CS112 - Homework #6 Solution

1. An organization has N employees where N is a large number. Each employee has one of three possible job classifications and changes classifications (independently) according to a Markov Chain with transition probabilities

$$\begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.2 & 0.6 & 0.2 \\ 0.1 & 0.4 & 0.5 \end{bmatrix}$$

What percentage of employees are in each classification?

Solution. *Solving* $\pi = \pi \mathbf{P}$

$$\begin{aligned} \pi_0 &= 0.7\pi_0 + 0.2\pi_1 + 0.1\pi_2 \\ \pi_1 &= 0.2\pi_0 + 0.6\pi_1 + 0.4\pi_2 \\ \pi_2 &= 0.1\pi_0 + 0.2\pi_1 + 0.5\pi_2 \\ \pi_1 + \pi_2 + \pi_3 &= 1. \end{aligned}$$

$$\pi_1 = 0.35,\ \pi_2 = 0.41,\ \pi_3 = 0.24. \ \textit{Therefore}\ \ \pi = [0.35 \ \ 0.41 \ \ 0.24].$$

2. On any given day Gary is either cheerful (C), so-so (S) or glum (G). If he is cheerful today, then he will be C, S or G tomorrow with probabilities 0.5, 0.4 and 0.1, respectively. If he is feeling so-so today, then he will be C, S or G tomorrow with probabilities 0.3, 0.4 and 0.3. If he is glum today, then he will be C, S or G tomorrow with probabilities 0.2, 0.3, 0.5.

   (a) Find the transition probability matrix $\mathbf{P}$.

(b) In the long run, what proportion of time is Gary in each of the three 'moods'?

Solution. *We have a 3 state Markov Chain where,*

- **State 0 (C)** *Gary is cheerful*
- **State 1 (S)** *Gary is so-so.*
- **State 2 (G)** *Gary is glum.*

a)

$$\mathbf{P} = \begin{bmatrix} 0.5 & 0.4 & 0.1 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.3 & 0.5 \end{bmatrix}$$

b) Solving $\pi = \pi\mathbf{P}$

$$\begin{aligned} \pi_0 &= 0.5\pi_0 + 0.3\pi_1 + 0.2\pi_2 \\ \pi_1 &= 0.4\pi_0 + 0.4\pi_1 + 0.3\pi_2 \\ \pi_2 &= 0.1\pi_0 + 0.3\pi_1 + 0.5\pi_2 \\ \pi_1 + \pi_2 + \pi_3 &= 1. \end{aligned}$$

$$\pi_1 = \frac{21}{62}, \ \pi_2 = \frac{23}{62}, \ \pi_3 = \frac{18}{62}. \ \ \textit{Therefore} \ \ \pi = [\frac{21}{62} \ \ \frac{23}{62} \ \ \frac{18}{62}].$$

3. Suppose that whether or not it rains today depends on previous weather conditions through the last two days. That is, suppose that if it has rained for the past two days, then it will rain tomorrow with probability 0.7; if it rained today but not yesterday, then it will rain tomorrow with probability 0.5; if it rained yesterday but not today, then it will rain tomorrow with probability 0.4; if it has not rained in the past two days, then it will rain tomorrow with probability 0.2.

   (a) Define appropriate states in order to make the above model a Markov Chain.

   (b) Find the transition probability matrix $\mathbf{P}$ for the states defined in part (a).

