UCLA - Henry Samueli School of Engineering
Department of Computer Science
CS112 - Homework Set 3
Due date Wednesday, January 30th

The homework must be submitted online in electronic form using the PDF format. Verify that the file is readable before submitting, broken files and late submissions will not be graded. Comment your solutions step by step. Please indicate your complete name and UID in the headings.

1

Suppose that two cards are drawn from an ordinary deck, one-by-one, at random and with replacement (the drawn card is put back into the deck). Let X be the number of spades drawn. If an outcome of spades is denoted by s, and other outcomes are represented by t, then X is a real-valued function defined on sample space

$$S = \{(s,s),(s,t),(t,t),(t,s)\}$$

1.1

What is the probability for X = 1 , X = 2 , and X = 3 ?

1.2

Given that the first card is spade, what is the probability of X = 1 and X = 2 ?

1.3

Given that one card is spade, what is the probability of X = 1 and X = 2 ?

2

UCLA hospital reopen after renovation. Let X be the number of births in a hospital until the first boy is born.

2.1

What is the probability for $X = 1$, $X = 5$, and $X = 10$?

2.2

When the first boy is born, what is the probability that the number of girls born is equal or greater than boys?

3

From 18 potential women jurors and 28 potential men jurors, a jury of 12 is chosen at random. Let X be the number of women selected.

3.1

Find f, the pmf of X

3.2

What is the probability that all jurors are female?

3.3

What is the probability that half of the jurors are men?

4

A box contains 5 tennis balls, of which 3 are new.

4.1

Suppose that 2 balls are selected randomly, what is the probability that they are both new?

4.2

From (1), suppose these balls are used and returned to the box. If 2 other 2 balls are picked what is the probability that they are both new.

5

Around UCLA there are 40 taxis, labelled from 1 to 40. Four taxis arrive at random at a UCLA turnaround to pick up passengers. Assuming that each taxi only visit UCLA once per day. That is after leaving UCLA a taxi will not return.

5.1

What is the probability that the label of at least one of the taxis is less than 5 ?

5.2

What is the probability that the labels of at least two taxis is less than 5 ?

6

The hats of n persons are thrown into a box. The persons then pick up their hats at random (i.e. so that every assignment of the hats to the persons is equally likely).

6.1

What is the probability that every person gets his or her hat back?

6.2

What is the probability that the first m persons who picked hats get their own hats back?

6.3

What is the probability that everyone among the first m persons to pick up the hats gets back a hat belonging to one of the last m persons to pick up the hats?

7

A communication channel inverts a bit with probability ϵ. Suppose that you are receiving the output (bit by bit) and you want to decode the original message (or what this most likely was). What mapping function do you use ? consider the possible different values of ϵ. When is it impossible to decode the original bits? [Hint: Consider the conditional probability of the input given the output]

8

Alice, Bob, and Caroll play a ping-pong tournament. Alice and Bob play the first game. The third player plays next against the winner of that game. The tournament ends when some player wins two successive games. Let a tournament history be the list of game winners, so for example ACBAA corresponds to the tournament where Alice won games 1, 4, and 5, Caroll won game 2, and Bob won game 3. We are told that every possible tournament history that consists of k games has probability $\frac{1}{2^k}$, and that a tournament history consisting of an innite number of games has zero probability. Demonstrate that this assignment of probabilities dened a legitimate probability law.

9

A computer is defective and completes a task with probability p. You now submit a batch of n tasks and you want to know the probability that not all of them terminated correctly.

10

There are 10 workstations, each of which in average sends 10 emails per hour distributed accordingly to a Poisson distribution. There have been some problems with the mail server that now crashes if it receives more than 1 email per hour. what is the probability of the server crashing? Comment your answer.

