Review of PDEs

What makes ODEs ordinary is that they only involve derivatives in one variable!

$$\frac{dx}{dt} = f(x,t)$$

derivative in t

Partial differential equations (PDEs) involve derivatives in more than one variable:

Example: For a variable $T = T(x,t)$

$$\frac{\partial T}{\partial t} = D \frac{\partial^2 T}{\partial x^2}$$

The diffusion equation (will return to this)

constant

time derivative (rate of change of T in time)

spatial derivative (i.e. slope of T in space)

Remember: when you are considering derivates in multiple variables, use partial deriv. ∂

"curly"

→ We might consider derivatives in more than one variable because change can be caused by:

→ Time-dependent processes at one location ($\frac{\partial T}{\partial t}$)

→ Spatially-dependent processes which involve change over space ($\frac{\partial^2 T}{\partial x^2}$)

→ We want to understand and model the relationship between these derivates, typically in time and space, and we can do so by combining what we learned about solving coupled systems of ODEs with some new tricks

Classification of PDES

→ Most PDEs of interest in Earth Science involve 2nd order derivatives.

→ They can generally be classified into three categories, with diff num methods to solve each type

For a variable $u = u(x,t)$

$u_x = \partial u / \partial x \quad u_{xx} = \frac{\partial^2 u}{\partial x^2} \quad u_t = \frac{\partial u}{\partial t}$

Consider the most general PDE:

$$\boxed{G = Au_{xx} + Bu_{xt} + Cu_{tt} + Du_x + Eu_t + Fu}$$

If $G=0$, PDE is homogenous

Types of PDEs

① Parabolic : $B^2 - 4AC = 0$

Example) $Au_{xx} = u_t$ → Diffusion Equation

$B = C = 0 \rightarrow B^2 - 4AC = 0$

② Hyperbolic : $B^2 - 4AC > 0$

Example) $u_{xx} - Cu_{tt} = 0$ → Wave Equation

$A=1 \quad B=0 \rightarrow B^2 - 4AC > 0$

③ Elliptic Equations: $B^2-4AC<0$

Example: Poisson's/Laplace's Equation

$$u_{xx}+u_{yy}=0 \qquad A=C=1$$
$$B=0$$
$$B^2-4AC<0$$

→ Some combination of these PDEs is used in some way or another to describe many processes in Earth science.
space and time-dependent

→ We will extend what we have learned about ODE numerical methods to solve PDEs using numerical methods.

Advection Equation

- If we have a flow of the background fluid, mobile things can be transported

J: amount of some stuff (heat, chemicals, biology, etc.)

chain rule!

$$\frac{dJ}{dt} = \frac{\partial J}{\partial t} + \frac{\partial J}{\partial x}\boxed{\frac{\partial x}{\partial t}}$$

→ velocity of frame of reference (i.e. fluid flow)

$$\boxed{\frac{\partial J}{\partial t} + u\frac{\partial J}{\partial x} = S(x,t)}$$

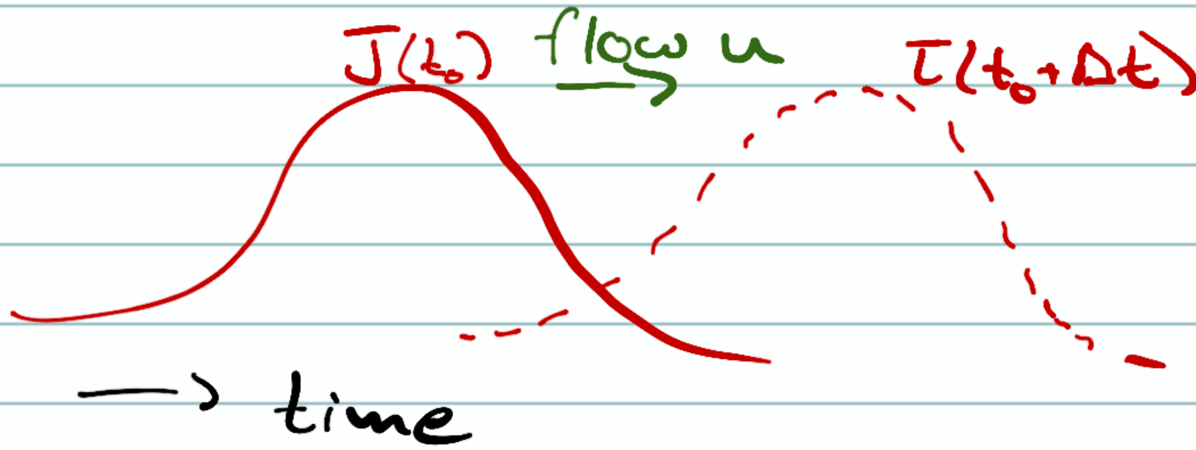
Advection Equation!

