- Rework the for-loops into while loops.
- The inner loop only changes p: prove whatever that loop does separately.
- Include as much information into the invariant of the outer loop.
  What exactly do we know about the values in $L[0 \ldots pos)$.
- A complete proof guarantees that list L keeps all original values!

A second sort algorithm: InsertionSort

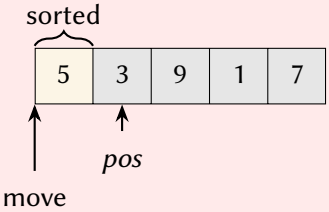


Algorithm InsertionSort(L):

Input: List $L[0 \ldots N)$ of N values.

1: **for** $pos := 1$ **to** $N - 1$ **do**
2: $v := L[pos]$.
3: Move all values $w \in L[0 \ldots pos)$ with $v > w$ one to the right.
4: $L[p] := v$.

A second sort algorithm: InsertionSort



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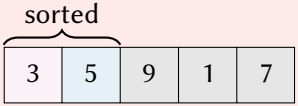

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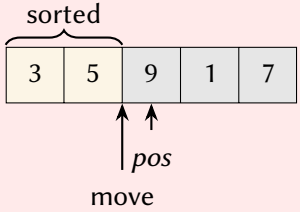

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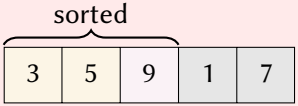

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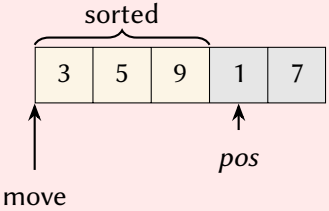

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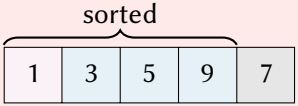

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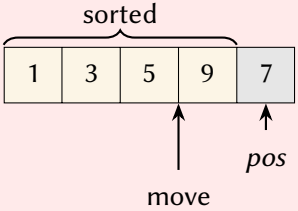

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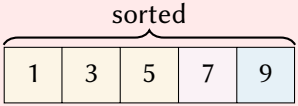

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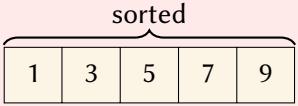

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Runtime complexity of InsertionSort

A good estimate: number of comparisons and exchanges of list values.

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Comparisons: $\leq pos$.
Changes: $\leq pos$.

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Runtime complexity of InsertionSort

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When does InsertionSort have N^2 comparisons and changes?

Reverse-ordered array: every next array is moved to the start of the list.

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When does InsertionSort have less than N^2 comparisons and changes?

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When does InsertionSort have less than N^2 comparisons and changes?

Ordered array: N comparisons and changes as every value stays in place.

A second sort algorithm: InsertionSort

Algorithm InsertionSort(L):
Input: List $L[0 \ldots N)$ of N values, $L = \mathcal{L}$.

1: $pos := 1$.
2: **while** $pos \neq N$ **do**
   /* inv: $L[0 \ldots pos)$ is ordered and L holds the same values as $\mathcal{L}$, bf: $N - pos$. */
3: $v := L[pos]$.
4: $p := pos$.
5: **while** $p > 0$ **and** $v < L[p-1]$ **do**
   /* inv: $F = L[0 \ldots p)$ is ordered, $S = L[p+1 \ldots pos+1)$ is ordered, all values in F are smaller than the values in S, all values in S are larger than v, and the values in F, S, $[v]$, and $L[pos+1 \ldots, N)$ are exactly the values in $\mathcal{L}$, bf: p. */
6: $L[p] := L[p-1]$.
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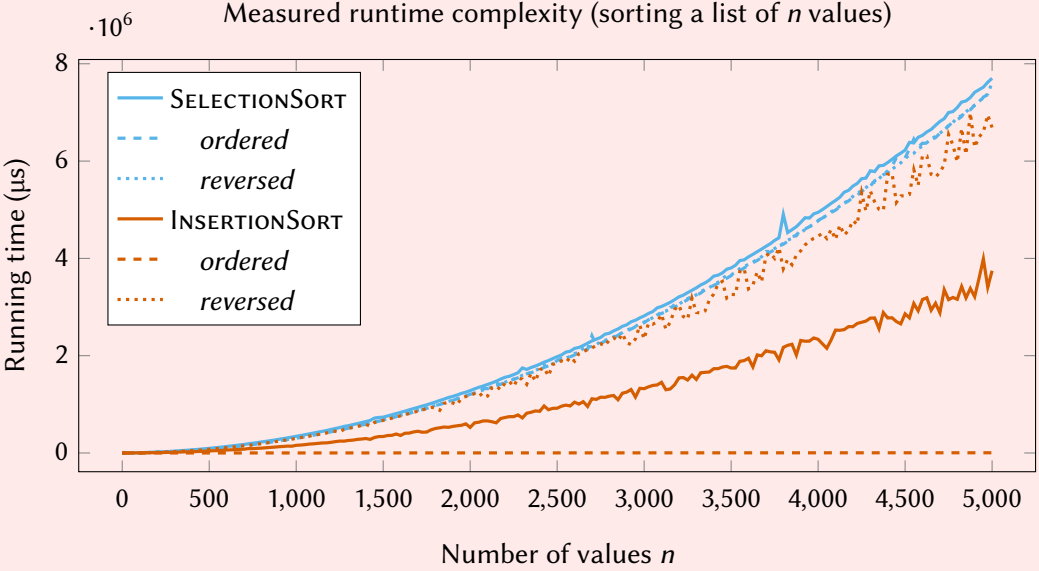
A summary of basic sorting

	Comparisons	Changes	Memory
SelectionSort	$\Theta(N^2)$	$\Theta(N)$	$\Theta(1)$
InsertionSort	$O(N^2)$	$O(N^2)$	$\Theta(1)$

A summary of basic sorting



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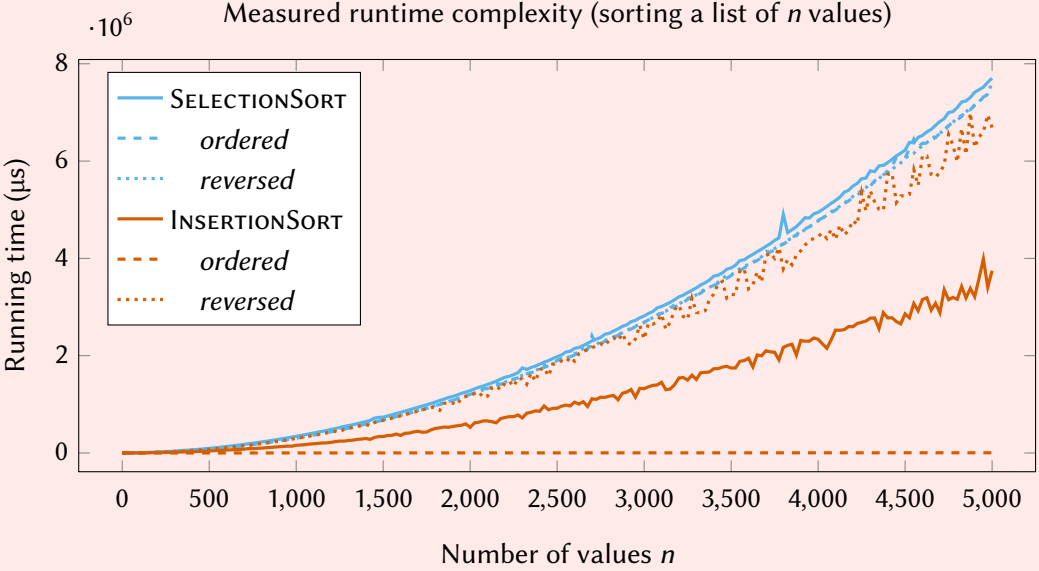

Toward faster sorting

The issue with SelectionSort and InsertionSort

- The algorithms do not perform “global reorderings”.
- The algorithms sort one element at a time.
  E.g., small elements at the end of the list are moved to the front one at a time.

MergeSortR: Sorting using divide-and-conquer

Divide-and-conquer

Divide Turn problem into smaller subproblems.

Conquer Solve the smaller subproblems using *recursion*.

Combine Combine the subproblem solutions into a final solution.

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BinarySearchR is a divide-and-conquer algorithm.

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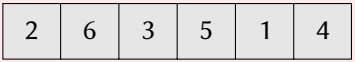
Merge two sorted halves together to obtain the result.

MergeSortR: High-level overview

Algorithm MergeSortR($L[start \ldots end]$)**:**



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Algorithm MergeSortR($L[start \ldots end)$):

1: **if** $end - start > 1$ **then**

6: **else return** L.

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1: **if** $end - start > 1$ **then**

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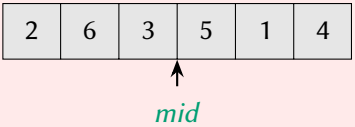

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Algorithm MergeSortR($L[start \ldots end)$):

1: **if** $end - start > 1$ **then**
2: $mid := (end - start)$ div 2.
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4: $L_2 :=$ MergeSortR($L[mid \ldots end)$).

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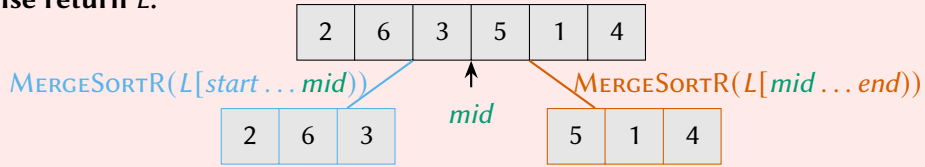

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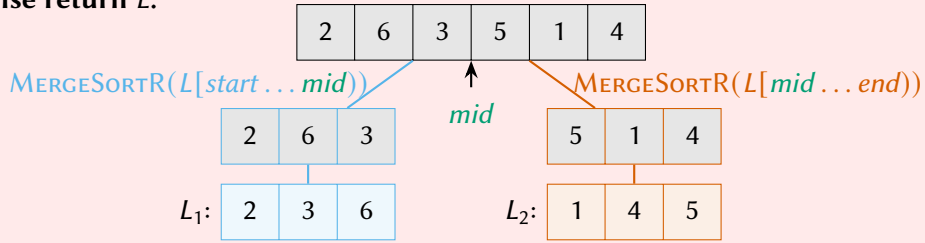

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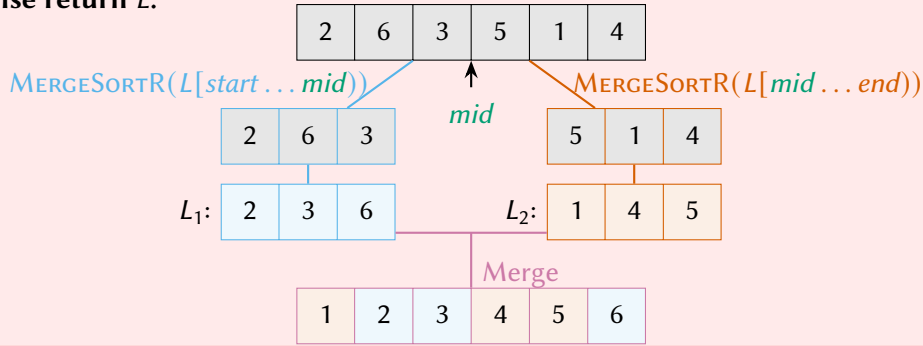