Solution. *If we let the state at day n depend only on whether or not it is raining on day n the given model is not a Markov Chain. Hence we transform the model into a Markov Chain by saying that the state depends on whether or not it rained on the previous day and on the current day. Hence we define:*

- **State 0 (RR)** *Rained yesterday, Rained today*
- **State 1 (DR)** *Didn't rain yesterday, Rained today.*
- **State 2 (RD)** *Rained yesterday, Didn't rain today.*
- **State 3 (DD)** *Didn't rain yesterday, Didn't rain today.*

b) For this 4 state Markov Chain the transition probability matrix is

$$P = \begin{bmatrix} 0.7 & 0 & 0.3 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.4 & 0 & 0.6 \\ 0 & 0.2 & 0 & 0.8 \end{bmatrix}$$

4. Suppose that coin 1 has probability 0.7 of coming up heads, and coin 2 has probability 0.6 of coming up heads. If the coin flipped today comes up heads, then we select coin 1 to flip tomorrow, and if it comes up tails, then we select coin 2 to flip tomorrow. If the coin initially flipped is equally likely to be coin 1 or coin 2, then what is the probability that the coin flipped on the third day after the initial flip is coin 1?

Solution. *Let the state be the number of coin flipped on that day. Then the transition probability matrix is,*

$$P = \begin{bmatrix} 0.7 & 0.3 \\ 0.6 & 0.4 \end{bmatrix}$$

and so,

$$P^2 = \begin{bmatrix} 0.67 & 0.33 \\ 0.66 & 0.34 \end{bmatrix}$$

and

$$P^3 = \begin{bmatrix} 0.667 & 0.333 \\ 0.666 & 0.334 \end{bmatrix}$$

Hence, probability that coin 1 is flipped on third day is,

$$\frac{1}{2}\left[P_{11}^3 + P_{21}^3\right] = 0.6665$$

5. Three out of every four trucks on the road are followed by a car, while one out of every five cars is followed by a truck. What fraction of vehicles on the road are trucks?

   Solution. *Let the state be 0 if the vehicle is a car and 1 if a truck we obtain the following two state Markov chain with transition probability matrix:*

   $$P = \begin{bmatrix} 0.8 & 0.2 \\ 0.75 & 0.25 \end{bmatrix}$$

   The long run proportion of cars and trucks are obtained by solving for π. Solving the equation we get $\pi = [\frac{15}{19} \;\; \frac{4}{19}]$ or 4 out of 19 vehicles are trucks.

6. Consider the homogenous Markov Chain whose state diagram is given in figure 1

   (1) Find P, the probability transition matrix.

   (2) Under what conditions if any will the chain be irreducible and aperiodic.

   (3) Solve for the equilibrium probability vector π.

   (4) What is the mean recurrence time for state E_2.

   (5) For what values of α and p will we have $\pi_1 = \pi_2 = \pi_3$? (Give a physical interpretation of this case.

1-p

1 2

1

p α

3

1-α

Figure 1: Markov Chain for Problem 6

Solution. *1)*

$$\mathbf{P} = \begin{bmatrix} 0 & 1-p & p \\ 1 & 0 & 0 \\ 0 & \alpha & 1-\alpha \end{bmatrix}$$

2) The Markov Chain is ireducible and aperiodic for all $0 < p \leq 1$ *and* $0 < \alpha \leq 1$ *except* $\alpha = p = 1$*. This is because if wither p or alpha is zero the chain ceases to become irreducible and if* $\alpha = p = 1$ *then it becomes periodic with period = 3.*

3) We solve for $\pi = \pi\mathbf{P}$.

$$\pi_1 = \pi_2$$
$$\pi_2 = (1-p)\pi_1 + \alpha\pi_3$$
$$\pi_1 + \pi_2 + \pi_3 = 1.$$

Thus, $\pi_1 = \pi_2 = \frac{\alpha}{p+2\alpha}$
and $\pi_3 = \frac{p}{p+2\alpha}$

4)

$$\mu_2 = \frac{1}{\pi_2} = \frac{p+2\alpha}{\alpha} = 2 + \frac{p}{\alpha}$$

5) We need, $\alpha = p$. **Interpretation** *Since each visit to state 1 is followed by exactly one visit to state 2 and vice-versa for all* p, α *we have* $\pi_1 = \pi_2$ *always. Also,* p $(= \alpha)$ *of the time we go to state 3 and the average number of steps (or mean time) spent in state 3 per visit is* $\frac{1}{1-(1-\alpha)} = \frac{1}{\alpha}$*. Thus, the average number of viits to state 3 per visit to state 1 is* $\alpha.\frac{1}{\alpha} = 1$.

