Prompts for Mathematics for Machine Learning: Linear Algebra

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July 2025

- System of linear equations definition.
- The number of solutions to a system of linear equations in terms of the number of linear equations in the system.
- Matrix definition.
- Matrix row definition.
- Matrix column definition.
- Long vector representation of a matrix.
- Sum of two matrices.
- Multiplication of two matrices.
- Multiplication of two matrices in terms of the dot product of rows and columns.
- Necessary and sufficient conditions for two matrices to be conformable for multiplication.
- Properties of matrix multiplication and addition with respect to commutativity, associativity, and distributivity.
- Hadamard product of two matrices.
- Identity matrix definition.
- Multiplication of a matrix with the identity matrix.
- Inverse matrix definition.
- Names of matrices that possess an inverse.
- Names of matrices that do not possess an inverse.
- Is the inverse matrix for a non-singular matrix unique?

- Inverse of a 2×2-matrix.
- Transpose of a matrix definition.
- Properties of the inverse and transpose of matrices (refer to notes).
- Symmetric matrix definition.
- Square matrix definition.
- Sum and product of symmetric matrices.
- Multiplication of a matrix by a scalar.
- Properties of multiplication of matrices with scalars with respect to associativity, distributivity, and the transpose of the matrix.
- Representation of a system of linear equations in matrix form.
- Let $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\boldsymbol{x} \in \mathbb{R}^n$. Describe the product $\mathbf{A}\boldsymbol{x}$ in terms of linear combinations.
- General approach for solving the matrix equation $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$.
- Are the general solutions and particular solutions to matrix equations in the form $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ unique?
- Elementary row operations.
- The effect of elementary row operations on the solution set of a matrix equation.
- Pivot element of a matrix definition.
- Row echelon form of a matrix.
- Basic variables and free variables of a matrix in row echelon form.
- How to obtain a particular solution by using row echelon form.
- Reduced row echelon form of a matrix.
- How to obtain a particular solution by using reduced row echelon form.
- Gaussian elimination.
- The minus-1 trick for reading out the solutions $\boldsymbol{x}$ of a homogeneous system of linear equations $\mathbf{A}\boldsymbol{x} = \mathbf{0}$, where $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\boldsymbol{x} \in \mathbb{R}^n$. Why does this work?
- Calculating the inverse of a matrix using reduced row echelon form.
- Group definition.

- Abelian group definition.
- General linear group definition.
- Real-valued vector space definition.
- Outer product of two vectors.
- Inner product of two vectors.
- Vector subspace definition.
- How to determine whether a vector space is a vector subspace of another vector space.
- Let $U = (\mathcal{U}, +, \cdot)$ be a vector space. Describe the relationship between every subspace $U \subseteq (\mathbb{R}^n, +, \cdot)$ and the solution space of a homogeneous system of linear equations $\mathbf{A}\boldsymbol{x} = \mathbf{0}$ for $\boldsymbol{x} \in \mathbb{R}^n$.
- Linear combination definition.
- Linear independence definition.
- Properties which can be used to determine whether vectors are linearly independent (refer to notes).
- Consider a vector space V with k linearly independent vectors $\boldsymbol{b}_1, \dots, \boldsymbol{b}_k$ and m linear combinations

  $$\begin{aligned} \boldsymbol{x}_1 &= \sum_{i=1}^{k} \lambda_{i,1} \boldsymbol{b}_i, \\ &\vdots \\ \boldsymbol{x}_m &= \sum_{i=1}^{k} \lambda_{i,m} \boldsymbol{b}_i. \end{aligned}$$

  Defining $\mathbf{B} = [\boldsymbol{b}_1, \dots, \boldsymbol{b}_k]$ as the matrix whose columns are the linearly independent vectors $\boldsymbol{b}_1, \dots, \boldsymbol{b}_k$, we can write

  $$\boldsymbol{x}_j = \mathbf{B}\boldsymbol{\lambda}_j, \quad \boldsymbol{\lambda}_j = \begin{bmatrix} \lambda_{1,j} \\ \vdots \\ \lambda_{k,j} \end{bmatrix}, \quad j = 1, \dots, m.$$

  How can you test whether $\boldsymbol{x}_1, \dots, \boldsymbol{x}_m$ are linearly independent?
  Explain whether $\boldsymbol{x}_1, \dots, \boldsymbol{x}_m$ are linearly independent if $m > k$.
- Generating set definition.
- Span definition. Notation for span.