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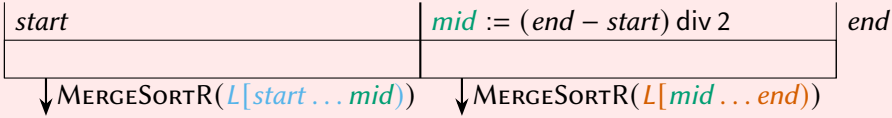

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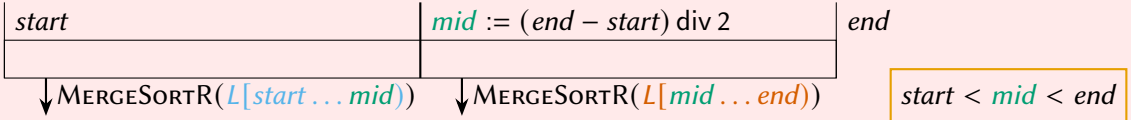

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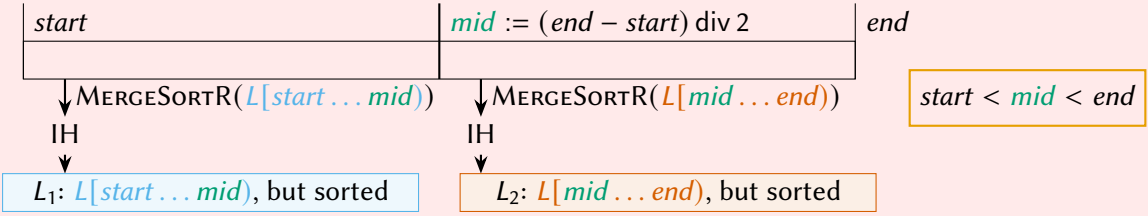

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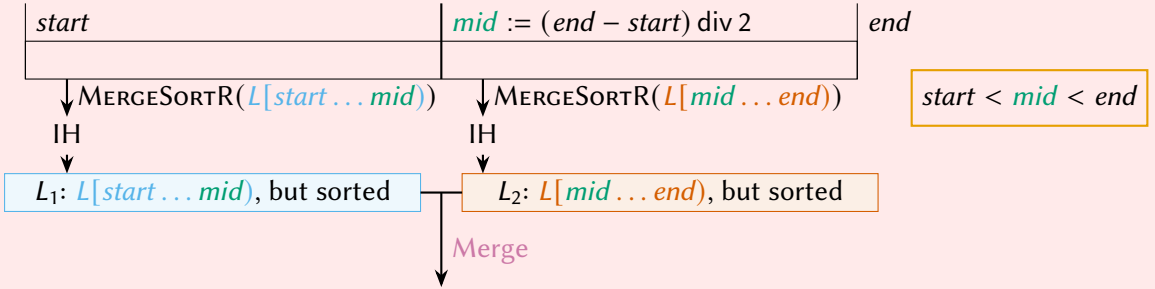

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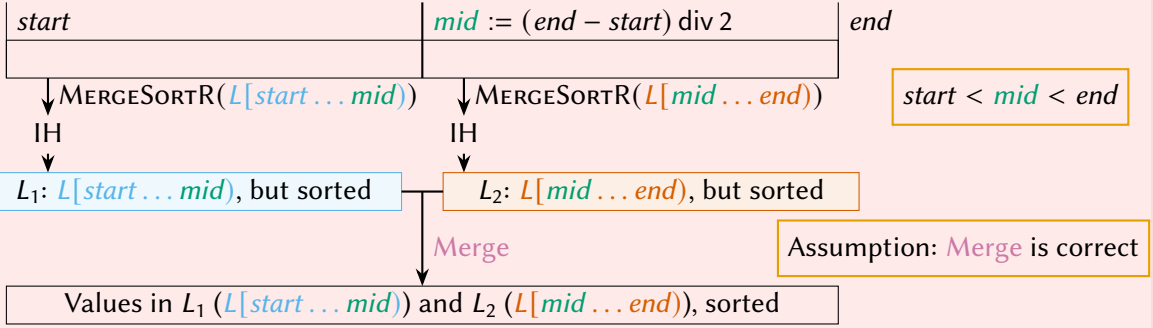

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1: R is a new array for $N_1 + N_2$ values.

10: **return** R.

Assumption: Merge is correct

Algorithm MERGE($L_1[0 \ldots N_1)$, $L_2[0 \ldots N_2)$):
Input: L_1 and L_2 are sorted.

1: R is a new array for $N_1 + N_2$ values.
2: $i_1, i_2 := 0, 0$.
3: **while** $i_1 < N_1$ **or** $i_2 < N_2$ **do**

10: **return** R.

Assumption: Merge is correct

Algorithm MERGE($L_1[0 \ldots N_1)$, $L_2[0 \ldots N_2)$):
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L_1:

2	3	6

L_2:

1	4	5

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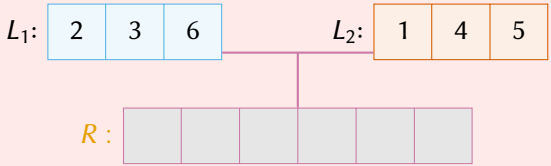

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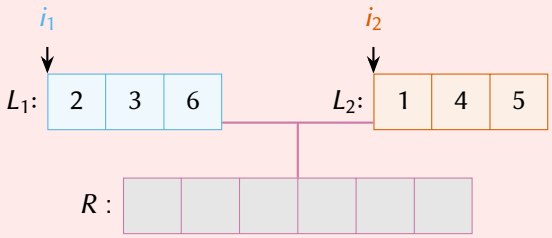

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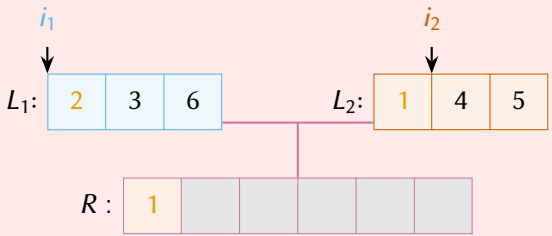

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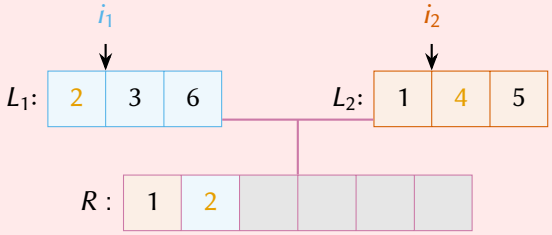

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2: $i_1, i_2 := 0, 0$.
3: **while** $i_1 < N_1$ **or** $i_2 < N_2$ **do**
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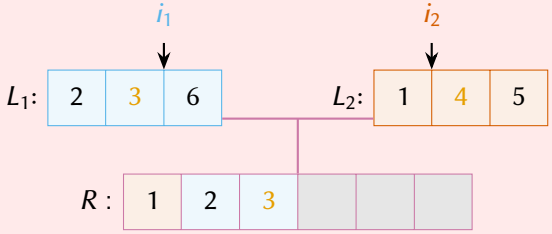

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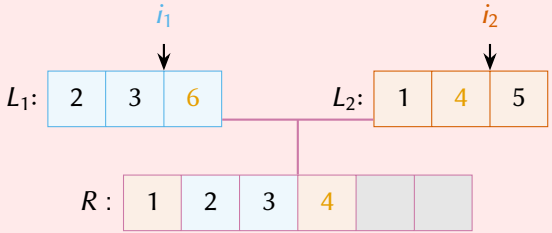

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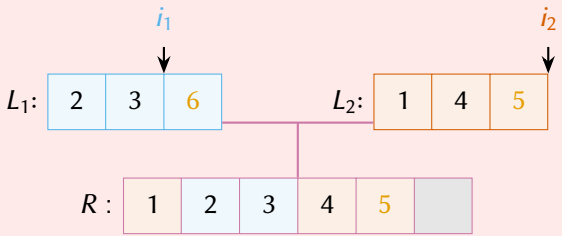

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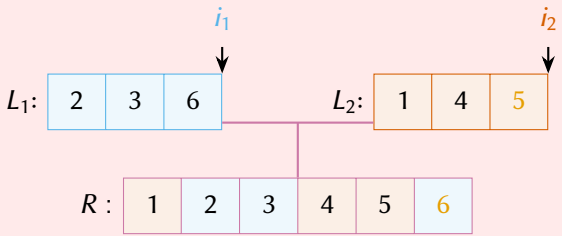

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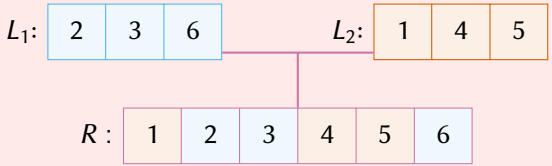

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Input: L_1 and L_2 are sorted.

1: R is a new array for $N_1 + N_2$ values.
2: $i_1, i_2 := 0, 0$.
3: **while** $i_1 < N_1$ **or** $i_2 < N_2$ **do**
   /* inv: $R[0 \ldots i_1 + i_2)$ has all values from $L_1[0 \ldots i_1)$ and $L_2[0 \ldots i_2)$, sorted. */
   /* bf: $(N_1 + N_2) - (i_1 + i_2)$. */
4: **if** $i_2 = N_2$ **or** ($i_1 < N_1$ **and** $L_1[i_1] < L_2[i_2]$) **then**
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MergeSortR: A complete example

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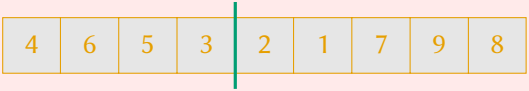

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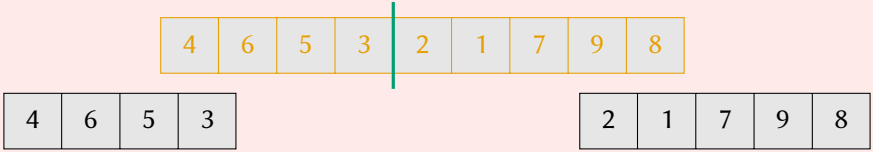

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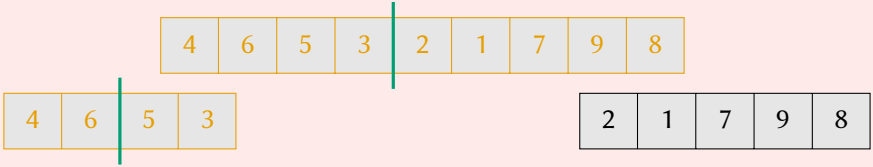

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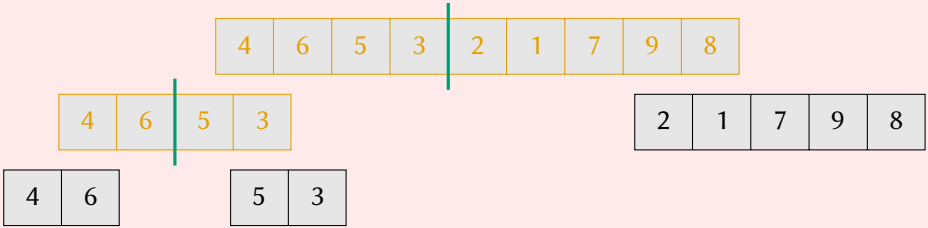

