Searching

SFWRENG 2CO3: Data Structures and Algorithms

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2-3 search trees: Toward balanced binary search trees

Balanced tree: any path from the root to a leaf has length $\lceil \log_2(N+1) \rceil$ (in terms of the number of nodes on the path).

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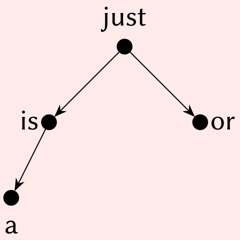
A *balanced* binary search tree with $N = 4$ nodes



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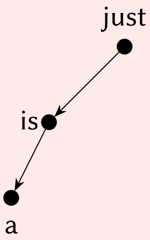

Consider removing “or”: *paths are now too long!*

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With a bit of flexibility, we can keep trees balanced-enough when adding or removing v by *only* making changes *locally* along a path from root to the node holding v.

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2-3 search trees

In a 2-3 tree, there are two types of nodes:

Two-nodes that hold one key value k and two children l and r.

Three-nodes that hold two key values k_1, k_2 and three children c_0, c_1, and c_2.

Furthermore, all leaf nodes in a 2-3 tree must have the *same* distance to the root.

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Two-nodes that hold one key value k and two children l and r.
l holds values $< k$ and r holds values $> k$.

Three-nodes that hold two key values k_1, k_2 and three children c_0, c_1, and c_2.
c_0 holds values $< k_1$, c_1 holds values $> k_1, < k_2$, and c_2 holds values $> k_2$.

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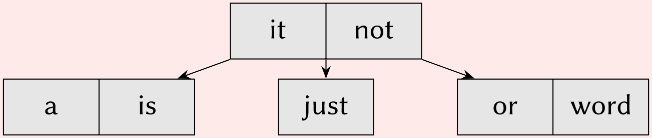

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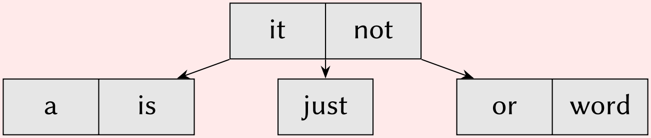

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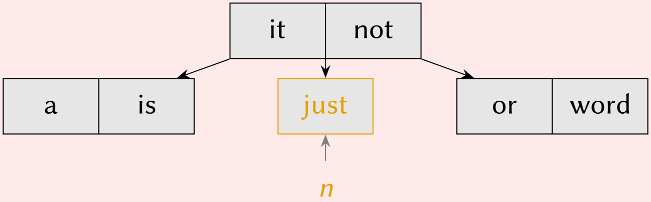

Consider adding "juice", "bee", and "zoo"

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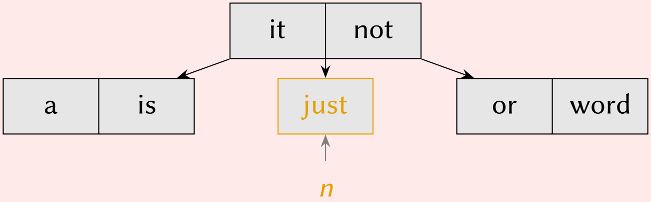

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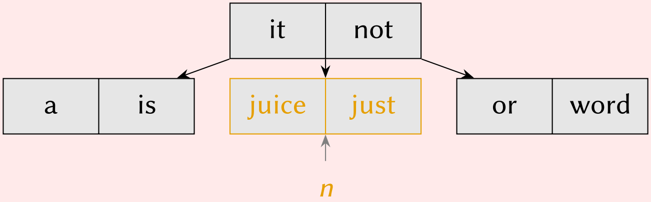

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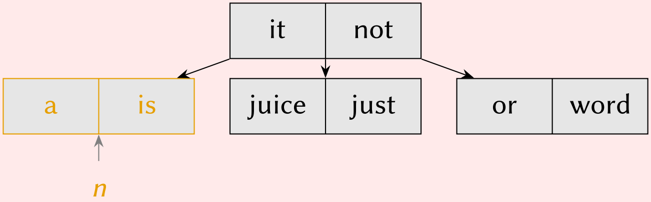

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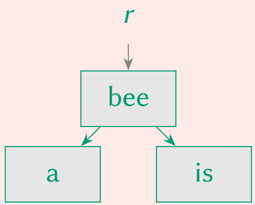

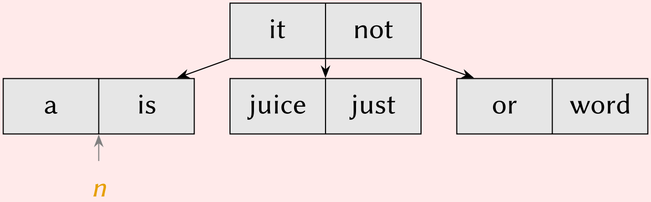

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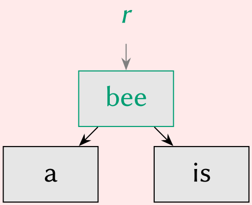

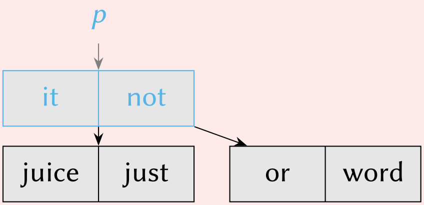

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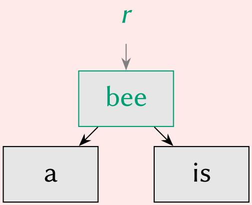

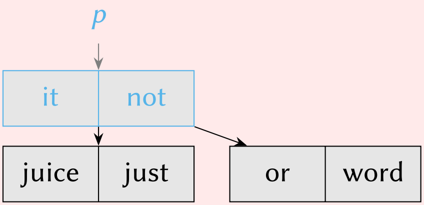

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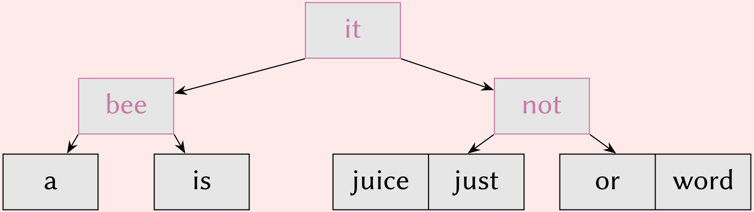

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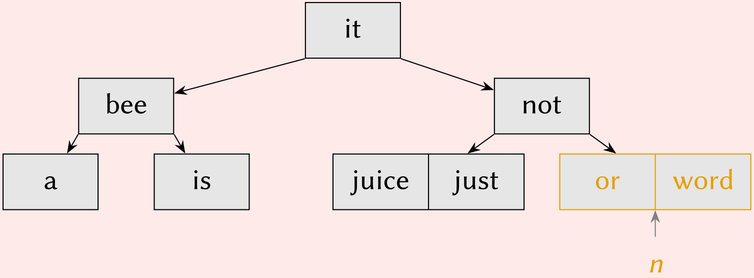

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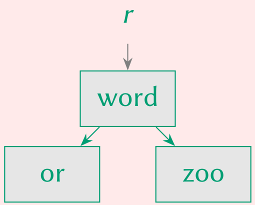

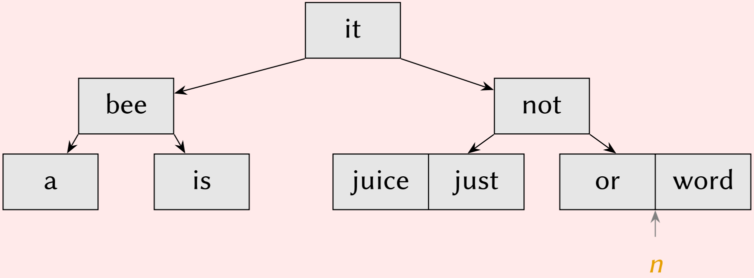

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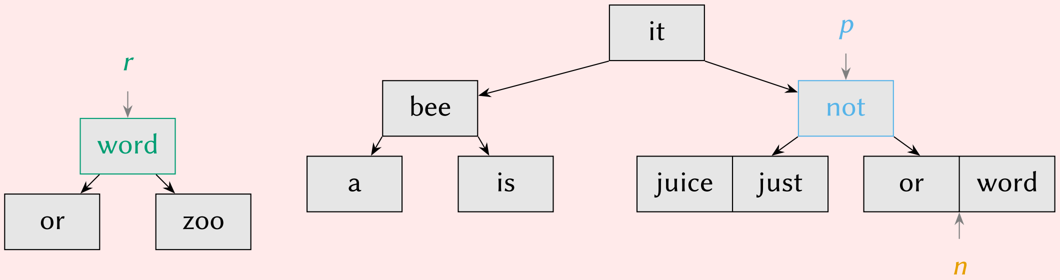

