Prompts for Mathematics for Machine Learning: Linear Algebra

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- System of linear equations definition.

- The number of solutions to a system of linear equations in terms of the number of linear equations in the system.

- Matrix definition.

- Matrix row definition.

- Matrix column definition.

- Long vector representation of a matrix.

- Sum of two matrices.

- Multiplication of two matrices.

- Multiplication of two matrices in terms of the dot product of rows and columns.

- Necessary and sufficient conditions for two matrices to be conformable for multiplication.

- Properties of matrix multiplication and addition with respect to commutativity, associativity, and distributivity.

- Hadamard product of two matrices.

- Identity matrix definition.

- Multiplication of a matrix with the identity matrix.

- Inverse matrix definition.

- Names of matrices that possess an inverse.

- Names of matrices that do not possess an inverse.

- Is the inverse matrix for a non-singular matrix unique?

- Inverse of a 2×2-matrix.

- Transpose of a matrix definition.

- Properties of the inverse and transpose of matrices (refer to notes).

- Symmetric matrix definition.

- Square matrix definition.

- Sum and product of symmetric matrices.

- Multiplication of a matrix by a scalar.

- Properties of multiplication of matrices with scalars with respect to associativity, distributivity, and the transpose of the matrix.

- Representation of a system of linear equations in matrix form.

- Let $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\boldsymbol{x} \in \mathbb{R}^n$. Describe the product $\mathbf{A}\boldsymbol{x}$ in terms of linear combinations.

- General approach for solving the matrix equation $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$.

- Are the general solutions and particular solutions to matrix equations in the form $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ unique?

- Elementary row operations.

- The effect of elementary row operations on the solution set of a matrix equation.

- Pivot element of a matrix definition.

- Row echelon form of a matrix.

- Basic variables and free variables of a matrix in row echelon form.

- How to obtain a particular solution by using row echelon form.

- Reduced row echelon form of a matrix.

- How to obtain a particular solution by using reduced row echelon form.

- Gaussian elimination.

- The minus-1 trick for reading out the solutions $\boldsymbol{x}$ of a homogeneous system of linear equations $\mathbf{A}\boldsymbol{x} = \mathbf{0}$, where $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\boldsymbol{x} \in \mathbb{R}^n$. Why does this work?

- Calculating the inverse of a matrix using reduced row echelon form.

- Group definition.

- Abelian group definition.

- General linear group definition.

- Real-valued vector space definition.

- Outer product of two vectors.

- Inner product of two vectors.

- Vector subspace definition.

- How to determine whether a vector space is a vector subspace of another vector space.

- Let $U = (\mathcal{U}, +, \cdot)$ be a vector space. Describe the relationship between every subspace $U \subseteq (\mathbb{R}^n, +, \cdot)$ and the solution space of a homogeneous system of linear equations $\mathbf{A}\boldsymbol{x} = \mathbf{0}$ for $\mathbf{A} \in \mathbb{R}^{m \times n}, \boldsymbol{x} \in \mathbb{R}^n$.

- Linear combination definition.

- Linear independence definition.

- Properties which can be used to determine whether vectors are linearly independent (refer to notes).

- Consider a vector space V with k linearly independent vectors $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_k$ and m linear combinations

$$\boldsymbol{x}_1 = \sum_{i=1}^{k} \lambda_{i,1} \boldsymbol{b}_i,$$

$$\vdots$$

$$\boldsymbol{x}_m = \sum_{i=1}^{k} \lambda_{i,m} \boldsymbol{b}_i.$$

  Defining $\mathbf{B} = [\boldsymbol{b}_1, \ldots, \boldsymbol{b}_k]$ as the matrix whose columns are the linearly independent vectors $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_k$, we can write

$$\boldsymbol{x}_j = \mathbf{B}\boldsymbol{\lambda}_j, \quad \boldsymbol{\lambda}_j = \begin{bmatrix} \lambda_{1,j} \\ \vdots \\ \lambda_{k,j} \end{bmatrix}, \quad j = 1, \ldots, m.$$

  How can you test whether $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m$ are linearly independent?
  Explain whether $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m$ are linearly independent if $m > k$.

