Augmented stability conditions

Antonios-Alexandros Robotis

Based on joint works with Daniel Halpern-Leistner and Jeffrey Jiang

Quick review

1. $\mathcal{D} = \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ for X a complex projective manifold
2. $\mathrm{Stab}(\mathcal{D}) = \mathrm{Stab}(X)$ – space of *stability conditions* $(Z, \mathcal{P})$ on $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$
   - *central charge*: $Z \in \mathrm{Hom}(K_0(X), \mathbf{C})$ which factors through $\mathrm{ch} : K_0(X) \to H^*_{\mathrm{alg}}(X)$.
   - $\mathcal{P} = \{\mathcal{P}(\phi)\}_{\phi \in \mathbf{R}}$ is a *slicing*, a categorical structure which refines the notion of bounded t-structure
   - $\mathcal{P}(\phi)$ category of *semistable objects* of phase $\phi \in \mathbf{R}$, and

$$Z(E) \in \mathbf{R}_{>0} \cdot \exp(i\pi\phi)$$

   - (*Bridgeland*) $\mathrm{Stab}(X) \to \mathrm{Hom}(H^*_{\mathrm{alg}}(X), \mathbf{C})$ given by $(Z, \mathcal{P}) \mapsto Z$ is a local homeo. $\mathrm{Stab}(X)$ is a $\mathbf{C}$-manifold modeled on $H^*_{\mathrm{alg}}(X; \mathbf{C})$.
   - Natural $\mathbf{C}$-action on $\mathrm{Stab}(X)$: $w \cdot (Z, \mathcal{P}) = (e^w \cdot Z, \mathcal{P}^w)$.

Quick review

1. $\mathcal{D} = \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ for X a complex projective manifold
2. $\mathrm{Stab}(\mathcal{D}) = \mathrm{Stab}(X)$ – space of *stability conditions* $(Z, \mathcal{P})$ on $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$
   - *central charge*: $Z \in \mathrm{Hom}(K_0(X), \mathbf{C})$ which factors through $\mathrm{ch} : K_0(X) \to H^*_{\mathrm{alg}}(X)$.
   - $\mathcal{P} = \{\mathcal{P}(\phi)\}_{\phi \in \mathbf{R}}$ is a *slicing*, a categorical structure which refines the notion of bounded t-structure
   - $\mathcal{P}(\phi)$ category of *semistable objects* of phase $\phi \in \mathbf{R}$, and

$$Z(E) \in \mathbf{R}_{>0} \cdot \exp(i\pi\phi)$$

   - (*Bridgeland*) $\mathrm{Stab}(X) \to \mathrm{Hom}(H^*_{\mathrm{alg}}(X), \mathbf{C})$ given by $(Z, \mathcal{P}) \mapsto Z$ is a local homeo. $\mathrm{Stab}(X)$ is a $\mathbf{C}$-manifold modeled on $H^*_{\mathrm{alg}}(X; \mathbf{C})$.
   - Natural $\mathbf{C}$-action on $\mathrm{Stab}(X)$: $w \cdot (Z, \mathcal{P}) = (e^w \cdot Z, \mathcal{P}^w)$.

Quick review

1. $\mathcal{D} = \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ for X a complex projective manifold
2. $\mathsf{Stab}(\mathcal{D}) = \mathsf{Stab}(X)$ – space of *stability conditions* $(Z, \mathcal{P})$ on $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$
   - *central charge*: $Z \in \mathrm{Hom}(K_0(X), \mathbf{C})$ which factors through $\mathrm{ch} : K_0(X) \to H^*_{\mathrm{alg}}(X)$.
   - $\mathcal{P} = \{\mathcal{P}(\phi)\}_{\phi \in \mathbf{R}}$ is a *slicing*, a categorical structure which refines the notion of bounded t-structure
   - $\mathcal{P}(\phi)$ category of *semistable objects* of phase $\phi \in \mathbf{R}$, and

$$Z(E) \in \mathbf{R}_{>0} \cdot \exp(i\pi\phi)$$

   - (*Bridgeland*) $\mathsf{Stab}(X) \to \mathrm{Hom}(H^*_{\mathrm{alg}}(X), \mathbf{C})$ given by $(Z, \mathcal{P}) \mapsto Z$ is a local homeo. $\mathsf{Stab}(X)$ is a $\mathbf{C}$-manifold modeled on $H^*_{\mathrm{alg}}(X; \mathbf{C})$.
   - Natural $\mathbf{C}$-action on $\mathsf{Stab}(X)$: $w \cdot (Z, \mathcal{P}) = (e^w \cdot Z, \mathcal{P}^w)$.

Quick review

1. $\mathcal{D} = \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ for X a complex projective manifold
2. $\mathrm{Stab}(\mathcal{D}) = \mathrm{Stab}(X)$ – space of *stability conditions* $(Z, \mathcal{P})$ on $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$
   - *central charge*: $Z \in \mathrm{Hom}(K_0(X), \mathbf{C})$ which factors through $\mathrm{ch} : K_0(X) \to H^*_{\mathrm{alg}}(X)$.
   - $\mathcal{P} = \{\mathcal{P}(\phi)\}_{\phi \in \mathbf{R}}$ is a *slicing*, a categorical structure which refines the notion of bounded t-structure
   - $\mathcal{P}(\phi)$ category of *semistable objects* of phase $\phi \in \mathbf{R}$, and

$$Z(E) \in \mathbf{R}_{>0} \cdot \exp(i\pi\phi)$$

   - (*Bridgeland*) $\mathrm{Stab}(X) \to \mathrm{Hom}(H^*_{\mathrm{alg}}(X), \mathbf{C})$ given by $(Z, \mathcal{P}) \mapsto Z$ is a local homeo. $\mathrm{Stab}(X)$ is a $\mathbf{C}$-manifold modeled on $H^*_{\mathrm{alg}}(X; \mathbf{C})$.
   - Natural $\mathbf{C}$-action on $\mathrm{Stab}(X)$: $w \cdot (Z, \mathcal{P}) = (e^w \cdot Z, \mathcal{P}^w)$.

Quick review

1. $\mathcal{D} = \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ for X a complex projective manifold
2. $\mathrm{Stab}(\mathcal{D}) = \mathrm{Stab}(X)$ – space of *stability conditions* $(Z, \mathcal{P})$ on $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$
   - *central charge*: $Z \in \mathrm{Hom}(K_0(X), \mathbf{C})$ which factors through $\mathrm{ch} : K_0(X) \to H^*_{\mathrm{alg}}(X)$.
   - $\mathcal{P} = \{\mathcal{P}(\phi)\}_{\phi \in \mathbf{R}}$ is a *slicing*, a categorical structure which refines the notion of bounded t-structure
   - $\mathcal{P}(\phi)$ category of *semistable objects* of phase $\phi \in \mathbf{R}$, and

$$Z(E) \in \mathbf{R}_{>0} \cdot \exp(i\pi\phi)$$

   - (*Bridgeland*) $\mathrm{Stab}(X) \to \mathrm{Hom}(H^*_{\mathrm{alg}}(X), \mathbf{C})$ given by $(Z, \mathcal{P}) \mapsto Z$ is a local homeo. $\mathrm{Stab}(X)$ is a $\mathbf{C}$-manifold modeled on $H^*_{\mathrm{alg}}(X; \mathbf{C})$.
   - Natural $\mathbf{C}$-action on $\mathrm{Stab}(X)$: $w \cdot (Z, \mathcal{P}) = (e^w \cdot Z, \mathcal{P}^w)$.

Quick review

1. $\mathcal{D} = \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ for X a complex projective manifold
2. $\mathrm{Stab}(\mathcal{D}) = \mathrm{Stab}(X)$ – space of *stability conditions* $(Z, \mathcal{P})$ on $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$
   - *central charge*: $Z \in \mathrm{Hom}(K_0(X), \mathbf{C})$ which factors through $\mathrm{ch} : K_0(X) \to H^*_{\mathrm{alg}}(X)$.
   - $\mathcal{P} = \{\mathcal{P}(\phi)\}_{\phi \in \mathbf{R}}$ is a *slicing*, a categorical structure which refines the notion of bounded t-structure
   - $\mathcal{P}(\phi)$ category of *semistable objects* of phase $\phi \in \mathbf{R}$, and
   $$Z(E) \in \mathbf{R}_{>0} \cdot \exp(i\pi\phi)$$
   - (*Bridgeland*) $\mathrm{Stab}(X) \to \mathrm{Hom}(H^*_{\mathrm{alg}}(X), \mathbf{C})$ given by $(Z, \mathcal{P}) \mapsto Z$ is a local homeo. $\mathrm{Stab}(X)$ is a $\mathbf{C}$-manifold modeled on $H^*_{\mathrm{alg}}(X; \mathbf{C})^{\vee}$.
   - Natural $\mathbf{C}$-action on $\mathrm{Stab}(X)$: $w \cdot (Z, \mathcal{P}) = (e^w \cdot Z, \mathcal{P}^w)$.

Quick review

1. $\mathcal{D} = \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ for X a complex projective manifold
2. $\mathrm{Stab}(\mathcal{D}) = \mathrm{Stab}(X)$ – space of *stability conditions* $(Z, \mathcal{P})$ on $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$
   - *central charge*: $Z \in \mathrm{Hom}(K_0(X), \mathbf{C})$ which factors through $\mathrm{ch} : K_0(X) \to H^*_{\mathrm{alg}}(X)$.
   - $\mathcal{P} = \{\mathcal{P}(\phi)\}_{\phi \in \mathbf{R}}$ is a *slicing*, a categorical structure which refines the notion of bounded t-structure
   - $\mathcal{P}(\phi)$ category of *semistable objects* of phase $\phi \in \mathbf{R}$, and

$$Z(E) \in \mathbf{R}_{>0} \cdot \exp(i\pi\phi)$$

   - (*Bridgeland*) $\mathrm{Stab}(X) \to \mathrm{Hom}(H^*_{\mathrm{alg}}(X), \mathbf{C})$ given by $(Z, \mathcal{P}) \mapsto Z$ is a local homeo. $\mathrm{Stab}(X)$ is a $\mathbf{C}$-manifold modeled on $H^*_{\mathrm{alg}}(X; \mathbf{C})^\vee$.
   - Natural $\mathbf{C}$-action on $\mathrm{Stab}(X)$: $w \cdot (Z, \mathcal{P}) = (e^w \cdot Z, \mathcal{P}^w)$.

Motivation from NMMP

In arXiv:2301.13168, Halpern-Leistner proposes *noncommutative minimal model program (NMMP)*

Heuristic (Optimistic)

Given $\sigma_0 = (Z_0, \mathcal{P}_0) \in \mathrm{Stab}(X)$, solving "canonical ODEs" in $H^*_{\mathrm{alg}}(X; \mathbf{C})$ with initial point Z_0 (+ initial conditions) gives paths $Z_t : [0, \infty) \to H^*_{\mathrm{alg}}(X; \mathbf{C})$ which lifts to $\sigma_t : [0, \infty) \to \mathrm{Stab}(X)$.

As $t \to \infty$, σ_t should give rise to semiorthogonal decompositions of $\mathcal{D}$.

Motivation from NMMP

In arXiv:2301.13168, Halpern-Leistner proposes *noncommutative minimal model program (NMMP)*

Heuristic (Optimistic)

Given $\sigma_0 = (Z_0, \mathcal{P}_0) \in \mathrm{Stab}(X)$, solving "canonical ODEs" in $H^*_{\mathrm{alg}}(X;\mathbf{C})^\vee$ with initial point Z_0 (+ initial conditions) gives paths $Z_t : [0,\infty) \to H^*_{\mathrm{alg}}(X;\mathbf{C})^\vee$ which lifts to $\sigma_t : [0,\infty) \to \mathrm{Stab}(X)$.

As $t \to \infty$, σ_t should give rise to semiorthogonal decompositions of $\mathcal{D}$.

Motivation from NMMP

In arXiv:2301.13168, Halpern-Leistner proposes *noncommutative minimal model program (NMMP)*

Heuristic (Optimistic)

Given $\sigma_0 = (Z_0, \mathcal{P}_0) \in \mathrm{Stab}(X)$, solving "canonical ODEs" in $H^*_{\mathrm{alg}}(X; \mathbf{C})$ with initial point Z_0 (+ initial conditions) gives paths $Z_t : [0, \infty) \to H^*_{\mathrm{alg}}(X; \mathbf{C})$ which lifts to $\sigma_t : [0, \infty) \to \mathrm{Stab}(X)$.

As $t \to \infty$, σ_t should give rise to semiorthogonal decompositions of $\mathcal{D}$.

Quasi-convergent paths

In arXiv:2401.00600, (with D. Halpern-Leistner and J. Jiang) we introduce *quasi-convergent paths* $\sigma_t : [0, \infty) \to \mathrm{Stab}(\mathcal{D})$.

Theorem (HL, J, R '23)

A generic quasi-convergent path σ_t gives a semiorthogonal decomposition $\mathcal{D} = \langle \mathcal{D}_1, \ldots, \mathcal{D}_n \rangle$ plus $\sigma_i \in \mathrm{Stab}(\mathcal{D}_i)/\mathbf{C}$ for $i = 1, \ldots, n$.

1. study growth of $\phi_t(E)$ – if for all $t \gg 0$, $\phi_t(E) < \phi_t(F)$, then $\mathrm{Hom}(F, E) = 0$.
2. $\mathcal{D}_1$ is generated by objects with ϕ_t growing "slowest" and $\mathcal{D}_n$ is generated by objects with ϕ_t growing "fastest."
3. resulting SOD + stability conditions depends only on $\sigma_t : [0, \infty) \to \mathrm{Stab}(\mathcal{D})/\mathbf{C}$.

Quasi-convergent paths

In arXiv:2401.00600, (with D. Halpern-Leistner and J. Jiang) we introduce *quasi-convergent paths* $\sigma_t : [0, \infty) \to \mathrm{Stab}(\mathcal{D})$.

Theorem (HL, J, R '23)

A generic quasi-convergent path σ_t gives a semiorthogonal decomposition $\mathcal{D} = \langle \mathcal{D}_1, \ldots, \mathcal{D}_n \rangle$ plus $\sigma_i \in \mathrm{Stab}(\mathcal{D}_i)/\mathbf{C}$ for $i = 1, \ldots, n$.

1. study growth of $\phi_t(E)$ – if for all $t \gg 0$, $\phi_t(E) < \phi_t(F)$, then $\mathrm{Hom}(F, E) = 0$.
2. $\mathcal{D}_1$ is generated by objects with ϕ_t growing "slowest" and $\mathcal{D}_n$ is generated by objects with ϕ_t growing "fastest."
3. resulting SOD + stability conditions depends only on $\sigma_t : [0, \infty) \to \mathrm{Stab}(\mathcal{D})/\mathbf{C}$.

Quasi-convergent paths

In arXiv:2401.00600, (with D. Halpern-Leistner and J. Jiang) we introduce *quasi-convergent paths* $\sigma_t : [0, \infty) \to \mathrm{Stab}(\mathcal{D})$.

Theorem (HL, J, R '23)

A generic quasi-convergent path σ_t gives a semiorthogonal decomposition $\mathcal{D} = \langle \mathcal{D}_1, \ldots, \mathcal{D}_n \rangle$ plus $\sigma_i \in \mathrm{Stab}(\mathcal{D}_i)/\mathbf{C}$ for $i = 1, \ldots, n$.

1. study growth of $\phi_t(E)$ – if for all $t \gg 0$, $\phi_t(E) < \phi_t(F)$, then $\mathrm{Hom}(F, E) = 0$.
2. $\mathcal{D}_1$ is generated by objects with ϕ_t growing "slowest" and $\mathcal{D}_n$ is generated by objects with ϕ_t growing "fastest."
3. resulting SOD + stability conditions depends only on $\sigma_t : [0, \infty) \to \mathrm{Stab}(\mathcal{D})/\mathbf{C}$.

Quasi-convergent paths

In arXiv:2401.00600, (with D. Halpern-Leistner and J. Jiang) we introduce *quasi-convergent paths* $\sigma_t : [0, \infty) \to \mathrm{Stab}(\mathcal{D})$.

Theorem (HL, J, R '23)

A generic quasi-convergent path σ_t gives a semiorthogonal decomposition $\mathcal{D} = \langle \mathcal{D}_1, \ldots, \mathcal{D}_n \rangle$ plus $\sigma_i \in \mathrm{Stab}(\mathcal{D}_i)/\mathbf{C}$ for $i = 1, \ldots, n$.

1. study growth of $\phi_t(E)$ – if for all $t \gg 0$, $\phi_t(E) < \phi_t(F)$, then $\mathrm{Hom}(F, E) = 0$.
2. $\mathcal{D}_1$ is generated by objects with ϕ_t growing "slowest" and $\mathcal{D}_n$ is generated by objects with ϕ_t growing "fastest."
3. resulting SOD + stability conditions depends only on $\sigma_t : [0, \infty) \to \mathrm{Stab}(\mathcal{D})/\mathbf{C}$.

Quasi-convergent paths

In arXiv:2401.00600, (with D. Halpern-Leistner and J. Jiang) we introduce *quasi-convergent paths* $\sigma_t : [0, \infty) \to \mathrm{Stab}(\mathcal{D})$.

Theorem (HL, J, R '23)

A generic quasi-convergent path σ_t gives a semiorthogonal decomposition $\mathcal{D} = \langle \mathcal{D}_1, \ldots, \mathcal{D}_n \rangle$ plus $\sigma_i \in \mathrm{Stab}(\mathcal{D}_i)/\mathbf{C}$ for $i = 1, \ldots, n$.

1. study growth of $\phi_t(E)$ – if for all $t \gg 0$, $\phi_t(E) < \phi_t(F)$, then $\mathrm{Hom}(F, E) = 0$.
2. $\mathcal{D}_1$ is generated by objects with ϕ_t growing "slowest" and $\mathcal{D}_n$ is generated by objects with ϕ_t growing "fastest."
3. resulting SOD + stability conditions depends only on $\sigma_t : [0, \infty) \to \mathrm{Stab}(\mathcal{D})/\mathbf{C}$.

Quasi-convergent paths cont.

$\mathcal{D} = \langle \mathcal{D}_1, \ldots, \mathcal{D}_n \rangle + \sigma_i \in \mathrm{Stab}(\mathcal{D}_i)/\mathbf{C}$ for $i = 1, \ldots, n$ is called a *polarised semiorthogonal decomposition*.

Theorem (HL, J, R '23)

Let $\mathcal{D}$ be smooth and proper (as a dg-category). Every polarised SOD $\langle \mathcal{D}_1, \ldots, \mathcal{D}_n | \sigma_1, \ldots, \sigma_n \rangle$ comes from a qc path.

The proof uses the gluing construction of Collins - Polishchuk.

Heuristic

Qc. paths should converge in a (partial) compactification of $\mathrm{Stab}(\mathcal{D})/\mathbf{C}$ to boundary points which correspond to polarised SODs (+ some additional data!)

Quasi-convergent paths cont.

$\mathcal{D} = \langle \mathcal{D}_1, \ldots, \mathcal{D}_n \rangle$ + $\sigma_i \in \mathrm{Stab}(\mathcal{D}_i)/\mathbf{C}$ for $i = 1, \ldots, n$ is called a *polarised semiorthogonal decomposition*.

Theorem (HL, J, R '23)

Let $\mathcal{D}$ be smooth and proper (as a dg-category). Every polarised SOD $\langle \mathcal{D}_1, \ldots, \mathcal{D}_n | \sigma_1, \ldots, \sigma_n \rangle$ comes from a qc path.

The proof uses the gluing construction of Collins - Polishchuk.

Heuristic

Qc. paths should converge in a (partial) compactification of $\mathrm{Stab}(\mathcal{D})/\mathbf{C}$ to boundary points which correspond to polarised SODs (+ some additional data!)

Quasi-convergent paths cont.