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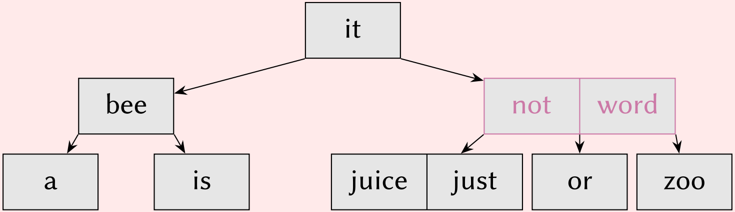

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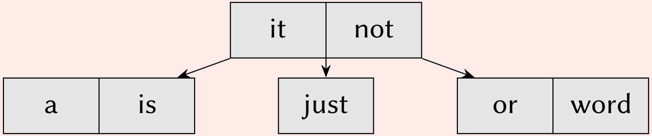

Deleting values from a 2-3 tree

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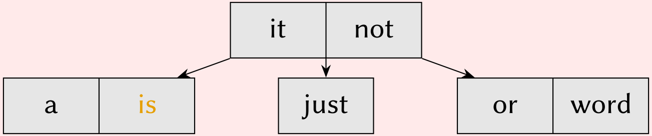

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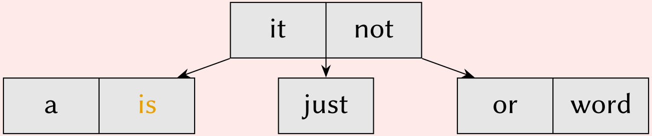

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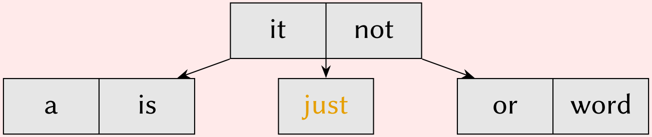

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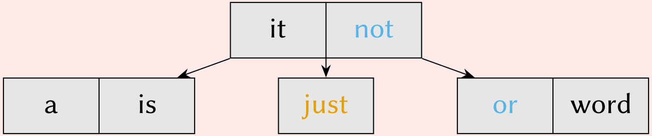

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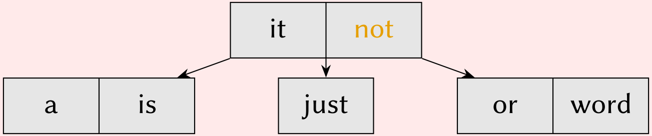

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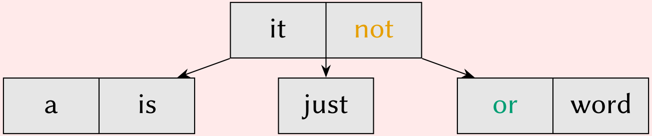

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3. Deleting an internal value.
   Complex: replace value by the succeeding value (a leaf value),
   remove that leaf value.

2-3 search trees in practice

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2-3 trees can be generalized to $(k - 2k)$-trees that are even compacter:
these $(k - 2k)$-trees are at the basis of external memory data structures,
e.g., B+trees that are widely used in file systems and large-scale databases.

From 2-3 trees to *left-leaning* red-black trees

Question: How can we simplify 2-3 trees?

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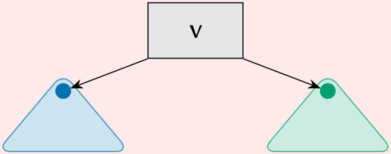

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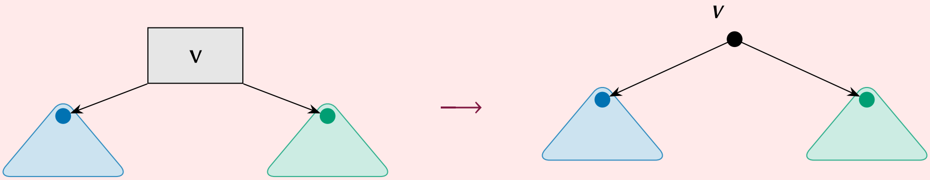

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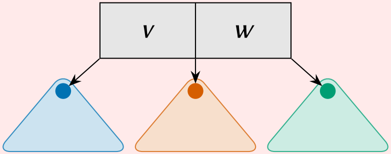

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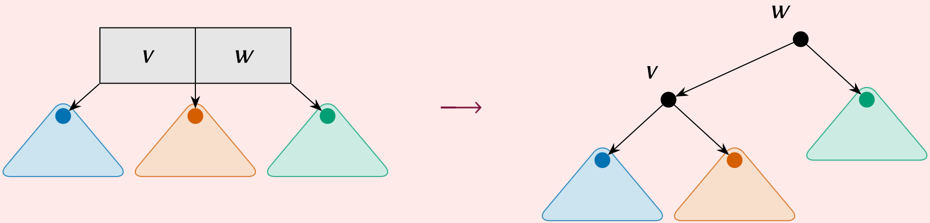

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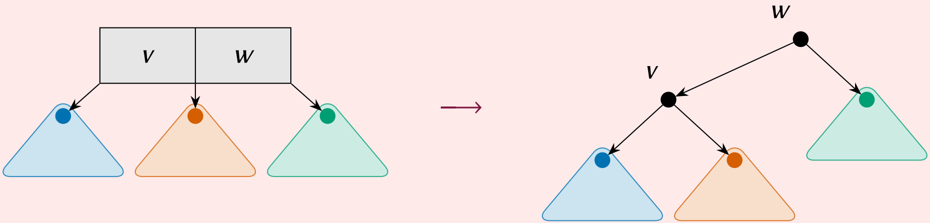

Reusing the addition and removal algorithms from 2-3 trees

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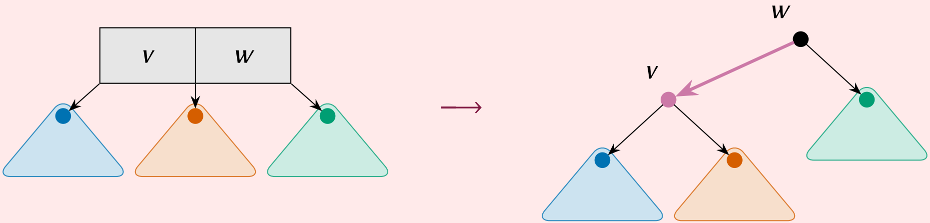

Reusing the addition and removal algorithms from 2-3 trees

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→ Mark the added left-leaning node (with the color red).

