GROUP THEORY AND QUANTUM FIELD THEORY

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1 Introduction

Group theory has origins dating back to the 19th century, primarily said to have been started by Cauchy and Galois independently. Nobody would have guessed the importance that group theory plays in explaining our current understanding of the nature of our world. The year 1905 sparked a revolution in physics after the discovery of the photoelectric effect, the theory of special relativity, explanations of Brownian motion, and the principle of mass-energy equivalence, all developed by Einstein, known as the *Annus merabilis* papers.

This lead to many different developments in quantum mechanics, lead by Bohr, Schrödinger, Heisenberg, de Broglie, Dirac, and many others. Although the theory was successful in many respects, it did not account for relativistic effects on particles. One of the major leading developments was by de Broglie in 1924, when he had the idea of describing elementary systems in terms of waves. In 1927, researchers attempted to apply quantum mechanics to fields rather than single particles, which was the birth of quantum field theories.

By the 1950's, a near complete theory of quantum electrodynamics had been formulated. A key idea that lead to the formulation of quantum field theory, which includes quantum electrodynamics, was the idea of symmetries in a system, and describing these symmetries by using groups. The formulation of gauge theory was started by considering local transformations of the phase of the wave function describing the state of a system, rather than global transformations. In the context of quantum electrodynamics, the phase of the wave function for a charged particle can be modified by a complex phase factor, which varies with position in spacetime, as opposed to a global modification by a constant factor Λ. This local phase transformation ensures that physical observables remain unchanged, and it led to the development of the electromagnetic gauge symmetry. In the 1970s, this idea was applied to quarks to formulate quantum chromodynamics.

2 Preliminaries

2.1 Representations

To make the study of groups easier, and to find useful properties of the groups, we can instead consider *representations* of groups. A representation, then, is a group G, together with a vector space V, such that

$$\rho : G \longrightarrow GL(V)$$

Is a group homomorphism; that is, $\rho(g)\rho(h) = \rho(gh)$. This means that the group structure is preserved by using the usual definition of matrix multiplication. If the matrix is of size $n \times n$, then we say ρ is an n-dimensional representation.

A unitary matrix U is a matrix such that $U^{-1} = U^{\dagger}$, or that the inverse is equal to the adjoint; the complex conjugate transpose of U (also called a **Hermitian** matrix), which is $U^{*t} = U^{\dagger}$. Given a represenation ρ, we say that ρ is equivalent to ρ' if their matrices are related by an equivalence transformation:

$$\rho'(g) = U^{-1}\rho(g)U \quad \forall g \in G.$$

Representations play an important role in understanding symmetries in physical systems. All groups being considered in our discussion of quantum electrondynamics and chromodynamics will be seen as their representations. It also turns out that they are a special class of groups, which we will discuss.

2.2 Lie Groups

The general classification of groups is an extremely difficult problem to solve. Being less specific, there are two kinds of groups: finite and continuous groups. Our focus will be primarily on continuous, or infinite groups, as they play the most central role in quantum field theories. In quantum mechanics, nearly all transformation operators are unitary. Although there may be infinitely many operations, it is true that two operations combine to give another single operation. There may be infinitely many unitary operations, but they act on finitely many parameters. We might then begin to think these operators form a group structure, and indeed they do. This is why it is beneficial to us to study the theory of continuous groups.

Abstractly, a lie group G is both a smooth manifold, and a group, such that group multiplication and group inverses are smooth maps. More concretely, it combines the notions of a group and a smooth manifold. A manifold can be thought of as a space that locally resembles Euclidean space. One example of this is the surface of a sphere. A sphere does not have the same geometry as a flat plane, but if we zoom in extremely close, it does start to resemble a plane. An example is zooming into the surface of the Earth. It is a sphere, yet everybody on Earth would not be able to visibly see

the curvature (unless they are extremely high above the ground), and say that the city they live in "looks" flat. To be smooth means that it has derivatives of every order, everywhere on the manifold. This means that the manifold locally looks similar enough to a vector space, so that we may apply calculus to it. A lie group combines these two ideas, where the lie group G is both a group, and a smooth manifold.

A typical example of such a group is the group of rotations in 3 dimensions. It is typically denoted by $SO(3)$, as it is referred to as the special orthogonal group of degree 3. Consider a set of basis vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ in $\mathbb{R}^3$. A linear transformation that is a rotation of these vectors would require that all distances and angles between them stay the same. This requires that for any matrix $M \in SO(3)$,

$$M^{-1} = M^t$$

which gives rise to a group $O(3)$, the group of proper and improper rotations, or rotations with determinant ± 1. In order to get the group $SO(3)$, one must also require that $det(M) = 1$, eliminating rotations that are also reflections.

