2.16) Linear mappings

$\Phi(\lambda x) = \lambda \Phi(x)$

$\Phi(x+y) = \Phi(x) + \Phi(y)$

for all $x, y \in V$, $\lambda \in \mathbb{R}$ where $\Phi : V \to W$

$\Rightarrow \Phi(\underline{0}_V) = \underline{0}_W$

Proof:

let $\underline{v} \in V$

$\Phi(\underline{0}_V) = \Phi(0\underline{v}) = 0\Phi(\underline{v}) = \underline{0}_W$

a)

let $f, g \in L^1([a,b])$, $\lambda \in \mathbb{R}$

$$\Phi(\lambda f) = \int_a^b \lambda f(x)\,dx = \lambda \int_a^b f(x)\,dx = \lambda \Phi(x)$$

$$\Phi(f+g) = \int_a^b f(x) + g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx = \Phi(f) + \Phi(g)$$

$\Rightarrow \Phi$ is a linear mapping.

b)

let $f, g \in C^1$, $\lambda \in \mathbb{R}$

$\Phi(\lambda f) = (\lambda f)' = \lambda f' = \lambda \Phi(f)$

$\Phi(f+g) = (f+g)' = f' + g' = \Phi(f) + \Phi(g)$

$\Rightarrow \Phi$ is a linear mapping.

c)

consider $x = 0 \in \mathbb{R}$

$\Phi(0) = \cos(0) = 1 \neq 0$

0 does not map onto 0.

$\Rightarrow \Phi$ is not a linear mapping.

d)

let $\underline{x}, \underline{y} \in \mathbb{R}^3$, $\lambda \in \mathbb{R}$

$$\Phi(\lambda \underline{x}) = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 3 \end{bmatrix}(\lambda \underline{x}) = \lambda\left(\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 3 \end{bmatrix}\underline{x}\right) = \lambda \Phi(\underline{x})$$

$$\Phi(\underline{x}+\underline{y}) = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 3 \end{bmatrix}(\underline{x}+\underline{y}) = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 3 \end{bmatrix}\underline{x} + \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 3 \end{bmatrix}\underline{y} = \Phi(\underline{x}) + \Phi(\underline{y})$$

$\Rightarrow \Phi$ is a linear mapping.

e)

let $\underline{x}, \underline{y} \in \mathbb{R}^2$, $\lambda \in \mathbb{R}$

$$\Phi(\lambda \underline{x}) = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}(\lambda \underline{x}) = \lambda\left(\begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}\underline{x}\right) = \lambda \Phi(\underline{x})$$

$$\Phi(\underline{x} + \underline{y}) = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} (\underline{x} + \underline{y}) = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \underline{x} + \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \underline{y} = \Phi(\underline{x}) + \Phi(\underline{y})$$

$\Rightarrow \Phi$ is a linear mapping.