- Generating set definition.

- Span definition. Notation for span.

- Relationship between generating sets and vector subspaces.

- Basis definition.

- Let $V = (\mathcal{V}, +, \cdot)$ be a vector space and let a basis of V be $\mathcal{B}$, where $\mathcal{B} \subseteq \mathcal{V}, \mathcal{B} \neq \emptyset$. Describe the following properties.

  - $\mathcal{B}$ as the minimal generating set of V.
  - $\mathcal{B}$ as the maximal linearly independent set of vectors in V.

- Does every vector space possess a basis?

- Relate the number of basis vectors for different bases for the same vector space.

- Dimension (for finite-dimensional vector spaces) definition. Notation for dimension of a vector space.

- Let U, V be vector spaces and $U \subseteq V$. Describe necessary and sufficient conditions for $U = V$ in terms of $\dim(U)$ and $\dim(V)$.

- Intuition of dimension as the number of independent directions in a vector space.

- How to find a basis of a subspace $U = \text{span}[\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m] \subseteq \mathbb{R}^n$.

- Rank of a matrix definition. Notation for rank of a matrix.

- Relationship between $\text{rk}(\mathbf{A})$ and $\text{rk}(\mathbf{A}^{\mathsf{T}})$.

- Describe the subspace spanned by the columns of $\mathbf{A} \in \mathbb{R}^{m \times n}$. Names for this subspace.

- Describe the subspace spanned by the rows of $\mathbf{A} \in \mathbb{R}^{m \times n}$. How can the basis of this subspace be found?

- Necessary and sufficient conditions for a matrix to possess an inverse in terms of the rank.

- Let $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\boldsymbol{b} \in \mathbb{R}^m$. Describe necessary and sufficient conditions for solutions to exist to the linear equation system $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ in terms of rank.

- Let $\boldsymbol{A} \in \mathbb{R}^{m \times n}$. What does $\dim(\ker(\boldsymbol{A}))$ equal to?

- Full rank definition.

- Rank deficient definition.

- Map definition.

- Define the following with respect to mappings.

- – Domain.
- – Codomain.
- Linear mapping definition.
- Interpretation of a matrix as a collection of vectors.
- Interpretation of a matrix as a linear mapping.
- Injective mapping definition.
- Surjective mapping definition.
- Bijective mapping definition.
- What kind of mappings possess an inverse mapping?
- Isomorphism definition.
- Endomorphism definition.
- Automorphism definition.
- Identity mapping definition.
- Let V, W be finite-dimensional isomorphic vector spaces. Describe necessary and sufficient conditions for V and W to be isomorphic in terms of $\dim(V)$ and $\dim(W)$.
- Justify the representation of $\mathbb{R}^{m \times n}$ with $\mathbb{R}^{mn}$.
- Consider vector spaces V, W, X.
  - – Let $\Phi : V \to W$ and $\Psi : W \to X$ be linear mappings. Describe what kind of mapping $\Psi \circ \Phi$ is.
  - – Let $\Phi : V \to W$ be an isomorphism. Describe what kind of mapping Φ^{-1} is.
  - – Let $\phi : V \to W$ and $\Psi : V \to W$ be linear mappings and $\lambda, \mu \in \mathbb{R}$. Describe what kind of mapping $\lambda \Phi + \mu \Psi$ is.
- Ordered basis (of a finite-dimensional vector space) definition.
- Ordered list (tuple) notation.
- Unordered list (set) notation.
- Notation for a matrix with column vectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$.
- Coordinate representation of vectors given an ordered basis.
- Transformation matrix definition.