Another group we should consider is the group of unitary matrices of size n, denoted $U(n)$. The set of groups $U(n)$ is actually quite analogous to the set of groups $O(n)$. The condition that the matrices be unitary is very similar to the condition in $O(n)$ that the matrices be orthogonal. The difference is that the field over which elements in the matrix of $O(n)$ reside are in $\mathbb{R}$, while the elements in the matrix of $U(n)$ reside in $\mathbb{C}$. The fact that it is not just the transpose, but complex conjugate transpose, is so that the inner product is preserved and is a real number, as if

$$\langle v, w \rangle = \sum v^{*t} w$$

then for a unitary matrix U, the inner product of the transformations is

$$\begin{aligned} \langle Uv, Uw \rangle &= \sum (Uv)^{*t} Uw \\ &= \sum v^{*t} U^{*t} Uw \\ &= \sum v^{*t} (Id) w \\ &= \sum v^{*t} w \end{aligned}$$

So just as the inner product is preserved for matrices in $O(n)$ for real field elements, the inner product is preserved for matrices in $U(n)$ for complex field elements. To get the group $SU(n)$, called the *special linear group*, we must also require that $\forall U \in SU(n), det(U) = 1$. The groups $U(1)$, $SU(2)$ and $SU(3)$ will be of particular interest to us, as they are the symmetries that are used in the description of the electric, weak, and strong force respectively.

As an example, the group $U(1)$ can be represented as a map,

$$\begin{aligned} \rho : \mathbb{R} &\longrightarrow \mathbb{C}^* \\ x &\longmapsto e^{i2\pi x} \end{aligned}$$

As ρ maps all real numbers to complex numbers with a magnitude of 1.

A continuous group is called *compact* if all parameters are bounded, and their domain of variation is closed. This essentially means that the topological space of the group is "small", and does not extend to infinity. All the groups we have previously discussed are compact. This notion will be useful in using the fact that the strong force has an $SU(3)$ symmetry in our analysis of the force.

2.3 Lagrangian Mechanics

In Newtonian mechanics, we only use three postulates, known as Newton's laws, to derive all of our equations of motion. In Lagrangian mechanics, we instead only consider one: that the action of the system must be minimized (or maximized). This is called the Principle of Least Action. Instead of considering laws concerning forces, we consider energies of the system. The Lagrangian is defined as

$$\mathcal{L} = T - V$$

where T is the kinetic energy of a system, and V is the potential energy. The action of a system is typically defined as the integral of the Lagrangian from an initial time t_0 to a final time t_f:

$$S = \int_{t_0}^{t_f} \mathcal{L}\, dt$$

It turns out that this idea is very fundamental. It was key in the formulation of not only quantum mechanics, but quantum field theory, as the principle held even for systems of particles. Suppose we have a path $y(t)$ that minimizes the action S, and another path $\bar{y}(t)$ that is very close to the path $y(t)$. Then

$$\bar{y}(t) = y(t) + \varepsilon\eta(t)$$

where ϵ is a very small parameter. The Lagrangian of this new function $\bar{y}(t)$ can be related to the Lagrangian of the path $y(t)$:

$$\mathcal{L}(t, \bar{y}, \frac{d\bar{y}}{dt}) = \mathcal{L}(t, y + \varepsilon\eta, \epsilon\frac{d\eta}{dt})$$

We will not go through the entire derivation, but from this assumption, using calculus of variations, we can eventually find the Euler-Lagrange equation:

$$\frac{\partial \mathcal{L}}{\partial y} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial y'} = 0$$

and this equation can be used to derive the equations of motion of any system where the kinetic and potential energies are known. Although this was originally used in classical mechanics, it is also central to the descriptions of both quantum electrodynamics and chromodynamics. The Langrangian of the respective systems are invariant under both global and local transformations due to their $U(1)$ and $SU(3)$ symmetries.

Hamilton soon after came up with a slightly different method, called Hamiltonian mechanics, where the Hamiltonian is defined as

$$H = T + V$$

so we add the energies instead of subtracting. Typically in quantum mechanics, we consider the Hamiltonian instead of the Lagrangian, although both formulations are equivalent. It simply depends on which formulation leads to cleaner formulas and less complex expressions.

