2.(2)

<u>Basis of U_1</u>

$$\begin{array}{c}\begin{array}{ccc}\underline{c_1} & \underline{c_2} & \underline{c_3}\end{array}\\ \begin{bmatrix}1&2&-1\\1&-1&1\\-3&0&-1\\1&-1&1\end{bmatrix}\end{array}\begin{matrix}\\ -R_1\\ +3R_1\\ -R_1\end{matrix} \rightsquigarrow \begin{bmatrix}1&2&-1\\0&-3&2\\0&6&-4\\0&-3&2\end{bmatrix}\begin{matrix}\\ \cdot -\frac{1}{3}\\ +2R_2\\ -R_2\end{matrix} \rightsquigarrow \begin{bmatrix}1&2&-1\\0&1&-\frac{2}{3}\\0&0&0\\0&0&0\end{bmatrix}$$

$\Rightarrow U_1 = \text{span}\left[\underline{c_1}, \underline{c_2}\right]$

<u>Basis of U_2</u>

$$\begin{array}{c}\begin{array}{ccc}\underline{c_4} & \underline{c_5} & \underline{c_6}\end{array}\\ \begin{bmatrix}-1&2&-3\\-2&-2&6\\2&0&-2\\1&0&-1\end{bmatrix}\end{array}\begin{matrix}\text{swap with } R_4\\ \\ \\ \\ \end{matrix} \rightsquigarrow \begin{bmatrix}1&0&-1\\-2&-2&6\\2&0&-2\\-1&2&-3\end{bmatrix}\begin{matrix}\\ +2R_1\\ -2R_1\\ +R_1\end{matrix} \rightsquigarrow \begin{bmatrix}1&0&-1\\0&-2&4\\0&0&0\\0&2&-4\end{bmatrix}\begin{matrix}\\ \cdot -\frac{1}{2}\\ \\ +R_2\end{matrix} \rightsquigarrow \begin{bmatrix}1&0&-1\\0&1&-2\\0&0&0\\0&0&0\end{bmatrix}$$

$\Rightarrow U_2 = \text{span}\left[\underline{c_4}, \underline{c_5}\right]$

<u>Basis of $(U_1 \cap U_2)$</u>

Let B be the basis of $(U_1 \wedge U_2)$

$W = \text{span}(B)$

$\Rightarrow W$ satisfies $(W \subseteq U_1) \wedge (W \subseteq U_2)$

$$\begin{array}{c}\begin{array}{cccc}\underline{c_1} & \underline{c_2} & \underline{c_4} & \underline{c_5}\end{array}\\ \begin{bmatrix}1&2&-1&2\\1&-1&-2&-2\\-3&0&2&0\\1&-1&1&0\end{bmatrix}\end{array}\begin{matrix}\\ -R_1\\ +3R_1\\ -R_1\end{matrix} \rightsquigarrow \begin{bmatrix}1&2&-1&2\\0&-3&-1&-4\\0&6&-1&6\\0&-3&2&-2\end{bmatrix}\begin{matrix}\\ \cdot -\frac{1}{3}\\ +2R_2\\ -R_2\end{matrix} \rightsquigarrow \begin{bmatrix}1&2&-1&2\\0&1&\frac{1}{3}&\frac{4}{3}\\0&0&-3&-2\\0&0&3&2\end{bmatrix}\begin{matrix}\\ \\ \cdot -\frac{1}{3}\\ +R_3\end{matrix}$$

$$\rightsquigarrow \begin{bmatrix}1&2&-1&2\\0&1&\frac{1}{3}&\frac{4}{3}\\0&0&1&\frac{2}{3}\\0&0&0&0\end{bmatrix}$$

$\Rightarrow \dim(W) = 1$

There exists some $\underline{w} \in W \setminus \{\underline{0}\}$ such that

$\underline{w} = \alpha_1 \underline{c_1} + \alpha_2 \underline{c_2}$

$\underline{w} = \alpha_4 \underline{c_4} + \alpha_5 \underline{c_5}$

where $\alpha_1, \alpha_2, \alpha_4, \alpha_5 \in \mathbb{R}$

$\Rightarrow \alpha_1 \underline{c_1} + \alpha_2 \underline{c_2} - \alpha_4 \underline{c_4} - \alpha_5 \underline{c_5} = \underline{0}$

α_5 is a free variable $\because$ $\underline{c_5}$ is linearly dependent from $\underline{c_1}, \underline{c_2}, \underline{c_4}$.

Let $\alpha_5 = 1$

$\alpha_1 \underline{c_1} + \alpha_2 \underline{c_2} - \alpha_4 \underline{c_4} = \underline{c_5}$

$$
\begin{array}{c}
\begin{array}{cccc} \underline{c_1} & \underline{c_2} & \underline{-c_4} & \end{array} \\
\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 1 & -1 & 2 & -2 \\ -3 & 0 & -2 & 0 \\ 1 & -1 & -1 & 0 \end{array}\right]
\end{array}
\begin{array}{l} \\ \\ -R_1 \\ +3R_1 \\ -R_1 \end{array}
\rightsquigarrow
\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & -3 & 1 & -4 \\ 0 & 6 & 1 & 6 \\ 0 & -3 & -2 & -2 \end{array}\right]
\begin{array}{l} \\ \cdot -\frac{1}{3} \\ +2R_2 \\ -R_2 \end{array}
\rightsquigarrow
\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 1 & -\frac{1}{3} & \frac{4}{3} \\ 0 & 0 & 3 & -2 \\ 0 & 0 & -3 & 2 \end{array}\right]
\begin{array}{l} \\ \\ \cdot \frac{1}{3} \\ +R_3 \end{array}
\rightsquigarrow
\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 1 & -\frac{1}{3} & \frac{4}{3} \\ 0 & 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 0 & 0 \end{array}\right]
\begin{array}{l} -R_3 \\ +\frac{1}{3}R_3 \\ \\ \\ \end{array}
$$

$$
\rightsquigarrow
\left[\begin{array}{ccc|c} 1 & 2 & 0 & \frac{8}{3} \\ 0 & 1 & 0 & \frac{10}{9} \\ 0 & 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 0 & 0 \end{array}\right]
\begin{array}{l} -2R_2 \\ \\ \\ \\ \end{array}
\rightsquigarrow
\left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{4}{9} \\ 0 & 1 & 0 & \frac{10}{9} \\ 0 & 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 0 & 0 \end{array}\right]
$$

$\Rightarrow \alpha_1 = \frac{4}{9}, \ \alpha_2 = \frac{10}{9}, \ \alpha_4 = -\frac{2}{3}$

$$
\Rightarrow \underline{w} = \frac{4}{9}\begin{bmatrix} 1 \\ 1 \\ -3 \\ 1 \end{bmatrix} + \frac{10}{9}\begin{bmatrix} 2 \\ -1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{24}{9} \\ \frac{-6}{9} \\ -\frac{12}{9} \\ -\frac{6}{9} \end{bmatrix}
$$

$\Rightarrow [4, -1, -2, -1]^T$ is a basis of $(U_1 \wedge U_2)$.