CS112 - Homework Set 1
Due date Friday, January 18th

The homework must be submitted online in electronic form using the PDF format. Verify that the file is readable before submitting, broken files and late submissions will not be graded. Please indicate your complete name and UID in the headings.

A.

Find the Laplace Transform of:
1. $f(x) = e^{ax}$
2. $f(x) = x^2$
3. $f(x) = 4x^2 - 3x + 7$
4. $f(x) = (x-1)^2$

B.

Sum the series: $\sum n^3$

C.

Using partial fractions decompose:
1. $F^*(s) = \frac{s+3}{(s-2)(s+1)}$
2. $F^*(s) = \frac{1}{(s^2+1)(s^2+2s+1)}$

D.

Consider the previous results as the Laplace transform $\mathcal{L}(f(x))$ of some f(x), find f(x) using the tables

E.

Solve the following differential equation:
$x(t) = \frac{dx(t)}{dt} + 2\frac{d^2x(t)}{dt^2}$
$x(0) = 1$

A.

1.

$$f(x) = e^{ax}$$
$$F^*(s) = \int_0^\infty e^{ax}e^{sx}dx$$

Integrating:

$$\int_0^\infty e^{-(s-a)x}dx = \frac{1}{s-a}$$

Note that this result is nothing but the exponential shift.

2.

$$f(x) = x^2$$
$$F^*(s) = \int_0^\infty x^2e^{sx}dx$$

Using integration by parts

$$[-\frac{x^2}{(s)}e^{-(s)x}]_0^\infty + \frac{2}{s}\int_0^\infty xe^{-(s)x}dx$$
$$0 + \frac{2}{s}\left([-\frac{x}{(s)}e^{-sx}]_0^\infty + \int_0^\infty e^{-sx}dx = \frac{1}{(s)^2}\right)$$
$$0 + \frac{2}{s}\left(0 + \frac{1}{s^2}\right) = \frac{2}{s^3}$$

Note that from the tables we know that the Laplace transform for $f(x) = x^n$ is equal to $L(s) = \frac{n!}{s^{n+1}}$.

3.

$$f(x) = 4x^2 - 3x + 7$$

Because of the linearity property we are allowed to calculate the transform of each term independently from the others. Hence:

$$F^*(s) = \int_0^\infty 4x^2e^{sx}dx - \int_0^\infty 3xe^{sx}dx + \int_0^\infty 7e^{sx}dx$$

From the tables and from the previous problem we know the transform of all the terms. The first term is exactly the previous exercise (multiplied by 4). The second term is the function $f(x) = x$ (multiplied by 3) and the last one is the unit function (multiplied by 7).

$$F^*(s) = \frac{4}{(s)^3} + \frac{3}{(s)^2} + \frac{7}{(s)}$$

4.

$$f(x) = (x-1)^2$$

By expanding the power:

$$f(x) = x^2 - 2x + 1$$

Applying the linearity property:

$$F^*(s) = \frac{2}{(s)^3} + \frac{2}{(s)^2} + \frac{1}{s}$$

B.

Sum the series: $\sum_{i=1}^{n} n^3$
This sum can be derived directly from pag. 1 of the cheat sheet given during discussion session (and available on course web), which contains also an outline of the demonstration. More material and a proof by induction can be found online at:
`http://www.proofwiki.org/wiki/Sum_of_Sequence_of_Cubes`

C.

1.

$$F^*(s) = \frac{s+3}{(s-2)(s+1)}$$

The setting for this case is:

$$\frac{s+3}{(s-2)(s+1)} = \frac{A}{(s-2)} + \frac{B}{(s+1)}$$

Using the cover-up method we find $A = 5/3$ and $B = -2/3$

$$\frac{5}{3(s-2)} - \frac{2}{3(s+1)}$$

2.

$$F^*(s) = \frac{1}{(s^2+1)(s^2+2s+1)}$$

In this case we first need to reduce the second factor of the denominator, which can be recognized as the square of the polynomial $(s+1)$. Therefore:

$$\frac{1}{(s^2+1)(s^2+2s+1)} = \frac{1}{(s^2+1)(s+1)^2}$$

The resulting partial fraction has one repeated term and one factor of degree two. The set-up for this scenario is:

$$\frac{1}{(s^2+1)(s+1)^2} = \frac{A}{(s+1)} + \frac{B}{(s+1)^2} + \frac{Cs+D}{(s^2+1)}$$