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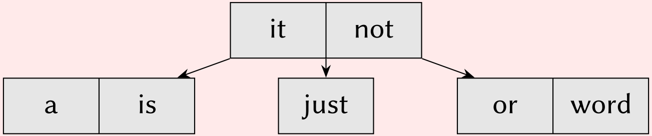
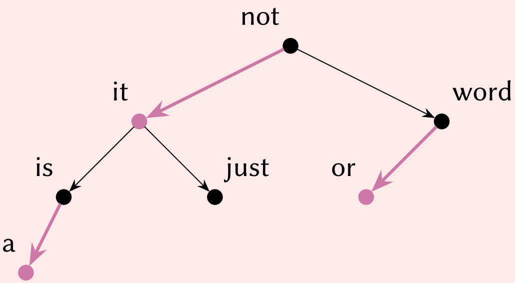
A 2-3 tree



An equivalent *left-leaning* red-black tree



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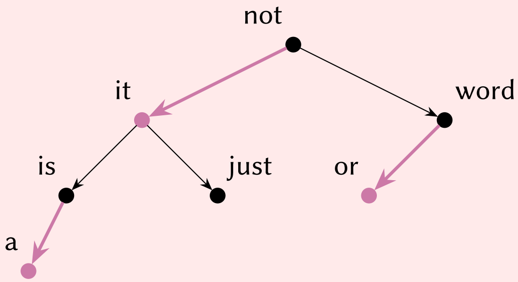

Some usefull properties

1. Every path from root to leaf has at-most $\log_2(N)$ unmarked nodes.
2. Every path from root to leaf has the same number of *unmarked* nodes.
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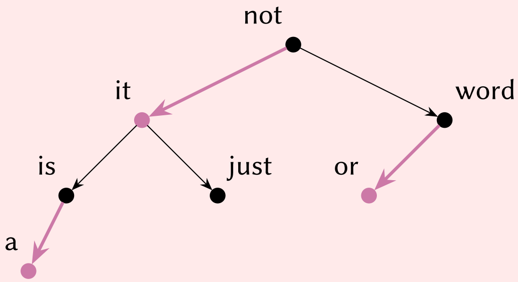

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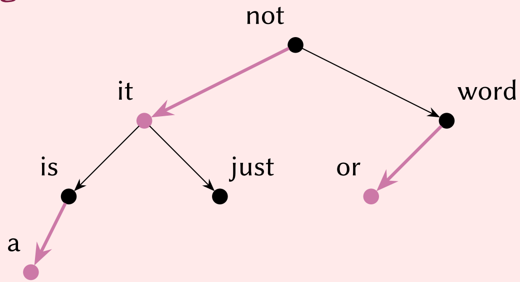
Adding to *left-leaning* red-black trees



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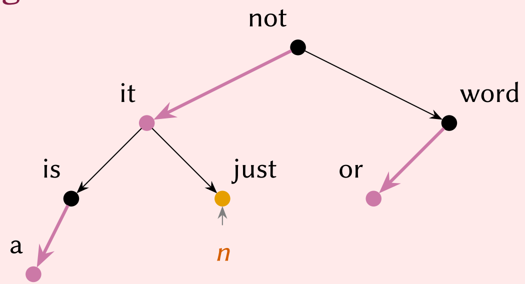

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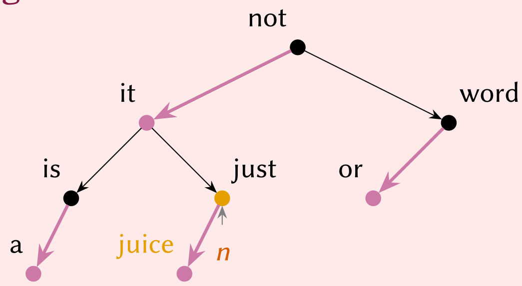

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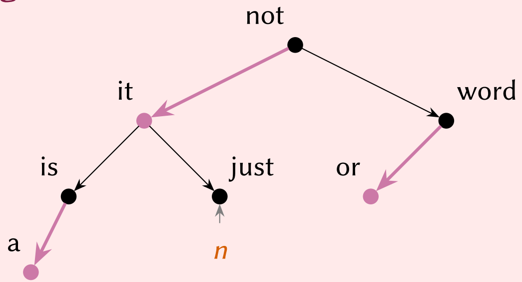
2. We can turn node *n* into a *three-node*.

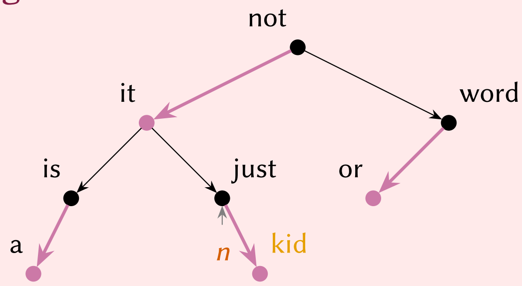
Adding to *left-leaning* red-black trees



Consider adding "juice", "kid", "ism", "bee", and "now"

1. Search for "kid": we find the leaf *two-node* *n* holding "just".

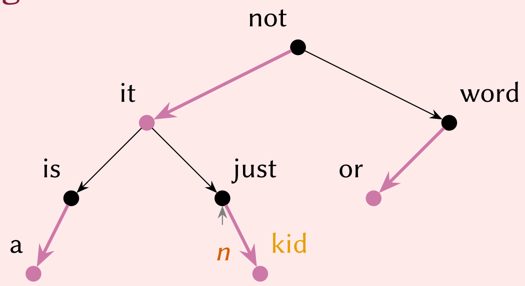
Adding to *left-leaning* red-black trees



Consider adding “juice”, “kid”, “ism”, “bee”, and “now”

1. Search for “kid”: we find the leaf *two-node* *n* holding “just”.
2. We can turn node *n* into a *three-node*, but simply adding “kid” will not do so!

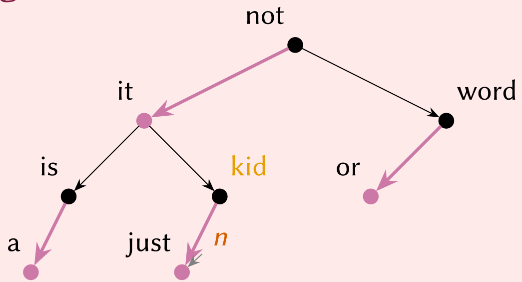
Adding to *left-leaning* red-black trees



Consider adding “juice”, “kid”, “ism”, “bee”, and “now”

1. Search for “kid”: we find the leaf *two-node* *n* holding “just”.
2. We can turn node *n* into a *three-node*, but simply adding “kid” will not do so!
3. We can *rotate left* around *n* to make a proper three-node.

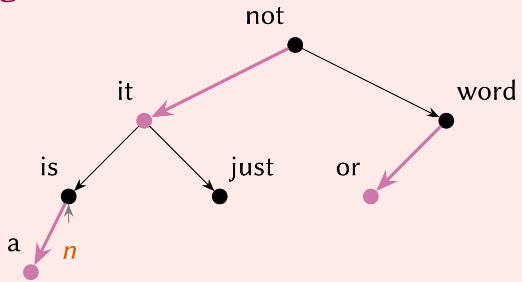
Adding to *left-leaning* red-black trees



Consider adding “juice”, “kid”, “ism”, “bee”, and “now”

1. Search for “kid”: we find the leaf *two-node* *n* holding “just”.
2. We can turn node *n* into a *three-node*, but simply adding “kid” will not do so!
3. We can *rotate left* around *n* to make a proper three-node.

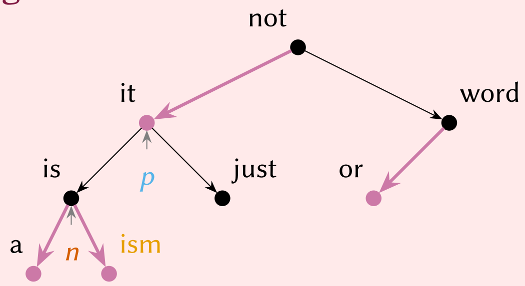
Adding to *left-leaning* red-black trees



Consider adding “juice”, “kid”, “ism”, “bee”, and “now”

1. Search for “ism”: we find the node *n* holding “is” (part of a *three-node*).

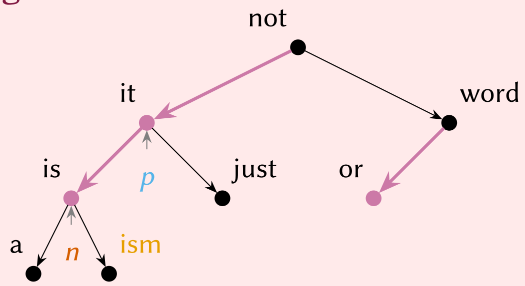
Adding to *left-leaning* red-black trees



Consider adding "juice", "kid", "ism", "bee", and "now"

1. Search for "ism": we find the node *n* holding "is" (part of a *three-node*).
2. We can turn node *n* into a *three-node*, but simply adding "ism" will not do so!

