2.(2)

<u>Basis of U_1</u>

$$
\begin{array}{l}
\quad\ \ \underline{c_1}\quad \underline{c_2}\quad \underline{c_3} \\
\begin{bmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \\ -3 & 0 & -1 \\ 1 & -1 & 1 \end{bmatrix}
\begin{array}{l} \\ -R_1 \\ +3R_1 \\ -R_1 \end{array}
\rightsquigarrow
\begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 2 \\ 0 & 6 & -4 \\ 0 & -3 & 2 \end{bmatrix}
\begin{array}{l} \\ \cdot -\frac{1}{3} \\ +2R_2 \\ -R_2 \end{array}
\rightsquigarrow
\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\end{array}
$$

$\Rightarrow U_1 = \text{span}\left[\underline{c_1}, \underline{c_2}\right]$

<u>Basis of U_2</u>

$$
\begin{array}{l}
\quad\ \ \underline{c_4}\quad \underline{c_5}\quad \underline{c_6} \\
\begin{bmatrix} -1 & 2 & -3 \\ -2 & -2 & 6 \\ 2 & 0 & -2 \\ 1 & 0 & -1 \end{bmatrix}
\begin{array}{l} \text{swap with } R_4 \\ \\ \\ \\ \end{array}
\rightsquigarrow
\begin{bmatrix} 1 & 0 & -1 \\ -2 & -2 & 6 \\ 2 & 0 & -2 \\ -1 & 2 & -3 \end{bmatrix}
\begin{array}{l} \\ +2R_1 \\ -2R_1 \\ +R_1 \end{array}
\rightsquigarrow
\begin{bmatrix} 1 & 0 & -1 \\ 0 & -2 & 4 \\ 0 & 0 & 0 \\ 0 & 2 & -4 \end{bmatrix}
\begin{array}{l} \\ \cdot -\frac{1}{2} \\ \\ +R_2 \end{array}
\rightsquigarrow
\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\end{array}
$$

$\Rightarrow U_2 = \text{span}\left[\underline{c_4}, \underline{c_5}\right]$

<u>Basis of $(U_1 \cap U_2)$</u>

Let B be the basis of $(U_1 \cap U_2)$

$W = \text{span}(B)$

$\Rightarrow W$ satisfies $(W \subseteq U_1) \wedge (W \subseteq U_2)$

$$
\begin{array}{l}
\quad\ \ \underline{c_1}\quad \underline{c_2}\quad \underline{c_4}\quad \underline{c_5} \\
\begin{bmatrix} 1 & 2 & -1 & 2 \\ 1 & -1 & -2 & -2 \\ -3 & 0 & 2 & 0 \\ 1 & -1 & 1 & 0 \end{bmatrix}
\begin{array}{l} \\ -R_1 \\ +3R_1 \\ -R_1 \end{array}
\rightsquigarrow
\begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & -3 & -1 & -4 \\ 0 & 6 & -1 & 6 \\ 0 & -3 & 2 & -2 \end{bmatrix}
\begin{array}{l} \\ \cdot -\frac{1}{3} \\ +2R_2 \\ -R_2 \end{array}
\rightsquigarrow
\begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & 1 & \frac{1}{3} & \frac{4}{3} \\ 0 & 0 & -3 & -2 \\ 0 & 0 & 3 & 2 \end{bmatrix}
\begin{array}{l} \\ \\ \cdot -\frac{1}{3} \\ +R_3 \end{array}
\end{array}
$$

$$
\rightsquigarrow
\begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & 1 & \frac{1}{3} & \frac{4}{3} \\ 0 & 0 & 1 & \frac{2}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}
$$

$\Rightarrow \dim(W) = 1$

There exists some $\underline{w} \in W \setminus \{\underline{0}\}$ such that

$\underline{w} = \alpha_1 \underline{c_1} + \alpha_2 \underline{c_2}$

$\underline{w} = \alpha_4 \underline{c_4} + \alpha_5 \underline{c_5}$

where $\alpha_1, \alpha_2, \alpha_4, \alpha_5 \in \mathbb{R} \setminus \{0\}$

$\Rightarrow \alpha_1 \underline{c_1} + \alpha_2 \underline{c_2} - \alpha_4 \underline{c_4} - \alpha_5 \underline{c_5} = \underline{0}$

α_5 is a free variable $\because \underline{c_5}$ is a linearly dependent vector.

Let $\alpha_5 = 1$

$\alpha_1 \underline{c_1} + \alpha_2 \underline{c_2} - \alpha_4 \underline{c_4} = \underline{c_5}$

$$
\begin{array}{c}
\begin{array}{ccc} \underline{c_1} & \underline{c_2} & \underline{-c_4} \end{array} \\
\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 1 & -1 & 2 & -2 \\ -3 & 0 & -2 & 0 \\ 1 & -1 & -1 & 0 \end{array}\right]
\end{array}
\begin{array}{l} \\ -R_1 \\ +3R_1 \\ -R_1 \end{array}
\rightsquigarrow
\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & -3 & 1 & -4 \\ 0 & 6 & 1 & 6 \\ 0 & -3 & -2 & -2 \end{array}\right]
\begin{array}{l} \\ \cdot -\frac{1}{3} \\ +2R_2 \\ -R_2 \end{array}
\rightsquigarrow
\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 1 & -\frac{1}{3} & \frac{4}{3} \\ 0 & 0 & 3 & -2 \\ 0 & 0 & -3 & 2 \end{array}\right]
\begin{array}{l} \\ \\ \cdot \frac{1}{3} \\ +R_3 \end{array}
\rightsquigarrow
\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 1 & -\frac{1}{3} & \frac{4}{3} \\ 0 & 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 0 & 0 \end{array}\right]
\begin{array}{l} -R_3 \\ +\frac{1}{3}R_3 \\ \\ \\ \end{array}
$$

$$
\rightsquigarrow
\left[\begin{array}{ccc|c} 1 & 2 & 0 & \frac{8}{3} \\ 0 & 1 & 0 & \frac{10}{9} \\ 0 & 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 0 & 0 \end{array}\right]
\begin{array}{l} -2R_2 \\ \\ \\ \\ \end{array}
\rightsquigarrow
\left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{4}{9} \\ 0 & 1 & 0 & \frac{10}{9} \\ 0 & 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 0 & 0 \end{array}\right]
$$

$\Rightarrow \alpha_1 = \frac{4}{9}, \ \alpha_2 = \frac{10}{9}, \ \alpha_4 = -\frac{2}{3}$

$$
\Rightarrow \underline{w} = \frac{4}{9}\begin{bmatrix} 1 \\ 1 \\ -3 \\ 1 \end{bmatrix} + \frac{10}{9}\begin{bmatrix} 2 \\ -1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{24}{9} \\ \frac{-6}{9} \\ -\frac{12}{9} \\ -\frac{6}{9} \end{bmatrix}
$$

$\Rightarrow [4, -1, -2, -1]^T$ is a basis of $(U_1 \cap U_2)$.