CS112 Homework 8
Queueing preliminaries

Q1. Consider a taxi station where taxis and customers arrive in a poisson process with respective rates of one and two per minute. A taxi will wait no matter how many taxis are waiting. But if an arriving customer does not find a taxi he leaves. Find
a) the average number of taxis waiting
b) the proportion of arriving customers that get taxis.

Solution:
Let the state be the number of taxis waiting. Then we get a birth-death process with $\lambda = 1$ and $\mu = 2$. Also, this can be thought of as an M/M/1 system where being serviced is equivalent to waiting for a customer. Therefore:

a) Average number of taxis waiting $= \frac{1}{\mu - \lambda} = 1$.

b) The proportion of arriving customers tht get a taxi is the proportion of arriving customers that find at least one taxi waiting. This is equivalent to the proportion of time the system is **not** in state 0. This is equal to $1 - P_0 = 1 - (1 - \frac{\lambda}{\mu}) = \frac{1}{2}$.

Q2. In a population growth model, suppose that immigration is allowed only when the population is 0, 1, 2, 3, or 4 (births can occur when the population size is 1, 2,..., unlimited). Assume a birth rate per individual = 1, death rate per individual = 2, immigration rate = 4 (when allowed),
(a) set up "flow balance" equations for the p_i, using the "circle one more state each time" shortcut helpful in analyzing birth-death models.
(b) calculate p_0.
(c) calculate the equilibrium fraction-of-time during which immigration is allowed.

Solution:
a) The flow balance equations are as follows:
$p_k(\theta + i\lambda) = p_{k+1}(i+1)\mu$
$p_k \lambda i = p_{k+1}(i+1)\mu$
given that $\theta = 4, \lambda = 1, \mu = 2$.
(b) We find that $p_1 = 2p_0,\ p_2 = p_3 = \frac{5}{2}p_0,\ p_4 = \frac{35}{16}p_0,\ p_5 = \frac{7}{4}p_0$.
And, for $k \geq 6,\ p_k = \frac{35}{4k}(\frac{\lambda}{\mu})^{k-5}p_0$.
$\Rightarrow p_0(1 + 2 + 5/2 + 5/2 + 35/16 + 7/4 + \frac{35}{4}\sum_{k=6}^{\infty}\frac{1}{k}(\frac{\lambda}{\mu})^{k-5}) = 1$
$\Rightarrow p_0(11.9375 + 280(ln2 - \sum_{k=1}^{5}\frac{1}{k2^k})) = 1$ (Because $\sum_{k=1}^{\infty} x^k = -ln(1-x)$)
So, $p_0 = 0.0756$.
(c) P(immigration is allowed) $= p_0 + p_1 + p_2 + p_3 + p_4 = 0.77$.

Q3. Customers arrive at a single service facility at a Poisson rate of 40 per hour. When two or fewer customers are present, a single attendant operates at the facility and service time

for each customer is exponentially distributed with a mean value of two minutes. however, when there are more than two customers present, the attendant is joined by an assistant and the mean service time reduces to one minute. Assuming a system capacity of four customers:
a) What proportion of time are both attendants free?
b) Each man is to receive a salary proportional to the amount of time they work. If they have to split 100 dollars between them how should this money be split?
c) What is the average amount of time a customer spends in the system and what is the average waiting time? *Hint: Use Little's Result*

Solution:
This is a 5 state birth death process, each state being the number of customers in service. λ is 40 for all states, while μ is 30,30, 60 and 60 respectively. Writing balance equations and solving we get, $p_0 = 81/493$.

a) Fraction of time both are free is simply p_0.
b) The first server works for (1-p_0) of the time and the second works for ($p_3 + p_4$) of the time. So the second works for ($p_3 + p_4$)/ (1-p_0) of the time the first server works. This is approximately 0.414. So, 1.414 X = 100, X = 70.72 and 100 - x = 29.28.
c) T = N/λ = (940/493)/[40(429/493)] = 0.054 hours = 3.286 minutes. And waiting time, W = $\frac{N_q}{\lambda}$ = 1.286 minutes.

Q4. Consider a sequential-service system consisting of two servers, A and B. Arriving customers will enter this system only if server A is free. If a customer does enter, he is immediately serviced by server A. When service by A is completed he goes to server B if B is free, or if B is busy, he leaves the system. Upon completion of service at server B, the cutomer departs. Assuming that the Poisson arrival rate is two customers per hour and that A nd B serve customers at a rate of 4 per hour and 2 per hour respectively,
a) What proportion of customers enter the system?
b) What proportion of entering customers receive service from B?
c) What is the average number of customers in the system?
d) What is the average amount of time an entering customer spends in the system?

Solution:
There are four states, 0, A, B, AB signifying which of the servers are free. The balance equations are:
$2p_0 = 2p_B, 4p_A = 2p_0 + 2p_{AB}, 4p_B = 4p_A + 4p_{AB}, 6p_{AB} = 2p_B$. Solving, we get, $p_0 = 3/9, p_A = 2/9, p_B = 3/9, p_{AB} = 1/9$.
a) $p_0 + p_B = 2/3$.
b) By conditioning on whether the customer entered when the system was in state 0 or in state B, we get probability = 1/2 + (1/2)(2/6) = 2/3.
c) Average number in system = $p_A + p_B + 2p_{AB} = 7/9$.
d) Effective arrival rate = $\lambda(p_0 + p_B)$ = 4/3 per hour. Using Little's formula, we get T =

$(7/9)/(4/3) = 7/12$.We could also solve this by conditioning on when the customer arrived. We'd get $1/2(1/4 + 1/2) + 1/2(1/4 + (2/6)(1/2)) = 7/12$.

Q5. Derive the probability density function (pdf) of the waiting time for an M/M/1 queue in terms ofthe sytem parameters λ and μ.

[Hint: Try to find the transform of the waiting time W*(s) first. To find the transform, condition on the number of customers an arrival finds in the system and then uncondition. i.e. Find the conditional W*(s) given the arrival finds k customers in the system, then uncondition.]

Solution:
The waiting time for an arrival is simply the time it spends in the queue waiting for the customers in front of it to be served. If an arrival finds k customers in the system, it will have to wait for all k of them to be served. The distribution of this time is the sum of k iid Random Variables each with exponential distribution with parameter μ. Instead of a lengthy convolution we take the Laplace transform of waiting time W*(s) by multiplying the LTs of the k RVs.
W*(s| arrival finds k customers in system) = $[B*(s)]^k$
Unconditioning, we get,

$$\begin{aligned} W*(s) &= \sum_{k=0}^{\infty}[B*(s)]^k p_k \\ &= \sum_{k=0}^{\infty}[B*(s)]^k(1-\rho)\rho^k \\ &= (1-\rho)+\sum_{k=1}^{\infty}[B*(s)]^k(1-\rho)\rho^k \\ &= (1-\rho)+(1-\rho)\frac{B*(s)\rho}{1-B*(s)\rho} \\ &= (1-\rho)+(1-\rho)\frac{\frac{\mu\rho}{s+\mu}}{\frac{s+\mu-\rho\mu}{s+\mu}} \\ &= (1-\rho)+\rho\frac{\mu(1-\rho)}{s+\mu(1-\rho)} \\ \Leftrightarrow w(y) &= (1-\rho)u_0(t)+\rho(1-\rho)\mu e^{-\mu(1-\rho)t} \end{aligned}$$

W(y) is a mixed distribution as there is an impulse at t = 0 and a continuous distribution from 0 to ∞.