CS112 Homework 7
Continuous Time Markov Chains

Homeworks are due **03/6/13**

Q1. Consider two machines maintained by a single repairman. Machine i functions for an exponential amount of time with rate μ_i before breaking down, $i = 1, 2$. The repair times for either machine are exponential with parameter μ. Can we analyze this as a birth-death process? If not, how can we analyze it?

Solution: This is not a birth-death process since we need more information than the number of machines working. If we draw the state diagram we find that we need five states and there are transitions to states other than adjacent states. We can let the states be defined as follows:

- b : Both machines are working.
- 1 : 1 is working, 2 is down
- 2 : 2 is working, 1 is down
- 0_1 : Both are down and 1 is being serviced
- 0_2 : Both are down and 2 is being serviced

Then we find,

- $v_b = \mu_1 + \mu_2$
- $v_1 = \mu_1 + \mu$
- $v_2 = \mu_2 + \mu$
- $v_{0_1} = \mu$
- $v_{0_2} = \mu$

And,

- $P_{b,1} = \frac{\mu_1}{\mu_1+\mu_2} = 1 - P_{b,2}$
- $P_{1,b} = \frac{\mu}{\mu_1+\mu} = 1 - P_{1,0_2}$
- $P_{2,b} = \frac{\mu}{\mu_2+\mu} = 1 - P_{2,0_1}$
- $P_{0_1,1} = P_{0_2,2} = 1$

Q2. Consider a birth-death system in which $\lambda_k = \lambda$ and $\mu_k = k\mu$ for $k \geq 0$. For all k find the difference differential equations for $P_k(t) = P[\text{k in system at time t}]$.

Solution:
$\frac{dP_i}{dt} = OUT_i - IN_i$ at each node i of the graph. For i=0 there is only one out and one in term, and $i = 1, 2, 3, \ldots \infty$ cases look alike with two outs and two ins.

$$\frac{dP_k(t)}{dt} = -(\lambda + k\mu)P_k(t) + \lambda P_{k-1}(t) + (k+1)\mu P_{k+1}(t) \qquad k \geq 1$$

$$\frac{dP_0(t)}{dt} = -\lambda P_0(t) + \mu P_1(t) \quad k = 0$$

Q3. Consider a system in which the birth rate decreases and the death rate increases as he number in system increases as follows: $\lambda_k = (K - k)\lambda$ for$k \leq K$ and 0 otherwise. And $\mu_k = k\mu$ for $k \leq K$ and 0 otherwise. For all k find the difference differential equations for $P_k(t) = P$[k in system at time t].

Solution:
The difference-differential equations are

$$\begin{aligned}
\frac{dP_0(t)}{dt} &= \mu P_1(t) - K\lambda P_0(t) & k = 0 \\
\frac{dP_k(t)}{dt} &= \lambda(K - k + 1)P_{k-1}(t) + (k+1)\mu P_{k+1}(t) - [(K-k)\lambda + k\mu]P_k(t) & k = 1, 2, \ldots, K-1 \\
\frac{dP_k(t)}{dt} &= \lambda P_{K-1}(t) - K\mu] P_K(t) & k = K
\end{aligned}$$

Q4. A small barbershop has room for atmost two customers (including the one being served). Potential customers arrive at a Poisson rate of three per hour, and the successive service times are independent exponential random variables with mean 0.25 hours. What is:
a) The average number of customers in the shop?
b) The proportion of potential customers that get a hair cut?
c) If the barber could work twice as fast how much more business would he do?
d) If the barber hired another barber who works just as fast as him then how much more business would the barber do?

Solution:
a) Here, $\lambda = 3$ and $\mu = 4$. The average number of customers in the shop $E[k] = \sum_{k=0}^{\infty} kp_k$. We first have to solve for p_k. We find that $p_1 = \frac{\lambda}{\mu}p_0 \; and \; p_2 = \frac{\lambda}{\mu}p_1$. Solving we get, $p_0 = \frac{1-\frac{\lambda}{\mu}}{1-(\frac{\lambda}{\mu})^3} = \frac{16}{37}$.

$$\overline{N} = \sum_{k=0}^{\infty} kp_k = 1.\frac{12}{37} + 2.\frac{9}{37} = \frac{30}{37}$$

b) The proportion of customers that get a haircut is simply those that arrive when we are in state 0 or state 1 since when we are in state 2 the customers that arrive cannot enter the shop and have to leave. Hence the answer is $p_0 + p_1 = \frac{16}{37} + \frac{12}{37} = \frac{28}{37} = 0.756$.
c) Since the barber works twice as fast μ increases to 8. Using the same balance equations but with a different μ we solve to get, $p_0 = \frac{64}{97}, \; p_1 = \frac{24}{97}, \; p_2 = \frac{9}{97}$. Therefore the answer is $p_0 + p_1 = \frac{88}{97} = 0.907$.

