UCLA - Henry Samueli School of Engineering
Department of Computer Science
CS112 - Homework Set 3
Due date Wednesday, January 30th

The homework must be submitted online in electronic form using the PDF format. Verify that the file is readable before submitting, broken files and late submissions will not be graded. Comment your solutions step by step. Please indicate your complete name and UID in the headings.

1

The professor asks two students to participate to a probability experiment. The first student chooses arbitrarily a number between 0 and 10 and a second student guesses in what interval this number is.

1.1

Suppose the second student guess the number is less or equal to 5. What is the probability that he guessed right?

1.2

Suppose, from 1.1, that the student was right. He now guesses that the number is greater or equal to 4. What is the probability that he is right again?

1.3

Suppose the third student comes to class and, without knowing the previous answers, he guesses a random number between 0 and 10. What is the probabilty that he guesses right ?

2

Let X denote the lifetime of a computer in years. The density function of X is given by

$$f(x) = \begin{cases} \frac{1}{15}e^{-x/15} & \text{for} \quad 0 \leq x < \infty \\ 0 & \mathit{otherwise} \end{cases}$$

2.1

What is the probability that a computer breaks in less than 5 years ?

2.2

What is the probability that 5 computers last more than 5 years ?

2.3

What is the probability that out of 5 computers at least one of them last more than 5 years ?

2.4

What is the probability that out of 10 computers at least 5 last more than 5 years.

3

You are waiting for the UCLA shuttle bus. The interval between two arriving buses is uniformly distributed between 10 to 20 minutes.

3.1

Suppose you are really unlucky. One bus arrives but it is too full for you to get in and you have to wait for the next one. What is the probability that the next bus arrives within 14 minutes ?

3.2

Suppose your friend arrives after you have been waiting for 14 minutes and no bus has yet come. What is the probability that a bus arrives in the next 3 minutes?

3.3

Suppose you arrive at the bus stop at a random time. What is the probability that you arrive somewhere in between two consecutive bus arrivals that are 16 minutes apart from each other; in other words a bus arrives before you and a second bus arrives 16 minutes after, while you arrived sometime in between.

4

Suppose you want to draw a random size circle centered at O(0,0) on a x-y plane.

4.1

You select a random point A between (1,0) and (0,0). Then you draw a circle using the segment OA as radius. What is the probability that your circle size is less than $\frac{1}{4}\pi$?

4.2

You first draw a circle with radius 1, centered in O=(0,0). You choose randomly a point B within this circle. Using $\overline{OB}$ as radius you draw a circle centered in O. What is the probability that your circle size is less than $\frac{1}{4}\pi$?

5

Suppose you host a popular web page. The interval between two consecutive user visits to your web page is a exponential distribution with average 10 users per minute.

5.1

What is the probability that no user visits your website in the last 10 second?

5.2

Suppose your website needs to perform a scheduled maintanance. Assuming the request for your website remain unchanged before, during, and after the maintainance. After rebooting the server, what is the probability that the 3rd users visit your website within the first 12 seconds ?

6

A computer system consists of n subsystems, each of which has an exponential distribution of lifetime with parameter λ_i , $i = 1, 2, , n$. Each subsystem is independent but the whole computer system fails if any of the subsystems do.

6.1

Find the distribution of the minimum of these lifetimes.

6.2

Compute the probability that a given component j, (j can be $1, 2, ..., n$) is the cause of the system failure.

7

A batch of 10 tasks is sent for computation to one of UCLA's supercomputers. You know from previous experiments that the completing time of each task follows an exponential distribution of parameter $\lambda = \frac{1}{6}$ (expressed in minutes). (NOTE: for the following you do not need to write the exact result just to find an algebraic expression for it)

7.1

What is the probability that the computation is completed in less than an hour?

8

Let X and Y be continuous random variables with ranges $0 < x < \infty$, $0 < y < \infty$ and joint density function:

$$f(x,y) = \begin{cases} xe^{-x(y+1)} & \text{for} \quad 0 < x < \infty, 0 < y < \infty \\ 0 & \text{for} \quad otherwise. \end{cases}$$

Are X and Y independent?