1

1.1

$$P(X=1)=\frac{|(s,t),(t,s)|}{|S|}=\frac{2}{4}$$

$$P(X=2)=\frac{|(s,s)|}{|S|}=\frac{1}{4}$$

$$P(X=3)=\frac{|0|}{|S|}=0$$

1.2

$$P(X=1)=\frac{|(s,t)|}{|(s,t),(s,s)|}=\frac{1}{2}$$

$$P(X=2)=\frac{|(s,s)|}{|(s,t),(s,s)|}=\frac{1}{2}$$

1.3 c

$$P(X=1)=\frac{|(s,t),(t,s)|}{|(s,t),(s,s),(t,s)|}=\frac{2}{3}$$

$$P(X=2)=\frac{|(s,s)|}{|(s,t),(s,s),(t,s)|}=\frac{1}{3}$$

2

2.1

$$P(X=1)=\frac{1}{2}$$

$$P(X=5)=(\frac{1}{2})^4(\frac{1}{2})=(\frac{1}{2})^5$$

$$P(X=10)=(\frac{1}{2})^9(\frac{1}{2})=(\frac{1}{2})^{10}$$

2.2

$$p(2) + p(3) + p(4) + ... + p(\infty) = 1 - p(1) = \frac{1}{2}$$

3

3.1

$$F(x) = \frac{C_x^{18} C_{12-x}^{28}}{C_{12}^{46}}$$

3.2

$$F(12) = \frac{C_{12}^{18}}{C_{12}^{46}}$$

3.3

$$F(0) = \frac{C_{12}^{28}}{C_{12}^{46}}$$

4

4.1

Let X be a random variable define as the number of new ball selected.

$$F(x) = \frac{C_x^3 C_{2-x}^2}{C_2^5}$$

$$F(2) = \frac{C_2^3}{C_2^5} = \frac{3}{10} = 0.3$$

4.2

Let X be a random variable define as the number of new ball selected in the first round. Let Y be a random variable define as the number of new ball selected in the second round.

$$P(Y = 2) = P(Y = 2|X = 0)P(X = 0) + P(Y = 2|X = 1)P(X = 1) + P(Y = 2|X = 2)P(X = 2)$$
$$= 0.3 * 0.1 + 0.1 * 0.6 + 0 * 0.3 = 0.09$$

5

5.1

$$P(\text{At least one taxi number is less than 5}) = 1 - P(\text{No taxi number is less than 5})$$
$$= 1 - \frac{C_4^{35}}{C_4^{40}}$$

5.2

$$P(\text{At least one taxi number is less than 5}) = 1 - P(\text{No taxi number is less than 5})$$
$$-P(\text{One taxi number is less than 5}) = 1 - \frac{C_4^{35}}{C_4^{40}} - \frac{C_3^{35} C_1^5}{C_4^{40}}$$

6

6.1

consider the sample space of all possible hat assignments. It has n! elements (n hat selections for the rst person, after that n-1 for the second, etc.), with every singleelement event equally likely (hence having probability 1/n!). The question is to calculate the probability of a single-element event, so the answer is 1/n!

6.2

consider the same sample space and probability as in the solution of (a). The probability of an event with (n-m)! elements (this is how many ways there are to disribute the remaining n-m hats after the first m are assigned to their owners) is (n-m)!/n!

6.3

there are m! ways to distribute m hats among the first m persons, and (n-m)! ways to distribute the remaining n-m hats. The probability of an event with m!(n-m)! elements is m!(n-m)!/n!.

7

For $\epsilon < 1/2$ you keep the symbol that you received. For $\epsilon > 1/2$ you invert the symbol that you received. For $\epsilon = 1/2$ input and output are independent therefore a message cannot be decoded meaningfully.

8

The probability of an event is $1/2^k$ times the number of finite histories contained in the event. The probability of the event consisting of one or both infinite histories is 0. We have to show that this probability law satisfise the three probability axioms. It clearly satisfies non-negativity and additivity. To check normalization, we have to verify that the probabilities of all tournament histories sum up to 1. Start by noticing that two of the histories are infinite and have probability 0. Each one of the remaining histories has some nite length $k \leq 2$ and probability $1/2^k$. Hence, summing all probabilities we get:

$$2 \cdot 0 + \sum_{k=2}^{\infty} 2 \cdot \frac{1}{2^k} = \sum_{k=2}^{\infty} \frac{1}{2^{k-1}} = \sum_{k=0}^{\infty} \frac{1}{2^{k+1}} = \frac{1}{2} \sum_{k=0}^{\infty} \frac{1}{2^{k+1}} = \frac{1}{2} \cdot \frac{1}{1 - 1/2} = 1$$

9

If a tasks ends with probability p, the probability of having k tasks in a row ending correctly is geometrically distributed. What we are looking for in this case is the cumulative distribution of the geometric random variable since what we are looking for is the distribution of $P[X < n]$

10

The activity of each machine can be modeled with the poisson distribution of intensity $\lambda = 10$. We know that it is possible to combine poisson distributions together by summing their intensities. Therefore, all the machines can be modeled as a single random variable of intensity $\lambda = 100$. Given this distribution, we know that we are looking for the probability of $P(X > 1) = 1 - P(X = 0) - P(X = 1)$