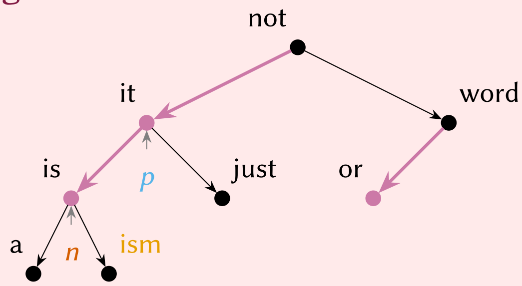
Adding to *left-leaning* red-black trees



Consider adding “juice”, “kid”, “ism”, “bee”, and “now”

1. Search for “ism”: we find the node *n* holding “is” (part of a *three-node*).
2. We can turn node *n* into a *three-node*, but simply adding “ism” will not do so!
3. Push color toward parent *p* of *n*: now marked nodes “touch” each other.

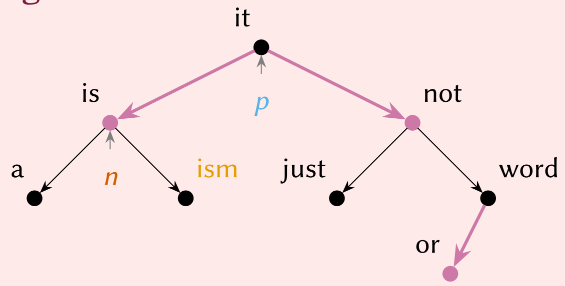
Adding to *left-leaning* red-black trees



Consider adding “juice”, “kid”, “ism”, “bee”, and “now”

1. Search for “ism”: we find the node *n* holding “is” (part of a *three-node*).
2. We can turn node *n* into a *three-node*, but simply adding “ism” will not do so!
3. Push color toward parent *p* of *n*: now marked nodes “touch” each other.
4. We can *rotate right* around the parent of *p* toward fixing the marked nodes.

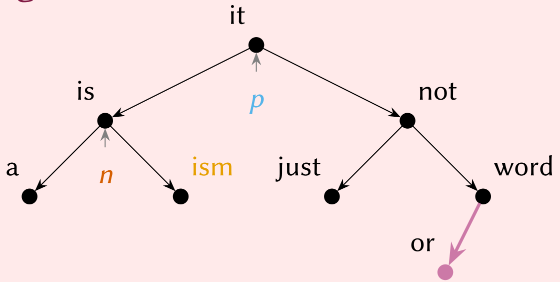
Adding to *left-leaning* red-black trees



Consider adding “juice”, “kid”, “ism”, “bee”, and “now”

1. Search for “ism”: we find the node n holding “is” (part of a *three-node*).
2. We can turn node n into a *three-node*, but simply adding “ism” will not do so!
3. Push color toward parent p of n: now marked nodes “touch” each other.
4. We can *rotate right* around the parent of p toward fixing the marked nodes.

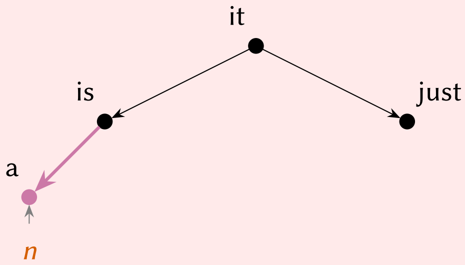
Adding to *left-leaning* red-black trees



Consider adding "juice", "kid", "ism", "bee", and "now"

1. Search for "ism": we find the node n holding "is" (part of a *three-node*).
2. We can turn node n into a *three-node*, but simply adding "ism" will not do so!
3. Push color toward parent p of n: now marked nodes "touch" each other.
4. We can *rotate right* around the parent of p toward fixing the marked nodes.
5. Push color toward parent of p (roots stay unmarked).

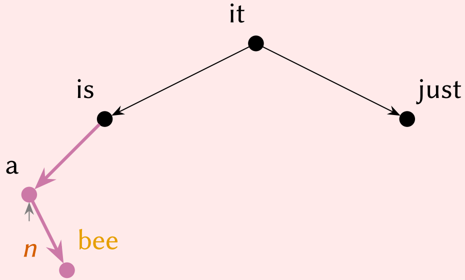
Adding to *left-leaning* red-black trees



Consider adding “juice”, “kid”, “ism”, “bee”, and “now”

1. Search for “bee”: we find the node *n* holding “a” (part of a *three-node*).

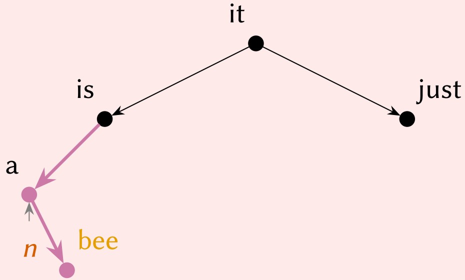
Adding to *left-leaning* red-black trees



Consider adding "juice", "kid", "ism", "bee", and "now"

1. Search for "bee": we find the node *n* holding "a" (part of a *three-node*).
2. Simply adding "bee" invalidates the entire structure.

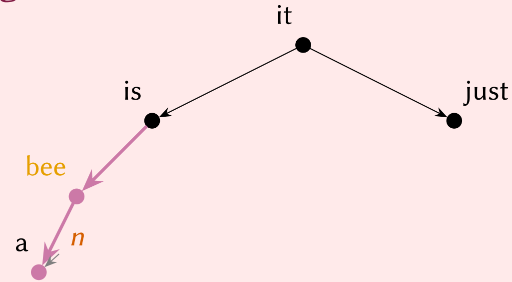
Adding to *left-leaning* red-black trees



Consider adding “juice”, “kid”, “ism”, “bee”, and “now”

1. Search for “bee”: we find the node *n* holding “a” (part of a *three-node*).
2. Simply adding “bee” invalidates the entire structure.
3. We can *rotate left* around *n* to turn this case into a previous case!

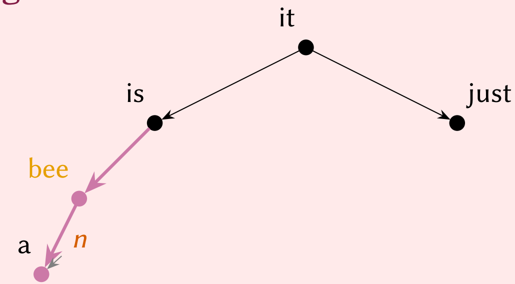
Adding to *left-leaning* red-black trees



Consider adding "juice", "kid", "ism", "bee", and "now"

1. Search for "bee": we find the node *n* holding "a" (part of a *three-node*).
2. Simply adding "bee" invalidates the entire structure.
3. We can *rotate left* around *n* to turn this case into a previous case!

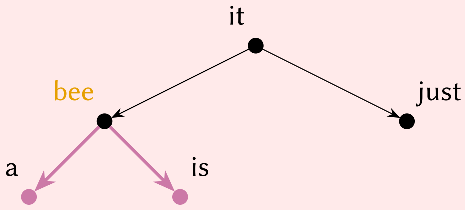
Adding to *left-leaning* red-black trees



Consider adding "juice", "kid", "ism", "bee", and "now"

1. Search for "bee": we find the node *n* holding "a" (part of a *three-node*).
2. Simply adding "bee" invalidates the entire structure.
3. We can *rotate left* around *n* to turn this case into a previous case!
4. Marked nodes "touch" each other: we *rotate right* around the node holding "is".

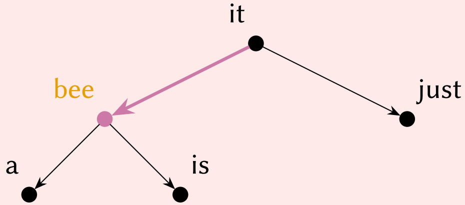
Adding to *left-leaning* red-black trees



Consider adding "juice", "kid", "ism", "bee", and "now"

1. Search for "bee": we find the node *n* holding "a" (part of a *three-node*).
2. Simply adding "bee" invalidates the entire structure.
3. We can *rotate left* around *n* to turn this case into a previous case!
4. Marked nodes "touch" each other: we *rotate right* around the node holding "is".