$\mathcal{D} = \langle \mathcal{D}_1, \ldots, \mathcal{D}_n \rangle$ + $\sigma_i \in \mathrm{Stab}(\mathcal{D}_i)/\mathbf{C}$ for $i = 1, \ldots, n$ is called a *polarised semiorthogonal decomposition*.

Theorem (HL, J, R '23)

Let $\mathcal{D}$ be smooth and proper (as a dg-category). Every polarised SOD $\langle \mathcal{D}_1, \ldots, \mathcal{D}_n | \sigma_1, \ldots, \sigma_n \rangle$ comes from a qc path.

The proof uses the gluing construction of Collins - Polishchuk.

Heuristic

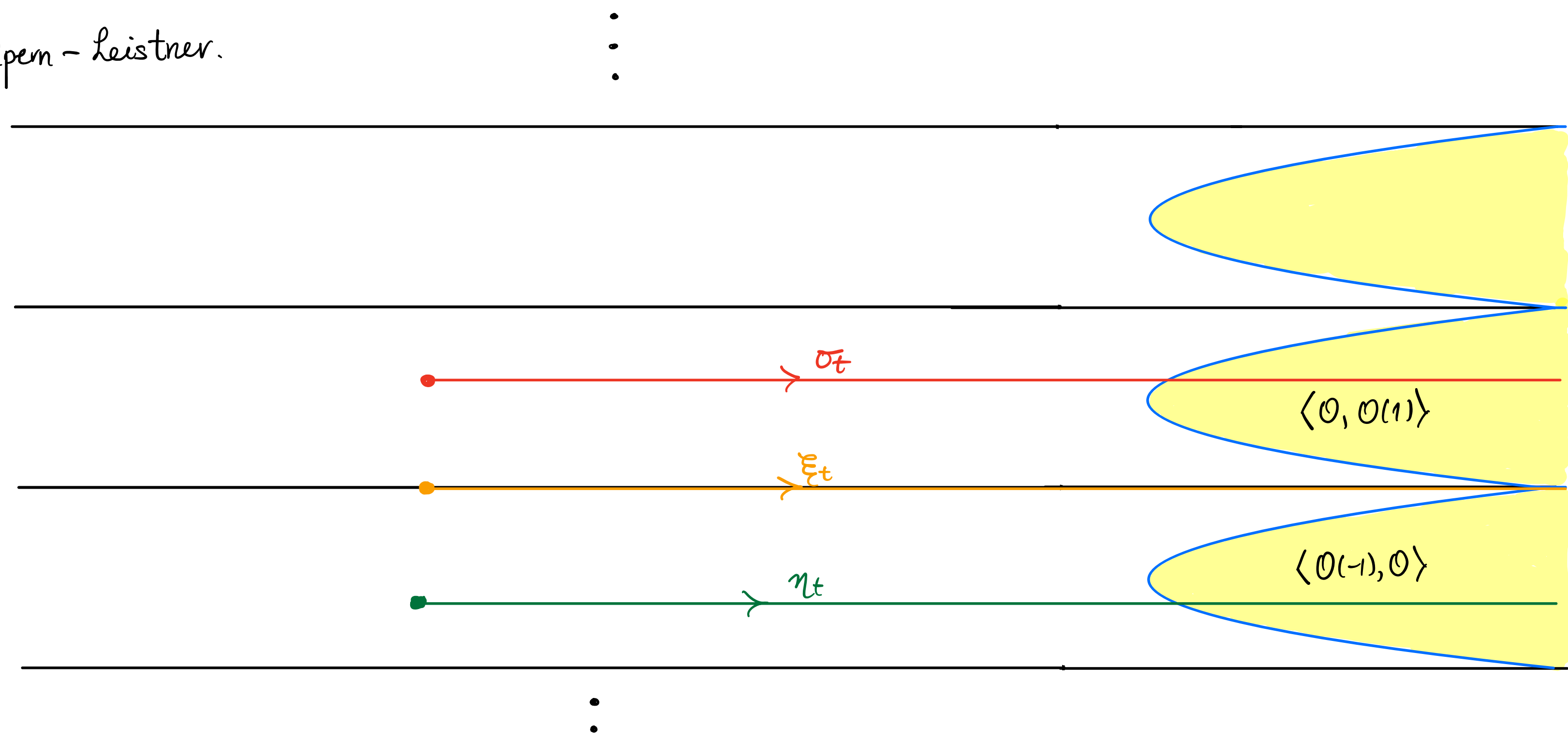
Qc. paths should converge in a (partial) compactification of $\mathrm{Stab}(\mathcal{D})/\mathbf{C}$ to boundary points which correspond to polarised SODs (+ some additional data!)

The case of $\mathbf{P}^1$

The case of $\mathbf{P}^1$ gives a good overview of general phenomena:

$\mathrm{Stab}(\mathbb{P}^1)/\mathbb{C} \cong \mathbb{C}$ (Okada)

Picture: Halpern-Leistner.



$\sigma_t \rightsquigarrow \langle \mathcal{O}, \mathcal{O}(1) \rangle$

$\eta_t \rightsquigarrow \langle \mathcal{O}(-1), \mathcal{O} \rangle$

$\xi_t \rightsquigarrow$??

Coordinates on the stability manifold

1. Given $\sigma \in \mathrm{Stab}(X)$, choose σ-stable $E_1, \ldots, E_n$ in $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ such that $\{\mathrm{ch}(E_i)\}_{i=1}^n$ is basis of $H^*_{\mathrm{alg}}(X; \mathbf{C})$.
2. Bridgeland's Theorem $\Rightarrow \tau \mapsto (Z_\tau(E_1), \ldots, Z_\tau(E_n)) \in (\mathbf{C}^*)^n$ is a coordinate system around σ.
3. Put $\log Z_\tau(E_i) := \log|Z_\tau(E_i)| + i\pi\phi_\tau(E_i)$.

$$\tau \mapsto (\log Z_\tau(E_1), \ldots, \log Z_\tau(E_n)) \in \mathbf{C}^n$$

logarithmic coordinates

4. $\forall\, w \in \mathbf{C}$, $\log Z_{w\cdot\tau}(E_i) = \log Z_\tau(E_i) + w$ so

$$(\log Z_\tau(E_1), \ldots, \log Z_\tau(E_n)) \mapsto (\log Z_\tau(E_1) + w, \ldots, \log Z_\tau(E_n) + w)$$

5. Conclusion: $\mathrm{Stab}(\mathcal{D})/\mathbf{C}$ is locally modeled on $\mathbf{C}^n/\mathbf{C}$.

Coordinates on the stability manifold

1. Given $\sigma \in \mathrm{Stab}(X)$, choose σ-stable $E_1, \ldots, E_n$ in $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ such that $\{\mathrm{ch}(E_i)\}_{i=1}^n$ is basis of $H^*_{\mathrm{alg}}(X; \mathbf{C})$.
2. Bridgeland's Theorem $\Rightarrow \tau \mapsto (Z_\tau(E_1), \ldots, Z_\tau(E_n)) \in (\mathbf{C}^*)^n$ is a coordinate system around σ.
3. Put $\log Z_\tau(E_i) := \log|Z_\tau(E_i)| + i\pi\phi_\tau(E_i)$.

$$\tau \mapsto (\log Z_\tau(E_1), \ldots, \log Z_\tau(E_n)) \in \mathbf{C}^n$$

logarithmic coordinates

4. $\forall\, w \in \mathbf{C}$, $\log Z_{w\cdot\tau}(E_i) = \log Z_\tau(E_i) + w$ so

$$(\log Z_\tau(E_1), \ldots, \log Z_\tau(E_n)) \mapsto (\log Z_\tau(E_1) + w, \ldots, \log Z_\tau(E_n) + w)$$

5. Conclusion: $\mathrm{Stab}(\mathcal{D})/\mathbf{C}$ is locally modeled on $\mathbf{C}^n/\mathbf{C}$.

Coordinates on the stability manifold

1. Given $\sigma \in \mathrm{Stab}(X)$, choose σ-stable $E_1, \ldots, E_n$ in $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ such that $\{\mathrm{ch}(E_i)\}_{i=1}^n$ is basis of $H^*_{\mathrm{alg}}(X; \mathbf{C})$.
2. Bridgeland's Theorem $\Rightarrow \tau \mapsto (Z_\tau(E_1), \ldots, Z_\tau(E_n)) \in (\mathbf{C}^*)^n$ is a coordinate system around σ.
3. Put $\log Z_\tau(E_i) := \log|Z_\tau(E_i)| + i\pi\phi_\tau(E_i)$.

$$\tau \mapsto (\log Z_\tau(E_1), \ldots, \log Z_\tau(E_n)) \in \mathbf{C}^n$$

logarithmic coordinates

4. $\forall\, w \in \mathbf{C}$, $\log Z_{w\cdot\tau}(E_i) = \log Z_\tau(E_i) + w$ so

$$(\log Z_\tau(E_1), \ldots, \log Z_\tau(E_n)) \mapsto (\log Z_\tau(E_1) + w, \ldots, \log Z_\tau(E_n) + w)$$

5. Conclusion: $\mathrm{Stab}(\mathcal{D})/\mathbf{C}$ is locally modeled on $\mathbf{C}^n/\mathbf{C}$.

Coordinates on the stability manifold

1. Given $\sigma \in \mathrm{Stab}(X)$, choose σ-stable $E_1, \ldots, E_n$ in $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ such that $\{\mathrm{ch}(E_i)\}_{i=1}^n$ is basis of $H^*_{\mathrm{alg}}(X; \mathbf{C})$.
2. Bridgeland's Theorem $\Rightarrow \tau \mapsto (Z_\tau(E_1), \ldots, Z_\tau(E_n)) \in (\mathbf{C}^*)^n$ is a coordinate system around σ.
3. Put $\log Z_\tau(E_i) := \log|Z_\tau(E_i)| + i\pi\phi_\tau(E_i)$.

$$\tau \mapsto (\log Z_\tau(E_1), \ldots, \log Z_\tau(E_n)) \in \mathbf{C}^n$$

   logarithmic coordinates

4. $\forall\, w \in \mathbf{C}$, $\log Z_{w\cdot\tau}(E_i) = \log Z_\tau(E_i) + w$ so

$$(\log Z_\tau(E_1), \ldots, \log Z_\tau(E_n)) \mapsto (\log Z_\tau(E_1) + w, \ldots, \log Z_\tau(E_n) + w)$$

5. Conclusion: $\mathrm{Stab}(\mathcal{D})/\mathbf{C}$ is locally modeled on $\mathbf{C}^n/\mathbf{C}$.

Coordinates on the stability manifold

1. Given $\sigma \in \mathrm{Stab}(X)$, choose σ-stable $E_1, \ldots, E_n$ in $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ such that $\{\mathrm{ch}(E_i)\}_{i=1}^{n}$ is basis of $H^*_{\mathrm{alg}}(X; \mathbf{C})$.
2. Bridgeland's Theorem $\Rightarrow \tau \mapsto (Z_\tau(E_1), \ldots, Z_\tau(E_n)) \in (\mathbf{C}^*)^n$ is a coordinate system around σ.
3. Put $\log Z_\tau(E_i) := \log|Z_\tau(E_i)| + i\pi\phi_\tau(E_i)$.

$$\tau \mapsto (\log Z_\tau(E_1), \ldots, \log Z_\tau(E_n)) \in \mathbf{C}^n$$

   logarithmic coordinates

4. $\forall\, w \in \mathbf{C}$, $\log Z_{w\cdot\tau}(E_i) = \log Z_\tau(E_i) + w$ so

$$(\log Z_\tau(E_1), \ldots, \log Z_\tau(E_n)) \mapsto (\log Z_\tau(E_1) + w, \ldots, \log Z_\tau(E_n) + w)$$

5. Conclusion: $\mathrm{Stab}(\mathcal{D})/\mathbf{C}$ is locally modeled on $\mathbf{C}^n/\mathbf{C}$.

Summary

1. Want to (partially) compactify $\mathrm{Stab}(\mathcal{D})/\mathbf{C}$ with points corresponding to polarised SODs obtained as limits of "quasi-convergent" paths
2. The local model of $\mathrm{Stab}(\mathcal{D})/\mathbf{C}$ is $\mathbf{C}^n/\mathbf{C}$ so we consider first the problem of compactifying there

Summary

1. Want to (partially) compactify $\mathrm{Stab}(\mathcal{D})/\mathbf{C}$ with points corresponding to polarised SODs obtained as limits of "quasi-convergent" paths
2. The local model of $\mathrm{Stab}(\mathcal{D})/\mathbf{C}$ is $\mathbf{C}^n/\mathbf{C}$ so we consider first the problem of compactifying there

Multiscale lines

$$\mathbf{C}^n/\mathbf{C} \iff \{(\mathbf{P}^1, \infty, dz, p_1, \ldots, p_n) \,|\, p_i \neq \infty \,\forall\, i\}^{\cong}$$

Proof:

- $\mu \in \mathrm{Aut}(\mathbf{P}^1)$: $\infty \mapsto \infty \Rightarrow \mu(z) = az + b$
- $\mu^*(dz) = dz \Rightarrow a = 1$

We compactify $\mathbf{C}^n/\mathbf{C}$ by introducing a "new" moduli space of marked genus 0 curves, called *multiscale lines* (inspired by Bainbridge - Chen - Gendron - Grushevsky - Möller).

Note: dz on $\mathbf{P}^1$ is characterised up to a scalar as a meromorphic differential with an order 2 pole at ∞; i.e, $dz \in \Gamma(\Omega_{\mathbf{P}^1}(2\infty)) = \mathbf{C}$.

Multiscale lines

$$\mathbf{C}^n/\mathbf{C} \iff \{(\mathbf{P}^1, \infty, dz, p_1, \ldots, p_n) \mid p_i \neq \infty \, \forall \, i\}^{/\cong}$$

Proof:

- $\mu \in \mathrm{Aut}(\mathbf{P}^1)$: $\infty \mapsto \infty \Rightarrow \mu(z) = az + b$
- $\mu^*(dz) = dz \Rightarrow a = 1$

We compactify $\mathbf{C}^n/\mathbf{C}$ by introducing a "new" moduli space of marked genus 0 curves, called *multiscale lines* (inspired by Bainbridge - Chen - Gendron - Grushevsky - Möller).

Note: dz on $\mathbf{P}^1$ is characterised up to a scalar as a meromorphic differential with an order 2 pole at ∞; i.e, $dz \in \Gamma(\Omega_{\mathbf{P}^1}(2\infty)) = \mathbf{C}$.

Multiscale lines

$$\mathbf{C}^n/\mathbf{C} \iff \{(\mathbf{P}^1, \infty, dz, p_1, \ldots, p_n) \mid p_i \neq \infty \,\forall\, i\}/\cong$$

Proof:

- $\mu \in \mathrm{Aut}(\mathbf{P}^1)$: $\infty \mapsto \infty \Rightarrow \mu(z) = az + b$
- $\mu^*(dz) = dz \Rightarrow a = 1$

We compactify $\mathbf{C}^n/\mathbf{C}$ by introducing a "new" moduli space of marked genus 0 curves, called *multiscale lines* (inspired by Bainbridge - Chen - Gendron - Grushevsky - Möller).

Note: dz on $\mathbf{P}^1$ is characterised up to a scalar as a meromorphic differential with an order 2 pole at ∞; i.e, $dz \in \Gamma(\Omega_{\mathbf{P}^1}(2\infty)) = \mathbf{C}$.

Multiscale lines

$$\mathbf{C}^n/\mathbf{C} \iff \{(\mathbf{P}^1, \infty, dz, p_1, \ldots, p_n) \mid p_i \neq \infty \, \forall \, i\}^{/\cong}$$

Proof:

- $\mu \in \mathrm{Aut}(\mathbf{P}^1)$: $\infty \mapsto \infty \Rightarrow \mu(z) = az + b$
- $\mu^*(dz) = dz \Rightarrow a = 1$

We compactify $\mathbf{C}^n/\mathbf{C}$ by introducing a "new" moduli space of marked genus 0 curves, called *multiscale lines* (inspired by Bainbridge - Chen - Gendron - Grushevsky - Möller).

Note: dz on $\mathbf{P}^1$ is characterised up to a scalar as a meromorphic differential with an order 2 pole at ∞; i.e, $dz \in \Gamma(\Omega_{\mathbf{P}^1}(2\infty)) = \mathbf{C}$.

Multiscale lines

$$\mathbf{C}^n/\mathbf{C} \iff \{(\mathbf{P}^1, \infty, dz, p_1, \ldots, p_n) \mid p_i \neq \infty \, \forall \, i\}/_{\cong}$$

Proof:

- $\mu \in \mathrm{Aut}(\mathbf{P}^1)$: $\infty \mapsto \infty \Rightarrow \mu(z) = az + b$
- $\mu^*(dz) = dz \Rightarrow a = 1$

We compactify $\mathbf{C}^n/\mathbf{C}$ by introducing a “new” moduli space of marked genus 0 curves, called *multiscale lines* (inspired by Bainbridge - Chen - Gendron - Grushevsky - Möller).

Note: dz on $\mathbf{P}^1$ is characterised up to a scalar as a meromorphic differential with an order 2 pole at ∞; i.e, $dz \in \Gamma(\Omega_{\mathbf{P}^1}(2\infty)) = \mathbf{C}$.

Multiscale lines

$$\mathbf{C}^n/\mathbf{C} \iff \{(\mathbf{P}^1, \infty, dz, p_1, \ldots, p_n) \mid p_i \neq \infty \, \forall \, i\}_{/\cong}$$

Proof:

- $\mu \in \mathrm{Aut}(\mathbf{P}^1)$: $\infty \mapsto \infty \Rightarrow \mu(z) = az + b$
- $\mu^*(dz) = dz \Rightarrow a = 1$

We compactify $\mathbf{C}^n/\mathbf{C}$ by introducing a "new" moduli space of marked genus 0 curves, called *multiscale lines* (inspired by Bainbridge - Chen - Gendron - Grushevsky - Möller).

Note: dz on $\mathbf{P}^1$ is characterised up to a scalar as a meromorphic differential with an order 2 pole at ∞; i.e, $dz \in \Gamma(\Omega_{\mathbf{P}^1}(2\infty)) = \mathbf{C}$.

Multiscale lines

Definition (Part I)

An n-marked *multiscale line* is $(\Sigma, p_\infty, \preceq, \omega_\bullet, p_1, \ldots, p_n)$ where

1. Σ is a nodal genus 0 curve over $\mathbf{C}$
2. p_∞ is a special "top" point on the curve
3. $\preceq$ is a level structure
4. $p_1, \ldots, p_n$ are marked points which may collide with each other but not with p_∞

+ conditions: e.g. $p_1, \ldots, p_n$ are all on components furthest from p_∞ and all such components are at the same "level" ...

Multiscale lines

Definition (Part I)

An n-marked *multiscale line* is $(\Sigma, p_\infty, \preceq, \omega_\bullet, p_1, \ldots, p_n)$ where

1. Σ is a nodal genus 0 curve over $\mathbf{C}$
2. p_∞ is a special "top" point on the curve
3. $\preceq$ is a level structure
4. $p_1, \ldots, p_n$ are marked points which may collide with each other but not with p_∞

+ conditions: e.g. $p_1, \ldots, p_n$ are all on components furthest from p_∞ and all such components are at the same "level" ...

Multiscale lines

Definition (Part I)

An n-marked *multiscale line* is $(\Sigma, p_\infty, \preceq, \omega_\bullet, p_1, \ldots, p_n)$ where

1. Σ is a nodal genus 0 curve over $\mathbf{C}$
2. p_∞ is a special "top" point on the curve
3. $\preceq$ is a level structure
4. $p_1, \ldots, p_n$ are marked points which may collide with each other but not with p_∞

+ conditions: e.g. $p_1, \ldots, p_n$ are all on components furthest from p_∞ and all such components are at the same "level" ...