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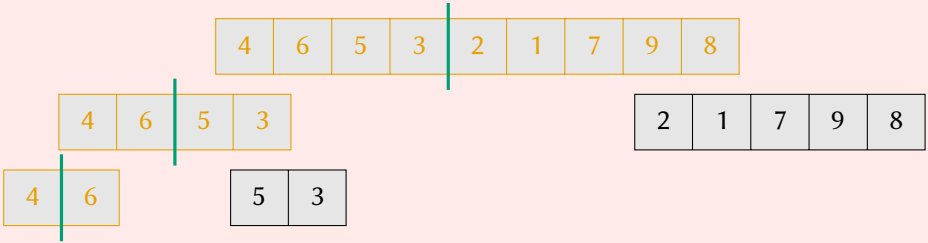

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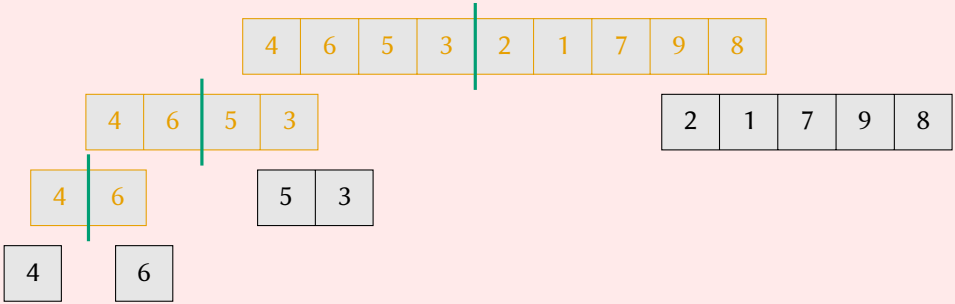

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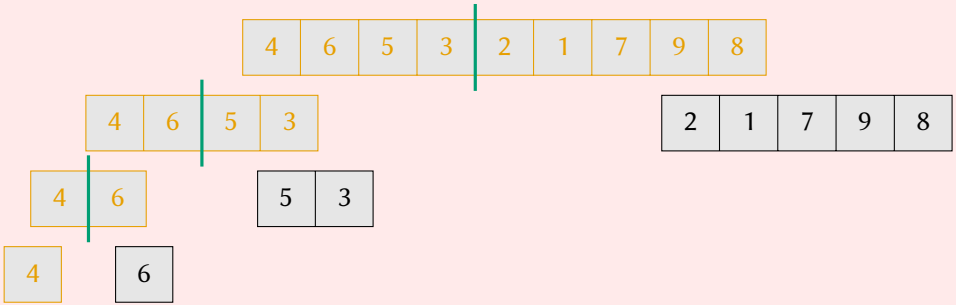

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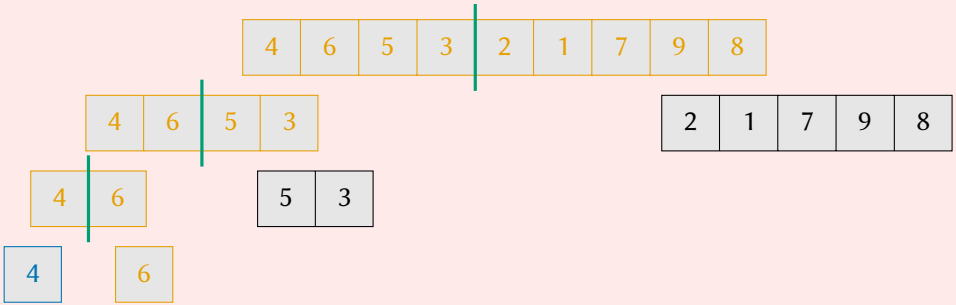

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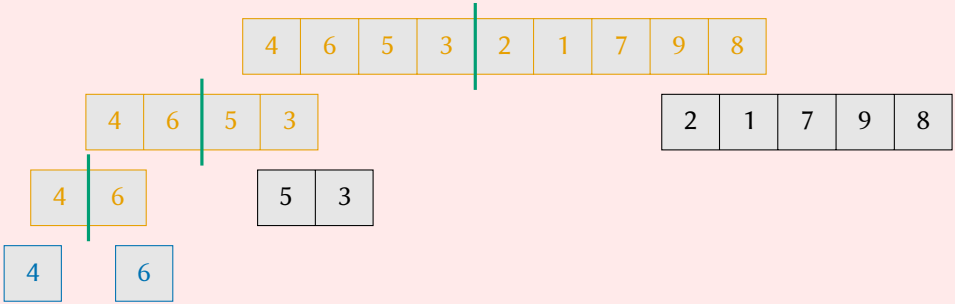

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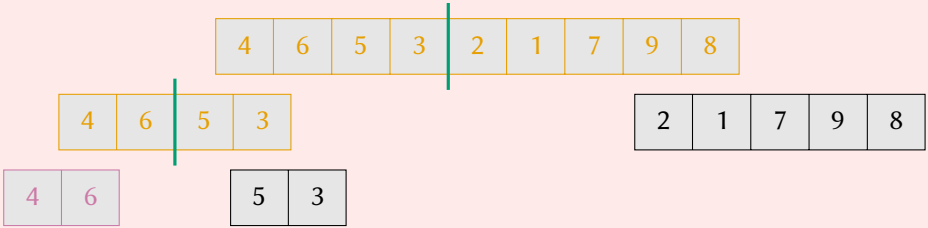

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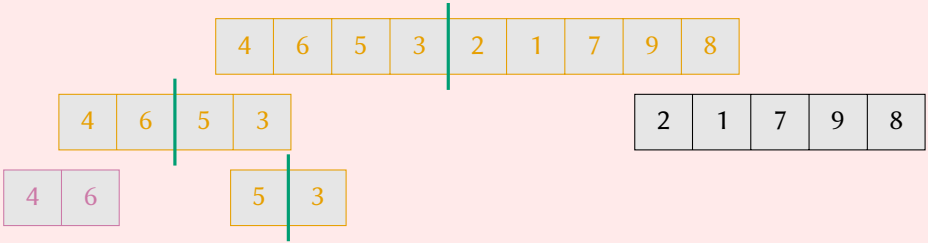

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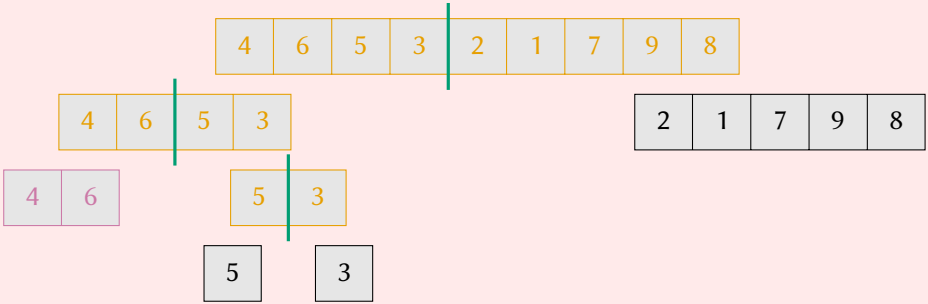

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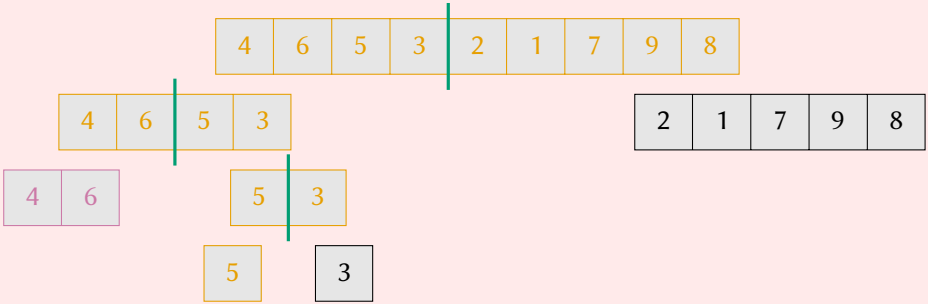

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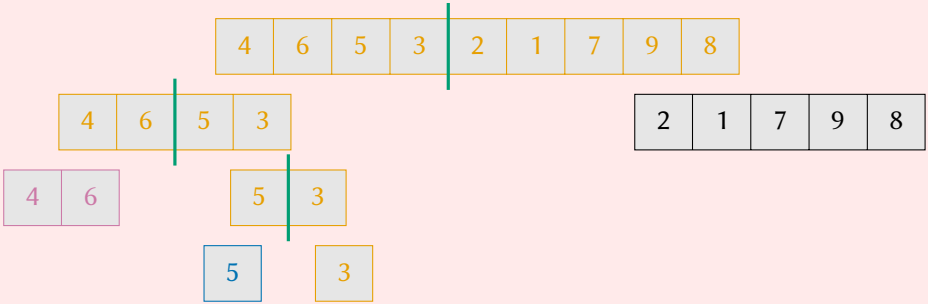

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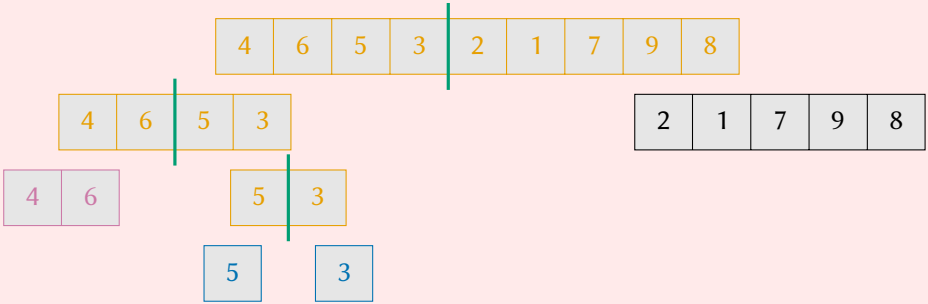

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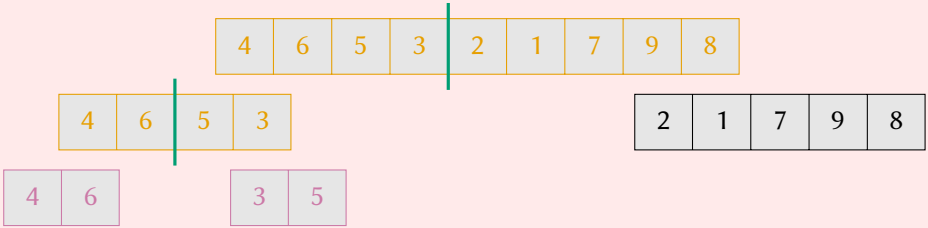

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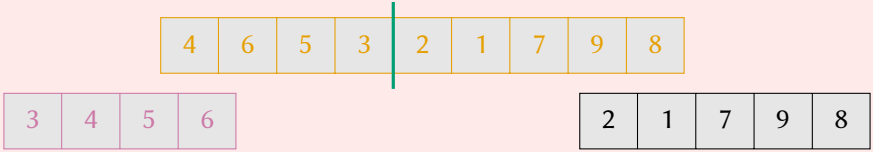