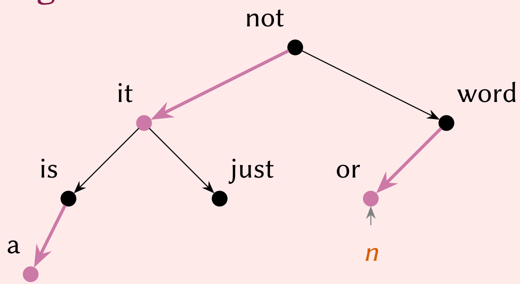
Adding to *left-leaning* red-black trees



Consider adding "juice", "kid", "ism", "bee", and "now"

1. Search for "bee": we find the node *n* holding "a" (part of a *three-node*).
2. Simply adding "bee" invalidates the entire structure.
3. We can *rotate left* around *n* to turn this case into a previous case!
4. Marked nodes "touch" each other: we *rotate right* around the node holding "is".
5. Push color toward parent.

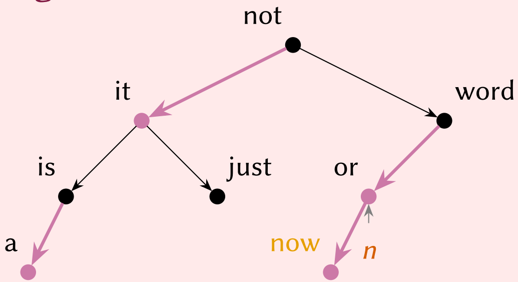
Adding to *left-leaning* red-black trees



Consider adding “juice”, “kid”, “ism”, “bee”, and “now”

1. Search for “nor”: we find the node *n* holding “or” (part of a *three-node*).

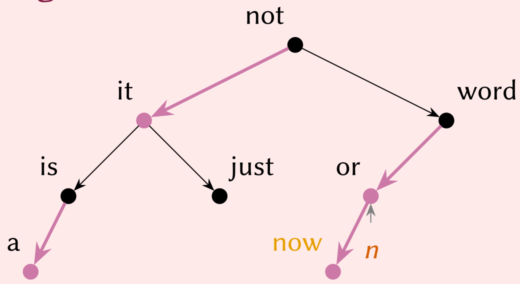
Adding to *left-leaning* red-black trees



Consider adding “juice”, “kid”, “ism”, “bee”, and “now”

1. Search for “nor”: we find the node *n* holding “or” (part of a *three-node*).
2. Simply adding “nor” invalidates the entire structure.

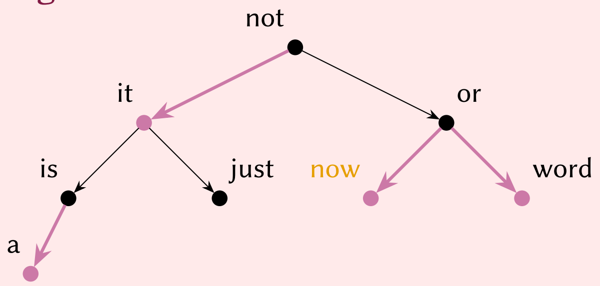
Adding to *left-leaning* red-black trees



Consider adding “juice”, “kid”, “ism”, “bee”, and “now”

1. Search for “nor”: we find the node *n* holding “or” (part of a *three-node*).
2. Simply adding “nor” invalidates the entire structure.
3. This is a previous case: *rotate right*.

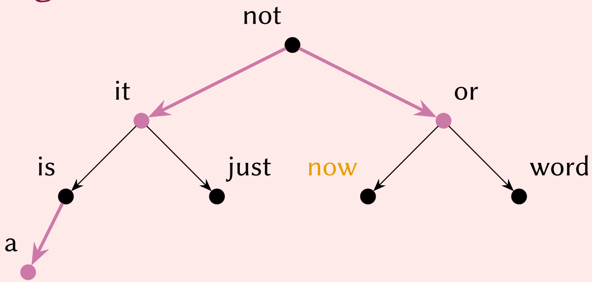
Adding to *left-leaning* red-black trees



Consider adding “juice”, “kid”, “ism”, “bee”, and “now”

1. Search for “nor”: we find the node *n* holding “or” (part of a *three-node*).
2. Simply adding “nor” invalidates the entire structure.
3. This is a previous case: *rotate right*.

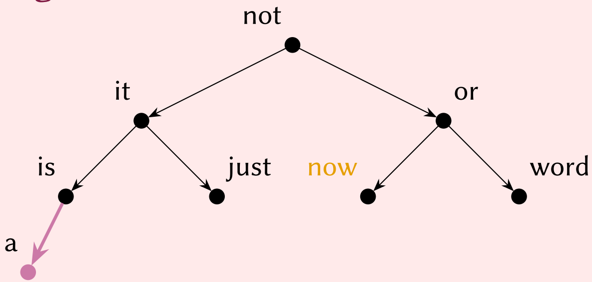
Adding to *left-leaning* red-black trees



Consider adding "juice", "kid", "ism", "bee", and "now"

1. Search for "nor": we find the node *n* holding "or" (part of a *three-node*).
2. Simply adding "nor" invalidates the entire structure.
3. This is a previous case: *rotate right*.
4. Push color up.

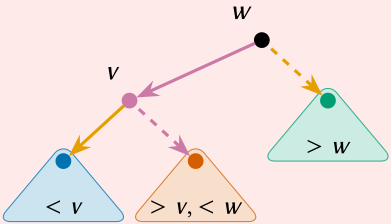
Adding to *left-leaning* red-black trees



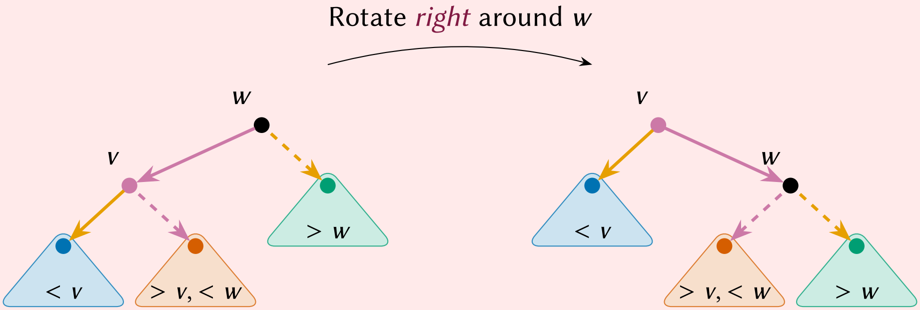
Consider adding "juice", "kid", "ism", "bee", and "now"

1. Search for "nor": we find the node *n* holding "or" (part of a *three-node*).
2. Simply adding "nor" invalidates the entire structure.
3. This is a previous case: *rotate right*.
4. Push color up.
5. Push color up (roots stay unmarked).

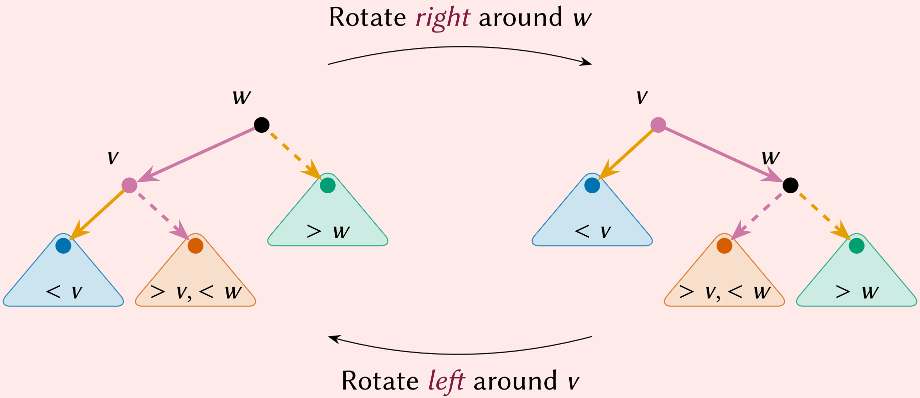
The *rotate left* and *rotate right* operations