Multiscale lines

Definition (Part I)

An n-marked *multiscale line* is $(\Sigma, p_\infty, \preceq, \omega_\bullet, p_1, \ldots, p_n)$ where

1. Σ is a nodal genus 0 curve over $\mathbf{C}$
2. p_∞ is a special "top" point on the curve
3. $\preceq$ is a level structure
4. $p_1, \ldots, p_n$ are marked points which may collide with each other but not with p_∞

+ conditions: e.g. $p_1, \ldots, p_n$ are all on components furthest from p_∞ and all such components are at the same "level" ...

Multiscale lines

Definition (Part I)

An n-marked *multiscale line* is $(\Sigma, p_\infty, \preceq, \omega_\bullet, p_1, \ldots, p_n)$ where

1. Σ is a nodal genus 0 curve over $\mathbf{C}$
2. p_∞ is a special "top" point on the curve
3. $\preceq$ is a level structure
4. $p_1, \ldots, p_n$ are marked points which may collide with each other but not with p_∞

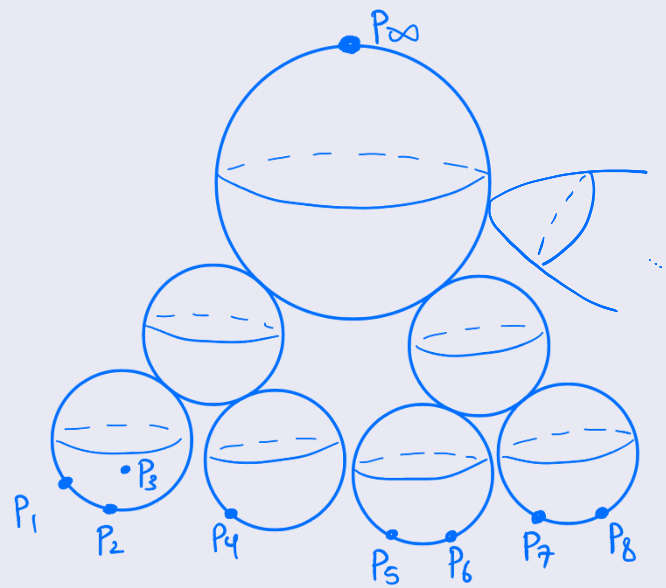
+ conditions: e.g. $p_1, \ldots, p_n$ are all on components furthest from p_∞ and all such components are at the same "level" ...

Multiscale lines

Definition (Part I)

An n-marked *multiscale line* is $(\Sigma, p_\infty, \preceq, \omega_\bullet, p_1, \ldots, p_n)$ where

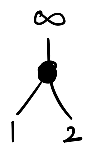
1. Σ is a nodal genus 0 curve over $\mathbf{C}$
2. p_∞ is a special "top" point on the curve
3. $\preceq$ is a level structure
4. $p_1, \ldots, p_n$ are marked points which may collide with each other but not with p_∞

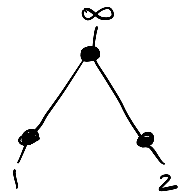


+ conditions: e.g. $p_1, \ldots, p_n$ are all on components furthest from p_∞ and all such components are at the same "level" ...

Some Combinatorial types:

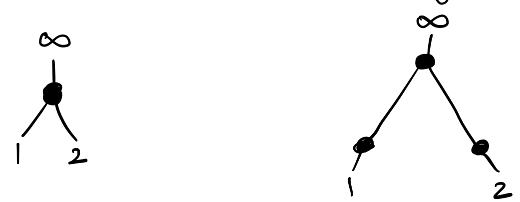
$n=2$.



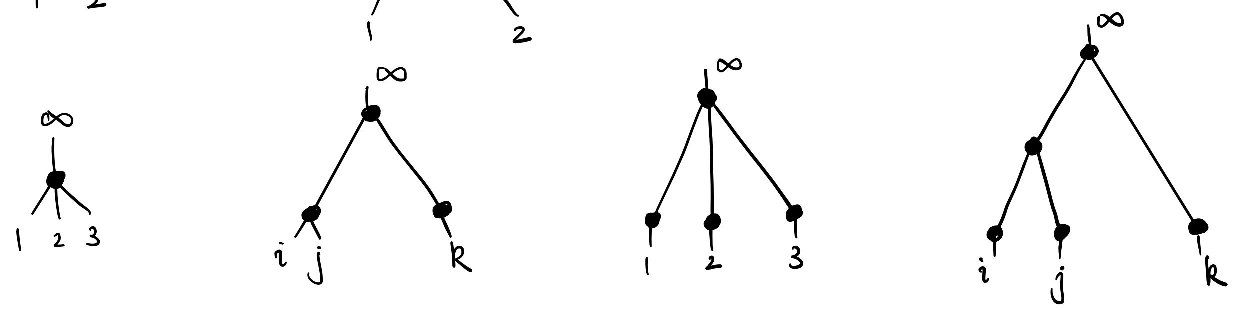


Some Combinatorial types:

$\underline{n=2.}$



$\underline{n=3.}$

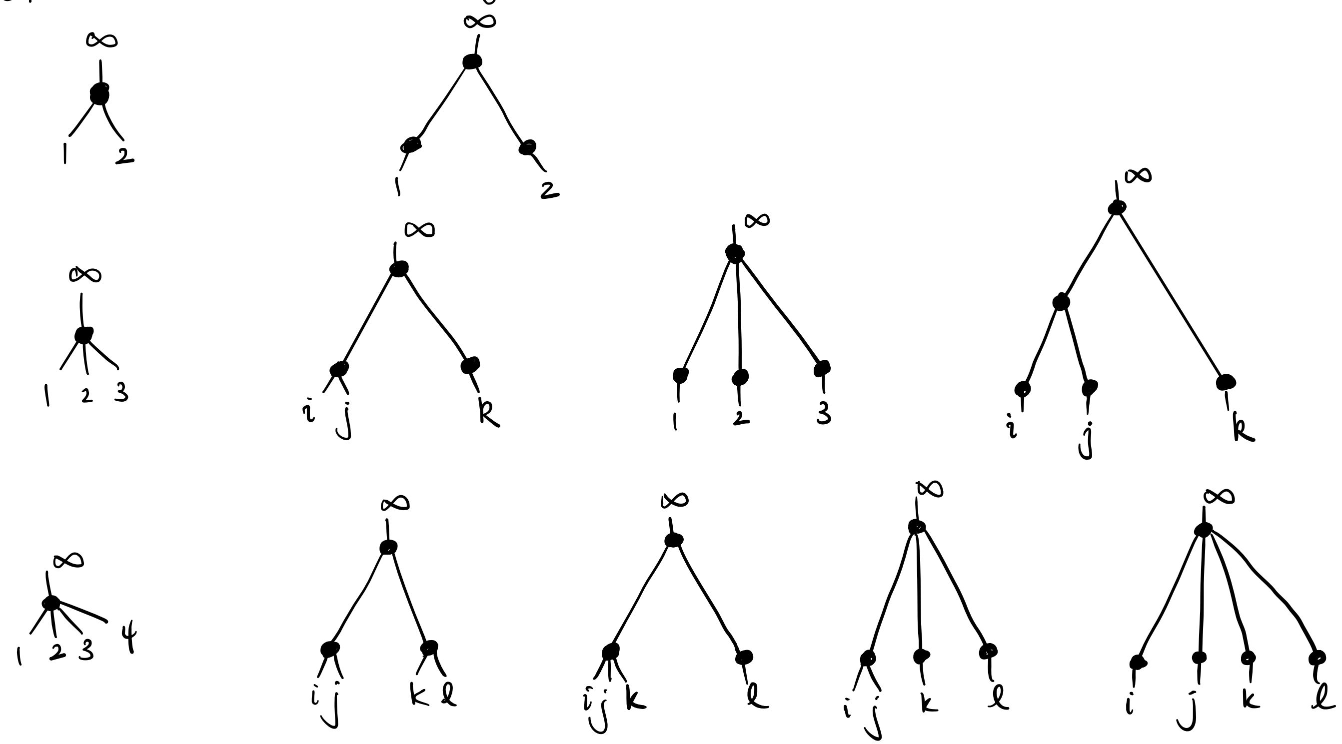


Some Combinatorial types:

$n = 2.$

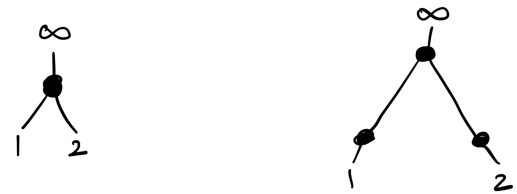
$n = 3.$

$n = 4.$

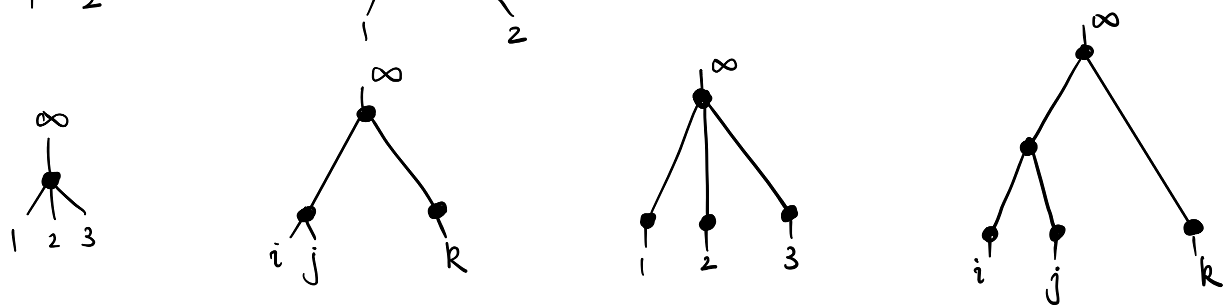


Some Combinatorial types:

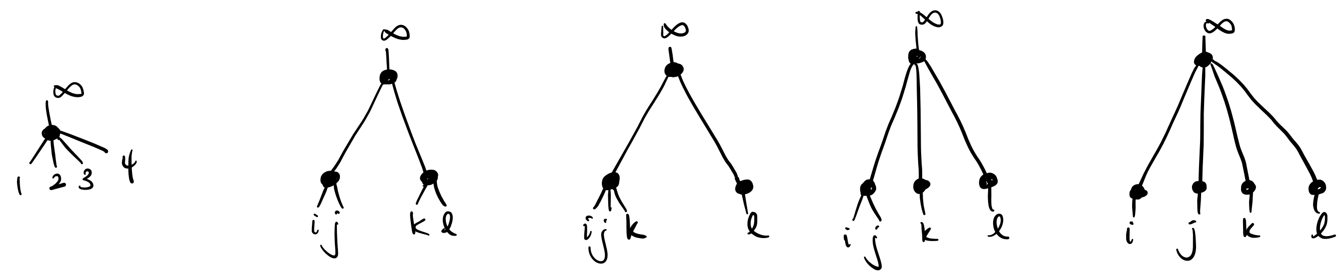
$\underline{n=2.}$

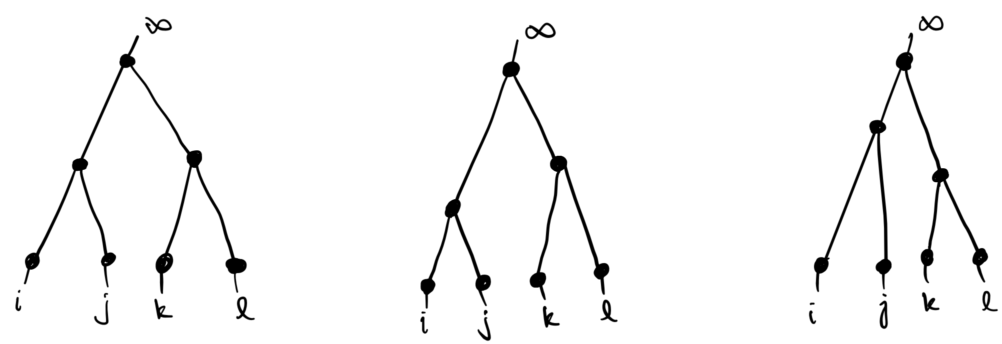


$\underline{n=3.}$



$\underline{n=4.}$





etc.

Multiscale lines

Definition (Part II)

... and $\omega_\bullet = \{\omega_v\}_{v \in V(\Sigma)} \rightsquigarrow$ meromorphic 1-forms, one per component Σ_v, with unique order 2 pole at the node connecting Σ_v to the root (resp. at p_∞ if $v = \text{root}$)

When $\Sigma = \mathbf{P}^1$, a multiscale line is just $(\mathbf{P}^1, \infty, \omega = \lambda dz, p_1, \ldots, p_n)$ for $\lambda \in \mathbf{C}^* \rightsquigarrow \{\text{irred. } n-\text{marked multiscale lines}\} = \mathbf{C}^n/\mathbf{C}$.

For each $v \in V(\Sigma)$, ω_v on $\Sigma_v \longleftrightarrow \lambda_v \in \mathbf{C}^*$.

Multiscale lines

Definition (Part II)

... and $\omega_\bullet = \{\omega_v\}_{v \in V(\Sigma)} \rightsquigarrow$ meromorphic 1-forms, one per component Σ_v, with unique order 2 pole at the node connecting Σ_v to the root (resp. at p_∞ if $v =$ root)

When $\Sigma = \mathbf{P}^1$, a multiscale line is just $(\mathbf{P}^1, \infty, \omega = \lambda dz, p_1, \ldots, p_n)$ for $\lambda \in \mathbf{C}^* \rightsquigarrow \{\text{irred. } n-\text{marked multiscale lines}\} = \mathbf{C}^n/\mathbf{C}$.

For each $v \in V(\Sigma)$, ω_v on $\Sigma_v \longleftrightarrow \lambda_v \in \mathbf{C}^*$.

Multiscale lines

Definition (Part II)

... and $\omega_\bullet = \{\omega_v\}_{v \in V(\Sigma)} \rightsquigarrow$ meromorphic 1-forms, one per component Σ_v, with unique order 2 pole at the node connecting Σ_v to the root (resp. at p_∞ if $v = \text{root}$)

When $\Sigma = \mathbf{P}^1$, a multiscale line is just $(\mathbf{P}^1, \infty, \omega = \lambda dz, p_1, \ldots, p_n)$ for $\lambda \in \mathbf{C}^* \rightsquigarrow \{\text{irred. } n - \text{marked multiscale lines}\}^{/\cong} = \mathbf{C}^n/\mathbf{C}$.

For each $v \in V(\Sigma)$, ω_v on $\Sigma_v \longleftrightarrow \lambda_v \in \mathbf{C}^*$.

Multiscale lines

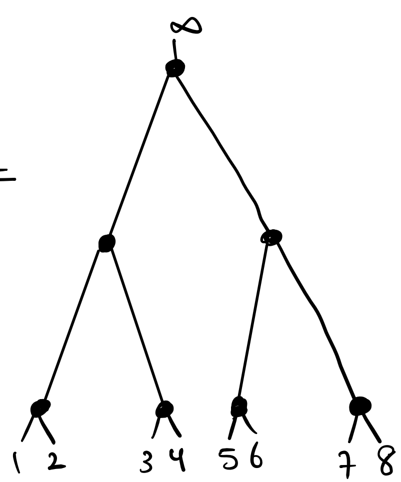
Definition (Part II)

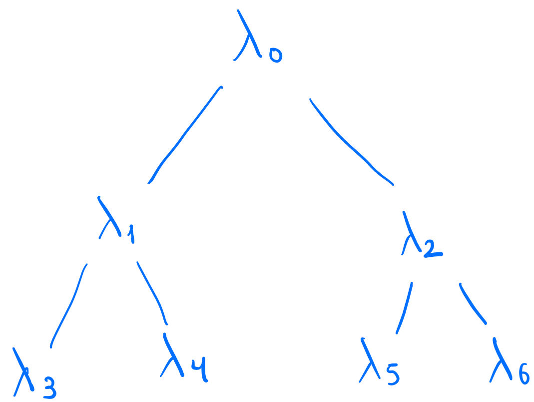
... and $\omega_\bullet = \{\omega_v\}_{v \in V(\Sigma)} \rightsquigarrow$ meromorphic 1-forms, one per component Σ_v, with unique order 2 pole at the node connecting Σ_v to the root (resp. at p_∞ if $v = \text{root}$)

When $\Sigma = \mathbf{P}^1$, a multiscale line is just $(\mathbf{P}^1, \infty, \omega = \lambda dz, p_1, \ldots, p_n)$ for $\lambda \in \mathbf{C}^* \rightsquigarrow \{\text{irred. } n - \text{marked multiscale lines}\} = \mathbf{C}^n/\mathbf{C}$.

For each $v \in V(\Sigma)$, ω_v on $\Sigma_v \longleftrightarrow \lambda_v \in \mathbf{C}^*$.

$\Gamma(\Sigma) =$





$\in (\mathbb{C}^*)^{V(\Sigma)}$

Equivalence

Definition

A $\mathbf{C}$-projective (resp. $\mathbf{R}$-oriented) iso. of multiscale lines $f : \Sigma \to \Sigma'$ is an iso. of curves that preserves level structures and marked points s.t.

$$f^*(\omega'_v) = c_v \omega_v \quad \text{for } c_v \in \mathbf{C}^* \text{ (resp. } \mathbf{R}_{>0}\text{)}$$

and $c_v = c_w$ if $v \sim w$ and $c_v = 1$ for v a bottom comp. of Σ.

Equivalence

Definition

A $\mathbf{C}$-projective (resp. $\mathbf{R}$-oriented) iso. of multiscale lines $f : \Sigma \to \Sigma'$ is an iso. of curves that preserves level structures and marked points s.t.

$$f^*(\omega'_v) = c_v \omega_v \text{ for } c_v \in \mathbf{C}^* \text{ (resp. } \mathbf{R}_{>0})$$

and $c_v = c_w$ if $v \sim w$ and $c_v = 1$ for v a bottom comp. of Σ.

