Mathematics for Machine Learning
Chapter 2: Linear Algebra
Exercises

2-1) a)

- Closure

$a * b = ab + a + b \in \mathbb{R}$ $\because$ closure of $\mathbb{R}$ under $+$ and $\cdot$

Assume $a * b = -1$
$\Rightarrow ab + a + b = -1$
$\Rightarrow ab + a + b + 1 = 0$
$\Rightarrow (a+1)(b+1) = 0$
$\Rightarrow a = -1$ or $b = -1$
This contradicts $a, b \in \mathbb{R} \setminus \{-1\}$ $\therefore$ $a * b \neq -1$.

$\Rightarrow a * b \in \mathbb{R} \setminus \{-1\}$
$\mathbb{R} \setminus \{-1\}$ is closed under $*$.

- Commutativity

$a * b = ab + a + b$

$b * a = ba + b + a$
$\quad = ab + a + b$

$\Rightarrow a * b \equiv b * a$
$\mathbb{R} \setminus \{-1\}$ is commutative under $*$.

- Associativity

$$
\begin{aligned}
(a * b) * c &= (ab + a + b) * c \\
&= (ab + a + b)c + (ab + a + b) + c \\
&= abc + ac + bc + ab + a + b + c \\
&= abc + ab + ac + bc + a + b + c
\end{aligned}
$$

$$
\begin{aligned}
a * (b * c) &= a * (bc + b + c) \\
&= a(bc + b + c) + a + (bc + b + c) \\
&= abc + ab + ac + a + bc + b + c \\
&= abc + ab + ac + bc + a + b + c
\end{aligned}
$$

$\Rightarrow (a * b) * c \equiv a * (b * c)$
$\mathbb{R} \setminus \{-1\}$ is associative under $*$.

- Neutral element

$a * e = e * a = a$ where $e \in \mathbb{R} \setminus \{-1\}$
$\Rightarrow ae + a + e = a$
$\Rightarrow ae + e = 0$
$\Rightarrow e(a+1) = 0$
$\Rightarrow e = 0 \in \mathbb{R} \setminus \{-1\}$
Ignore $a = -1$ $\because$ $a \in \mathbb{R} \setminus \{-1\}$

$\Rightarrow$ The neutral element exists.

- Inverse element

$a * a^{-1} = a^{-1} * a = e \quad$ where $a^{-1} \in \mathbb{R} \setminus \{-1\}$

$\Rightarrow a a^{-1} + a + a^{-1} = 0$

$\Rightarrow a^{-1}(a+1) + a = 0$

$\Rightarrow a^{-1} = \dfrac{-a}{a+1} \in \mathbb{R} \setminus \{-1\}$

$\Rightarrow$ The inverse element exists for all $a \in \mathbb{R} \setminus \{-1\}$.

- $(\mathbb{R} \setminus \{-1\}, *)$ is an Abelian group. $\square$

2.1) b)

$(3 * x) * x = 15$

$\Rightarrow (3x + 3 + x) * x = 15$

$\Rightarrow (4x + 3) * x = 15$

$\Rightarrow (4x^2 + 3x) + (4x + 3) + x = 15$

$\Rightarrow 4x^2 + 8x - 12 = 0$

$\Rightarrow x^2 + 2x - 3 = 0$

$\Rightarrow (x+3)(x-1) = 0$

$\Rightarrow x = -3$ or $x = 1$. $\square$