Vector

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$$

In physics, vector has a direction and magnitude, like velocity of a moving object.

In data analysis, a series of discrete data can be expressed as a vector (let's say there are *m* data points, then you have a *m*-dimensional vector)

In computational model, model state can be expressed as a vector (e.g. chemical concentrations in box models, dynamic height in geophys fluid dynamics, with m being the number of grid points)

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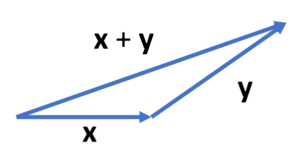
Vector

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

When there are more then two vectors, they can add together to form another vector



2

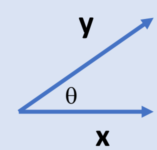
Vector

Vectors can multiply together: **inner product produces a scalar (remember "row times column")**

$$\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^T \mathbf{y} = \begin{bmatrix} x_1 x_2 \cdots x_m \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

$$= x_1 y_1 + x_2 y_2 + \cdots x_m y_m$$

In a 2D example,



$$\mathbf{x} \cdot \mathbf{y} = |\mathbf{x}||\mathbf{y}|cos\theta$$

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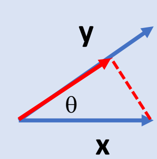
Vector

Vectors can multiply together: **inner product produces a scalar (remember "row times column")**

The projection of **x** onto **y** is:

$$|\mathbf{x}|cos\theta = \frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{y}|}$$

In a 2D example,



$$\mathbf{x} \cdot \mathbf{y} = |\mathbf{x}||\mathbf{y}|cos\theta$$

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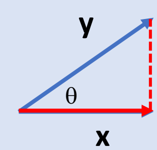
Vector

Vectors can multiply together: **inner product produces a scalar (remember "row times column")**

The projection of **y** onto **x** is:

$$|\mathbf{y}|cos\theta = \frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{x}|}$$

In a 2D example,



$$\mathbf{x} \cdot \mathbf{y} = |\mathbf{x}||\mathbf{y}|cos\theta$$

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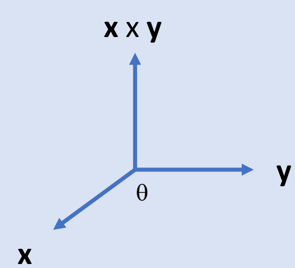
Vector

Vectors can multiply in a different way: **cross product produces a vector (remember "the right hand rule")**

$$\mathbf{x} \times \mathbf{y} = |\mathbf{x}||\mathbf{y}|sin\theta\hat{\mathbf{z}}$$



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Vector

Vectors can multiply together in another different ways: **outer product produces a matrix (again, remember "row times column")**

$$\mathbf{x}\mathbf{y}^T = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \begin{bmatrix} y_1 \; y_2 \; \cdots \; y_m \end{bmatrix} = \begin{bmatrix} x_1y_1 & x_1y_2 & \cdots & x_1y_m \\ x_2y_1 & x_2y_2 & \cdots & x_2y_m \\ & & \vdots & \\ x_my_1 & x_my_2 & \cdots & x_my_m \end{bmatrix}$$

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Inner product and (co)variance

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$$

In data analysis, vector can contain a collection of measurements (e.g. mean August temperature of Atlanta from last 140 years)

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Inner product and (co)variance

$$\mathbf{x}' = \begin{bmatrix} x_1 - \overline{x} \\ x_2 - \overline{x} \\ \vdots \\ x_m - \overline{x} \end{bmatrix} = \begin{bmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_m \end{bmatrix}$$

Let’s subtract mean of **x** from all elements, so **x** becomes **x’**

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Inner product and (co)variance

$$\mathbf{x}' \cdot \mathbf{x}' = |\mathbf{x}'|^2 = \Sigma x_n'^2 = (N-1)\sigma^2 = (N-1)\sigma_{xx}$$

Inner product of **x’** with itself the square of the **”length” of the x’ vector.**

It is equal to the **variance (σ^2 or σ_{xx})** of the data contained in vector **x** multiplied by N-1

The “length” |**x’**| is also called the **L2 norm**

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Inner product and (co)variance

$$\mathbf{x}' \cdot \mathbf{y}' = \Sigma x'_n y'_n = (N-1)\sigma_{xy}$$

Inner product of **x’** and **y’** is:

(1) the **covariance** of the data contained in vector **x** and **y** (multiplied by N-1)

(2) the **projection** of **x** onto **y** (multiplied by |**y’**|)

(3) Proportional to the **correlation** between **x** and **y**

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Inner product and (co)variance

$$\mathbf{x}' \cdot \mathbf{y}' = \Sigma x'_n y'_n = (N-1)\sigma_{xy}$$

Inner product of **x’** and **y’** is:

(1) the **covariance** of the data contained in vector **x** and **y** (multiplied by N-1)

(2) the **projection** of **x** onto **y** (multiplied by |**y’**|)

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Matrix-vector product and $\mathbf{Ax} = \mathbf{b}$

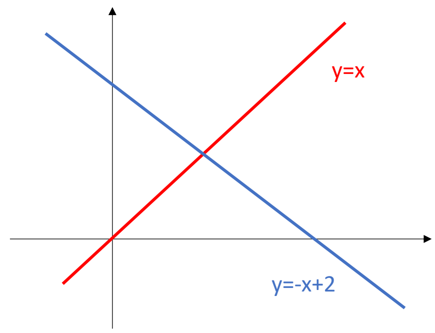
A (2 x 2) example

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

A **x** = **b**

The "row" point of view: $\boxed{x - y = 0}$

Intersection of two lines $\boxed{x + y = 2}$



The problem **Ax** = **b** is solvable only if **the two lines intersect at a point**

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Matrix-vector product and $\mathbf{Ax} = \mathbf{b}$

Inverse matrix **A**$^{-1}$ $\quad \mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$

If inverse matrix of **A** exists, we have

In MATLAB it is **inv** function. Also the backslash operation "A\b" does it for you…

In Python, **np.linalg.inv** (or **scipy.linalg.inv**)

```
% create a random matrix A and a vector b
A=rand(2,2);
b=rand(2,1);
% solve A^-1 b
x=inv(A)*b
```

```
x = 2×1
   -1.1415
    1.9246
```

```
x=A\b
```

```
x = 2×1
   -1.1415
    1.9246
```

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Matrix-vector product and $\mathbf{Ax} = \mathbf{b}$

What if **A** is not a square matrix?

Let's say there are too many (m) equations than (n) unknowns.

In this case, there is no unique solution **x** that can satisfy all equations in **Ax** = **b** in the exact sense

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Matrix-vector product and $\mathbf{Ax} = \mathbf{b}$

Consider **A** is (m x 2) matrix, **x** is (2 x 1) and **b** is (m x 1) and m is much larger than 2.

In this case, **A x** is unlikely equal to **b** in the exact sense.

However, we could look for the approximate solution **x** that minimizes the magnitude of the misfit |**Ax** - **b**| (aka least square method).

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Pseudoinverse of non-square matrix A

$$\mathbf{Ax} = \mathbf{b}$$

← Solution to this equation does not exist because of non-square A

$$\mathbf{A^T A x} = \mathbf{A^T b}$$

$$\mathbf{x} = \boxed{(\mathbf{A^T A})^{-1}\mathbf{A^T}}\mathbf{b}$$

This is the pseudoinverse of A

However, pseudoinverse provides a solution that minimizes the discrepancies between **Ax** and **b**

The solution, x, is the best-fit solution to Ax = b in the "Least Square" sense!

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