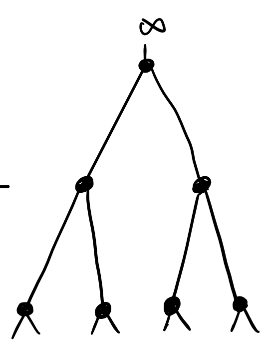
Equivalence

Definition

A **C**-projective (resp. **R**-oriented) iso. of multiscale lines $f : \Sigma \to \Sigma'$ is an iso. of curves that preserves level structures and marked points s.t.

$$f^*(\omega'_v) = c_v \omega_v \quad \text{for } c_v \in \mathbf{C}^* \text{ (resp. } \mathbf{R}_{>0})$$

and $c_v = c_w$ if $v \sim w$ and $c_v = 1$ for v a bottom comp. of Σ.

	up to isomorphism		
$\Gamma(\Sigma) =$ 	$\lambda_0 \in \mathbb{C}^*$ $(\lambda_1, \lambda_2) \in (\mathbb{C}^*)^2$ $(\lambda_3, \lambda_4, \lambda_5, \lambda_6) \in (\mathbb{C}^*)^4$		

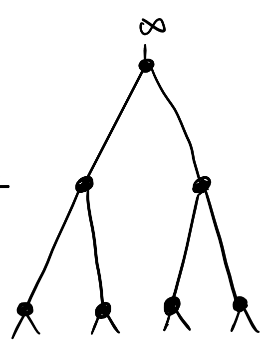
Equivalence

Definition

A $\mathbf{C}$-projective (resp. $\mathbf{R}$-oriented) iso. of multiscale lines $f : \Sigma \to \Sigma'$ is an iso. of curves that preserves level structures and marked points s.t.

$$f^*(\omega'_v) = c_v \omega_v \quad \text{for } c_v \in \mathbf{C}^* \text{ (resp. } \mathbf{R}_{>0}\text{)}$$

and $c_v = c_w$ if $v \sim w$ and $c_v = 1$ for v a bottom comp. of Σ.

	up to isomorphism	up to $\mathbb{C}$-projective iso.	
	$\lambda_0 \in \mathbb{C}^*$	$[\lambda_0] \in \mathbb{P}^0$	
$\Gamma(\Sigma) =$ 	$(\lambda_1 \ , \ \lambda_2) \in (\mathbb{C}^*)^2$	$[\lambda_0 : \lambda_1] \in \mathbb{P}^1$	
	$(\lambda_3, \lambda_4, \lambda_5, \lambda_6) \in (\mathbb{C}^*)^4$	$(\lambda_3, \lambda_4, \lambda_5, \lambda_6) \in (\mathbb{C}^*)^4$	

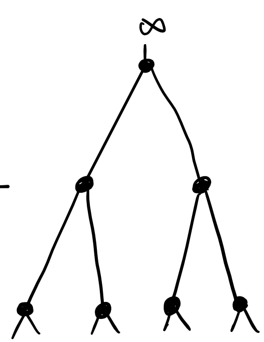
Equivalence

Definition

A **C**-projective (resp. **R**-oriented) iso. of multiscale lines $f : \Sigma \to \Sigma'$ is an iso. of curves that preserves level structures and marked points s.t.

$$f^*(\omega'_v) = c_v \omega_v \quad \text{for } c_v \in \mathbf{C}^* \text{ (resp. } \mathbf{R}_{>0})$$

and $c_v = c_w$ if $v \sim w$ and $c_v = 1$ for v a bottom comp. of Σ.

	up to isomorphism	up to $\mathbb{C}$-projective iso.	up to $\mathbb{R}$-oriented iso.
	$\lambda_0 \in \mathbb{C}^*$	$[\lambda_0] \in \mathbb{P}^0$	$\lambda_0 \in \mathbb{C}^*/\mathbb{R}_{>0} \cong S^1$
$\Gamma(\Sigma) =$ 	$(\lambda_1, \lambda_2) \in (\mathbb{C}^*)^2$	$[\lambda_0 : \lambda_1] \in \mathbb{P}^1$	$(\lambda_0, \lambda_1) \in (\mathbb{C}^*)^2/\mathbb{R}_{>0}$
	$(\lambda_3, \lambda_4, \lambda_5, \lambda_6) \in (\mathbb{C}^*)^4$	$(\lambda_3, \lambda_4, \lambda_5, \lambda_6) \in (\mathbb{C}^*)^4$	$(\lambda_3, \lambda_4, \lambda_5, \lambda_6) \in (\mathbb{C}^*)^4$

Examples

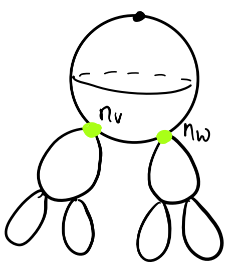
Up to $\mathbb{C}$ - projective isomorphism . . .

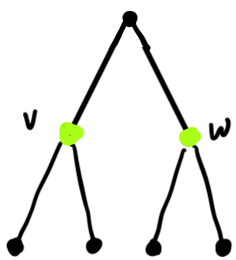
$$\frac{\int_\gamma \omega_{root}}{\int_\eta \omega_{root}}, \quad \frac{\int_\beta \omega_w}{\int_\alpha \omega_v} \quad \text{are defined.}$$

Examples

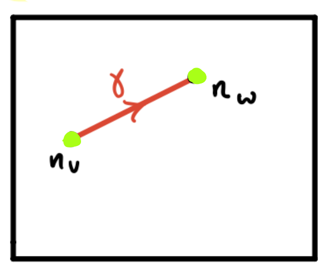
Up to $\mathbb{C}$-projective isomorphism...

$$\frac{\int_\gamma \omega_{root}}{\int_\eta \omega_{root}}, \quad \frac{\int_\beta \omega_w}{\int_\alpha \omega_v} \quad \text{are defined.}$$





Up to $\mathbb{R}$-oriented isom., more functions are defined



$\Sigma_{root} \setminus \{p_\infty\}$

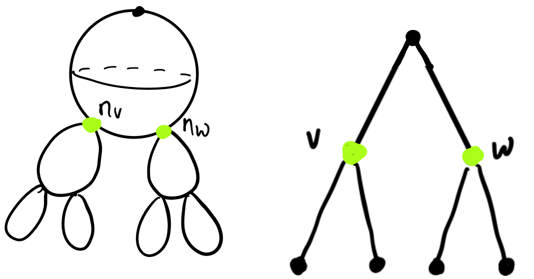
$$p(v,w) := \frac{\int_\gamma \omega_{root}}{\left|\int_\gamma \omega_{root}\right|} \in S^1$$

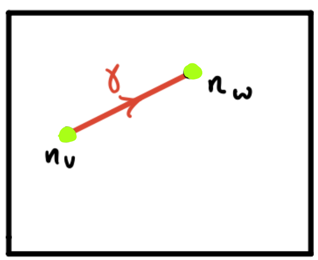
Examples

Up to $\mathbb{C}$-projective isomorphism...

$$\frac{\int_\gamma \omega_{root}}{\int_\eta \omega_{root}}, \quad \frac{\int_\beta \omega_w}{\int_\alpha \omega_v} \quad \text{are defined.}$$

Up to $\mathbb{R}$-oriented isom., more functions are defined





$\Sigma_{root} \setminus \{p_\infty\}$

$$p(v,w) := \frac{\int_\gamma \omega_{root}}{\left|\int_\gamma \omega_{root}\right|} \in S^1$$

Heuristic

A multiscale line Σ up to real-oriented isomorphism gives its dual tree the structure of a level graph with *angles* between edges

Moduli spaces

1. $\mathcal{A}_n := \{\mathbf{C}-\text{proj. iso. classes of } n\text{-marked multiscale lines}\}$
2. $\mathbf{C}^n/\mathbf{C} = \mathcal{A}_n^\circ \subset \mathcal{A}_n$ is the set of irreducible multiscale lines
3. Coordinates on $\mathcal{A}_n$ are constructed using the integral functions from the last slide.

Theorem (Halpern-Leistner, R.)

$\mathcal{A}_n$ is a compact complex algebraic manifold containing $\mathbf{C}^n/\mathbf{C}$ as an open dense subset. The boundary $D := \mathcal{A}_n \setminus \mathcal{A}_n^\circ$ is snc.

The set of $\mathbf{R}$-oriented iso. classes of n-marked multiscale lines is in canonical bijection with the real oriented blowup of $\mathcal{A}_n$ along D, denoted $\mathcal{A}_n^{\mathbf{R}}$.

$\mathcal{A}_n^{\mathbf{R}}$ will be the *local model* for partial compactification of $\mathrm{Stab}(\mathcal{D})/\mathbf{C}$.

Moduli spaces

1. $\mathcal{A}_n := \{\mathbf{C}-\text{proj. iso. classes of } n\text{-marked multiscale lines}\}$
2. $\mathbf{C}^n/\mathbf{C} = \mathcal{A}_n^\circ \subset \mathcal{A}_n$ is the set of irreducible multiscale lines
3. Coordinates on $\mathcal{A}_n$ are constructed using the integral functions from the last slide.

Theorem (Halpern-Leistner, R.)

$\mathcal{A}_n$ is a compact complex algebraic manifold containing $\mathbf{C}^n/\mathbf{C}$ as an open dense subset. The boundary $D := \mathcal{A}_n \setminus \mathcal{A}_n^\circ$ is snc.

The set of $\mathbf{R}$-oriented iso. classes of n-marked multiscale lines is in canonical bijection with the real oriented blowup of $\mathcal{A}_n$ along D, denoted $\mathcal{A}_n^{\mathbf{R}}$.

$\mathcal{A}_n^{\mathbf{R}}$ will be the *local model* for partial compactification of $\mathrm{Stab}(\mathcal{D})/\mathbf{C}$.

Moduli spaces

1. $\mathcal{A}_n := \{\mathbf{C}-\text{proj. iso. classes of } n\text{-marked multiscale lines}\}$
2. $\mathbf{C}^n/\mathbf{C} = \mathcal{A}_n^\circ \subset \mathcal{A}_n$ is the set of irreducible multiscale lines
3. Coordinates on $\mathcal{A}_n$ are constructed using the integral functions from the last slide.

Theorem (Halpern-Leistner, R.)

$\mathcal{A}_n$ is a compact complex algebraic manifold containing $\mathbf{C}^n/\mathbf{C}$ as an open dense subset. The boundary $D := \mathcal{A}_n \setminus \mathcal{A}_n^\circ$ is snc.

The set of $\mathbf{R}$-oriented iso. classes of n-marked multiscale lines is in canonical bijection with the real oriented blowup of $\mathcal{A}_n$ along D, denoted $\mathcal{A}_n^{\mathbf{R}}$.

$\mathcal{A}_n^{\mathbf{R}}$ will be the *local model* for partial compactification of $\mathrm{Stab}(\mathcal{D})/\mathbf{C}$.

Moduli spaces

1. $\mathcal{A}_n := \{\mathbf{C} - \text{proj. iso. classes of } n\text{-marked multiscale lines}\}$
2. $\mathbf{C}^n/\mathbf{C} = \mathcal{A}_n^\circ \subset \mathcal{A}_n$ is the set of irreducible multiscale lines
3. Coordinates on $\mathcal{A}_n$ are constructed using the integral functions from the last slide.

Theorem (Halpern-Leistner, R.)

$\mathcal{A}_n$ is a compact complex algebraic manifold containing $\mathbf{C}^n/\mathbf{C}$ as an open dense subset. The boundary $D := \mathcal{A}_n \setminus \mathcal{A}_n^\circ$ is snc.

The set of $\mathbf{R}$-oriented iso. classes of n-marked multiscale lines is in canonical bijection with the real oriented blowup of $\mathcal{A}_n$ along D, denoted $\mathcal{A}_n^{\mathbf{R}}$.

$\mathcal{A}_n^{\mathbf{R}}$ will be the *local model* for partial compactification of $\mathrm{Stab}(\mathcal{D})/\mathbf{C}$.

Moduli spaces

1. $\mathcal{A}_n := \{\mathbf{C} - \text{proj. iso. classes of } n\text{-marked multiscale lines}\}$
2. $\mathbf{C}^n/\mathbf{C} = \mathcal{A}_n^\circ \subset \mathcal{A}_n$ is the set of irreducible multiscale lines
3. Coordinates on $\mathcal{A}_n$ are constructed using the integral functions from the last slide.

Theorem (Halpern-Leistner, R.)

$\mathcal{A}_n$ is a compact complex algebraic manifold containing $\mathbf{C}^n/\mathbf{C}$ as an open dense subset. The boundary $D := \mathcal{A}_n \setminus \mathcal{A}_n^\circ$ is snc.

The set of $\mathbf{R}$-oriented iso. classes of n-marked multiscale lines is in canonical bijection with the real oriented blowup of $\mathcal{A}_n$ along D, denoted $\mathcal{A}_n^{\mathbf{R}}$.

$\mathcal{A}_n^{\mathbf{R}}$ will be the *local model* for partial compactification of $\mathrm{Stab}(\mathcal{D})/\mathbf{C}$.

Moduli spaces

1. $\mathcal{A}_n := \{\mathbf{C} -$ proj. iso. classes of n-marked multiscale lines$\}$
2. $\mathbf{C}^n/\mathbf{C} = \mathcal{A}_n^\circ \subset \mathcal{A}_n$ is the set of irreducible multiscale lines
3. Coordinates on $\mathcal{A}_n$ are constructed using the integral functions from the last slide.

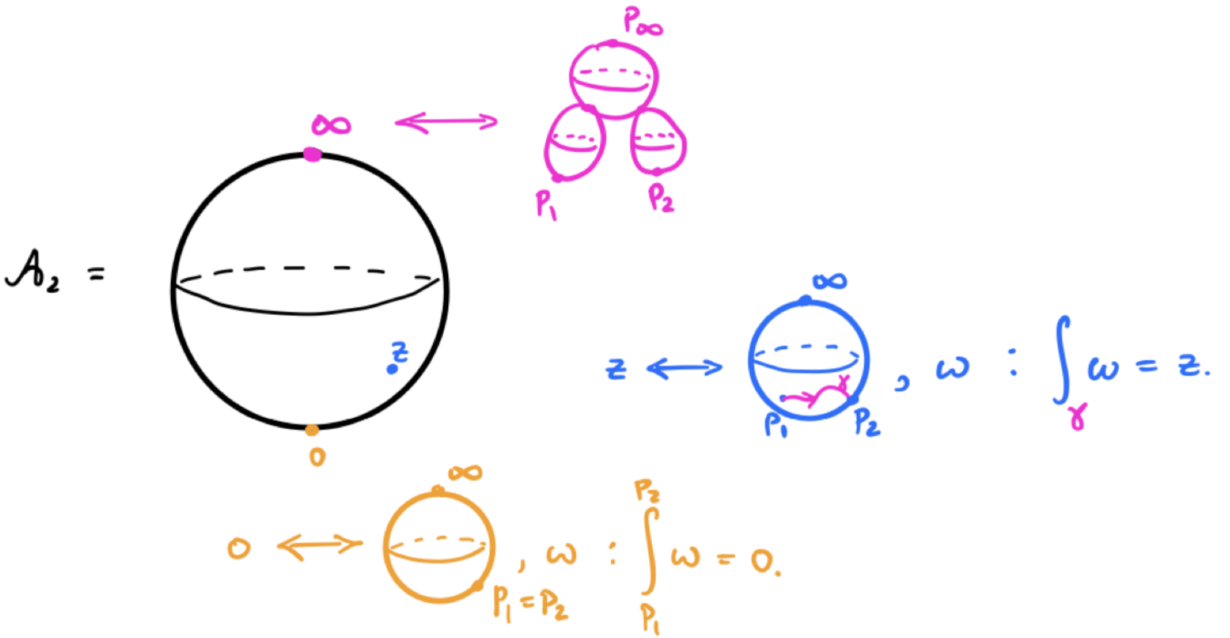
Theorem (Halpern-Leistner, R.)

$\mathcal{A}_n$ is a compact complex algebraic manifold containing $\mathbf{C}^n/\mathbf{C}$ as an open dense subset. The boundary $D := \mathcal{A}_n \setminus \mathcal{A}_n^\circ$ is snc.

The set of $\mathbf{R}$-oriented iso. classes of n-marked multiscale lines is in canonical bijection with the real oriented blowup of $\mathcal{A}_n$ along D, denoted $\mathcal{A}_n^{\mathbf{R}}$.

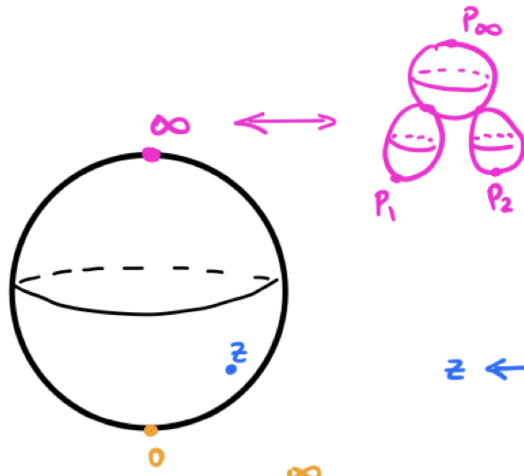
$\mathcal{A}_n^{\mathbf{R}}$ will be the *local model* for partial compactification of $\mathrm{Stab}(\mathcal{D})/\mathbf{C}$.

$\mathcal{A}_2$ and $\mathcal{A}_2^{\mathbf{R}}$



$\mathcal{A}_2$ and $\mathcal{A}_2^{\mathbb{R}}$

$\mathcal{A}_2 =$



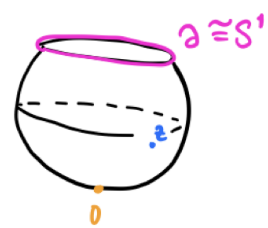
$z \longleftrightarrow$ 

$, \ \omega : \int_\gamma \omega = z.$

$0 \longleftrightarrow$ 

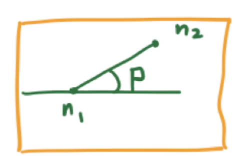
$, \ \omega : \int_{p_1}^{p_2} \omega = 0.$

$\mathcal{A}_2^{\mathbb{R}} =$



Coordinate on $\partial \ni$ 

$$\varphi(p_1, p_2) = \frac{\int_{n_1}^{n_2} \omega_{root}}{\left| \int_{n_1}^{n_2} \omega_{root} \right|} \in S^1$$



$\mathbb{C} = \mathbb{P}^1_{root} \setminus \{p_\infty\}$

Multiscale decompositions

Definition (Multiscale decomposition)

A *multiscale decomposition* $\mathcal{D} = \langle \mathcal{D}_\bullet \rangle_\Sigma$ is:

1. an un-marked multiscale line $(\Sigma, p_\infty, \preceq, \omega_\bullet)$ and
2. thick triangulated subcategories $\mathcal{D}_{\leq v}$ for each bottom $v \in V(\Sigma)$

such that $\mathcal{D}_{\leq v}$ and $\mathcal{D}_{\leq w}$ satisfy relations encoded by $\mathfrak{p}(v,w) \in S^1$.

Heuristic

A multiscale decomposition is a categorical structure interpolating between a *filtration* and a *semiorthogonal decomposition* of $\mathcal{D}$.

Multiscale decompositions

Definition (Multiscale decomposition)

A *multiscale decomposition* $\mathcal{D} = \langle \mathcal{D}_\bullet \rangle_\Sigma$ is:

1. an un-marked multiscale line $(\Sigma, p_\infty, \preceq, \omega_\bullet)$ and
2. thick triangulated subcategories $\mathcal{D}_{\leq v}$ for each bottom $v \in V(\Sigma)$

such that $\mathcal{D}_{\leq v}$ and $\mathcal{D}_{\leq w}$ satisfy relations encoded by $\mathfrak{p}(v, w) \in S^1$.

Heuristic

A multiscale decomposition is a categorical structure interpolating between a *filtration* and a *semiorthogonal decomposition* of $\mathcal{D}$.

Multiscale decompositions

Definition (Multiscale decomposition)

A *multiscale decomposition* $\mathcal{D} = \langle \mathcal{D}_\bullet \rangle_\Sigma$ is:

1. an un-marked multiscale line $(\Sigma, p_\infty, \preceq, \omega_\bullet)$ and
2. thick triangulated subcategories $\mathcal{D}_{\leq v}$ for each bottom $v \in V(\Sigma)$

such that $\mathcal{D}_{\leq v}$ and $\mathcal{D}_{\leq w}$ satisfy relations encoded by $\mathfrak{p}(v, w) \in S^1$.

Heuristic

A multiscale decomposition is a categorical structure interpolating between a *filtration* and a *semiorthogonal decomposition* of $\mathcal{D}$.

Multiscale decompositions

Definition (Multiscale decomposition)

A *multiscale decomposition* $\mathcal{D} = \langle \mathcal{D}_\bullet \rangle_\Sigma$ is:

1. an un-marked multiscale line $(\Sigma, p_\infty, \preceq, \omega_\bullet)$ and
2. thick triangulated subcategories $\mathcal{D}_{\leq v}$ for each bottom $v \in V(\Sigma)$

such that $\mathcal{D}_{\leq v}$ and $\mathcal{D}_{\leq w}$ satisfy relations encoded by $\mathfrak{p}(v, w) \in S^1$.

Heuristic

A multiscale decomposition is a categorical structure interpolating between a *filtration* and a *semiorthogonal decomposition* of $\mathcal{D}$.

Multiscale decompositions

Definition (Multiscale decomposition)

A *multiscale decomposition* $\mathcal{D} = \langle \mathcal{D}_\bullet \rangle_\Sigma$ is:

1. an un-marked multiscale line $(\Sigma, p_\infty, \preceq, \omega_\bullet)$ and
2. thick triangulated subcategories $\mathcal{D}_{\leq v}$ for each bottom $v \in V(\Sigma)$

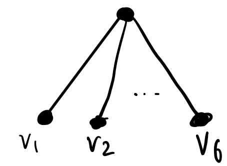
such that $\mathcal{D}_{\leq v}$ and $\mathcal{D}_{\leq w}$ satisfy relations encoded by $\mathfrak{p}(v, w) \in S^1$.

