Serre Chapter 1

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Generalities on linear representations

1 Definitions

Let V be a vector space over the field $\mathbb{C}$ and let $GL(V)$ be the group of isomorphisms of V onto itself. By definition every element of $GL(V)$ is an invertible linear transformation whose inverse is also a linear transformation. When V has a finite basis (e_i) of n elements, each linear map $a : V \to V$ can be defined by a square matrix (a_{ij}) of order n. Its coefficients are the unique complex numbers such that:

$$a(e_j) = \sum_i a_{ij} e_i$$

Definition 1. *Suppose G is a finite group with identity element 1, and composition $(s, t) \mapsto st$. A linear representation of G in V is a homomorphism ρ from G to $GL(V)$. So we associate each element $s \in G$ with some $\rho(s) \in GL(V)$ (often instead ρ_s) such that the following holds:*

$$\rho(st) = \rho(s) \cdot \rho(t) \textit{ for } s, t \in G$$

When ρ is given we say that V is a representation space of G, or simply a representation of G.

Moving forward we restrict our attention to V of finite dimension. And in this case we say that $n := \dim V$ is the degree of the representation. Let (e_i) be a basis of V and let R_s be the matrix of ρ_s with respect to this basis. We have:

$$\det R_s \neq 0, \qquad R_{st} = R_s \cdot R_t, \qquad \text{if } s, t \in G$$

If we let $r_{ij}(s)$ be the coefficients of the matrix R_s then the homomorphism condition becomes:

$$r_{ik}(s) = \sum_j r_{ij}(s) \cdot r_{jk}(s)$$

So the homomorphism ρ may be identified using linear maps or their matrix representations in a given basis that satisfy this condition.

Definition 2. *Let ρ, ρ' be representations of the same group G in vector spaces V, V'. These representations are said to be similar (or isomorphic) if there exists a linear isomorphism $\tau : V \to V'$ such that:*

$$\tau \circ \rho(s) = \rho'(s) \circ \tau \qquad \textit{for all } s \in G$$

In matrix form this corresponds to an invertible matrix T such that:

$$T \cdot R_s = R'_s \cdot T \qquad \textit{for all } s \in G$$

which is also written $R'_s = T \cdot R_s \cdot T^{-1}$.

In some sense this is a way of identifying two representations as we may relabel each element of $x \in V$ by $\tau(x) \in V'$ and preserve relationships between the different maps/matrices. Note that in particular this implies that ρ and ρ' have the same degree.

2 Basic Examples

(a) A representation of degree 1 of a group G is a homomorphism $\rho : G \to C^*$ where C^* is the multiplicative group of the nonzero complex numbers. Since each element in G has finite order, the values $\rho(s)$ of ρ are roots of unity. In particular, if we take $\rho(s) = 1$ for all $s \in G$ then we call this the unit (or trivial) representation.

(b) Let g be the order of G, and let V be a vector space of dimension g, with a basis $(e_t)_{t \in G}$ indexed by $t \in G$. For $s \in G$, let ρ_s be the linear map of $V \to V$ that sends e_s to e_{st}. This defines a linear representation called the regular representation of G. Its degree is the order of G. Note that $e_s = \rho_s(e_1)$ so the images of e_1 form a basis of V. If we have a representation W of G containing a vector w such that $\rho_s(w)$, $s \in G$ forms a basis of W then W is isomorphic to the regular representation with isomorphism $e_s \mapsto \rho_s(w)$.

(c) Now if G acts on a finite set X then for each $s \in G$ there is a permutation of $x \mapsto sx$ of X satistfying:

$$1x = x, \quad s(tx) = (st)x \quad \text{if } s, t \in G,\, x \in X$$

Now let V be a vector space with a basis $(e_x)_{x \in X}$. For $s \in G$ let ρ_s be the linear map of V into V. which sends $e_x \to e_{sx}$. This linear representation of G is called the permutation representation associated with X.

3 Subrepresentations

Given a linear representation $\rho : G \to GL(V)$, and a subspace W of V such that W is G-invariant (or stable under G). I.e. $x \in W$ implies $\rho_s x \in W$, for all $s \in G$. The restriction ρ_s^W of ρ_s to W is then still an automorphism of W with $\rho_{st}^W = \rho_s^W \cdot \rho_t^W$. Thus $\rho^W : G \to GL(V)$ is a linear representation of G in W known as a subrepresentation of V.

Now we proceed with a worked example. Take for V the regular representation of G and let W be the 1 dimensional subspace spanned by $x = \sum_{s \in G} e_s$. Since $\rho_s : e_t \to e_s t$ and $sG = G$ since every element in G can just be written as $s(s^{-1}g)$. Thus, $\rho_s(x) = x$ for all $s \in G$. This means W is a subrepresentation of V isomorphic to the unit representation.

Theorem 1. *Let $\rho : G \to GL(V)$ be a linear representation of G in V and let W be a vector subspace of V stable under G. Then there exists a complement W^0 of W in V which is stable under G.*