d) Now since both barbers cannot work on the same customer, we use $\mu = 4$ but the transition from state 2 to state 1 is at the rate 2μ instead of μ. With the new balance

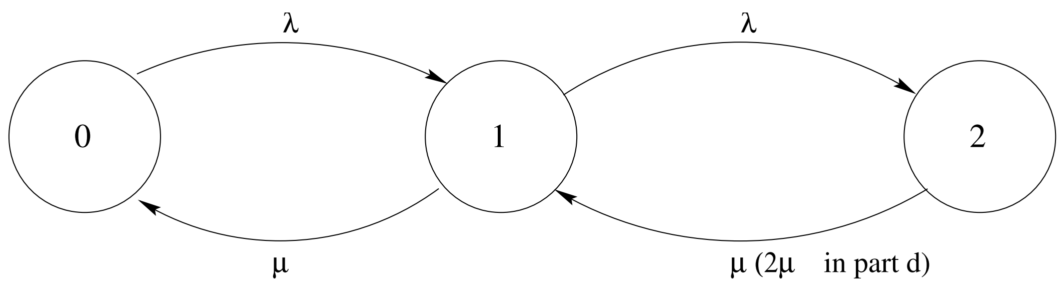

Figure 1: Transition diagram for question 4

equations we solve to get, $p_0 = \frac{32}{65},\ p_1 = \frac{24}{65},\ p_2 = \frac{9}{65}$. Therefore the answer is $p_0 + p_1 = \frac{56}{65}$ $= 0.861$.

Q5. A single repairperson looks after both machine 1 and 2. Each time it is repaired, machine i stays up for an exponential time with rate $\lambda_i,\ i = 1,\ 2$. When machine i fails, it requires an exponentially distributed amount of work with rate μ_i to complete its repair. The repairperson will always service machine 1 when it is down. For instance, if machine 1 fails while 2 is being repaired, then the repairperson will immediately stop work on machine 2 and start on 1. What proportion of time is machine 2 down?

Solution:

There are 4 states.
State 0: no machines are down
State 1: machine one is down and two is up
State 2: machine one is up and two is down
State 3: both machines are down.

The balance equations are as follows:

$$\begin{aligned}
(\lambda_1 + \lambda_2)P_0 &= \mu_1 P_1 + \mu_2 P_2 \\
(\mu_1 + \lambda_2)P_1 &= \lambda_1 P_0 \\
(\lambda_1 + \mu_2)P_2 &= \lambda_2 P_0 + \mu_1 P_3 \\
\mu_1 P_3 &= \lambda_2 P_1 + \lambda_1 P_2 \\
P_0 + P_1 + P_2 + P_3 &= 1
\end{aligned}$$

The proportion of time machine 2 is down is $\pi_2 + \pi_3$.

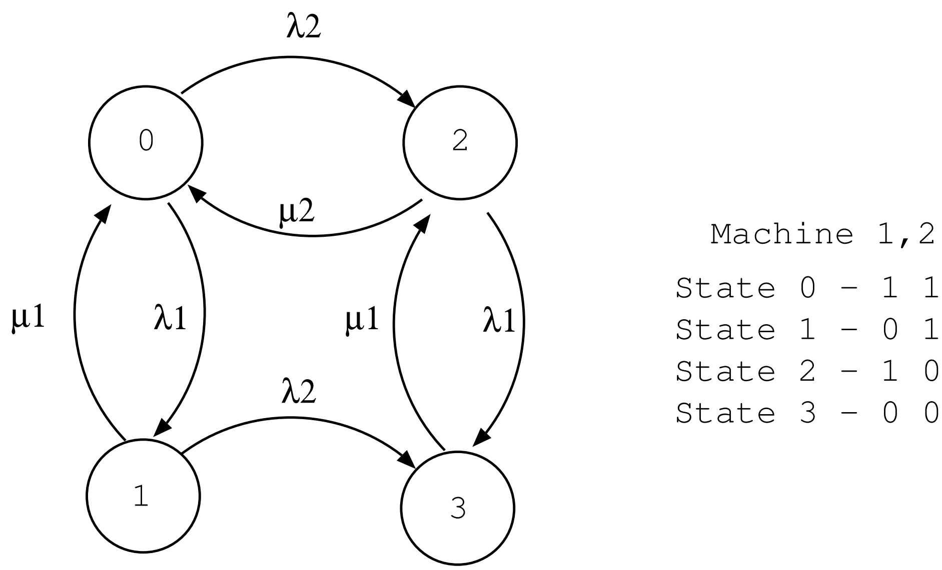

Figure 2: Transition diagram for question 5