Heuristic

A multiscale decomposition is a categorical structure interpolating between a *filtration* and a *semiorthogonal decomposition* of $\mathcal{D}$.

When $\Sigma \cong \mathbf{P}^1$, a multiscale decomp. is trivial, just $\mathcal{D} = \langle \mathcal{D} \rangle_{\mathbf{P}^1}$.

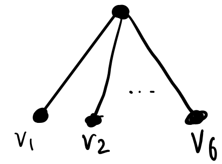
When $\Sigma \cong \mathbf{P}^1$, a multiscale decomp. is trivial, just $\mathcal{D} = \langle \mathcal{D} \rangle_{\mathbf{P}^1}$. Next, let

$\Gamma(\Sigma) =$ 

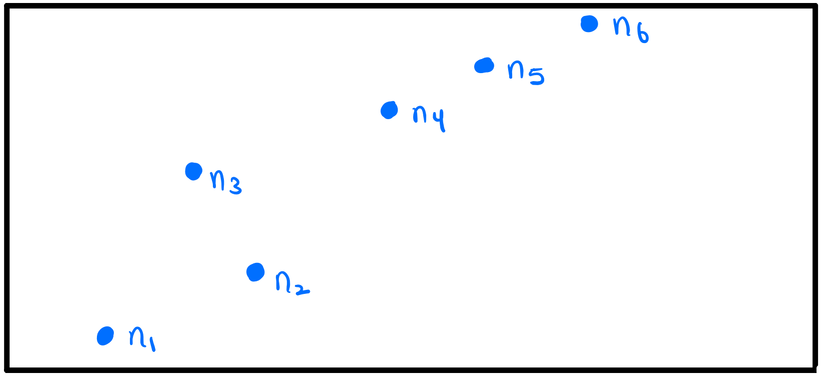
Consider the following configurations of nodes on $\Sigma_{\text{root}} \setminus p_\infty = \mathbf{C}$:

When $\Sigma \cong \mathbf{P}^1$, a multiscale decomp. is trivial, just $\mathcal{D} = \langle \mathcal{D} \rangle_{\mathbf{P}^1}$. Next, let

$$\Gamma(\Sigma) =$$

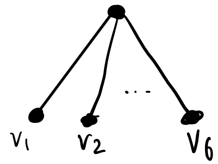


Consider the following configurations of nodes on $\Sigma_{\mathrm{root}} \setminus p_\infty = \mathbf{C}$:

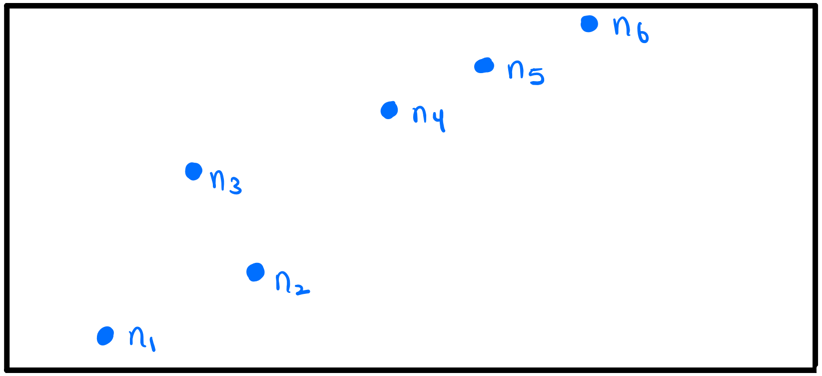


$\Im\mathfrak{p}(v_i, v_j) > 0 \Rightarrow \mathrm{Hom}(\mathcal{D}_{\leq v_j}, \mathcal{D}_{\leq v_i}) = 0$; get sod $\mathcal{D} = \langle \mathcal{D}_{\leq v_1}, \ldots, \mathcal{D}_{\leq v_6} \rangle$.

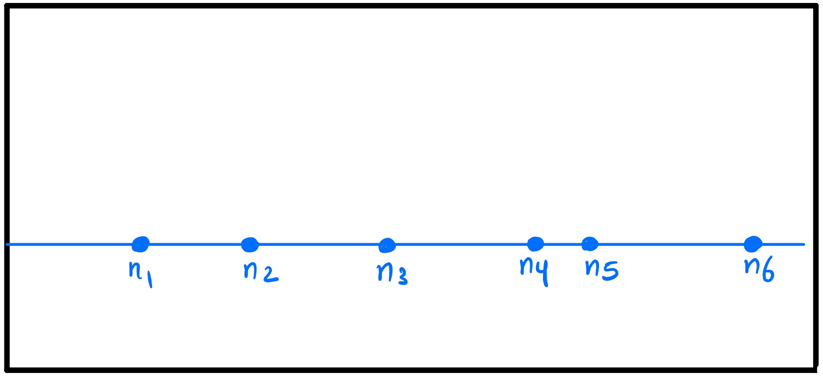
When $\Sigma \cong \mathbf{P}^1$, a multiscale decomp. is trivial, just $\mathcal{D} = \langle \mathcal{D} \rangle_{\mathbf{P}^1}$. Next, let

$\Gamma(\Sigma) =$ 

Consider the following configurations of nodes on $\Sigma_{\mathrm{root}} \setminus p_\infty = \mathbf{C}$:

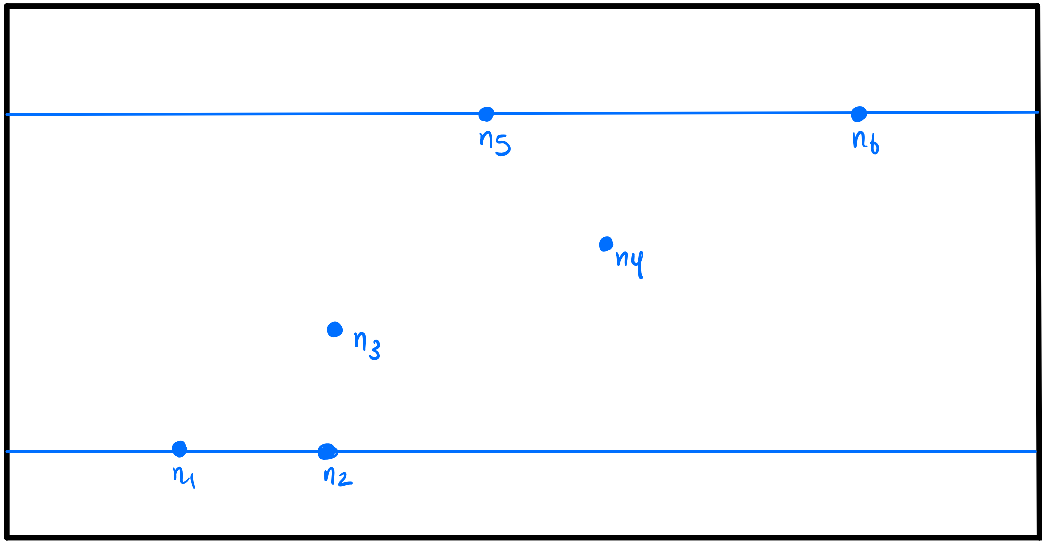


$\Im \mathfrak{p}(v_i, v_j) > 0 \Rightarrow \mathrm{Hom}(\mathcal{D}_{\leq v_j}, \mathcal{D}_{\leq v_i}) = 0$; get sod $\mathcal{D} = \langle \mathcal{D}_{\leq v_1}, \ldots, \mathcal{D}_{\leq v_6} \rangle$.

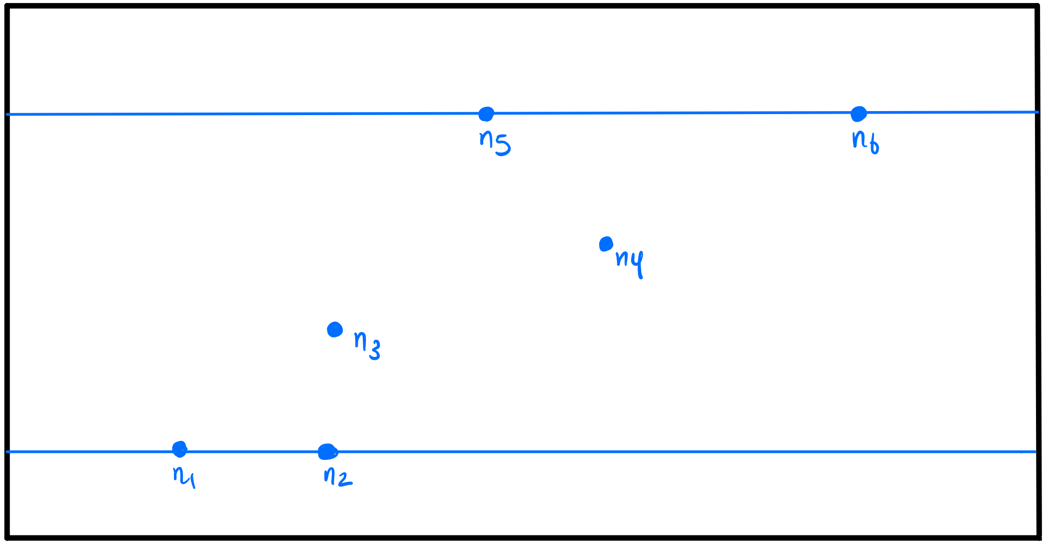


$\mathfrak{p}(v_i, v_j) = 1 \Rightarrow \mathcal{D}_{\leq v_i} \subsetneq \mathcal{D}_{\leq v_j}$; get filt. $0 \subsetneq \mathcal{D}_{\leq v_1} \subsetneq \cdots \subsetneq \mathcal{D}_{\leq v_6} = \mathcal{D}$.

Next, consider the following configuration:



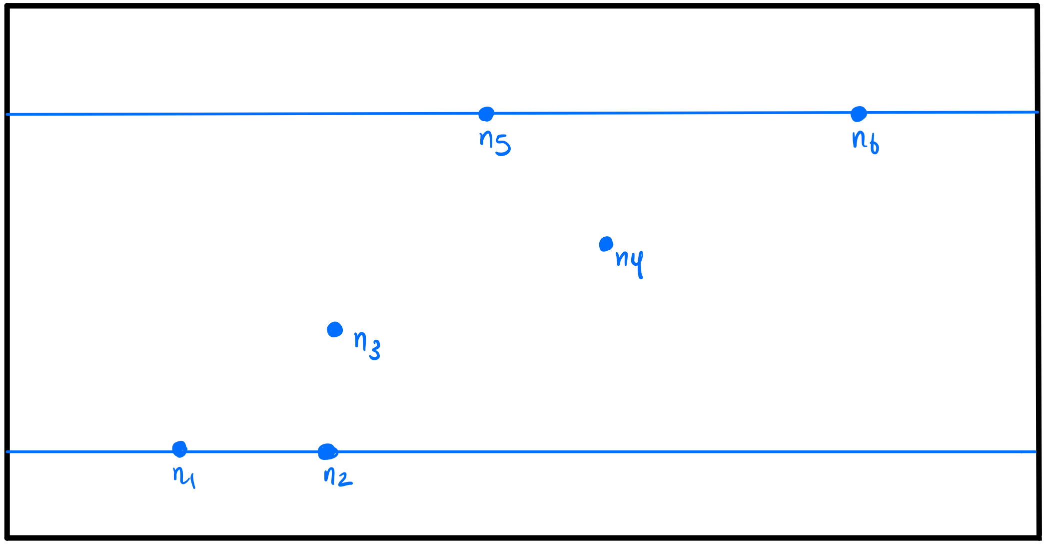
Next, consider the following configuration:



We get a filtered SOD,

$$\mathcal{D} = \langle \mathcal{D}_{\leq v_1} \subsetneq \mathcal{D}_{\leq v_2}, \mathcal{D}_{\leq v_3}, \mathcal{D}_{\leq v_4}, \mathcal{D}_{\leq v_5} \subsetneq \mathcal{D}_{\leq v_6} \rangle.$$

Next, consider the following configuration:



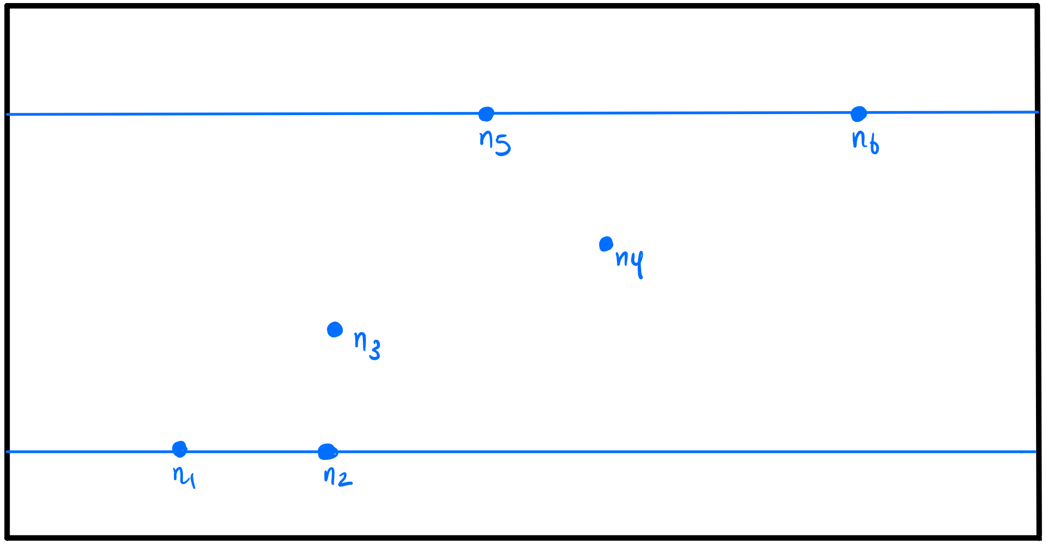
We get a filtered SOD,

$$\mathcal{D} = \langle \mathcal{D}_{\leq v_1} \subsetneq \mathcal{D}_{\leq v_2}, \mathcal{D}_{\leq v_3}, \mathcal{D}_{\leq v_4}, \mathcal{D}_{\leq v_5} \subsetneq \mathcal{D}_{\leq v_6} \rangle.$$

Each $\mathcal{D}_{\leq v}$ has *associated graded* category $\mathrm{gr}_v(\mathcal{D}_\bullet)$ obtained by quotienting by categories "to the left"

-
-

Next, consider the following configuration:



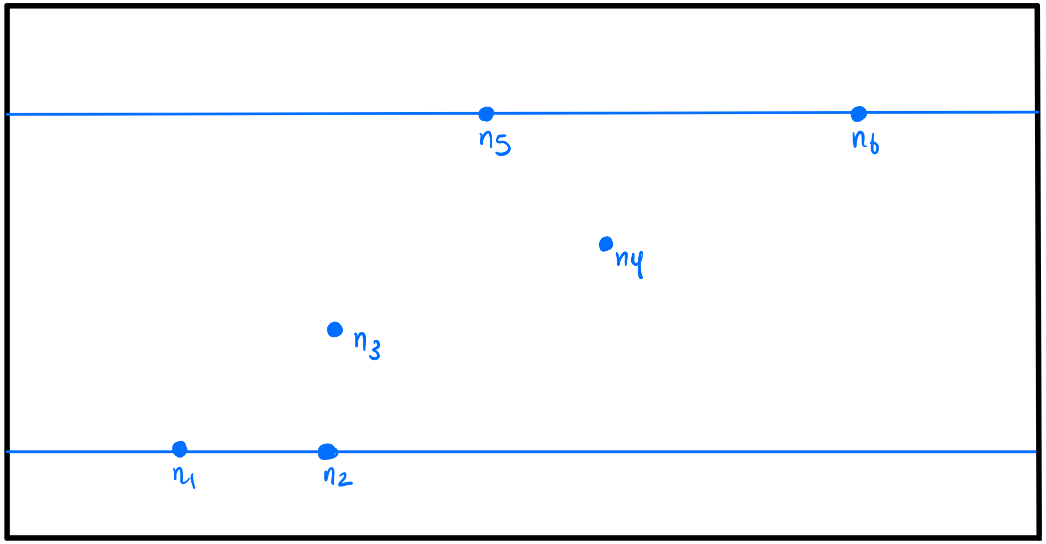
We get a filtered SOD,

$$\mathcal{D} = \langle \mathcal{D}_{\leq v_1} \subsetneq \mathcal{D}_{\leq v_2}, \mathcal{D}_{\leq v_3}, \mathcal{D}_{\leq v_4}, \mathcal{D}_{\leq v_5} \subsetneq \mathcal{D}_{\leq v_6} \rangle.$$

Each $\mathcal{D}_{\leq v}$ has *associated graded* category $\mathrm{gr}_v(\mathcal{D}_\bullet)$ obtained by quotienting by categories "to the left"

- $\mathrm{gr}_{v_4}(\mathcal{D}_\bullet) = \mathcal{D}_{\leq v_4}$, $\mathrm{gr}_{v_5}(\mathcal{D}_\bullet) = \mathcal{D}_{\leq v_5}$
-

Next, consider the following configuration:



We get a filtered SOD,

$$\mathcal{D} = \langle \mathcal{D}_{\leq v_1} \subsetneq \mathcal{D}_{\leq v_2}, \mathcal{D}_{\leq v_3}, \mathcal{D}_{\leq v_4}, \mathcal{D}_{\leq v_5} \subsetneq \mathcal{D}_{\leq v_6} \rangle.$$

Each $\mathcal{D}_{\leq v}$ has *associated graded* category $\mathrm{gr}_v(\mathcal{D}_\bullet)$ obtained by quotienting by categories "to the left"

- $\mathrm{gr}_{v_4}(\mathcal{D}_\bullet) = \mathcal{D}_{\leq v_4}$, $\mathrm{gr}_{v_5}(\mathcal{D}_\bullet) = \mathcal{D}_{\leq v_5}$
- $\mathrm{gr}_{v_6}(\mathcal{D}_{\leq v_6}) = \mathcal{D}_{\leq v_6} / \mathcal{D}_{\leq v_5}$, etc.

Augmented stability conditions

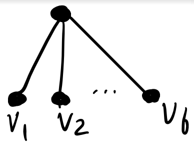
Definition (Augmented stability condition)

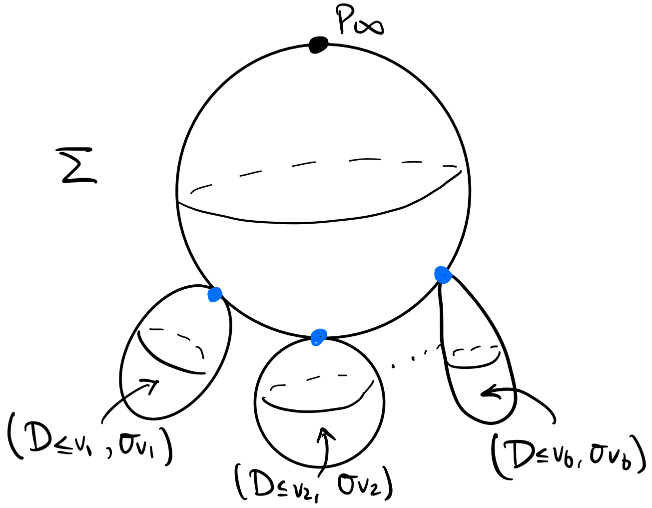
An *augmented stability condition* $\sigma = \langle \mathcal{D}_\bullet | \sigma_\bullet \rangle_\Sigma$ is a multiscale decomposition $\mathcal{D} = \langle \mathcal{D}_\bullet \rangle_\Sigma$ such that $\mathrm{gr}_v(\mathcal{D}_\bullet)$ is equipped with $\sigma_v \in \mathrm{Stab}(\mathrm{gr}_v(\mathcal{D}_\bullet))/\mathbf{C}$ for each $v \in V(\Sigma)_{\mathrm{bot}}$.