Note that the cover-up method does not apply to this set-up so we need to solve a system of equations. By removing the denominator we obtain:

$$1 = A(s^2+1)(s+1) + B(s^2+1) + (Cs+D)(s+1)^2$$

Algebraically we can derive:

$$1 = As^3 + As^2 + As + A + Bs^2 + B + Cs^3 + 2Cs^2 + Cs + Ds^2 + 2Ds + D$$

Grouping the powers of s we obtain:

$$1 = (A+C)s^3 + (A+B+2C+D)s^2 + (A+C+2D)s + (A+B+D)s^0$$

Now we can express the equivalence of the coefficients of the right side with those of the left side with the following system of equations:

$$\begin{cases} A+C=0 \\ A+B+2C+D=0 \\ A+C+2D=0 \\ A+B+D=1 \end{cases}$$

which results in:

$$\begin{cases} A = \frac{1}{2} \\ B = \frac{1}{2} \\ C = -\frac{1}{2} \\ D = 0 \end{cases}$$

The result is then:

$$F^*(s) = \frac{1}{2(s+1)} + \frac{1}{2(s+1)^2} - \frac{s}{2(s^2+1)}$$

D.

1.

$$F^*(s) = \frac{5}{3(s-2)} - \frac{2}{3(s+1)}$$

The transform of a shifted unit function is $\frac{1}{s-a}$, in this case the denominators $(s-2)$ and $(s+1)$ tell us that the two original functions are shifted respectively of 2 and -1. Furthermore, the two are multiplied by the constants $\frac{5}{3}$ and $\frac{2}{3}$. Hence:

$$f(t) = \frac{5}{3}e^{2t} - \frac{2}{3}e^{-t}$$

2.

$$F^*(s) = \frac{1}{2(s+1)} + \frac{1}{2(s+1)^2} + \frac{s}{2(s^2+1)}$$

Again, we start with the first term which is the transform of a shifted unit function (multiplied by $\frac{1}{2}$). The second term can be recognized to be the shift of the function t (multiplied by $\frac{1}{2}$). Ultimately, the last term can be found in the tables as the Laplace transform of the function $f(t) = cos(t)$. Hence the result is:

$$\frac{1}{2}e^{-t} + te^{-t} - \frac{cos(t)}{2}$$

E.

Solve the following differential equation:

$$x(t) = \frac{dx(t)}{dt} + 2\frac{d^2x(t)}{dt^2}$$
$$x(0) = 1 \ x'(0) = \frac{1}{2}$$

This problem can be solved by expressing the equation in terms of the Laplace transform of the original function. Namely, the unknown function $x(t)$ has transform $X(S)$ and its derivatives have a transform known from the table to be respectively $L\{x'(t)\} = sX(s) - x(0)$ and $L\{x''(t)\} = s^2X(s) - sx(0) - x'(0)$. Substituting in the formula we obtain:

$$X(s) = sX(s) - x(0) + 2[s^2X(s) - sx(0) - x'(0)]$$

Before simplifying we can plug in the values of the initial conditions:

$$X(s) = sX(s) - 1 + 2[s^2X(s) - s - \frac{1}{2}]$$

which leads to:

$$X(s)[2s^2 + S - 1] = 2s$$
$$X(s) = \frac{2s}{(2s-1)(s+1)}$$

Which can be solved using partial fractions:

$$\frac{2s}{(2s-1)(s+1)} = \frac{A}{(2s-1)} + \frac{B}{(s+1)}$$

with $A = \frac{2}{3}$ and $B = \frac{2}{3}$

$$\frac{2s}{(2s-1)(s+1)} = \frac{2}{3(2s-1)} + \frac{2}{3(s+1)}$$
$$\frac{2s}{(2s-1)(s+1)} = \frac{1}{3(s-\frac{1}{2})} + \frac{2}{3(s+1)}$$

Which from the table is equal to:

$$x(t) = \frac{2}{3}e^{\frac{1}{2}t} + \frac{2}{3}e^{-t}$$