fluid velocity

source of stuff

This form assumes u is constant, but it can also be written in more general flux form

$$\boxed{\frac{\partial J}{\partial t} + \frac{\partial}{\partial x}(uJ) = S(x,t)}$$



Advection equation describes how things are transported by <u>advection</u> of background flow

Discretizing the advection equation

Consider $J_i^k = J(x_i, t_k)$

k ← time index

i ← space index

$i \in [1, \ldots, n]$

$k \in [1, \ldots, m]$

Diffusion Equation

→ Typically represents the action of small-scale processes to "mix" a quantity down-gradient (i.e. moving from greater to lower)

$$\boxed{\frac{\partial J}{\partial t} = D \frac{\partial^2 J}{\partial x^2}}$$

diffusivity

$D > 0$

important

This is the time-dependent version of the 1D diffusion in the Earth we considered before

- Often use to describe heat conduction
- Can also model spreading / diffusion of material in a porous matrix (i.e. contaminant in an aquifer)

How to discretize?

In time: $\frac{\partial J}{\partial t} = \frac{J_i^{k+1} - J_i^{k}}{\Delta t}$

Advection-Diffusion Equation

→ Combines two processes:

$$\boxed{\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} = D\frac{\partial^2 T}{\partial x^2}}$$

Often used to capture things carried in a turbulent geophysical flow (e.g. ocean, atmosphere) because small-scale turbulence is hard to model, but has the effect of mixing things up on small scales and transporting them on large scales

Consider two discretizations:

FE in time, CFE in advection and diffussion

$$\frac{T_i^{k+1} - T_i^k}{\Delta t} + \frac{u}{2\Delta x}\left(T_{i+1}^k - T_{i-1}^k\right) = \frac{D}{\Delta x^2}\left(T_{i+1}^k - 2T_i^k + T_{i-1}^k\right)$$

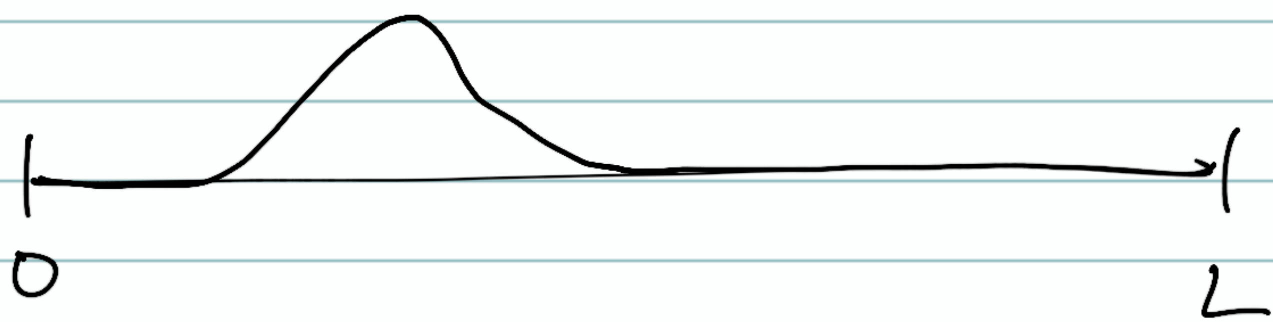
The Wave Equation

→ Reminder, the wave equation is an example of a hyperbolic PDE:

$$\frac{\partial^2 u}{\partial t^2} - \underbrace{\alpha^2}_{\text{wave speed}} \frac{\partial^2 u}{\partial x^2} = 0 \qquad u = u(x,t)$$

With some boundary conditions:

$$u(0,t) = a \qquad u(L,t) = b$$



And initial condition on both u and $\frac{\partial u}{\partial t}$

$$u(x,0) = f(x) \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x)$$

→ So, we know that when α is constant in time and space, the wave equation has a solution of a traveling wave

i.e. $u(x,t) = f(x-at) + g(x+at)$

→ However this is not so simple if α is not constant for all x, t.

→ In Earth sciences, we frequently consider waves that travel through heterogeneous media:

Atmospheric gravity waves
Seismic waves
Ocean waves
Porosity waves etc etc

→ Such problems cannot be solved analytically for arbitrary $\alpha(x,t)$

A numerical method: discretize u_{xx}, u_{tt} terms.

$x_i = a + i\Delta x$

$t_k = k\Delta t$

Then use centered difference formulas for both terms.