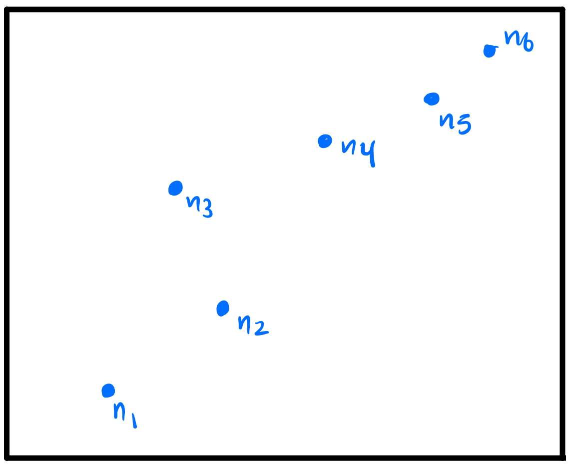
Augmented stability conditions

Definition (Augmented stability condition)

An *augmented stability condition* $\sigma = \langle \mathcal{D}_\bullet | \sigma_\bullet \rangle_\Sigma$ is a multiscale decomposition $\mathcal{D} = \langle \mathcal{D}_\bullet \rangle_\Sigma$ such that $\mathrm{gr}_v(\mathcal{D}_\bullet)$ is equipped with $\sigma_v \in \mathrm{Stab}(\mathrm{gr}_v(\mathcal{D}_\bullet))/\mathbf{C}$ for each $v \in V(\Sigma)_{\mathrm{bot}}$.

Here is a generic – $\forall v, w,\ \mathfrak{p}(v,w) \notin \mathbf{R}$ – example when $\Gamma(\Sigma) =$ 



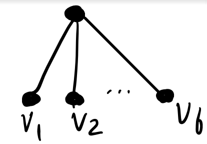


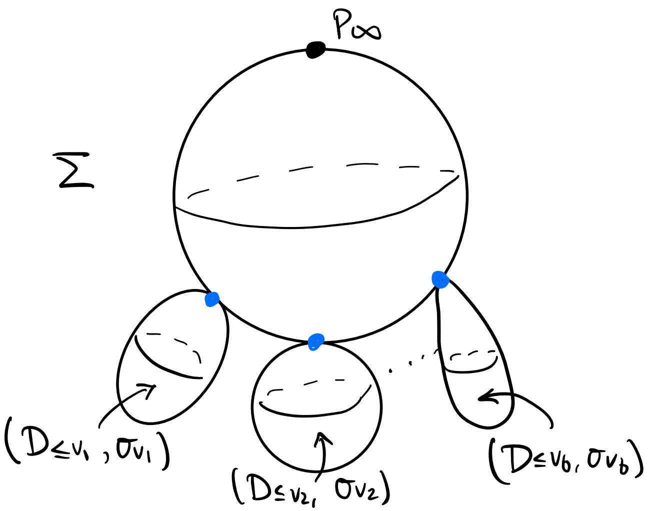
Augmented stability conditions

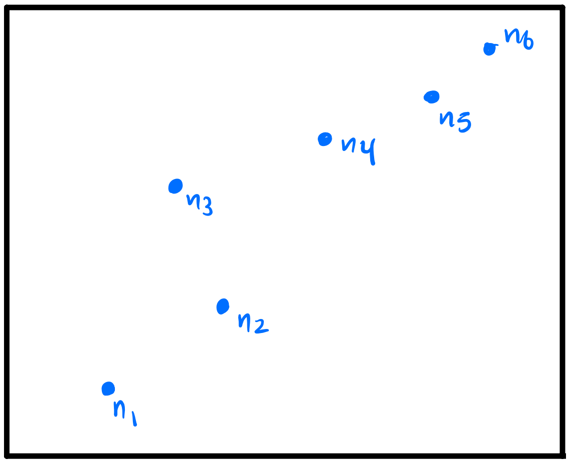
Definition (Augmented stability condition)

An *augmented stability condition* $\sigma = \langle \mathcal{D}_\bullet | \sigma_\bullet \rangle_\Sigma$ is a multiscale decomposition $\mathcal{D} = \langle \mathcal{D}_\bullet \rangle_\Sigma$ such that $\mathrm{gr}_v(\mathcal{D}_\bullet)$ is equipped with $\sigma_v \in \mathrm{Stab}(\mathrm{gr}_v(\mathcal{D}_\bullet))/\mathbf{C}$ for each $v \in V(\Sigma)_{\mathrm{bot}}$.

Here is a generic – $\forall v, w,\ \mathfrak{p}(v,w) \notin \mathbf{R}$ – example when $\Gamma(\Sigma) =$







$\mathcal{D} = \langle \mathcal{D}_{\le v_1}, \ldots, \mathcal{D}_{\le v_6} \rangle$

$+ \quad \sigma_i \in \mathrm{Stab}(\mathcal{D}_{\le v_i}')/\mathbb{C}$

$i = 1, \ldots, 6$

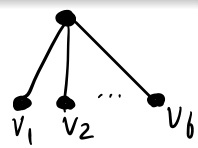
(polarised SOD + Σ)

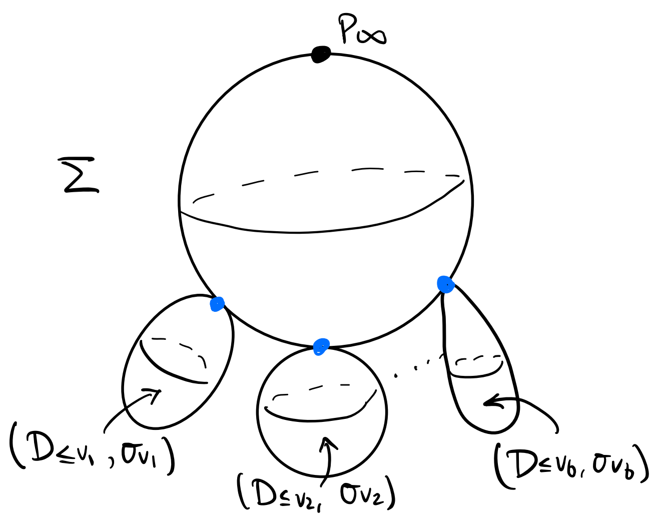
Augmented stability conditions

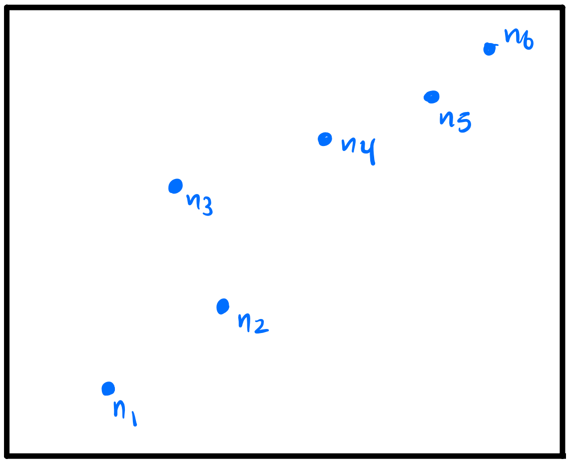
Definition (Augmented stability condition)

An *augmented stability condition* $\sigma = \langle \mathcal{D}_\bullet | \sigma_\bullet \rangle_\Sigma$ is a multiscale decomposition $\mathcal{D} = \langle \mathcal{D}_\bullet \rangle_\Sigma$ such that $\mathrm{gr}_v(\mathcal{D}_\bullet)$ is equipped with $\sigma_v \in \mathrm{Stab}(\mathrm{gr}_v(\mathcal{D}_\bullet))/\mathbf{C}$ for each $v \in V(\Sigma)_{\mathrm{bot}}$.

Here is a generic – $\forall v, w,\ \mathfrak{p}(v,w) \notin \mathbf{R}$ – example when $\Gamma(\Sigma) =$







$\mathcal{D} = \langle \mathcal{D}_{\le v_1}, \dots, \mathcal{D}_{\le v_6} \rangle$

$+\ \sigma_i \in \mathrm{Stab}(\mathcal{D}_{\le v_i})/\mathbb{C}$

$i = 1, \dots, 6$

(polarised SOD + Σ)

$\mathrm{Stab}(\mathcal{D})/\mathbf{C}$ is identified with the set of points in $\mathcal{A}\,\mathrm{Stab}(\mathcal{D})$ of the form

$$\langle \mathcal{D} \mid \sigma \in \mathrm{Stab}(\mathcal{D})/\mathbf{C} \rangle_{\mathbf{P}^1}.$$

Main Theorem

$\mathcal{A}\,\mathrm{Stab}(\mathcal{D}) :=$ set of augmented stability conditions.

Theorem (Halpern-Leistner, R.)

There is a Hausdorff topology on $\mathcal{A}\,\mathrm{Stab}(\mathcal{D})$ such that

1. $\mathrm{Stab}(\mathcal{D})/\mathbf{C}$ *is an open subspace;*
2. *around any generic "admissible" boundary point $\sigma \in \mathcal{A}\,\mathrm{Stab}(\mathcal{D})$, $\exists$ open U homeomorphic to an open nbhd of a boundary point in $\mathcal{A}_n^{\mathbf{R}}$;*
3. *generic quasi-convergent paths (with a few mild hypotheses) converge to their corresponding polarised SODs; and*
4. $\mathrm{Aut}(\mathcal{D})$*-action on* $\mathrm{Stab}(\mathcal{D})/\mathbf{C}$ *extends continuously to* $\mathcal{A}\,\mathrm{Stab}(\mathcal{D})$.

Main Theorem

$\mathcal{A}\,\mathrm{Stab}(\mathcal{D})$:= set of augmented stability conditions.

Theorem (Halpern-Leistner, R.)

There is a Hausdorff topology on $\mathcal{A}\,\mathrm{Stab}(\mathcal{D})$ *such that*

1. $\mathrm{Stab}(\mathcal{D})/\mathbf{C}$ *is an open subspace;*
2. *around any generic "admissible" boundary point* $\sigma \in \mathcal{A}\,\mathrm{Stab}(\mathcal{D})$, $\exists$ *open* U *homeomorphic to an open nbhd of a boundary point in* $\mathcal{A}_n^{\mathbf{R}}$*;*
3. *generic quasi-convergent paths (with a few mild hypotheses) converge to their corresponding polarised SODs; and*
4. $\mathrm{Aut}(\mathcal{D})$*-action on* $\mathrm{Stab}(\mathcal{D})/\mathbf{C}$ *extends continuously to* $\mathcal{A}\,\mathrm{Stab}(\mathcal{D})$.

Main Theorem

$\mathcal{A}\,\mathrm{Stab}(\mathcal{D}) :=$ set of augmented stability conditions.

Theorem (Halpern-Leistner, R.)

There is a Hausdorff topology on $\mathcal{A}\,\mathrm{Stab}(\mathcal{D})$ such that

1. $\mathrm{Stab}(\mathcal{D})/\mathbf{C}$ *is an open subspace;*
2. *around any generic "admissible" boundary point $\sigma \in \mathcal{A}\,\mathrm{Stab}(\mathcal{D})$, $\exists$ open U homeomorphic to an open nbhd of a boundary point in $\mathcal{A}_n^{\mathbf{R}}$;*
3. *generic quasi-convergent paths (with a few mild hypotheses) converge to their corresponding polarised SODs; and*
4. $\mathrm{Aut}(\mathcal{D})$*-action on* $\mathrm{Stab}(\mathcal{D})/\mathbf{C}$ *extends continuously to* $\mathcal{A}\,\mathrm{Stab}(\mathcal{D})$.

Main Theorem

$\mathcal{A}\,\mathrm{Stab}(\mathcal{D}) :=$ set of augmented stability conditions.

Theorem (Halpern-Leistner, R.)

There is a Hausdorff topology on $\mathcal{A}\,\mathrm{Stab}(\mathcal{D})$ such that

1. $\mathrm{Stab}(\mathcal{D})/\mathbf{C}$ *is an open subspace;*
2. *around any generic "admissible" boundary point $\sigma \in \mathcal{A}\,\mathrm{Stab}(\mathcal{D})$, $\exists$ open U homeomorphic to an open nbhd of a boundary point in $\mathcal{A}_n^{\mathbf{R}}$;*
3. *generic quasi-convergent paths (with a few mild hypotheses) converge to their corresponding polarised SODs; and*
4. $\mathrm{Aut}(\mathcal{D})$*-action on* $\mathrm{Stab}(\mathcal{D})/\mathbf{C}$ *extends continuously to* $\mathcal{A}\,\mathrm{Stab}(\mathcal{D})$.

Main Theorem

$\mathcal{A}\,\mathrm{Stab}(\mathcal{D}) :=$ set of augmented stability conditions.

Theorem (Halpern-Leistner, R.)

There is a Hausdorff topology on $\mathcal{A}\,\mathrm{Stab}(\mathcal{D})$ such that

1. $\mathrm{Stab}(\mathcal{D})/\mathbf{C}$ *is an open subspace;*
2. *around any generic "admissible" boundary point $\sigma \in \mathcal{A}\,\mathrm{Stab}(\mathcal{D})$, $\exists$ open U homeomorphic to an open nbhd of a boundary point in $\mathcal{A}_n^{\mathbf{R}}$;*
3. *generic quasi-convergent paths (with a few mild hypotheses) converge to their corresponding polarised SODs; and*
4. $\mathrm{Aut}(\mathcal{D})$*-action on* $\mathrm{Stab}(\mathcal{D})/\mathbf{C}$ *extends continuously to* $\mathcal{A}\,\mathrm{Stab}(\mathcal{D})$.

Main Theorem

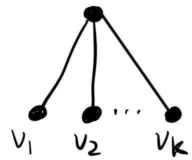
$\mathcal{A}\,\mathrm{Stab}(\mathcal{D})$:= set of augmented stability conditions.

Theorem (Halpern-Leistner, R.)

There is a Hausdorff topology on $\mathcal{A}\,\mathrm{Stab}(\mathcal{D})$ *such that*

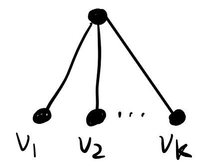
1. $\mathrm{Stab}(\mathcal{D})/\mathbf{C}$ *is an open subspace;*
2. *around any generic "admissible" boundary point* $\sigma \in \mathcal{A}\,\mathrm{Stab}(\mathcal{D})$, $\exists$ *open* U *homeomorphic to an open nbhd of a boundary point in* $\mathcal{A}_n^{\mathbf{R}}$*;*
3. *generic quasi-convergent paths (with a few mild hypotheses) converge to their corresponding polarised SODs; and*
4. $\mathrm{Aut}(\mathcal{D})$*-action on* $\mathrm{Stab}(\mathcal{D})/\mathbf{C}$ *extends continuously to* $\mathcal{A}\,\mathrm{Stab}(\mathcal{D})$.

Manifold with corners property

1. Take $\sigma = \langle \mathcal{D}_\bullet | \sigma_\bullet \rangle_\Sigma$ with Σ generic. For simplicity, let $\Gamma(\Sigma) =$ 

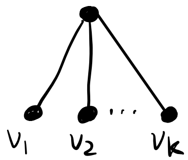
2.
3.
4.

Manifold with corners property

1. Take $\sigma = \langle \mathcal{D}_\bullet | \sigma_\bullet \rangle_\Sigma$ with Σ generic. For simplicity, let $\Gamma(\Sigma) =$ 

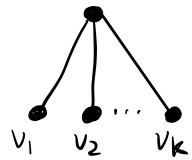
2. $\mathcal{D} = \langle \mathcal{D}_{\leq v_1}, \ldots, \mathcal{D}_{\leq v_k} \rangle \rightsquigarrow H^*(\mathcal{D}) = \bigoplus_{i=1}^k H^*(\mathcal{D}_{\leq v_i})$.
3.
4.

Manifold with corners property

1. Take $\sigma = \langle \mathcal{D}_\bullet | \sigma_\bullet \rangle_\Sigma$ with Σ generic. For simplicity, let $\Gamma(\Sigma) =$ 

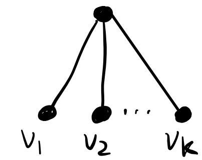
2. $\mathcal{D} = \langle \mathcal{D}_{\leq v_1}, \ldots, \mathcal{D}_{\leq v_k} \rangle \rightsquigarrow H^*(\mathcal{D}) = \bigoplus_{i=1}^k H^*(\mathcal{D}_{\leq v_i})$.
3. $\forall\, i = 1, \ldots, k$ choose σ_{v_i}-stable objects $E_\bullet^{(i)} \subseteq \mathrm{Ob}(\mathcal{D}_{\leq v_i})$ such that $E_\bullet^{(i)}$ is basis of $H^*(\mathcal{D}_{\leq v_i})_{\mathbf{Q}}$.
4.

Manifold with corners property

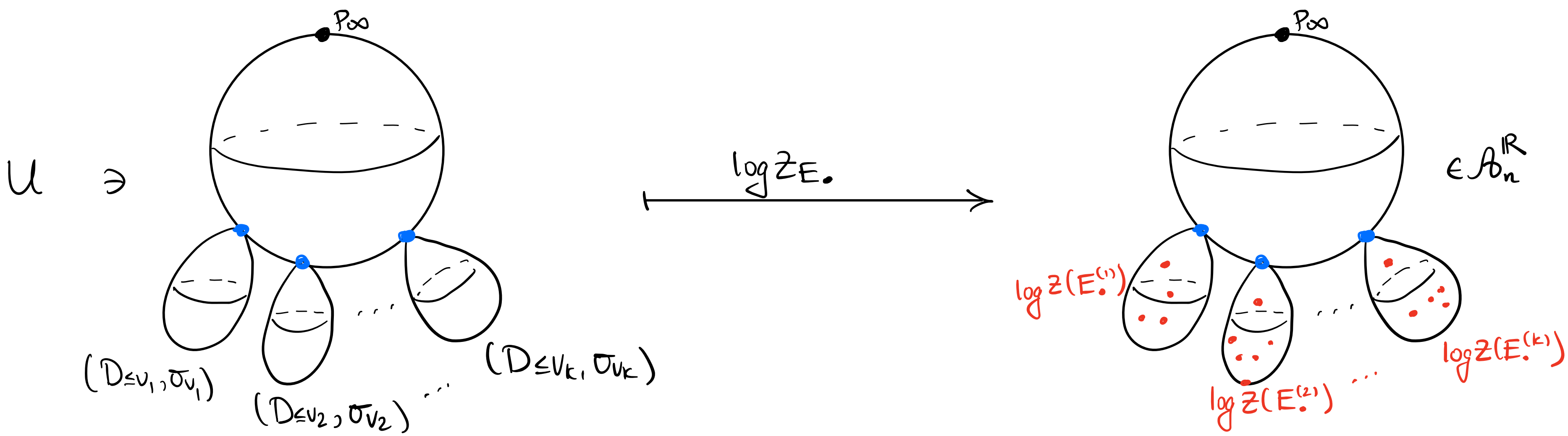
1. Take $\sigma = \langle \mathcal{D}_\bullet | \sigma_\bullet \rangle_\Sigma$ with Σ generic. For simplicity, let $\Gamma(\Sigma) =$ 

2. $\mathcal{D} = \langle \mathcal{D}_{\leq v_1}, \ldots, \mathcal{D}_{\leq v_k} \rangle \rightsquigarrow H^*(\mathcal{D}) = \bigoplus_{i=1}^k H^*(\mathcal{D}_{\leq v_i})$.
3. $\forall\, i = 1, \ldots, k$ choose σ_{v_i}-stable objects $E_\bullet^{(i)} \subseteq \mathrm{Ob}(\mathcal{D}_{\leq v_i})$ such that $E_\bullet^{(i)}$ is basis of $H^*(\mathcal{D}_{\leq v_i})_{\mathbf{Q}}$.
4. $\exists\, U \ni \sigma$ on which $E_\bullet^{(i)}$ are all stable and the map $\log Z_{E_\bullet} : U \to \mathcal{A}_n^{\mathbf{R}}$ is a local homeo.