9

Let X and Y be continuous random variables with ranges $0 < x < \infty$, $0 < y < \infty$ and joint density function:

$$f(x,y) = \begin{cases} c(x^2 + y^2) & \text{for} \quad 0 < x < 1, 0 < y < 1 \\ 0 & \text{for} \quad otherwise. \end{cases}$$

9.1

Find c

9.2

Find the probability of $0.5 < X < 2and - 1 < Y < 0.5$

10

Let the random variable N represent the number of stars in any region of space of volume V. We model this as a being homogeneous across space, i.e., this distribution holds for a volume anywhere in space. Assume that N is a Poisson random variable with pmf

$$p_{N(n)} = \frac{e^{-\rho V}(\rho V)^n}{n!}, n = 0, 1, 2, ...,$$

where ρ is the density of stars in space. We choose an arbitrary point in space and define the random variable X to be the distance from the chosen point to the nearest star. Find the pdf of X in terms of ρ.

1

1.1

The selected number is a uniform distribution between 0 and 10.

$$f(x) = \left\{ \begin{array}{ccc} \frac{1}{10} & \text{for} & 0 \leq x \leq 10 \\ 0 & \text{for} & otherwise \end{array} \right.$$

$$\int_0^5 \frac{1}{10} dx = \frac{5-0}{10} = \frac{5}{10}$$

1.2

$$P(4 \leq x | x \leq 5) = \frac{\int_4^5 \frac{1}{10} dx}{0.5} = \frac{1}{5}$$

1.3

(suppose for instance he guessed 4)

$$P(x = 4) = \int_4^4 \frac{1}{10} dx = 0$$

2

2.1

$$\int_0^5 f(x) = 1 - e^{-5/15}$$

2.2

$$(\int_0^5 f(x))^5 = (1 - e^{-5/15})^5$$

2.3

At least one of them last for more than 5 years = 1 - No computer last for more than 5 years

$$1-(\int_5^\infty f(x))^5 = 1-(1-\int_0^5 f(x))^5 = 1-(1-(1-e^{-5/15}))^5$$
$$= 1-(e^{-5/15})^5 = 1-e^{-25/15}$$

2.4

Let Y be the number of computer last for more than 5 years. Applying binomial distribution (10,p), where $p = e^{-5/15}$ obtained from 2.1

$$P(5 \le Y \le 10) = \sum_{i=5}^{10} e^{-5i/15}(1-e^{-5/15})^{10-i}$$

3

3.1

$$f(x) = \begin{cases} \frac{1}{10} & \text{for} \quad 10 \le x \le 20 \\ 0 & \text{for} \quad o.w. \end{cases}$$

$$P(x \le 14) = \frac{4}{10}$$

3.2

$$F(x \le 17 | x \le 14) = \frac{\int_{14}^{17} f(x)dx}{1-0.4} = \frac{0.3}{0.6} = \frac{1}{2}$$

3.3

Intuition: While arriving at random. it is more likely to arrive between a longer interval than the shorter interval.

$$\frac{\int_{10}^{16} xdx}{\int_{10}^{20} xdx} = \frac{\frac{1}{2}(256-100)}{\frac{1}{2}(400-100)} = \frac{156}{300} = 0.52$$

4

4.1

$$\text{Circle size less than} \frac{1}{2}\pi = \text{radius less than } 0.5$$

$$P(radius < 0.5) = 0.5$$

4.2

$$\text{Circle size less than} \frac{1}{2}\pi = \text{point B is selected with the 0.5 circle}$$

$$P(\text{point B is selected with the 0.5 circle}) = \frac{(\frac{1}{2})^2\pi}{1^2\pi} = 0.25$$

5

5.1

$$\text{10 user per minute} = 1/6 \text{ user per second}$$

$$1 - \int_{10}^{\infty} \frac{1}{6} e^{-\frac{1}{6}x} dx = e^{-\frac{10}{6}}$$

5.2

Modeling using Poission distribution

$$P(\text{The 3rd users visit your website within the first 12 seconds}) =$$

$$= 1 - P(\text{No visit in the first 12 seconds})$$

$$-P(\text{One visit in the first 12 seconds}) - P(\text{Two visits in the first 12 seconds})$$

$$= 1 - \frac{e^{-2}3^0}{0!} - \frac{e^{-2}3^1}{1!} - \frac{e^{-2}3^2}{2!}$$

6

6.1

Let X_i be a random variable that denotes the lifetime of the $i-th$ component. We consider $Y = min(X_1, X_2, ..., X_n)$. For each real y, $Y < y$ iff and only if none of $X_1 > y, X_2 > y, ..., X_n > y$ holds. Hence:

$$P[Y < y] = 1 - P[X_1 > y, X_2 > y, ..., X_n > y] =$$
$$= 1 - P[X_1 > y]P[X_2 > y]...P[X_n > y]$$