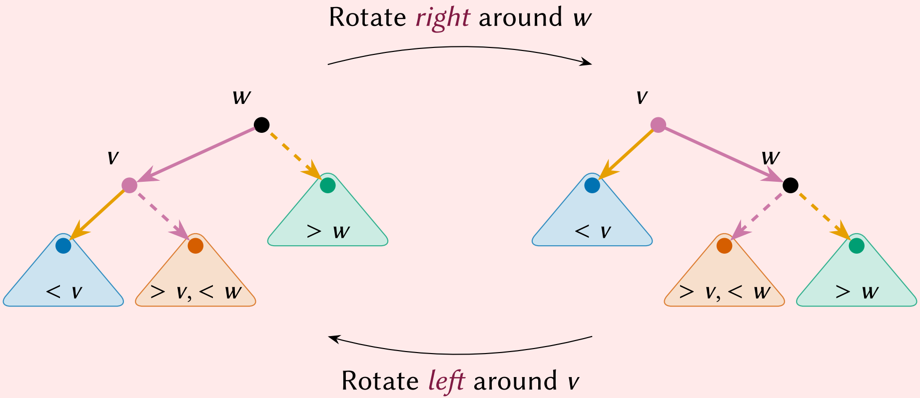
The *rotate left* and *rotate right* operations



The *rotate left* and *rotate right* operations



The *rotate left* and *rotate right* operations

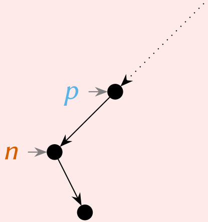


Rotate operations *affect node markings*.

Can be implemented using *only* pointer manipulation.

Removing values from *left-leaning* red-black trees (sketch)

Consider a minimum value v at node n with parent p



Removing values from *left-leaning* red-black trees (sketch)

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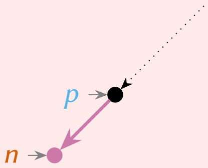
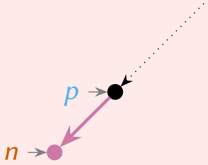


- n is marked and has no children.

Removing values from *left-leaning* red-black trees (sketch)

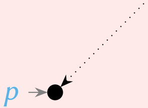
Consider a minimum value v at node n with parent p



- n is marked and has no children.
  Simple: Removing has zero consequences on the structure of the tree.

Removing values from *left-leaning* red-black trees (sketch)

Consider a minimum value v at node n with parent p



- n is marked and has no children.
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Removing values from *left-leaning* red-black trees (sketch)

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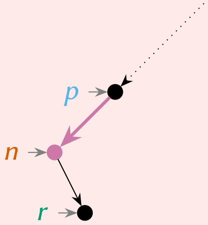
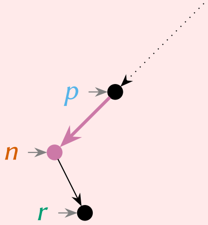


- n is marked and has no children.
  Simple: Removing has zero consequences on the structure of the tree.
- n is marked and has one (right) child node r.

Removing values from *left-leaning* red-black trees (sketch)

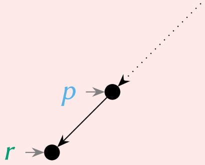
Consider a minimum value v at node n with parent p



- n is marked and has no children.
  Simple: Removing has zero consequences on the structure of the tree.
- n is marked and has one (right) child node r.
  Simple: Replace n by r, which has zero consequences on the structure of the tree.

Removing values from *left-leaning* red-black trees (sketch)

Consider a minimum value v at node n with parent p



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Removing values from *left-leaning* red-black trees (sketch)

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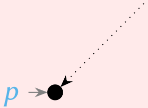
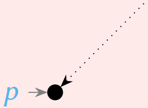


- n is marked and has no children.
  Simple: Removing has zero consequences on the structure of the tree.
- n is marked and has one (right) child node r.
  Simple: Replace n by r, which has zero consequences on the structure of the tree.
- n is *not* marked.

Removing values from *left-leaning* red-black trees (sketch)

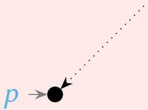
Consider a minimum value v at node n with parent p



- n is marked and has no children.
  Simple: Removing has zero consequences on the structure of the tree.
- n is marked and has one (right) child node r.
  Simple: Replace n by r, which has zero consequences on the structure of the tree.
- n is *not* marked $\longrightarrow$ Complex: Removing n invalidates the structure of the tree.

Removing values from *left-leaning* red-black trees (sketch)

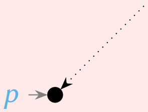
Consider a minimum value v at node n with parent p



- n is marked and has no children.
  Simple: Removing has zero consequences on the structure of the tree.
- n is marked and has one (right) child node r.
  Simple: Replace n by r, which has zero consequences on the structure of the tree.
- n is *not* marked $\longrightarrow$ Complex: Removing n invalidates the structure of the tree.
  Idea: Ensure that n is marked.

Removing values from *left-leaning* red-black trees (sketch)

Consider a minimum value v at node n with parent p

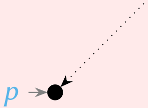


Idea: Ensure that n is marked.

- We can *introduce* marked nodes at the root of the tree.
- We can push marked nodes down the tree using *rotates* toward the minimum value.

Removing values from *left-leaning* red-black trees (sketch)

Consider a minimum value v at node n with parent p



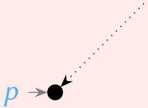
Idea: Ensure that n is marked.

- We can *introduce* marked nodes at the root of the tree.
- We can push marked nodes down the tree using *rotates* toward the minimum value.

We have seen the reverse while *adding* values.

Removing values from *left-leaning* red-black trees (sketch)

Consider a minimum value v at node n with parent p

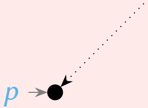


Generalization: Remove arbitrary values.

- Replace arbitrary values by their successor.
- Removing successor: generalize the methods to remove the minimum from a tree.

Removing values from *left-leaning* red-black trees (sketch)

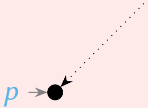
Consider a minimum value v at node n with parent p



Removal is possible with only local tree modifications along the path from root to value.

Removing values from *left-leaning* red-black trees (sketch)

Consider a minimum value v at node n with parent p



Removal is possible with only local tree modifications along the path from root to value.

Many minute details to deal with in a plethora of cases.

Conclusion: Left-leaning red-black trees

Some usefull properties (that we can maintain)

1. Every path from root to leaf has at-most $\log_2(N)$ unmarked nodes.
2. Every path from root to leaf has the same number of *unmarked* nodes.
3. No marked nodes “touch” each other.

Conclusion: Left-leaning red-black trees

Some usefull properties (that we can maintain)

1. Every path from root to leaf has at-most $\log_2(N)$ unmarked nodes.
2. Every path from root to leaf has the same number of *unmarked* nodes.
3. No marked nodes "touch" each other.

Paths from root to leafs have length *at-most* $2\log_2(N)$:
all operations of interest in worst-case $\Theta(\log_2(N))$.

Final notes on binary search trees

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In practice, one typically uses ordinary red-back trees:
Very similar, just *more cases* to consider when adding or removing values.