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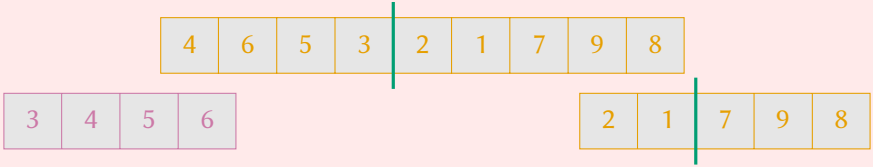

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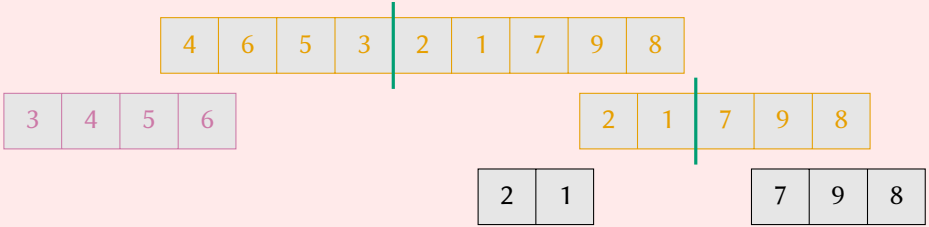
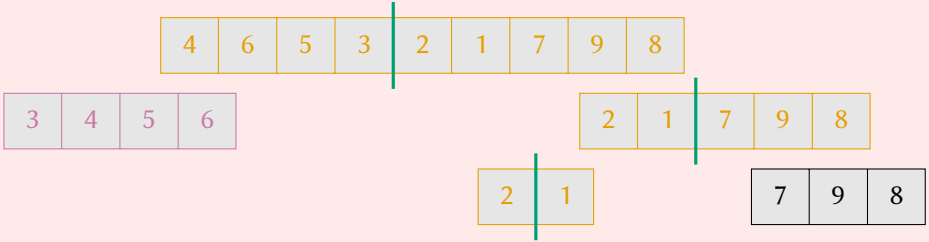
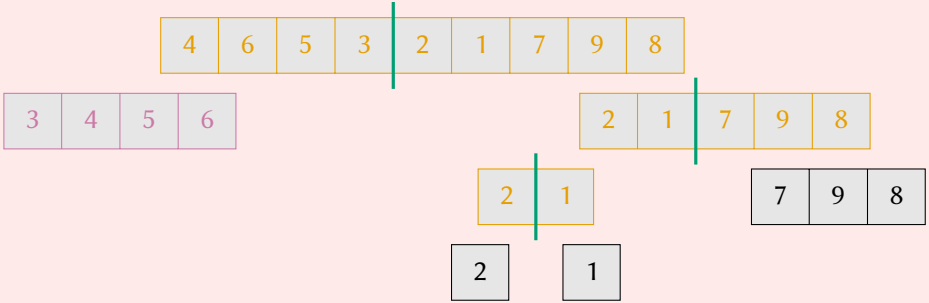


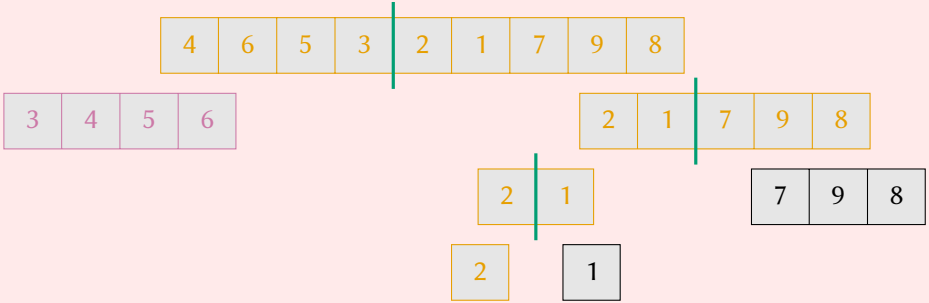
MergeSortR: A complete example



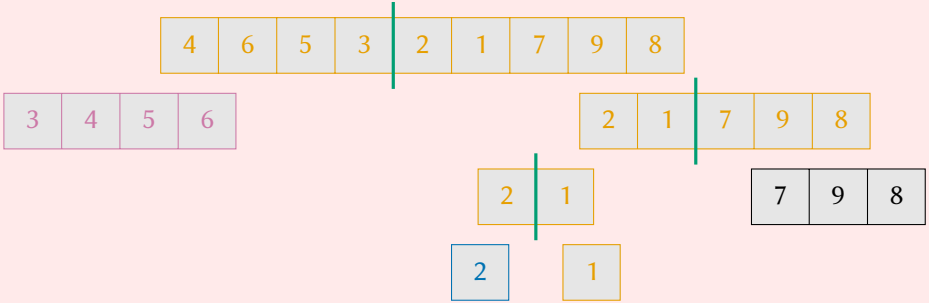
MergeSortR: A complete example



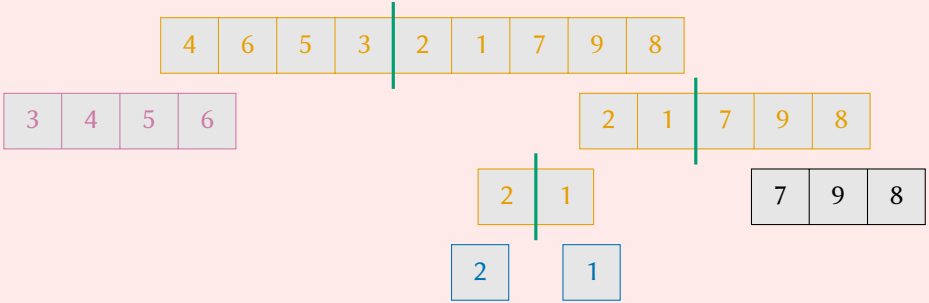
MergeSortR: A complete example



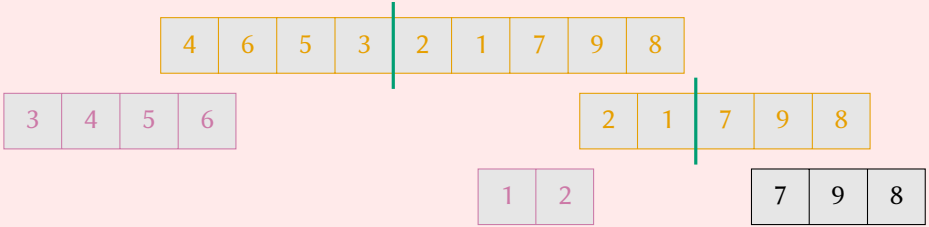
MergeSortR: A complete example



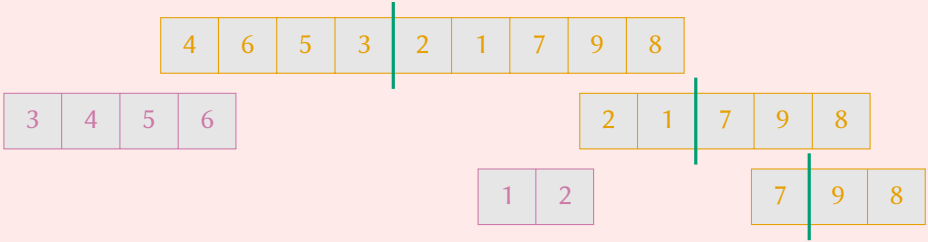
MergeSortR: A complete example



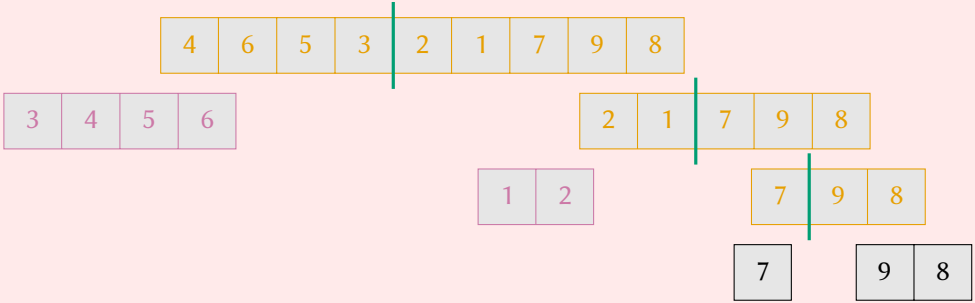
MergeSortR: A complete example



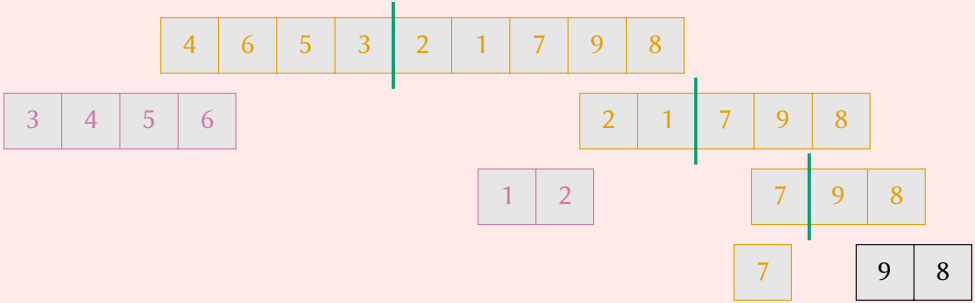
MergeSortR: A complete example



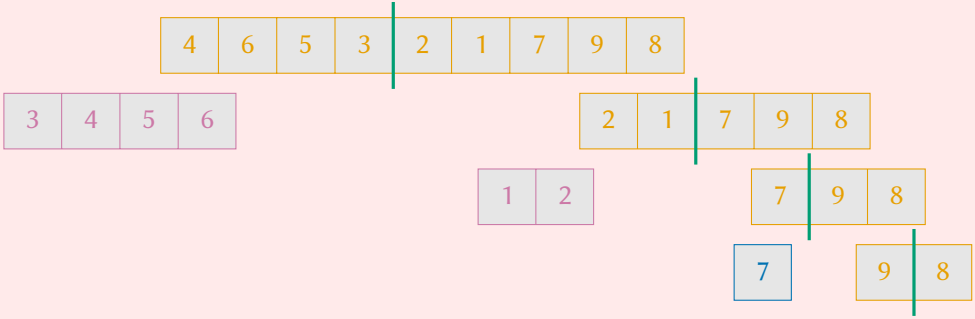
MergeSortR: A complete example



MergeSortR: A complete example



MergeSortR: A complete example



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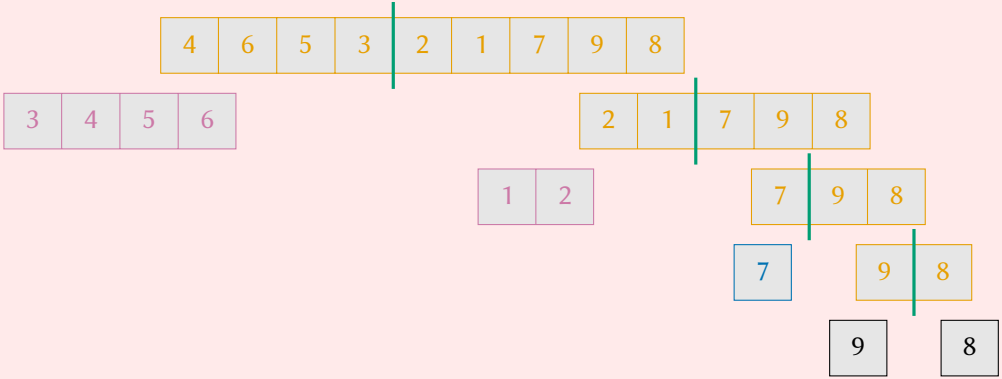
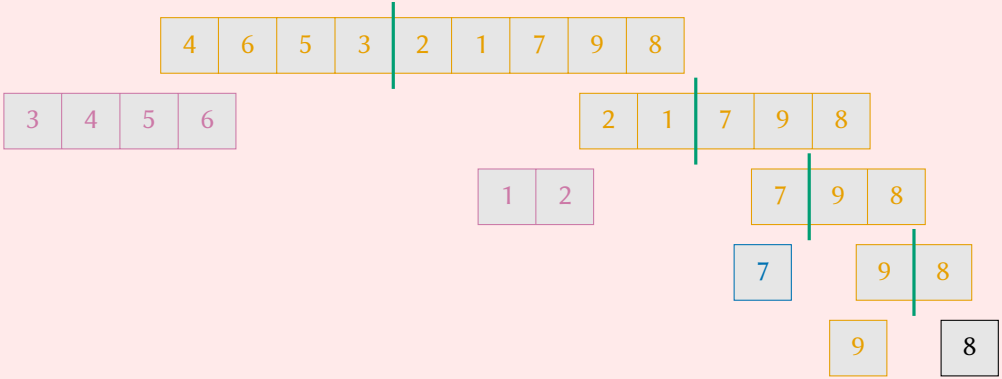


