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- General form of a system of linear equations.

$$a_{1,1}x_1 + \dots + a_{1,n}x_n = b_1$$
$$\vdots$$
$$a_{m,1}x_1 + \dots + a_{m,n}x_n = b_m$$

where $a, b \in \mathbb{R}$

$x_1, \dots, x_n \in \mathbb{R}$ are unknowns

- There are either no solutions, n solutions, or an infinite number of solutions to a system of linear equations in n unknowns.

- A matrix is a rectangular array of elements organised into rows and columns.

- A matrix row is a matrix of size $1 \times n$.

- A matrix column is a vector with m elements.

- Long vector representation of a matrix.

A matrix of size $m \times n$ can be represented as a vector with mn elements.

- Example

$$A = \begin{bmatrix} a_{1,1} & \dots & a_{1,n} \\ \vdots & & \vdots \\ a_{m,1} & \dots & a_{m,n} \end{bmatrix} \in \mathbb{R}^{m \times n}$$

A can be represented using a vector formed by augmenting a vector with the columns of A.

$$\begin{bmatrix} a_{1,1} & \dots & a_{m,1} & \dots & a_{1,j} & \dots & a_{m,j} & \dots & a_{1,n} & \dots & a_{m,n} \end{bmatrix}^T \in \mathbb{R}^{mn}$$

- Sum of two matrices.

Two matrices must have the same size for them to be conformable for addition.

- Let $A, B, C \in \mathbb{R}^{m \times n}$

$$A + B = C.$$

$$c_{i,j} = a_{i,j} + b_{i,j}$$

- Multiplication of two matrices.

- Let $A \in \mathbb{R}^{m \times n}$

$B \in \mathbb{R}^{p \times q}$.

The multiplication AB is valid if and only if $n = p$.

If $n = p$, then $AB \in \mathbb{R}^{m \times q}$.

- Let $\underline{x} \in \mathbb{R}^n$.

$$A\underline{x} = \begin{bmatrix} a_{1,1} & \dots & a_{1,n} \\ \vdots & & \vdots \\ a_{m,1} & \dots & a_{m,n} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$R_i = \begin{bmatrix} a_{i,1} & \dots & a_{i,n} \end{bmatrix}$ Matrix row

$\underline{c}_j = \begin{bmatrix} a_{1,j} & \dots & a_{m,j} \end{bmatrix}$ Matrix column

$$A\underline{x} = \sum_{j=1}^{n} x_j \underline{c}_j$$

$$= \begin{bmatrix} \sum_{j=1}^{n} a_{1,j} x_j \\ \vdots \\ \sum_{j=1}^{n} a_{m,j} x_j \end{bmatrix}$$

$$= \begin{bmatrix} R_1 \underline{x} \\ \vdots \\ R_m \underline{x} \end{bmatrix}$$

- Let $n = p$

  $\underline{v}_1, \dots, \underline{v}_q$ be the columns of B such that $B = [\underline{v}_1, \dots, \underline{v}_q]$.

$AB = [A\underline{v}_1, \dots, A\underline{v}_q]$

A matrix can be considered to be a collection of column vectors.

– Properties of matrix addition and scalar multiplication.

Let $A, B, C \in \mathbb{R}^{m \times n}$

$\lambda, \mu \in \mathbb{R}$

- $A + B = B + A$
- $(A + B) + C = A + (B + C)$
- $\lambda(A + B) = \lambda A + \lambda B$
- $(\lambda + \mu) A = \lambda A + \mu A$

– Properties of matrix multiplication.

Let $A \in \mathbb{R}^{m \times n}$

$B, C \in \mathbb{R}^{p \times q}$.

- Let $n = p$.

Matrix multiplication is non-commutative.

AB does not necessarily equal BA.

- Let $n = p$

  $r = q$.

$(AB)C = A(BC)$

- Let $n = p$

$A(B + C) = AB + AC$

– Hadamard product of two matrices.

Let $A, B \in \mathbb{R}^{m \times n}$

C = Hadamard product of A and B, $C \in \mathbb{R}^{m \times n}$

$c_{i,j} = a_{i,j} b_{i,j}$

– Identity matrix.

Let $A, A^{-1} \in \mathbb{R}^{n \times n}$

A be non-singular.

A^{-1} is called the multiplicative inverse of A if and only if $A^{-1}A = AA^{-1} = I_n$.

- A square matrix that possesses an inverse is called non-singular / non-degenerate.
- A square matrix that does not possess an inverse is called singular / degenerate.
- The inverse matrix for a non-singular matrix is unique.
- A linear isomorphism can be represented with a non-singular matrix.