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Set	`std::set`	`java.util.TreeSet`
Dictionary	`std::map`	`java.util.TreeMap`
Set (duplicates)	`std::multiset`	
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Variants of search trees are used *everywhere*: file systems, database systems, …

Faster sets and dictionaries: beyond $\log_2(N)$

Consider the following variant of WordCount

Algorithm GradeCount(*stream*):

Input: *stream* is a sequence of grades, each in $0, \dots, 10$.

```
grades := [0 | 0 ≤ i ≤ 10].
for all grade g from stream do
    grades[g] := grades[g] + 1.
output each pair (i ↦ grades[i]), 0 ≤ i ≤ 10.
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Result: output a histogram of the grades in *stream*.

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2: **for all** grade g from *stream* **do**

3: $\quad \mathit{grades}[g] := \mathit{grades}[g] + 1$.

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grades is an *array* that essentially serves as a *dictionary*
in which *grades* are keys and a grade-count is the associated value.

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Worst-case complexity only $\Theta(|stream|)$.

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$L[0 \ldots 10)$:

0:	
1:	
2:	
3:	
4:	
5:	
6:	
7:	
8:	
9:	

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First character	$h(v)$
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‘e’, ‘o’, ‘y’	4
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'j', 't'	9

w	$h(w)$
"a"	
"word"	
"is"	
"just"	
"or"	
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$L[0 \ldots 10)$:

0:	a	
1:		
2:	word	
3:		
4:	or	
5:		
6:		
7:		
8:	is	it? $\leftarrow$ a *collision*!
9:	just	

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0:	a
1:	
2:	word
3:	not
4:	or
5:	
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  For example, the number of strings is infinite.
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Hash tables

A *hash table* is a data structure that uses a *hash function* that maps *values* to array positions that can *hold that value*.

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We will look at two main flavors of hash tables:

- **Chaining** Use a linked list to store *collisions*.
- **Linear probing** Store *collisions* consecutively in the array.

The *uniform hashing* assumption

Let $h : \mathcal{K} \to \{0, \ldots, N-1\}$ be a hash function.
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Some settings allow a collision-free hash function: *perfect hashing*.
For example: the hash function $h(i) = i$ we used in GradeCount.

Hashing with chaining

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Contains value v Look up the linked list S at $L[h(v)]$,
search v in S (e.g., using a LinearSearch variant).

Adding value v Look up the linked list S at $L[h(v)]$,
add v to S if $v \notin S$ (sets do not have duplicates).

Removing value v Look up the linked list S at $L[h(v)]$,
remove v from S if $v \in S$.

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w	$h(\mathrm{w})$
"a"	
"word"	
"is"	
"just"	
"or"	
"it"	
"not"	

$L[0 \ldots 7)$:

0:	`@null`
1:	`@null`
2:	`@null`
3:	`@null`
4:	`@null`
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“word”	
“is”	
“just”	
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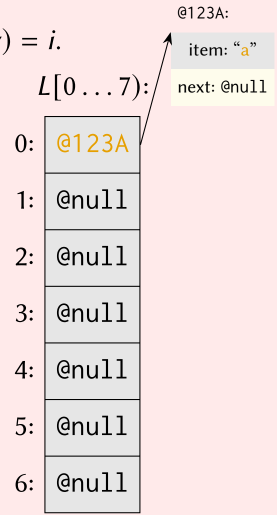

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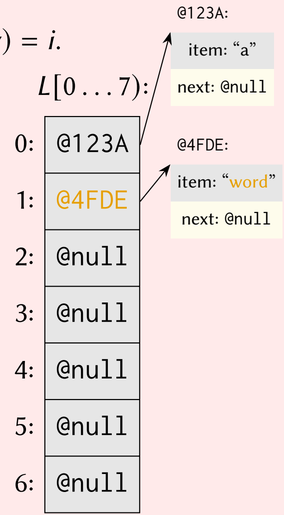

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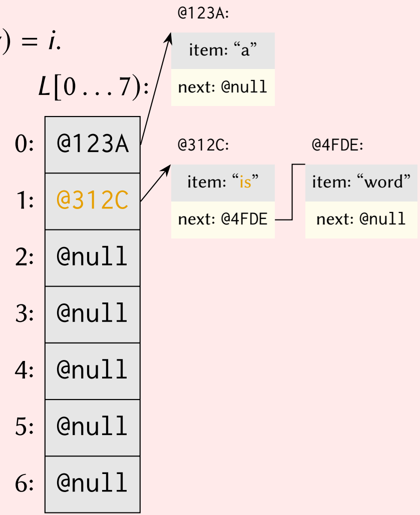

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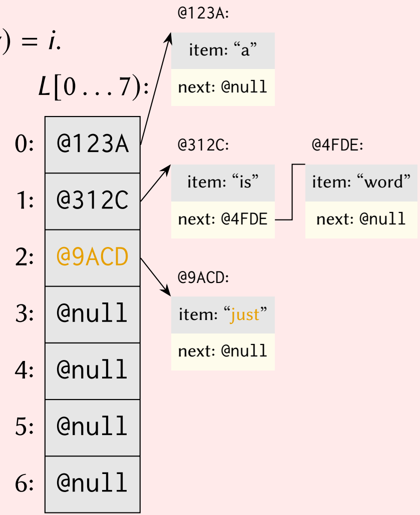

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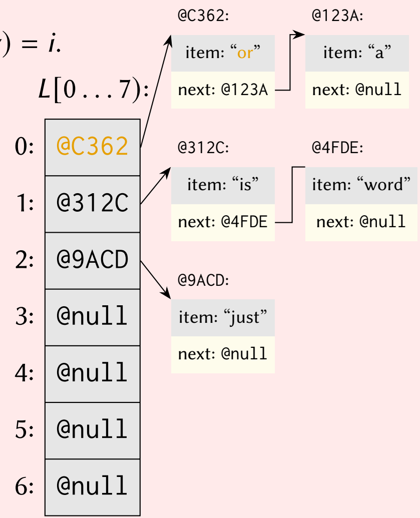