MergeSortR: A complete example



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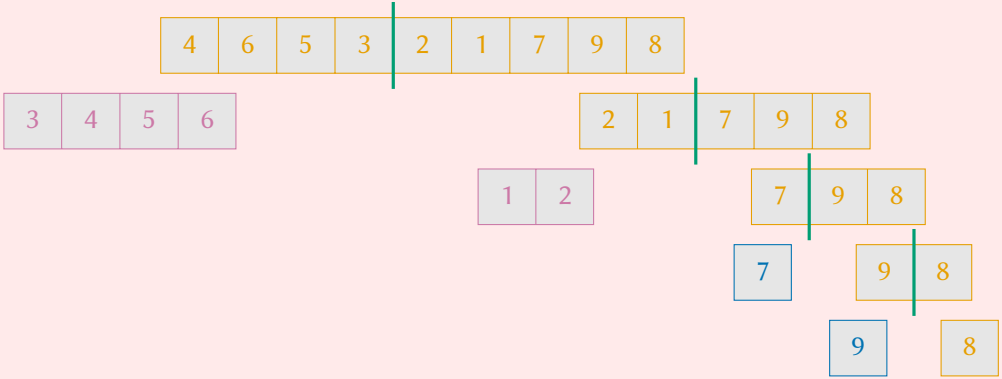
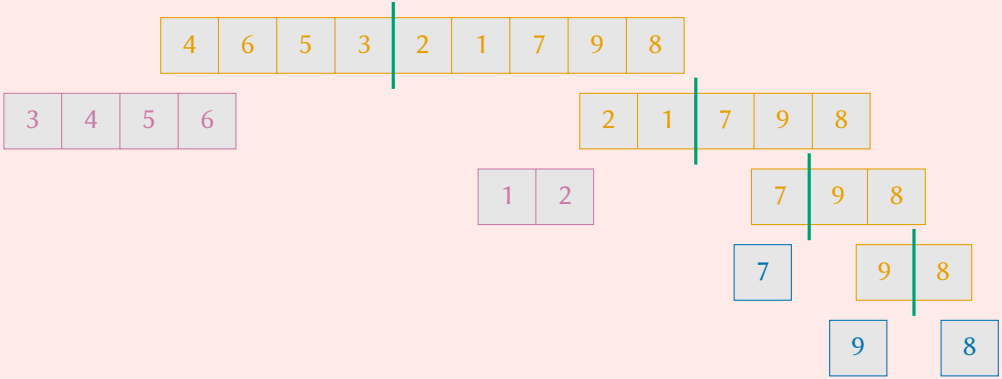


MergeSortR: A complete example



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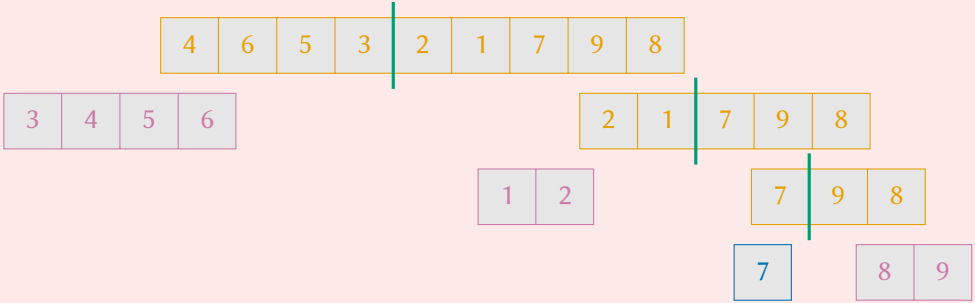
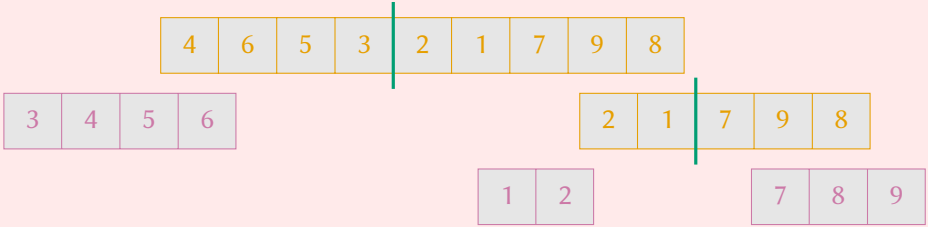


MergeSortR: A complete example



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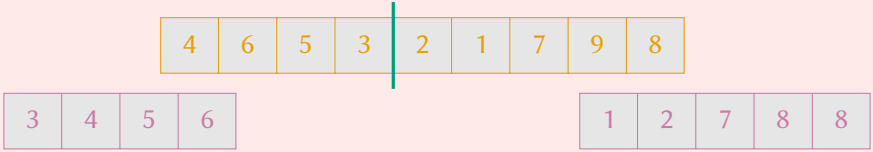

MergeSortR: A complete example

The complexity of MergeSortR

Plan

1. First, determine the complexity of a Merge call.
2. Then we can look at MergeSortR.

The complexity of MergeSortR

Algorithm Merge($L_1[0 \ldots N_1)$, $L_2[0 \ldots N_2)$):
Input: L_1 and L_2 are sorted.

1: R is a new array for $N_1 + N_2$ values.
2: $i_1, i_2 := 0, 0$.
3: **while** $i_1 < N_1$ **or** $i_2 < N_2$ **do**
4: **if** $i_2 = N_2$ **or** ($i_1 < N_1$ **and** $L_1[i_1] < L_2[i_2]$) **then**
5: $R[i_1 + i_2] := L_1[i_1]$.
6: $i_1 := i_1 + 1$.
7: **else**
8: $R[i_1 + i_2] := L_2[i_2]$.
9: $i_2 := i_2 + 1$.
10: **return** R.

The complexity of MergeSortR

Algorithm Merge($L_1[0 \ldots N_1)$, $L_2[0 \ldots N_2)$):
Input: L_1 and L_2 are sorted.

```
1:  R is a new array for N1 + N2 values.
2:  i1, i2 := 0, 0.
3:  while i1 < N1 or i2 < N2 do
4:      if i2 = N2 or (i1 < N1 and L1[i1] < L2[i2]) then
5:          R[i1 + i2] := L1[i1].
6:          i1 := i1 + 1.
7:      else
8:          R[i1 + i2] := L2[i2].
9:          i2 := i2 + 1.
10: return R.
```

Comparisons: $< N_1 + N_2$.
Changes: $N_1 + N_2$.

The complexity of MergeSortR

Algorithm MergeSortR($L[start \ldots end)$):

1: **if** $end - start > 1$ **then**
2: $mid := (end - start)$ div 2.
3: $L_1 :=$ MergeSortR($L[start \ldots mid)$).
4: $L_2 :=$ MergeSortR($L[mid \ldots end)$).
5: **return** Merge(L_1, L_2). N comparisons and changes.
6: **else return** L.

The complexity of MergeSortR

Algorithm MergeSortR($L[start \ldots end)$):

1: **if** $end - start > 1$ **then**
2: $\quad mid := (end - start)$ div 2.
3: $\quad L_1 :=$ MergeSortR($L[start \ldots mid)$).
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5: $\quad$ **return** Merge(L_1, L_2). N comparisons and changes.
6: **else return** L.

} Recursive case.

} Base case.

The complexity of MergeSortR

Algorithm MergeSortR($L[start \ldots end)$):

1: **if** $end - start > 1$ **then**
2: $\quad mid := (end - start)$ div 2.
3: $\quad L_1 := \text{MergeSortR}(L[start \ldots mid))$.
4: $\quad L_2 := \text{MergeSortR}(L[mid \ldots end))$.
5: $\quad$ **return** Merge(L_1, L_2). N comparisons and changes.
6: **else return** L.

} Recursive case.

} Base case.

The runtime complexity of MergeSortR(L, $start$, end) with $N = end - start$ is

$$T(N) = \begin{cases} 1 & \text{if } N \leq 1; \\ T\left(\left\lfloor \frac{N}{2} \right\rfloor\right) + T\left(\left\lceil \frac{N}{2} \right\rceil\right) + N & \text{if } N > 1. \end{cases}$$

The complexity of MergeSortR

Algorithm MergeSortR($L[start \ldots end)$):

1: **if** $end - start > 1$ **then**
2: $\quad mid := (end - start)$ div 2.
3: $\quad L_1 :=$ MergeSortR($L[start \ldots mid)$).
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} Recursive case.

} Base case.

The runtime complexity of MergeSortR(L, $start$, end) with $N = end - start$ is

$$T(N) = \begin{cases} 1 & \text{if } N \leq 1; \\ 2T\left(\frac{N}{2}\right) + N & \text{if } N > 1. \end{cases}$$

Assumption: N is a power-of-two.

The complexity of MergeSortR

$$T(N) = \begin{cases} 1 & \text{if } N \leq 1; \\ 2T\left(\frac{N}{2}\right) + N & \text{if } N > 1. \end{cases}$$

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How can we determine that $T(N) = f(N)$ for a closed-form $f(N)$?

The complexity of MergeSortR

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We can use induction!

The complexity of MergeSortR

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Assumption: N is a power-of-two.

How can we determine that $T(N) = f(N)$ for a closed-form $f(N)$?

We can use induction!?

We need to know $f(N)$ to formalize an induction hypothesis!

The complexity of MergeSortR

$$T(N) = \begin{cases} 1 & \text{if } N \leq 1; \\ 2T\left(\frac{N}{2}\right) + N & \text{if } N > 1. \end{cases}$$

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Recurrence tree for $T(N)$

N

The complexity of MergeSortR

$$T(N) = \begin{cases} 1 & \text{if } N \leq 1; \\ 2T\left(\frac{N}{2}\right) + N & \text{if } N > 1. \end{cases}$$

Assumption: N is a power-of-two.