Manifold with corners property

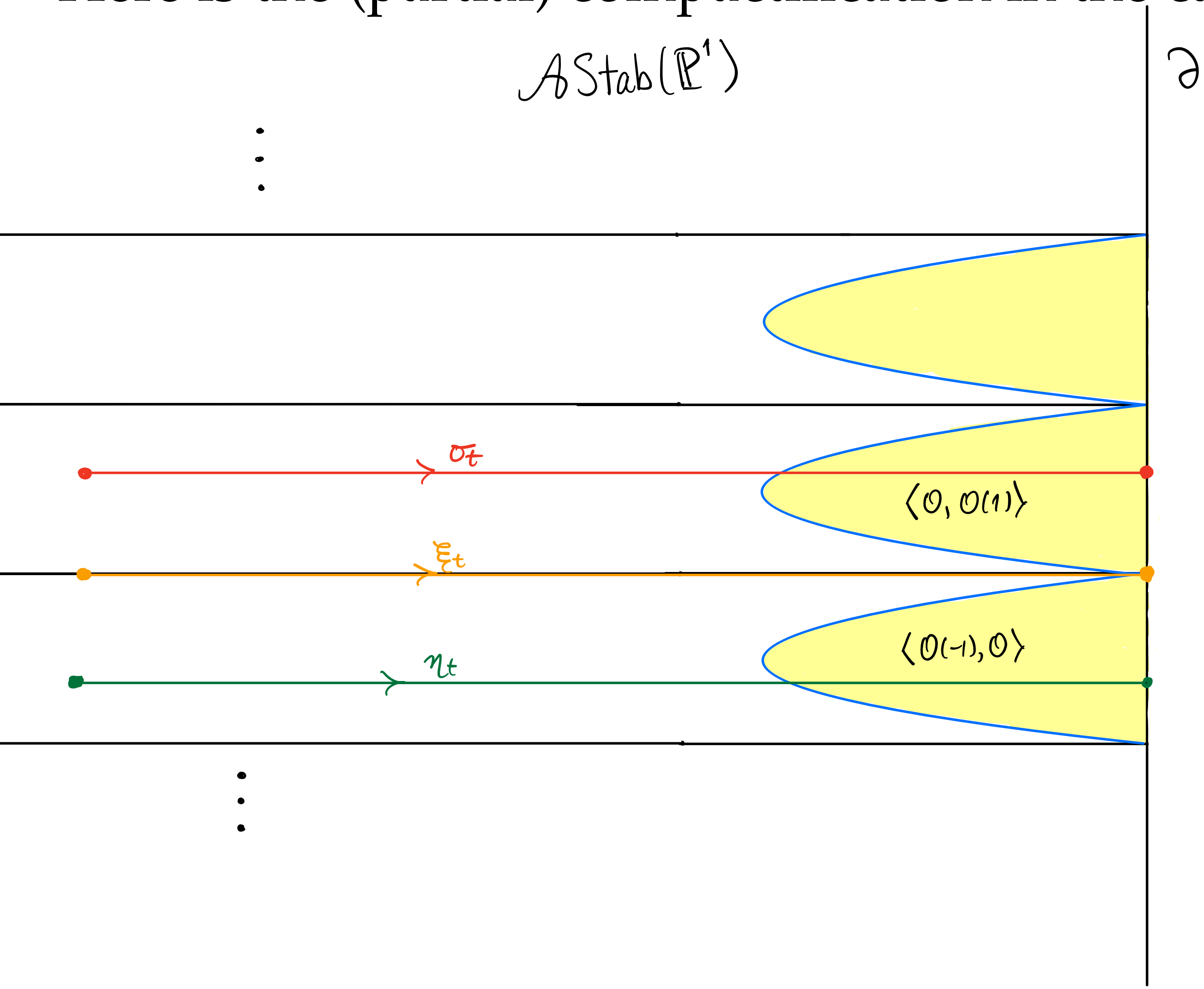
1. Take $\sigma = \langle \mathcal{D}_\bullet | \sigma_\bullet \rangle_\Sigma$ with Σ generic. For simplicity, let $\Gamma(\Sigma) =$ 

2. $\mathcal{D} = \langle \mathcal{D}_{\leq v_1}, \ldots, \mathcal{D}_{\leq v_k} \rangle \rightsquigarrow H^*(\mathcal{D}) = \bigoplus_{i=1}^k H^*(\mathcal{D}_{\leq v_i})$.
3. $\forall\, i = 1, \ldots, k$ choose σ_{v_i}-stable objects $E_\bullet^{(i)} \subseteq \mathrm{Ob}(\mathcal{D}_{\leq v_i})$ such that $E_\bullet^{(i)}$ is basis of $H^*(\mathcal{D}_{\leq v_i})_{\mathbf{Q}}$.
4. $\exists\, U \ni \sigma$ on which $E_\bullet^{(i)}$ are all stable and the map $\log Z_{E_\bullet} : U \to \mathcal{A}_n^{\mathbf{R}}$ is a local homeo.



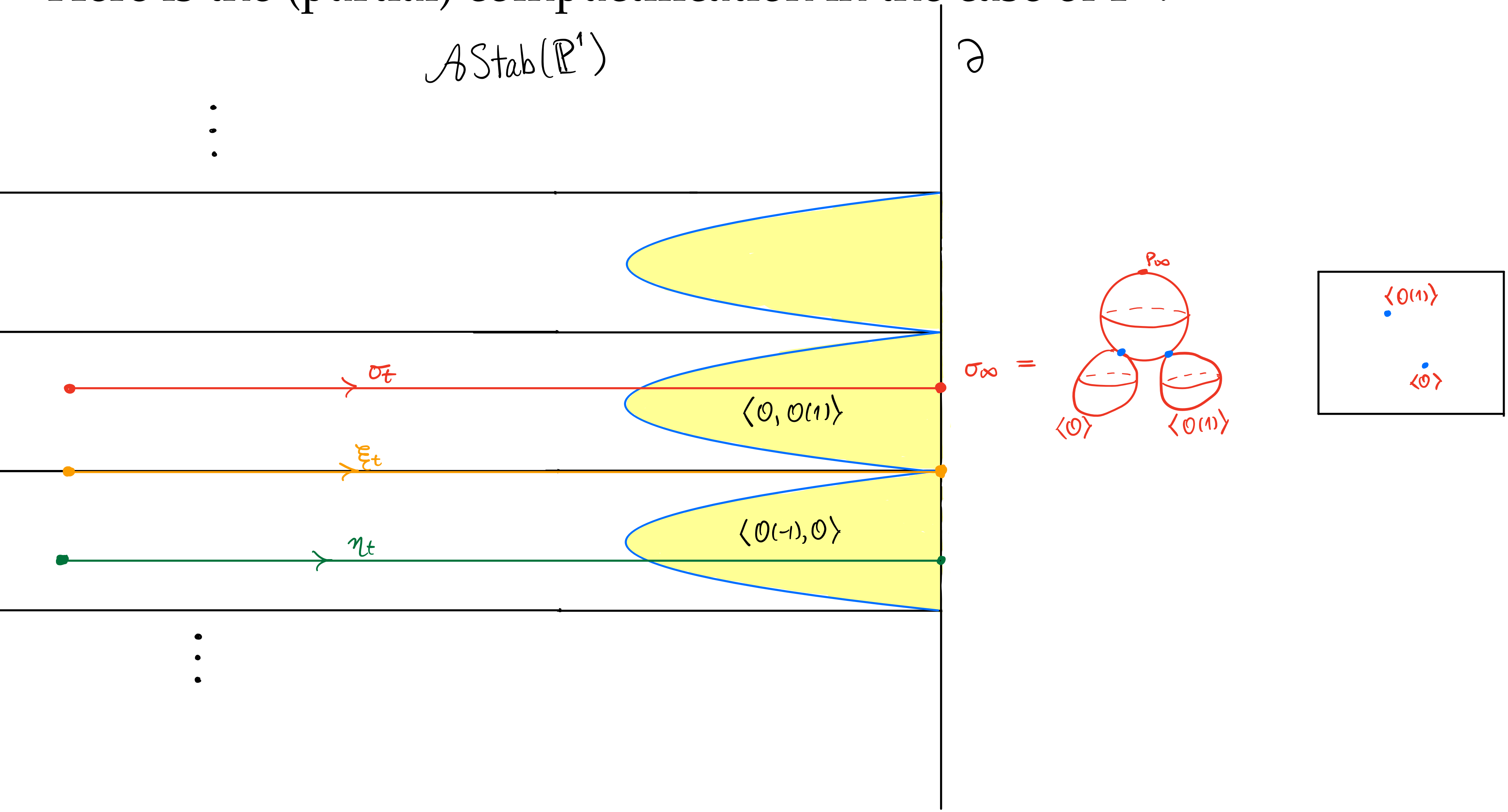
$\mathbf{P}^1$ revisited

Here is the (partial) compactification in the case of $\mathbf{P}^1$.



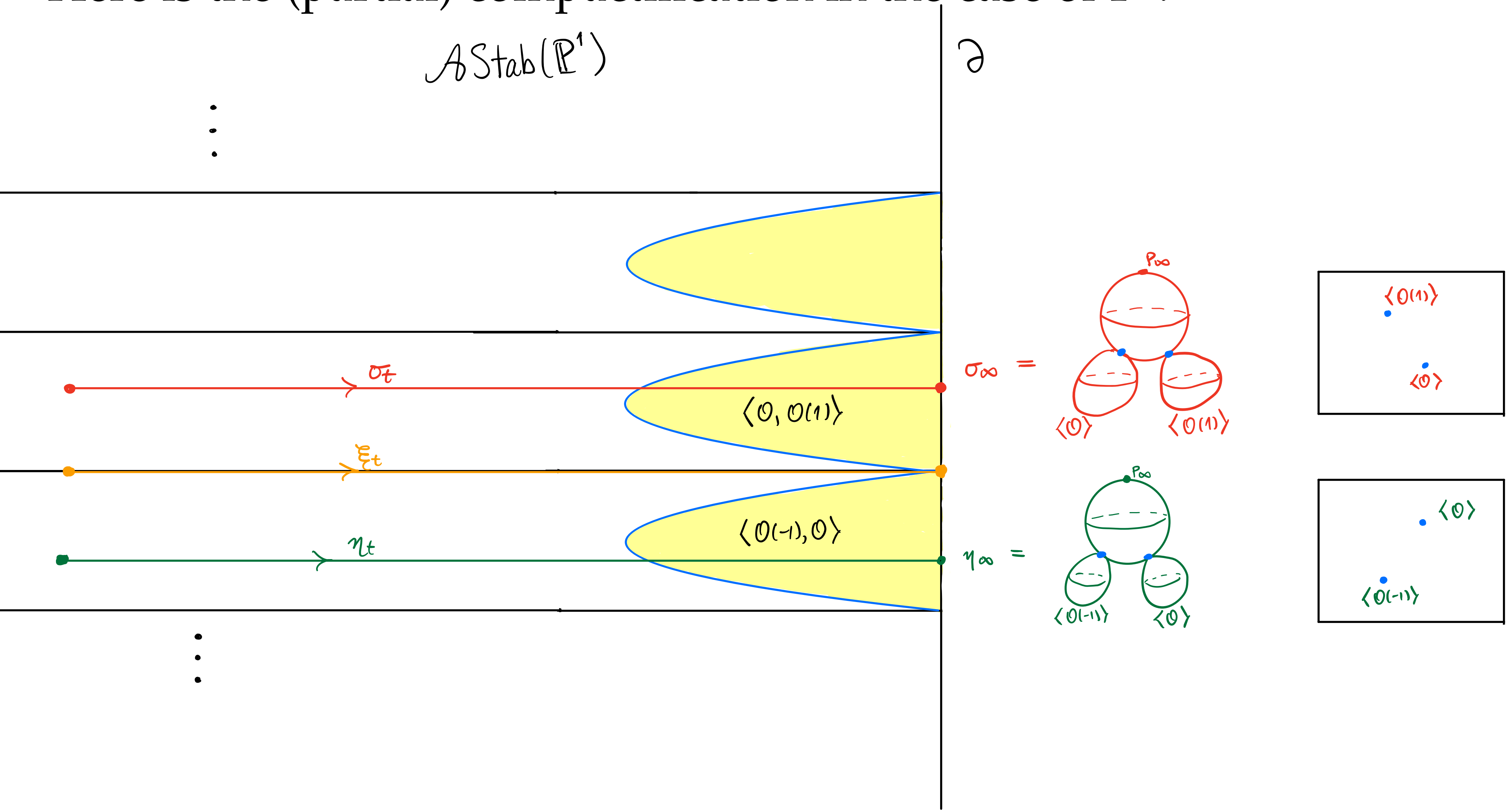
$\mathbf{P}^1$ revisited

Here is the (partial) compactification in the case of $\mathbf{P}^1$.

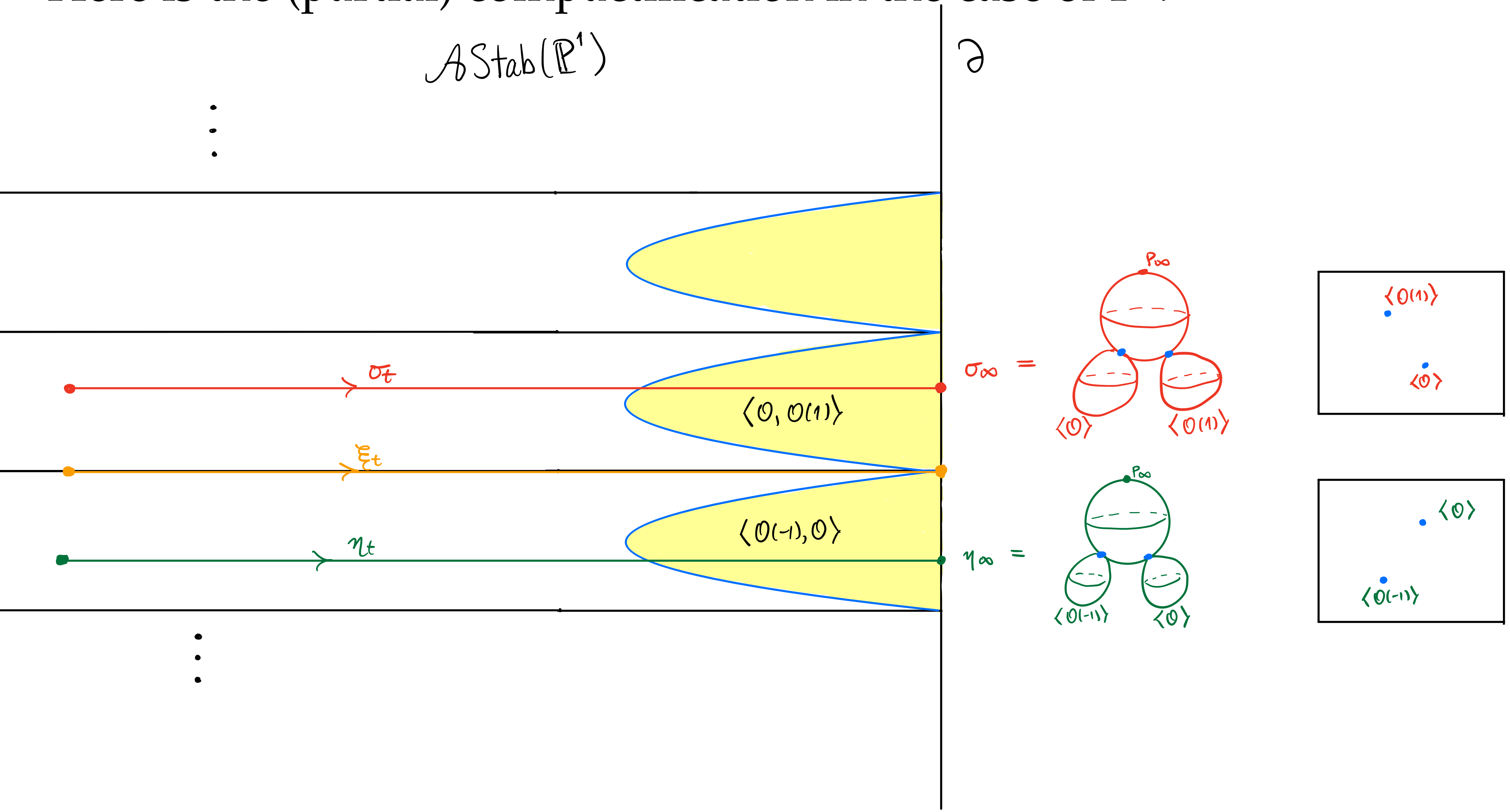


$\mathbf{P}^1$ revisited

Here is the (partial) compactification in the case of $\mathbf{P}^1$.



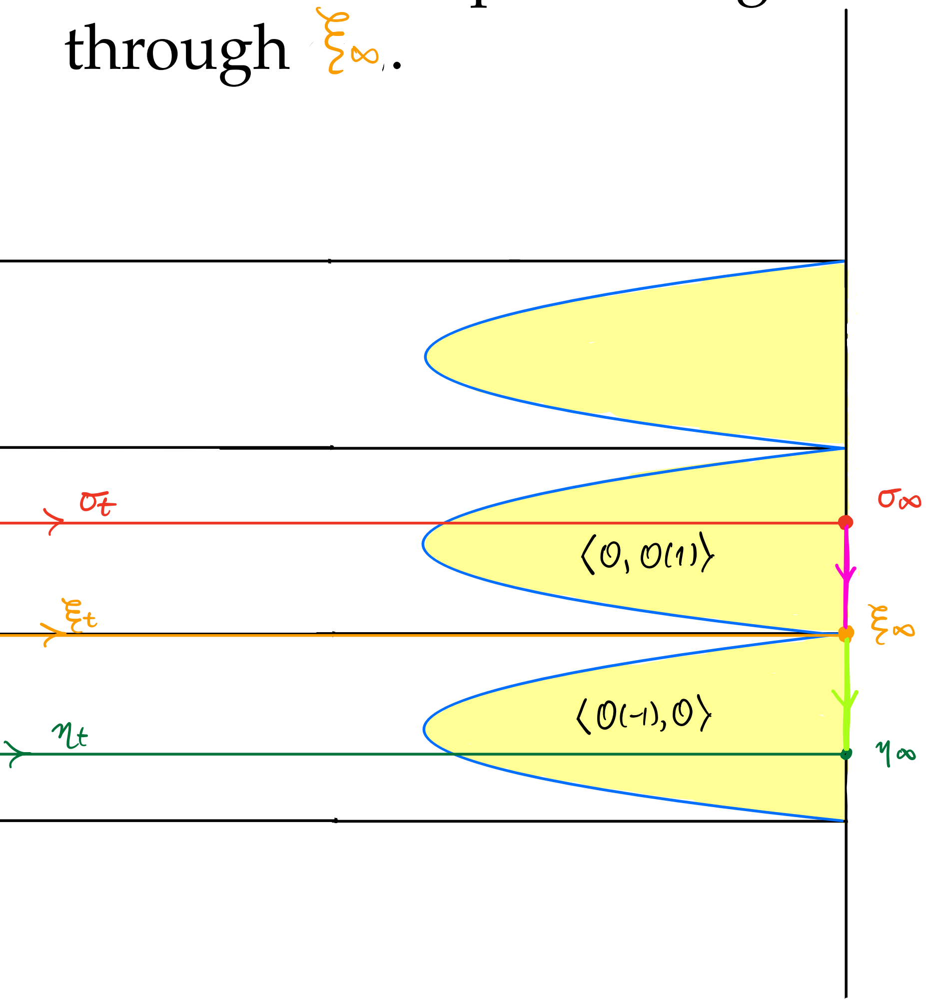
Here is the (partial) compactification in the case of $\mathbf{P}^1$.

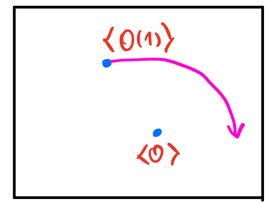


Question: what is the limiting point of ξ_t as $t \to \infty$?