7. For the discrete time Markov Chains in Figure 3:

   (a) Which states are transient?

   (b) Which states are absorbing?

   (c) Do the stationary state probabilities exist? in the case of F and G, does the answer depend on the exact values of the parameters p and q?

   Solution. *(A) states 1 and 2 are transient. Stationary state prob. vector is* $\pi = [0, 0, 1]$

   (B) state 3 is transient. $\pi = [4/7, 3/7, 0]$

   (C) no transient states. $\pi = [1/4, 1/2, 1/4]$

   (D) states 1 and 2 are transient. $\pi = [0, 0, 1/2, 1/2]$

   (E) state 3 is transient. There is no limiting state prob. vector independent of initial conditions.

   (F) All states are transient if $p \geq q$ *and no states are transient if* $p < q$*. Also, the stationary state probabilities exist if and only if the latter is the case.*

(G) The condition for stationary state probabilities to exist is $2p < q$. This can be seen as follows.

It is easy to see that the following are balance equations for the model:

$$\begin{aligned} \pi_0 &= \pi_1 q \\ \pi_0 + \pi_1 p &= \pi_2 q \\ \pi_1 + \pi_2 p &= \pi_3 q \end{aligned}$$

etc.

Summing all these equations we get:

$$2p \sum_{i=0}^{\inf} \pi_i = q \sum_{i=1}^{\inf} \pi_i$$

If there is a stationary state probability distribution, then $\sum \pi_i = 1$ and the above equation becomes:

$$2p = q(1 - \pi_0)$$

or

$$\pi_0 = 1 - \frac{2p}{q}$$

Clearly this requires $2p < q$ so that $\pi_0 > 0$.

8. Each morning an individual leaves his house and goes for a run. He is equally likely to leave either from his front or back door. Upon leaving the house, he chooses a pair of running shoes (or goes running barefoot if there are no shoes at the door from which he departed). On his return, he is equally likely to enter, and leave his running shoes, either by the front or back door. If he owns a total of k pairs of running shoes, what proportion of the time does he run barefooted?

   Solution. *Idea: Observe the person every day while running.*

   - *State:*

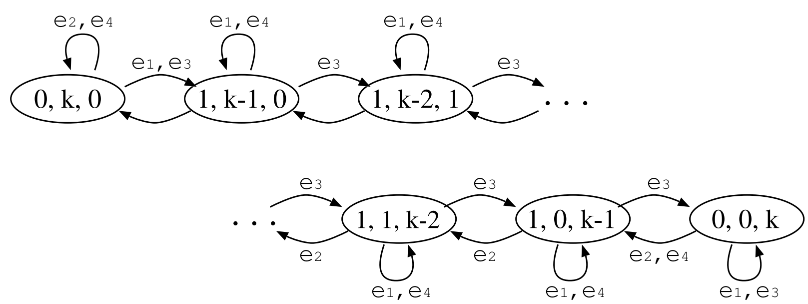

Figure 2: Discrete Time Markov Chain for Problem 2.

  - v_1 *indicates if barefooted (0) or not (1);*
  - v_2 *# shoes in Front door;*
  - v_3 *# shoes in Back door;*

- *Events: What may happen between two consecutive observations:*
  - e_1 *enter by Front door, and leave from Front door;*
  - e_2 *enter by Front door, and leave from Back door;*
  - e_3 *enter by Back door, and leave from Front door;*
  - e_4 *enter by Back door, and leave from Back door;*

Since the individual is equally likely to leave/enter by Front of Back door, every event e_i *has probability 1/4.*

Figure 2 show the Markov chain for this problem.

To get the proportion of days the individual run barefooted, we just solve the Markov chain to get the steady state probabilities and sum the probabilities associated to states $(0, k, 0)$ *and* $(0, 0, k)$.