2.4 Quantum Mechanics

As stated, the notion of the Lagrangian or Hamiltonian can be extended into the context of quantum mechanics. In quantum mechanics, we do not consider a position function of a particle, as particles do not have a definite position or momentum until measured, and measuring one leaves uncertainty in the other due to the Heisenberg uncertainty principle. Instead, we consider the *wave function* of the particle, $\Psi(\mathbf{x}, t)$, where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

which are the three spacial coordinates. Due to the fact that particles do not have a definite position or momentum until observed, the typical interpretation considers the square of the wave function $|\Psi|^2$ as the probability density of finding the particle at a particular position. We can derive all wanted quantities of a system from only the position and momentum, such as kinetic energy, potential energy, or angular momentum. Because of this, we instead consider operators on the wave function. Again, due to uncertainty in position, we consider the expectation value of either position or momentum. The expectation is analogous to the mean, or average, of a finite set of (x, y) points, where $y(x)$ denotes the number of times a measured quantity is seen with respect to x, which is

$$\langle x \rangle = \frac{\sum_{j=0}^{\infty} xy(x)}{y} = \sum_{j=0}^{\infty} xP(x)$$

where $P(x)$ denotes the probability of a given x. Then in quantum mechanics, the expectations of position and momentum are defined as:

$$\langle x \rangle = \int \Psi^*(x)\Psi \quad \text{and}$$
$$\langle p \rangle = \int \Psi^*(\frac{\hbar}{i}\frac{\partial}{\partial x})\Psi$$

where the operator (x) represents position, and the operator $(\frac{\hbar}{i}\frac{\partial}{\partial x})$ represents momentum. This is why such values are called operators; they *operate* on the wave function.

We will now introduce common notation in quantum mechanics, for clarity. A vector α is denoted as $|\alpha\rangle$, and is called a ket. This is bra-ket notation. The inner product of two vectors $|\alpha\rangle$ and $|\beta\rangle$, where $|\alpha\rangle = (a_1, a_2, ..., a_n)$ and $|\beta\rangle = (b_1, b_2, ..., b_n)$, is

$$\langle \alpha | \beta \rangle = a_1^* b_1 + a_2^* b_2 + ... + a_n^* b_n.$$

Matrix multiplication is typically not commutative. Then the commutator of two matrices A and B is defined as

$$[A, B] = AB - BA.$$

Typically, we do not consider vectors with real entries, but instead functional entries. Then the vector $|\Psi\rangle$, called the state vector, contains all the necessary information of the system the wave describes.

We can now begin to see the role unitary matrices play in quantum descriptions. If a symmetry operation O is carried out on $|\Psi\rangle$, such as a rotation, then since the inner product must remain invariant under these operations, it immediately follows that O is unitary. Let O act on $|\Psi\rangle$ as follows:

$$|\Psi\rangle \longrightarrow |\Psi'\rangle = O|\Psi\rangle$$

then the inner product of state vector $|\Psi'\rangle$ with itself is

$$\begin{aligned}
\langle\Psi'|\Psi'\rangle &= \langle O\Psi|O\Psi\rangle \\
&= \int (O\Psi)^* O\Psi \\
&= \int \Psi^* O^* O\Psi \\
&= \int \Psi^* (Id)\Psi \\
&= \int \Psi^*\Psi \\
&= \langle\Psi|\Psi\rangle.
\end{aligned}$$

This implies that any operator that is unitary leaves the inner product the same. Operators are used to give us expectation values of a desired quantity of a quantum system. This means that any unitary operator leaves the inner product invariant.

We discussed that $|\Psi|^2 = \langle\Psi|\Psi\rangle$ gives us the probability of seeing a particle in a particular position. The culmination of our results allows us to interpret that any unitary operator leaves the probabilities unchanged. We have already discussed how rotations of our coordinate system are unitary. It is also true that time translations, or looking at the state of the wave function at two times, $\Psi(t_0)$ and $\Psi(t)$, is unitary as well. Unitarity is necessary so that the probability density of Ψ is always 1.

These discussions would soon lead us to Noether's Theorem. We may consider that the energies before and after be the same. If we consider the expectation value of the energy, defined as $\langle\Psi|H|\Psi\rangle$, and consider the influence of the operator U as $\langle U\Psi|H|U\Psi\rangle$, we would eventually find that there must be a real, invariant quantity associated to the symmetry exhibited by U. This is what Noether's Theorem states. By considering each operator, we would find conservation of energy, momentum, angular momentum, and as we shall soon see, the conservation of charge.

3 Quantum Field Theory

3.1 U(1) Symmetry

The description of quantum chromodynamics, and its dependence on $SU(3)$ symmetry, was heavily inspired by the fact that electrodynamics was described so successfully by its $U(1)$ symmetries. It has commonly thought that electrodynamics is one of the most successful physical theories to date, with experimental agreement higher than one part in a billion for some certain measurements. To better understand where the idea for the description of quarks came from, we will first study how the $U(1)$ group plays a role in the description of other quantum field theories.

In the standard model, there are many different kinds of particles. There are fermions with half-integer spin, and bosons with integer spin. Spin can, in a vague sense, be thought of as intrinsic angular momentum of a particle (but it is not quite that; the details are unimportant to our discussion). Bosons are force mediators between particles. The photon is the boson that mediates the electromagnetic force. We would like to derive the photon as a consequence of the gauge invariance of a particular Lagrangian.