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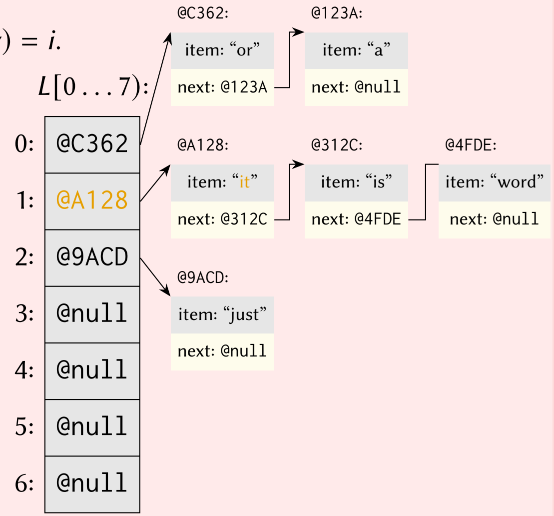

Hashing with chaining

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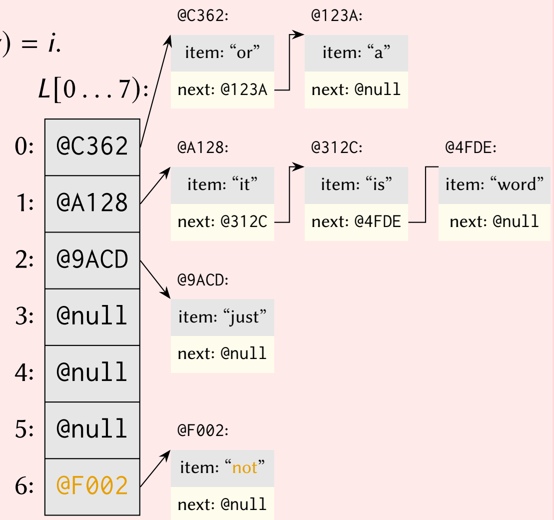

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- For somewhat decent hash functions and $N > M$,
  adding and removing values are $\Theta(1)$ in practice.
- Bad hash functions exist.
  For example, the hash function we used in our examples.

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“just”	
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“or”	
“not”	
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$L[0 \dots 7)$:

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$L[0 \dots 7]$:

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6:	“not”	Occupied!

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At wrong positions!

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How to remove a value at position j?

Removing values breaks consecutive sequences of non-free positions!

Option 1 reinsert all values in non-free positions following position j.

Option 2 set $L[j] := \textsc{Removed}$ with $\textsc{Removed}$ a special-purpose value. When searching: $\textsc{Removed}$ is unequal to any value.

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Option 1 is costlier during removal, but cheaper afterwards.

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We reinsert these three values.

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- To find an existing value (*removing*), we expect to inspect at-most $\frac{1}{\alpha} \ln\left(\frac{1}{1-\alpha}\right)$ positions.

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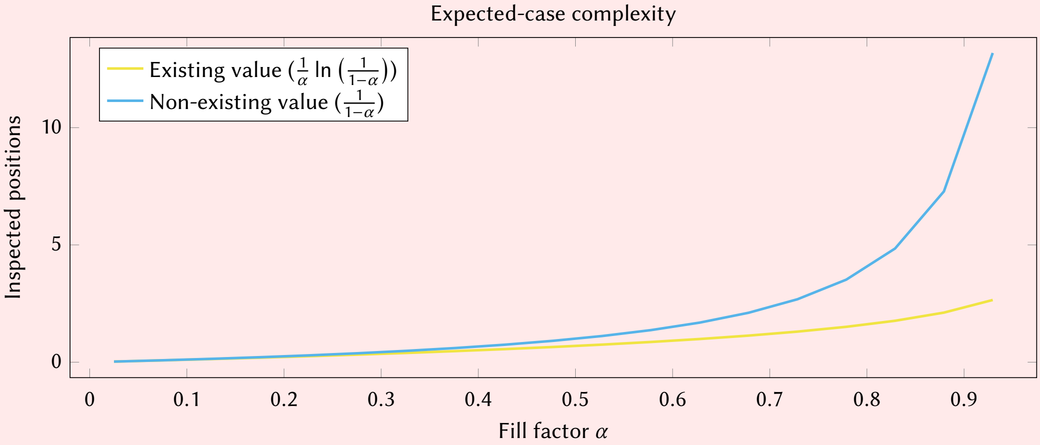

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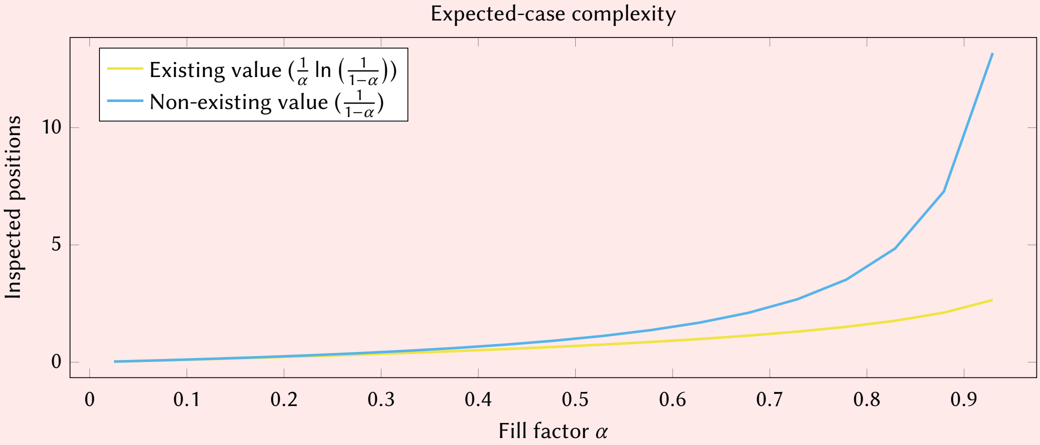

For somewhat decent hash functions and $N \ggg M$,
adding and removing values are $\Theta(1)$ in practice.

Hash tables and functions in practice

Hash tables provide a balance between memory usage and runtime cost:

- With mostly-empty tables (high memory usage), collisions are expected to be rare (low runtime cost).
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This requires a *family of hash functions* $h_N : \mathcal{K} \to \{0, \ldots, N-1\}$.

Let M be the *maximum size* of arrays in your system.
Let $h : \mathcal{K} \to \{0, \ldots, M-1\}$ be a hash function. One way to obtain h_N, $0 \leq N \leq M$, is via

$$h_N(i) = h(i) \bmod N.$$

Final notes on hash tables

Most dynamic hash tables are implemented on top of dynamic arrays using *chaining*.
Linear probing is especially usefull for *constant tables*.

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	C++	Java
Set	`std::unordered_set` (C++11)	`java.util.HashSet`
Dictionary	`std::unordered_map` (C++11)	`java.util.HashMap`
Set (duplicates)	`std::unordered_multiset` (C++11)	
Dictionary (duplicates)	`std::unordered_multimap` (C++11)	

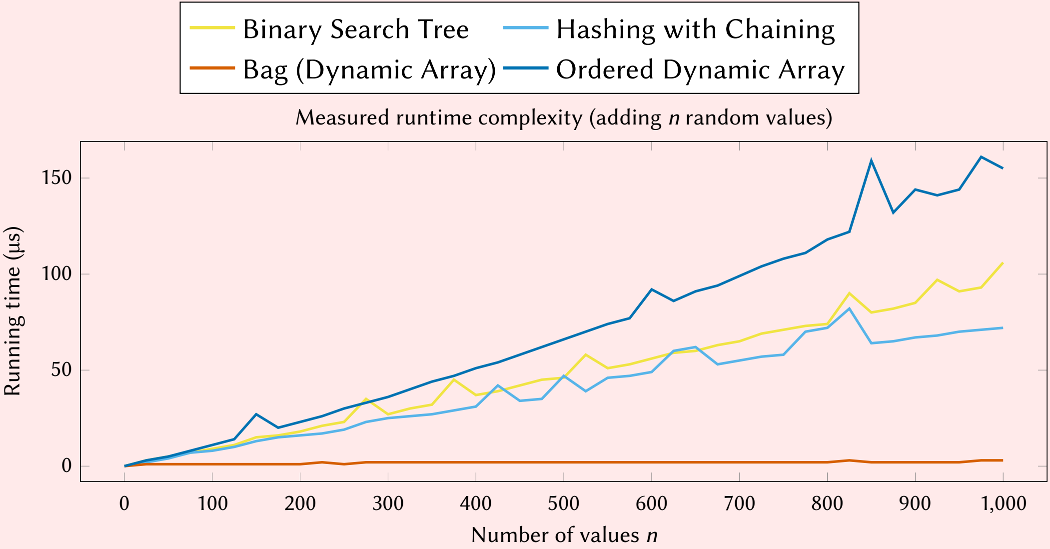
Sets and dictionaries in practice

	Cost	Ordered	Principle
Dynamic Arrays	$\Theta(N)$	No	
Ordered Dynamic Array[a]	$\Theta(\log_2(N))$, $\Theta(N)$	Yes	BinarySearch
Binary Search Trees	$\Theta(\log_2(N))$	Yes	Red-Black Trees.
Hash Tables	Expected $\Theta(1)$[b]	No	Chaining.

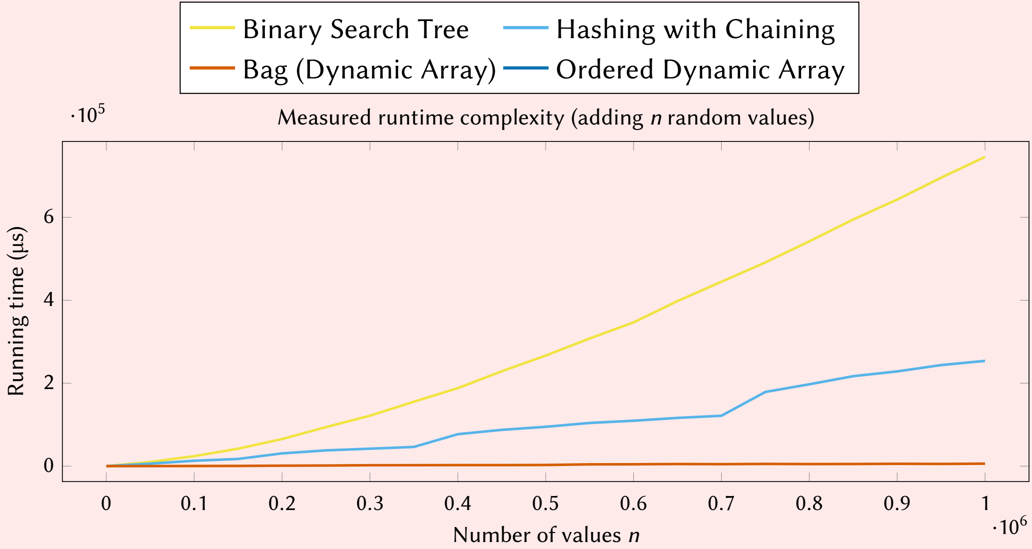
[a] Supported in C++23 via `std::flat_set` (set), `std::flat_map` (dictionary), `std::flat_multiset` (set, with duplicates), and `std::flat_multimap` (dictionary, with duplicates).

[b] For somewhat decent hash functions and large enough hash table.

Sets and dictionaries in practice



Sets and dictionaries in practice



Sets and dictionaries in practice