

2.9) a)

$$A = \left\{ \underline{x} = \lambda \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \mu \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad \lambda, \mu \in \mathbb{R} \right\}$$

$$\Rightarrow A = \left\{ \underline{x} \in \mathbb{R}^3 : \underline{x} = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad c_1, c_2 \in \mathbb{R} \right\}$$

$$A \subseteq \mathbb{R}^3$$

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & -1 \end{bmatrix} \xrightarrow{-R_1, -R_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \xrightarrow{+R_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow [1, 1, 1]^T$  and  $[0, 1, -1]^T$  are linearly independent.

$\Rightarrow [1, 1, 1]^T$  and  $[0, 1, -1]^T$  act as a basis of  $A$  as  $A$  is equivalent to the set of all linear combinations of these vectors.

NOTE: If a vector space has a basis, then the set of vectors contains  $\underline{0}$ , is closed under the outer operation (scalar multiplication), and is closed under the inner operation (vector addition).

Proof: Consider basis vectors  $\underline{b}_1, \dots, \underline{b}_m$ , the set of vectors  $V = \{ \underline{x} = \lambda_1 \underline{b}_1 + \dots + \lambda_m \underline{b}_m, \lambda_1, \dots, \lambda_m \in \mathbb{R} \}$ , and the vector space  $V = (V, +, \cdot)$ .

Zero vector:

$$\lambda_1 = \dots = \lambda_m = 0 \Rightarrow \underline{x} = \underline{0} \Rightarrow \underline{0} \in V$$

Closure under outer operation:

Let  $k \in \mathbb{R}$

Consider  $k\underline{x}$

$$k\underline{x} = (k\lambda_1)\underline{b}_1 + \dots + (k\lambda_m)\underline{b}_m, \quad k\lambda_1, \dots, k\lambda_m \in \mathbb{R}$$

$$\Rightarrow k\underline{x} \in V$$

Closure under inner operation:

$$\text{Let } \underline{x}_1 = \mu_1 \underline{b}_1 + \dots + \mu_m \underline{b}_m$$

$$\underline{x}_2 = \varphi_1 \underline{b}_1 + \dots + \varphi_m \underline{b}_m$$

consider  $\underline{x}_1 + \underline{x}_2$

$$\underline{x}_1 + \underline{x}_2 = (\mu_1 + \varphi_1)\underline{b}_1 + \dots + (\mu_m + \varphi_m)\underline{b}_m, \quad (\mu_1 + \varphi_1), \dots, (\mu_m + \varphi_m) \in \mathbb{R}$$

$$\Rightarrow \underline{x}_1 + \underline{x}_2 \in V$$

$\Rightarrow A$  is a subspace of  $\mathbb{R}^3$ .

2.9) b)

$$B = \left\{ \underline{x} = \lambda^2 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \lambda \in \mathbb{R} \right\}$$

$$= \left\{ \underline{x} \in \mathbb{R}^3 : \underline{x} = c \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, c \in \mathbb{R}, c \geq 0 \right\}$$

$$B \subseteq \mathbb{R}^3$$

zero vector

$$\underline{0} \in B$$

$$\text{Example: } \lambda = 0 \Rightarrow \underline{x} = \underline{0}$$

closure under scalar multiplication

Proof by counterexample

$$\text{let } k = -1 \in \mathbb{R}$$

$$\text{consider } k\underline{x} = -\lambda^2 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad -\lambda^2 \notin \{c \in \mathbb{R} : c \geq 0\}$$

$$\Rightarrow k\underline{x} \notin B$$

$B$  is not closed under scalar multiplication.

$\Rightarrow B$  is not a subspace of  $\mathbb{R}^3$ .

2.9) c)

$$C \subseteq \mathbb{R}^3$$

$$C = \left\{ \underline{x} \in \mathbb{R}^3 : \underline{x} = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix}, \quad \xi_1 - 2\xi_2 + 3\xi_3 = \gamma \right\}$$

Zero vector

$$\underline{0} \in C \Rightarrow \gamma = (0) - 2(0) + 3(0) = 0$$

$\Rightarrow C$  is not a subspace of  $\mathbb{R}^3$  if  $\gamma \neq 0$ .

For  $\gamma = 0$

closure under scalar multiplication

Let  $k \in \mathbb{R}$ .

$$\text{Consider } k\underline{x} = k \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} k\xi_1 \\ k\xi_2 \\ k\xi_3 \end{bmatrix}, \quad k\xi_1 - 2k\xi_2 + 3k\xi_3 = k(\xi_1 - 2\xi_2 + 3\xi_3) = k(0) = 0$$

$$\Rightarrow k\underline{x} \in C$$

$C$  is closed under scalar multiplication.

closure under vector addition

$$\text{Let } \underline{x}_1 = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} \in C \text{ and } \underline{x}_2 = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix} \in C.$$

$$\text{Consider } \underline{x}_1 + \underline{x}_2 = \begin{bmatrix} \lambda_1 + \mu_1 \\ \lambda_2 + \mu_2 \\ \lambda_3 + \mu_3 \end{bmatrix},$$

$$(\lambda_1 + \mu_1) - 2(\lambda_2 + \mu_2) + 3(\lambda_3 + \mu_3) = (\lambda_1 - 2\lambda_2 + 3\lambda_3) + (\mu_1 - 2\mu_2 + 3\mu_3) = (0) + (0) = 0$$

$$\Rightarrow \underline{x}_1 + \underline{x}_2 \in C$$

$C$  is closed under vector addition.

$\Rightarrow C$  is a subspace of  $\mathbb{R}^3$  for  $\gamma = 0$ .

2.9) d)

$$D \subseteq \mathbb{R}^3$$

Zero vector

Consider  $(0, 0, 0)$

$$\Rightarrow \xi_1 = \xi_2 = 0 \in \mathbb{R}, \quad \xi_3 = 0 \in \mathbb{Z}$$

so  $\underline{0} \in D$

closure of  $D$  under scalar multiplication

Proof by counterexample.

Consider  $(0, 1, 0) \in D$ ,  $\frac{1}{2} \in \mathbb{R}$

$$\frac{1}{2}(0, 1, 0) = (0, \frac{1}{2}, 0) \notin D \quad \because \xi_2 = \frac{1}{2} \notin \mathbb{Z}.$$

$\Rightarrow D$  is not closed under scalar multiplication.