2.9) a)

$$A = \left\{ \underline{x} = \lambda \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \mu^3 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad \lambda, \mu \in \mathbb{R} \right\}$$

$$\Rightarrow A = \left\{ \underline{x} \in \mathbb{R}^3 : \underline{x} = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad c_1, c_2 \in \mathbb{R} \right\}$$

$A \subseteq \mathbb{R}^3$

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{matrix} \\ -R_1 \\ -R_1 \end{matrix} \rightsquigarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{matrix} \\ \\ +R_2 \end{matrix} \rightsquigarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow$ $[1, 1, 1]^T$ and $[0, 1, -1]^T$ are linearly independent.

$\Rightarrow$ $[1, 1, 1]^T$ and $[0, 1, -1]^T$ act as a basis of A as A is equivalent to the set of all linear combinations of these vectors.

NOTE: If a vector space has a basis, then the set of vectors contains $\underline{0}$, is closed under the outer operation (scalar multiplication), and is closed under the inner operation (vector addition).

Proof: Consider basis vectors $\underline{b}_1, \dots, \underline{b}_m$, the set of vectors $\mathcal{V} = \{ \underline{x} = \lambda_1 \underline{b}_1 + \dots + \lambda_m \underline{b}_m, \ \lambda_1, \dots, \lambda_m \in \mathbb{R} \}$, and the vector space $V = (\mathcal{V}, +, \cdot)$.

Zero vector:

$\lambda_1 = \dots = \lambda_m \Rightarrow \underline{x} = \underline{0} \Rightarrow \underline{0} \in \mathcal{V}$

Closure under outer operation:

Let $k \in \mathbb{R}$

Consider $k\underline{x}$

$k\underline{x} = (k\lambda_1)\underline{b}_1 + \dots + (k\lambda_m)\underline{b}_m, \qquad k\lambda_1, \dots, k\lambda_m \in \mathbb{R}$

$\Rightarrow k\underline{x} \in \mathcal{V}$

Closure under inner operation:

Let $\underline{x}_1 = \mu_1 \underline{b}_1 + \dots + \mu_m \underline{b}_m$

$\underline{x}_2 = \varphi_1 \underline{b}_1 + \dots + \varphi_m \underline{b}_m$

Consider $\underline{x}_1 + \underline{x}_2$

$\underline{x}_1 + \underline{x}_2 = (\mu_1 + \varphi_1)\underline{b}_1 + \dots + (\mu_m + \varphi_m)\underline{b}_m, \qquad (\mu_1 + \varphi_1), \dots, (\mu_m + \varphi_m) \in \mathbb{R}$

$\Rightarrow \underline{x}_1 + \underline{x}_2 \in V$

$\Rightarrow$ A is a subspace of $\mathbb{R}^3$.

2.9) b)

$$B = \left\{ \underline{x} = \lambda^2 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \lambda \in \mathbb{R} \right\}$$

$$= \left\{ \underline{x} \in \mathbb{R}^3 : \underline{x} = c \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad c \in \mathbb{R}, c \geq 0 \right\}$$

$B \subseteq \mathbb{R}^3$

<u>Zero vector</u>

$\underline{0} \in B$

Example: $\lambda = 0 \Rightarrow \underline{x} = \underline{0}$

<u>Closure under scalar multiplication</u>

Proof by counterexample

Let $k = -1 \in \mathbb{R}$

Consider $k\underline{x} = -\lambda^2 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad -\lambda^2 \notin \{c \in \mathbb{R} : c \geq 0\}$

$\Rightarrow k\underline{x} \notin B$

B is not closed under scalar multiplication.

$\Rightarrow B$ is not a subspace of $\mathbb{R}^3$.

2.9) c)

$C \subseteq \mathbb{R}^3$

$$C = \left\{ \underline{x} \in \mathbb{R}^3 : \underline{x} = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix}, \quad \xi_1 - 2\xi_2 + 3\xi_3 = \gamma \right\}$$

<u>Zero vector</u>

$\underline{0} \in C \Rightarrow \gamma = (0) - 2(0) + 3(0) = 0$

$\Rightarrow C$ is not a subspace of $\mathbb{R}^3$ if $\gamma \neq 0$.

<u>For $\gamma = 0$</u>

<u>closure under scalar multiplication</u>

Let $k \in \mathbb{R}$.

Consider $k\underline{x} = k \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} k\xi_1 \\ k\xi_2 \\ k\xi_3 \end{bmatrix}$

$k\xi_1 - 2k\xi_2 + 3k\xi_3 = k(\xi_1 - 2\xi_2 + 3\xi_3) = k(0) = 0$

$\Rightarrow k\underline{x} \in C$

C is closed under scalar multiplication.

<u>closure under vector addition</u>

Let $\underline{x}_1 = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} \in C$ and $\underline{x}_2 = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix} \in C$.

Consider $\underline{x}_1 + \underline{x}_2 = \begin{bmatrix} \lambda_1 + \mu_1 \\ \lambda_2 + \mu_2 \\ \lambda_3 + \mu_3 \end{bmatrix}$

$(\lambda_1 + \mu_1) - 2(\lambda_2 + \mu_2) + 3(\lambda_3 + \mu_3) = (\lambda_1 - 2\lambda_2 + 3\lambda_3) + (\mu_1 - 2\mu_2 + 3\mu_3) = (0) + (0) = 0$

$\Rightarrow \underline{x}_1 + \underline{x}_2 \in C$

C is closed under vector addition.

$\Rightarrow C$ is a subspace of $\mathbb{R}^3$ for $\gamma = 0$.

2.9) d)

$D \subseteq \mathbb{R}^3$

<u>Zero vector</u>

Consider $(0,0,0)$

$\Rightarrow \xi_1 = \xi_2 = 0 \in \mathbb{R}, \quad \xi_3 = 0 \in \mathbb{Z}$

so $\underline{0} \in D$

<u>Closure of D under scalar multiplication</u>

Proof by counterexample.

Consider $(0,1,0) \in D, \quad \frac{1}{2} \in \mathbb{R}$

$\frac{1}{2}(0,1,0) = (0, \frac{1}{2}, 0) \notin D \quad \because \xi_2 = \frac{1}{2} \notin \mathbb{Z}.$

$\Rightarrow D$ is not closed under scalar multiplication.