First, we will discuss a couple notations and definitions. A tensor can be thought of as a multidimensional array. For our purposes, it suffices to think of the tensors we see as a matrix of matrices; that is, each entry in the tensor T is a matrix M_{ij}, where the position of M is denoted by its indices. For example, to find the entry in the 2^{nd} row and 3^{rd} column of a T, we would reference the matrix M_{23}. If a tensor is written as $T^{\mu\nu}$, this means the tensor is in contravariant form. The indices μ and ν are spacetime indices that go over the four dimensions of spacetime. As quantum field theory is a relativistic theory, we consider space and time to not be separate, but one four-dimensional object.

The action that generates the electron field's Dirac equation is given by

$$S = \int \psi^*(i\gamma^\mu \partial_\mu - mc^2)\psi \; d^4x$$

and the global symmetry for this system is

$$\psi \longmapsto e^{i\theta}\psi$$

Here, the gauge group is $U(1)$, as we have seen that any mapping to $e^{i\theta}$ is a $U(1)$ transformation, where θ is a constant. This describes rotations of the phase angle of the electron field. Gauge theories aim to localize the symmetry, by replacing the constant θ by a function $\theta(x)$, where the rotation in phase space is dependent on the spacetime location x. This requirement brings an additional term where the Lagrangian of the equation is no longer invariant. To fix this, we must introduce a covariant derivative:

$$D_\mu = \partial_\mu - i\frac{e}{\hbar}A_\mu$$

where A_μ represents a vector potential with respect to the electromagnetic field, and e is a charge. Adding a Lagrangian for A_μ in terms of the field strength tensor, we then get the Lagrangian density

for the gauge field A_μ:

$$\mathcal{L} = \psi^*(i\gamma^\mu D_\mu - mc^2)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

where $F^{\mu\nu}$ is the contravariant electromagnetic field tensor, described by $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$, and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, the covariant tensor. By considering local symmetries instead of global of the Lagrangian, we can then derive conserved quantities, such as charge. Because the group $U(1)$ has a single generator, this implies that the force is mediated by a single gauge boson, known as the photon.

3.2 SU(3) Symmetry and The Strong Force

Quarks have quite a different nature than particles such as leptons (particles that do not interact with the strong force). Quarks come with what we call color charge. It is not color in the usual sense, but rather an arbitrary label for a different type of charge unrelated to the electric charge. There are three color charges; blue, red, and green. It turns out that we can 'rotate' the colors in color space without changing any of the underlying physics. We can imagine the color charges as a vector

$$\begin{bmatrix} R \\ G \\ B \end{bmatrix}$$

where each index represents the corresponding color charge. We can then imagine a rotation operator in color space. This operation essentially turns one color of quark into another. This operator must be unitary as previously shown. It turns out that there are 8 different generators of the $SU(3)$ group, which, as analogous to quantum electrodynamics, relates to the bosons of the strong force. This means that there are are 8 force mediators of quarks, known as gluons. Due to the nature of the $SU(3)$ group, these gluons also carry a color charge themselves, unlike photons, which have no charge.

In fact, what is more interesting is that not only does the color charge exhibit $SU(3)$ symmetry, but as does the flavor of the quarks. Flavor is a term used to differentiate different kinds of quarks, and is called a quantum number of the particle. There are six flavors: up, down, charm, strange, top, and bottom. Most particles we see and interact with are made of up and down quarks. For example, a proton is comprised of two up quarks and one down quark (as well as the gluons between them). The flavors symmetries are used to describe the interactions between different flavors of quarks.

With our current description, there were problems with the theory, one of which was confinement; that a free quark has never been observed. What brought the theory together was the idea of gauge invariance, as similarly applied to quantum electrodynamics. Instead of considering global transformations, we instead would consider local ones, where the transformation is dependent on the spacetime location. With this new approach, the non-abelianness of $SU(3)$ explained the concept of confinement. The theory with local $SU(3)$ symmetry's dynamics is described by the Yang-Mills Lagrangian.

3.3 Conclusion

Group theory plays a fundamental role in our current understanding of the particles mediating our universe. Although these groups were discovered independently of physics research, they ended up being integral to the descriptions of the forces, and fundamental in explanations of how the force interacts with the universe. Our realization of $U(1)$ symmetries in the electromagnetic force allowed us to apply the same train of thought to the strong force and obtain $SU(3)$ symmetries of quarks, as well as $SU(2)$ symmetries of the weak force, and the more complex symmetries of the group $SU(2) \times U(1)$ of the electroweak force. It still remains a mystery as to the unification of gravity along with the other three fundamental forces, and being able to describe it in terms of some symmetries of the universe.

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