Recurrence tree for $T(N)$

N

Number	Cost
$1 = 2^0$	N

The complexity of MergeSortR

$$T(N) = \begin{cases} 1 & \text{if } N \leq 1; \\ 2T\left(\frac{N}{2}\right) + N & \text{if } N > 1. \end{cases}$$

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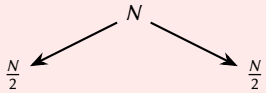
Recurrence tree for $T(N)$



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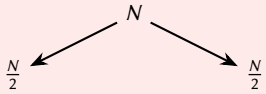

Number	Cost
$1 = 2^0$	N
$2 = 2^1$	$\frac{N}{2}$

The complexity of MergeSortR

$$T(N) = \begin{cases} 1 & \text{if } N \leq 1; \\ 2T\left(\frac{N}{2}\right) + N & \text{if } N > 1. \end{cases}$$

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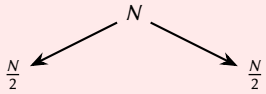

Number	Cost	Total
$1 = 2^0$	N	$1N = N$
$2 = 2^1$	$\frac{N}{2}$	$2\frac{N}{2} = N$

The complexity of MergeSortR

$$T(N) = \begin{cases} 1 & \text{if } N \leq 1; \\ 2T\left(\frac{N}{2}\right) + N & \text{if } N > 1. \end{cases}$$

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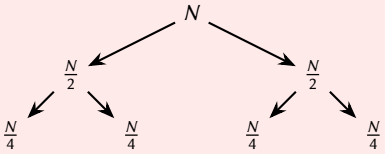

Number	Cost	Total
$1 = 2^0$	N	$1N = N$
$2 = 2^1$	$\frac{N}{2}$	$2\frac{N}{2} = N$
$4 = 2^2$	$\frac{N}{4}$	$4\frac{N}{4} = N$

The complexity of MergeSortR

$$T(N) = \begin{cases} 1 & \text{if } N \leq 1; \\ 2T\left(\frac{N}{2}\right) + N & \text{if } N > 1. \end{cases}$$

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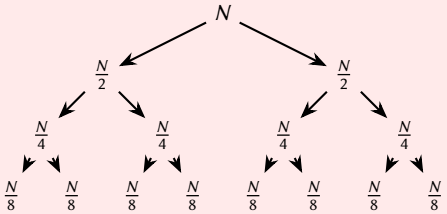

Number	Cost	Total
$1 = 2^0$	N	$1N = N$
$2 = 2^1$	$\frac{N}{2}$	$2\frac{N}{2} = N$
$4 = 2^2$	$\frac{N}{4}$	$4\frac{N}{4} = N$
$8 = 2^3$	$\frac{N}{8}$	$8\frac{N}{8} = N$

The complexity of MergeSortR

$$T(N) = \begin{cases} 1 & \text{if } N \leq 1; \\ 2T\left(\frac{N}{2}\right) + N & \text{if } N > 1. \end{cases}$$

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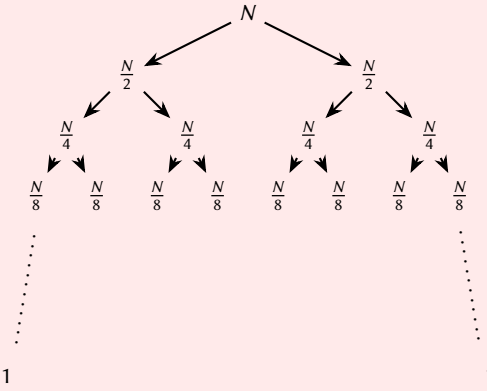
Recurrence tree for $T(N)$



Number	Cost	Total
$1 = 2^0$	N	$1N = N$
$2 = 2^1$	$\frac{N}{2}$	$2\frac{N}{2} = N$
$4 = 2^2$	$\frac{N}{4}$	$4\frac{N}{4} = N$
$8 = 2^3$	$\frac{N}{8}$	$8\frac{N}{8} = N$
$\vdots$	$\vdots$	$\vdots$
2^i	$\frac{N}{2^i}$	$2^i\frac{N}{2^i} = N$
$\vdots$	$\vdots$	$\vdots$

The complexity of MergeSortR

$$T(N) = \begin{cases} 1 & \text{if } N \leq 1; \\ 2T\left(\frac{N}{2}\right) + N & \text{if } N > 1. \end{cases}$$

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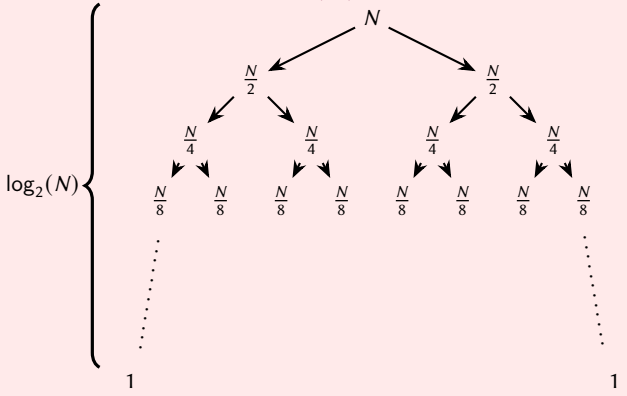

Number	Cost	Total
$1 = 2^0$	N	$1N = N$
$2 = 2^1$	$\frac{N}{2}$	$2\frac{N}{2} = N$
$4 = 2^2$	$\frac{N}{4}$	$4\frac{N}{4} = N$
$8 = 2^3$	$\frac{N}{8}$	$8\frac{N}{8} = N$
$\vdots$	$\vdots$	$\vdots$
2^i	$\frac{N}{2^i}$	$2^i\frac{N}{2^i} = N$
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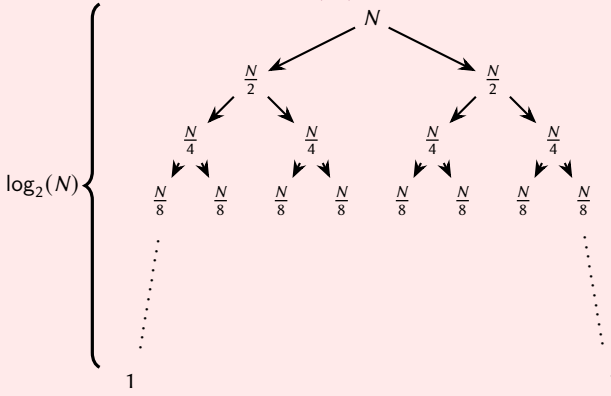
Recurrence tree for $T(N)$



Number	Cost	Total
$1 = 2^0$	N	$1N = N$
$2 = 2^1$	$\frac{N}{2}$	$2\frac{N}{2} = N$
$4 = 2^2$	$\frac{N}{4}$	$4\frac{N}{4} = N$
$8 = 2^3$	$\frac{N}{8}$	$8\frac{N}{8} = N$
$\vdots$	$\vdots$	$\vdots$
2^i	$\frac{N}{2^i}$	$2^i\frac{N}{2^i} = N$
$\vdots$	$\vdots$	$\vdots$
		+
		$\Theta(N \log_2(N))$

The complexity of MergeSortR

$$T(N) = \begin{cases} 1 & \text{if } N \leq 1; \\ 2T\left(\frac{N}{2}\right) + N & \text{if } N > 1. \end{cases}$$

Assumption: N is a power-of-two.

Can do without a power-of-two assumption?

The complexity of MergeSortR

$$T(N) = \begin{cases} 1 & \text{if } N \leq 1; \\ 2T\left(\frac{N}{2}\right) + N & \text{if } N > 1. \end{cases}$$

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Can do without a power-of-two assumption?

For any N, we have $2^{\lfloor \log_2(N) \rfloor} \leq N \leq 2^{\lceil \log_2(N) \rceil}$.

The complexity of MergeSortR

$$T(N) = \begin{cases} 1 & \text{if } N \leq 1; \\ 2T\left(\frac{N}{2}\right) + N & \text{if } N > 1. \end{cases}$$

Assumption: N is a power-of-two.

Can do without a power-of-two assumption?

For any N, we have $2^{\lfloor \log_2(N) \rfloor} \leq N \leq 2^{\lceil \log_2(N) \rceil}$.

The assumption provides lower and upper bounds that are off by a small factor

→ Typically good enough to understand the complexity of your code.

The complexity of MergeSortR

$$T(N) = \begin{cases} 1 & \text{if } N \leq 1; \\ T\left(\left\lfloor \frac{N}{2} \right\rfloor\right) + T\left(\left\lceil \frac{N}{2} \right\rceil\right) + N & \text{if } N > 1. \end{cases}$$

Can we prove $T(N) = \Theta(N \log_2(N))$ *exactly*?

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This induction becomes messy due to terms $\left\lfloor \frac{N}{2} \right\rfloor$ and $\left\lceil \frac{N}{2} \right\rceil$.

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$$\log_2(2+1) - 1 = \log_2(2) + (\log_2(3) - \log_2(2)) - 1 \approx 1 + (1.6 - 1) - 1) = \log_2(2) - 0.4.$$

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For *big enough* values of i and c_2, $i > B$, this is certainly true!

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Trick: make sure we always have big values of i.

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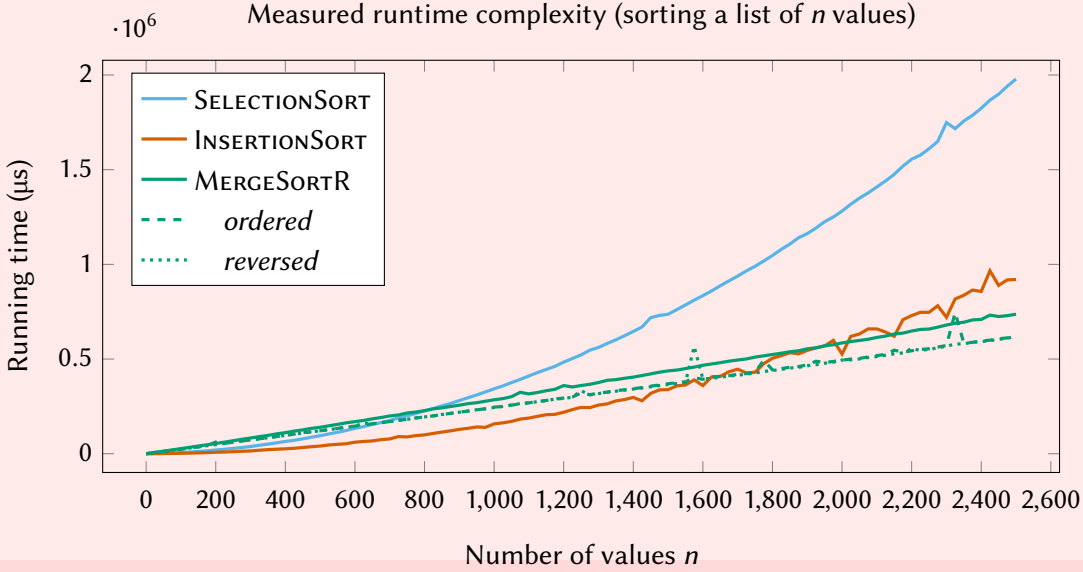
There are also *standard solutions* that you can use: the Master Theorem.