- Consider the linear mapping $\Phi : V \to W$, ordered bases

$$B = (\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n), \quad \tilde{B} = (\tilde{\boldsymbol{b}}_1, \ldots, \tilde{\boldsymbol{b}}_n)$$

  of V, ordered bases

$$C = (\boldsymbol{c}_1, \ldots, \boldsymbol{c}_m), \quad \tilde{C} = (\tilde{\boldsymbol{c}}_1, \ldots, \tilde{\boldsymbol{c}}_m)$$

  of W, and the transformation matrix A_Φ that maps coordinates with respect to B onto coordinates with respect to C.
  Describe how the transformation matrix $\tilde{A}_\Phi$ that maps coordinates with respect to $\tilde{B}$ onto coordinates with respect to $\tilde{C}$ can be found.

- Equivalent matrices definition. What does it mean for matrices to be equivalent?

- Similar matrices definition. What does it mean for matrices to be similar?

- Equal linear transformations definition.

- Image definition.

- Kernel definition.

- Consider a linear mapping $\Phi : V \to W$, where V, W are vector spaces.

  - Explain why $\ker(\Phi)$ is never empty.
  - What is $\mathrm{Im}(\Phi)$ a subspace of?
  - What is $\ker(\Phi)$ a subspace of?
  - Necessary and sufficient conditions for Φ to be injective in terms of $\ker(\Phi)$.

- Consider $\mathbf{A} \in \mathbb{R}^{m \times n}$ and a linear mapping $\Phi : \mathbb{R}^n \to \mathbb{R}^m, \boldsymbol{x} \mapsto \mathbf{A}\boldsymbol{x}$.

  - Column space definition. Relate the column space of $\mathbf{A}$ with $\mathbb{R}^m$.
  - Relate $\mathrm{rk}(\mathbf{A})$ with $\mathrm{Im}(\Phi)$.

- Rank-nullity theorem.

- Consider vector spaces V, W and a linear mapping $\Phi : V \to W$.

  - Describe $\ker(\Phi)$ if $\dim(\mathrm{Im}(\Phi)) < \dim(V)$.
  - Let $\mathbf{A}_\Phi$ be the transformation matrix of Φ with respect to an ordered basis and $\dim(\mathrm{Im}(\Phi)) < \dim(V)$. How many solutions does the system of linear equations $\mathbf{A}_\Phi \boldsymbol{x} = \mathbf{0}$ have under these conditions?
  - Under what conditions does the equivalence statement

    $$\Phi \text{ is injective} \iff \Phi \text{ is surjective} \iff \Phi \text{ is bijective}$$

    hold?

- Affine subspace definition.

- Direction space (of an affine subspace) definition.

- Support point (of an affine subspace) definition.

- Hyperplane definition.

- Consider two affine subspaces $L = \boldsymbol{x}_0 + U$ and $\tilde{L} = \tilde{\boldsymbol{x}}_0 + \tilde{U}$ of a vector subspace V. Describe necessary and sufficient conditions for $L \subseteq \tilde{L}$.

- Representation of affine subspaces using parametric equations.

- Describe the relationship between every affine subspace and the solutions to the system of linear equations $\mathbf{A}\boldsymbol{\lambda} = \boldsymbol{x}$ for $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\boldsymbol{\lambda} \in \mathbb{R}^n$, and $\boldsymbol{x} \in \mathbb{R}^m$.

- Describe the relationship between k-dimensional affine subspaces and solutions to inhomogeneous systems of linear equations in the form $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$, where $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\boldsymbol{b} \in \mathbb{R}^m$, and $\mathrm{rk}(\mathbf{A}) = n - k$.

- Affine mapping definition.

- Translation vector (of an affine mapping) definition.

- Consider two vector spaces V, W, a linear mapping $\Phi : V \to W$, $\boldsymbol{a} \in W$, and the affine mapping

$$\phi : V \to W$$
$$\boldsymbol{x} \mapsto \boldsymbol{a} + \Phi(\boldsymbol{x})$$

from V to W.

  - An affine mapping is the composition of a linear mapping and a translation. Are the components of the equivalent composite mapping uniquely determined?
  - Let $\phi' : W \to X$ be an affine mapping. Is the composition $\phi' \circ \phi$ affine?
  - Describe some properties that would be preserved by ϕ if the mapping is bijective.