Path along the boundary

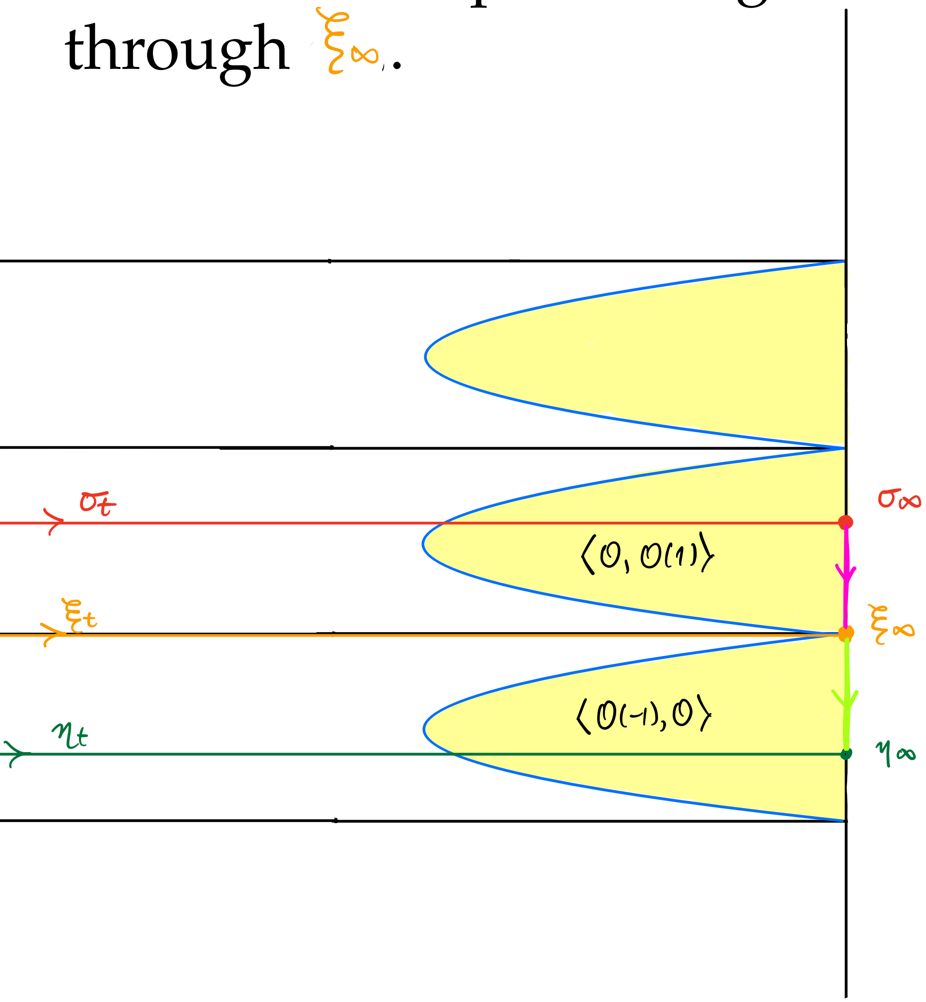
We consider a path along the boundary from σ_∞ to η_∞, which passes through ξ_∞.

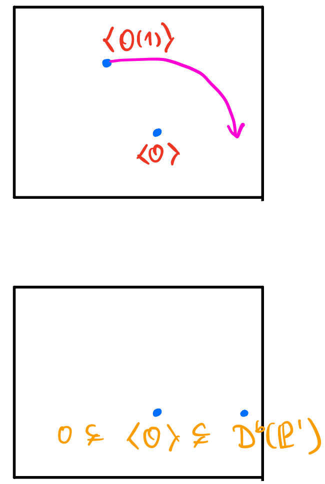




Path along the boundary

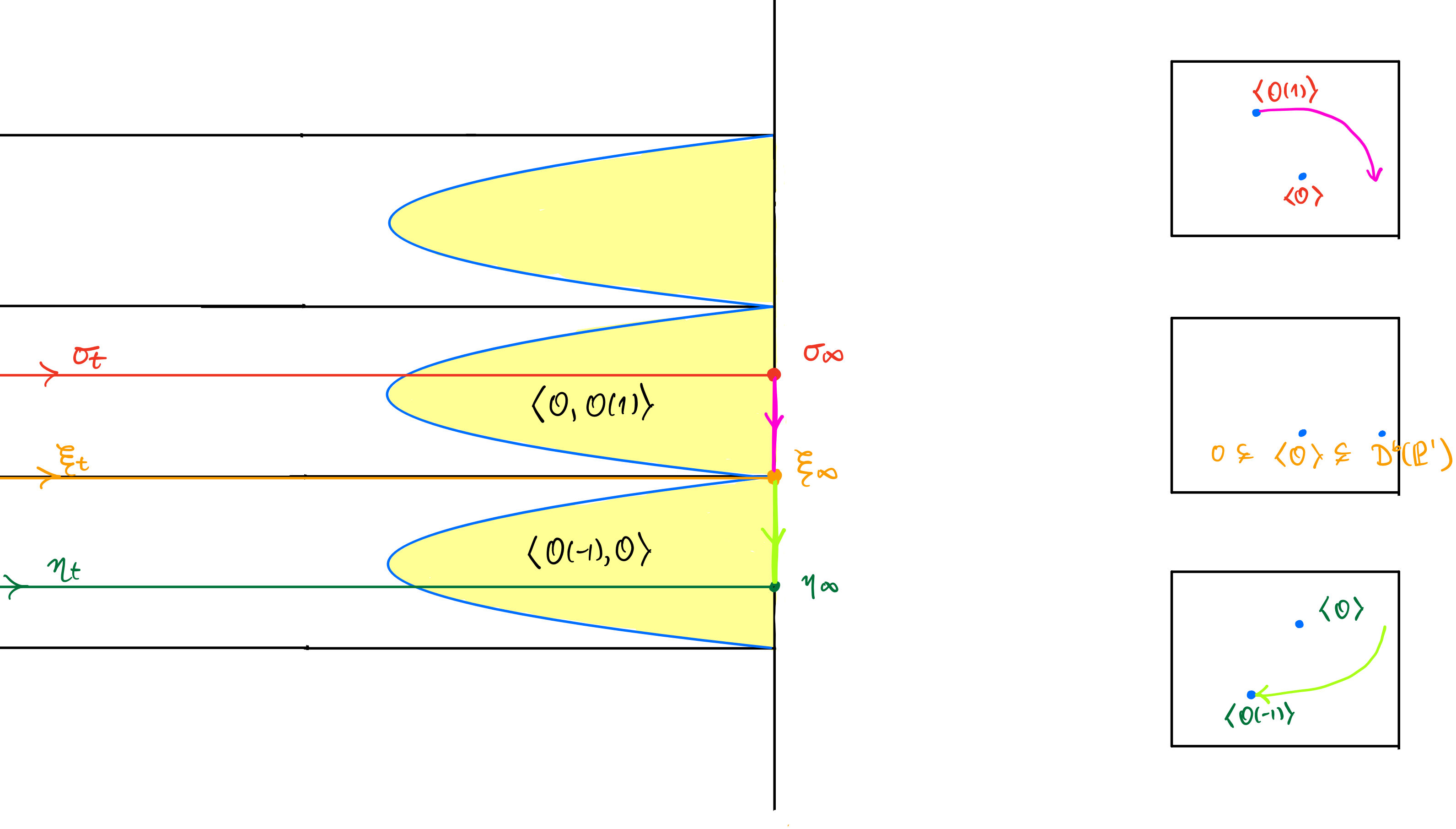
We consider a path along the boundary from σ_∞ to η_∞, which passes through ξ_∞.





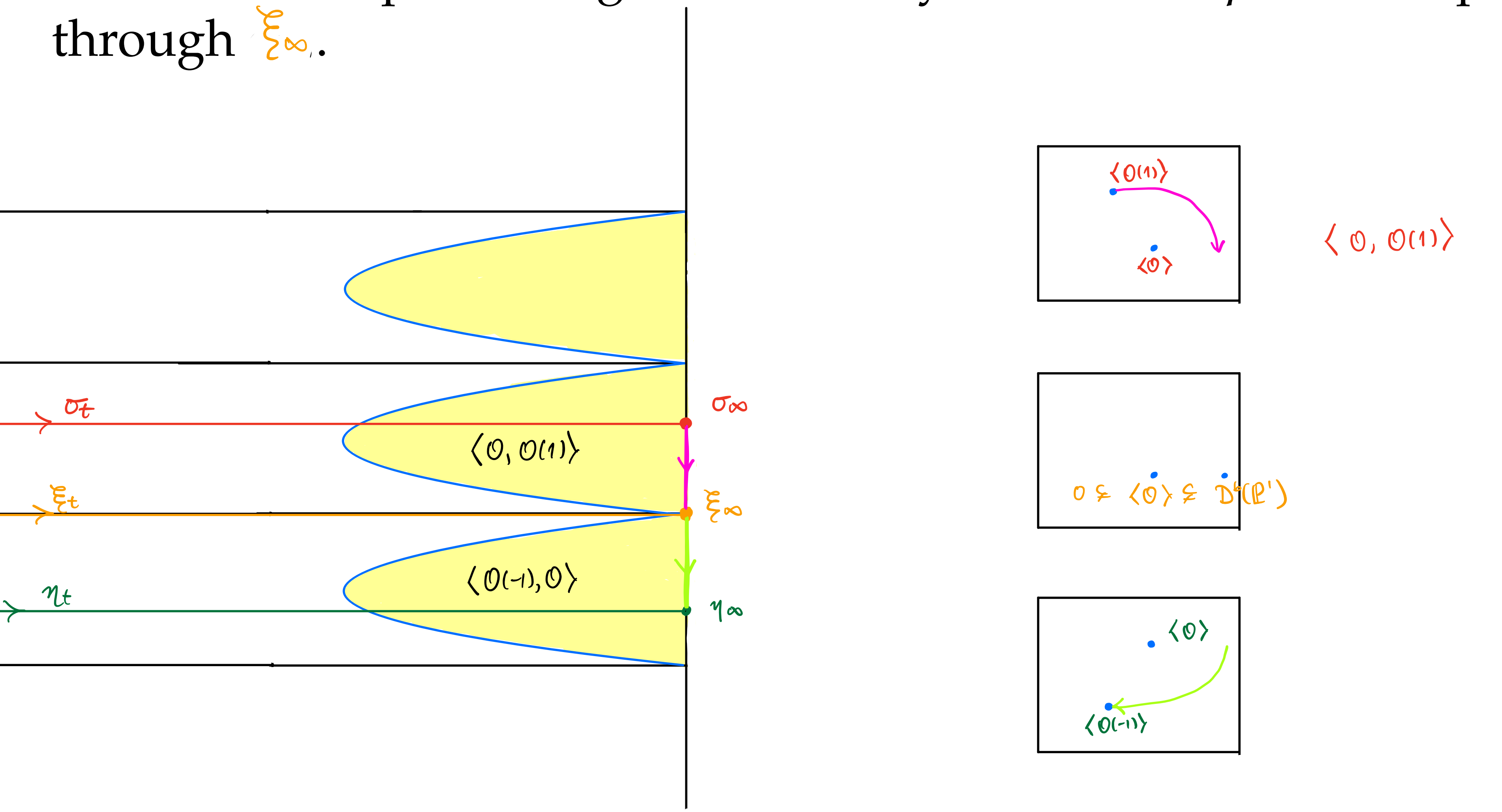
Path along the boundary

We consider a path along the boundary from σ_∞ to η_∞, which passes through ξ_∞.



Path along the boundary

We consider a path along the boundary from σ_∞ to η_∞, which passes through ξ_∞.

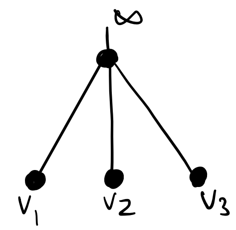


The boundary point $\lim_{t\to\infty} \xi_t$ is a *degenerate* semiorthogonal decomposition, i.e. an admissible *filtration* $0 \subsetneq \langle \mathcal{O} \rangle \subsetneq D^b_{\mathrm{coh}}(\mathbf{P}^1)$.

1. In $\mathbf{P}^1$ ex., moving along boundary mutates the SOD. This is a general feature.
2.
3.
4.
5.
6.

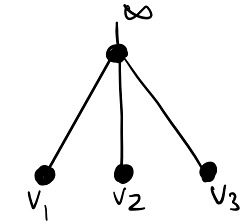
1. In $\mathbf{P}^1$ ex., moving along boundary mutates the SOD. This is a general feature.
2. Pirozhkov '20 classifies SODs of $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathbf{P}^2)$: coarsenings of mutations of $\langle \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2) \rangle$.
3.
4.
5.
6.

1. In $\mathbf{P}^1$ ex., moving along boundary mutates the SOD. This is a general feature.
2. Pirozhkov '20 classifies SODs of $\mathrm{D}^b_{\mathrm{coh}}(\mathbf{P}^2)$: coarsenings of mutations of $\langle \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2) \rangle$.
3. $\mathcal{A}\,\mathrm{Stab}(\mathbf{P}^2)_\Gamma$ is the stratum such that $\Gamma(\Sigma) =$



4.
5.
6.

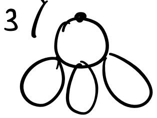
1. In $\mathbf{P}^1$ ex., moving along boundary mutates the SOD. This is a general feature.
2. Pirozhkov '20 classifies SODs of $D^b_{coh}(\mathbf{P}^2)$: coarsenings of mutations of $\langle \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2)\rangle$.
3. $\mathcal{A}\,\mathrm{Stab}(\mathbf{P}^2)_\Gamma$ is the stratum such that $\Gamma(\Sigma) =$

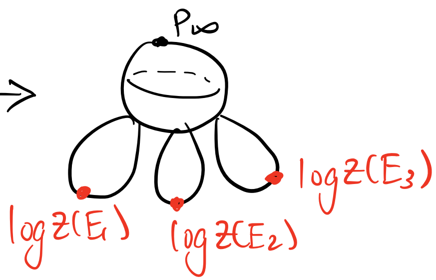


4. The map

$$\mathcal{A}\mathrm{Stab}(\mathbb{P}^2)_\Gamma \longrightarrow S_\Gamma \subseteq \mathcal{A}_3^{\mathbb{R}}$$

$$\langle E_1, E_2, E_3 \mid \sigma_1, \sigma_2, \sigma_3 \rangle \longmapsto$$

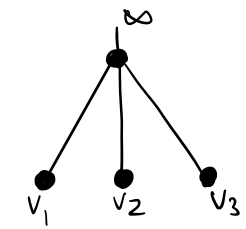




is a conn. cover

5.
6.

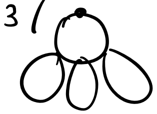
1. In $\mathbf{P}^1$ ex., moving along boundary mutates the SOD. This is a general feature.
2. Pirozhkov '20 classifies SODs of $\mathrm{D}^b_{\mathrm{coh}}(\mathbf{P}^2)$: coarsenings of mutations of $\langle \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2)\rangle$.
3. $\mathcal{A}\,\mathrm{Stab}(\mathbf{P}^2)_\Gamma$ is the stratum such that $\Gamma(\Sigma) =$

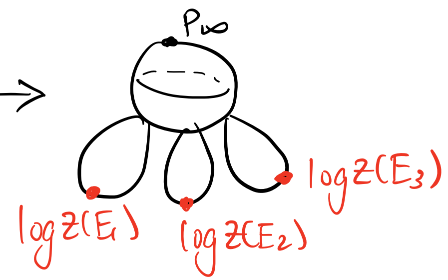


4. The map

$$\mathcal{A}\mathrm{Stab}(\mathbb{P}^2)_\Gamma \longrightarrow S_\Gamma \subseteq \mathcal{A}_3^{\mathbb{R}}$$

$$\langle E_1, E_2, E_3 \mid \sigma_1, \sigma_2, \sigma_3 \rangle \longmapsto$$

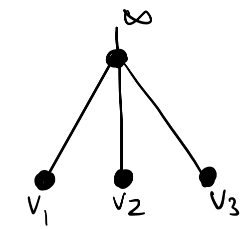




is a conn. cover

5. $S_\Gamma \simeq \mathrm{Conf}_3(\mathbf{C})$ and $\mathcal{A}\,\mathrm{Stab}(\mathbf{P}^2)_\Gamma \to S_\Gamma$ is univ. cover
6.

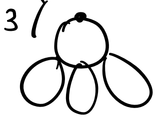
1. In $\mathbf{P}^1$ ex., moving along boundary mutates the SOD. This is a general feature.
2. Pirozhkov '20 classifies SODs of $\mathrm{D}^b_{\mathrm{coh}}(\mathbf{P}^2)$: coarsenings of mutations of $\langle \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2)\rangle$.
3. $\mathcal{A}\,\mathrm{Stab}(\mathbf{P}^2)_\Gamma$ is the stratum such that $\Gamma(\Sigma) =$

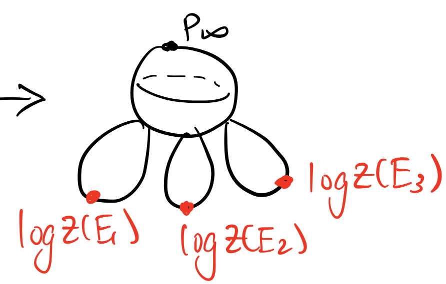


4. The map

$$\mathcal{A}\mathrm{Stab}(\mathbb{P}^2)_\Gamma \longrightarrow S_\Gamma \subseteq \mathcal{A}_3^{\mathbb{R}}$$

$$\langle E_1, E_2, E_3 \mid \sigma_1, \sigma_2, \sigma_3 \rangle \longmapsto$$





is a conn. cover

5. $S_\Gamma \simeq \mathrm{Conf}_3(\mathbf{C})$ and $\mathcal{A}\,\mathrm{Stab}(\mathbf{P}^2)_\Gamma \to S_\Gamma$ is univ. cover
6. Given $\gamma \in \pi_1(S_\Gamma) \longleftrightarrow b \in \mathfrak{B}_3$,

$$\gamma \cdot \langle \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2)\rangle_\Sigma = \langle E_1, E_2, E_3\rangle_\Sigma$$

where E_1, E_2, E_3 is obtained by mutation along b.

Perspective

Proposition (Informal)

Connected components of strata in $\partial\mathcal{A}\,\mathrm{Stab}$ correspond to equivalence classes of SODs up to mutation.

This gives us a revised:

Heuristic

Given $\sigma_0, \tau_0 \in \mathrm{Stab}(X)/\mathbf{C}$ and corresponding paths σ_t and τ_t, one hopes σ_t and τ_t converge to points in the same connected component of $\partial\mathcal{A}\,\mathrm{Stab}(X)$, giving a canonical mutation class of SOD for $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$.

Perspective

Proposition (Informal)

Connected components of strata in $\partial\mathcal{A}\,\mathrm{Stab}$ correspond to equivalence classes of SODs up to mutation.

This gives us a revised:

Heuristic

Given $\sigma_0, \tau_0 \in \mathrm{Stab}(X)/\mathbf{C}$ and corresponding paths σ_t and τ_t, one hopes σ_t and τ_t converge to points in the same connected component of $\partial\mathcal{A}\,\mathrm{Stab}(X)$, giving a canonical mutation class of SOD for $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$.

Perspective

Proposition (Informal)

Connected components of strata in $\partial\mathcal{A}\,\mathrm{Stab}$ correspond to equivalence classes of SODs up to mutation.

This gives us a revised:

Heuristic

Given $\sigma_0, \tau_0 \in \mathrm{Stab}(X)/\mathbf{C}$ and corresponding paths σ_t and τ_t, one hopes σ_t and τ_t converge to points in the same connected component of $\partial\mathcal{A}\,\mathrm{Stab}(X)$, giving a canonical mutation class of SOD for $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$.

Thank you!

Thank you for listening!