2.1) a)

<u>Closure</u>

$a * b = ab + a + b \in \mathbb{R}$ $\because$ closure of $\mathbb{R}$ under addition and multiplication.

Assume $a * b = -1$

$\Rightarrow ab + a + b = -1$

$\Rightarrow ab + a + b + 1 = 0$

$\Rightarrow (a+1)(b+1) = 0$

$\Rightarrow a = -1$ or $b = -1$

This contradicts $a, b \in \mathbb{R} \setminus \{-1\}$ $\therefore$ $ab + a + b \neq -1$.

$\Rightarrow a * b \in \mathbb{R} \setminus \{-1\}$.

$(\mathbb{R} \setminus \{-1\}, *)$ is closed.

<u>Associativity</u>

$$\begin{aligned}(a * b) * c &= (ab + a + b) * c \\ &= (ab + a + b)c + (ab + a + b) + c \\ &= abc + ab + ac + bc + a + b + c\end{aligned}$$

$$\begin{aligned}a * (b * c) &= a * (bc + b + c) \\ &= a(bc + b + c) + a + (bc + b + c) \\ &= abc + ab + ac + bc + a + b + c\end{aligned}$$

$\Rightarrow (a * b) * c \cong a * (b * c)$

$(\mathbb{R} \setminus \{-1\}, *)$ is associative.

<u>Neutral element</u>

$a * e = a$

$\Rightarrow ae + a + e = a$

$\Rightarrow ae + e = 0$

$\Rightarrow e(a+1) = 0$

$\Rightarrow e = 0 \in \mathbb{R} \setminus \{-1\}$

Ignore $a = -1$ $\because$ $a \in \mathbb{R} \setminus \{-1\}$

$$\begin{aligned}0 * a &= (0)(a) + 0 + a \\ &= a\end{aligned}$$

$\Rightarrow (\mathbb{R} \setminus \{-1\}, *)$ has neutral element $e = 0$.

<u>Inverse element</u>

$a * a^{-1} = e$

$\Rightarrow aa^{-1} + a + a^{-1} = 0$

$\Rightarrow a^{-1} = \dfrac{-a}{a+1} \in \mathbb{R} \setminus \{-1\}$ This exists for all $a \in \mathbb{R} \setminus \{-1\}$.

$$a * \frac{-a}{a+1} = \frac{-a^2}{a+1} + a - \frac{a}{a+1}$$

$$= \frac{-a^2 + a^2 + a - a}{a + 1}$$

$$= 0$$

$\Rightarrow$ The inverse element exists for all elements in $(\mathbb{R} \setminus \{-1\}, *)$.

<u>Commutativity</u>

$a * b = ab + a + b$

$b * a = ba + b + a$

$\quad\quad = ab + a + b$

$\Rightarrow a * b \equiv b * a$

$(\mathbb{R} \setminus \{-1\}, *)$ is associative.

<u>Conclusion</u>

$(\mathbb{R} \setminus \{-1\}, *)$ is an Abelian group.

2.1) b)

$3 * x * x = 15$

$\Rightarrow (3x + 3 + x) * x = 15$

$\Rightarrow (4x + 3) * x = 15$

$\Rightarrow (4x^2 + 3x) + (4x + 3) + x = 15$

$\Rightarrow 4x^2 + 8x - 12 = 0$

$\Rightarrow x^2 + 2x - 3 = 0$

$\Rightarrow (x + 3)(x - 1) = 0$

$\Rightarrow x = -3 \quad$ or $\quad x = 1$