- Relationship between generating sets and vector subspaces.
- Basis definition.
- Let $V = (\mathcal{V}, +, \cdot)$ be a vector space and let a basis of V be $\mathcal{B}$, where $\mathcal{B} \subseteq \mathcal{V}, \mathcal{B} \neq \emptyset$. Describe and explain the properties of basis with respect to the following.
  - $\mathcal{B}$ as the minimal generating set of V.
  - $\mathcal{B}$ as the maximal linearly independent set of vectors in V.
  - Linear combinations.
- Does every vector space possess a basis?
- Relate the number of basis vectors for different bases for the same vector space.
- Dimension (for finite-dimensional vector spaces) definition. Notation for dimension of a vector space.
- Let U, V be vector spaces and $U \subseteq V$. Describe necessary and sufficient conditions for $U = V$ in terms of $\dim(U)$ and $\dim(V)$.
- Intuition of dimension as the number of independent directions in a vector space.
- How to find a basis of a subspace $U = \text{span}[\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m] \subseteq \mathbb{R}^n$.
- Rank of a matrix definition. Notation for rank of a matrix.
- Relationship between $\text{rk}(\mathbf{A})$ and $\text{rk}(\mathbf{A}^\top)$.
- Describe the subspace spanned by the columns of $\mathbf{A} \in \mathbb{R}^{m \times n}$. Names for this subspace.
- Describe the subspace spanned by the rows of $\mathbf{A} \in \mathbb{R}^{m \times n}$. How can the basis of this subspace be found?
- Necessary and sufficient conditions for a matrix to possess an inverse in terms of the rank.
- Let $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\boldsymbol{b} \in \mathbb{R}^m$. Describe necessary and sufficient conditions for solutions to exist to the linear equation system $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ in terms of rank.
- Let $\boldsymbol{A} \in \mathbb{R}^{m \times n}$. What does $\dim(\ker(\boldsymbol{A}))$ equal to?
- Full rank definition.
- Rank deficient definition.
- Map definition.

- Define the following with respect to mappings.
  - Domain.
  - Codomain.
- Linear mapping definition.
- Interpretation of a matrix as a collection of vectors.
- Interpretation of a matrix as a linear mapping.
- Injective mapping definition.
- Surjective mapping definition.
- Bijective mapping definition.
- What kind of mappings possess an inverse mapping?
- Isomorphism definition.
- Endomorphism definition.
- Automorphism definition.
- Identity mapping definition.
- Let V, W be finite-dimensional isomorphic vector spaces. Describe necessary and sufficient conditions for V and W to be isomorphic in terms of $\dim(V)$ and $\dim(W)$.
- Justify the representation of $\mathbb{R}^{m\times n}$ with $\mathbb{R}^{mn}$.
- Consider vector spaces V, W, X.
  - Let $\Phi : V \to W$ and $\Psi : W \to X$ be linear mappings. Describe what kind of mapping $\Psi \circ \Phi$ is.
  - Let $\Phi : V \to W$ be an isomorphism. Describe what kind of mapping Φ^{-1} is.
  - Let $\phi : V \to W$ and $\Psi : V \to W$ be linear mappings and $\lambda, \mu \in \mathbb{R}$. Describe what kind of mapping $\lambda\Phi + \mu\Psi$ is.
- Ordered basis (of a finite-dimensional vector space) definition.
- Ordered list (tuple) notation.
- Unordered list (set) notation.
- Notation for a matrix with column vectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$.
- Coordinate representation of vectors given an ordered basis.
- Transformation matrix definition.

