2.15) a)

F

- Zero vector

Consider $(0, 0, 0) \in \mathbb{R}^3$

$0 + 0 - 0 = 0$

$\Rightarrow \underline{0} \in F$

- Closure under scalar multiplication

Let $[x, y, z]^T \in F$, $\lambda \in \mathbb{R}$

$$\lambda \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \lambda x \\ \lambda y \\ \lambda z \end{bmatrix}$$

$\lambda x + \lambda y - \lambda z = \lambda (x + y - z) = \lambda (0) = 0$

$\therefore$ F is closed under scalar multiplication.

- Closure under vector addition

Let $[x_1, y_1, z_1]^T, [x_2, y_2, z_2]^T \in F$

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix}$$

$(x_1 + x_2) + (y_1 + y_2) - (z_1 + z_2)$

$= (x_1 + y_1 - z_1) + (x_2 + y_2 - z_2)$

$= 0 + 0$

$= 0$

$\therefore$ F is closed under vector addition.

$\Rightarrow$ F is a subspace of $\mathbb{R}^3$.

G

- Zero vector

Let $a = b = 0 \in \mathbb{R}$

$\Rightarrow (a - b, a + b, a - 3b) = (0, 0, 0)$

$\Rightarrow \underline{0} \in G$

- Closure under scalar multiplication

Let $(a - b, a + b, a - 3b) \in G$, $\lambda \in \mathbb{R}$

$$\lambda \begin{bmatrix} a - b \\ a + b \\ a - 3b \end{bmatrix} = \begin{bmatrix} (\lambda a) - (\lambda b) \\ (\lambda a) + (\lambda b) \\ (\lambda a) - 3(\lambda b) \end{bmatrix}$$

$\lambda a, \lambda b \in \mathbb{R}$

$\therefore$ G is closed under scalar multiplication.

- Closure under vector addition

Let $(a_1 - b_1, a_1 + b_1, a_1 - 3b_1), (a_2 - b_2, a_2 + b_2, a_2 - 3b_2) \in G$

$$\begin{bmatrix} a_1 - b_1 \\ a_1 + b_1 \\ a_1 - 3b_1 \end{bmatrix} + \begin{bmatrix} a_2 - b_2 \\ a_2 + b_2 \\ a_2 - 3b_2 \end{bmatrix} = \begin{bmatrix} (a_1 + a_2) - (b_1 + b_2) \\ (a_1 + a_2) + (b_1 + b_2) \\ (a_1 + a_2) - 3(b_1 + b_2) \end{bmatrix}$$

$(a_1 + a_2), (b_1 + b_2) \in \mathbb{R}$

$\therefore$ G is closed under vector addition.

$\Rightarrow G$ is a subspace of $\mathbb{R}^3$.

b)

Let $\underline{x} = (a-b,\ a+b,\ a-3b) \in G$

Substitute into conditions for F

$(a-b) + (a+b) - (a-3b) = 0$

$\Rightarrow a + 3b = 0 \quad \Rightarrow \quad a = -3b$

$\Rightarrow \underline{x} = (-3b-b,\ -3b+b,\ -3b-3b) = (-4b, -2b, -6b)$

$\Rightarrow (F \cap G) = \text{span}\{[2,1,3]^T\}$ (scale $\underline{x}$ by $-\frac{1}{2}$)

c)

<u>F</u>

Hyperplane in $\mathbb{R}^3 \Rightarrow \dim(F) = 2$

$(0,0,0) \in F$

$1 + 0 - 1 = 0 \Rightarrow (1,0,1) \in F$

$0 + 1 - 1 = 0 \Rightarrow (0,1,1) \in F$

$\Rightarrow F$ contains direction vectors $[1,0,1]^T, [0,1,1]^T$.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{matrix} \\ \\ -R_1 \end{matrix} \rightsquigarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{matrix} \\ \\ -R_2 \end{matrix} \rightsquigarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$\therefore [1,0,1]^T$ and $[0,1,1]^T$ are linearly independent.

$\Rightarrow$ Basis of $F = \{[1,0,1]^T, [0,1,1]^T\}$

<u>G</u>

$$G = \left\{ a\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + b\begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix} \middle| \ a, b \in \mathbb{R} \right\}$$

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & -3 \end{bmatrix} \begin{matrix} \\ -R_1 \\ -R_1 \end{matrix} \rightsquigarrow \begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 0 & -2 \end{bmatrix} \begin{matrix} \\ \cdot\frac{1}{2} \\ +R_2 \end{matrix} \rightsquigarrow \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$\therefore [1,1,1]^T$ and $[-1,1,-3]^T$ are linearly independent.

$\Rightarrow$ Basis of $G = \{[1,1,1]^T, [-1,1,-3]^T\}$

$\underline{F \cap G}$

$$\begin{array}{cccc} \underline{c_1} & \underline{c_2} & \underline{c_3} & \underline{c_4} \end{array}$$

$$\begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & -3 \end{bmatrix}\begin{array}{l} \\ \\ -R_1 \end{array} \rightsquigarrow \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & -2 \end{bmatrix}\begin{array}{l} \\ \\ -R_2 \end{array} \rightsquigarrow \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & -3 \end{bmatrix}\begin{array}{l} \\ \\ \cdot -1 \end{array} \rightsquigarrow \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$\dim(F+G) = 3$

$\dim(F \cap G) = \dim(F) + \dim(G) - \dim(F+G) = 2+2-3 = 1$

let $\underline{w} \in (F \cap G)$

$\underline{w} = \alpha_1 \underline{c_1} + \alpha_2 \underline{c_2} \in F$

$\underline{w} = \alpha_3 \underline{c_3} + \alpha_4 \underline{c_4} \in G$

$\Rightarrow \alpha_1 \underline{c_1} + \alpha_2 \underline{c_2} - \alpha_3 \underline{c_3} = \alpha_4 \underline{c_4}$

α_4 is a free variable $\because$ $\underline{c_4}$ is linearly dependent on $\{\underline{c_1}, \underline{c_2}, \underline{c_3}\}$.

set $\alpha_4 = 1$.

$$\begin{array}{cccc} \underline{c_1} & \underline{c_2} & -\underline{c_3} & \underline{c_4} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & -1 \\ 0 & 1 & -1 & 1 \\ 1 & 1 & -1 & -3 \end{array}\right]\begin{array}{l} \\ \\ -R_1 \end{array} \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & -2 \end{array}\right]\begin{array}{l} \\ \\ -R_2 \end{array} \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -3 \end{array}\right]\begin{array}{l} +R_3 \\ +R_3 \\ \\ \end{array} \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -3 \end{array}\right]$$

$\Rightarrow \alpha_1 = -4, \ \alpha_2 = -2, \ \alpha_3 = -3$

$$\Rightarrow \underline{w} = -4\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - 2\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \\ -6 \end{bmatrix}$$

$\Rightarrow (F \cap G) = \text{span}\{[2, 1, 3]^T\}$ (scale $\underline{w}$ by $-\frac{1}{2}$)