The performance of MergeSortR



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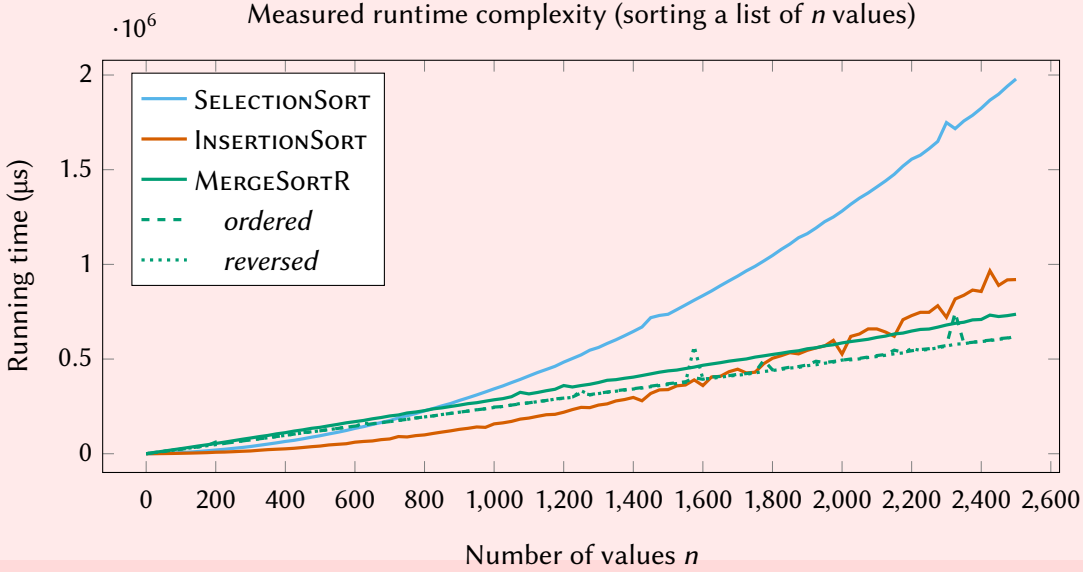

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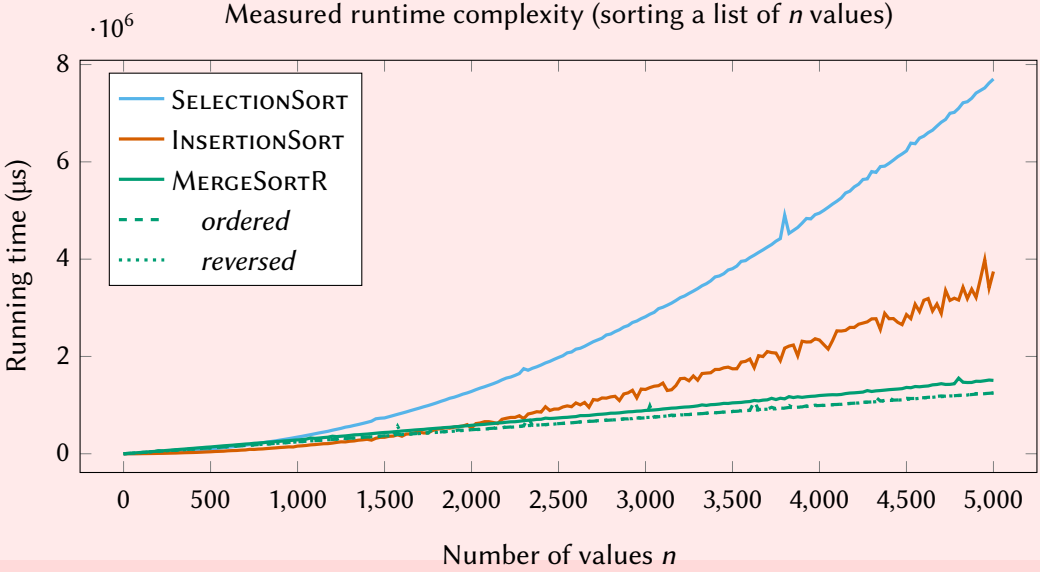

The performance of MergeSortR

MergeSortR should be much better than SelectionSort and InsertionSort:
Especially on big lists.

Concern: MergeSortR has big constants.

- Each Merge makes new arrays.
- A lot of recursive calls that only get us to arrays of size one.

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Especially on big lists.

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Can we finetune MergeSortR to reduce these constants?

- Each Merge makes new arrays.
  Idea: make a single target array to merge into.
- A lot of recursive calls that only get us to arrays of size one.
  Idea: switch from top-down (big-to-small arrays) to bottom-up (small-to-big arrays),
  we can do so using a loop instead of recursion!

MergeSort: A better MergeSortR

Algorithm MergeSort($L[0 \ldots N)$)**:**

1: R is a new array for N values.
2: $sl := 1$. The current *sorted length* of blocks in L.
3: **while** $sl \leq N$ **do**
4: $i := 0$.
5: **while** $i < N$ **do**
Conceptually: Merge $L[i \ldots i + sl)$ and $L[i + sl \ldots i + 2sl)$ into $R[i \ldots i + 2sl)$.
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7: $i := i + 2sl$.
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9: Switch the role of L and R.

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Careful: N does not have to be a multiple of $2sl$.
6: MergeInto(L, i, $\min(i + sl, N)$, $\min(i + 2sl, N)$, R).
7: $i := i + 2sl$.
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9: Switch the role of L and R.

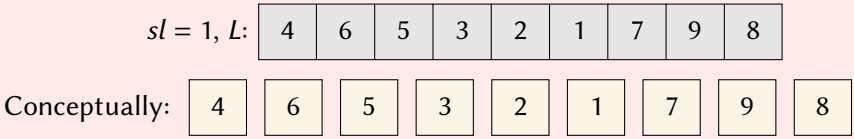
MergeSort: A better MergeSortR

Algorithm MergeInto($S[0 \ldots N)$, *start*, *mid*, *end*, $T[0 \ldots N)$):
Input: $0 \leq start \leq mid \leq end \leq N$ and
$S[start \ldots mid)$ and $S[mid \ldots end)$ are sorted.

1: $i_1, i_2 := start, mid$.
2: **while** $i_1 < mid$ **or** $i_2 < end$ **do**
3: **if** $i_2 = end$ **or** $(i_1 < mid$ **and** $S[i_1] < S[i_2])$ **then**
4: $T[i_1 + i_2] := S[i_1]$.
5: $i_1 := i_1 + 1$.
6: **else**
7: $T[i_1 + i_2] := S[i_2]$.
8: $i_2 := i_2 + 1$.

MergeSort: A better MergeSortR

MergeSort: A better MergeSortR



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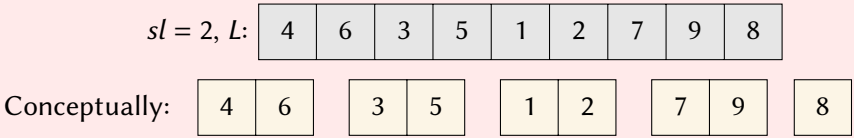
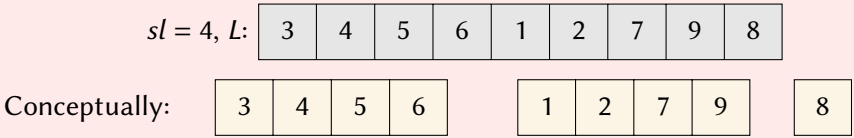


MergeSort: A better MergeSortR



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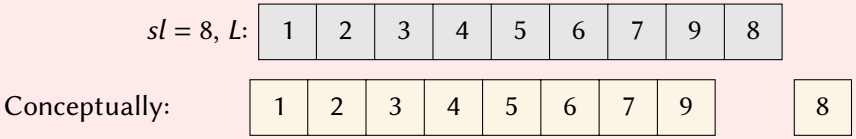

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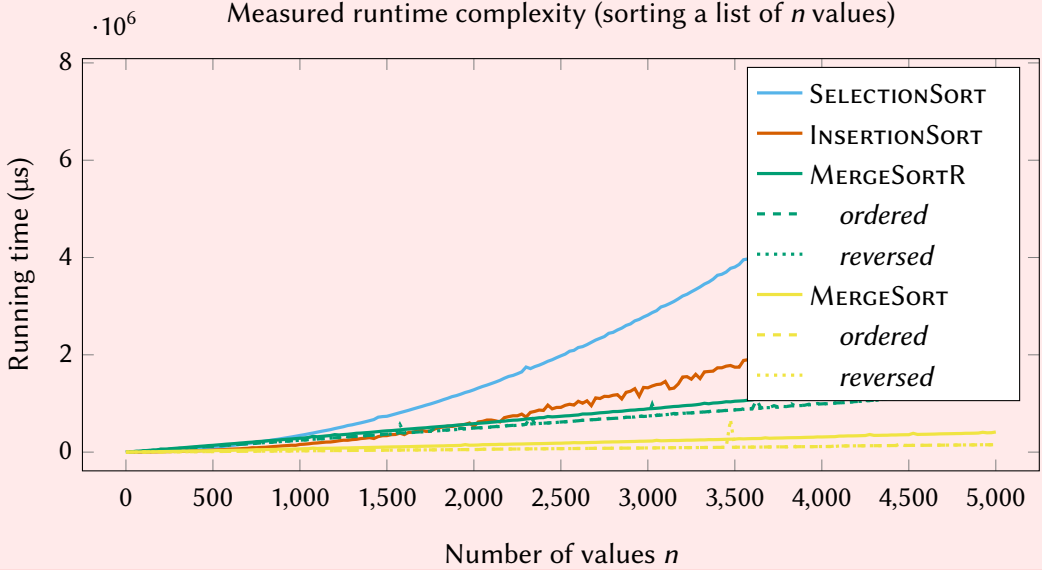

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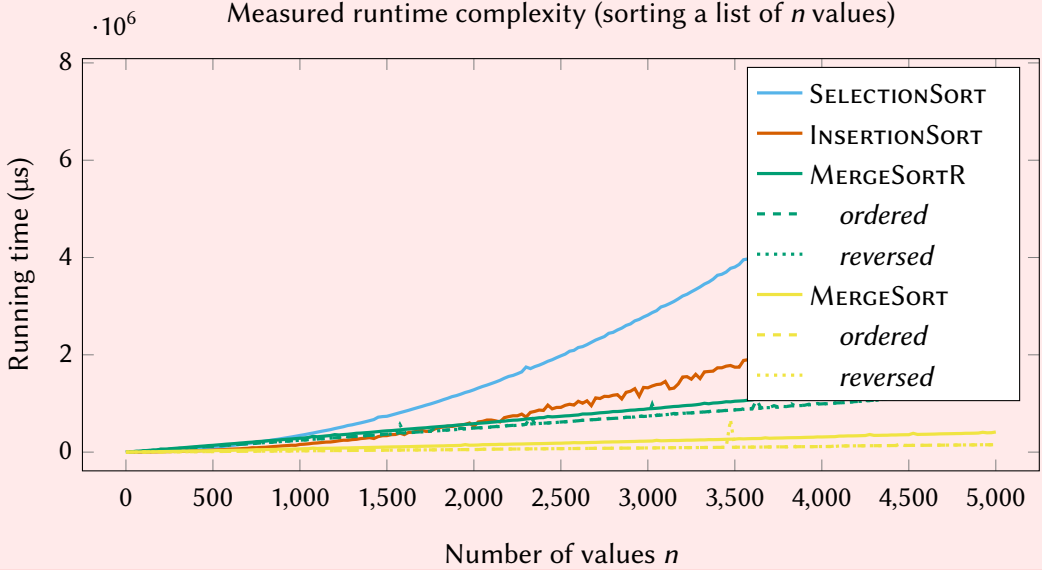

