Ordinary differential equations: review

What is an ODE?

Remember

$$\left.\frac{dx}{dt}\right|_{t_0} = \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t}$$

Draw graph rep

In essence:

rate of change = process + process + process

ODEs are a central part of modeling in the Earth sciences, because change is everywhere in the Earth system and we are often interested in determining why something changed and predicting how it will change in the future.

quant of interest

In terms of equations: $\dfrac{dx}{dt} = f(x,t)$

time

describes a generic ODE

process causing change

→ You took a whole class on how to solve these equations exactly, but what if you can't?

→ Generally, for most interesting Earth science problems $f(x,t)$ is complicated and may involve many nonlinear terms

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So, how to numerically solve

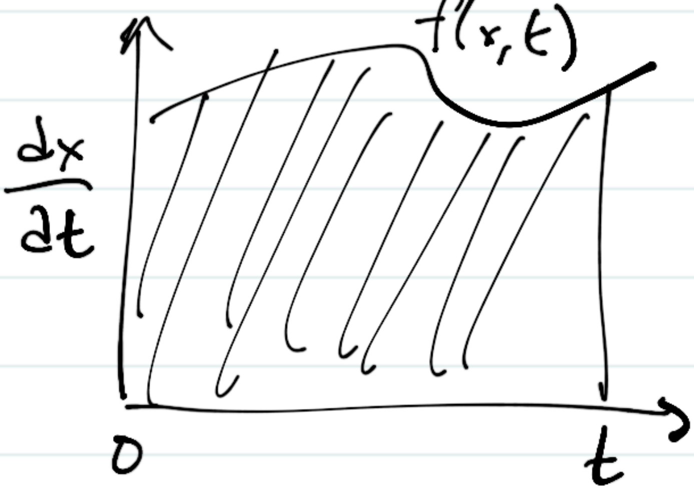
$$\frac{dx}{dt} = f(x,t) \quad ?$$

$$x(t=0) = x_0$$

Recall what you learned about numerical integration, how could you solve numerically for $x(t)$?

$$\int_0^t \frac{dx}{dt}\,dt = \underline{\underline{\int_0^t f(x,t)\,dt}}$$

We can solve this!



Apply the midpoint method to this problem:

$$\left.\frac{dx}{dt}\right|_{t=t_i} = \sum_{i=0}^{n-1} f(x_i, t_i)\,\Delta t$$

problem

→ However, if we don't know x_i yet, we need to start at the only place where we do know x, which is $t=0$ $\quad \underline{x(t=0)=x_0}$

initial condition

→ Again, we must discretize the solution in time:

$$\left.\frac{dx}{dt}\right|_{t=t_i} \approx \frac{x(t_i+\Delta t)-x(t_i)}{\Delta t} = f(x(t_i), t_i)$$

$$\boxed{\begin{gathered}\left.\frac{dx}{dt}\right|_{t=t_i} \approx \frac{x(t_i+\Delta t)-x(t_i)}{\Delta t} = f(x(t_i),t_i) \\ x(t=0)=x_0\end{gathered}}$$

Forward-Euler Method

If we rewrite this method

$$\boxed{x(t_i+\Delta t) = x(t_i) + f(x(t_i), t_i)\,\Delta t}$$

This gives us a way to solve for $x(t)$ by taking time steps from the known initial condition through the duration of time we want a solution for.

Note: the RHS of this equation only uses info from the past time step t_i

We can turn this into an algorithm for solving a general ODE

$x(1) = x_{init}$ ← initial condition

n = number of time steps to use

$$\Delta t = \frac{t_f - t_i}{n}$$

for $i = 1, \ldots, n$

$\quad t = t_i + i\Delta t$

can be coded here or defined elsewhere for a more general algorithm

$\quad x(i+1) = x(i) + f(x(i), t)\Delta t$

end

The smaller Δt → time step the more accurate this solution will be.

Since as $\Delta t \to 0$, this goes to the exact definition of a derivative

We can also make up a slightly different method:

Again discretize $\frac{dx}{dt} \approx \frac{x(t_i+\Delta t)-x(t_i)}{\Delta t}$

But now use information from the current time step for $f(x,t)$

$$\frac{x(t_i+\Delta t)-x(t_i)}{\Delta t} = f(x(t_i+\Delta t), t+\Delta t)$$

Rewrite this

$$\boxed{\underbrace{x(t_i+\Delta t)-f(x(t_i+\Delta t),t+\Delta t)\Delta t}_{\text{We have to solve this part for } x(t_i+\Delta t)} = \overset{\text{known}}{x(t_i)}}$$

Backward Euler method

We have to solve this part for $x(t_i+\Delta t)$.
Can be hard depending on how complex $f(x,t)$ is

Example: Nuclear Decay

$N \rightarrow$ concentration of a radioactive element

Rate of change in concentration related to current concentration:

$$\frac{dN}{dt} = -\lambda N \qquad N(t=0) = N_0$$

$\rightarrow \lambda$ is the decay constant which is related to half life by $t_{HL} = \frac{\ln(2)}{\lambda}$

$\rightarrow$ This can be solved analytically

$$\int_0^N \frac{dN}{N} = -\lambda \int_0^t dt$$

$$\ln(N) = -\lambda t \rightarrow \boxed{N(t) = N_0 e^{-\lambda t}}$$

We'll use this exact solution as a benchmark to compare against our two methods.

Forward-Euler Method

for $i = 1, \dots, n$

$$N(i+1) = N(i) + \left(-\lambda N(i)\right)\Delta t.$$

end

Backward-Euler Method

$$\frac{N(t+\Delta t) - N(t)}{\Delta t} = -\lambda N(t+\Delta t)$$

$$N(t+\Delta t) + \lambda N(t+\Delta t)\Delta t = N(t)$$

$$(1+\lambda\Delta t)\,N(t+\Delta t) = N(t)$$

$$N(t+\Delta t) = \frac{N(t)}{1+\lambda \Delta t}$$ ← $f(x,t)$ is simple enough that this is easy to solve

for $i=1, \ldots, n$

$N(i+1) = N(i) / (1+\lambda \Delta t)$

end

→ Compare each method to exact solution for different time step size

Slightly more complicated

Radiocarbon content of one-box biosphere
(S+K p. 40)

$$\frac{dM}{dt} = \underbrace{-kM}_{\text{decay}} + P$$

P ← production rate

Possibilities:

$P = \text{constant}$

$P = P_0 + b\sin(\omega t)$ periodic due to sunspots (e.g.)