Therefore:

$$F_Y(x) = 1 - (e^{-\lambda_1 y}e^{-\lambda_2 y}....e^{-\lambda_n y})$$
$$F_Y(x) = 1 - e^{-(\lambda_1+\lambda_2+...+\lambda_n)y}$$

6.2

If component j is the cause of the system to fail, then it must have shorter lifetime than any other components.

$$P[X > x] = P[X_1 > X_j, X_2 > X_j, ...X_{j-1} > X_j, X_{j+1} > X_j, ..., X_n > X_j] =$$
$$\int_0^\infty f_{X_j}(x) \prod_{i \neq j} P[X_i > x]dx$$

we know that the product is equal to:

$$e^{-(\lambda_1+\lambda_2+...\lambda_{i-1}+\lambda i+1+...+\lambda_n)x}$$

therefore

$$\int_0^\infty f_{X_j}(x)e^{-(\lambda_1+\lambda_2+...\lambda_{i-1}+\lambda i+1+...+\lambda_n)x}dx$$
$$\int_0^\infty \lambda_j e^{-\lambda_j x} \cdot e^{-(\lambda_1+\lambda_2+...\lambda_{i-1}+\lambda i+1+...+\lambda_n)x}dx$$
$$\int_0^\infty \lambda_j e^{-(\lambda_1+\lambda_2+...\lambda_n)x}dx$$
$$\lambda_j \int_0^\infty e^{-(\lambda_1+\lambda_2+...\lambda_n)x}dx$$
$$\frac{\lambda_j}{\sum_i \lambda_i}$$

7

A sum of n exponential random variables of parameter λ is equal to an Erlang distribution of parameter (n, λ).

$$P(X < 60) = \frac{\int_0^{60} \lambda^{10} x^9 e^{-\frac{1}{6}} dx}{(9)!}$$

8

$$f_X(x) = \int_0^{\infty} x e^{x(y+1)} dy = x \int_0^{\infty} e^{x(y+1)} = e^{-x}$$

$$f_Y(y) = \int_0^{\infty} x e^{x(y+1)} dx = \frac{1}{(y+1)^2}$$

$f_{XY} \neq f_X(x) f_Y(y)$ therefore the answer is no.

9

9.1

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{(X,Y)}(x, y) dx dy = 1$$

$$\int_0^1 \int_0^1 c(x^2 + y^2)(x, y) dx dy = 1$$

$$c \int_0^1 \int_0^1 (x^2 + y^2)(x, y) dx dy = 1$$

$$c \int_0^1 (\frac{x^3}{3} + y^2 x)|_0^1 dy = 1$$

$$c \int_0^1 (\frac{1}{3} + y^2) dy = 1$$

$$c(\frac{1}{3} y + \frac{y^3}{3})|_0^1 = 1$$

$$c \frac{2}{3} = 1$$

$$c = \frac{3}{2}$$

9.2

$$P(0.5 < X < 2 and - 1 < Y < 0.5) = \int_{0.5}^{2} \int_{-1}^{0.5} f_{XY}(x, y) dx dy$$

$$\int_{0.5}^{2} \int_{-1}^{0.5} 1.5(x^2 + y^2)(x, y) dx dy = \frac{1}{4}$$

10

The trick in this problem, as in many others, is to find a way to connect events regarding X with events regarding N. In our case, for $x \geq 0$:

$$F_X(x) = P\{X \leq x\} = 1 - P\{X > x\}$$

$$= 1 - P\{No\, stars\, within\, distance\, x\}$$

$$= 1 - P\{N = 0\, in\, sphere\, of\, radius\, x\, centered\, at\, origin\}$$

$$= 1 - e^{-\rho \frac{4}{3} \pi x^3}$$

We obtain the pdf $f_X(x)$ by differentiating the cdf $F_X(x)$:

$$f_X(x) = 4\pi\rho x^2 e^{-\rho \frac{4}{3} \pi x^3}$$