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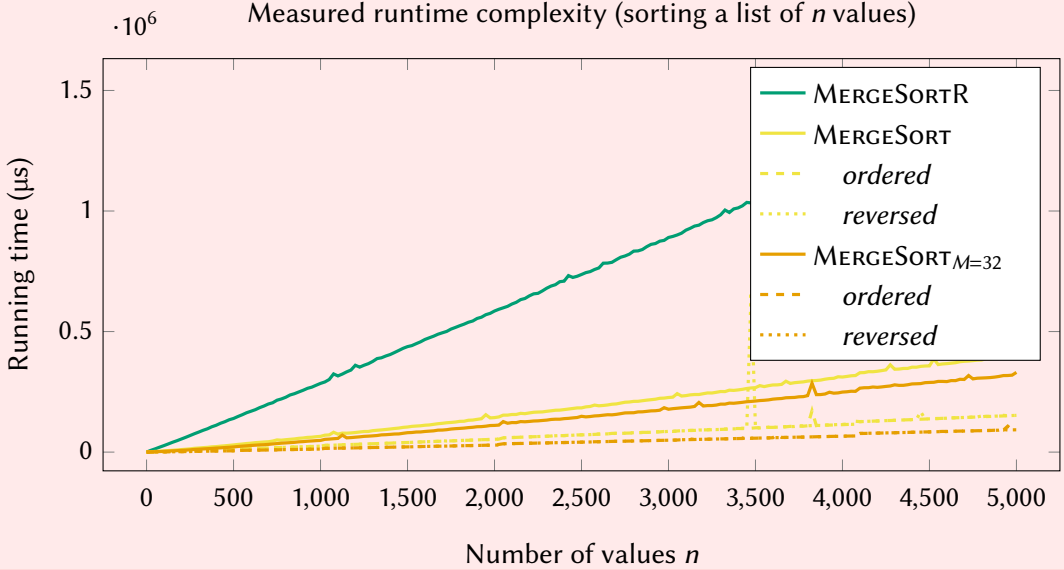

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- Runtime complexity: $\Theta(N \log_2(N))$ comparisons and changes;
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The power of MergeSort

The MergeSort algorithm is at the basis of many large-scale sort algortihms:

- multi-threaded sorting (GiB),
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The power of Merge

The Merge algorithm is flexible: you can easily change it to

- compute the *union* (without duplicates) of two sorted list;
- compute the *intersection* of two sorted list;
- compute the *difference* of two sorted list;
- compute a *join* of two tables (if sorted on the join attributes).

Final notes on MergeSort

	C++	Java
MergeSort	`std::stable_sort`	`java.util.Arrays.sort` (usually)
Merge	`std::merge`	

Merge-like	`std::set_union`
	`std::set_intersection`
	`std::set_difference`
	`std::set_symmetric_difference`	

(related)	`std::inplace_merge`	

Intermezzo: Recurrence trees

In a recurrence tree

- nodes labeled N represent a *function call* with "input size N";
- the children of a node represent *recursive calls*;
- per node, we can determine *the work* within that call (besides recursion);
- per depth, we can determine the *total work for that depth*;
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We already saw two examples: BinarySearchR and MergeSortR.

Intermezzo: Recurrence trees

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$$fib(N) = \begin{cases} 1 & \text{if } N = 1 \text{ or } N = 2; \\ fib(N-1) + fib(N-2) & \text{if } N > 2. \end{cases}$$

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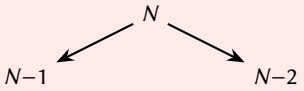

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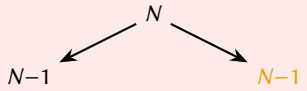

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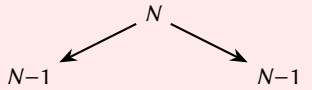

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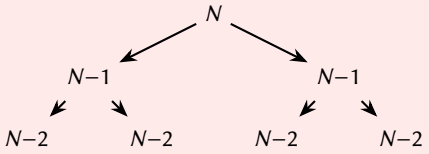

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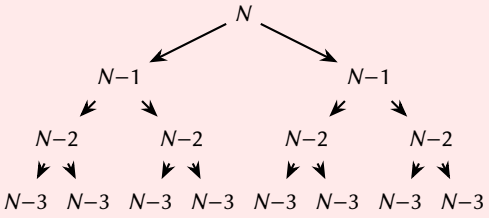

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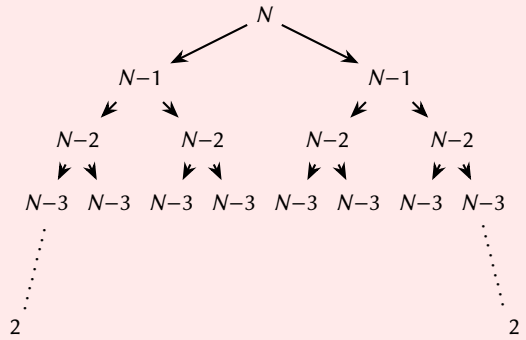

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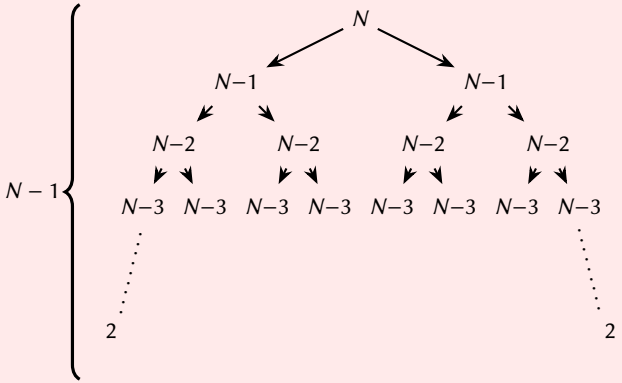

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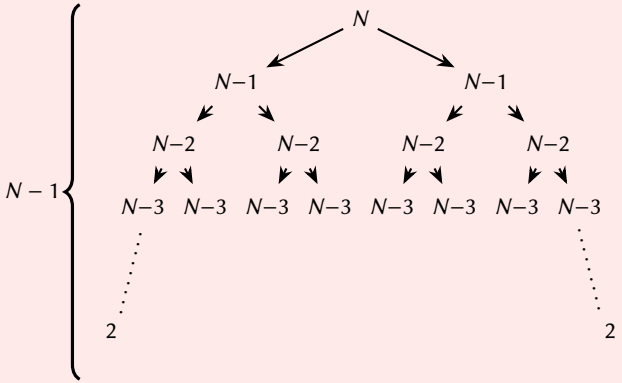

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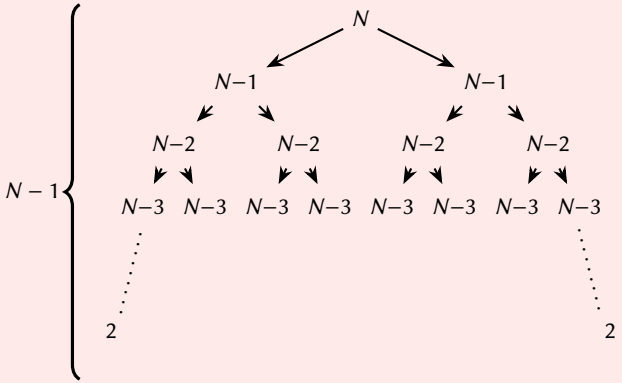

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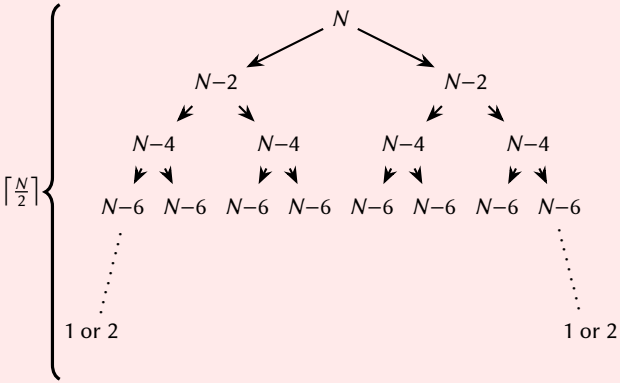

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Intermezzo: Recurrence trees

Example: the *Fibonacci numbers*

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Via recurrence trees, we have proven that:

$$2^{\lceil \frac{N}{2} \rceil} \leq fib(N) \leq 2^N.$$

Intermezzo: The Master Theorem

Let $T(N)$ be a *recurrence* of the form

$$T(N) = \begin{cases} \textit{constant} & \text{if } \textit{base case}; \\ aT\left(\frac{N}{b}\right) + f(N) & \text{if } \textit{recursive case}, \end{cases}$$

with $a \geq 1$, $b > 1$, and we can read $\frac{N}{b}$ also as $\lceil \frac{N}{b} \rceil$ or $\lfloor \frac{N}{b} \rfloor$.

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Someone else has already proved this—so we can reuse the result!

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Example: Runtime complexity of BinarySearchR

$$T(N) = \begin{cases} 4 & \text{if } N = 1; \\ T\left(\frac{N}{2}\right) + 8 & \text{if } N > 1. \end{cases}$$

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Case 2 yields: $T(N) = \Theta(N^{\log_2(1)} \log^1(N)) = \log(N)$.

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Example: Runtime complexity of MergeSortR

$$T(N) = \begin{cases} 1 & \text{if } N = 1; \\ T\left(\lfloor \frac{N}{2} \rfloor\right) + T\left(\lceil \frac{N}{2} \rceil\right) + N & \text{if } N > 1. \end{cases}$$

Intermezzo: The Master Theorem

Let $T(N)$ be a *recurrence* of the form

$$T(N) = \begin{cases} \textit{constant} & \text{if } \textit{base case}; \\ aT\left(\frac{N}{b}\right) + f(N) & \text{if } \textit{recursive case}, \end{cases}$$

with $a \geq 1$, $b > 1$, and we can read $\frac{N}{b}$ also as $\lceil \frac{N}{b} \rceil$ or $\lfloor \frac{N}{b} \rfloor$. We have the following

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We have $a = 2$, $b = 2$, $f(N) = N = \Theta(N) = N^{\log_2(2)}$.

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Case 2 yields: $T(N) = \Theta(N^{\log_2(2)} \log^1(N)) = \Theta(N \log(N))$.

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A third example

$$T(N) = \begin{cases} 1 & \text{if } N = 1; \\ 7T\left(\lfloor \frac{N}{4} \rfloor\right) + N & \text{if } N > 1. \end{cases}$$

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We have $a = 7$, $b = 4$, $f(N) = N = ON^{\log_4(7)-\epsilon}$.

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Case 1 yields: $T(N) = \Theta(N^{\log_4(7)}) \approx \Theta(N^{1.40367\ldots})$.

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A fourth example

$$T(N) = \begin{cases} 1 & \text{if } N = 1; \\ 2T\left(\lfloor \frac{N}{2} \rfloor\right) + N^3 & \text{if } N > 1. \end{cases}$$

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Case 3 yields: $T(N) = \Theta(N^3)$.

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Feel free to use the Master Theorem, we will provide a copy during the final exam.