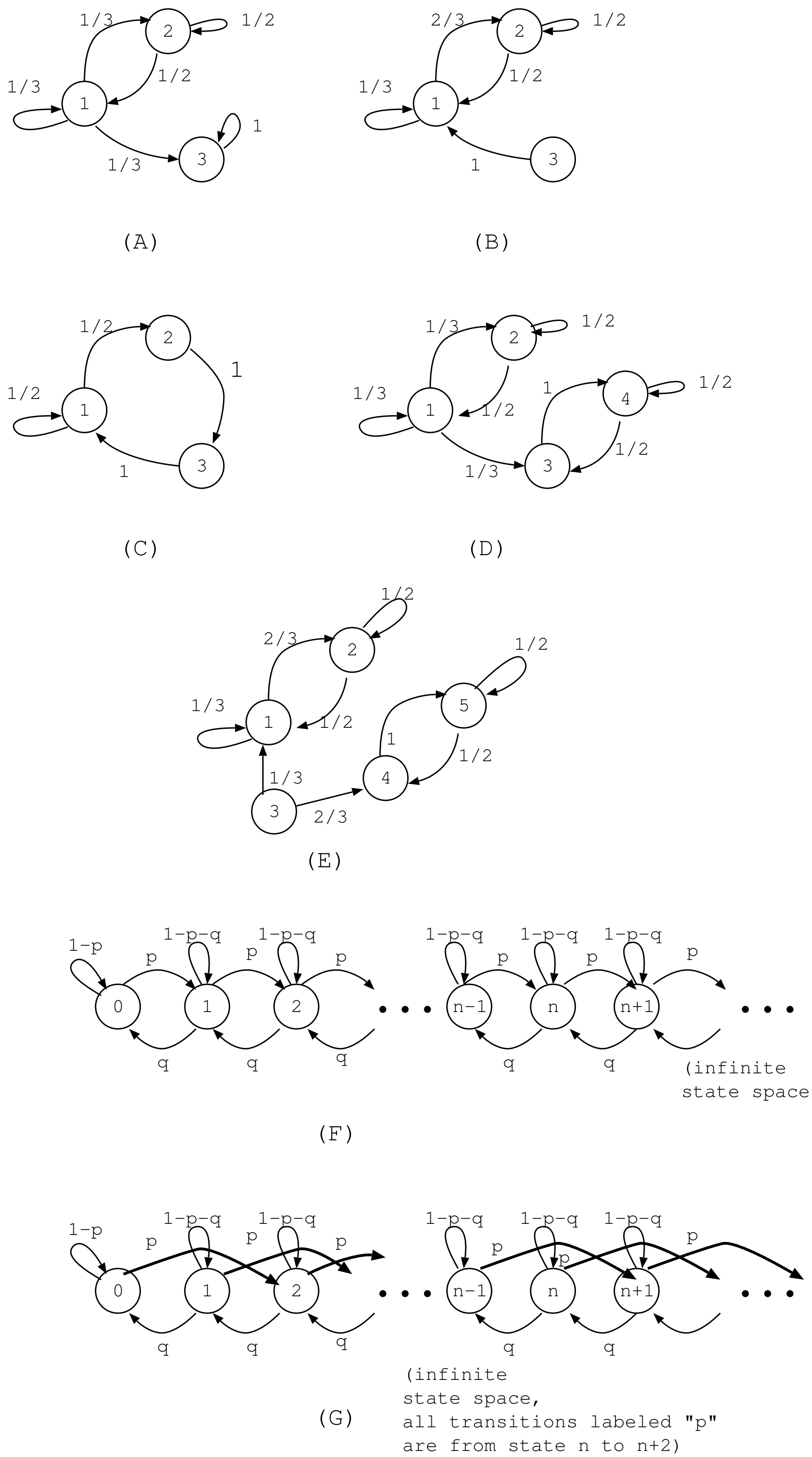

Figure 3: Discrete Time Markov Chains for Problem 1.