2.13 a.

$$\begin{array}{ccc} c_1 & c_2 & c_3 \\ \end{array}$$
$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & -2 & -1 \\ 2 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix} \begin{matrix} \\ -R_1 \\ -2R_1 \\ -R_1 \end{matrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & -2 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \\ \cdot -\frac{1}{2} \\ +\frac{1}{2}R_2 \\ \\ \end{matrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\text{rank}(A_1) = 2$

$\text{nullity}(A_1) = 3 - \text{rank}(A_1) = 3 - 2 = 1$ (rank-nullity theorem)

$\Rightarrow \dim(U_1) = 1$

$$\begin{array}{ccc} c_4 & c_5 & c_6 \\ \end{array}$$
$$\begin{bmatrix} 3 & -3 & 0 \\ 1 & 2 & 3 \\ 7 & -5 & 2 \\ 3 & -1 & 2 \end{bmatrix} \begin{matrix} \cdot\frac{1}{3} \\ \\ \\ \\ \end{matrix} \rightsquigarrow \begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & 3 \\ 7 & -5 & 2 \\ 3 & -1 & 2 \end{bmatrix} \begin{matrix} \\ -R_1 \\ -7R_1 \\ -3R_1 \end{matrix} \rightsquigarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 3 & 3 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{bmatrix} \begin{matrix} \\ \cdot\frac{1}{3} \\ -\frac{2}{3}R_2 \\ -\frac{2}{3}R_2 \end{matrix} \rightsquigarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\text{rank}(A_2) = 2$

$\text{nullity}(A_2) = 3 - \text{rank}(A_2) = 3 - 2 = 1$ (rank-nullity theorem)

$\Rightarrow \dim(U_2) = 1$

b.

U_1 basis: $\{ [1, 1, 2, 1]^T, [0, -2, 1, 0]^T \}$

U_2 basis: $\{ [3, 1, 7, 3]^T, [-3, 2, -5, -1]^T \}$

c.

$$\begin{array}{cccc} c_1 & c_2 & c_4 & c_5 \\ \end{array}$$
$$\begin{bmatrix} 1 & 0 & 3 & -3 \\ 1 & -2 & 1 & 2 \\ 2 & 1 & 7 & -5 \\ 1 & 0 & 3 & -1 \end{bmatrix} \begin{matrix} \\ -R_1 \\ -2R_1 \\ -R_1 \end{matrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 3 & -3 \\ 0 & -2 & -2 & 5 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{matrix} \\ \text{swap with } R_3 \\ \\ \\ \end{matrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 3 & -3 \\ 0 & 1 & 1 & 1 \\ 0 & -2 & -2 & 5 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{matrix} \\ \\ +2R_2 \\ \\ \end{matrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & 0 & 3 & -3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{matrix} \\ \\ \cdot\frac{1}{7} \\ -\frac{2}{7}R_3 \end{matrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 3 & -3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\dim(U_1 + U_2) = 3$

$\dim(U_1 \wedge U_2) = \dim(U_1) + \dim(U_2) - \dim(U_1 + U_2) = 2 + 2 - 3 = 1$

Let $\underline{w} \in (U_1 \wedge U_2)$

There exists solutions to

$\underline{w} = \alpha_1 \underline{c_1} + \alpha_2 \underline{c_2}$

$\underline{w} = \alpha_4 \underline{c_4} + \alpha_5 \underline{c_5}$

where $\alpha \in \mathbb{R}$

$\Rightarrow \alpha_1 \underline{c_1} + \alpha_2 \underline{c_2} - \alpha_5 \underline{c_5} = \alpha_4 \underline{c_4}$

$\underline{c_4}$ is linearly dependent on $\underline{c_1}, \underline{c_2}, \underline{c_5}$ $\Rightarrow \alpha_4$ is a free variable.

set $\alpha_4 = 1$

$$\begin{array}{c} \begin{array}{cccc} \underline{c_1} & \underline{c_2} & \underline{-c_5} & \underline{c_4} \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 3 & 3 \\ 1 & -2 & -2 & 1 \\ 2 & 1 & 5 & 7 \\ 1 & 0 & 1 & 3 \end{array}\right] \end{array} \begin{array}{l} \\ \\ -R_1 \\ -2R_1 \\ -R_1 \end{array} \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 3 & 3 \\ 0 & -2 & -5 & -2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & -2 & 0 \end{array}\right] \text{swap with } R_3 \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 3 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & -2 & -5 & -2 \\ 0 & 0 & -2 & 0 \end{array}\right] \begin{array}{l} \\ \\ +2R_2 \\ \\ \end{array}$$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 3 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & -7 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right] \begin{array}{l} \\ \\ \cdot -\frac{1}{7} \\ -\frac{2}{7}R_3 \end{array} \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 3 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\alpha_1 = 3, \ \alpha_2 = 1, \ \alpha_5 = 0$

$$\Rightarrow \underline{w} = 3 \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 7 \\ 3 \end{bmatrix}$$

Basis of $(U_1 \cap U_2)$: $[3, 1, 7, 3]^T$