- Consider the linear mapping $\Phi : V \to W$, ordered bases

$$B = (\boldsymbol{b}_1, \dots, \boldsymbol{b}_n), \quad \tilde{B} = (\tilde{\boldsymbol{b}}_1, \dots, \tilde{\boldsymbol{b}}_n)$$

  of V, ordered bases

$$C = (\boldsymbol{c}_1, \dots, \boldsymbol{c}_m), \quad \tilde{C} = (\tilde{\boldsymbol{c}}_1, \dots, \tilde{\boldsymbol{c}}_m)$$

  of W, and the transformation matrix A_Φ that maps coordinates with respect to B onto coordinates with respect to C.
  Describe how the transformation matrix $\tilde{A}_\Phi$ that maps coordinates with respect to $\tilde{B}$ onto coordinates with respect to $\tilde{C}$ can be found.

- Equivalent matrices definition. What does it mean for matrices to be equivalent?

- Similar matrices definition. What does it mean for matrices to be similar?

- Equal linear transformations definition.

- Image definition.

- Kernel definition.

- Consider a linear mapping $\Phi : V \to W$, where V, W are vector spaces.
  - Explain why $\ker(\Phi)$ is never empty.
  - What is $\mathrm{Im}(\Phi)$ a subspace of?
  - What is $\ker(\Phi)$ a subspace of?
  - Necessary and sufficient conditions for Φ to be injective in terms of $\ker(\Phi)$.

- Consider $\mathbf{A} \in \mathbb{R}^{m \times n}$ and a linear mapping $\Phi : \mathbb{R}^n \to \mathbb{R}^m, \boldsymbol{x} \mapsto \mathbf{A}\boldsymbol{x}$.
  - Column space definition. Relate the column space of $\mathbf{A}$ with $\mathbb{R}^m$.
  - Relate $\mathrm{rk}(\mathbf{A})$ with $\mathrm{Im}(\Phi)$.

- Rank-nullity theorem.

- Consider vector spaces V, W and a linear mapping $\Phi : V \to W$.
  - Describe $\ker(\Phi)$ if $\dim(\mathrm{Im}(\Phi)) < \dim(V)$.
  - Let $\mathbf{A}_\Phi$ be the transformation matrix of Φ with respect to an ordered basis and $\dim(\mathrm{Im}(\Phi)) < \dim(V)$. How many solutions does the system of linear equations $\mathbf{A}_\Phi \boldsymbol{x} = \mathbf{0}$ have under these conditions?
  - Under what conditions does the equivalence statement

$$\Phi \text{ is injective} \iff \Phi \text{ is surjective} \iff \Phi \text{ is bijective}$$

    hold?

- Affine subspace definition.
- Direction space (of an affine subspace) definition.
- Support point (of an affine subspace) definition.
- Hyperplane definition.
- Consider two affine subspaces $L = \boldsymbol{x}_0 + U$ and $\tilde{L} = \tilde{\boldsymbol{x}}_0 + \tilde{U}$ of a vector subspace V. Describe necessary and sufficient conditions for $L \subseteq \tilde{L}$.
- Representation of affine subspaces using parametric equations.
- Let $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\boldsymbol{\lambda} \in \mathbb{R}^n$, and $\boldsymbol{x} \in \mathbb{R}^m$. Describe the solutions to the system of linear equations $\mathbf{A}\boldsymbol{\lambda} = \boldsymbol{x}$ in terms of affine subspaces.
- Describe the relationship between k-dimensional affine subspaces and solutions to inhomogeneous systems of linear equations in the form $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$, where $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\boldsymbol{b} \in \mathbb{R}^m$, and $\mathrm{rk}(\mathbf{A}) = n - k$.
- Affine mapping definition.
- Translation vector (of an affine mapping) definition.
- Consider two vector spaces V, W, a linear mapping $\Phi : V \to W$, $\boldsymbol{a} \in W$, and the affine mapping

  $$\begin{aligned} \phi : V &\to W \\ \boldsymbol{x} &\mapsto \boldsymbol{a} + \Phi(\boldsymbol{x}) \end{aligned}$$

  from V to W.

  - An affine mapping is the composition of a linear mapping and a translation. Are the components of the equivalent composite mapping uniquely determined?
  - Let $\phi' : W \to X$ be an affine mapping. Is the composition $\phi' \circ \phi$ affine?
  - Describe some properties that would be preserved by ϕ if the mapping is bijective.